

Numerical exploration of a Penning trap

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<https://github.com/christopheblomsen/Project-3>

Abstract

This paper covers the numerical solutions of a Penning trap. The Penning trap is a device that confine particles inside a device using electric and magnetic field. In this report, we start with deriving the equations of motion for charged particles confined in the Penning trap. Before we build the simulation, we derive the general solution and find upper and lower bounds on the displacement of the particle from the centre of the trap. Next, we look at the specific case of Ca^+ ions and how they would behave in said Penning trap with the configurations $B_0 = 1.00\text{T}$, $V_0 = 25.0\text{mV}$ and $d = 500\mu\text{m}$. We accomplish this by using two different numerical methods, namely the Forward-Euler and the Runge-Kutta method implemented with object-oriented programming. The results are used to investigate the effect of the Coulomb interaction between the particles in the trap. The possibility of energy exchange between the particles allows the particles to spread out over a lager domain both in real space and in phase space. Since the particles display periodic motion we expect to find resonance phenomena. We study a time dependent sinusoidal electric field with different amplitudes and frequencies. At $\omega_V \approx 2.2 \text{ MHz}$ we find that most of the ions are escaping the trap during our simulation time of $500\mu\text{s}$. When we simulate the same system with Coulomb interactions we find that this frequency is split up, and shifted to a slightly lower value. The codes used for this project can be found at .

1 Introduction

Physical experiments often depend on the possibility of isolating particles. Examples include experiments where we want to measure the mass or charge of the trapped particles or collide the particles with other particles to see what comes out. One way of isolating and storing the particles, i.e. trapping them, is a Penning trap.

We want to study the physics of Penning traps using simulations of Ca^+ -ions. For many such particles, we will look into the effects of Coulomb interactions on the motion of the particles in the trap. Since the motion of the particles inside the trap is periodic, we can also find resonance phenomena. The particles might be more susceptible to gaining energy from the oscillating electric field for specific angular frequencies. We will look for resonance frequencies for a Penning trap and see how the Coulomb interactions effect the resonance.

