MATH 1336: Calculus III

Sections 5.2 & 5.3: Convergence Behavior of Infinite Series

Infinite Series Convergence Bathtub Analogy:

When thinking about convergence -vs- divergence of infinite series, we can draw an analogy between filling up a bathtub with successive amounts of water according to some specific pattern:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots$$

If the bathtub doesn't overflow, that must mean that the amount of water we are adding each time goes to zero:

Figure 1: Diagram showing a bathtub that is being filled with successive amounts of water going to zero and does not overflow.

Bathtub Theorem:

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

INCONCLUSIVE CASE!

If $\lim_{n\to\infty} a_n = 0$, we cannot conclude *anything* about the convergence behavior of $\sum_{n=1}^{\infty} a_n$.

If the amount of water we add each time doesn't go to zero, then the bathtub will definitely overflow:

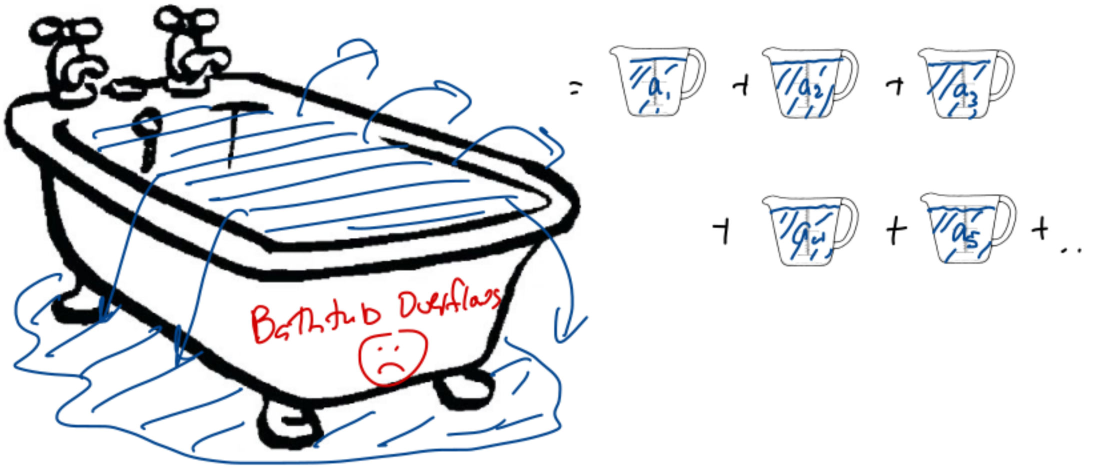

Figure 2: Diagram showing a bathtub that is being filled with successive equal amounts of water will overflow.

Test for Divergence (Theorem 5.8):

If $\lim_{n\to\infty} a_n$ DNE -**OR**- $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 5.7:

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are:

(i) $\sum ca_n = c \sum a_n$

(ii) $\sum (a_n + b_n) = \sum a_n + \sum b_n$

(iii) $\sum (a_n - b_n) = \sum a_n - \sum b_n$

Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$$

converges to $S = \dfrac{a}{1-r}$ if $|r| < 1$
diverges if $|r| \geq 1$.

Harmonic Series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

diverges

Problems for Group Work:

Be sure to fully justify your reasoning as a part of your solutions.
The answers are upside-down on the bottom of this page.

1. Determine the convergence/divergence of the following series:

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k^n}, \quad k > 1$

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{4 \cdot 5^n - 5 \cdot 4^n}{6^n}$

   c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

   d) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{n}{n+1}\right)$

   e) $\displaystyle\sum_{n=1}^{\infty} (-1)^{2n}$

Answers: **Problem 1:** a) Converge, b) Converge, c) Diverge, d) Diverge, e) Diverge,