Section 8.3 - Part 1: Integral Test

Section 8.3 - Tests for Series with POSITIVE Terms:

The tests that we will develop in this section can only be applied to series with POSITIVE terms: $a_n > 0$
$\Rightarrow$ verifying and stating that $a_n > 0$ is an important part of the argument when using these tests!

Key Idea:

Knowing that $a_n > 0$ means that the sequence of partial sums, $\{s_n\}$, is *increasing*.
If we can also show that $\{s_n\}$ is *bounded*, then by the **MST**, it must converge.

We just need ways to find an upper bound!

p-Series:

The **p-series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

- convergent if $p > 1$
- divergent if $p \leq 1$

The Integral Test:

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.
Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent if-and-only-if the improper integral $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent:

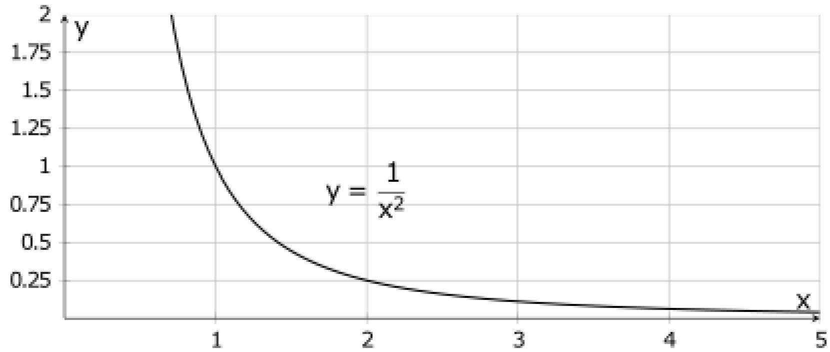
(i) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Integral Test Examples:

1. Use an area argument, referencing the figure shown below, to convince yourself that

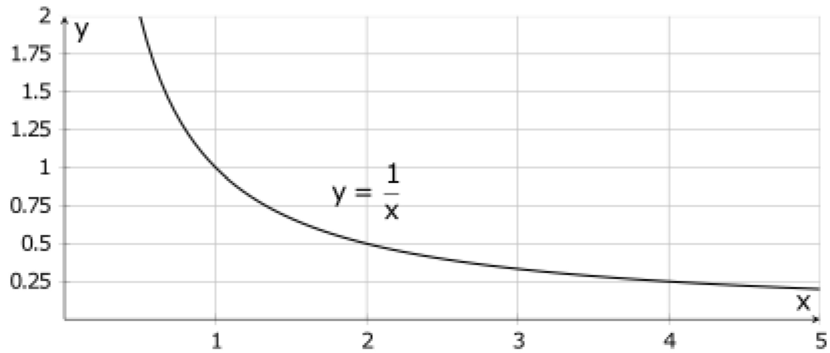
$$1 + \int_1^\infty \frac{dx}{x^2} > \sum_{n=1}^\infty \frac{1}{n^2}$$



Now show that the improper integral $\int_1^\infty \frac{dx}{x^2}$ converges, and by the Integral Test, so does $\sum_{n=1}^\infty \frac{1}{n^2}$.

2. Use an area argument, referencing the figure shown below, to convince yourself that

$$1 + \int_1^\infty \frac{dx}{x} > \sum_{n=1}^\infty \frac{1}{n}$$



In one of the pre-class videos, we discovered that the Harmonic Series, $\sum_{n=1}^\infty \frac{1}{n}$, diverges to ∞.

Use this fact to argue that $\int_1^\infty \frac{dx}{x}$ must also diverge.

Problems for Group Work:

Be sure to fully justify your reasoning as a part of your solutions.
The answers are upside-down on the bottom of this page.

1. Determine the convergence/divergence of the following p-series:

   (a) $\sum_{n=1}^\infty \frac{1}{n^3}$

   (b) $\sum_{n=1}^\infty \frac{1}{n^1}$

   (c) $\sum_{n=1}^\infty \frac{1}{n^{1.00001}}$

2. If the improper integral $\int_5^\infty \frac{dx}{x^p}$ converges, which of the following series <u>must</u> converge?

   A) $\sum_{n=1}^\infty \frac{1}{n^{p+1}}$

   B) $\sum_{n=5}^\infty \frac{1}{n^{p+1}}$

   C) $\sum_{n=1}^\infty \frac{1}{n^{p-1}}$

D) $\sum_{n=5}^{\infty} \frac{1}{n^{p-1}}$

E) Both A and B

F) Both C and D

3. Determine the convergence/divergence of the following series using the tools we currently have:

a) $\sum_{n=1}^{\infty} \left[\frac{5}{n(n+1)} - \left(-\frac{1}{2}\right)^n \right]$

b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

Answers:

Problem ??: a) Converge, b) Diverge, c) Converge **Problem ??:** E,
Problem ??: a) Converge, b) Converge if $p > 1$, Diverge if $p \leq 1$