What are the consequences of having differing slit widths during the double slit experiment?

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15th September 2018

Abstract

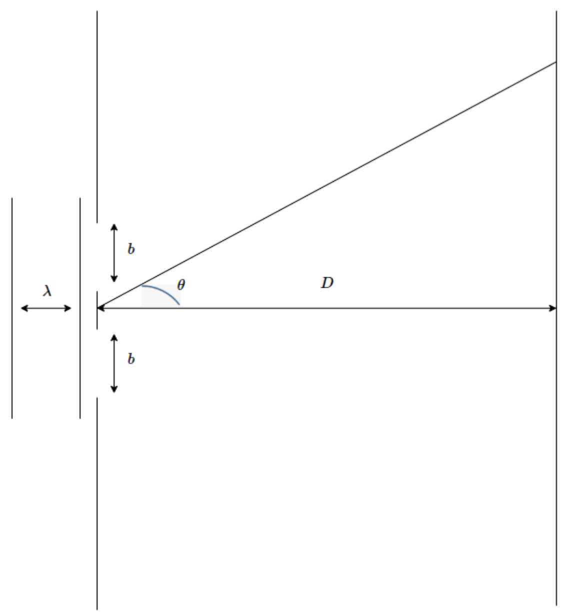
Young's Double Slit experiment is a well known investigation in quantum physics, but is traditionally performed using two slits of equal width, and uses the small wavelength of light to its advantage, e.g. by performing approximations, to yield a simple yet reliable theory of interference. This paper attempts to extend the theory to other waves, such as sound waves, that do not have such small wavelengths, and also attempts to generalise the theory further to account for differing slit widths rather than restricting itself to two sliths of the same width.

Contents

1 Zero intensity points

One of the most important parts of the standard Double Slit theory is the formula for finding the points at which the total intensity is zero. occurs. The formula relies on the fact that the path difference to any point on the screen from the two slits can be well approximated by $\Delta x = \frac{1}{2} D \sin\theta$. (See **Figure 1** below)

Figure 1: Double slit experiment with two sliths of equal width



The Double Slit Theory relies on the fact that the angle to any given point on the screen from both slits is approximately the same, because the distance to the screen D is much greater than the distance between the slits. However, when waves with larger wavelengths, e.g. sound waves, are used, this approximation is no longer justified. Therefore, this investigation will not use the same method, and, as will be seen later, will rely on the widths of the slits and the width of the object separating them instead. Let ω be the angular frequency of the waves. By virtue of the relationship $v = \lambda\frac{\omega}{2\pi}$, it is trivial to see that $\omega = 2\pi\frac{v}{\lambda}$, where v is the propagation speed of the waves. Since the two slits have equal widths, the amplitudes of the waves emitted from both slits are equal. If the wavelength of the waves used is much larger than the slit widths, the effects of diffraction will be negligible, and so the superposition S of the two waves can be modeled by the equation

$$S = \sin\omega t + \sin(\omega t + \phi). \tag{1}$$

The phase shift ϕ is given by $\phi = \frac{2\pi}{\lambda}\Delta x$, and zero intensity requires $S = 0$ for all $t \in \mathbb{R}$, hence we have

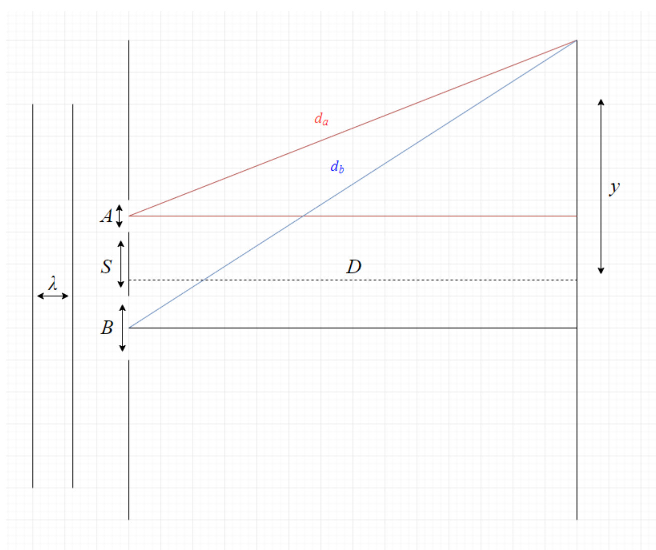
$$\phi = (2n-1)\pi, n \in \mathbb{Z} \implies \frac{1}{2}D\sin\theta = \frac{(2n-1)\lambda}{2},$$

which implies

$$D\sin\theta = (2n-1)\lambda \tag{2}$$

This is the famous *Double Slit Formula*. However, there are two issues, as the model in (1) is a simplification. Firstly, it does not account for the fact that the path difference of the two slits to the point will have an impact on the amplitudes of the two waves; the waves emitted from the slit closer to the point will have a larger amplitude. Of course, when the experiment is performed with light, this path difference is neglible compared to the distance to the screen, but when performed with e.g sound waves, it is no longer neglibile, and must be taken into consideration. We will use the *Law of inverse squares* (Berg 2018), and assume that the amplitude is inversely proportional to the square of the distance from the source to the screen. Let d_a and d_b be the distances from the slits to the screen. Let the two slits have widths A and B. The distance from the slits to the screen is D. The width of the separator between the slits is S, and the wavelength of the waves is λ. (See **Figure 2** below)

Figure 2: Double slit experiment with two sliths of different widths



The dashed line in **Figure 2** above defines the points that are equidistant from the bottom of slit A and the top of slit B as seen in the diagram. The distance from the top of slit A to the bottom of slit B is Then, the model in (1) becomes

$$S = \frac{1}{d_a^2} \sin \omega t + \frac{1}{d_b^2} \sin(\omega t + \phi), \tag{3}$$

Secondly, we must take into account that the two slits may not be same width. We will assume that the amplitude of the waves emitted from the slits is proportional to the width of the slits. For justification, note that a slit twice as large will collect twice as much sound, light, etc. Let the two slits have widths A and B respectively, and the model becomes

$$S = \frac{A}{d_a^2} \sin \omega t + \frac{B}{d_b^2} \sin(\omega t + \phi). \tag{4}$$

For there to be a solution to $S = 0$ for all $t \in \mathbb{R}$, the two sine waves must have equal amplitudes. So, we have

$$\frac{A}{d_a^2} = \frac{B}{d_b^2} \implies \frac{A}{B} = \frac{d_a^2}{d_b^2}. \tag{5}$$

The condition in (5) ensures the waves have the same amplitude. For the total intensity to be zero, they must also have a phase shift of π. Since the phase shift ϕ is given by $\phi = \frac{2\pi}{\lambda}\Delta x$, and the path difference is given by $\Delta x = |d_b - d_a|$, the phase shift can be expressed as

$$\phi = \frac{2\pi}{\lambda} |d_b - d_a| \tag{6}$$

Then, letting $\phi = (2n-1)\pi$, we get

$$\frac{2\pi}{\lambda}|d_b - d_a| = (2n-1)\pi,$$

which implies

$$|d_b - d_a| = \frac{2n-1}{2}\lambda \tag{7}$$

So, to summarise, the conditions for a total intensity of zero are:

$$\begin{cases} \frac{A}{B} = \frac{d_a^2}{d_b^2} \\ |d_b - d_a| = \frac{2n-1}{2}\lambda \end{cases} \tag{8}$$

