MATH 1336: Calculus III

Section 2.4: Cross Product

Cross Product Definition:

The **cross product** of two vectors, $\vec{a} = \langle a_1, a_2, a_3 \rangle, \vec{b} = \langle b_1, b_2, b_3 \rangle$ results in a vector that is orthogonal to both $\vec{a}$ and $\vec{b}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\mathbf{k}} = (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}$$

The direction of the vector $\vec{a} \times \vec{b}$ is determined using the **right hand rule**, and it's magnitude is given by $||\vec{a} \times \vec{b}|| = ||\vec{a}||\ ||\vec{b}|| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Right Hand Rule:

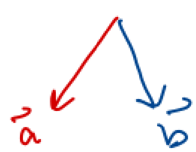

Figure 1: Figure used to demonstrate the right hand rule.

To determine the direction of $\vec{a} \times \vec{b}$, point the fingers of your *right hand* in the direction of $\vec{a}$, then curl them toward $\vec{b}$. Your thumb will then point in the direction of $\vec{a} \times \vec{b}$!

Note that order matters! (See property 1 on the next page.)

Example 1:

Find $\vec{a} \times \vec{b}$ and verify that it is perpendicular to $\vec{a}$ and $\vec{b}$.

$$\vec{a} = \langle 1, 2, 3 \rangle, \qquad \vec{b} = \langle 4, 5, 6 \rangle$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\hat{\mathbf{k}} = [(2)(6)-(3)(5)]\hat{\mathbf{i}} - [(1)(6)-(3)(4)]\hat{\mathbf{j}} + [(1)(5)-(2)(4)]\hat{\mathbf{k}}$$

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = (12 - 15)\hat{\mathbf{i}} - (6 - 12)\hat{\mathbf{j}} + (5 - 8)\hat{\mathbf{k}} = \langle -3, 6, -3 \rangle$$

To verify that the result is perpendicular to both $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$, use the dot product!

$$(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot \vec{\boldsymbol{a}} = \langle -3, 6, -3 \rangle \cdot \langle 1, 2, 3 \rangle = -3 + 12 - 9 = 0$$

$$(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot \vec{\boldsymbol{b}} = \langle -3, 6, -3 \rangle \cdot \langle 4, 5, 6 \rangle = -12 + 30 - 18 = 0$$

Algebraic Properties of the Cross Product:

$\vec{a}, \vec{b}, \vec{c}$ are all vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, and s is a scalar

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(s\vec{a}) \times \vec{b} = s(\vec{a} \times \vec{b}) = \vec{a} \times (s\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Geometric Properties of the Cross Product:

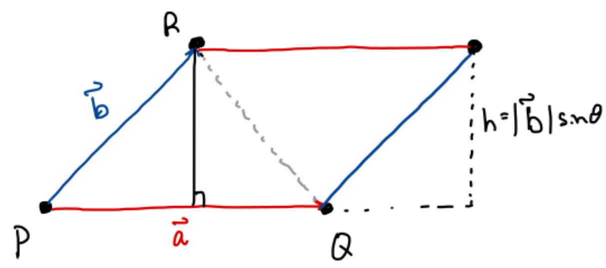

Figure 2: Diagram illustrating the area of a parallelogram determined by vectors a and b.

The area of the parallelogram determined by $\vec{a}$ and $\vec{b}$ is given by

$$A_{\text{parallelogram}} = ||\vec{a} \times \vec{b}|| = ||\vec{a}||\ ||\vec{b}|| \sin\theta$$

The area of the triangle PQR is half of the area of the parallelogram:

$$A_{\text{triangle}} = \frac{1}{2}||\vec{a} \times \vec{b}||$$

These areas would be zero if the vectors were parallel:

Vectors $\vec{a}$ and $\vec{b}$ are parallel if-and-only-if $\vec{a} \times \vec{b} = \vec{0}$.

Torque:

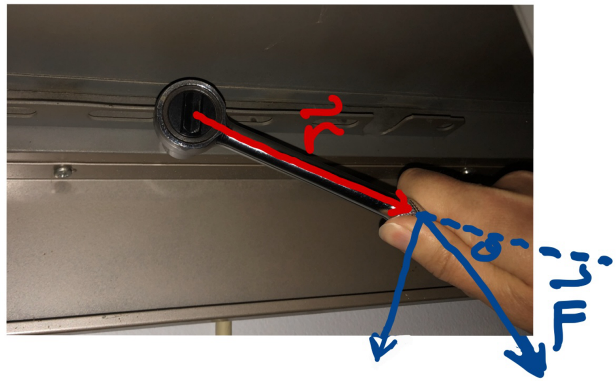

Figure 3: Diagram illustrating applied torque using a ratchet on a bolt under Dr. Cole's desk.

The **torque**, $\vec{\boldsymbol{\tau}}$, from applying a force $\vec{\boldsymbol{F}}$ to a rigid body at a point given by position vector $\vec{\boldsymbol{r}}$ is calculated using:

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$$
$$||\vec{\boldsymbol{\tau}}|| = ||\vec{\boldsymbol{r}}||\ ||\vec{\boldsymbol{F}}|| \sin\theta$$

$$||\vec{\boldsymbol{\tau}}|| = (\text{length of wrench}) * (\text{component of force perpendicular to wrench})$$

"righty-tighty, lefty-loosy" rule $\Leftrightarrow$ right hand rule for cross products!

Vector Problems for Group Work:

Problem 1: Evaluate the following cross products. Do the results agree with your intuition?

a) $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$

b) $\hat{\mathbf{j}} \times \hat{\mathbf{k}}$

c) $\langle 1,1,0\rangle \times \langle -1,1,0\rangle$

Problem 2: Compute the area of the triangle determined by $\vec{\boldsymbol{a}} = \langle 1,-2,6\rangle$ and $\vec{\boldsymbol{b}} = \langle 4,3,-1\rangle$.

Problem 3: Let $\vec{\boldsymbol{a}} = s\vec{\boldsymbol{b}} + t\vec{\boldsymbol{c}}$, where s and t are scalars. Show that $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = 0$.

Problem 4: Vector, Scalar, or Nonsense?
Some of the expressions below are meaningful, and some are not. Determine which expressions make sense, and if the result is a scalar or a vector:

a) $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}})$

b) $\vec{a} \times (\vec{b} \cdot \vec{c})$

c) $\vec{a} \times (\vec{b} \times \vec{c})$

d) $\vec{a} \cdot (\vec{b} \cdot \vec{c})$

e) $(\vec{a} \cdot \vec{b}) \times (\vec{c} \cdot \vec{d})$

f) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

Problem 5: (Fun Geometry Challenge!)

Use vector methods to show that any angle inscribed on a semicircle is a right angle.

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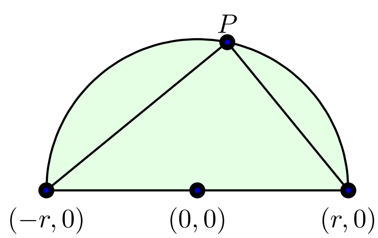

Figure 4: Diagram for Problem 5.