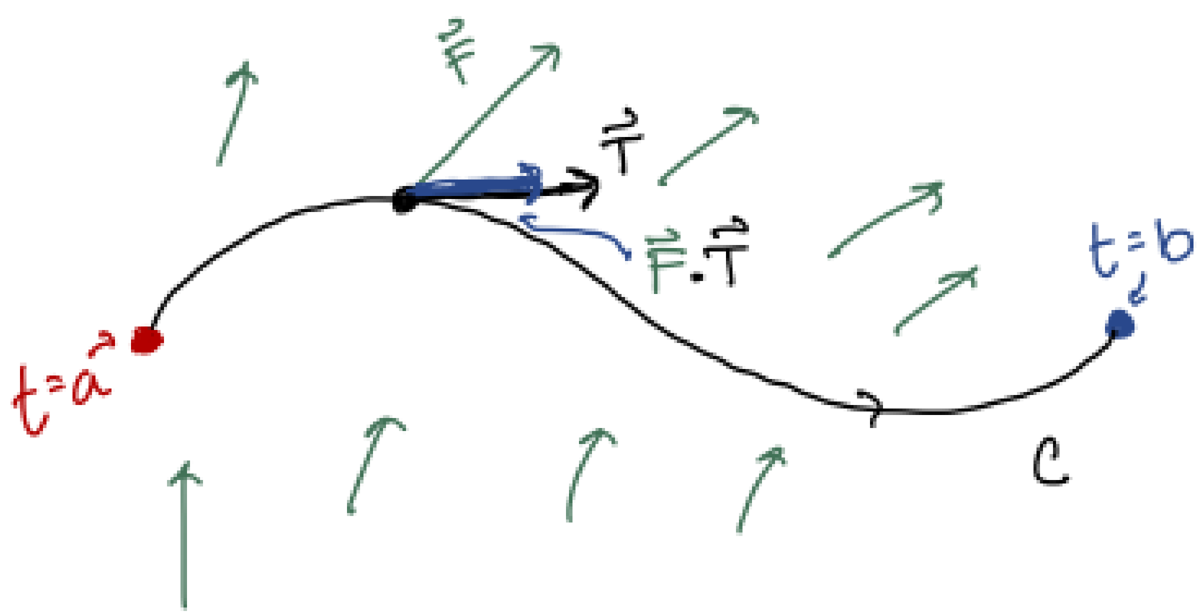
6.2 - Part 2: Line Integrals over Vector Fields



A curve C can be parametrized using a **vector function**:

$$\vec{r}(t) = \langle x(t), y(t) \rangle, \qquad a \le t \le b$$

Tangent Vector:

$$\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$$

Unit Tangent Vector:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Work:

$$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos\theta = F_D |\vec{D}|$$

Line Integrals over Vector Fields: Work

This type of line integral represents the work done by a force $\vec{F}$ to move a particle along the curve C from $t = a$ to $t = b$:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_C \vec{F} \cdot \vec{T}\; ds$$

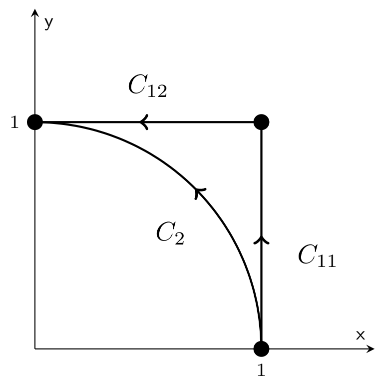
If $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ and $d\vec{r} = \langle dx, dy\rangle$, then

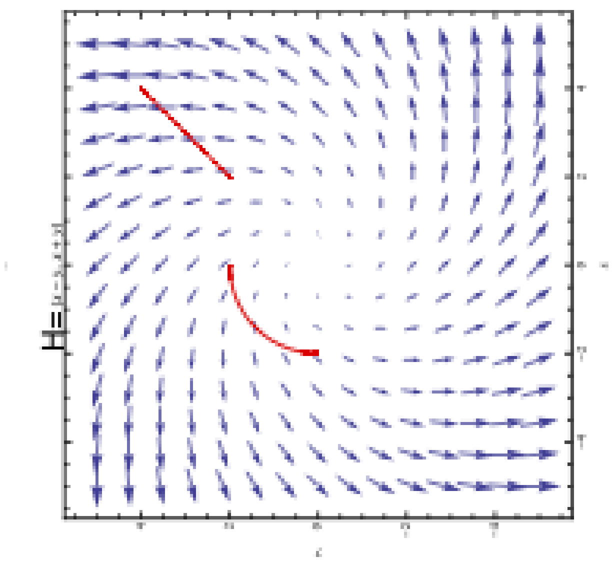
$$\int_C \vec{F}\cdot d\vec{r} = \int_C P(x,y)\ dx + Q(x,y)\ dy = \int_a^b P(x(t),y(t))\ x'(t) + Q(x(t),y(t))\ y'(t)\ dt$$