2 Constructive interference

Another important part of the Double Slit Theory is the position of the points at which total constructive interference occurs. Modifying the *Double Slit Formula* (2), we obtain the condition

$$D\sin\theta = 2n\lambda. \tag{9}$$

However, once again, this formula does not generalise to waves with larger wavelengths and/or sliths with differing widths. Using the formula for the phase shift from (6) and the condition $\phi = 2n\pi$, we obtain the condition

$$|d_b - d_a| = n\lambda. \tag{10}$$

Notice that for zero intensity, there were two conditions:

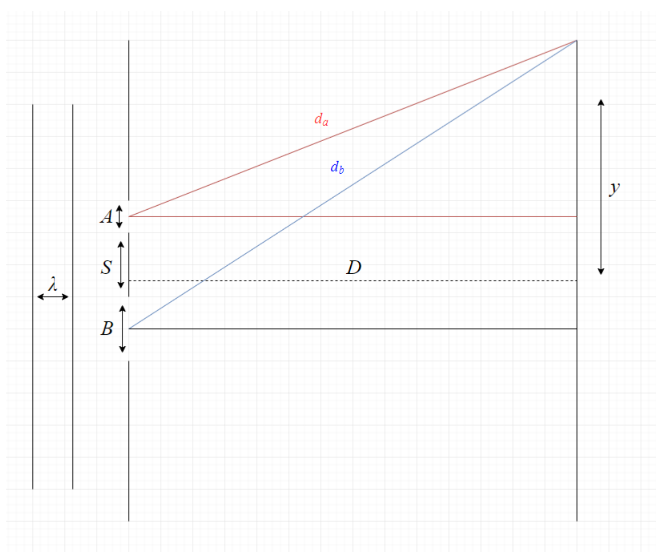
1 The amplitudes must equal (5).

2 The phase shift must be a half integer multiple of π (7).

However, for total constructive interference, the first condition is not necessary, so there is only the condition in (10).

3 Finding an expression for the distances from the slits to a given point

Recall the diagram from **Figure 2**:



The distance from the top of slit A to the bottom of slit B is given by

$$A + B + S.$$

Therefore, the distance from the top of slit A to the midpoint of the two slits is

$$\frac{A + B + S}{2}.$$

Subtracting the distance from the top of slit A to the middle of slit A ($\frac{A}{2}$) gives the distance from the middle of slit A to the midpoint of the two slits:

$$\frac{A + B + S}{2} - \frac{A}{2} = \frac{B + S}{2}.$$

Subtracting the above from y gives the length of one of the sides of the red triangle, the other having length D. Hence, d_a is given by

$$d_a = \sqrt{D^2 + \left(y - \frac{B + S}{2}\right)^2}. \tag{11}$$

The distance from slit B to the point can be found likewise, but adding instead of subtracting:

$$d_b = \sqrt{D^2 + \left(y + \frac{A + S}{2}\right)^2}. \tag{12}$$

4 Finding an expression for the phase shift at a given point

If the path difference Δx of two waves and their wavelength λ is known, it is easy to find their phase shift:

$$\phi = \frac{2\pi}{\lambda}\Delta x. \tag{13}$$

In the last section, the distances d_a and d_b from the slits A and B, respectively, to a given point were found. Then, the path difference is simply given by

$$\Delta x = |d_b - d_a|,$$

which, when incorporating the results from (11) and (12), yields

$$\Delta x = \left| \sqrt{D^2 + \left(y + \frac{A+S}{2}\right)^2} - \sqrt{D^2 + \left(y - \frac{B+S}{2}\right)^2} \right|$$

Then, using the formula from (13), we obtain

$$\phi = \frac{2\pi}{\lambda} \left| \sqrt{D^2 + \left(y + \frac{A+S}{2}\right)^2} - \sqrt{D^2 + \left(y - \frac{B+S}{2}\right)^2} \right| \tag{14}$$

5 Generalised formula for the intensity at any point

It would certainly be interesting to derive a formula for the intensity at any given point. To obtain the total intensity, we will add the amplitudes contributed by both slits, and square the total amplitude. Since intensity is proportional to the square of amplitude, and we are only interested in relative/arbitrary units, this is a legitimate method of obtaining the intensity.

5.1 Theoretical derivation

The amplitude coming from the slit with width A a distance d_a away from the point will contribute an amplitude

$$amplitude_a = \frac{A}{d_a^2}\sin\omega t.$$

Likewise, the other slit, with a width of B a distance d_b away from the point will contribute an amplitude

$$ampltiude_b = \frac{B}{d_b^2}\sin(\omega t + \phi).$$

So, adding them together, the amplitude of the superposition S is given by

$$S = \frac{A}{d_a^2}\sin\omega t + \frac{B}{d_b^2}\sin(\omega t + \phi).$$

Squaring the above expression gives the intensity I at a given point:

$$I = \left(\frac{A}{d_a^2} \sin \omega t + \frac{B}{d_b^2} \sin(\omega t + \phi) \right)^2 . \tag{15}$$

Notice, however, that the intensity seems to vary with time. However, if the frequency of the wave is very high, as is usually the case with light and/or sound waves, the human ear will not detect this variation; instead, a person would observe an *average intensity*. Furthermore, the intensity is given by the square of a sum of sine functions. The sine function is periodic, so after one full period, the intensity will have taken all the vales it will ever take. So, we will find the average intensity over one full period. During one period, ωt goes from 0 to 2π. Therefore, t goes from 0 to $\frac{2\pi}{\omega}$, and hence, we need to find the average intensity with respect to time over the inerval $\left[0, \frac{2\pi}{\omega}\right]$. The average value of some function f over some interval $[x_0, x_1]$ can be calculated by dividing the area under the curve on that interval by the length of the interval:

$$\text{avg}(f(x), [x_0, x_1]) = \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} f(x)\ dx.$$

So, the average intensity at a given point will be given by

$$I_{avg} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \left(\frac{A}{d_a^2} \sin \omega t + \frac{B}{d_b^2} \sin(\omega t + \phi) \right)^2\ dt.$$

Expanding the integrand and using the following facts:

$$\begin{cases} \int \Big[f(x) + g(x)\Big] \; dx = \int f(x) \; dx + \int g(x) \; dx, \\ \int c \times f(x) \; dx = c \int f(x) \; dx, c \in \mathbb{R}, \end{cases}$$

we obtain the following expression for the intensity:

$$\begin{aligned} I_{avg} = \frac{\omega}{2\pi} \left(\frac{A^2}{d_a^4} \int_0^{\frac{2\pi}{\omega}} \sin^2 \omega t \; dt \right) \\ + \frac{\omega}{2\pi} \left(\frac{2AB}{d_a^2 d_b^2} \int_0^{\frac{2\pi}{\omega}} \sin \omega t \sin(\omega t + \phi) \; dt \right) \\ + \frac{\omega}{2\pi} \left(\frac{B^2}{d_b^4} \int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t + \phi) \; dt \right) \end{aligned} \tag{16}$$

