MATH 2330: Multivariable Calculus

Section 5.1: Double Integrals over Rectangular Regions

Double Integral Definition:

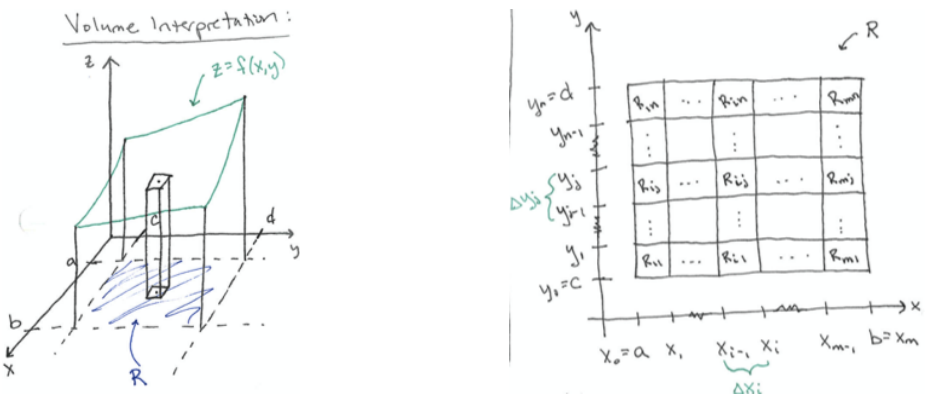

Figure 1: Double Integral Definition Diagrams

A **double integral over a rectangle** $R : [a, b] \times [c, d]$ is defined as:

$$\iint_R f(x,y)\ dA = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A_{ij}$$

Key Idea:

- Split the rectangle into mn sub-rectangles of area $\Delta A_{ij} = \Delta x_i \Delta y_j$
- Pick a sample point from each sub-rectangle: $\left(x_{ij}^*, y_{ij}^*\right)$
- Evaluate the function at each sample point: $f\left(x_{ij}^*, y_{ij}^*\right)$
- Calculate the volume of a column: $V_{ij} = f\left(x_{ij}^*, y_{ij}^*\right) \Delta A_{ij}$
- Add up the volumes of the columns to estimate the total volume: $V \approx V_{11} + V_{12} + \ldots + V_{mn}$

Fubini's Theorem:

If $f(x,y)$ is continuous on $R = \{(x,y) | a \le x \le b, c \le y \le d\}$ then:

$$\iint_R f(x,y)\ dA = \int_c^d \int_a^b f(x,y)\ dx\ dy = \int_a^b \int_c^d f(x,y)\ dy\ dx$$

Key Ideas:

- Order of Operations: work from the inside out (like parentheses!)

- We can swap the order of integration in the iterated integrals
- The final result should be a number, no x's or y's should remain
- Partial Integration:
  - $\int_a^b f(x,y)\,dx$: hold y fixed and integrate wrt x.
  - $\int_c^d f(x,y)\,dy$: hold x fixed and integrate wrt y.

Partial Integration - Visual POV:

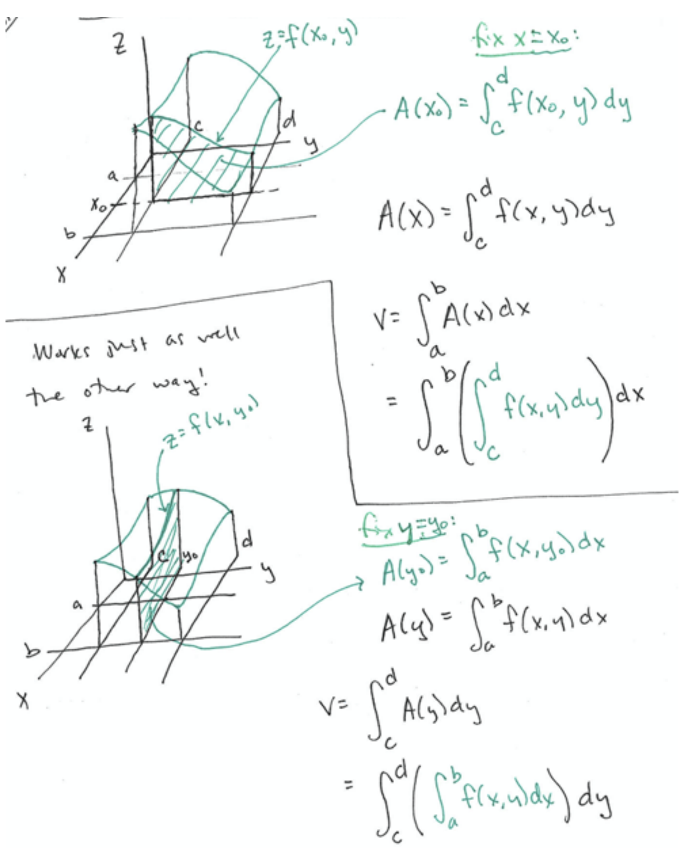

Figure 2: Partial Integration - Visual POV Diagrams

Examples:

Example 0: Evaluate $$\iint_R 3+4x\;dA, \qquad R=[0,1]\times[0,2]$$

Example 1: Evaluate $$\iint_R x^3+y^5\;dA, \qquad R=[-2,2]\times[0,2]$$

Example 2: Evaluate $\displaystyle\int_0^1 \int_{\frac{1}{2}}^2 ye^{xy}\ dy\ dx$

Example 3: Evaluate $\displaystyle\iint_R \frac{\ln y}{xy}\ dA,\qquad R = [1,3]\times[1,5]$

Example 4: Evaluate $\displaystyle\iint_R \sqrt{1-y^2}\ dA,\qquad R = [0,2]\times[-1,1]$

Example 5: True or False: $\displaystyle\iint_R \cos\left(2\pi(y^2+x)\right)\ dA = 0,\qquad R = [0,1]\times[0,1]$