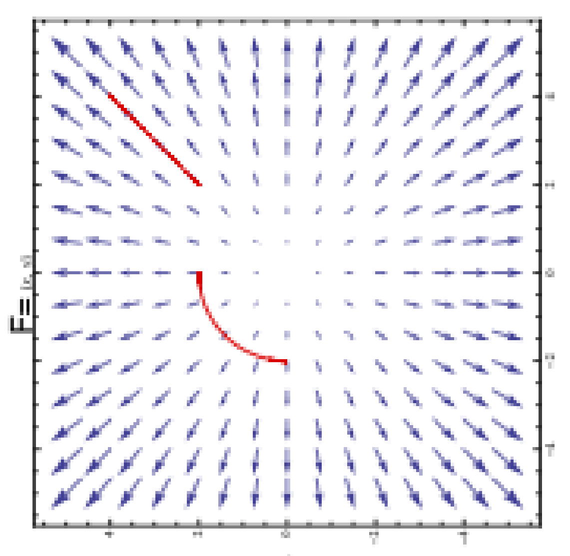
Discussion: Geometric Implications of $W = \displaystyle\int_C \vec{F}\cdot d\vec{r}$ *(See diagrams on the next page)*

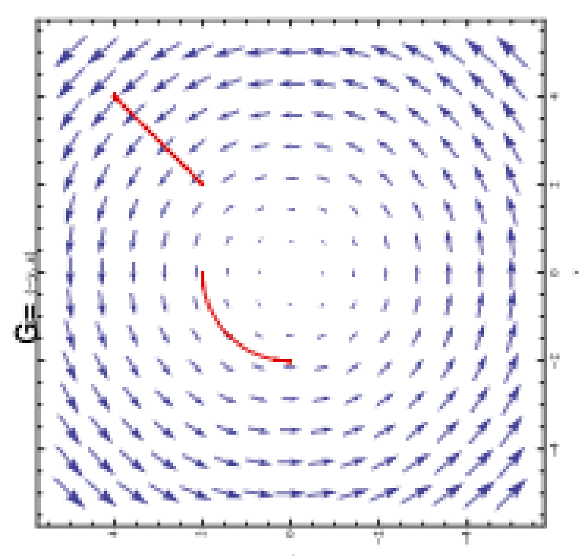
Example:

Example 1: We will integrate $\vec{F} = \langle x^2, -xy\rangle$ and $\vec{G} = \langle y, x+2y\rangle$ along the paths shown below, which both start at $(1,0)$ and end at $(0,1)$.
Note that $C_1 = C_{11} + C_{12}$ is made up of two line segments, and C_2 is an arc of a circle.




H=


F=


G=

6.3 - The Fundamental Theorem of Calculus for Line Integrals (FTCFLI)

Path Dependence Terminology:

Consider a line integral over a vector field from a starting point A to and ending point B:

$$\int_C \vec{F} \cdot d\vec{r}$$

If the value of the line integral changes depending on what path is taken from A to B, then we say that it is **path dependent**.
If the value of the line integral is the same for any path between A and B, then we say that it is **path independent**.

0FTCFLI:

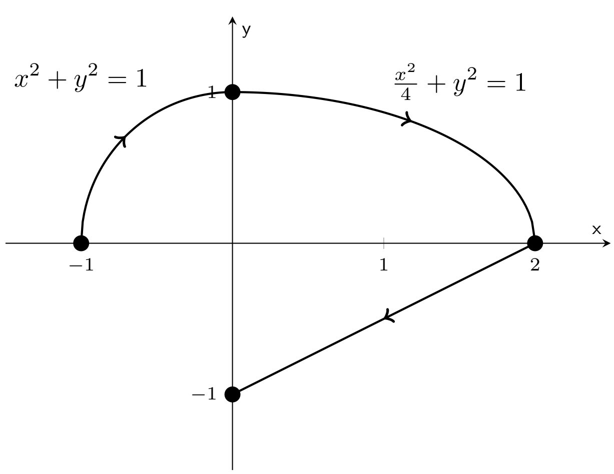
If f is differentiable and $\vec{\nabla} f$ is continuous on a curve C parametrized as $\vec{r}(t)$ for $a \leq t \leq b$, then

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

This means that if $\vec{F} = \vec{\nabla} f$ for some potential function f then the line integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

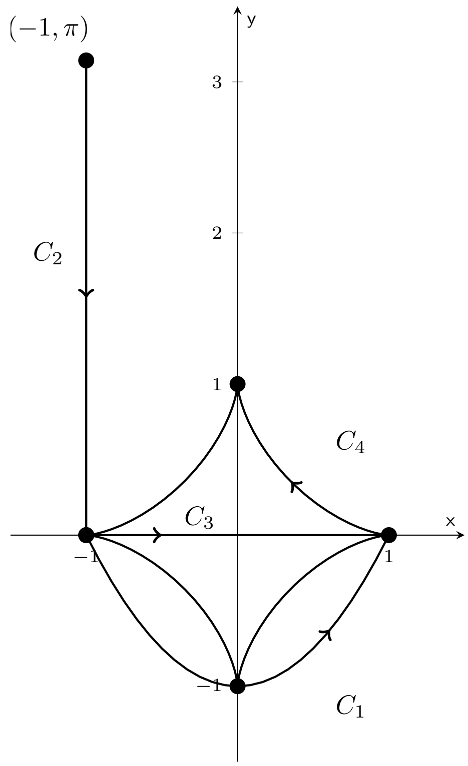
Example:

Example 2: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle 2xy, x^2 \rangle$ and C shown below:



Group Work:

Evaluate the line integral $\int_C 2x\cos y\ dx - x^2 \sin y\ dy$ for the curves described below.



(a) C_1: parabola $y = x^2 - 1$ from $(-1,0)$ to $(1,0)$

(b) C_2: line segment from $(-1,\pi)$ to $(-1,0)$

(c) C_3: x axis from $(-1,0)$ to $(1,0)$

(d) C_4: the astroid $\vec{r}(t) = \left\langle \cos^3 t, \sin^3 t \right\rangle$ for $0 \le t \le 2\pi$