MATH 1336: Calculus III

Section 5.3, Part 2: Integral Test

Integral & Comparison Tests - Tests for Series with POSITIVE Terms:

These tests can only be applied to series with POSITIVE terms: $a_n > 0$
$\Rightarrow$ verifying and stating that $a_n > 0$ is an important part of the argument when using these tests!

Key Idea:

Knowing that $a_n > 0$ means that the sequence of partial sums, $\{s_n\}$, is *increasing*.
If we can also show that $\{s_n\}$ is *bounded*, then by the **MCT**, it must converge.

We just need ways to find an upper bound!

p-Series:

The **p-series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

- convergent if $p > 1$
- divergent if $p \leq 1$

The Integral Test (Theorem 5.9):

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

(i) f is continuous

(ii) f is decreasing

(iii) $f(n) = a_n$ for $n \geq N$

Then

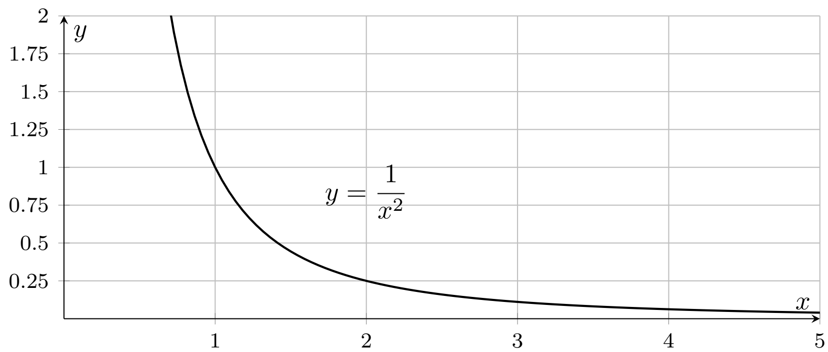
$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_N^{\infty} f(x)\, dx$$

have the **same convergence behavior:**
either they both converge or they both diverge.

Integral Test Examples:

1. Use an area argument, referencing the figure shown below, to convince yourself that

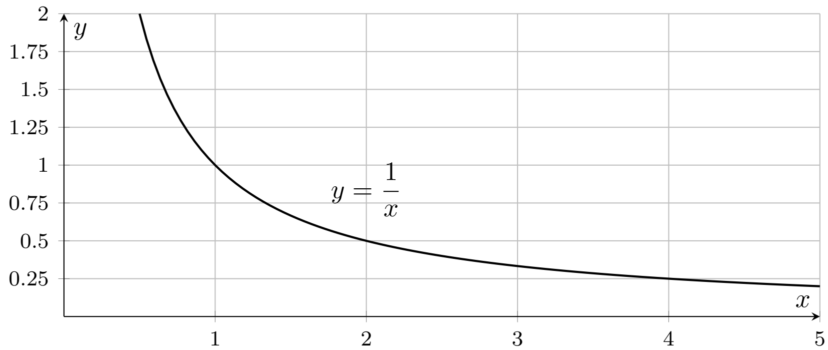
$$1+\int_1^\infty \frac{dx}{x^2} > \sum_{n=1}^\infty \frac{1}{n^2}$$



Now show that the improper integral $\int_1^\infty \frac{dx}{x^2}$ converges, and by the Integral Test, so does $\sum_{n=1}^\infty \frac{1}{n^2}$.

2. Use an area argument, referencing the figure shown below, to convince yourself that

$$1 + \int_1^\infty \frac{dx}{x} > \sum_{n=1}^\infty \frac{1}{n}$$



In one of the pre-class videos, we discovered that the Harmonic Series, $\sum_{n=1}^\infty \frac{1}{n}$, diverges to ∞.

Use this fact to argue that $\int_1^\infty \frac{dx}{x}$ must also diverge.

Problems for Group Work:

Be sure to fully justify your reasoning as a part of your solutions.
The answers are upside-down on the bottom of this page.

1. Determine the convergence/divergence of the following p-series:

   (a) $\sum_{n=1}^\infty \frac{1}{n^3}$

   (b) $\sum_{n=1}^\infty \frac{1}{n^1}$

   (c) $\sum_{n=1}^\infty \frac{1}{n^{1.00001}}$

2. If the improper integral $\int_5^\infty \frac{dx}{x^p}$ converges, which of the following series must converge?

   A) $\sum_{n=1}^\infty \frac{1}{n^{p+1}}$

   B) $\sum_{n=5}^\infty \frac{1}{n^{p+1}}$

   C) $\sum_{n=1}^\infty \frac{1}{n^{p-1}}$

   D) $\sum_{n=5}^\infty \frac{1}{n^{p-1}}$

E) Both A and B

F) Both C and D

3. Determine the convergence/divergence of the following series using the tools we currently have:

a) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{5}{n(n+1)} - \left(-\frac{1}{2}\right)^n\right]$

b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

oblem 1: a) Converge, b) Diverge, c) Converge **Problem 2:** E,**Problem 3:** a) Converge, b) Converge if $p > 1$, Diverge if $p \leq 1$