CMSE 821: Numerical Methods for Differential Equations

Fall 2025 Syllabus

Instructor: Dr. Andrew Christlieb

Course Information

- **Course Title:** CMSE 821: Numerical Methods for Differential Equations
- **Meeting Times:** Mondays and Wednesdays, 10:20 AM – 11:40 AM
- **Location:** TBA
- **Term Dates:** August 25, 2025 – December 7, 2025
- **Final Exam Week:** December 8 – December 12, 2025

Instructor Information

- **Name:** Dr. Andrew Christlieb
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- **Email:** christli@msu.edu
- **Office Hours:** Mondays and Wednesdays, 1:00 PM – 2:00 PM

Course Description

This graduate-level course provides a rigorous introduction to numerical methods for solving partial differential equations. Topics include:

- Finite difference methods for linear and nonlinear PDEs
- Time stepping methods for evolutionary PDEs (explicit, implicit, IMEX)
- Iterative solvers and numerical linear algebra
- Stability, accuracy, and convergence analysis
- Introduction to multigrid methods and nonlinear dynamics

Prerequisites

Familiarity with differential equations, linear algebra, and and programming is assumed.

Textbooks

Required Textbook

- Randall J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*, SIAM, 2007.

Supplementary Texts

- Ascher, U. and Petzold, L., *Computer Methods for ODEs and DAEs*
- Trefethen, L. and Bau, D., *Numerical Linear Algebra*
- Trottenberg, U., Oosterlee, C., and Schuller, A., *Multigrid*
- Haller, G., *Modeling Nonlinear Dynamics from Equations and Data*
- Boscarino, S., Pareschi, L., and Russo, G., *Implicit-Explicit Methods for Evolutionary PDEs*

Grading

Homework Assignments (4 total)	40%
Final Project	20%, due on the day of the final
Midterm Exam	20%
Final Exam	20%

Grading Scale

- 4.0: 91% and above
- 3.5: 85–90%
- 3.0: 80–84%
- 2.5: 75–79%
- 2.0: 74% and below

Important Dates (Fall 2025)

- Classes Begin: Monday, August 25
- University Holiday (Labor Day): Monday, September 1
- Open Adds End: Friday, August 29
- Last Day for Refund: Thursday, September 18
- Middle of Semester: Monday, October 13 (Last day to drop with no grade)
- Fall Break: Monday–Tuesday, October 20–21
- Thanksgiving Break: Thursday–Friday, November 27–28

- Classes End: Sunday, December 7
- Final Exams: December 8–12
- Commencement: December 12–14

Assignments and Exams

- Homework will be assigned approximately every two weeks before they are due will include both analytical and computational components.
- Final project, see end of this document. It is due on the day of the final exam.
- Midterm exam will take place in late October (exact date Oct 15th 2025).
- Final exam will be held during the university-assigned final exam period. This exam will clover the whole crouse, as it counts as one of your CMSE qualifying exams.

Academic Integrity

All students are expected to adhere to MSU's policies on academic honesty. Collaboration on homework is allowed, but all submitted work must be your own.

Accommodations

Students requiring accommodations should work with the Resource Center for Persons with Disabilities (RCPD) and notify the instructor early in the semester.

Draft Final Project Description DUE on the Final Exam. You start work on this after the mid term, October 16th!

This final project include solving one of each of the key PDE's discussed in this class, elliptic, parabolic and hyperbolic. It also involves using some form of iterative solvers, either multi-grid or a Krylov method. Your solution must be at least second order accurate in space and third order in time. Implementing and demonstrating 4th order in space will earn a 15% extra credit. This can be done with a short time run, small T.

You will inherently be making use of IMEX methods in this project, implicit explicit methods. This is because your fluid model will use an explicit update and your heat and elliptic solves will be solved implicitly. Hyperbolic models are notoriously hard to solve implicitly with out adding a gibs phenomenon and elliptic equations already bring in iteration so it is natural to avoid small time step restrictions associated with explicit solutions of heat equations by using an implicit formulation for heat as well.

You must implement each of the methods your self. Hence, when we talk about Krylov or multi-grid solvers, it would be good to look at solving a heat equation and elliptic equations in 2D with Krylov or multi-grid solvers, so that you have these tools ready for the final project. Parts of these topics are covered in numerical linear algebra and your book for this class. Your NLA book has more information.

