MATH 2330: Multivariable Calculus

Chapter 6 - Part 3

6.3 - The Fundamental Theorem of Calculus for Line Integrals (FTCFLI), Part 2:

FTCFLI:

If f is differentiable and $\vec{\nabla} f$ is continuous on a curve C parametrized as $\vec{r}(t)$ for $a \leq t \leq b$, then

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

This means that if $\vec{F} = \vec{\nabla} f$ for some potential function f then the line integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

Properties of Conservative Vector Fields:

$\vec{F} = \langle P(x,y), Q(x,y) \rangle$ is conservative on a region R in the plane (with no holes) if:

A: $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed path C

B: $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints, not on the path itself

C: $\vec{F} = \vec{\nabla} f$ for some potential function f

D: The components satisfy the **component test**: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Problems for Group Work:

Problem 1: The following vector fields are conservative. For each one, find the potential function, f such that $\vec{F} = \vec{\nabla} f$:

a) $\vec{F} = \langle 3xy^2, 3x^2y \rangle$

b) $\vec{F} = y\sin(xy)\,\hat{\mathbf{\imath}} + x\sin(xy)\,\hat{\mathbf{\jmath}}$

c) $\vec{F} = (2x+y)\,\hat{\mathbf{\imath}} + (x+3y^2)\,\hat{\mathbf{\jmath}}$

d) $\vec{F} = \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle$

Problem 2: Which, if any, of the following vector fields are conservative?

a) $\vec{F} = xy\,\hat{\mathbf{\imath}} - 2xy\,\hat{\mathbf{\jmath}}$

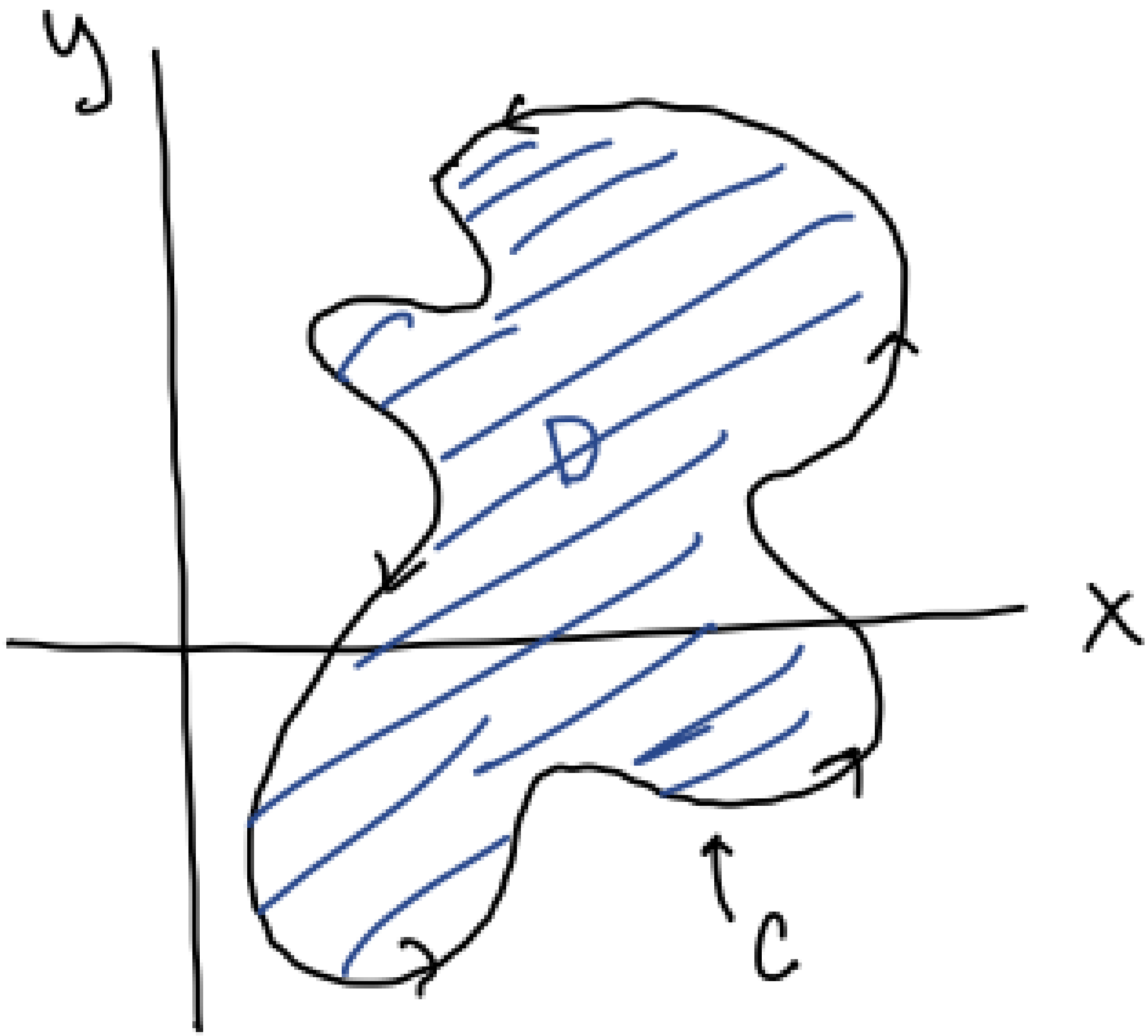
b) $\vec{F} = (2xy + \cos(2y))\,\hat{\mathbf{\imath}} + (x^2 - 2x\sin(2y))\,\hat{\mathbf{\jmath}}$

c) $\vec{F} = (3x - 5y)\,\hat{\mathbf{\imath}} + (7y - 5x)\,\hat{\mathbf{\jmath}}$

Problem 3: Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ for each of the vector fields from Problem 2, for the curve C: $x^2 + y^2 = 1$, oriented counter-clockwise.

6.4 - Green's Theorem:

Green's Theorem:



For any vector field $\vec{F} = \langle P, Q \rangle$ that has continuous first partials:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\ dx + Q\ dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA,$$

where C is a **simple closed curve** that is **piecewise smooth** and **positively oriented** that encloses the region D.

- **simple closed curve:** only intersects itself at the start/end points
- **piecewise smooth:** can be broken up into pieces without corners
- **positively oriented:** draw the arrows on C sot that the enclosed region D is always on the left as you go around C.

Area Formulas:

Green's Theorem can be used to calculate the area of the region D by choosing P and Q so that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$. The most popular options are shown below:

$$\text{Area}(D) = \oint_C x\ dy = -\oint_C y\ dx = \frac{1}{2}\oint x\ dy - y\ dx.$$

Use whichever formula makes the problem the easiest!

Examples:

Example 1: **Making the tedious...less tedious!**
Evaluate $\oint_C x^2 y\ dx + (x^3 + 2xy^2)\ dy$, where C encloses the region between the unit circle and the circle with radius 2 in quadrants IV and I.

Example 2: **Making the Impossible Possible!**
Evaluate the line integral

$$\oint_C \left(2y + \sqrt{1+x^5}\right)\ dx + \left(5x - e^{y^2}\right)\ dy,$$

where C is the "unit square" connecting the points $(-\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$, and $(\frac{1}{2}, -\frac{1}{2})$

Example 3: **Area of the Astroid!**
Use Green's Theorem to find the area enclosed by the astroid: $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$, $0 \le t \le 2\pi$.