The first integral is simple to evalute:

$$\begin{aligned} & \int_0^{\frac{2\pi}{\omega}} \sin^2 \omega t \; dt \\ &= \int_0^{\frac{2\pi}{\omega}} \frac{1 - \cos 2\omega t}{2} \; dt \\ &= \frac{1}{2} \int_0^{\frac{2\pi}{\omega}} \Big(1 - \cos 2\omega t \Big) \; dt \\ &= \frac{1}{2} \left(t - \frac{\sin 2\omega t}{2\omega} \right) \Bigg|_0^{\frac{2\pi}{\omega}} \\ &= \frac{1}{2} \times \frac{2\pi}{\omega} \\ &= \frac{\pi}{\omega}. \end{aligned} \tag{17}$$

For the second integral, we will first find the indefinite integral:

$$
\begin{aligned}
&\int \sin \omega t \sin(\omega t + \phi)\ dt \\
&= \int \frac{-\cos(2\omega t + \phi) + \cos(-\phi)}{2}\ dt \\
&= \frac{1}{2} \int \Big(\cos(\phi) - \cos(2\omega t + \phi) \Big)\ dt \\
&\mathit{Let}\ u = \omega t + \phi \implies dt = \frac{du}{\omega} \\
&\frac{1}{2\omega} \int \Big(\cos\phi - \cos(2u - \phi) \Big)\ dt \\
&= \frac{1}{2\omega} \left(u \cos\phi - \frac{1}{2} \sin(2u - \phi) \right) + C \\
&= \frac{1}{2\omega} \left((\omega t + \phi) \cos\phi - \frac{1}{2} \sin(2\omega t + \phi) \right) + C \\
&= \frac{1}{4\omega} \Big[2(\omega t + \phi) \cos\phi - \sin(2\omega t + \phi) \Big] + C
\end{aligned}
$$

Now, we can evaluate the integral at the boundaries:

$$
\begin{aligned}
&\int_0^{\frac{2\pi}{\omega}} \sin \omega t \sin(\omega t + \phi)\ dt \\
&= \frac{1}{4\omega} \Big(2(\omega t + \phi) \cos\phi - \sin(2\omega t + \phi) \Big) \Big|_0^{\frac{2\pi}{\omega}} \\
&= \frac{1}{4\omega} \Big(4\pi \cos\phi + 2\phi \cos\phi - \sin(4\pi + \phi) - 2\phi \cos\phi + \sin\phi \Big) \\
&= \frac{1}{4\omega} \times 4\pi \cos\phi \\
&= \frac{\pi}{\omega} \cos\phi
\end{aligned} \tag{18}
$$

For the third and last integral, we will perform a simple u subsitution and find the indefinite integral:

$$Let\ u = \omega t + \phi \implies dt = \frac{du}{\omega} \implies \frac{1}{\omega} \int \sin^2 u \, du$$

Now, the procedure is simple and similar to that of the first integral, and is left to the reader. The third integral evaluates as follows:

$$\int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t + \phi) \, dt = \frac{\pi}{\omega}. \tag{19}$$

Now, using the expression from (16) and the calculated integrals from (17), (18), and (19), we can calculate the average intensity at any given point:

$$I_{avg} = \frac{\omega}{2\pi} \left[\frac{A^2}{d_a^4} \times \frac{\pi}{\omega} + \frac{2AB}{d_a^2 d_b^2} \times \frac{\pi}{\omega} \cos\phi + \frac{B^2}{d_b^4} \times \frac{\pi}{\omega} \right]$$
$$= \frac{1}{2} \left[\frac{A^2}{d_a^4} + \frac{2AB}{d_a^2 d_b^2} \cos\phi + \frac{B^2}{d_b^4} \right].$$

However, since we are using arbitrary units, the factor of $\frac{1}{2}$ is of no importance, so we can obtain the formula

$$I_{avg} = \frac{A^2}{d_a^4} + \frac{2AB}{d_a^2 d_b^2} \cos\phi + \frac{B^2}{d_b^4}.$$

However, this assumes that the waves are formed at the slits. In reality, the waves are formed first at the source, and then reformed at the slits due to diffraction. Let the perpendicular distance from the source to the slits be

k. Then, assuming the waves had an initial amplitude $A_{initial}$ at the source, once they reach the slits, they will have an amplitude $\frac{A_{initial}}{k^2}$, due to the *Law of Inverse Squares* (Berg 2018). Incorporating this into the expression for intensity above gives

$$I_{avg} = \frac{1}{k^2}\left(\frac{A^2}{d_a^4} + \frac{2AB}{d_a^2 d_b^2}\cos\phi + \frac{B^2}{d_b^4}\right). \tag{20}$$

5.2 Comparing the theory to experimental results

To test the theory, sound waves were used. It was expected that the theory would fit the experimental well when the wavelength used was much larger than the slit widths (corresponding to a low frequency), and that they would not agree very well when the wavelength used was not much larger than the slit widths.

5.2.1 The experiment

A single speaker was placed in a classroom, a distance k away from a doorway of width 87.5 cm. In practice, k was never varied; it was kept constant at 244 cm. (A 1 metre ruler was used for all distance measurements) A tall (taller than the doorway) piece of cardboard was used to define the slits. It was approximately 9 cm wide. By placing the cardboard such that its center was in the center of the doorway, two slits of equal width could be produced; moving it to the left or to the right would make one slit larger than the other; single slit experiments were performed by not using the cardboard at all.

Unfortunately, it was difficult to find a suitable location for the experiment that allowed for a screen like in the model. Instead, the sound intensity was measured at a distance of 248 cm from the doorway. The first measurement was done 192 cm 'to the right' of the midpoint, and measurements were made all the way to the midpoint, and then all the way to the point 192 cm 'to the left' of the midpoint. On the floor, there were small white circular dots, with a spacing of 24 cm. This made it easy to measure the intensity at intervals of 24 cm. Using the relationship $v = \lambda f \implies f = \frac{v}{\lambda}$, where $v = 340\,\mathrm{m\,s^{-2}}$, it was trivial to calculate which frequency of sound to use. Once the frequency had been determined, an online tone generator (Szynalski n.d.) was used to send monochromatic (only one wavelength) sound waves through the doorway. A mobile app (Abc Apps 2018) was used to record the sound intensity. However, the mobile app used gave values in dB. According to (Georgia State University n.d.), β dB is equivalent to an intensity I by the following equation:

$$\beta = 10\log_{10}\frac{I}{I_0},$$

where I_0, the 'stanard threshold of hearing' is equal to $10 \times 10^{-12}\,\mathrm{W\,m^{-2}}$ (Georgia State University n.d.). Solving for I yields

$$I = I_0 \times 10^{\frac{\beta}{10}}$$

The measured dB values were converted to watts per square metre using the formula above, and then plotted against the distance from the midpoint.