Your code must be your own. You may program in Python, C++ or Matlab. However, as you will have parallel computing next term, I highly recommended C++. The final project could serve as a baseline code to accelerate in your parallel computing class.

Simulating Driven Cavity Flow with Thermal Gradient in an L-Shaped Domain

The final project is to simulate thermally driven cavity flow governed by the incompressible Navier–Stokes equations with thermal coupling via the Boussinesq approximation. The domain is a unit square cavity $\Omega = [0,1]^2$ with a cut-out square of size $[0,0.25] \times [0,0.25]$ removed from the lower-left corner, resulting in an L-shaped geometry.

Project Goals

Students are expected to:

1. Discretize the below system using appropriate finite difference or finite volume methods.
2. Resolve the velocity, pressure, and temperature fields within the L-shaped domain.
3. Accurately capture the thermal boundary layer and driven vortex structures.
4. Explore variations in Rayleigh and Reynolds numbers and report how they affect convection rolls and temperature stratification.
5. Visualize vorticity, streamlines, and isotherms.

Governing Equations

Let $\mathbf{u}(\mathbf{x},t)$ be the velocity field, $p(\mathbf{x},t)$ the pressure, and $T(\mathbf{x},t)$ the temperature. The equations are:

1. **Mass conservation (Incompressibility):**

$$\nabla \cdot \mathbf{u} = \mathbf{0}$$

2. **Momentum equation:**

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla P + \mu \nabla^2 \mathbf{u} + \mathbf{f}$$

where $\mathbf{f} = \rho \mathbf{g}$ is the buoyancy force. Under the Boussinesq approximation, replace all ρ by a reference density, ρ_0, except in the buoyancy force,

$$\mathbf{f} = \rho_0 \beta (T - T_{ref}) \mathbf{g} \ .$$

This gives,

$$\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \nu \nabla^2 \mathbf{u} + \beta (T - T_{ref}) \mathbf{g}$$

Here:

- ρ_0 is the reference density
- μ is the dynamic viscosity
- $\mathbf{g}$ is gravitational acceleration (vector)
- $\nu = \rho_0^{-1} \mu$ is the kinematic viscosity
- β is the thermal expansion coefficient
- T_{ref} is a reference temperature
- $p = \frac{P}{\rho_0}$ is the kinematic pressure.

3. **Thermal energy (Advection-diffusion) equation:**

$$\rho_0 c_p \left(\frac{\partial T}{\partial t} + u \cdot \nabla T \right) = \kappa \nabla^2 T + Q$$

where:

- c_p is the specific heat at constant pressure
- κ is the thermal conductivity
- Q is the internal heat generation (optional)

In 2D, $\mathbf{x} = (x, y)$, You are to solve this model on an L-shaped domain with no heat generation.

Initial Conditions

At time $t = 0$, assume:

$$\begin{aligned} \mathbf{u}(x, y, 0) &= (0, 0)(m/s) \quad \text{(fluid initially at rest)} \\ T(x, y, 0) &= 300K \quad \text{(uniform initial temperature)} \\ p(x, y, 0) &= 0\text{m}^2/\text{s}^2 \quad \text{(reference pressure)} \end{aligned}$$

Note: Some smoothing or initialization techniques may be needed near discontinuities at thermal boundaries for numerical stability.

Boundary Conditions

- **Top lid ($y = 1m$, $0m \leq x \leq 1m$):**
  - Velocity: $\mathbf{u} = (0.1, 0)(m/s)$ (moving lid)
  - Temperature: $T = 300K$ (cooling)
- **Vertical wall of the interior cut-out ($x = 0.25m$, $0 \leq y \leq 0.25m$):**
  - Velocity: $\mathbf{u} = (0, 0)(m/s)$ (no-slip)
  - Temperature: $T = 301K$ (heating)
- **All other boundaries (including inner and outer cavity walls):**
  - Velocity: $\mathbf{u} = (0, 0)(m/s)$ (no-slip)
  - Thermal: $\frac{\partial T}{\partial \mathbf{n}} = 0$ (insulating)

Physical Parameters

using the laboratory conditions:

