MATH 1336: Calculus III

Section 1.2: Calculus with Parametric Curves, Part 2

Integral Calculus with Parametric Curves:

Consider a curve C on the interval $a \le x \le b$ with parametric equations

$$x = x(t), \quad y = y(t), \quad \alpha \le t \le \beta$$

We would like to extend the concepts of **area** and **arclength** from Integral Calculus to the new setting of parametric curves.

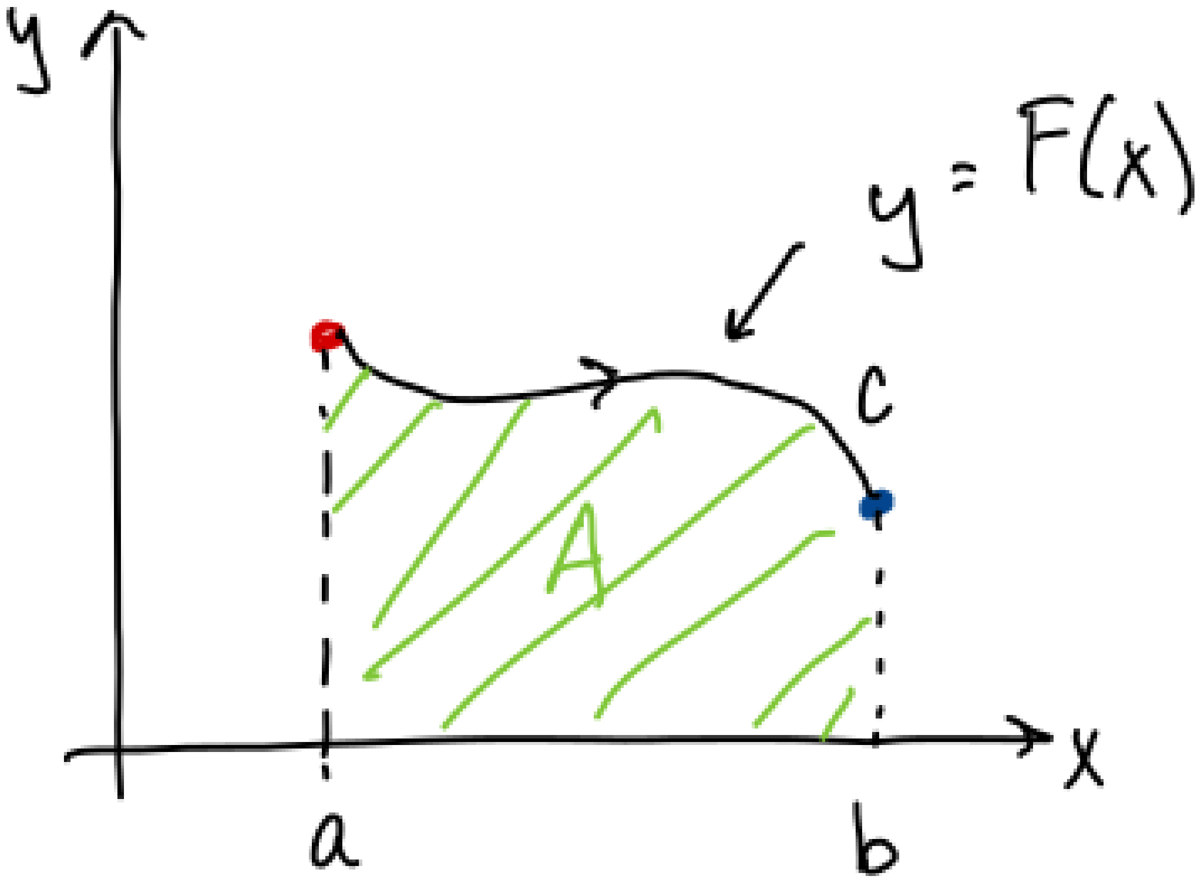

Figure 1: Diagram illustrating finding the area under a parametric curve.

Area:
To calculate the area under the curve $y = F(x)$ from $x = a$ to $x = b$:

$$A = \int_a^b F(x)\, dx = \int_a^b y\, dx = \int_\alpha^\beta y(t)\, x'(t)\, dt$$

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The set-up above is for a curve that is parametrized from left to right:

$$a = x(\alpha), \;\; b = x(\beta).$$

If the curve is parametrized from right to left, the limits of integration should be swapped so that the lower limit of integration corresponds to the left endpoint of the interval, $x = a$, and the upper limit of integration corresponds to the right endpoint of the interval, $x = b$.

