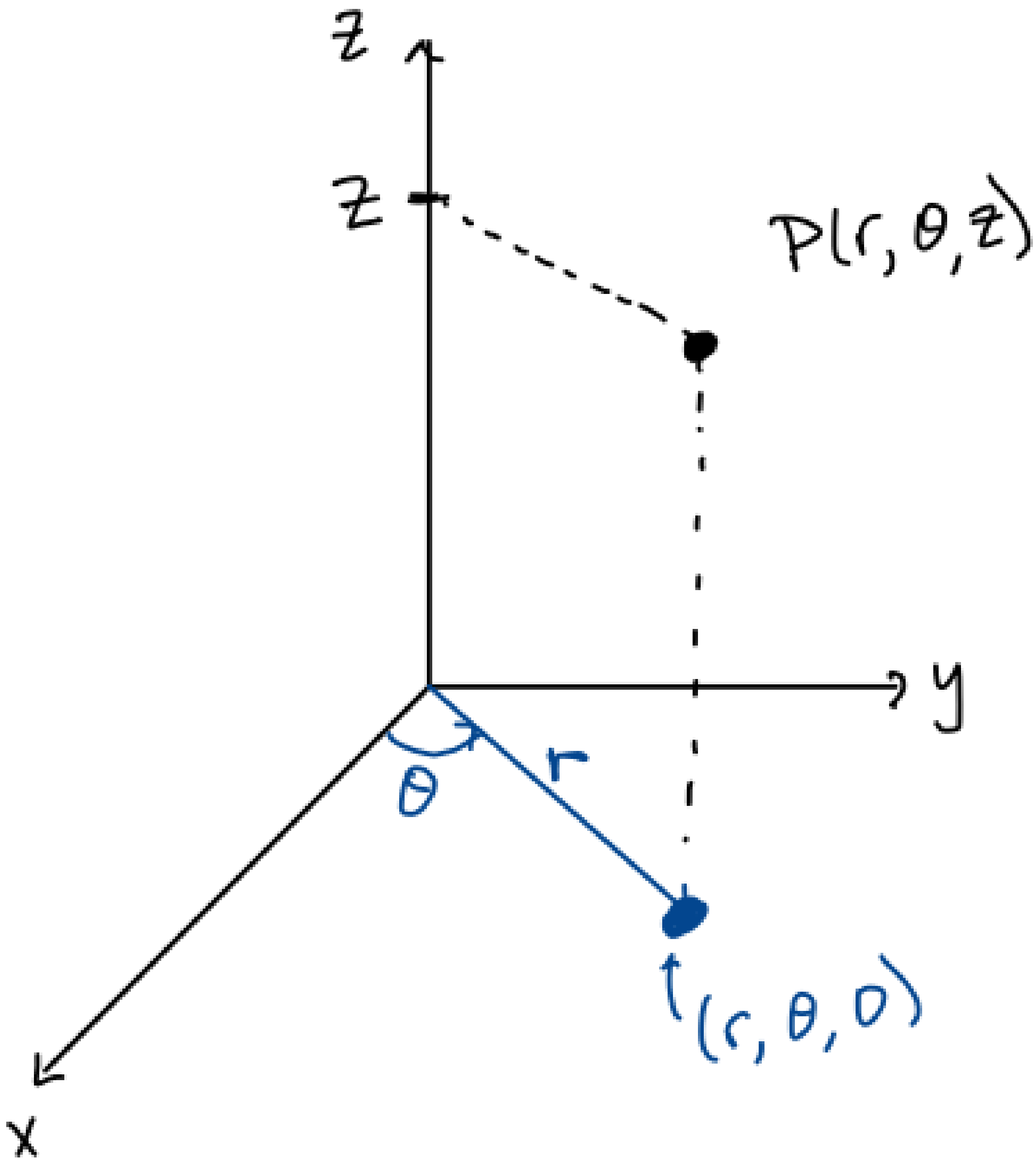
Cylindrical Coordinates



Equations:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \qquad x^2 + y^2 = r^2$$

Triple Integral Setup:

Integrate with respect to z first, and be sure to replace all x's and y's.

