Problem Set: Elliptic PDEs and Finite Difference Methods

CMSE 821: Numerical Methods for PDEs

Due Sep. 22nd, 2025

Problem 1: 1D Poisson Equation with a Gaussian Source

Consider the one-dimensional Poisson equation on the domain $x \in [0, 1]$:

$$-u''(x) = f(x), \quad \text{with} \quad u(0) = u(1) = 0,$$

where $f(x) = e^{-(x-0.5)^2/\sigma^2}$ is a Gaussian centered at $x = 0.5$.

- **a)** Derive a **fourth-order accurate finite difference approximation** to $-u''(x)$ using a uniform grid.
- **b)** Write the resulting **linear system** in matrix form $A\mathbf{u} = \mathbf{f}$. Describe the structure of the matrix A.
- **c)** Explain how **Dirichlet boundary conditions** are incorporated into your scheme.
- **d)** Estimate the **truncation error** and justify the fourth-order accuracy.
- **e)** Derive the **Jacobi iteration formula** for solving the linear system $A\mathbf{u} = \mathbf{f}$ obtained from your fourth-order scheme. Analyze its convergence behavior your method.
- **f)** Implement the scheme in code (Python, MATLAB, or C++). Plot the numerical solution and compare it to an analytical or highly resolved reference solution.
- **g)** Perform a **convergence study** using mesh refinement. Plot the error norms to verify fourth-order accuracy.

Problem 2: 1D geometric multi-grid

- **a)** Implement geometric multi-grid in 1D for problem 1.
- **b)** Analyze the convergence of your geometric multi-grid using a discreet Fourier series approach.
- **c)** Implement the scheme in code (Python, MATLAB, or C++). Plot the numerical solution and compare it to an analytical or highly resolved reference solution.

d) Perform a **convergence study** using mesh refinement. Plot the error norms to verify fourth-order accuracy.

Problem 3: 2D Poisson Equation on an L-shaped Domain

Consider the two-dimensional Poisson equation:

$$-\Delta u(x, y) = f(x, y),$$

on an L-shaped domain defined as:

$$\Omega = [0, 1]^2 \setminus [0.5, 1] \times [0.5, 1].$$

Use the right-hand side:

$$f(x, y) = \sin(\pi x) \sin(\pi y).$$

Apply the following boundary conditions:

- **Dirichlet:** $u = 0$ on the outer boundaries of Ω.
- **Neumann:** $\dfrac{\partial u}{\partial n} = 0$ on the inner boundaries of the removed square.

a) Derive a **second-order accurate finite difference scheme** for this problem on a uniform Cartesian grid adapted to the L-shaped domain.

b) Describe the treatment of **Dirichlet and Neumann boundary conditions** within your finite difference scheme.

c) Implement a **geometric multigrid solver**. Clearly explain the steps of smoothing, restriction, and prolongation.

d) Visualize the numerical solution and comment on how the **boundary conditions influence** the result.

e) Analyze and discuss the **convergence behavior** of the multigrid method.

Problem 4: Krylov Method (Conjugate Gradient)

Continue with the 2D Poisson problem from Problem 2.

a) Implement the **Conjugate Gradient (CG)** method to solve the linear system obtained from the discretization.

b) Compare the **convergence behavior** of the CG method with that of the geometric multigrid solver.

c) Investigate the effect of **preconditioning** on the CG method. Consider diagonal and incomplete Cholesky preconditioners.

d) Provide and interpret **plots of error norms versus iteration count** for both methods.

Problem 5: 2D Helmholtz Equation with a Compact Source

Consider the Helmholtz equation on a square domain $\Omega = [0,1]^2$:

$$-\Delta u(x,y) + \lambda u(x,y) = f(x,y), \quad \text{in } \Omega,$$

with homogeneous Dirichlet boundary conditions:

$$u(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega.$$

Let $\lambda > 0$ be a given constant. The source term $f(x,y)$ is compactly supported and centered at the midpoint $(0.5, 0.5)$ of the domain, and is defined as:

$$f(x,y) = \begin{cases} \exp\left(-\frac{r^2}{\sigma^2}\right), & r \leq R, \\ 0, & r > R, \end{cases}$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$, $R = 0.25$, and σ is a parameter controlling the sharpness of the Gaussian.

a) Derive a **second-order accurate finite difference scheme** to discretize the Helmholtz equation on a uniform Cartesian grid.

b) Implement the **Gauss-Seidel iterative method** to solve the resulting linear system.

c) Analyze the **stability and convergence** of the Gauss-Seidel method.

d) Solve the same problem using a **geometric multigrid** method. Compare the results.

e) Comment on differences in accuracy, iteration count, and performance.

