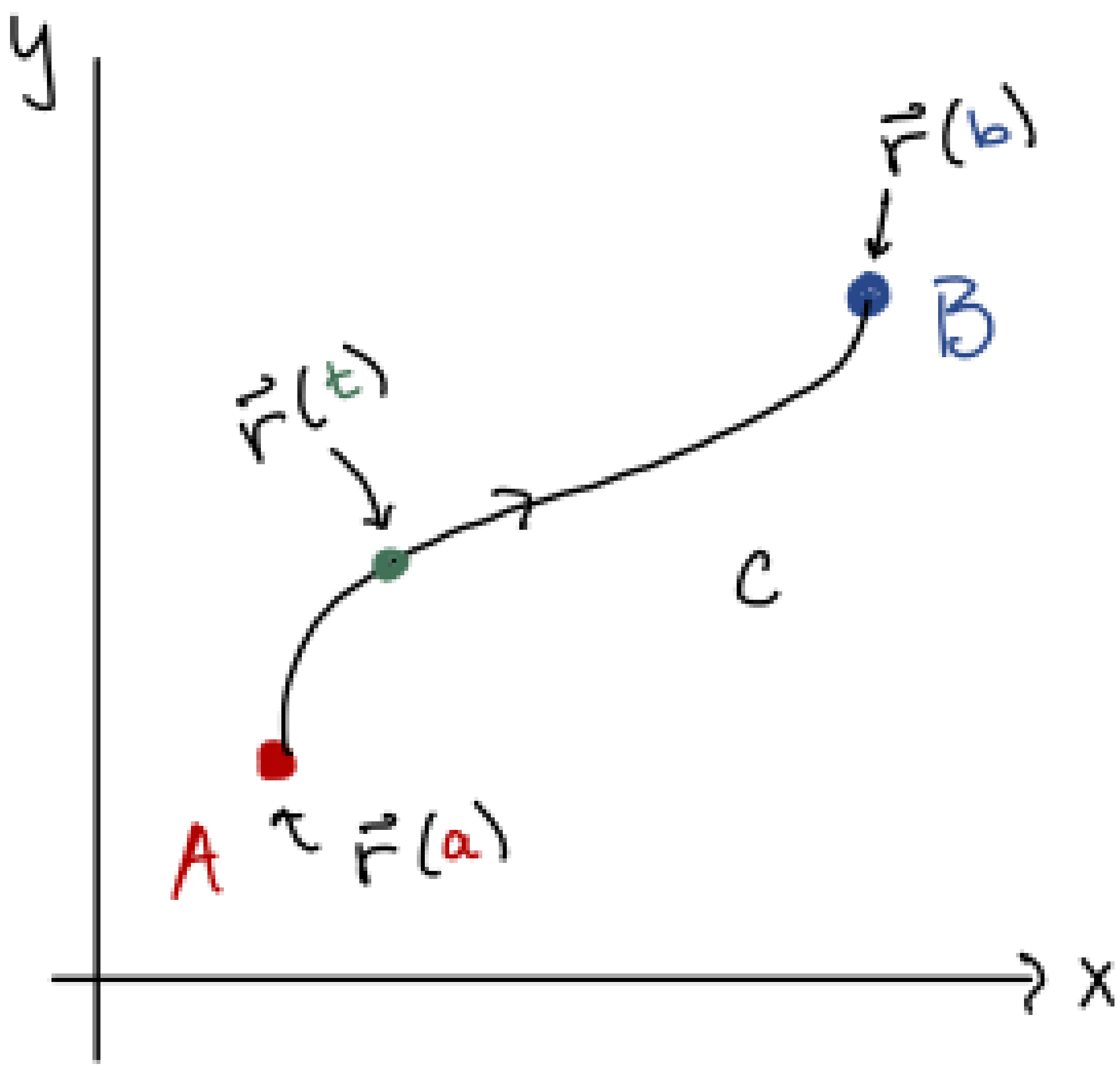
6.2 - Part 1: Line Integrals wrt Arclength

Parametric Curves Review (See OpenStax Calculus, Volume 3: 1.1-1.2, 3.1, 3.3):



A curve C can be parametrized using a **vector function**:

$$\vec{r}(t) = \langle x(t), y(t) \rangle, \qquad a \le t \le b$$

Tangent Vector:

$$\vec{r}\,'(t) = \langle x'(t), y'(t)) \rangle$$

Unit Tangent Vector:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Arclength:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b |\vec{r}\,'(t)| dt = \int_C \; ds$$

Line Integral of a Scalar Function with respect to Arclength:

$$\int_C f(x, y) \; ds$$

denotes an integral whose domain of integration is a curve C from some starting point A to an ending point B. To evaluate, we want to rewrite the integral in terms of the parameter t, using the distance element $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$:

$$\int_C f(x, y) \; ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example:

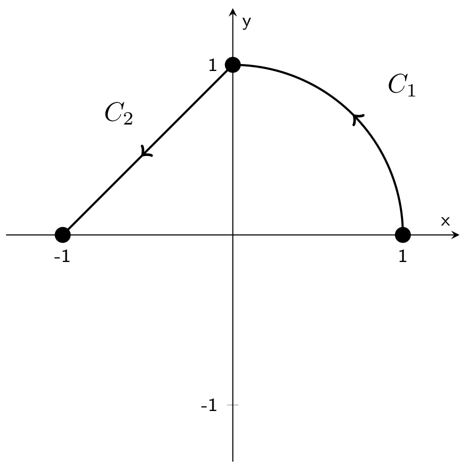
Example 1: Evaluate the line integral shown below, where C : left half of the circle $x^2+y^2 = 16$, counterclockwise.

$$\int_C (x^2 + y^2) \; ds$$

Followup Discussion: Does this represent something that we can visualize?

Group Work:

Evaluate the line integral $\int_C f(x,y)\ ds$ for $f(x,y) = x + y$ for the curve $C = C_1 + C_2$ shown below, which is made up of a circular arc and a line segment.



Note:

$$\int_C f(x,y)\ ds = \int_{C_1} f(x,y)\ ds + \int_{C_2} f(x,y)\ ds$$

Hint: Start by parametrizing the curves C_1 and C_2.

Line Integrals with respect to x or y:

Given a curve C with parametrization $\vec{r}(t) = \langle x(t), y(t) \rangle, \qquad a \le t \le b$

$$\int_C f(x,y)\ dx = \int_a^b f(x(t), y(t))\ x'(t)\ dt$$

$$\int_C f(x,y)\ dy = \int_a^b f(x(t), y(t))\ y'(t)\ dt$$

As it turns out, these often occur together, so we develop a shorthand:

$$\int_C P(x,y)\ dx + \int_C Q(x,y)\ dy = \int_C P(x,y)\ dx + Q(x,y)\ dy$$

Example:

Example 2: Integrate $\int_C y^2\ dx + x\ dy$ for C : along the curve $x = 4 - y^2$ from $(3, -1)$ to $(0, 2)$.

6.1 - Vector Fields:

Vector Field Definitions & Terminology:

A **vector field**, $\vec{F}$ is a function that assigns a *vector* to each point in the domain:

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

P, Q are called the **component functions**, and are *scalar* functions.

Recall from section 4.6, we defined the **gradient** $\vec{\nabla} f = \langle f_x, f_y \rangle$. The vector field defined by $\vec{\nabla} f$ is called the **Gradient Field:**

$$\vec{F} = \vec{\nabla} f.$$

A vector field $\vec{F}$ is called **conservative** if

$$\vec{F} = \vec{\nabla} f.$$

for some scalar function f, which is called the **potential function** for $\vec{F}$.