They were then normalised by dividing each intensity value by the maximum. For each measured value, a predicted value (according to the formula in (20)) was generated, and a graphw as plotted from these values as well. Once again, the predicted values were normalised. The recorded dB values and their value after conversion to SI units were also recorded, along with the dB and SI values predicted, in a text file. Below is the Python code used to record the results:

```
import math
import plotly.offline as py
import plotly.graph_objs as go

import numpy
print("Application started")

n = 1000
v = 340
D = 248
k = 244
l = 39
A_0 = 39
B_0 = 39
S = 9
```

```python
def is_number(string):
    try:
        float(string)
        return True
    except ValueError:
        return False


print("Use default parametres? ('yes' or 'no')")
res = input()
if (res.startswith('n')):
    print(f"Wave propagation speed (Default: {v} for 
        sound waves)")
    res = input()
    if is_number(res):
        v = float(res)

    print(f"Number of sources used in the approximation
         (Default: {n})")
    res = input()
    if is_number(res):
        n = float(res)
```

```
print(f"Distance from slits to screen in cm (
    default: {D})")
res = input()
if is_number(res):
     D = float(res)

print(f"Distance from source to slits in cm (
    default: {k})")
res = input()
if is_number(res):
     k = float(res)

print(f"Wavelength in cm (default: {l})")
res = input()
if is_number(res):
     l = float(res)

print(f"Width of slit A in cm (default: {A_0})")
res = input()
if is_number(res):
     A_0 = float(res)
```

```
    print(f"Width of slit B in cm (default: {B_0})")
    res = input()
    if is_number(res):
        B_0 = float(res)

    print(f"Width of separator in cm (default: {S})")
    res = input()
    if is_number(res):
        S = float(res)


def convert_decibel_to_intensity(dB):
    return 10 ** (-12) * 10 ** (dB / 10)


def convert_intensity_to_decibel(intensity):
    # Multiplying my 10^(12) is
    # almost certainly faster than dividing by 10^(-12)
    return 10*math.log10(intensity * 10**(12))


def d_a(x):
    return math.sqrt(D**2 + (x - (B_0 + S) / 2)**2)
```

```python
def d_b(x):
    return math.sqrt(D**2 + (x + (A_0 + S) / 2)**2)


def phi(x):
    return abs(math.pi / l * (d_b(x) - d_a(x)))


def I(x):
    return (1/k**2) * (A_0**2 / (d_a(x))**4 + 2*A_0*B_0
        *math.cos(phi(x)) / ((d_a(x))**2 * (d_b(x))**2)
        + B_0**2 / (d_b(x))**4)


def omega(x):
    return 2 * math.pi / l * v


def phi_a(x, i):
    return 2*math.pi / l * abs(d_a(x) - math.sqrt(D**2
        + (x - i * A_0 / n - (B_0 + S) / 2)**2))
```

```python
def phi_b(x, i):
    return 2 * math.pi / l * abs(d_b(x) - math.sqrt(D
        **2 + (x + i*A_0 / n + (A_0 + S) / 2)**2))


def A(x):
    A = 0
    i = -(n - 1) / 2
    while (i <= (n - 1) / 2):
        A += math.sin(phi_a(x, i))
        i += 1
    A = A * A_0 / n
    return A


def B(x):
    B = 0
    i = -(n - 1) / 2
    while (i <= (n - 1) / 2):
        B += math.sin(phi_b(x, i))
        i += 1
```

```
    B = B * B_0 / n
    return B


def I_d(x):
    return (A(x) + B(x)) ** 2


print("Input your values - Type 'done' when finished")

# Lists for measured values
X = []
Y = []

# Lists for predicted values
tx = []
ty = []

# Lists for predicted values accounting for diffraction
diffx = []
diffy = []

print("Created lists")
```

```
values = []
points = []

value = input()

if ('.txt' in value):
    data_filename = value
    data_file = open(f"data/experimental/{data_filename
        }")
    data_lines = data_file.readlines()

    i = 3
    while (i < len(data_lines)):
        X.append(float(data_lines[i].split(' ')[0]))
        Y.append(convert_decibel_to_intensity(
            float(data_lines[i].split(' ')[6])))
        i += 1

else:
        # Convert values to floats and add to list —>
            values array
```

```
    while value != 'done':
        values.append([float(value.split(',')[0]),
            float(value.split(',')[1])])
        value = input()

    for point in values:
        points.append([point[0],
            convert_decibel_to_intensity(point[1])])
print("Type the filename to save the data to")
filename = input()

# Define layout for plot
layout = go.Layout(
    title=f"D={D}, k={k},  ={l}, A={A_0}, B={B_0}, S={
        S}",
    xaxis=dict(
        title="Distance from center (cm)",
    ),
    yaxis=dict(
        title="Intensity"
    )
)
print("Defined layout")
```

```python
# Create separate lists for measured x and y values
for point in points:
    X.append(point[0])
    Y.append(point[1])

print("Generating theoretical prediction...")
# Generate a theoretical prediction
# with and without accounting for diffraction
# of the y value for each x value measured
  experimentally
for value in X:
    tx.append(value)
    ty.append(I(value))
    diffx.append(value)
    diffy.append(I_d(value))
print("Below are the lists:")
print(f"Y: {Y}")
print(f"ty: {ty}")
print(f"diffy: {diffy}")

print("Saving data to text file...")
```

```python
print("Name of file to save only experimental data to")
experiment_filename = input()

experimental_file = open(f"data/experimental/{filename
    }.txt", "w")

print(f"Saved experimental data to data/experimental/{
    experiment_filename}")

# Save experimental data to separate text file

# Write parametres
experimental_file.write("%s\n" %
                        f"D={D}, k={k}, lambda={l}, A={
                            A_0}, B={B_0}, S={S}"
                        )

# Write header 'Experimental'
experimental_file.write("\n%s\n" %
                        'Experimental'
                        )
# Loop over measured values and save them to text file
i = 0
```

```
while i < len(X):
    experimental_file.write("%s\n" %
                            f'{X[i]}      {values[i][1]} dB       {Y[i]}'
                            )
    i += 1



# Save data to text file:
thefile = open(f'data/{filename}.txt', 'w')

# Write parametres
thefile.write("%s\n" %
              f"D={D}, k={k}, lambda={l}, A={A_0}, B={B_0}, S={S}"
              )

print("Saving experimental data")

# Write header 'Experimental'
thefile.write("\n%s\n" %
              'Experimental'
              )
```

```
# Loop over measured values and save them to text file
i = 0
while i < len(X):
    thefile.write("%s\n" %
                  f'{X[i]}      {values[i][1]} dB      
                    {Y[i]}'
                  )
    i += 1

print("Saving theoretical data without diffraction")

# Write header 'Theory'
thefile.write('\n%s\n' % 'Theory')

# Loop over predicted values and save them to text file
i = 0
while i < len(X):
    thefile.write("%s\n" %
                  f'{tx[i]}      {
                    convert_intensity_to_decibel(ty[i
                    ])} dB      {ty[i]}')
    i += 1
```

```
print("Saving theoretical values with diffraction")

# Write header 'Theory with diffraction'
thefile.write('\n%s\n' % 'Theory with diffraction')

# Loop over predicted values accounting for diffraction
    and save
# them to text file
i = 0
while i < len(X):
    thefile.write("%s\n" %
                  f'{diffx[i]}   {
                      convert_intensity_to_decibel(diffy
                      [i])} dB')
    i += 1

print("Normalising experimental values...")
# Normalise the measured y-values to fit in [0, 1]
y_max = max(Y)
i = 0
while i < len(Y):
    Y[i] = Y[i] / y_max
    i += 1
```

```
print("Normalising theoretical values")
# Normalise the theoretically predicted y-values to fit
    in [0, 1]
i = 0
ty_max = max(ty)
while i < len(ty):
    ty[i] = ty[i] / ty_max
    i += 1

# Normalise the predicted y-values accounting for
   diffraction to fit in[0, 1]
i = 0
diffy_max = max(diffy)
while i < len(diffy):
    diffy[i] = diffy[i] / diffy_max
    i += 1

print("Saving normalised experimental data")

# Write header 'Normalised experimental'
thefile.write("\n%s\n" %
              'Normalised Experimental'
```

```
                    )
# Loop over measured values and save them to text file
i = 0
while i < len(X):
    thefile.write("%s\n" %
                        f'{X[i]}      {values[i][1]} dB      
                          {Y[i]}'
                        )
    i += 1

print("Saving normalised theoretical values with no 
   diffraction")

# Write header 'Normalised theory'
thefile.write('\n%s\n' % 'Normalised Theory')

# Loop over predicted values and save them to text file
i = 0
while i < len(X):
    thefile.write("%s\n" %
                        f'{tx[i]}      {
                          convert_intensity_to_decibel(ty[i
                          ])} dB      {ty[i]}')
```

```
    i += 1

print("Saving normalised theoretical values with 
    diffraction")

# Write header 'Normalised theory with diffraction'
thefile.write('\n%s\n' % 'Normalised Theory with 
    diffraction')

# Loop over predicted values accounting for diffraction
     and
# save them to text file
i = 0
while i < len(X):
    thefile.write("%s\n" %
                  f'{diffx[i]}      {
                     convert_intensity_to_decibel(diffy
                     [i])} dB      {diffy[i]}')
    i += 1

print("Creating traces...")

# Create a trace 'Experimental' for the measured values
```

