MATH 1336: Calculus III

Section 5.4, Part 1: Comparison Tests

Integral & Comparison Tests - Tests for Series with POSITIVE Terms:

These tests can only be applied to series with POSITIVE terms: $\boldsymbol{a_n > 0}$
$\Rightarrow$ verifying and stating that $\boldsymbol{a_n > 0}$ is an important part of the argument when using these tests!

Comparison Test:

(i) If $\sum b_n$ is convergent and $0 \leq a_n \leq b_n$ for all $n \geq N$,
then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $0 \leq b_n \leq a_n$ for all $n \geq N$,
then $\sum a_n$ is also divergent.

Limit Comparison Test:

Let $a_n, b_n \geq 0$ for all $n \geq 1$.

(i) If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c$, **and** $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$
have the **same convergence behavior**.

(ii) If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, **and** $\sum b_n$ converges,
then $\sum a_n$ also converges.

(iii) If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, **and** $\sum b_n$ diverges,
then $\sum a_n$ also diverges.

Note 1 - What to compare with?
Both tests rely on comparison with series for which we already know the convergence behavior:

- Geometric Series
- Harmonic Series
- p-Series
- Series we can test another way...

Note 2 - $\boldsymbol{n \geq N}$
We don't *always* have to start with $n = 1$. Convergence is really about the tails, or end behavior, of the terms.
If we can find a comparison that only holds for $n \geq 500$, for example, the tests still work!

Motivating Example (pre-class video):

Intuition: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ should have the **same convergence behavior**

We *proved* it by observing that:
$0 < \dfrac{1}{3^n+1} < \dfrac{1}{3^n}$ for $n \geq 1$. Then we used the n^{th} partial sum formula for Geometric Series to show that $0 < \displaystyle\lim_{n\to\infty} s_n < \frac{1}{2}$, so the sequence of partial sums $\{s_n\}$ is *bounded*.

We also know that $\{s_n\}$ is increasing, since $\dfrac{1}{3^n+1} > 0$, so the series must converge by the **MCT**.

The Comparison Tests provide a faster way to use the same intuition that can also be applied to more situations!

Examples:

We will work through the following examples together.

Do the series listed below converge, diverge, or are we unable to determine the convergence behavior given the tools that we have?

Example 1: $$\sum_{n=1}^{\infty} \frac{1}{3^n + 1}$$

Example 2: $$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$$

Example 3: $$\sum_{n=1}^{\infty} \frac{1}{n!}$$

Example 4: $$\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$$

Problems for Group Work:

Be sure to fully justify your reasoning as a part of your solutions.
The answers are upside-down on the bottom of this page.

For Problems 1-4, use either the Comparison Test or the Limit Comparison Test to determine whether the series is convergent or divergent. If neither of the tests can be used, explain why.

Problem 1: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, 3^n}$

Problem 2: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}+1}$

Problem 3: $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Problem 4: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+300}$

Answers: **Problem 1:** Converge, **Problem 2:** Diverge, **Problem 3:** Cannot Determine, **Problem 4:** Converge