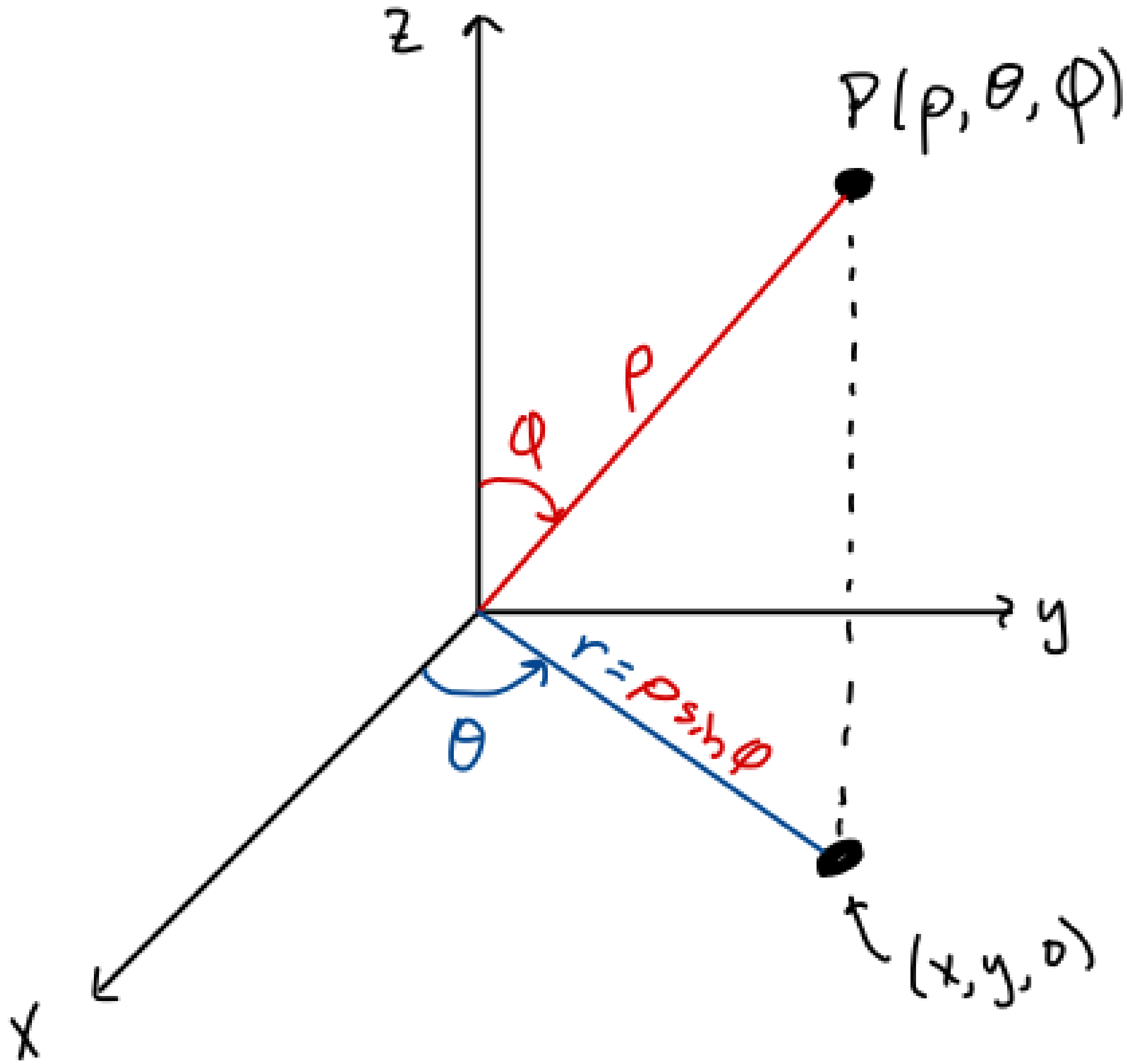
$$\iiint_E f(x,y,z)\ dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz \right) dA$$

$$= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f\left(r\cos\theta, r\sin\theta, z\right) r dz dr d\theta$$

Volume Element: $dV = r\ dz\ dr\ d\theta$

Example 1: Revisit finding the volume of the "Ice Cream Cone" solid region bounded above by $x^2 + y^2 + z^2 = 4$ and below by $x^2 + y^2 = z^2$, where $z \geq 0$.

