MATH 2330: Multivariable Calculus

Chapter 6 - Part 4: Line Integral Strategy

Types of Line Integrals:

Line Integral of a Scalar Function with respect to Arclength:

$$\int_C f(x,y)\ ds$$

Interpretation: Net area under $z = f(x,y)$ above the curve C in the $xy-$plane.

Evaluate by:

1. Sketch & parametrize C
2. Replace $x = x(t), \quad y = y(t), \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
3. Integrate wrt t from a to b.

Line Integrals with respect to x and/or y:

$$\int_C P(x,y)\ dx + Q(x,y)\ dy$$

Evaluate by:

1. Sketch & parametrize C
2. Replace $x = x(t), \quad y = y(t), \quad dx = x'(t)\ dt, \quad dy = y'(t)\ dt$
3. Integrate wrt t from a to b.

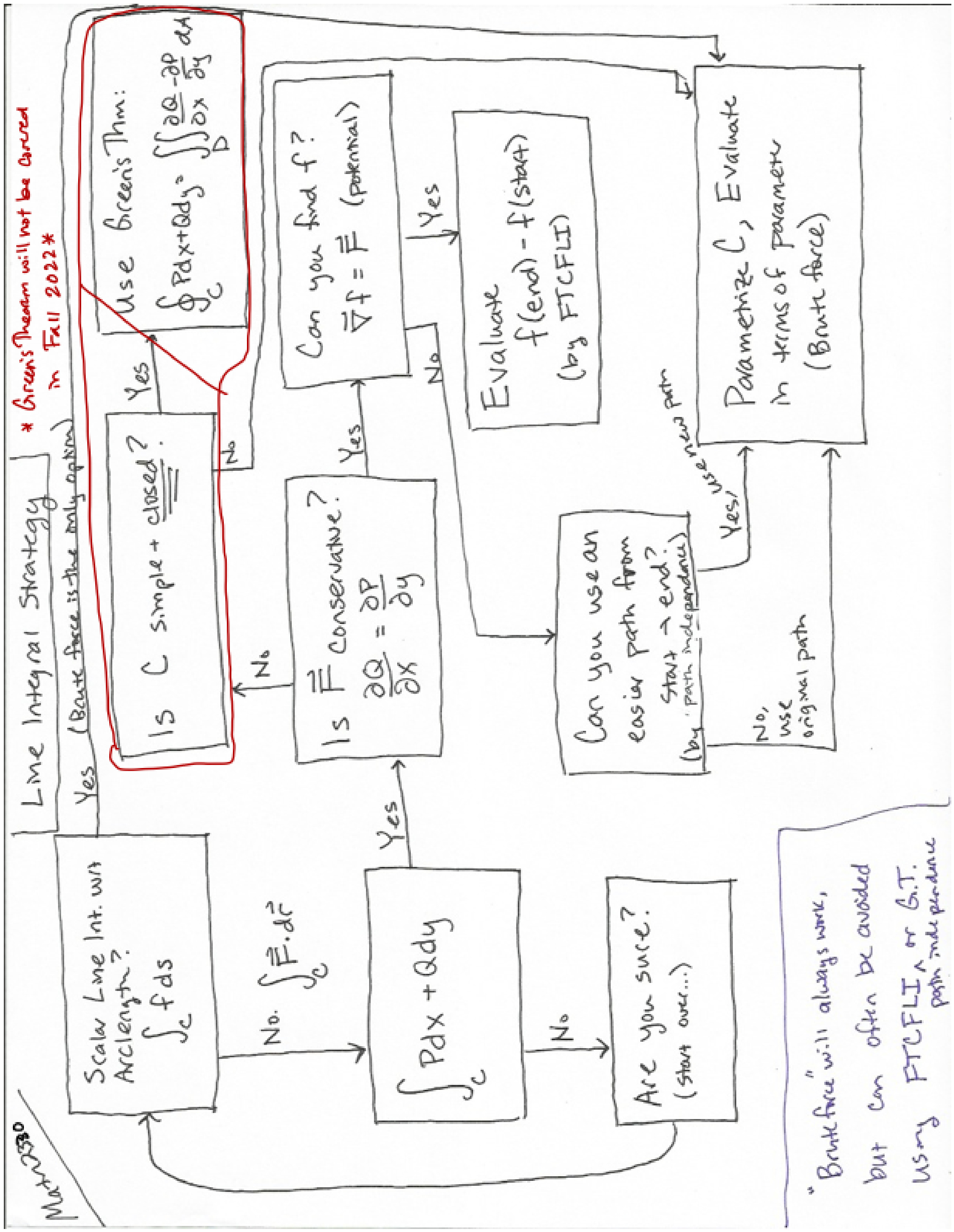
Line Integrals over Vector Fields:

$$\int_C \vec{F} \cdot d\vec{r}$$

Interpretation: Work done by a force $\vec{F}$ to move a particle along C from the starting point to the ending point.

Evaluate by:

1. Sketch & parametrize C
2. Replace with $\int_a^b \vec{F(t)} \cdot \vec{r}'(t)\ dt$


Math 2330
Line Integral Strategy
(Brute force is the only option)
Scalar Line Int. wrt Arclength? $\int_C f\,ds$
Yes
No. $\int_C \vec{F}\cdot d\vec{r}$
$\int_C P\,dx + Q\,dy$
No
Are you sure? (start over...)
Yes
Is $\vec{F}$ conservative? $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$
No
Is C simple + closed?
Yes
Use Green's Thm: $\oint_C P\,dx+Q\,dy = \iint_D \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dA$
No
* Green's Theorem will not be covered in Fall 2022 *
Yes
Can you find f? $\vec{\nabla} f = \vec{F}$ (potential)
Yes
Evaluate f(end) - f(start) (by FTCFLI)
No
Can you use an easier path from start → end? (by path independence)
Yes, use new path
No, use original path
Parametrize C, Evaluate in terms of parameter (Brute force)
"Brute force" will always work, but can often be avoided using FTCFLI or G.T. path independence

Miscellaneous Line Integral Problems

Problem 1. $\displaystyle\int_C \frac{x\ dx + y\ dy}{\sqrt{x^2+y^2}}$, $\quad C$: semi-circular arc of $x^2+y^2=4$ from $(2,0)$ to $(-2,0)$.

Problem 2. $\displaystyle\int_C x + 2y\ ds$, $\quad C$: parametrized by $\vec{r}(t) = \langle 2-3t, 4t-1\rangle$, $\quad 0 \le t \le 2$.

Problem 3. $\displaystyle\oint_C \left(x^2-y\right)\ dx + \left(y^2-x\right)\ dy$, $\quad C$: circle of radius 5 centered at the origin, counterclockwise.

Problem 4. $\displaystyle\int_C 1 + \frac{y}{3}\ ds$, $\quad C$: parametrized by $\vec{r}(t) = \left\langle 30\cos^3 t, 30\sin^3 t\right\rangle$, $\quad 0 \le t \le \pi/2$.

Answers: **Problem 1:** 0, **Problem 2:** 50, **Problem 3:** 0, **Problem 4:** 225