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3rd set - Large and Social Networks

Problem 1) 1) 1 central node
N-1 'leaf' nodes] star topology

link between <u>consecutive</u> leafs with probability p - independent

$P(\text{degree} = N-1) = 1$, for the central node (given, deterministic)

For the 'leaf' nodes:

$P(\text{degree} = 1) = \underbrace{(1-p)^2}_{\text{No link with the 2 neighboors}} \cdot \underbrace{1}_{\text{prob of link with central node}}$

$P(\text{degree} = 2) = (1-p) \cdot \underbrace{2p}_{\text{connection to 1 neighboor}} \cdot 1$

$P(\text{degree}=3) = p^2 \cdot 1$

↳ connection to 2 neighboors

2) Average degree of leaf node:

$$\Sigma\left[(\text{degree}=i)\cdot P(\text{degree}=i)\right]$$

$$= 1\cdot(1-p)^2\cdot 1 + 2\cdot 2p(1-p)\cdot 1 + 3p^2$$

$$= 1^2 - 2p + p^2 + 4p - 4p^2 + 3p^2$$

$$= 1 + 2p + \cancel{p^2} - \cancel{p^2}$$

$$= 2p + 1$$

total_avg_degree =

$$\frac{\overset{\text{leaf}}{(N-1)}(2p+1) + \overset{\text{central}}{1}(N-1)}{N} =$$

$$\frac{2pN - 2p + N - 1 + N - 1}{N} =$$

$$\frac{2pN + 2N - 2p - 2}{N} =$$

$$2\left(p + 1 - \frac{p+1}{N}\right)$$

3) Probability that 2 neighboors are also neighboors

for central node $c = 0$.

For leaf nodes - probability that consecutive neighboors are connected

So total $C = \frac{1}{N}\left(C_{central} + \underbrace{\sum_{i=1}^{N-1} C_{leaf}}_{\text{leaf nodes}}\right)$

$\Leftrightarrow \quad C = \frac{1}{N}\left(0 + (N-1)p\right) = \frac{N-1}{N}p$

$\simeq p$

because:

$C_{leaf} = \frac{p\overbrace{\binom{degree}{2}}^{\text{num of closed triplets}}}{\underbrace{\binom{degree}{2}}_{\text{num of posible triplets}}}$ so $\frac{p\binom{3}{2}}{\binom{3}{2}}$

4) Remove edges with probability $1-q$

$$P(\text{degree}=1) = (1-p)^2 \cdot q + 2p(1-p)\underbrace{(1-q)}_{\text{links removed}}$$

$$P(\text{degree}=2) = 2 \cdot p(1-p) \cdot q + p^2 \underbrace{(1-q)}$$

$$P(\text{degree}=3) = p^2 \cdot q$$

$$P(\text{degree}=N-1) = 1 \quad \text{as before}$$

Removing $1-q$ of $N-1$ links leaves the remaining q of $N-1$ links which is still a giant component element, unless: $q(N-1) < 1$

So: $q < \dfrac{1}{N-1}$ is the threshold

Problem 2)

a) red, blue create random graphs

average degree: $\langle k \rangle = p(N-1)$
condition for giant component: $\langle k \rangle = p(N-1) \geq 1$

$$\Leftrightarrow \boxed{p \geq \frac{1}{N-1}}$$

Average degree between red, blue nodes will be q, by definition, so for the mixed graph to have giant component:

$$q \cdot N \geq 1 \Rightarrow \boxed{q \geq \frac{1}{N}}$$

Both are the required minimal values (the equalities).

(b) The graphs (of only red/blue) are Erdös-Rényi so they are small world. So for the total graph to be small world we depend on the mixed graph, meaning on the values of q.

So, $q \geq \frac{1}{N}$ is a good condition, because

every node from each graph will connect to a node from the other graph.

and so the mixed graph is a mixed Random graph with:

$$\langle k \rangle = p(N-1) + q \cdot N$$

(c) Using the result from b, each node chooses neighboor to include with prob:

$$p = \frac{1}{\langle k \rangle} = \frac{1}{p(N-1)+qN}$$

If the starting red node is connected to a blue one we can get there with prob:

- $\frac{qN}{p(N-1)+qN}$ else, through another red node:

- $\frac{p(N-1)}{p(N-1)+qN} \cdot \frac{qN}{p(N-1)+qN} =$

$$\frac{p(N-1) \cdot qN}{(p(N-1)+qN)^2}$$

(d) We have the average degree from 6 and we know Erdös - Rényi are balanced, as node have around the same number of neighbours ⊛

from theory $p(x) \sim \frac{kx}{2|E|} \overset{⊛}{\approx} \frac{\langle k \rangle}{2|E|}$

$$2|E| \to 2N(p(N-1)+qN)$$

So the rate is ...