Section 8.7, Part 2: More Practice with Taylor & Maclaurin Series

Taylor/Maclaurin Series Summary:

Key Idea: When x is close to a and n is large: the n^{th} degree Taylor/Maclaurin polynomial should approximate $f(x)$ very well!

Definition for Taylor Series for $f(x)$ centered at a:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2$$

$$+ \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Definition for Maclaurin Series for $f(x)$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Taylor's Formula / Lagrange's Form of the Remainder:

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1},$$

Will NOT be covered in Spring 2023

Alternating Series Estimation Theorem Formula:

$$|R_n| = |S - S_n| \leq b_{n+1}$$

Selected Maclaurin Series with Radius of Convergence, R:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots, \quad R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots, \quad R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots, \quad R = \infty$$

Geometric Series: $\displaystyle \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots, \quad R = 1$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots, \quad R = 1$$

Taylor & Maclaurin Series Practice Problems

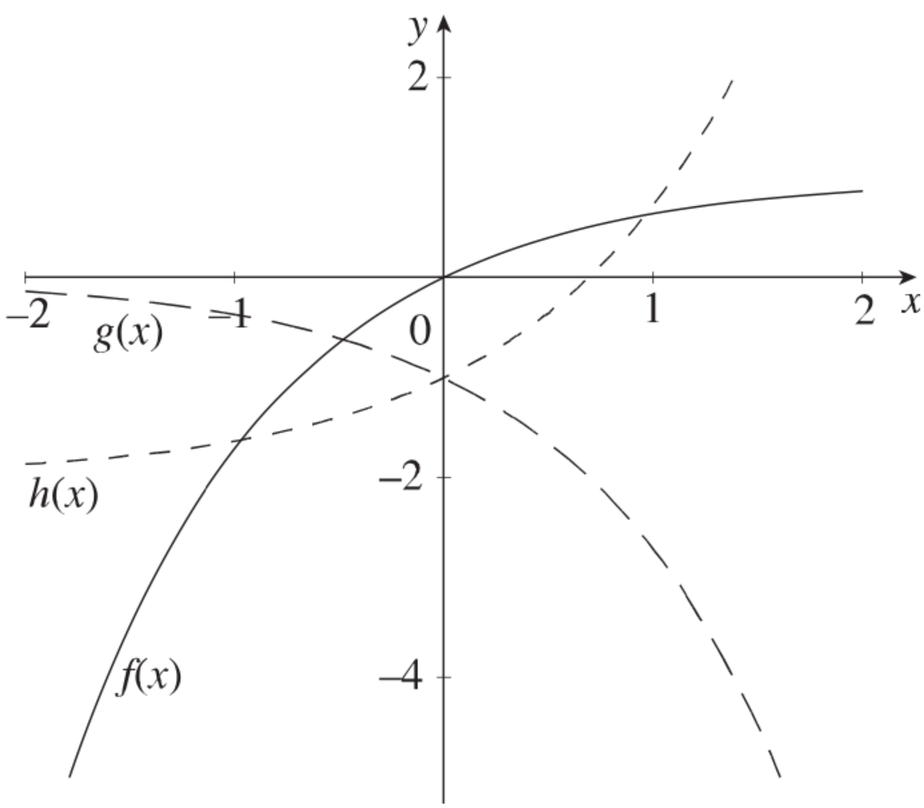
1. Find a Taylor series for $f(x) = e^{2x}$ at $a = 3$.
   Give your answer in the following formats:

   (a) The first four non-zero terms of the series, followed by a "$+\dots$"

   (b) Using summation notation

2. The graphs of f, g, and h are shown below. Explain why the series shown below cannot be the Maclaurin series for f, g, or h.

$$s(x) = -1 + 0.3x - 0.1x^2 + 0.08x^3 + \dots$$



Graph showing $f(x)$: a concave down increasing function that passes through the origin, $g(x)$: a concave down decreasing function that passes through the point $(0, -1)$, $h(x)$: a concave up increasing function that passes through the point $(0, -1)$.

3. Find the series representations for the following functions by any means possible:

(a) $f(x) = \dfrac{1}{1+x^4}$

(b) $f(x) = x^3 \sin x^2$