MATH 1336: Calculus III

Sections 3.1-3.3: Vector Functions, Space Curves, & Arclength

Vector Functions, Space Curves, & Arclength:

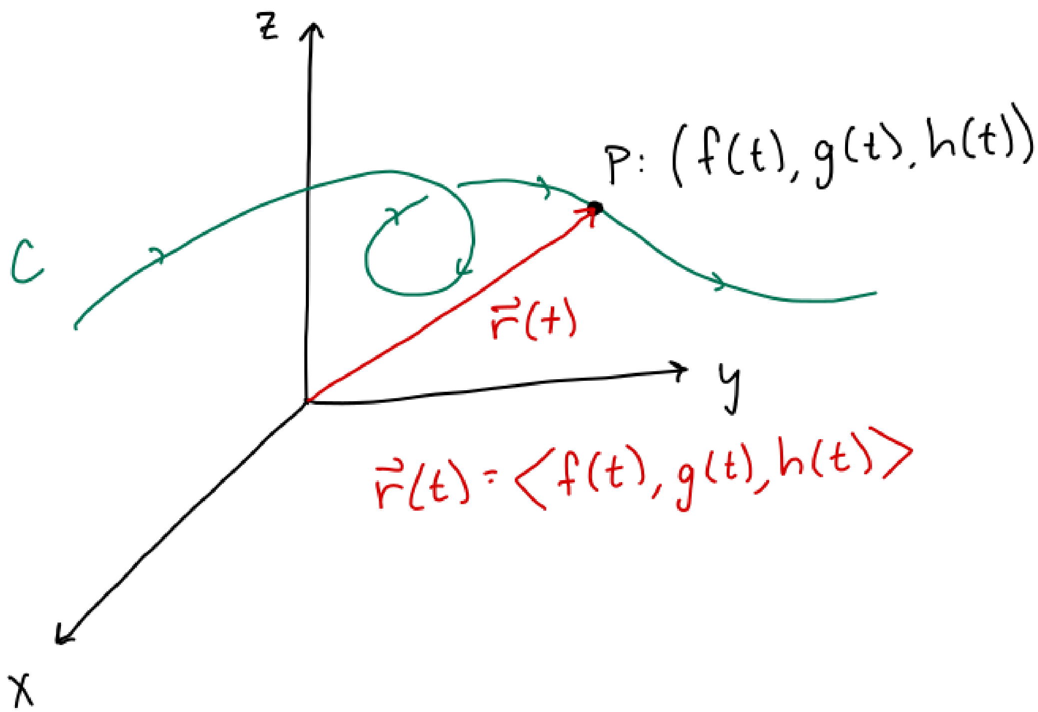

Figure 1: Diagram illustrating a general space curve in three dimensions.

A space curve in $\mathbb{R}^3$ can be parametrized as a **vector function** whose tip traces out the curve C:

$$\vec{\boldsymbol{r}}(t) = \langle f(t), g(t), h(t) \rangle$$

Tangent Vector:

$$\vec{\boldsymbol{r}}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Unit Tangent Vector:

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}\,'(t)}{\|\vec{\boldsymbol{r}}\,'(t)\|}$$

Definite Integrals:

$$\int_a^b \vec{\boldsymbol{r}}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$$

Arclength:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b \|\vec{\boldsymbol{r}}\,'(t)\| dt$$

Warmup: Parallel, Perpendicular, or Neither?

For each of the following, determine whether the given items are parallel, perpendicular, or neither:

Warmup 1: The plane with the standard equation $2x - y - 5z = 0$ and each of the planes listed below:

$$\begin{aligned} -6x + 3y + 15z &= 3 \\ x + 2y &= -4 \\ 2x - 5z &= -1 \end{aligned}$$

Warmup 2: The line with vector equation $\vec{r}(t) =< 5 - 3t, 2 + 4t, 4 - 2t >$ and each of the planes listed below:

$$\begin{aligned} -4.5x + 6y - 3z &= -21 \\ 3x + 5y - 2z &= -30 \\ 4x + 7y + 8z &= 11 \end{aligned}$$

Space Curve Examples:

Example 1: We will revisit the example from the pre-class video in a Mathematica demonstration: Answer the following question about the vector function:

$$\vec{\boldsymbol{r}}(t) = \left\langle e^{2t}, t^2 - t, \ln t \right\rangle = e^{2t}\,\hat{\mathbf{\imath}} + (t^2 - t)\,\hat{\mathbf{\jmath}} + \ln t\,\hat{\mathbf{k}}$$

a) What is the domain of $\vec{\boldsymbol{r}}(t)$?

b) Find a tangent vector at the point where $t = 0.2$.

c) Find an equation for the tangent line to $\vec{\boldsymbol{r}}(t)$ at the point where $t = 0.2$.

d) What is the unit tangent vector $\vec{\boldsymbol{T}}(t)$ for $\vec{\boldsymbol{r}}(t)$? At $t = 0.2$?

Example 2: Find $\vec{\boldsymbol{r}}(t)$ if $\vec{\boldsymbol{r}}\,'(t) = \left\langle e^t, t^2, \cos 2t \right\rangle$ and $\vec{\boldsymbol{r}}(0) = \langle 2, 1, -1 \rangle$.

Space Curve Problems:

Problem 1: Calculate the length along the helix from $t = 0$ to $t = 3\pi$. Does your answer make sense? Why or why not?

$$\vec{\boldsymbol{r}}(t) = \langle 2\cos t, 2\sin t, t \rangle$$

Problem 2: For the vector function shown below, show that $\vec{\boldsymbol{T}}\,'(t) \perp \vec{\boldsymbol{T}}(t)$.

$$\vec{\boldsymbol{r}}(t) = \langle 2\cos t, 2\sin t, t \rangle$$

Problem 3: Prove that $\vec{\boldsymbol{T}}\,'(t) \perp \vec{\boldsymbol{T}}(t)$ in general.
Hint: Start from the fact that $\vec{\boldsymbol{T}}(t)$ is a unit vector.

Problem 4: Find vector functions that represents the curves of intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$.