- Domain size: $L = 1$ m
- Lid velocity: $U = 0.1$ m/s
- Kinematic viscosity: $\nu = 0.01$ m^2/s $= \frac{\mu}{\rho_0}$
- Reference density of air: $\rho_0 = 1.2$ kg/m^3 at standard temperature and pressure
- Thermal diffusivity: $\alpha = 0.01$ m^2/s
- Thermal expansion coefficient: $\beta = 0.01$ 1/K
- Gravitational acceleration: $g = 10$ m/s^2 – Note this is rounded up for simplicity.
- Temperature difference: $\Delta T = 1$ K (e.g., $T_{\text{hot}} = 301$, $T_{\text{cold}} = 300$). **We note that the temperature filed is "non-dimensional", and these are non-dimensional values. Here $T_{ref} = 300.5$.**
- k_B is **Boltzmann's constant**, a fundamental constant from statistical mechanics that relates temperature to energy per particle, with a value of $k_B \approx 1.38 \times 10^{-23}$ J/K.

Assignment

Step 1 Non-dimensionalize the equations - 10% of the grade

Use the following definitions:

- $\mathbf{x} = L\hat{\mathbf{x}}$
- $\mathbf{u} = U\hat{\mathbf{u}}$
- $T = \Delta T \hat{T} + T_{cold}$,

and non-dimensionalize the equations such that the following physical constants appear in the system:

- Re is the **Reynolds number**, defined as
$$\mathrm{Re} = \frac{UL}{\nu}$$
where U is the characteristic velocity (e.g., lid velocity), L is the characteristic length (e.g., cavity width), and ν is the kinematic viscosity.

- Pr is the **Prandtl number**, defined as
$$\mathrm{Pr} = \frac{\nu}{\alpha}$$
where ν is the kinematic viscosity and α is the thermal diffusivity,
$$\alpha = \frac{\kappa}{\rho_0 c_p}\ .$$
It measures the relative thickness of momentum and thermal boundary layers.

- Ra is the **Rayleigh number**, given by
$$\mathrm{Ra} = \frac{g\beta(T_{\mathrm{hot}} - T_{\mathrm{cold}})L^3}{\nu\alpha}$$
where g is gravitational acceleration, β is the thermal expansion coefficient, and the numerator quantifies buoyancy. It characterizes the strength of natural convection.

- $\hat{\mathbf{y}}$ is the **unit vector in the vertical direction**, used to indicate that buoyancy forces act upward (in the $+y$ direction) in the momentum equation.

Dimensionless governing equations

You will arrive at the dimensionless governing equations are:

$$\begin{aligned}
\nabla \cdot \hat{\mathbf{u}} &= 0 \quad \text{(incompressibility)} \\
\frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} &= -\nabla \hat{p} + \frac{1}{\mathrm{Re}} \nabla^2 \hat{\mathbf{u}} + \mathrm{Ra}\mathrm{Pr}\, \hat{T}\, \hat{\mathbf{y}} \quad \text{(momentum)} \\
\frac{\partial \hat{T}}{\partial t} + \hat{\mathbf{u}} \cdot \nabla \hat{T} &= \frac{1}{\mathrm{Re}\mathrm{Pr}} \nabla^2 \hat{T} \quad \text{(thermal transport)}
\end{aligned}$$

We note that the self constantan Kinematic pressure comes form taking a divergence of the momentum, pressure Poisson equation:

$$\nabla^2 \hat{p} = \nabla \cdot \left(-\hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} + \frac{1}{\mathrm{Re}} \nabla^2 \hat{\mathbf{u}} + \mathrm{Ra}\mathrm{Pr}\, \hat{T}\, \hat{\mathbf{y}} \right)$$

Initial Conditions

At time $t = 0$, assume:

$$\begin{aligned}
\hat{\mathbf{u}}(x, y, 0) &= (0, 0) \quad \text{(fluid initially at rest)} \\
\hat{T}(x, y, 0) &= 0 \quad \text{(uniform initial temperature)} \\
\hat{p}(x, y, 0) &= 0 \quad \text{(reference pressure)}
\end{aligned}$$

Note: Some smoothing or initialization techniques may be needed near discontinuities at thermal boundaries for numerical stability.