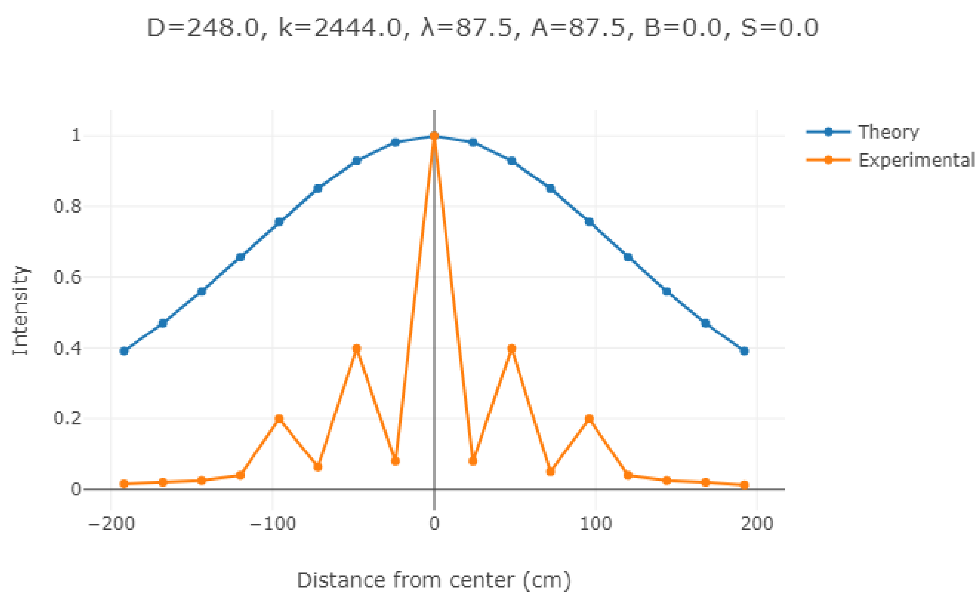
```python
experimental = go.Scatter(
    x=X,
    y=Y,
    name="Experimental"
)
# Create a trace 'Theory' for the theoretically
  predicted values
theory = go.Scatter(
    x=tx,
    y=ty,
    name='Theory'
)

# Create a trace 'Theory with diffraction' for the
  predicted values
# accounting for diffraction
diff_theory = go.Scatter(
    x=diffx,
    y=diffy,
    name='Theory with diffraction'
)
print(f"Saving data to file data/{filename}")
```

```python
print(f"Finished saving data to data/{filename} !")

print("Creating data from traces...")

# Create the data from the traces
data = [theory, diff_theory, experimental]

print("Creating figures using data and layout...")

# Create the figure using the data and layout
fig = go.Figure(data=data, layout=layout)

print("Plotting the figure and saving to html file")

# Plot the figure and save to html file
py.plot(fig, filename=f'plots/{filename}.html')

print(f"Plot saved to plots/{filename}.html ")
```

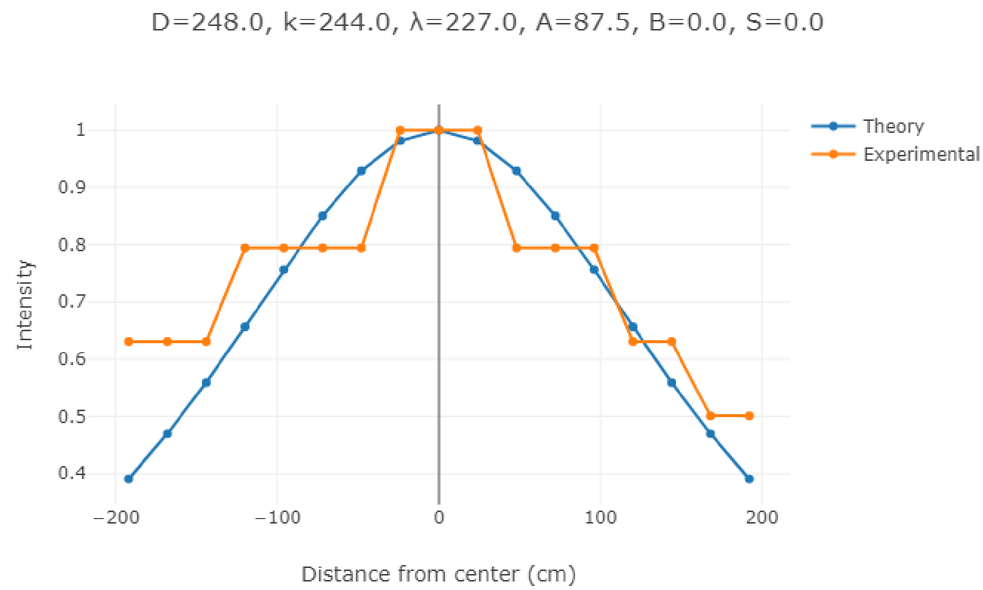
5.2.2 Experimental results

Six cases in particular were tested. One single-slit experiment, one double-slit experiment with two slits of equal widths, and one experiment where one slit was twice as wide as the other. For two of these cases, the experiment was performed first with a wavelength approximately as large as the largest slit, and afterwards with a wavelength much larger than the largest slit width. The third case was only investigated with a large wavelength.

