MATH 1336: Calculus III

Section 6.1: Intro to Power Series & Series Tests Practice, Round 3

Power Series Definition:

A series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

is called a **Power Series centered at** $x = a$.
(Power Series in $(x-a)$, Power Series about a)

Notes:

- A Power Series **always** converges at the center: $x = a$.
- Adopt the convention that $0^0 = 1$.
- By definition: $0! = 1$.

Power Series Convergence Theorem:

For a given Power Series, $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are three possibilities:

(i) The series converges *only* when $x = a$.

(ii) The series converges for all x.

(iii) There is some $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Radius & Interval of Convergence:

R is the **radius of convergence**, as described above.
The **interval of convergence**, I, is the interval of all the $x-$values where the series converges.

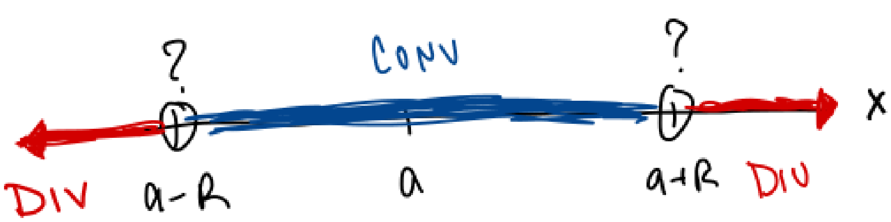

Figure 1: Interval of Convergence Diagram

Strategy:

1. Use the Ratio Test (or Root Test) to find the radius of convergence.
2. If you need to know the convergence behavior at the endpoints of the interval of convergence, use another series test!

Motivating Example (pre-class video):

Example 1: For which values of x does the following series converge? $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Using the Ratio Test, we found that:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}|x| = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^2|x| = |x| = L,$$

so the series converges when $|x| < 1$, diverges when $|x| > 1$, and the test is inconclusive for $|x| = 1$, indicating that the **Radius of Convergence** is $R = 1$. In order to determine the convergence at $x = \pm 1$, we must use other series tests: $\underline{x = 1:}$

$$\sum_{n=1}^{\infty}\frac{1^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

convergent $p-$series
$\Rightarrow$ series converges at $x = 1$

$\underline{x = -1:}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}, \qquad \text{so} \qquad \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges by A.S.T. or absolute convergence
$\Rightarrow$ series converges at $x = -1$ Therefore, the **Interval of Convergence** is $[-1, 1]$.

Examples we will work through together:

Find the radius and interval of convergence for the following series:

Example 2: $$\sum_{n=0}^{\infty} n!\, x^n$$

Example 3: $$\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$$

Example 4: $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

Problems for Group Work

Find the radius and interval of convergence for the following power series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+5)^n}{4^{n+1}}$

2. $\displaystyle\frac{(-5)^n(x-10)^n}{n!}$

3. $\displaystyle\sum_{n=0}^{\infty} e^n(x-2)^n$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n^2}$

5. $\displaystyle\sum_{n=0}^{\infty} x^n$

Series Testing Strategy Practice, Round 3 (All Series Tests!):

For each of the following series, state which test you would use to determine the convergence or divergence behavior, and explain why.

(You do not have to carry out the test in detail, but follow the argument long enough to make sure your reasoning would work.)

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$