Message passing and expectation propagation

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Overview

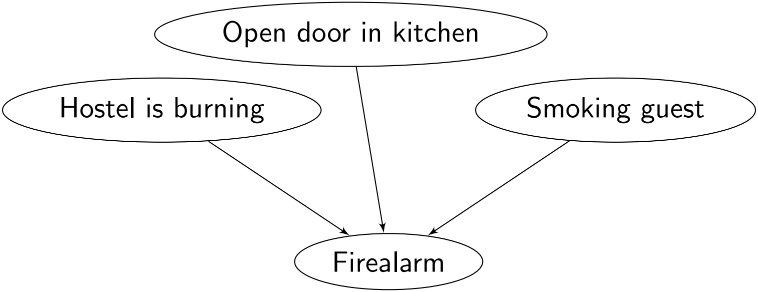
Graphical models: Bayesian networks and Markov random fields

$$p(X) = \prod_s f_s(X_s)$$

Inference in graphical models:

- Marginalization
- Maximum aposteriori estimation
- Posterior approximation

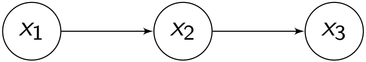
Motivation for marginalization



Factor graph

More simple and formally:

$$p(X) = p(x_1)p(x_2|x_1)p(x_3|x_1)$$



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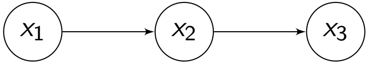
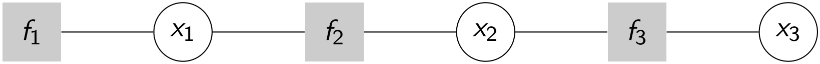


A corresponding factor graph: $f_1 = p(x_1)$, $f_2 = p(x_2|x_1)$, $f_2 = p(x_3|x_2)$



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Message passing

Message passing: Idea

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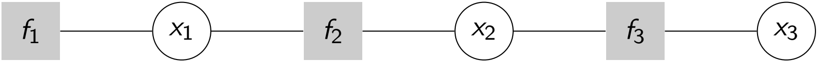

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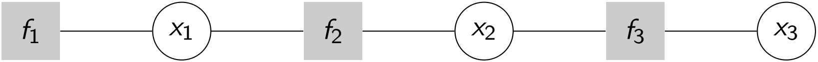

Marginalization naive:

$$p(x_2) = \sum_{x_1} \sum_{x_3} p(X) \in \mathcal{O}(k^n)$$

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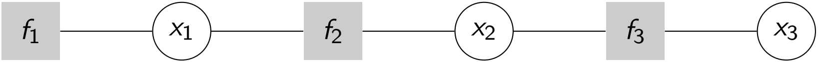

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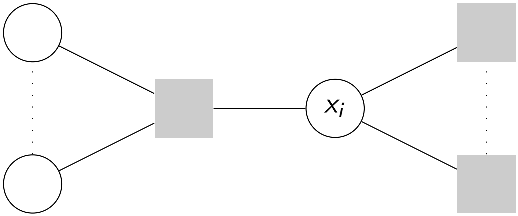
Marginalization advanced:

$$\begin{aligned} p(x_2) &= \sum_{x_1} \sum_{x_3} p(x_1) p(x_2 | x_1) p(x_3 | x_2) \\ &= \underbrace{\left[\sum_{x_1} p(x_1) p(x_2 | x_1) \right]}_{\mu_{f_2 \to x_2}} \cdot \underbrace{\left[\sum_{x_3} p(x_3 | x_2) \right]}_{\mu_{f_3 \to x_2}} \in \mathcal{O}(n \cdot k^2) \end{aligned}$$

Sum-product algorithm 1

General marginalization:

$$p(x_i) = \prod_{s \in ne(x_i)} \mu_{f_s \to x_i}(x_i)$$



Sum-product algorithm 2

Local messages through factor graph:

$$\mu_{f_s \to x_i}(x_i) = \sum_{\mathbf{x_s} \setminus x_i} f_s(x_i, X_s) \prod_{m \in ne(f_s) \setminus x_i} \mu_{x_m \to f_s}(x_m)$$

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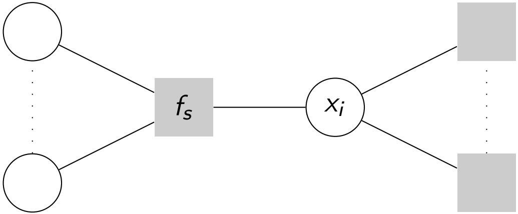

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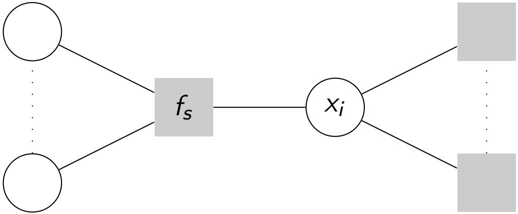
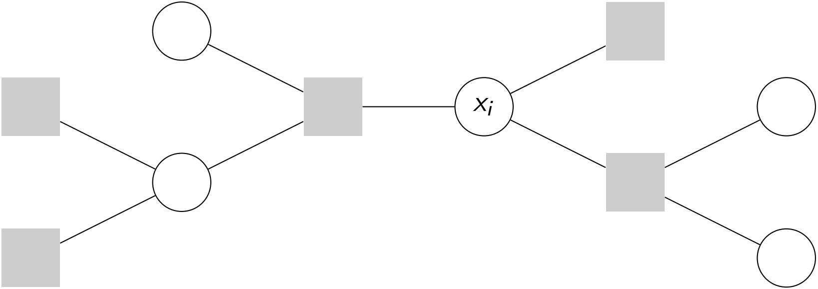


