MATH 1336: Calculus III

Chapter 8 Missing Section: Strategies for Testing Series

Series For Practice

Test several of the series listed below for convergence or divergence.
Start with Problems 1-8, then pick other series from the list that interest you.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n-1}{n^2+n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$

12. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\frac{n}{n^2+25}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty} \sin n$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

18. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

23. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

27. $\displaystyle\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n\sqrt{n}}$

30. $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

31. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

32. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

34. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$

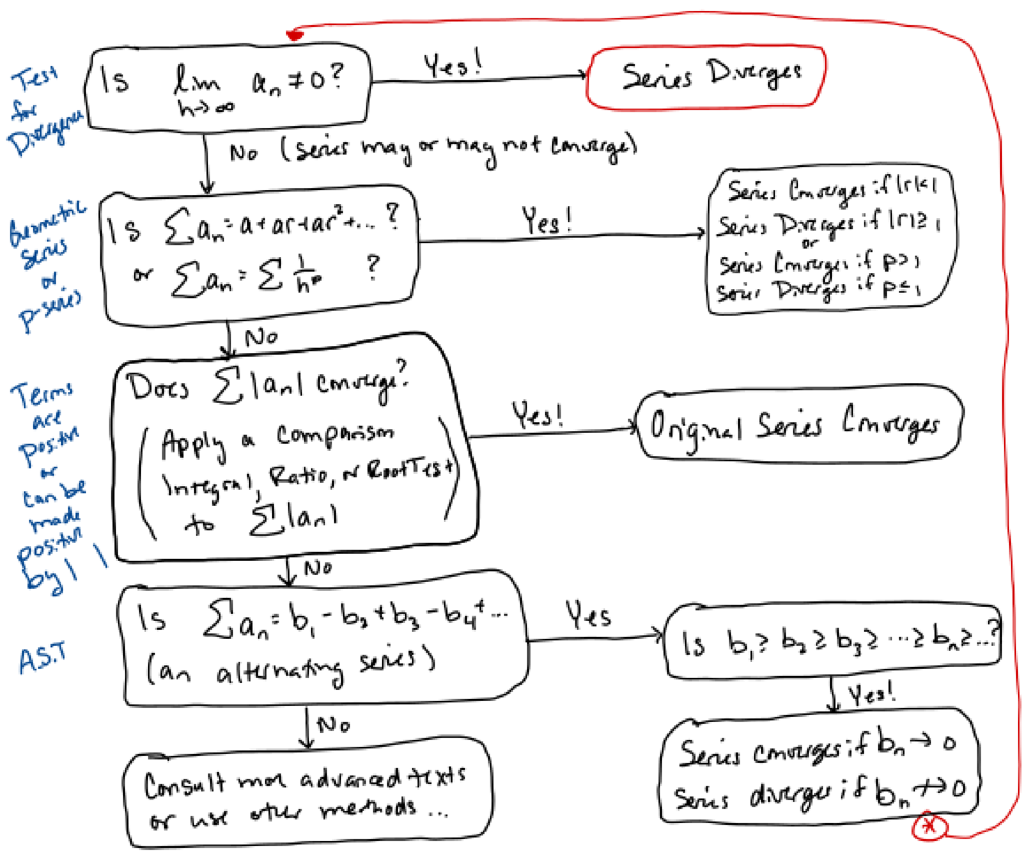
Solutions are posted on the Chapter 8 Objectives & Problems page on Canvas.

STRATEGY FOR TESTING SERIES

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. (The value of p should be chosen as in Section 8.3 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\Sigma\, (-1)^{n-1}b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).



Summary of Convergence Tests for Series

Math 1336 & 1337, Spring 2023, Dr. Cole

Test	Series	Convergence or Divergence	Comments
Test for Divergence	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$ (or DNE)	Inconclusive if $\lim_{n\to\infty} a_n = 0$. More work must be done!
Geometric Series	$\sum_{n=1}^{\infty} ar^{n-1}$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$. Diverges if $\lvert r\rvert \geq 1$.	Useful for comparison tests if the n^{th} term of a_n is similar to r^n.
$p-$Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges if $p > 1$. Diverges if $p \leq 1$.	Useful for comparison tests if the n^{th} term a_n is similar to $\frac{1}{n^p}$.
Integral	$\sum_{n=c}^{\infty} a_n, \quad c > 0$ $a_n = f(n)$	Converges if $\int_c^{\infty} f(x)\, dx$ Converges. Diverges if $\int_c^{\infty} f(x)\, dx$ Diverges.	The function f must be positive, decreasing, & you must be able to integrate $f(x)$ for $x \geq c$.
Comparison	$\sum a_n$ and $\sum b_n$ $0 \leq a_n \leq b_n$	$\sum b_n$ Converges $\Rightarrow \sum a_n$ Converges. $\sum a_n$ Diverges $\Rightarrow \sum b_n$ Diverges.	The comparison series is often a geometric or p-series, (or a series to which another test may be applied).
Limit Comparison	$\sum a_n$ and $\sum b_n$ $a_n, b_n > 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$	$\sum b_n$ Converges $\Rightarrow \sum a_n$ Converges. $\sum b_n$ Diverges $\Rightarrow \sum a_n$ Diverges.	To find b_n consider only the terms of a_n that have the greatest effect when n is large.
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, $b_n > 0$	Converges if (i): $b_{n+1} \leq b_n$ *and* (ii): $\lim_{n\to\infty} b_n = 0$	Applicable only to series with alternating terms. (Note that $(-1)^{n-1}$ is not the only quantity that can cause alternation. Carefully consider series including $\cos(n\pi)$, etc.)
Absolute Convergence	$\sum a_n$	$\sum \lvert a_n\rvert$ Converges $\Rightarrow \sum a_n$ Converges.	Useful for series containing both positive and negative terms. You may apply other tests to $\sum \lvert a_n\rvert$ since all of the terms will be positive.
Ratio	$\sum a_n$ $\lim_{n\to\infty} \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} = L$	Converges (absolutely) if $L < 1$. Diverges if $L > 1$ or $L = \infty$.	Inconclusive if $L = 1$. Test fails, try another test! Useful if a_n contains terms like: $n!$, n^p, or c^n.
Root	$\sum a_n$ $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L$	Converges (absolutely) if $L < 1$. Diverges if $L > 1$ or $L = \infty$.	Inconclusive if $L = 1$. Test fails, try another test! Useful if a_n contains n^{th} powers.