CMSE 821: Iterative Methods and Multigrid

Analysis Problems

Problems

Problem 1. Convergence of Richardson Iteration

Consider the Richardson iteration for solving $Au = f$:

$$u^{k+1} = u^k + \omega(f - Au^k),$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite.

(a) Show that the iteration matrix is $T(\omega) = I - \omega A$.
(b) Derive $\rho(T(\omega))$ in terms of the eigenvalues λ_i of A.
(c) Find the optimal relaxation parameter ω_{opt} minimizing $\rho(T(\omega))$.
(d) Prove that with this choice,

$$\rho_{\min} = \frac{\kappa(A) - 1}{\kappa(A) + 1}, \quad \kappa(A) = \lambda_{\max}/\lambda_{\min}.$$

Problem 2. Spectral Properties of Jacobi for Poisson

Let A be the standard 1D Poisson matrix (tridiagonal with entries $-1, 2, -1$).

(a) Show that the eigenvectors are $\phi_j(i) = \sin(\frac{\pi j i}{n+1})$.
(b) Derive the eigenvalues λ_j.
(c) For Jacobi, compute the amplification factor $\mu_j = 1 - \frac{\lambda_j}{2}$.
(d) Identify which error modes converge fastest and slowest.
(e) Explain how this motivates the use of coarse grids in multigrid.

Problem 3. SOR Optimality on a Model Problem

Consider SOR for the 2D Poisson problem on an $n \times n$ grid.

(a) Show that the spectral radius of Jacobi is $\rho_J = \cos(\frac{\pi}{n})$.
(b) Use Young's analysis to argue that the optimal relaxation is

$$\omega_{\text{opt}} \approx \frac{2}{1+\sqrt{1-\rho_J^2}}.$$

(c) Estimate the convergence factor ρ_{SOR} for $\omega = \omega_{\text{opt}}$.
(d) How does ρ_{SOR} behave as n increases? Why does this motivate multigrid?

Problem 4. Two-Grid Analysis for 1D Poisson

Consider the 1D Poisson problem discretized with mesh size h. Let T_h be the Jacobi operator, and P_{2h}^h, R_h^{2h} the prolongation and restriction.

(a) Decompose error into Fourier modes ϕ_j. Which modes are damped by Jacobi?
(b) Prove that modes not damped are representable on the coarse grid.

(c) Derive the two-grid operator

$$M_h = T_h^{\nu_2}\big(I - P_{2h}^h A_{2h}^{-1} R_h^{2h} A_h\big) T_h^{\nu_1}.$$

(d) Using Fourier analysis, argue that $\rho(M_h)$ is bounded independently of h.

Problem 5. Multigrid vs. Krylov Methods

Suppose we solve a large sparse SPD system with both multigrid and Conjugate Gradient (CG).

(a) Prove that CG minimizes the error in the A-norm over the Krylov space $\mathcal{K}_k(A, r^0)$.

(b) Argue why multigrid with an exact coarse solve can converge in a fixed number of cycles independent of n.

(c) Compare asymptotic convergence factors: when is multigrid superior to CG?

(d) Discuss practical scenarios where multigrid is used as a preconditioner for Krylov methods and why this can be optimal.