Suplmental Problem Set: Elliptic PDEs and Finite Difference Methods

CMSE 821: Numerical Methods for PDEs

Due Sep. 22nd, 2025, 5% Extra Credit Total Grade

Problem A: Consistency of a Second-Order Finite Difference Scheme

Consider the boundary value problem:

$$-u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

Let the finite difference scheme on a uniform grid with spacing h be:

$$\frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} = f(x_j), \quad j = 1, \dots, N-1.$$

a) Derive the local truncation error using Taylor expansions.

b) Show that the scheme is consistent of order $\mathcal{O}(h^2)$.

c) Discuss what changes (if any) would arise if non-uniform grid spacing were used.

Problem B: Stability of a Three-Point Scheme Using Maximum Principle

Consider the same BVP as in Problem A. Use the standard centered finite difference scheme.

a) Show that the coefficient matrix is strictly diagonally dominant.

b) Prove that the discrete solution satisfies a discrete maximum principle.

c) Use the maximum principle to argue that the scheme is stable in the maximum norm.

Problem C: Convergence via Lax Equivalence Theorem

Again consider:

$$-u''(x) = f(x), \quad u(0) = u(1) = 0.$$

Let u_h be the numerical solution obtained using a consistent finite difference scheme.

a) State the Lax Equivalence Theorem.

b) Given that the scheme is consistent and that the matrix is symmetric positive definite, explain why convergence follows.

c) Derive a bound for the error $\|u - u_h\|_\infty$ using the stability and consistency results.

Problem D: Von Neumann Stability Analysis (Linearized)

Let the boundary value problem be:

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = f(x), \quad u(0) = u(1) = 0,$$

with $a(x) > 0$ smooth and bounded. Let a finite difference scheme be constructed on a uniform grid using:

$$-\frac{1}{h^2}\left[a_{j+1/2}(u_{j+1} - u_j) - a_{j-1/2}(u_j - u_{j-1})\right] = f_j.$$

a) Derive the scheme and explain the meaning of $a_{j\pm 1/2}$.

b) Assume $a(x) \equiv 1$ and perform a von Neumann analysis.

c) What does the result suggest about the stability of the scheme?

Problem E: Convergence of Gauss–Seidel for a 1D Poisson Problem

Let the finite difference system for the 1D Poisson equation be solved with the Gauss–Seidel method.

a) Prove that the coefficient matrix A is symmetric and positive definite.

b) Show that the Gauss–Seidel iteration converges for this matrix.

c) Estimate the spectral radius of the iteration matrix and relate it to the convergence rate.