

Suplmental Problem Set: Elliptic PDEs and Finite Difference Methods

CMSE 821: Numerical Methods for PDEs

August 23, 2025

Problem A: Consistency of a Second-Order Finite Difference Scheme

Consider the boundary value problem:

$$-u''(x) = f(x), \quad x \in (0, 1), \quad u(0) = u(1) = 0.$$

Let the finite difference scheme on a uniform grid with spacing h be:

$$\frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} = f(x_j), \quad j = 1, \dots, N-1.$$

- a) Derive the local truncation error using Taylor expansions.
- b) Show that the scheme is consistent of order $\mathcal{O}(h^2)$.
- c) Discuss what changes (if any) would arise if non-uniform grid spacing were used.

Problem B: Stability of a Three-Point Scheme Using Maximum Principle

Consider the same BVP as in Problem A. Use the standard centered finite difference scheme.

- a) Show that the coefficient matrix is strictly diagonally dominant.
 - b) Prove that the discrete solution satisfies a discrete maximum principle.
 - c) Use the maximum principle to argue that the scheme is stable in the maximum norm.
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Problem C: Convergence via Lax Equivalence Theorem

Again consider:

$$-u''(x) = f(x), \quad u(0) = u(1) = 0.$$

Let u_h be the numerical solution obtained using a consistent finite difference scheme.

- a) State the Lax Equivalence Theorem.
 - b) Given that the scheme is consistent and that the matrix is symmetric positive definite, explain why convergence follows.
 - c) Derive a bound for the error $\|u - u_h\|_\infty$ using the stability and consistency results.
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Problem D: Von Neumann Stability Analysis (Linearized)

Let the boundary value problem be:

$$-\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) = f(x), \quad u(0) = u(1) = 0,$$

with $a(x) > 0$ smooth and bounded. Let a finite difference scheme be constructed on a uniform grid using:

$$-\frac{1}{h^2} [a_{j+1/2}(u_{j+1} - u_j) - a_{j-1/2}(u_j - u_{j-1})] = f_j.$$

- a) Derive the scheme and explain the meaning of $a_{j\pm 1/2}$.
 - b) Assume $a(x) \equiv 1$ and perform a von Neumann analysis.
 - c) What does the result suggest about the stability of the scheme?
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Problem E: Convergence of Gauss–Seidel for a 1D Poisson Problem

Let the finite difference system for the 1D Poisson equation be solved with the Gauss–Seidel method.

- a) Prove that the coefficient matrix A is symmetric and positive definite.
- b) Show that the Gauss–Seidel iteration converges for this matrix.
- c) Estimate the spectral radius of the iteration matrix and relate it to the convergence rate.