Initialization at leaf nodes: $\mu_{x \to f}(x) = 1$, $\mu_{f \to x}(x) = f(x)$

Message passing in action



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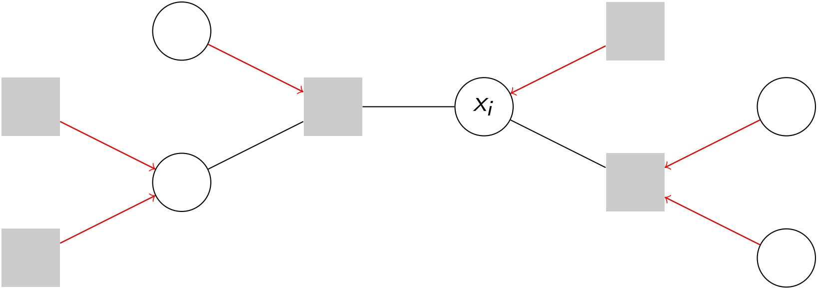

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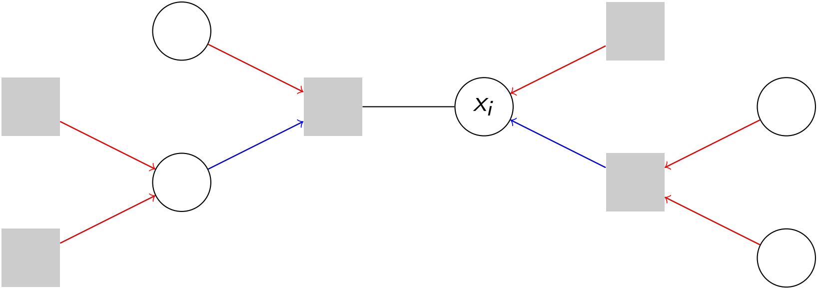

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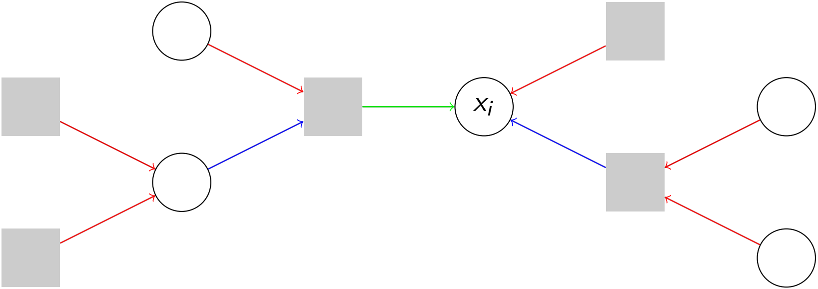

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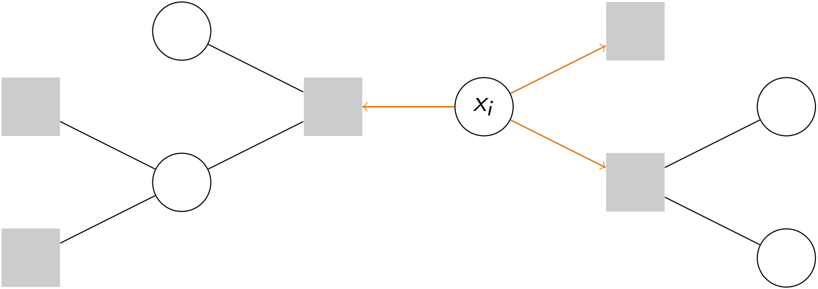

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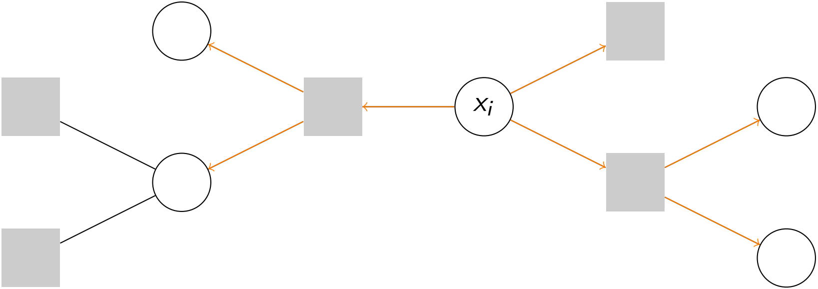
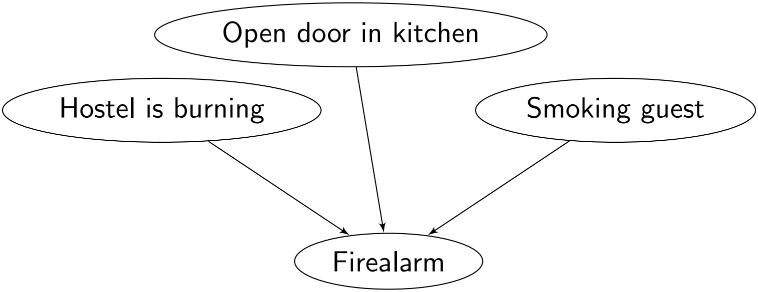