Problem 6: 1D Variable Coefficient Equation with Mixed Boundary Conditions

Solve the equation

$$-\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = f(x), \quad x \in [0,1],$$

with boundary conditions:

$$u(0) = 0, \quad \frac{du}{dx}(1) = 0,$$

where $f(x) = \sin(\pi x)$.

a) Derive a second-order finite difference scheme for this variable coefficient problem.

b) Incorporate the Neumann boundary condition at $x = 1$ using a one-sided difference approximation.

c) Solve the resulting linear system using a direct solver.

d) Perform a grid refinement study and comment on the observed order of accuracy.

Problem 7: 1D Poisson Equation with Robin Boundary Conditions

Consider:

$$-u''(x) = f(x), \quad x \in [0, 1],$$

with boundary conditions:

$$u'(0) + \alpha u(0) = 0, \quad u(1) = 1,$$

where $f(x) = x(1 - x)$ and $\alpha = 5$.

a) Derive a second-order finite difference method for this problem, properly handling the Robin condition at $x = 0$.

b) Implement the scheme and solve the system for several values of α.

c) Discuss the impact of the Robin boundary condition on the boundary layer behavior.

Problem 8: 3D Laplace Equation in a Unit Cube

Solve:

$$\Delta u(x, y, z) = 0, \quad \text{in } [0, 1]^3,$$

with the boundary conditions:

$$\begin{aligned} u(x, y, 0) = u(x, y, 1) &= 0, \\ u(x, 0, z) = u(x, 1, z) &= 0, \\ u(0, y, z) &= 0, \\ u(1, y, z) &= \sin(\pi y)\sin(\pi z). \end{aligned}$$

a) Derive the 7-point finite difference stencil for discretizing the Laplacian.

b) Solve the resulting system using the Conjugate Gradient method with diagonal preconditioning.

c) Visualize the solution on the plane $x = 0.5$ and discuss physical interpretation.

Problem 9: Equation-Free Conjugate Gradient Method

This problem explores a structure-aware Conjugate Gradient (CG) solver where the system matrix is not explicitly formed, but instead accessed through an operator.

Consider a 2D Poisson equation with Dirichlet boundary conditions:

$$-\Delta u(x, y) = f(x, y), \quad (x, y) \in [0, 1]^2, \quad u|_{\partial\Omega} = 0.$$

Let $f(x, y) = \sin(\pi x)\sin(\pi y)$. Rather than assembling the matrix A, define a matrix-free function $v \mapsto Av$ that uses a 5-point stencil on a uniform grid.

a) Define a function that applies the 5-point stencil to a given grid function v, treating it as an operator A.

b) Implement a Conjugate Gradient method using only this function (i.e., without explicitly forming the matrix).

c) Compare the result and performance of this equation-free CG method with a standard matrix-based CG solver.

d) Discuss the potential benefits of equation-free methods in large-scale or adaptive mesh contexts.

Problem 10: Nonlinear 1D Boundary Value Problem

Consider the nonlinear boundary value problem:

$$-\frac{d}{dx}\left((1 + u(x)^2)\frac{du}{dx}\right) = f(x), \quad x \in (0, 1), \quad u(0) = 0, \quad u(1) = 1,$$

where $f(x) = x(1 - x)$.

a) Derive a second-order finite difference discretization of the problem using central differences.

b) Implement a nonlinear solver such as Newton's method to solve the resulting discrete system.

c) Discuss the consistency of your discretization.

d) Investigate how the nonlinearity affects stability and convergence of the method.

e) Compare solutions for several boundary values at $x = 1$, and interpret the behavior.