

MATH 1336: Calculus III

Section 2.4: Cross Product

Cross Product Definition:

The **cross product** of two vectors, $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ results in a vector that is orthogonal to both \vec{a} and \vec{b} .

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

The direction of the vector $\vec{a} \times \vec{b}$ is determined using the **right hand rule**, and its magnitude is given by $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$, where θ is the angle between \vec{a} and \vec{b} .

Right Hand Rule:

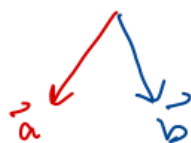


Figure 1: Figure used to demonstrate the right hand rule.

To determine the direction of $\vec{a} \times \vec{b}$, point the fingers of your *right hand* in the direction of \vec{a} , then curl them toward \vec{b} . Your thumb will then point in the direction of $\vec{a} \times \vec{b}$!

Note that order matters! (See property 1 on the next page.)

Example 1:

Find $\vec{a} \times \vec{b}$ and verify that it is perpendicular to \vec{a} and \vec{b} .

$$\vec{a} = \langle 1, 2, 3 \rangle, \quad \vec{b} = \langle 4, 5, 6 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \hat{k} = [(2)(6) - (3)(5)]\hat{i} - [(1)(6) - (3)(4)]\hat{j} + [(1)(5) - (2)(4)]\hat{k}$$

$$\vec{a} \times \vec{b} = (12 - 15)\hat{i} - (6 - 12)\hat{j} + (5 - 8)\hat{k} = \langle -3, 6, -3 \rangle$$

To verify that the result is perpendicular to both \vec{a} and \vec{b} , use the dot product!

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle -3, 6, -3 \rangle \cdot \langle 1, 2, 3 \rangle = -3 + 12 - 9 = 0$$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle -3, 6, -3 \rangle \cdot \langle 4, 5, 6 \rangle = -12 + 30 - 18 = 0$$

Algebraic Properties of the Cross Product:

$\vec{a}, \vec{b}, \vec{c}$ are all vectors in either \mathbb{R}^2 or \mathbb{R}^3 , and s is a scalar

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(s\vec{a}) \times \vec{b} = s(\vec{a} \times \vec{b}) = \vec{a} \times (s\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Geometric Properties of the Cross Product:

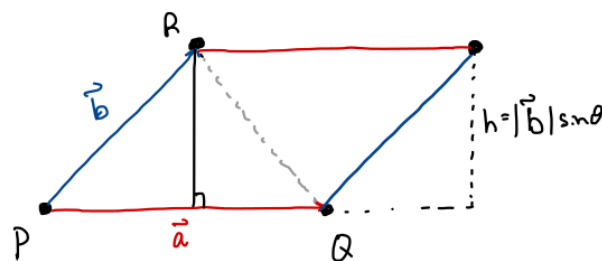


Figure 2: Diagram illustrating the area of a parallelogram determined by vectors \vec{a} and \vec{b} .

The area of the parallelogram determined by \vec{a} and \vec{b} is given by

$$A_{\text{parallelogram}} = \|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

The area of the triangle PQR is half of the area of the parallelogram:

$$A_{\text{triangle}} = \frac{1}{2} \|\vec{a} \times \vec{b}\|$$

These areas would be zero if the vectors were parallel:

Vectors \vec{a} and \vec{b} are parallel if-and-only-if $\vec{a} \times \vec{b} = \vec{0}$.

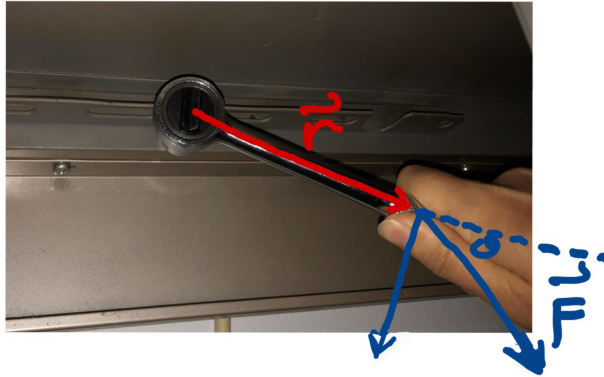


Figure 3: Diagram illustrating applied torque using a ratchet on a bolt under Dr. Cole's desk.

Torque:

The **torque**, $\vec{\tau}$, from applying a force \vec{F} to a rigid body at a point given by position vector \vec{r} is calculated using:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$||\vec{\tau}|| = ||\vec{r}|| ||\vec{F}|| \sin \theta$$

$$||\vec{\tau}|| = (\text{length of wrench}) * (\text{component of force perpendicular to wrench})$$

“righty-tighty, lefty-loosy” rule \Leftrightarrow right hand rule for cross products!

Vector Problems for Group Work:

Problem 1: Evaluate the following cross products. Do the results agree with your intuition?

- a) $\hat{i} \times \hat{j}$
- b) $\hat{j} \times \hat{k}$
- c) $\langle 1, 1, 0 \rangle \times \langle -1, 1, 0 \rangle$

Problem 2: Compute the area of the triangle determined by $\vec{a} = \langle 1, -2, 6 \rangle$ and $\vec{b} = \langle 4, 3, -1 \rangle$.

Problem 3: Let $\vec{a} = s\vec{b} + t\vec{c}$, where s and t are scalars. Show that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

Problem 4: Vector, Scalar, or Nonsense?

Some of the expressions below are meaningful, and some are not. Determine which expressions make sense, and if the result is a scalar or a vector:

a) $\vec{a} \cdot (\vec{b} \times \vec{c})$

b) $\vec{a} \times (\vec{b} \cdot \vec{c})$

c) $\vec{a} \times (\vec{b} \times \vec{c})$

d) $\vec{a} \cdot (\vec{b} \cdot \vec{c})$

e) $(\vec{a} \cdot \vec{b}) \times (\vec{c} \cdot \vec{d})$

f) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

Problem 5: (Fun Geometry Challenge!)

Use vector methods to show that any angle inscribed on a semicircle is a right angle.

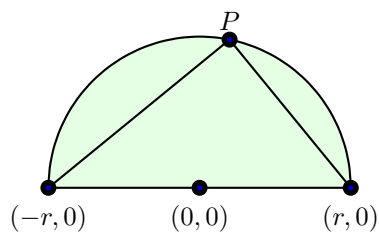


Figure 4: Diagram for Problem 5.