MATH 1336: Calculus III

Section 2.3: The Dot Product

Intro to the Dot Product:

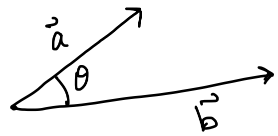


Figure 1: Diagram showing two vectors and the angle between them.

The **dot product** of two vectors, $\vec{\boldsymbol{a}} = \langle a_1, a_2, a_3 \rangle$, $\vec{\boldsymbol{b}} = \langle b_1, b_2, b_3 \rangle$ can be calculated using either formula below, where θ is the angle between the two vectors

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = a_1 b_1 + a_2 b_2 + a_3 b_3, \qquad \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = ||\vec{\boldsymbol{a}}||\ ||\vec{\boldsymbol{b}}||\ \cos\theta$$

Examples:

Calculate the following quantities, or explain why they don't make sense:

Example 1. $\langle 1, 2, 3 \rangle \cdot \langle 4, 5, 6 \rangle$

Example 2. $\langle 1, -2 \rangle \cdot \langle -1, 2 \rangle$

Example 3. $\langle 0, 1 \rangle \cdot \langle 1, 0 \rangle$

Example 4. $\langle 1, 2 \rangle \cdot \langle 1, 2 \rangle$

Example 5. $\langle 0, 1, 2 \rangle \cdot \langle 3, 4 \rangle$

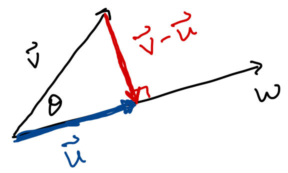


Figure 2: Diagram illustrating the projection of vector v onto vector w.

Discussion:

Geometric Implications of $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b} = ||\vec{a}||\ ||\vec{b}||\ \cos\theta$

Properties of the Dot Product:

Two vectors are **orthogonal** (perpendicular) if-and-only-if $\vec{a} \cdot \vec{b} = 0$.

$\vec{u}, \vec{v}, \vec{w}$ are all vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, and c is a scalar

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$
5. $\vec{0} \cdot \vec{u} = 0$

Projections:

Definition: The **vector projection** of $\vec{v}$ onto $\vec{w}$ is the vector labeled $\vec{u}$ on the diagram shown below.
It has the same initial point as $\vec{v}$ and $\vec{w}$, and represents the component of the vector $\vec{v}$ that acts in the direction of $\vec{w}$.
The length of this vector is called the **scalar projection** of $\vec{v}$ onto $\vec{w}$.

The **scalar projection** of $\vec{v}$ onto $\vec{w}$:

$$\text{comp}_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||}$$

The **vector projection** of $\vec{v}$ onto $\vec{w}$:

$$\text{proj}_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2}\vec{w}$$

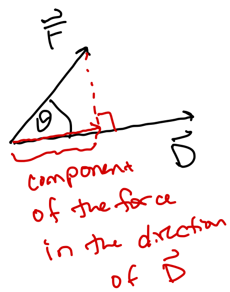


Figure 3: Diagram illustrating a Force vector projected onto a Displacement vector.

Work:

The **work** done by a force $\vec{\boldsymbol{F}}$ to move an object a distance $|\vec{\boldsymbol{D}}|$ is calculated using:

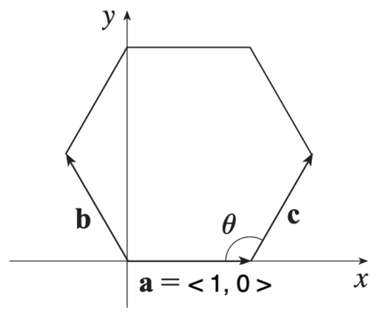
$$W = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{D}} = ||\vec{\boldsymbol{F}}||\ ||\vec{\boldsymbol{D}}||\ \cos\theta$$

Example 1: Show that the standard basis vectors are mutually orthogonal.

Example 2:
Suppose a constant force of magnitude 50 lbs. acts on a sled that is being pulled through the snow a distance of 20 ft, and the force is applied at an angle θ above the horizontal. How much work is done is $\theta = \pi/3$? $\theta = \pi/4$?

Group Work:

The Regular Hexagon



Consider the regular hexagon shown above.

1. Compute $||\vec{a}||, ||\vec{b}||,$ and $||\vec{c}||$.

2. What is the angle labeled θ ?

3. What is $\vec{a} \cdot \vec{c}$?

4. What is $\vec{a} \cdot \vec{b}$?

5. What are $\text{proj}_{\vec{a}}\vec{c}$ and $\text{proj}_{\vec{b}}\vec{c}$?

6. What is the $x-$component of $\vec{a} + \vec{b} + \vec{c}$?