MATH 1336: Calculus III

Section 6.4: Applications of Taylor Polynomials - Error Estimation

Taylor/Maclaurin Polynomial Approximation Key Ideas:

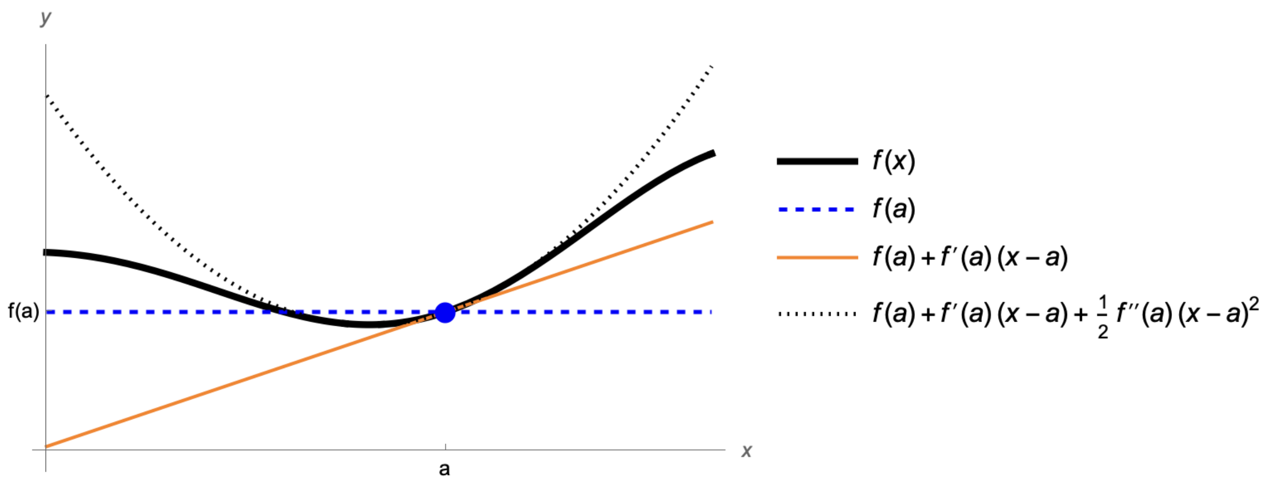


Figure 1: A graph showing a function and its zeroth, first, and second degree Taylor Polynomials, centered at x=a.

To build a polynomial of degree n that approximates the function $f(x)$ well near $x = a$, make sure that it matches the first n derivatives of f *exactly* at $x = a$.

The more derivatives we match, the better our approximation should be!

When x is close to a and n is large: the n^{th} degree Taylor/Maclaurin polynomial should approximate $f(x)$ very well!

Taylor Polynomials -vs- Taylor Series:

A Taylor Polynomial of degree n (or order n) $T_n(x)$ terminates after the $(x-a)^n$ term:

$$T_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

while a Taylor Series is a power series centered at $x = a$ that typically has infinitely many terms. A Taylor Series that is centered at $x = 0$ is given a special name: Maclaurin Series.

Error Estimation:

The methods shown below can be used to find an upper bound on the error due to approximating a function with only the first n terms of the Taylor Series. The key idea behind both methods is that the size of the error should be bounded by the *first term left off.*

Taylor's Forumla/Lagrange's Form of the Remainder:

The remainder can be expressed as shown below, where z is a number strictly between x and a

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

Taylor's Formula is applied on an *interval*, as shown below.

The goal is to find an upper bound (worst-case-scenario) for $|R_n|$ on the interval I.

Alternating Series Estimation Theorem (5.4):

Note that this can only be used if the Taylor/Maclaurin Series has alternating signs!

If $S = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is the sum of a convergent alternating series, then

