MATH 2330: Multivariable Calculus

4.4: Tangent Planes & Linear Approximation

Section 4.4 - Tangent Planes & Linear Approximation, Part 1:

Definitions & Terminology:

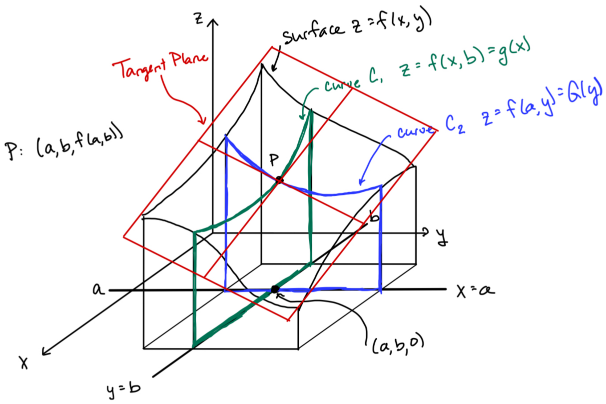


Figure 1: Tangent Plane Diagram

The **tangent plane** to the surface $z = f(x, y)$ at the point (a, b) is the plane defined by the tangent lines in the $x-$ and $y-$ directions at the point (a, b).

The tangent plane gives the most accurate planar approximation to the surface at the point (a, b).

Tangent Plane Equation:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Linearization of f at (a, b):

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Linear Approximation or Tangent Plane Approximation:

$$f(x, y) \approx L(x, y)$$

Example 1: Consider the function $f(x, y) = x^2y^3$.

(a) Find the equation of the tangent plane to $z = f(x, y)$ at the point $(3, 1, 9)$.

(b) Use Linear Approximation to estimate the value of $f(x, y)$ at the point $(2.5, 1.5)$.

(See Mathematica Demonstration!)

Section 4.4 - Tangent Planes & Linear Approximation, Part 2:

Definitions & Terminology:

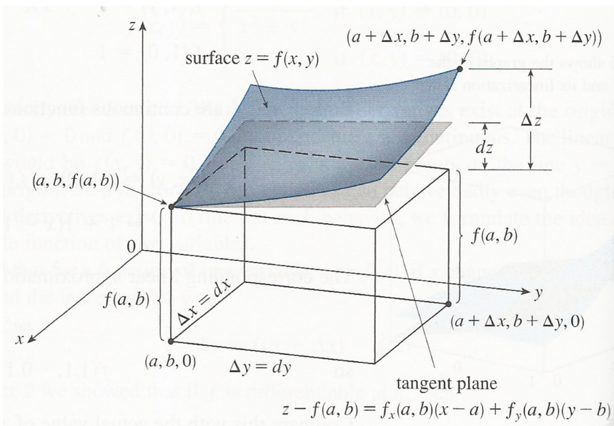


Figure 2: Tangent Planes & Differentials Diagram

Δz: the change in height on the *surface* $z = f(x, y)$ that results from moving away from (a, b) by Δx and Δy.

dz: the change in height on the *tangent plane*, also called the **total differential**:

$$dz = f_x(a, b)dx + f_y(a, b)dy$$

Differentiability:

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to 0$.

Key Idea: A function is differentiable at a given point if the tangent plane approximates the graph well near the point of tangency.

0.0.1 Theorem: (Sufficient Condition for Differentiability)

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

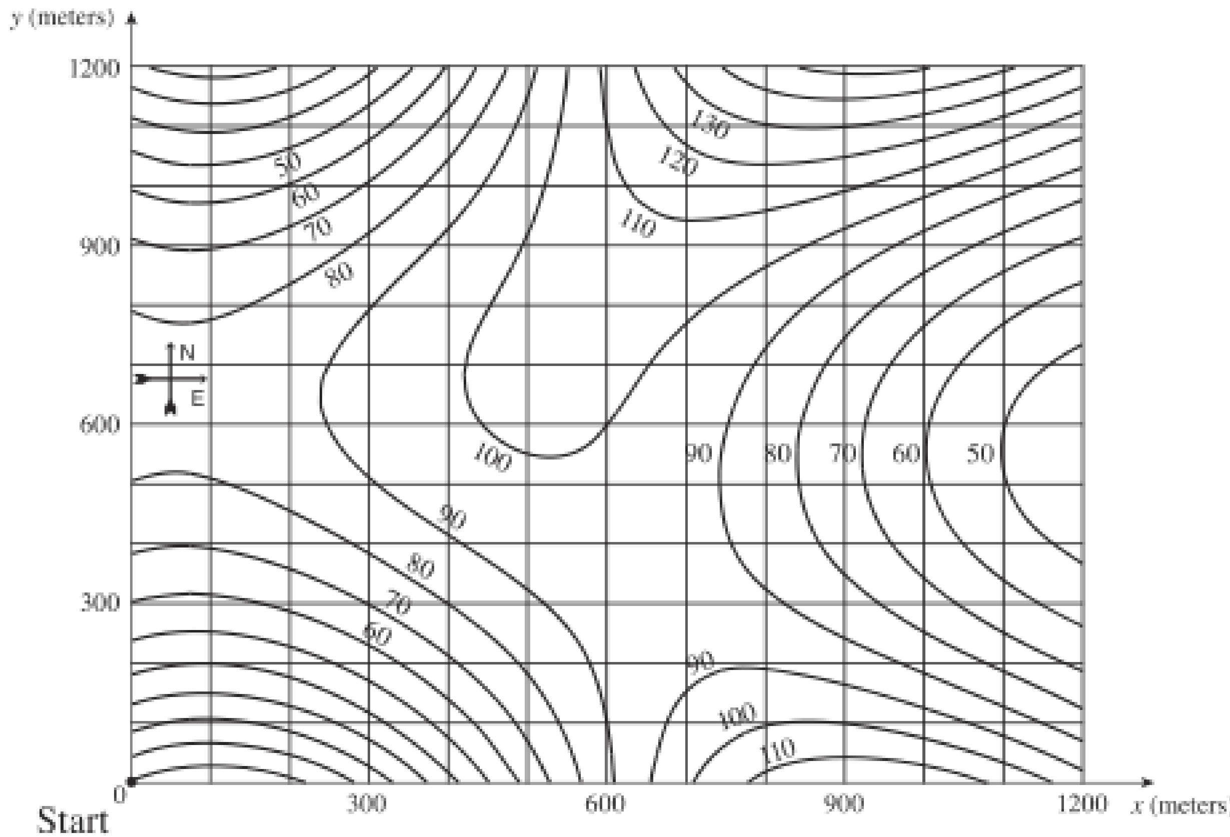
Example 2: Give an example of a surface that is not differentiable at the origin.

Example 3: Find the total differential of $f(x, y) = \cos(x^2 y)$

Group Work:

Back to the Park

The following is a map with curves of the same elevation of a region in Orangerock National Park:



We define the altitude function, $A(x, y)$, as the altitude at a point x meters east and y meters north of the origin ("Start").

Figure 3: Contour Plot for "Back to the Park" Group Work Activity

1. Estimate $A(300, 300)$ and $A(500, 500)$.

2. Estimate $A_x(300, 300)$ and $A_y(300, 300)$.

3. What do A_x and A_y represent in physical terms?

4. In which direction does the altitude increase most rapidly at the point $(300, 300)$?

5. Use your estimates of $A_x(300, 300)$ and $A_y(300, 300)$ to approximate the altitude at $(320, 310)$.