MATH 1336: Calculus III

Section 5.1, Part 2: More Fun with Sequences!

More Sequence Definitions:

A sequence $\{a_n\}$ is **increasing** for all $n \geq n_0$ if:
$a_n < a_{n+1}$ for all $n \geq n_0$.

A sequence $\{a_n\}$ is **decreasing** for all $n \geq n_0$ if:
$a_n > a_{n+1}$ for all $n \geq n_0$.

A sequence is called **monotone** (or **monotonic**) for all $n \geq n_0$ if:
it is *either* increasing *or* decreasing for all $n \geq n_0$.

A sequence is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$.

A sequence is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$.

If a sequence is bounded above and below, then we call it a **bounded sequence**.

If a sequence is not bounded, it is an **unbounded sequence**.

Monotone Convergence Thm. (MCT):

Every bounded, monotone sequence is convergent.

The MCT is important because it is the foundation for proving most of the series tests we will be using later in this chapter!

(In the previous textbook for this class, this was "Monotonic Sequence Theorem (MST)," which may still appear in my notes/solutions as we transition to the new textboook.)

Geometric Sequences:

A sequence of the form

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \dots, \quad ar^{n-1}, \quad \dots$$

is called a **geometric sequence**. a is the first term, and r is the **common ratio**. The nth term is $a_n = ar^{n-1}$.

A geometric sequence converges if $-1 < r \leq 1$, and diverges otherwise.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

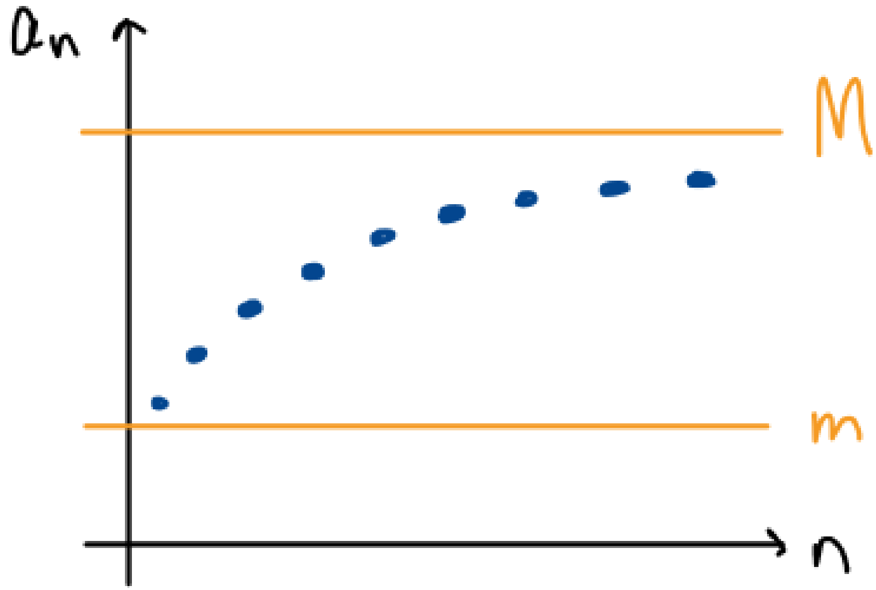

Figure 1: Illustration of a bounded increasing sequence, demonstrating the concept of the MCT.

Warm-up Problems:

Sketch graphs of sequences matching the descriptions below.

(a) a sequence that is decreasing and converges to a finite value

(b) a sequence that is bounded that does not converge

Examples:

1. **Challenge Problem from last time:**
   Determine the convergence of the sequence shown below, where $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$
   $a_n = \dfrac{(-3)^n}{n!}$

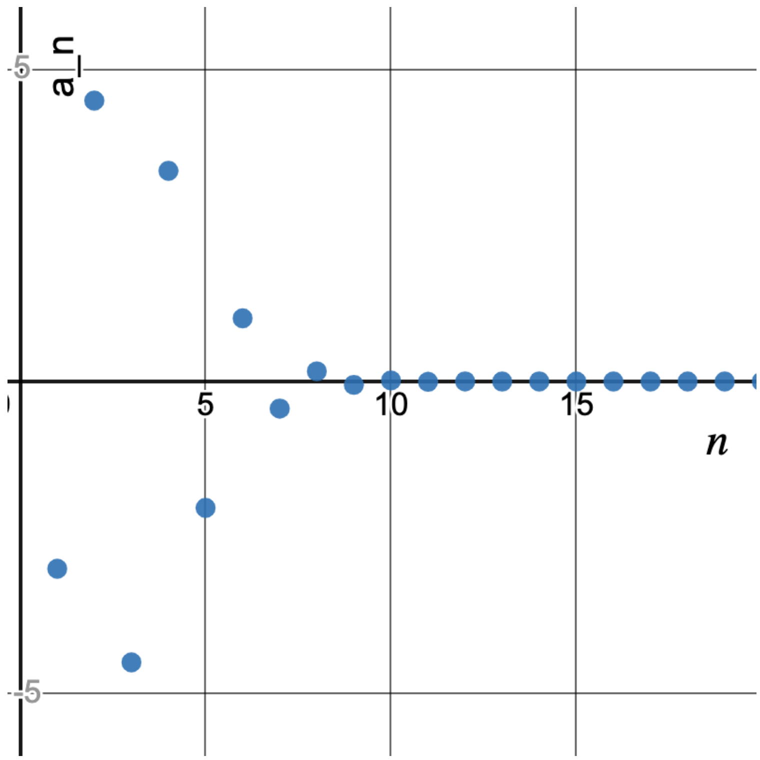

Figure 2: (Click here for desmos link)

Intuition & Plan: If we plot the first several terms of the sequence, we can build intuition that the sequence seems to converge to zero. If we can show that $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Today's Follow-up Question: Consider the sequence $|a_n| = \dfrac{3^n}{n!}$.
What, if anything, does the MCT say about this sequence? Can it help us determine the convergence of the original sequence?

2. Determine whether the sequence converges or diverges. If it converges, find the limit.
   $b_n = \ln(3n^2 + 1) - \ln(n^2 - 4)$

Problems for Group Work:

Is the sequence increasing, decreasing, or not monotone?
Is the sequence bounded?
Does the sequence converge or diverge?

1. $c_n = 4\left(\frac{-1}{3}\right)^n$

2. $d_n = 6^n$

3. $b_n = n + \frac{1}{n}$

4. $a_n = \frac{2n-3}{3n+4}$

The following sequences are geometric. Identify the common ratio, r. Does the sequence converge or diverge?

5.

$$2, \quad 6, \quad 18, \quad 54, \quad 162, \quad \ldots$$

6.

$$2, \quad 1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \quad \ldots$$

Transition Question:

What do you think will happen if we add up all of the (infinitely many) terms of the sequences shown in problems 5 & 6?