<u>**One slit.**</u> Below is a plot of the predicted results vs. the measured results for a small wavelength:

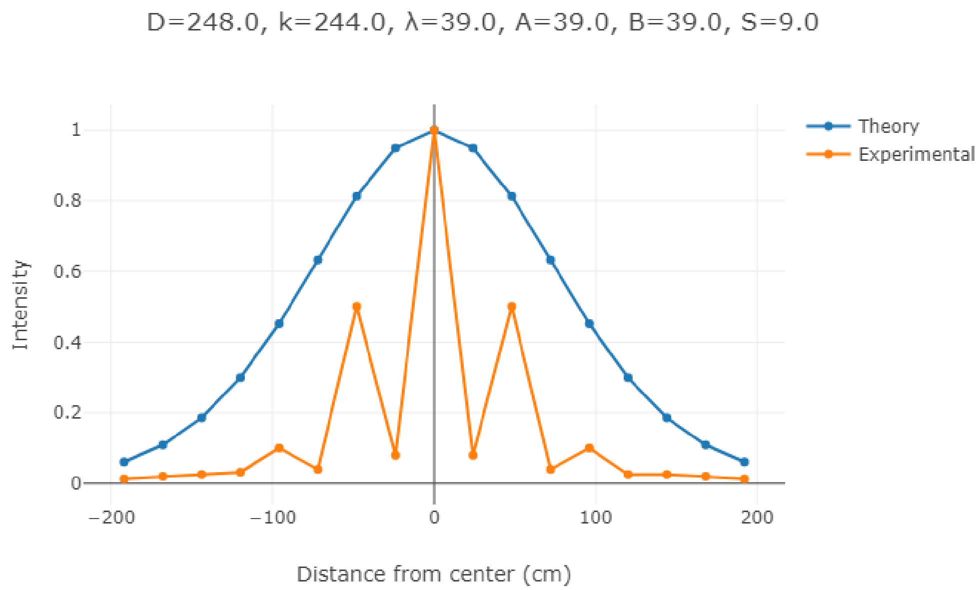


From the figure above, it can be seen that the theory does not fit the data. Now, let us compare theory and results when the wavelength is large:

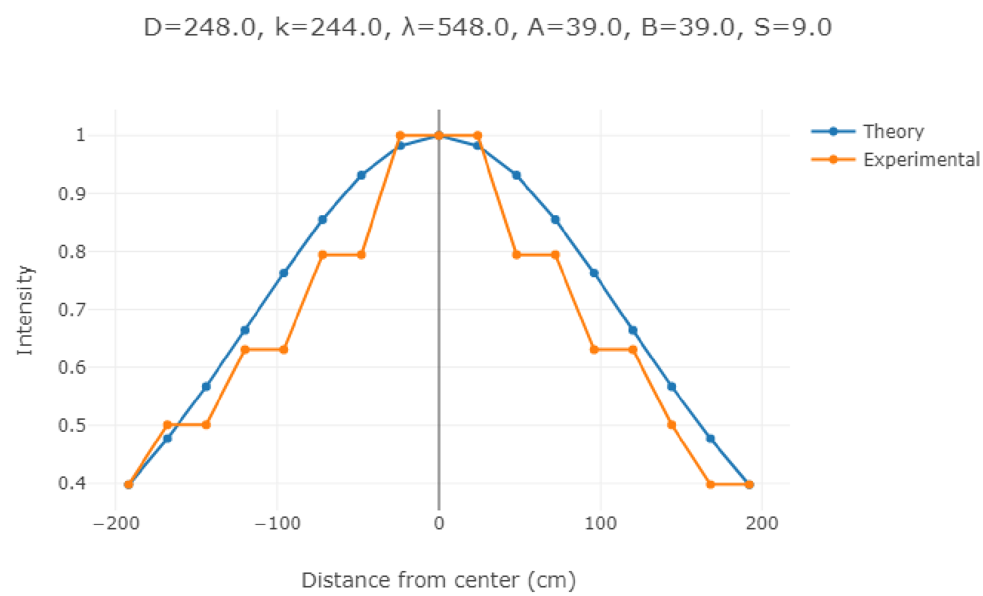


Clearly, in this case, the theory fit the measurements much better.

<u>Two slits of equal width.</u> Below is a plot of the predited results vs. the measured results for a small wavelength:

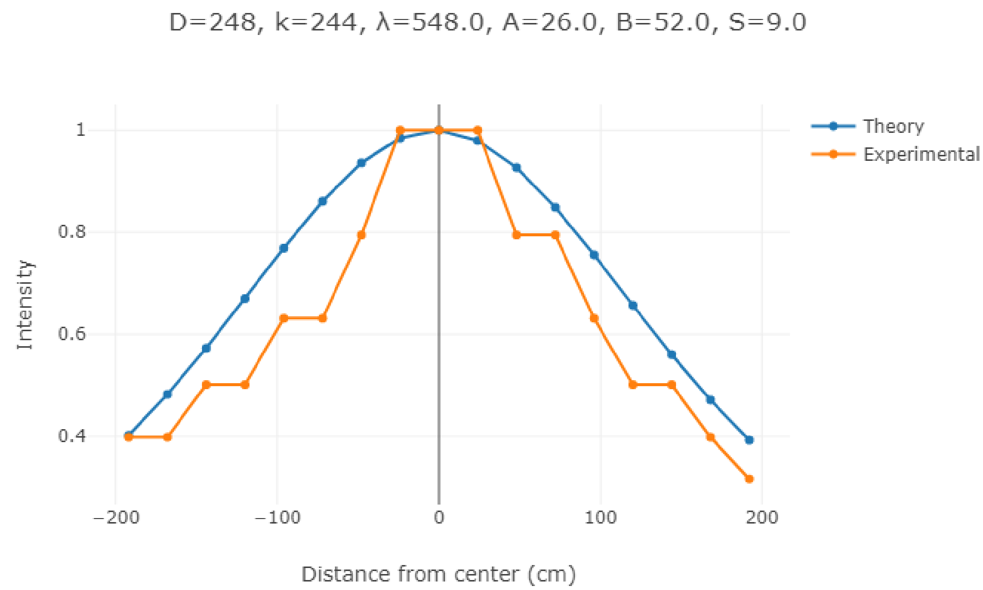


We can see that the theory most certainly does not fit the data. Now, let us compare the results for a large wavelength:



Once again, the theory is a much better fit when the wavelength is large.

<u>**One slit twice as wide as the other.**</u> Below are the results measured and the predicted results from the theory for two slits where one is twice as wide as the other:



Once again, the theory fits the data well. In **Section 1**, we said that the theory we have now developed would only provide an accurate description of the intensity curve if the wavelength was large because then the effects of diffration would be negligible. The results above seem to support this.

6 Accounting for diffraction

The intensity formula (20) derived in **Section 5** only accounts for interfernece between the waves. It assumes the waves travel as plane waves from the source to the slits, and then spreads out like a point source at the slits. In fact, as long as the slit wavelength is as large as or larger than the slit widths, there will be a diffraction pattern as well. As discussed in **Section 1**, if the wavelength is much larger than the slit widths, the effects of diffraction can be ignored. We will now attempt to extend our theory to account for diffraction as well.

6.1 Frauenhoffer diffraction

6.2 A mathematical derivation using Huygen's principle

Huygen's principle (citation) states that to explain diffraction, we may assume that when a wave undergoes diffraction, every point on the wavefront acts as a cirular point source; so there, will be infinitely many circular point sources. To obtain an expression for the effect on intenity caused by diffraction, we will assume that the wave consists of n point sources, and let n go to infinity.

