1 Linear Polynomial

A first degree polynomial is represented by the equation

$$f(x) = ax + b$$

Given our secret 42 and a random $a = 4$ we can represent a straight line

$$f(x) = 4x + 42 \tag{1}$$

Suppose we have a single point on this line, say $P = (-17, -26)$, we want to want to find an equation that describes all possible lines that include this point. Given point P, we solve for b:

$$\begin{aligned} -26 &= -17a + b \\ b &= -26 + 17a \end{aligned} \tag{2}$$

Now we may substitute b from equation 2 into equation 1.

$$\begin{aligned} f_a(x) &= ax - 26 + 17a \\ f_a(x) &= a(x + 17) - 26 \end{aligned} \tag{3}$$

Equation 3 describes all lines passing through point $P = (-17, -26)$.

2 Quadratic Polynomial

A second degree, or quadratic, polynomial is represented as

$$f(x) = ax^2 + bx + c \tag{4}$$

Given a secret $S = 42$, we can choose random values for a and b. Suppose we choose $a = 7$ and $b = 3$. We may then write

$$f(x) = 7x^2 + 3x + 42 \tag{5}$$

Three points must be given to describe a quadratic equation. Given only two points, we want to find all quadratic curves that pass through the two given points.

Suppose we have points $P_1 = (2, 76)$ and $P_2 = (5, 232)$ discovered by choosing two x values randomly and solving equation 5 for $f(x)$. We may find the equation describing all curves passing through P_1 and P_2 by solving the system of equations as we did in the case of the linear polynomial.

Substitutine P_1 and P_2 into equation 4 we can write

$$\begin{aligned} f(2) &= 4a + 2b + c \\ 76 &= 4a + 2b + c \\ c &= -4a - 2b + 76 \end{aligned} \tag{6}$$

and

$$\begin{aligned} f(5) &= 25a + 5b + c \\ 232 &= 25a + 5b + c \\ c &= -25a - 5b + 232 \end{aligned} \tag{7}$$

Since we know equations 6 and 7 are equal, we can solve for b in terms of a as follows:

$$\begin{aligned} -4a - 2b + 76 &= -25a - 5b + 232 \\ 5b - 2b &= 232 - 76 + 4a - 25a \\ 3b &= -21a + 156 \\ b &= -7a + 52 \end{aligned} \tag{8}$$

We can now substitute equation 8 into equation 6 to write:

$$\begin{aligned} c &= -4a - 2(-7a + 52) + 76 \\ &= -4a + 14a - 104 + 76 \\ c &= 10a - 28 \end{aligned} \tag{9}$$

Now that we have solved for both b and c in terms of a, we may rewrite equation 4 as follows:

$$\begin{aligned} f_a(x) &= ax^2 + (-7a + 52)x + 10a - 28 \\ &= ax^2 + -7ax + 52x + 10a - 28 \\ f_a(x) &= a(x^2 - 7x + 10) + 52x - 28 \end{aligned} \tag{10}$$

Equation 10 is the set of quadratic curves that pass through points $P_1 = (2, 76)$ and $P_2 = (5, 232)$.

3 Cubic Polynomial

A third degree, or cubic, polynomial is represented as:

$$f(x) = ax^3 + bx^2 + cx + d \tag{11}$$

As before, our secret $S = 42$. We then select random values for a, b, and c. Suppose we have $a = 1$, $b = 3$, and $c = 13$. We may write:

$$f(x) = x^3 + 3x^2 + 13x + 42 \tag{12}$$

Four points are required to identify a specific cubic equation. Suppose we have three points $P_1 = (-6, -144)$, $P_2 = (1, 59)$, $P_3 = (4, 206)$. We can find the set of cubic polynomials that contains each of P_1, P_2, and P_3 by solving the system of equations as before. We can write:

$$\begin{aligned} f(-6) &= a(-6)^3 + b(-6)^2 + c(-6) + d \\ -144 &= -216a + 36b - 6c + d \\ d &= 216a - 36b + 6c - 144 \end{aligned} \tag{13}$$

$$\begin{aligned} f(1) &= a + b + c + d \\ 59 &= a + b + c + d \\ d &= -a - b - c + 59 \end{aligned} \tag{14}$$

$$\begin{aligned} f(4) &= 4^3a + 4^2b + 4c + d \\ 206 &= 64a + 16b + 4c + d \\ d &= -64a - 16b - 4c + 206 \end{aligned} \tag{15}$$

We may now represent c in terms of a and b by simplifying the equality of equations 13 and 14:

$$\begin{aligned} 216a - 36b + 6c - 144 &= -a - b - c + 59 \\ 7c &= -217a + 35b + 203 \\ c &= -31a + 5b + 29 \end{aligned} \tag{16}$$

And now repeat for equations 14 and 15:

$$\begin{aligned} -a - b - c + 59 &= -64a - 16b - 4c + 206 \\ 3c &= -63a - 15b + 147 \\ c &= -21a - 5b + 49 \end{aligned} \tag{17}$$

Repeating once again to solve for b by simplifying the equality represented by the equations 16 and 17:

$$\begin{aligned} -31a + 5b + 29 &= -21a - 5b + 49 \\ 10b &= 10a + 20 \\ b &= a + 2 \end{aligned} \tag{18}$$

At this point, we may substitute equations 18, 16, and 14 into the general cubic polynomial formula 11. Terms with b and c must be replaced as needed in equations 16 and 14 so that we have an equation in terms of a and x only. The algebra is simple but long so here is the result without further ceremony:

$$f_a(x) = a(x^3 + x^2 - 26x + 24) + 2x^2 + 39x - 41 \tag{19}$$

4 Polynomials

This is the general form of a polynomial of degree k.

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_kx^k$$

Or, more generally…

$$f(x) = \sum_{k=0}^{n} a_kx^k$$