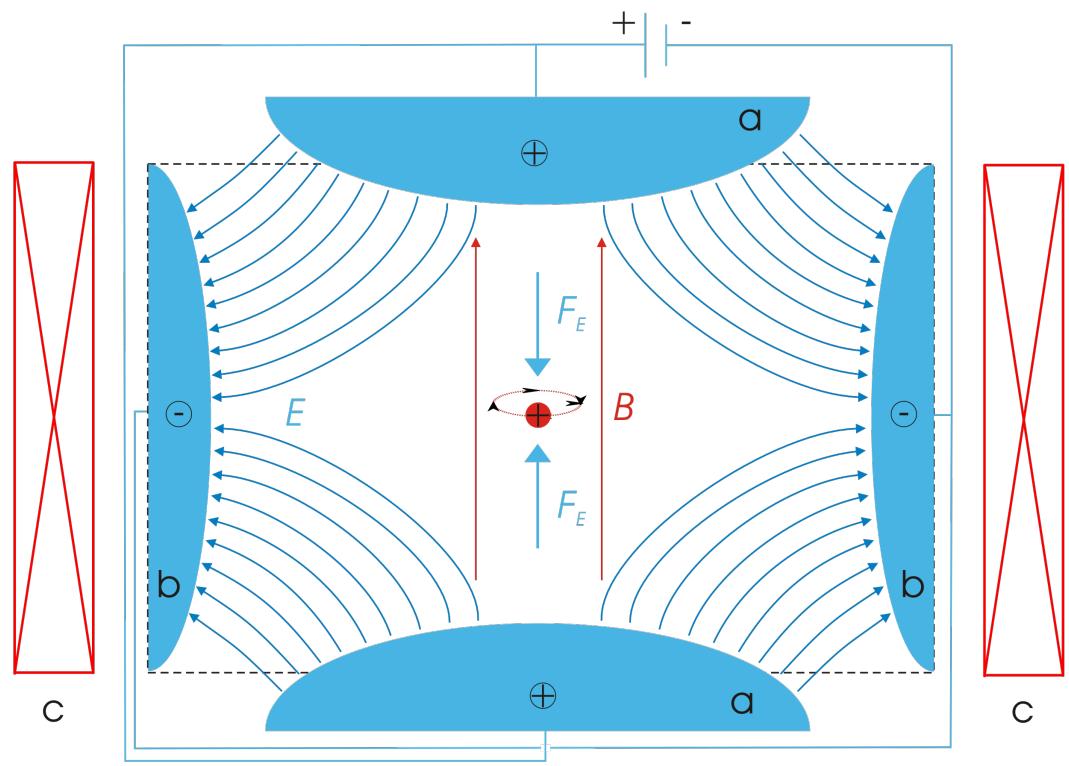


Figure 1: Illustration by Arian Kriesch

2 Theory

An illustration of a Penning trap is in [Figure 1](#). It consists of three electrodes, two caps and one ring. The caps and the ring have opposite charges. When the caps are positive, as shown in [Figure 1](#) they set up an electrical potential given by

$$V(x, y, z) = \frac{V_0}{2d^2} 2z^2 - x^2 - y^2. \quad (1)$$

Where V_0 indicates the strength of the potential and d is a measure of the size of the trap. This potential has an unstable equilibrium along a vertical line in the centre of the ring. The particles we will be studying are positive we need to prevent them from being pulled to the negatively charged ring electrode. To do this, we place the entire trap inside a magnet so that there is a constant magnetic field within the trap

$$\mathbf{B} = B_0 \mathbf{e}_z. \quad (2)$$

Since the Lorentz force on the particles is perpendicular to the magnetic field and their direction of motion, a sufficiently strong magnetic field will force the particles to orbit around the centre of the trap and not escape or hit the negative electrode.

With Newton's second law we can find the equations of motion for a positive particle within the trap. The force on the particle is the Lorentz force. Since we are only trapping single Ca^+ -ions, we can safely neglect gravity.

The Lorentz force is given by

$$\mathbf{F} = q\mathbf{E} + q\dot{\mathbf{r}} \times \mathbf{B}, \quad (3)$$

where q is the particle's charge, \mathbf{r} is the position vector and \mathbf{B} is the magnetic field vector.

The gradient of the electric potential V gives the electric field

$$\mathbf{E} = -\nabla V = -\frac{V_0}{d^2} (2\mathbf{e}_z - x\mathbf{e}_x - y\mathbf{e}_y), \quad (4)$$

and the cross product of the velocity and the \mathbf{B} -field can be written as

$$\dot{\mathbf{r}} \times \mathbf{B} = B_0 y \mathbf{e}_x - B_0 x \mathbf{e}_y. \quad (5)$$

Thus the equations of motion can be written as

$$\ddot{x} - \omega_0 \dot{y} - \frac{1}{2} \omega_z^2 x = 0, \quad (6)$$

$$\ddot{y} + \omega_0 \dot{x} - \frac{1}{2} \omega_z^2 y = 0, \quad (7)$$

$$\ddot{z} + \omega_z^2 z = 0, \quad (8)$$

where we have defined ω_0 and ω_z by

$$\omega_0 \equiv \frac{qB_0}{m}, \quad (9)$$

$$\omega_z^2 \equiv \frac{2qV_0}{md^2}, \quad (10)$$

$$(11)$$

for simplicity. We recognize that the motion of the particle in the z direction should be sinusoidal, since [Equation 10](#) is the equation for a harmonic oscillator. [Equation 6](#) and [Equation 7](#) are coupled, but can write them as one equation by defining

$$f(t) = x(t) + iy(t). \quad (12)$$

Taking the double derivative of this, we obtain

$$\ddot{f} = \ddot{x} + i\ddot{y}, \quad (13)$$

$$= \omega_0 \dot{y} + \frac{1}{2} \omega_z^2 x - i\omega_0 \dot{x} + i\frac{1}{2} \omega_z^2 y, \quad (14)$$

$$= \omega_0 (\dot{y} - i\dot{x}) + \frac{1}{2} \omega_z^2 (x + iy), \quad (15)$$

$$= -i\omega_0 \dot{f} + \frac{1}{2} \omega_z^2 f, \quad (16)$$

where we have inserted \ddot{x} and \ddot{y} from [Equation 6](#) and [Equation 7](#). The equation for f has analytical solutions on the form

$$f(t) = A_+ e^{-i(\omega_+ t + \phi_+)} + A_- e^{-i(\omega_- t + \phi_-)}, \quad (17)$$

where ϕ_{\pm} are initial phases, A_{\pm} are the initial amplitudes, and the angular frequencies ω_{\pm} are given by

$$\omega_{\pm} = \frac{\omega_0 \pm \sqrt{\omega_0^2 - 2\omega_z^2}}{2} \quad (18)$$

