Clarifying Clairaut's Theorem

Consider $f(x, y, z) = x^2 \cos\left(y^3 + z^2\right)$.

1. Why do we know that $f_{zyyxxx} = 0$ without doing any computation?

2. Do we also know, without doing any computation, that $f_{xyzzz} = 0$? Why or why not?

3. Suppose that $f_x = 3x + ay^2$, $f_y = bxy + 2y$, $f_y(1, 1) = 3$, and f has continuous mixed second partial derivatives f_{xy} and f_{yx}.

(a) Find values for a and b and thus equations for f_x and f_y. HINT What does Clairaut's Theorem say about the mixed partial derivatives of a function? When does the theorem apply?

(b) Can you find a function $F(x, y)$ such that $\dfrac{\partial F}{\partial x} = f_x$ in part (a)?

(c) Can you find a function $G(x, y) = F(x, y) + k(y)$ such that $\dfrac{\partial G}{\partial y} = f_y$ in part (a)? What is $k(y)$?

(d) What is $\dfrac{\partial G}{\partial x}$? Can you now find $f(x, y)$?