Let 1 be the amplitude of the wave as it enters the slits. Since we are using relative units, we will let $1 = 1$. If there are n sources, then the amplitude 1 of the wave must be 'split evenly' across the sources. Hence, each source will have amplitude $\frac{1}{n}$. The distance from the ith source to the point will be some distance $d_a(i)$, Each source will have a certain phase shift relative to the other sources. Arbitrarily, we will choose the source in the middle of the slit to be in phase, and assign a phase shift $\phi_a(i)$ to the ith source, where the $\frac{n-1}{2}$th source is at the top of the slit, the 0th source is in the middle of the slit, and the $-\frac{n-1}{2}$th source is at the bottom of the slit. The amplitude

A_i of each of these sources at a given point will be given by

$$A_i = \frac{1}{n \times d_a(i)^2} \sin(\omega t + \phi_a(i)).$$

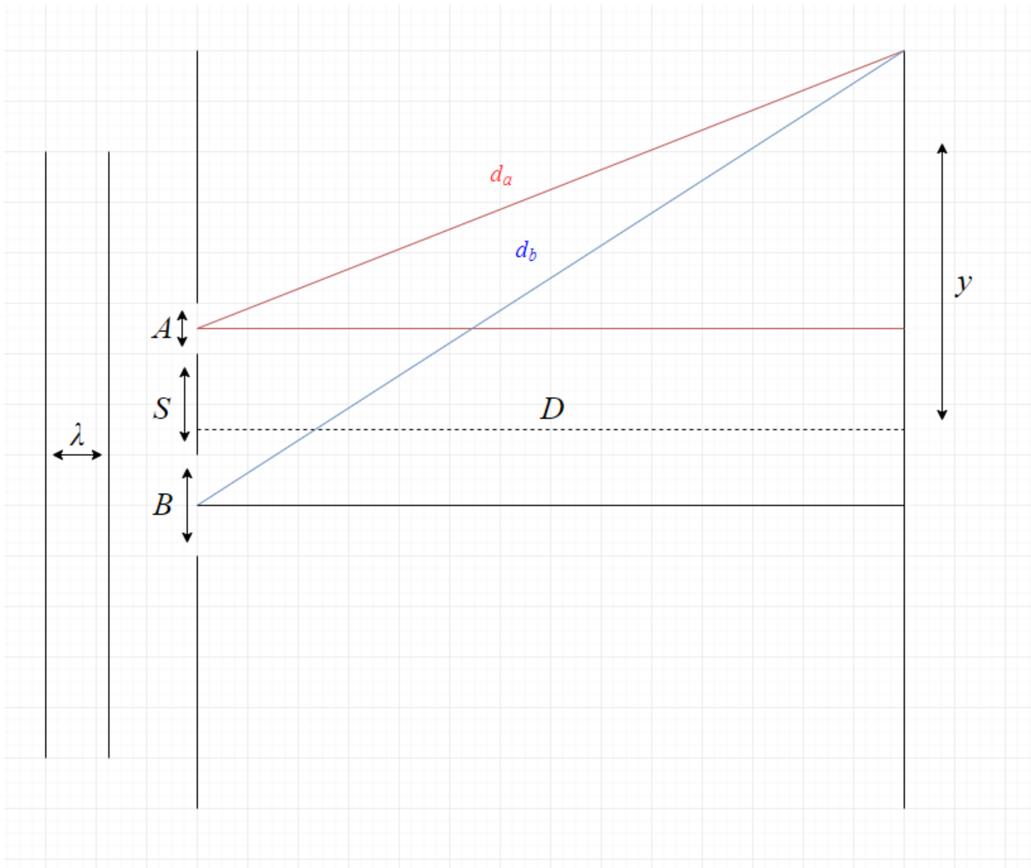
The total amplitude, given that there are n sources, at a given point will be the sum of the amplitudes of the sources:

$$A_d(n) = \sum_{i=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{n \times d_a(i)^2} \sin(\omega t + \phi_a(i)) \tag{21}$$

Now, we will attempt to derive an expression for $\phi_a(i)$.

6.2.1 Deriving an expression for $\phi_a(i)$

Recall the following diagram from **Section 1**:



The distance from the 0th source to the point is $d_a = \sqrt{D^2 + \left(y - \frac{B+S}{2}\right)^2}$. Here, D is the horizontal component of the distance, and $y - \frac{B+S}{2}$ is the vertical component. At the ith source, the vertical distance will be $y - \frac{B+S}{2} - \frac{iA}{n}$. Therefore, the distance from the ith source to a given point is given by

$$d_a(i) = \sqrt{D^2 + \left(y - \frac{iA}{n} - \frac{B+S}{2}\right)^2}.$$

Then, the path difference is given by

$$\Delta x_i = \left| \sqrt{D^2 + \left(y - \frac{B+S}{2}\right)^2} - \sqrt{D^2 + \left(y - \frac{iA}{n}\frac{B+S}{2}\right)^2} \right|.$$

Using the formula in (13), we obtain the following expression for the phase shift $\phi_a(i)$:

$$\phi_a(i) = \frac{2\pi}{\lambda} \left| \sqrt{D^2 + \left(y - \frac{B+S}{2}\right)^2} - \sqrt{D^2 + \left(y - \frac{iA}{n}\frac{B+S}{2}\right)^2} \right|. \quad (22)$$

6.2.2 Further derivation

Now, combining the formulas from (21) and (22), we obtain

$$A_d(n) = \sum_{i=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{n \times d_a(i)^2} \sin\left(\omega t + \frac{2\pi}{\lambda}\left(d_a - d_a(i)\right)\right).$$

We will now take the limit as n goes to infinity, yielding

$$A_d = \lim_{n\to\infty} \sum_{i=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{n \times d_a(i)^2} \sin\left[\omega t + \frac{2\pi}{\lambda}\left(d_a - d_a(i)\right)\right].$$

Let us now define a function Φ:

$$\Phi : x \to \frac{2\pi}{\lambda}\left(d_a - d_a(i)\right),$$

and a function f:

$$f : x \to \sin[\omega t + \Phi(x)].$$

If D is large relative to the slit widths, then, for any i and n, $d_a \approx d_a(i)$. Thus,

$$A_d = 1 \lim_{n\to\infty} \sum_{i=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{d_a^2} f\left(\frac{i}{n}\right)\frac{1}{n}.$$

Since d_a is a constant with respect to i, we can move it outside the summation, and since it is a constant with respect to n, we can move it outside the limit, yielding

$$A_d = \frac{1}{d_a^2} \lim_{n\to\infty} \sum_{i=-\frac{n-1}{2}}^{\frac{n-1}{2}} f\left(\frac{i}{n}\right)\frac{1}{n}.$$

Notice that $f\left(\frac{i}{n}\right)\frac{1}{n}$ is the area of a rectangle whose base is $\frac{1}{n}$ and whose height is $f\left(\frac{i}{n}\right)$. By summing from $i = -\frac{n-1}{2}$ to $\frac{n-1}{2}$, we are adding together the area of $\frac{n-1}{2} - \left(-\frac{n-1}{2}\right) + 1 = n - 1 + 1 = n$ of these rectangles. The combined length of all these rectangles will be $n \times \frac{1}{n} = 1$, and our summation limits are symmetric about 0, as, trivially, $-\frac{n-1}{2} = -\left(\frac{n-1}{2}\right)$. As n tends to infinity (corresponding to $\lim_{n\to\infty}$), our sum tends towards the following integral:

$$A_d = \frac{1}{d_a^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx.$$

Expanding $f(x)$ gives

$$A_d = \frac{1}{d_a^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} 1\sin\left[\omega t + \frac{2\pi}{\lambda}\left(d_a - \sqrt{D^2 + \left(y - \Gamma_0 x - \frac{B+S}{2}\right)^2}\right)\right].$$