The position in the xy -plane, contained in f , will grow unboundedly when ω_{\pm} is a complex number. For the particles to be trapped, we thus need ω_{\pm} to be real numbers. We achieve this when

$$\omega_0^2 - 2\omega_z^2 \geq 0. \quad (19)$$

Using [Equation 9](#) and [Equation 10](#) we can write this as

$$\frac{q}{m} \geq \frac{4V_0}{B_0^2 d^2}. \quad (20)$$

This indicates that the system is more stable when the particles have a low mass relative to their charge. Inertial effects will then be less prominent. We see that it is easier to trap particles when the magnetic field and size of the trap are large relative to the electric field. The increasing strength of the \mathbf{B} -field will make the particle orbits narrower. Increasing the size of the trap reduces the need for a strong \mathbf{B} -field, as the particles can have a wider orbit without collapsing onto the ring.

Assuming the particle is trapped, we can use [Equation 17](#) to find upper and lower bounds on the position of the particle given by its initial position.

We rewrite the positions $x(t) = \mathcal{R}(f(t))$, and $y(t) = \mathcal{I}(f(t))$ using Eulers formula

$$x(t) = A_+ \cos(\omega_+ t + \phi_+) + A_- \cos(\omega_- t + \phi_-) \quad (21)$$

$$y(t) = -A_- \sin(\omega_- t + \phi_-) - A_+ \sin(\omega_+ t + \phi_+). \quad (22)$$

The distance from the centre of the trap is $D^2 = x^2 + y^2$. Inserting the above expressions for $x(t)$ and $y(t)$ we find

$$D^2 = A_+^2 + A_-^2 + 2A_+ A_- \cos((\omega_+ - \omega_-)t + \varphi), \quad (23)$$

where $\varphi := \phi_+ - \phi_-$. We find the largest and smallest distance from the centre by maximizing and minimizing the cosine term. This yields an upper bound $D_{\max} = |A_+ + A_-|$ and a lower bound $D_{\min} = \sqrt{A_+^2 + A_-^2 - 2A_+ A_-} = |A_+ - A_-|$.

A Penning trap is not limited to trapping only single particles. We need to consider the Coulomb interactions between multiple particles that will be trapped. The electric field from N particles with charge $q > 0$ is the sum of each particle field. We write it as

$$\mathbf{E} = k_e \sum_{j=1}^N q \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|^3}, \quad (24)$$

where k_e is the Coulomb constant, \mathbf{r}_i is the particle position and ¹ r is an arbitrary position. The force on each particle from the electric field is $\mathbf{F}_i = q\mathbf{E}_i$, and the charge times the electric field in the particle's position.

We can write the equations of motion for particle number i as

$$\ddot{x} - \omega_0 \dot{y} - \frac{1}{2} \omega_z^2 x - \frac{k_e q^2}{m} \sum_{j \neq i} \frac{x_i - x_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} = 0, \quad (25)$$

$$\ddot{y} + \omega_0 \dot{x} - \frac{1}{2} \omega_z^2 y - \frac{k_e q^2}{m} \sum_{j \neq i} \frac{y_i - y_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} = 0, \quad (26)$$

$$\ddot{z} + \omega_z^2 z - \frac{k_e q^2}{m} \sum_{j \neq i} \frac{z_i - z_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} = 0, \quad (27)$$

Each particle can be thought of as a small current. Charges in motion will set up a weak magnetic field according to Ampere's law

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \quad (28)$$

where μ_0 is the magnetic permeability of the vacuum and ε_0 is the electric permittivity of the vacuum. We neglect this magnetic field's effect on the particles' motion. If many particles fly in opposite directions, we expect the contributions from each of them to cancel out. In comparison, the effect from many like-charged particles will add up. We, therefore, expect this to be a more prominent effect in our simulations.

3 Method

3.1 Numerical

Since calculating the analytical solution for N Ca⁺ ions is going to be hard, we will instead use our machines' power. We want to solve the equations of motion 25, 26 and 27 numerically.