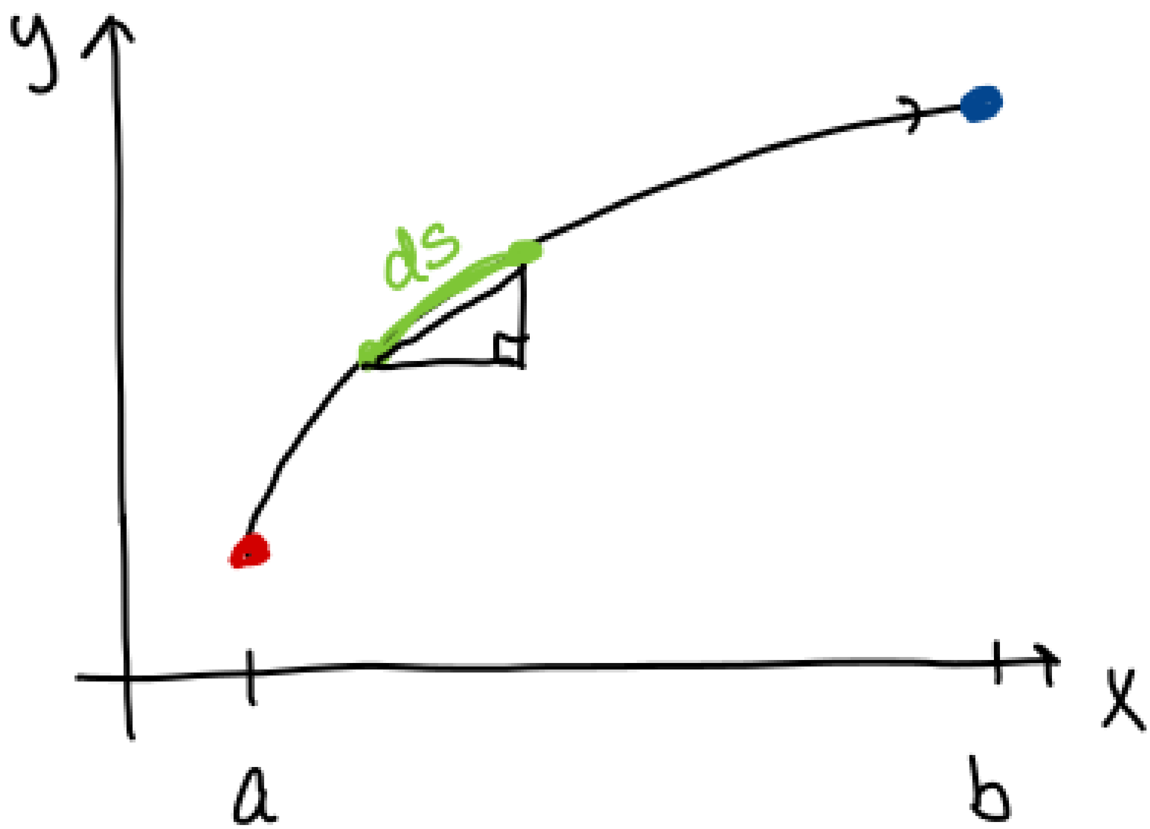

Figure 2: Diagram illustrating finding the length of a parametric curve.

Arclength:
To calculate the length, L, of the parametric curve from $t = \alpha$ to $t = \beta$, set up the following integral, where ds is the infinitesimal distance element along the curve:

$$L = \int ds = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note that the expression for ds comes from the Pythagorean theorem, and the idea that in the limit as each of the distance elements go to zero:

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

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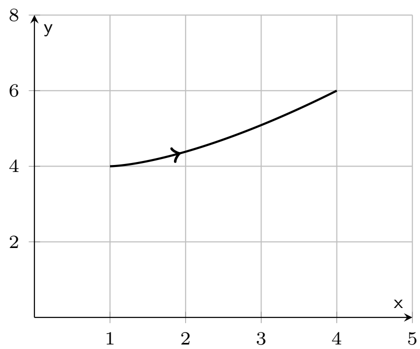
In-Class Problems:

We will work through Problem 1 together, then work on the remaining problems in small groups.

1. Consider the curve described by the following parametric equations, which is graphed on the axes below.

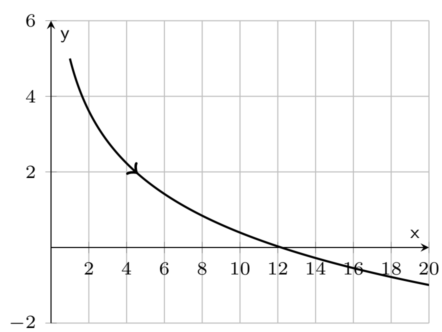
$$x = 1 + 3t^2, \qquad y = 4 + 2t^3, \qquad 0 \le t \le 1$$



(a) Set up and evaluate a definite integral to calculate the area under the curve between $t = 0$ and $t = 1$.
(b) Does your answer seem reasonable? Explain why or why not.
(c) Set up and evaluate a definite integral to calculate the length of the curve between $t = 0$ and $t = 1$.
(d) Does your answer seem reasonable? Explain why or why not.

2. Consider the curve described by the following parametric equations, which is graphed on the axes below.

$$x = e^t, \qquad y = 5 - 2t, \qquad 0 \le t \le 3$$



(a) Set up and evaluate a definite integral to calculate the area under the curve between $t = 0$ and $t = 3$.
(b) Does your answer seem reasonable? Explain why or why not.
(c) Set up, but **DO NOT** evaluate, a definite integral to calculate the length of the curve between $t = 0$ and $t = 3$.

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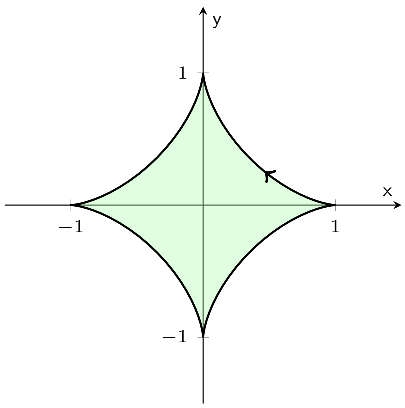
3. Consider the **Astroid** described by the following parametric equations, which is graphed on the axes below.

$$x = \cos^3(t), \qquad y = \sin^3(t), \qquad 0 \le t \le 2\pi$$



(a) Find the coordinates of the points corresponding to $t = 0$ and $t = \pi/2$.

(b) Set up and evaluate a definite integral to calculate L_1: the length of the part of the curve that lies in the first quadrant.

(c) Use the value that you calculated for L_1 to calculate the length of the entire Astroid.

(d) Does your answer seem reasonable? Explain why or why not.

(e) Set up, but **DO NOT** evaluate, a definite integral to calculate A_1: the area under the part of the curve that lies in the first quadrant.

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