Hence, we can rewrite the integral as follows:

$$A_d = \frac{1}{d_a^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\left[\omega t + \frac{2\pi}{\lambda}\left(\sqrt{D^2 + \left(y - \frac{B+S}{2}\right)^2} - \sqrt{D^2 + \left(y - u - \frac{B+S}{2}\right)^2}\right)\right]\,du. \tag{23}$$

Similar calculations for slit B, but adding a phase shift ϕ (14) to the sine function, yield a formula for the amplitude contributed by slit B:

$$B_d = \frac{1}{d_b^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\left[\omega t + \phi + \frac{2\pi}{\lambda}\left(\sqrt{D^2 + \left(y - \frac{B+S}{2}\right)^2} - \sqrt{D^2 + \left(y - u + \frac{A+S}{2}\right)^2}\right)\right]\,du. \tag{24}$$

6.2.3 Finding the intensity

As we did before without diffraction, to find the intensity, we will add the two amplitudes together, and square them:

$$I = \left(\frac{1}{d_a^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin \left[\omega t + \frac{2\pi}{\lambda} \left(\sqrt{D^2 + \left(y - \frac{B+S}{2} \right)^2} - \sqrt{D^2 + \left(y - u - \frac{B+S}{2} \right)^2} \right) \right] du \right.$$
$$\left. + \frac{1}{d_b^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin \left[\omega t + \phi + \frac{2\pi}{\lambda} \left(\sqrt{D^2 + \left(y - \frac{B+S}{2} \right)^2} - \sqrt{D^2 + \left(y - u + \frac{A+S}{2} \right)^2} \right) \right] du \right)^2 .$$

Unfortunately, none of these integrals can be evaluated analytically. Therefore, we will perform some approximations.

6.2.4 Approximations to simplify the intensity formula

Consider the mapping $z \to \sqrt{1+z}$. If z is small, then $\sqrt{1+z} \approx 1 + \frac{z}{2}$ (using the first two terms of the Maclaurin series of $\sqrt{1+z}$.) If D is large compared to $r := y - \frac{B+S}{2}$, and $u \in \left[-\frac{1}{2}, \frac{1}{2} \right]$, then

$$\begin{aligned}
\sqrt{D^2 + r^2} - \sqrt{D^2 + (r-u)^2} &= D \left[\sqrt{1 + \left(\frac{r}{D} \right)^2} - \sqrt{1 + \left(\frac{r-u}{D} \right)^2} \right] \\
&\approx D \left[1 + \frac{1}{2} \left(\frac{r}{D} \right)^2 - 1 - \frac{1}{2} \left(\frac{r-u}{D} \right)^2 \right] \\
&= \frac{u(2r-u)}{2D}.
\end{aligned}$$

If u is small in comparison to r, we can make further simplifications. If u is small compared to r, we have

$$r \gg u \implies y - \frac{B+S}{2} \gg u \implies y \gg 1 + \frac{B+S}{2}.$$

This can be accomplised by following:

1 $\frac{B+S}{2}$ can be made small by decreasing the width of the slits and/or the separator.

2 y is large far away from the midpoint on the screen.

Continuing, if $r \gg u$, then $\frac{u(2r-u)}{2D} \approx \frac{ru}{D}$. Now, let $p = y + \frac{A+S}{2}$, and all the same arguments can be made for the other slit. Thus, we can approximate our amplitude integrals as follows:

$$A_d \approx \frac{1}{d_a^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\left(\omega t + \frac{2\pi r u}{D\lambda}\right)\, du,$$

$$B_d \approx \frac{1}{d_b^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\left(\omega t + \phi\frac{2\pi p u}{D\lambda}\right)\, du$$

Both of these integrals can be evaluated analytically, as they are just sine functions with a constant period and a constant phase shift. Evaluating the integrals yields:

$$\begin{cases} A_d = \frac{D\lambda \sin(\omega t) \sin\left(\frac{r\pi}{D\lambda}\right)}{d_a^2 r\pi}, \\ B_d = \frac{D\lambda \sin(\omega t + \phi) \sin\left(\frac{r\pi}{D\lambda}\right)}{d_b^2 p\pi}. \end{cases}$$

Getting rid of constants such as π since we are using relative units we get the following expressions yields:

$$A_d = \frac{D\lambda \sin\left[\frac{\pi}{D\lambda}\left(y - \frac{B+S}{2}\right)\right]}{rd_a^2}\sin(\omega t), \tag{25}$$

$$B_d = \frac{D\lambda \sin\left[\frac{\pi}{D\lambda}\left(y + \frac{A+S}{2}\right)\right]}{pd_b^2}\sin(\omega t + \phi) \tag{26}$$

Then, expanding the square, we get

$$\begin{aligned} I = D^2\lambda^2 \Bigg[& \frac{\sin^2\left(\frac{r\pi}{D\lambda}\right)}{r^2 d_a^4}\sin^2(\omega t) \\ & + 2\frac{\sin\left(\frac{r\pi}{D\lambda}\right)\sin\left(\frac{p\pi}{D\lambda}\right)}{prd_a^2 d_b^2}\sin(\omega t)\sin(\omega t + \phi) \\ & + \frac{\sin^2\left(\frac{p\pi}{D\lambda}\right)}{p^2 d_b^4}\sin^2(\omega t + \phi)\Bigg]. \end{aligned}$$

Like in **Section 5**, our intensity depends on time. We will do as we did previously, finding an average value by integrating over one full period and dividing by the length of the period. Divind the resulting integral into three integrals, we obtain

$$\begin{aligned} I_{avg} = D^2\lambda^2 \Bigg[& \frac{\omega \sin^2\left(\frac{r\pi}{D\lambda}\right)}{\pi r^2 d_a^4}\int_0^{\frac{\omega}{2\pi}} \sin^2(\omega t)\ dt \\ & + \frac{\omega \sin\left(\frac{r\pi}{D\lambda}\right)\sin\left(\frac{p\pi}{D\lambda}\right)}{\pi prd_a^2 d_b^2}\int_0^{\frac{2\pi}{\omega}} \sin(\omega t)\sin(\omega t + \phi)\ dt \\ & + \frac{\omega \sin^2\left(\frac{p\pi}{D\lambda}\right)}{\pi p^2 d_b^4}\int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t + \phi)\ dt.\Bigg]. \end{aligned}$$

Evaluating the integrals yields

$$I_{avg} = D^2\lambda^2\left[\frac{\sin^2\left(\frac{r\pi}{D\lambda}\right)}{2r^2d_a^4} + \frac{\sin\left(\frac{r\pi}{D\lambda}\right)\sin\left(\frac{p\pi}{\lambda}\right)\cos\phi}{prd_a^2d_b^2} + \frac{\sin^2\left(\frac{p\pi}{D\lambda}\right)}{2p^2d_b^4}\right].$$

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