MATH 1336: Calculus III

Section 6.3, Part 1: Intro to Taylor & Maclaurin Series

Taylor & Maclaurin Series Key Idea:

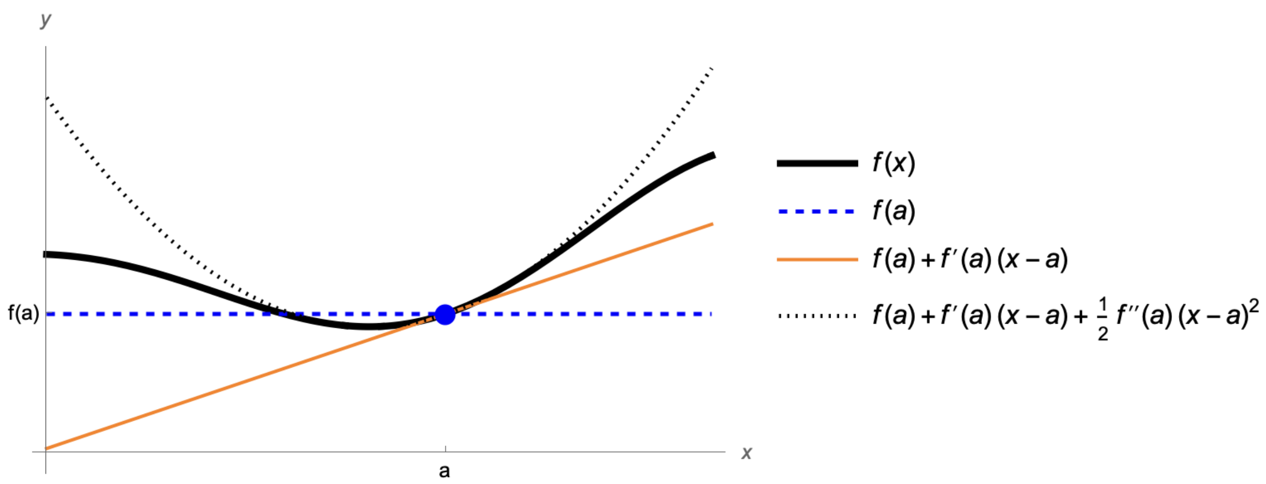

Figure 1: A graph showing a function and its zeroth, first, and second degree Taylor Polynomials, centered at x=a.

To build a polynomial of degree n that approximates the function $f(x)$ well near $x = a$, make sure that it matches the first n derivatives of f *exactly* at $x = a$.

The more derivatives we match, the better our approximation should be!

Taylor Polynomials -vs- Taylor Series:

A Taylor Polynomial of degree n (or order n) $T_n(x)$ terminates after the $(x-a)^n$ term:

$$T_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

while a Taylor Series is a power series centered at $x = a$ that typically has infinitely many terms. A Taylor Series that is centered at $x = 0$ is given a special name: Maclaurin Series.

Taylor Series for f(x) centered at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Maclaurin Series for f(x):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

Notation & Other Details:

- **Note:** We are "allowed" to represent f as the series above when x is within the radius of convergence, R, i.e.: $|x-a| < R$.
- $f^{(n)}(x)$ is the nth derivative of f with respect to x
- $f^{(0)}(x) = f(x)$
- $0! = 1$

Strategy for Finding Taylor/Maclaurin Series:

(1) Evaluate the first 4 or 5 derivatives of $f(x)$ at $x = a$.

(2) Look for a pattern that describes the coefficient of the $(x-a)^n$ term: $c_n = \dfrac{f^{(n)}(a)}{n!}$.

(3) If the "pattern" includes alternating signs, use either $(-1)^n$ or $(-1)^{n+1}$, depending on whether the first term is positive or negative.

Examples we will work through together:

Example 1: Find the Maclaurin Series for $f(x) = e^x$.

Example 2: Find the radius of convergence of the Maclaurin Series for $f(x) = e^x$.

Example 3: Find the Taylor Series for $f(x) = e^x$ centered at $x = 10$.

Problems for Group Work

Problem 1: Find the Maclaurin Series for $f(x) = \sin(x)$.

Problem 2: Use your answer to Problem **??** to discover the Maclaurin Series for $\cos(x)$.

Problem 3: Find the third order Taylor Polynomial, $T_3(x)$ for $f(x) = \ln(x^2)$, centered at $a = 1$.