Maximum aposteriori estimation



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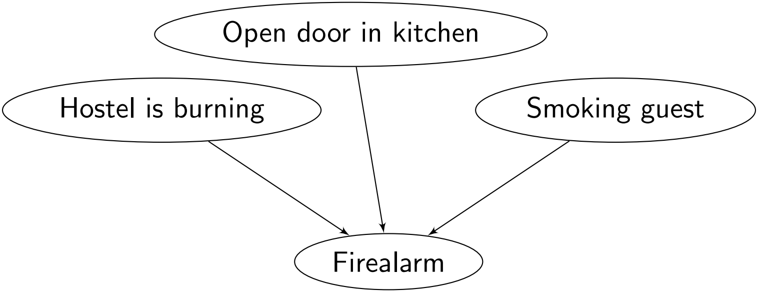

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Keep track of maximal X values!

Message passing: Discussion

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Expectation propagation

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Moment matching!

Expectation propagation: Minimize KL-divergence

Complete function:

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One factor at a time:

$$KL\Big(\prod_i f_i(X)\Big|\Big|f_i(X)\cdot\prod_{j\neq i} \tilde{f}_j(X)\Big) \qquad \text{simple, iterative, accurate}$$

Expectation propagation: Algorithm

Input: $\forall i : f_i(X)$

Initialization: $\forall i : \tilde{f}_i(X) = 1$

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Iteratively refine one factor at a time until convergence:

$$
\begin{aligned}
q^{\setminus j}(\theta) &= \prod_{i \neq j} \tilde{f}_i(\theta) \\
q^{new}(\theta) &\propto f_j \; q^{\setminus j}(\theta) \\
\tilde{f}_j &= \frac{1}{Z_j} \frac{q^{new}(\theta)}{q^{\setminus j}(\theta)}
\end{aligned}
$$

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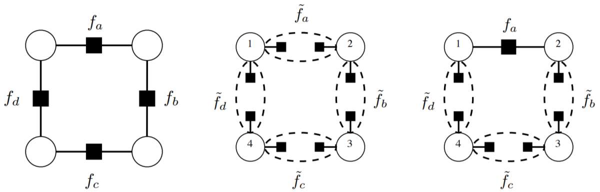

EP vs. VI

Expectation propagation: Minimize $KL(p||q) = \int p \ln \frac{p}{q}$

Variational inference: Minimize $KL(q||p) = \int q \ln \frac{q}{p}$

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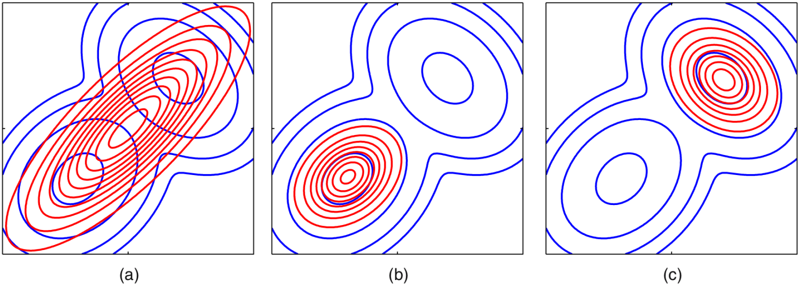
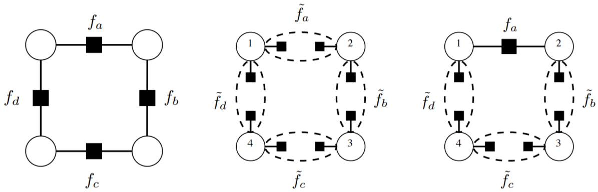


Figure taken from Bishop: Pattern Recognition and Machine Learning.

Thanks for your attention!
Questions?

Loopy BP as special case of EP



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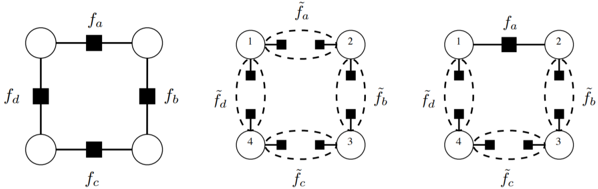

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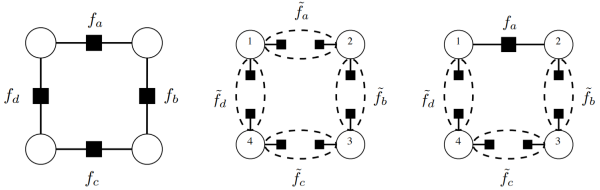

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