Boundary Conditions

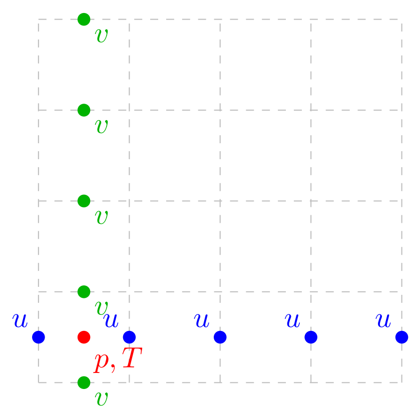
- **Top lid ($\hat{y} = 1,\, 0 \leq \hat{x} \leq 1$):**
  - Velocity: $\hat{\mathbf{u}} = (0.1, 0)$ (moving lid)
  - Temperature: $\hat{T} = 0$ (cooling)
- **Vertical wall of the interior cut-out ($\hat{x} = 0.25,\, 0 \leq \hat{y} \leq 0.25$):**
  - Velocity: $\hat{\mathbf{u}} = (0, 0)$ (no-slip)
  - Temperature: $\hat{T} = 1$ (heating)
- **All other boundaries (including inner and outer cavity walls):**
  - Velocity: $\hat{\mathbf{u}} = (0, 0)$ (no-slip)
  - Thermal: $\frac{\partial \hat{T}}{\partial \mathbf{n}} = 0$ (insulating)

Step 2: Grid Choice and Discretization - 10% of the grade

Choose a discretization method that enforce incompressibility and explain why you chose that method and how it works at a high level.

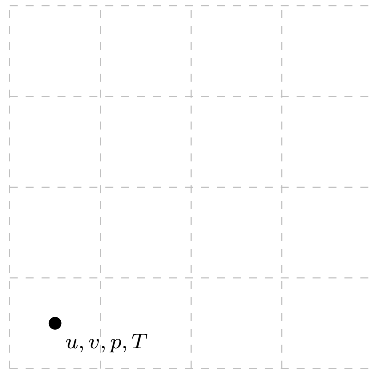
Students may choose to use either:

- A **staggered grid** (e.g., Marker-and-Cell (MAC) Method) to avoid pressure-velocity decoupling, or



MAC Grid Layout with Scalar Field T

- A **collocated grid** with appropriate stabilization techniques (e.g., Rhie–Chow interpolation or discrete divergence-gradient compatibility). The last idea of divergence-gradient compatibility is foundational in mimetic finite difference methods, compatible discretizations and finite volume methods with proper flux handling.



Collocated Grid Layout: All Variables at Cell Centers

This project must be implemented using a numerical solver of your own design, with structured or block-structured meshes supporting the non-rectangular cut-out geometry.

Step 3: Discretize the problem - 30% of the grade

Using an IMEX method in time that is 1st order, discretize the system using appropriate finite difference or finite volume methods to second order in space. **Prove that the method you have chosen satisfies the incompressibility condition at a discreet level**.

Step 4: Implement the method in your favorite language - 30% of the grade

Implement a 3rd order IMEX method with your spatial discretizations. You will have one elliptic solve, one hyperbolic update, and two parabolic solves. You should use your Multi-Grid code or Krylov Conjugate Gradient Code from the class to do these solves. If you can not get your own code to work, you may use a solve from a packages at a 15% deduction. **In practice, you should always use packages when they are available, but here we are learning about all the moving parts.**

As part of this step, do a refinement study to show that the method is 2nd order in space and 3rd order in time. Make sure to resolve the velocity, pressure, and temperature fields within the L-shaped domain and accurately capture the thermal boundary layer and driven vortex structures.

For an additional 15% extra credit, derive and implement a 4th order formulation and show that the method satisfies the incompressibility condition and us 4th order in space with a refinement study.

Step 4: Numerical study - 20% of the grade

Explore variations in Rayleigh and Reynolds numbers and report how they affect convection rolls and temperature stratification. Change the parameters and see how the results change. What happens to your time step as the dynamics change. Make sure to again resolve the velocity, pressure, and temperature fields within the L-shaped domain by running a range of resolutions. Develop visualizations for the vorticity, streamlines, and isotherms.

You must write up what you find and explain the results for full credit.

Final Note

This syllabus is subject to change with reasonable notice by the instructor.