Equations of Lines:

In $\mathbb{R}^2$ (the coordinate plane), we know that a point and a slope can be used to determine an equation for a line, using the point-slope formula:

$$y - y_1 = m(x - x_1)$$

We build on that idea to determine equations of lines in in $\mathbb{R}^3$: a point, $P_0 = (x_0, y_0, z_0)$ and a direction, $\vec{\boldsymbol{v}} = \langle a, b, c \rangle$, can be used to determine an equation of a line.

Let $\vec{\boldsymbol{r}}_0 = \langle x_0, y_0, z_0 \rangle$ be the position vector for the point P_0, $\vec{\boldsymbol{r}} = \langle x, y, z \rangle$ be the position vector for any general point $P = (x, y, z)$ on the line L, and let $\vec{\boldsymbol{v}} = \langle a, b, c \rangle$ be any vector parallel to the line.
Then the line can be described by any of the following types of equations:

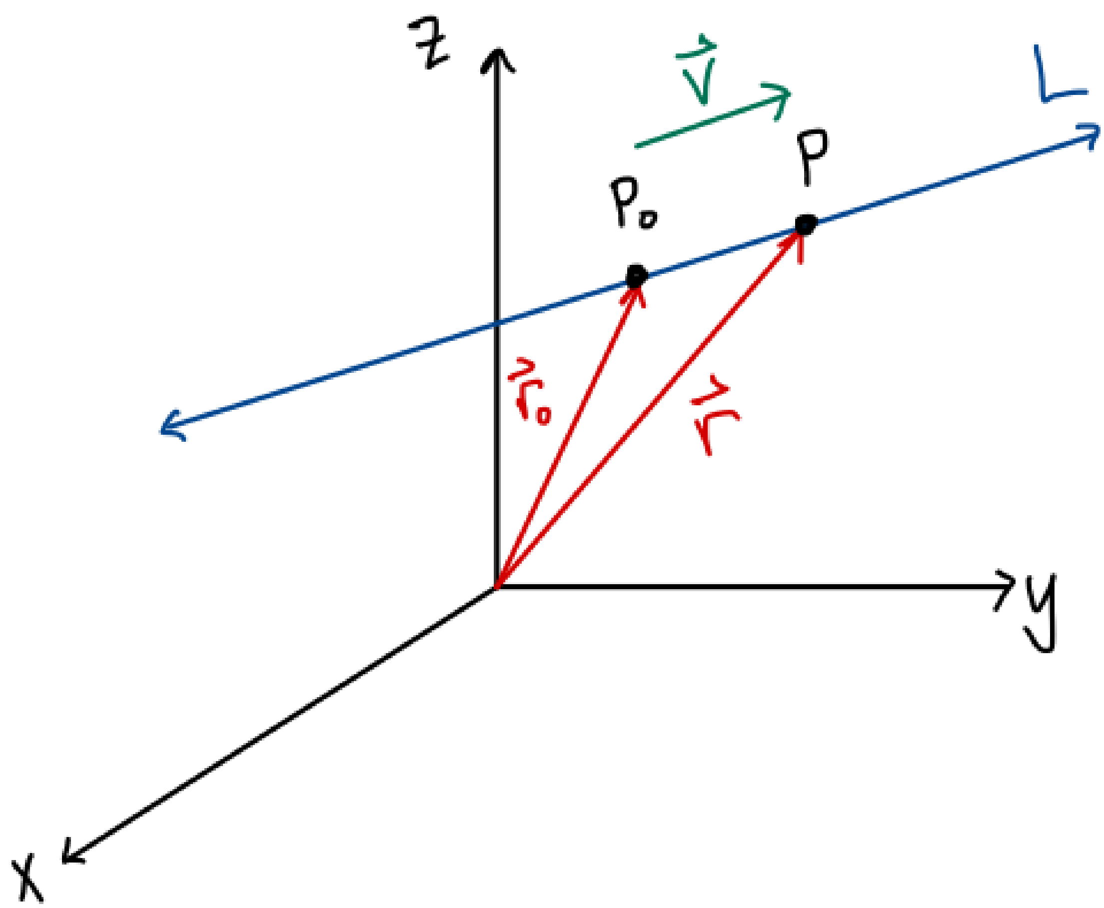

Figure 1: Diagram illustrating equations of lines in three dimensions.

Vector Equation:

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_0 + t\vec{\boldsymbol{v}}$$

Parametric Equations:

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

Symmetric Equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note that a, b, c are sometimes called the **direction numbers** of the line.

Example 1:

Find vector, parametric, and symmetric equations for the line that passes through the points $A = (0, 0, 4)$, and $B = (12, 8, 8)$.

In order to build our line equations, we need to find an initial vector $\vec{\boldsymbol{r}}_0$, and a vector $\vec{\boldsymbol{v}}$ parallel the the line. We can pick *either* one of the given points for $\vec{\boldsymbol{r}}_0$, so I decided to choose $\vec{\boldsymbol{r}}_0 = \langle 0, 0, 4\rangle$.

To find $\vec{\boldsymbol{v}}$, we can calculate the vector that points from A to B:

$$\vec{\boldsymbol{v}} = \overrightarrow{AB} = \langle 12 - 0, 8 - 0, 8 - 4\rangle = \langle 12, 8, 4\rangle$$

Vector Equation: $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_0 + t\vec{\boldsymbol{v}} = \langle 0, 0, 4\rangle + t\langle 12, 8, 4\rangle = \langle 12t, 8t, 4 + 4t\rangle$

Parametric Equations: $x = 12t, \qquad y = 8t, \qquad z = 4 + 4t$

Symmetric Equations: $\dfrac{x}{12} = \dfrac{y}{8} = \dfrac{z - 4}{4}$

Example 2:

Show that the lines below are **skew lines**: they are not parallel and do not intersect one another.

$$L_1: \qquad x-1=\frac{y+2}{3}=4-z, \qquad\qquad L_2: \qquad \frac{x}{2}=y-3=\frac{z+3}{4}$$

Lines Practice Problems:

Problem 1: Find parametric equations for the line that passes through $P=(3,1,4)$ and is parallel to the vector $\vec{\boldsymbol{v}}=-\hat{\mathbf{\imath}}+\hat{\mathbf{\jmath}}-2\hat{\mathbf{k}}$. Then find the points where the line passes through the coordinate planes and use the information to sketch the line.

Problem 2: Find the vector equation for the line that passes through the points $(-1,3,7)$ and $(4,2,-1)$.

Problem 3: Show that the lines

$$L_3: \qquad \frac{x-1}{2}=\frac{y+1}{1}=\frac{z-2}{4}, \qquad\qquad L_4: \qquad \frac{x+2}{4}=\frac{y}{-3}=\frac{z-\frac{1}{2}}{-1}$$

intersect, and find the point of intersection.