$$|R_n| = |S - S_n| \leq b_{n+1}$$

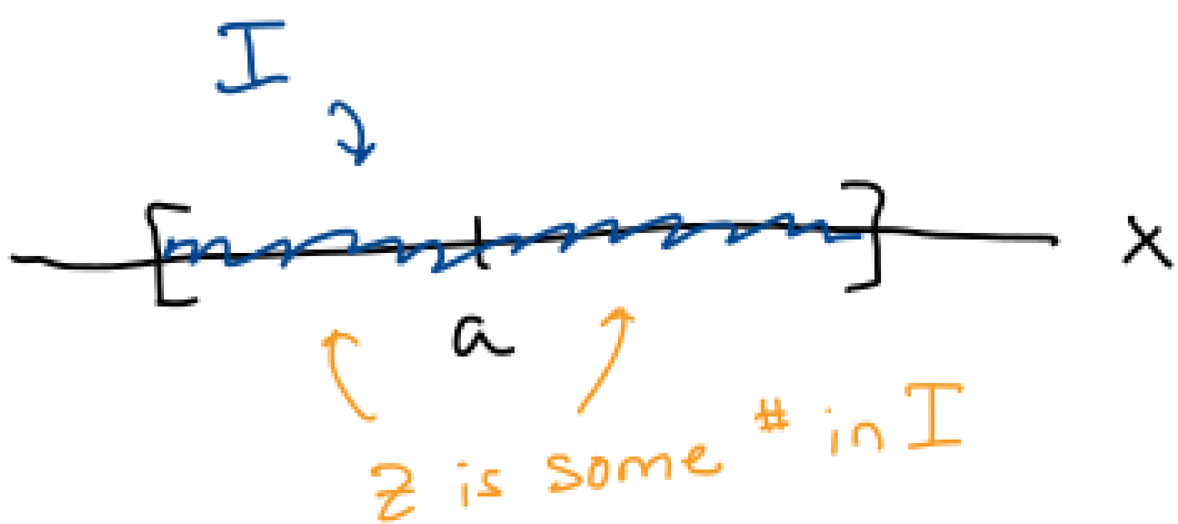


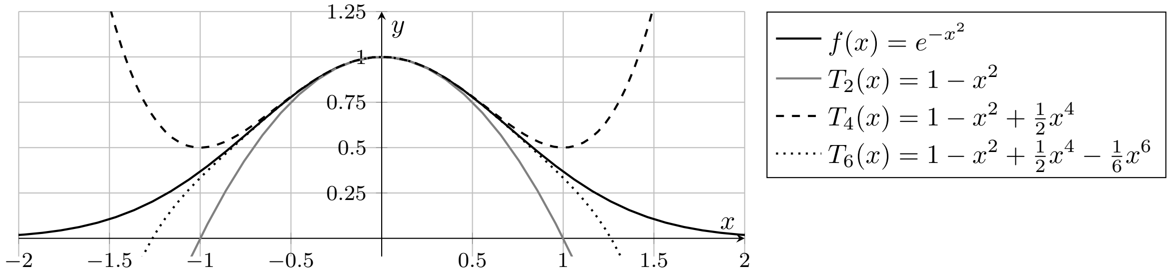
Figure 2: Figure illustrating the interval on which Taylor's Formula is applied.

Steps for using Taylor's Formula:

1. Find the $(n+1)^{st}$ derivative of f: $f^{(n+1)}(x)$
2. Find an upper bound on $|f^{(n+1)}(z)|$ for z in I
3. Find an upper bound on $\left|\frac{(x-a)^{n+1}}{(n+1)!}\right|$ for x in I

4. Multiply the values from steps 2 & 3

Example 2: The function $f(x) = e^{-x^2}$ and its second, fourth, and sixth degree Maclaurin Polynomials, $T_2(x), T_4(x), T_6(x)$, are shown on the graph below.



a) Should you expect $T_2(0)$ to be a good estimate of $f(0)$? Why or why not?

b) Should you expect $T_2(1)$ to be a good estimate of $f(1)$? Why or why not?

c) Use the graph shown above to estimate the accuracy of the approximations:
$f(1) \approx T_2(1)$, $f(1) \approx T_4(1)$, $f(1) \approx T_6(1)$

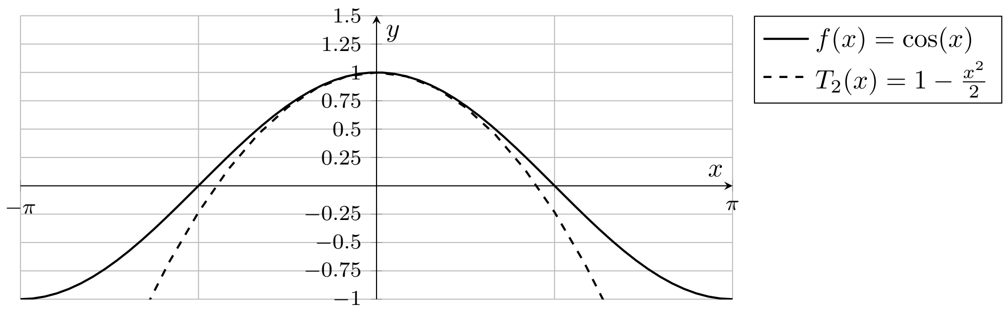
d) Use the Alternating Series Estimation/Remainder Theorem or Taylor's Formula to estimate the accuracy of the approximations:
$f(1) \approx T_2(1)$, $f(1) \approx T_4(1)$, $f(1) \approx T_6(1)$

e) Finally, we can achieve our goal from the beginning of Chapter 5! Use $T_6(x)$ to approximate the area under $f(x)$ between $x = 0$ and $x = 1$. Also, explain how to give an estimate of the accuracy of the approximation.

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 T_6(x)\,dx$$

Example 3: The function $f(x) = \cos(x)$ is plotted as a solid curve below.

Its second degree Maclaurin Polynomial, $T_2(x) = 1 - \dfrac{x^2}{2}$, is plotted with dashed curve below.



a) Should you expect $T_2(0)$ to be a good estimate of $f(0)$? Why or why not?

b) Should you expect $T_2(\frac{\pi}{2})$ to be a good estimate of $f(\frac{\pi}{2})$? Why or why not?

c) Use the graph shown above to estimate the accuracy of the approximation $f(\frac{\pi}{2}) \approx T_2(\frac{\pi}{2})$

d) Use the Alternating Series Estimation Theorem to estimate the accuracy of the approximation $f(\frac{\pi}{2}) \approx T_2(\frac{\pi}{2})$

e) Use Taylor's Formula (also known as Lagrange's form of the remainder term) to estimate the accuracy of the approximation $f(x) \approx T_2(x)$ on the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
Hint: Note that for $f(x) = \cos(x)$, $T_2(x) = T_3(x)$, so consider:

$$R_3(x) = \frac{f^{(4)}(z)}{4!}x^4$$