Section 8.1, Part 2: More Fun with Sequences!

More Sequence Definitions:

A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$.
A sequence $\{a_n\}$ is **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.
A sequence is called **monotonic** if it is *either* increasing or decreasing.

A sequence is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
A sequence is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$.
If a sequence is bounded above and below, then we call it a **bounded sequence**.

Monotonic Sequence Theorem (MST):

Every bounded, monotonic sequence is convergent.

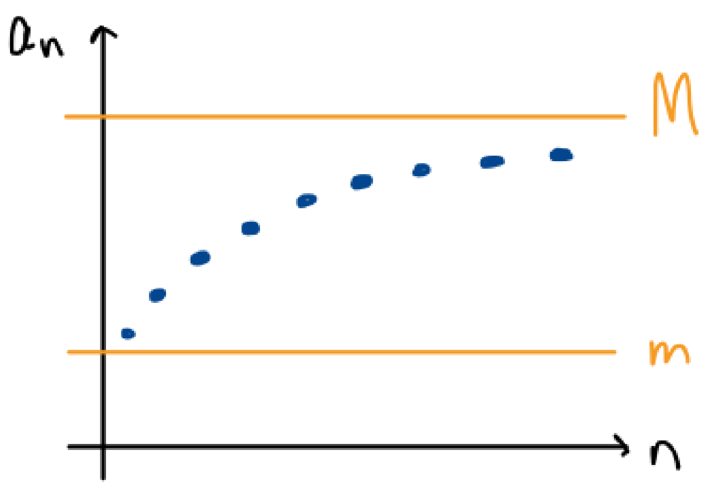

Figure 1: Illustration of a bounded increasing sequence, demonstrating the concept of the MST.

The MST is important because it is the foundation for proving most of the series tests we will be using later in *this chapter!*

Geometric Sequences:

A sequence of the form

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \ldots, \quad ar^{n-1}, \quad \ldots$$

is called a **geometric sequence**. a is the first term, and r is the **common ratio**. The nth term is $a_n = ar^{n-1}$.

A geometric sequence converges if $-1 < r \leq 1$, and diverges otherwise.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Warm-up Problems:

Sketch graphs of sequences matching the descriptions below.

(a) a sequence that is decreasing and converges to a finite value

(b) a sequence that is bounded that does not converge

Examples:

1. **Challenge Problem from last time:**
   Determine the convergence of the sequence shown below, where $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$
   $a_n = \dfrac{(-3)^n}{n!}$

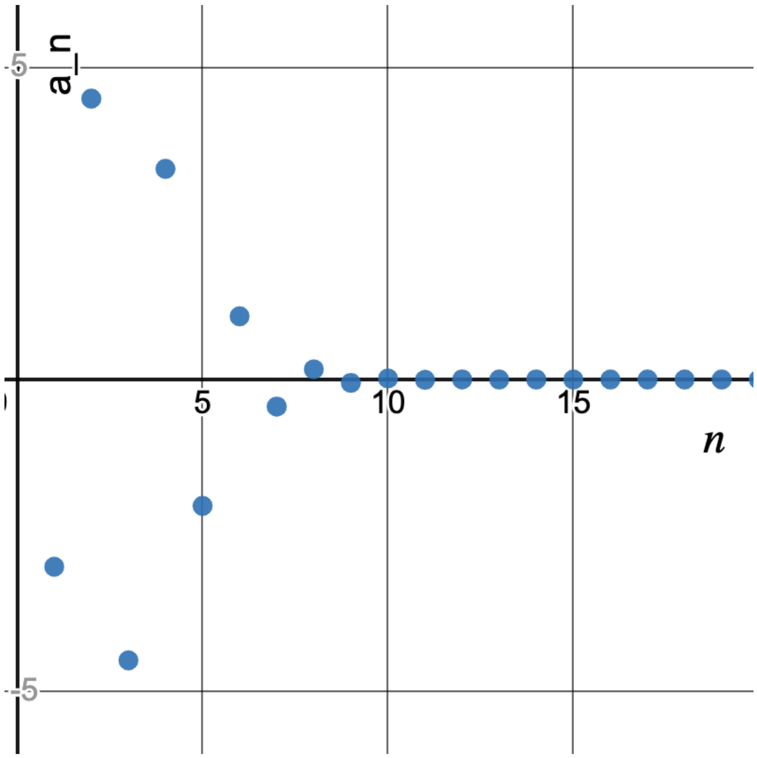

Figure 2: (Click here for desmos link)

Intuition & Plan: If we plot the first several terms of the sequence, we can build intuition that the sequence seems to converge to zero. If we can show that $\lim_{n\to\infty} |a_n| = 0$, then by Theorem 8.1.6, $\lim_{n\to\infty} a_n = 0$.

Today's Follow-up Question: Consider the sequence $|a_n| = \dfrac{3^n}{n!}$.
What, if anything, does the MST say about this sequence? Can it help us determine the convergence of the original sequence?

2. Determine whether the sequence converges or diverges. If it converges, find the limit.
   $b_n = \ln(3n^2 + 1) - \ln(n^2 - 4)$

Problems for Group Work:

Is the sequence increasing, decreasing, or not monotonic?
Is the sequence bounded?
Does the sequence converge or diverge?

1. $b_n = n + \dfrac{1}{n}$

2. $c_n = 4\left(\dfrac{-1}{3}\right)^n$

3. $d_n = 6^n$

4. $a_n = \dfrac{2n-3}{3n+4}$

The following sequences are geometric. Identify the common ratio, r. Does the sequence converge or diverge?

5. $$2, \quad 6, \quad 18, \quad 54, \quad 162, \quad \ldots$$

6. $$2, \quad 1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \quad \ldots$$