MATH 1336: Calculus III

Section 1.2: Calculus with Parametric Curves, Part 1

Differential Calculus with Parametric Curves:

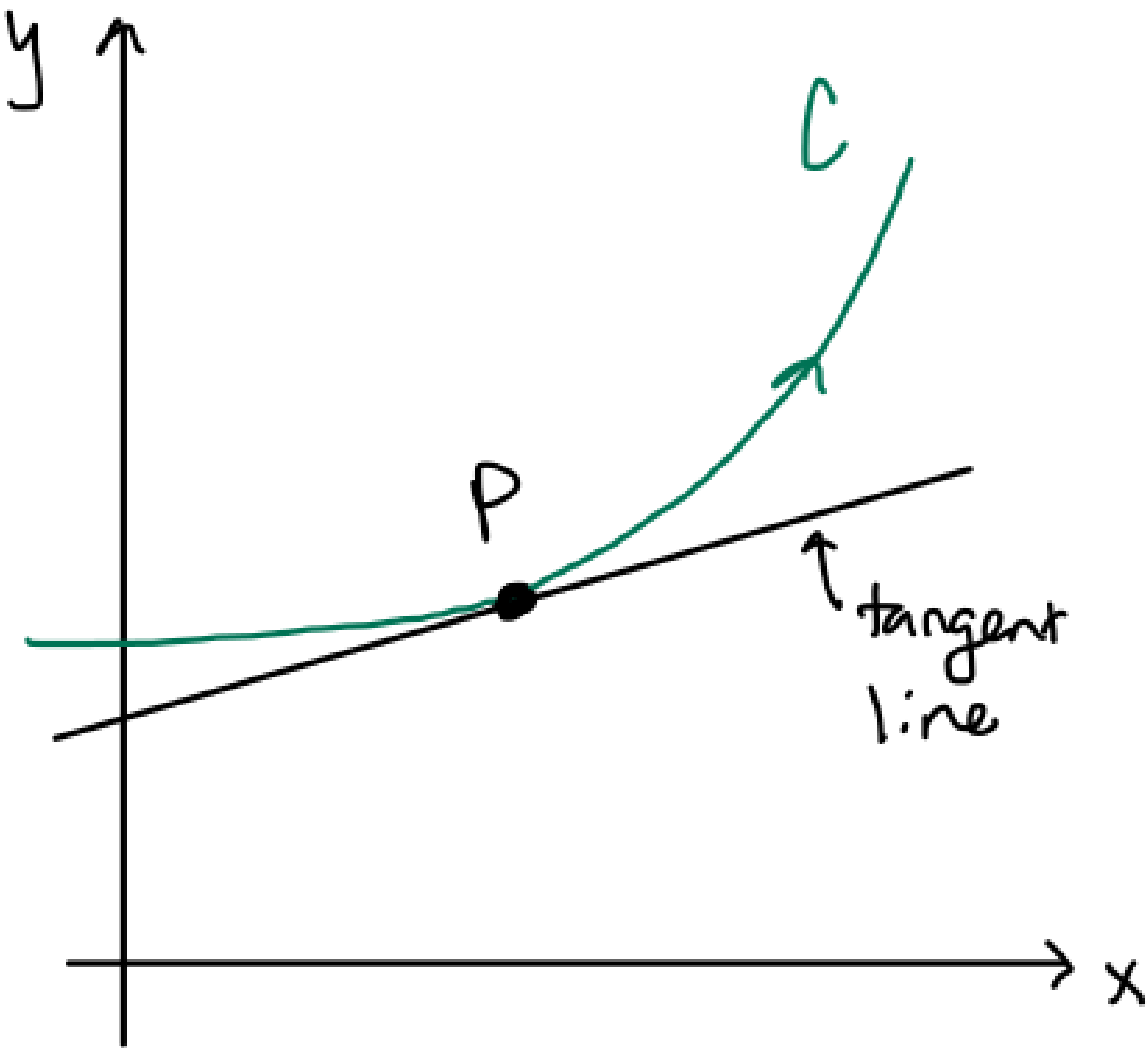


Figure 1: Figure showing a line tangent to a parametric curve at a point P.

Consider a point P on a parametric curve C that has coordinates given by

$$P : (x(t), y(t)).$$

We would like to extend the concepts of **slope** and **concavity** from Differential Calculus to the new setting of parametric curves.

Slope:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

Note that the tangent line will be: **horizontal** at points where $y'(t) = 0$,
vertical at points where $x'(t) = 0$.

To find an equation for the tangent line, you can use the the point-slope equation of a line:

$$y - y_1 = m(x - x_1)$$

Concavity:

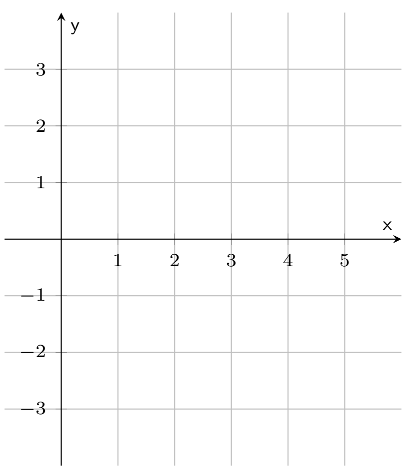
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{x'(t)}$$

Example:

Example 2: Consider the parametric curve

$$x = 3 - 2t - t^2, \quad y = t^3 + 3t^2 - 2, \quad -3 \le t \le 1.$$

We will use the following steps to draw a rough sketch of the curve without using a calculator or Mathematica.



(a) Determine the locations of any points on the curve where the tangent line is either horizontal or vertical.

(b) Plot and label the points found in part 2a, as well as the initial and terminal point.

(c) In order to connect the points from part 2b in the correct order, note that the curve must intersect itself. How can we find the locations of the intersection point(s)?

(Mathematica Demo: Parametric Calculus)

Problems for Group Work:

Problem 0: Introduce yourself to your partners and tell everyone about a new skill or hobby that you have learned/enjoyed recently.

Problem 1: As t varies, the following parametric equations trace out a line in the $xy-$plane

$$x = 2 + 3t, \qquad y = 4 + 7t$$

(a) Calculate $\frac{dy}{dx}$ to determine the slope of the line.

(b) Eliminate the parameter to find a Cartesian equation of the line.
Use this equation to check your work from the previous part of the problem.

(c) Calculate $\frac{d^2y}{dx^2}$. Does your result match with your intuition regarding the concavity of a line?

Problem 2: Consider the curve parametrized by

$$x = 3\cos(t), \qquad y = 2\sin(t), \qquad 0 \leq t \leq 2\pi.$$

(a) Sketch a graph of the curve, and indicate the direction in which the curve is traced as t increases.
Hint: What would the graph look like if the coefficients on the trig functions were the same?

(b) Find an equation for the line tangent to the curve at the point where $t = \pi/6$.

(c) Does your tangent line equation make sense, given the graph of the curve?

(d) Use calculus to find the points on the curve where the tangent line is horizontal or vertical. Compare your solutions with the graph to check your work.

(e) Find $\frac{d^2y}{dx^2}$. Determine for which values of t the curve is concave up.
Use the graph to check your work.

(f) Eliminate the parameter to find a Cartesian equation of the curve.