Spherical Coordinates



Variables:

- ρ: distance from the origin $\rho^2 = x^2 + y^2 + z^2$, $\rho \geq 0$
- θ: same angle as in polar/cylindrical coordinates, $0 \leq \theta \leq 2\pi$
- ϕ: angle measured down from the positive $z-$axis, $0 \leq \phi \leq \pi$

Equations:

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho\cos\phi, \qquad x^2 + y^2 + z^2 = \rho^2$$

Triple Integral Setup:

For a solid region E with bounds: $\rho:\ a$ to b, $\theta:\ \alpha$ to β, $\phi:\ c$ to d

$$\iiint_E f(x,y,z)\ dV = \int_c^d \int_\alpha^\beta \int_a^b f\left(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\right) \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

Volume Element: $dV = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$

Example 1: How would you describe a sphere of radius R using spherical coordinates? Top only? Bottom only? First octant only?

Example 2: Find the volume of the region described by $1 \leq x^2 + y^2 + z^2 \leq 9$ using a triple integral.

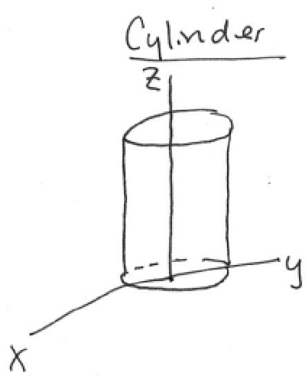
Example 3: Revisit finding the volume of the "Ice Cream Cone" solid region bounded above by $x^2+y^2+z^2 = 4$ and below by $x^2 + y^2 = z^2$, where $z \geq 0$.

Example 4: What does the following quantity represent? Calculate it for the "Ice Cream Cone" solid region described above.

$$\frac{\iiint_E z\ dV}{\iiint_E dV}$$

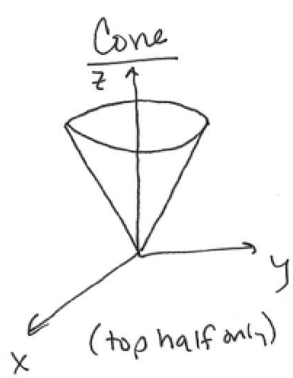
Equations of Some "Standard" Surfaces:

Cylinder of radius a:



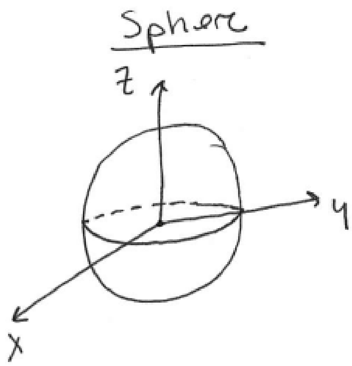
- **Rectangular Coordinates:** $x^2 + y^2 = a^2$
- **Cylindrical Coordinates:** $r = a$
- **Spherical Coordidnates:** (Not worth it!)

Cone (top half):



- **Rectangular Coordinates:** $z = \sqrt{x^2 + y^2}$
- **Cylindrical Coordinates:** $r = z$
- **Spherical Coordidnates:** $\phi = \frac{\pi}{4}$

Sphere of radius a:



- **Rectangular Coordinates:** $x^2 + y^2 + z^2 = a^2$
- **Cylindrical Coordinates:** $r^2 + z^2 = a^2$
- **Spherical Coordidnates:** $\rho = a$