3.1.1 Forward Euler

We start by deriving the Forward Euler method.

$$\frac{dx}{dt} = f(t, x) \quad (29)$$

⇓

$$\frac{x_{i+1} - x_i}{h} + \mathcal{O}(h) = f_i \quad (30)$$

$$x_{i+1} = x_i + h f_i + \mathcal{O}(h^2) \quad (31)$$

Where h is the step size, and $\mathcal{O}(h^2)$ is the remaining terms. If we truncate (31)

$$x_{i+1} = x_i + h f_i \quad (32)$$

This then has a truncation error of $\mathcal{O}(h^2)$ and a global error of $\mathcal{O}(h)$. However, we have a second-order differential equation. So using the same method but with

$$\frac{d^2 x}{dt^2} = f \left(t, x, \frac{dx}{dt} \right). \quad (33)$$

¹Except at $\mathbf{r} = \mathbf{r}_i$, since this is undefined.

We will instead have

$$v_{i+1} = v_i + hf_i \quad (34)$$

$$x_{i+1} = x_i + hv_i \quad (35)$$

Where it should be noted that you would get the Euler-Cromer method if in (35) you used v_{i+1} instead of v_i . The function f is, in our case, going to be the particle's acceleration, which we find using Newton's second law. So the full algorithm is as follows:

$$\mathbf{a}_i = \frac{\mathbf{F}_i}{m} \quad (36)$$

$$\mathbf{v}_{i+1} = \mathbf{v}_i + h\mathbf{a}_i \quad (37)$$

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h\mathbf{v}_i \quad (38)$$

$$(39)$$

Where \mathbf{a} is the 3D acceleration vector, \mathbf{F} is the 3D force vector, \mathbf{v} is the velocity vector, \mathbf{x} is the position vector, and m is the mass. We use the same mass since we will only have one type of ion in our trap.

3.1.2 Runge Kutta 4

The Runge-Kutta method is an extension of the forward Euler method described in 3.1.1. However, we do it four times instead of calculating the slope only once. The general approach is that for the first slope value k_1 is found using the 3.1.1. Then for the second slope value k_2 is the slope at the midpoint of the interval from t to $t + h$. Then we recalculate the midpoint slope using the k_2 value. Before we calculate the slope at $t + h$ to get the k_4 .

$$k_1 = f(t_n, x_n) \quad (40)$$

$$k_2 = f\left(t_n + \frac{h}{2}, x_n + h\frac{k_1}{2}\right) \quad (41)$$

$$k_3 = f\left(t_n + \frac{h}{2}, x_n + h\frac{k_2}{2}\right) \quad (42)$$

$$k_4 = f(t_n + h, x_n + hk_3) \quad (43)$$

Before the next step is calculated from

$$x_{n+1} = x_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) h. \quad (44)$$

Now for our specific case, we need to have one coefficient for the positions and the velocities. Then we need to calculate all the particles' coefficients before moving on to the next coefficient.

So our specific algorithm will have this shape:

```

1 // Use the forward Euler method to move all particles one step
2 evolve_forward_Euler(i, t);
3
4 // For all the particles
5 l1[n] = h * F(n, t);
6 k1[n] = h * v_[n];
7
8 // For all the particles
9 position[n+1] = position[n] + k1[n]/2.;
10 velocity[n+1] = velocity[n] + l1[n]/2.;
11
12 // For all particles

```

```

13  l2[n] = h * F(n, t + 0.5*h);
14  k2[n] = h * velocity[n+1];
15
16 // For all particles
17  position[n+1] = position[n] + k2[n] / 2.;
18  velocity[n+1] = velocity[n] + l2[n] / 2.;

19
20 // For all particles
21  l3[n] = h * F(n, t + 0.5*h);
22  k3[n] = h * velocity[n+1];
23
24 // For all particles
25  position[n+1] = position[n] + k3[n];
26  velocity[n+1] = velocity[n] + l3[n];

27
28 // For all particles
29  l4[n] = h * F(n, t + h);
30  k4[n] = h * velocity[n+1];
31
32 // For all particles
33  position[n+1] = position[n] + (k1[n] + 2.*k2[n] + 2.*k3[n] + k4[n]) /
   6.;
34  velocity[n+1] = velocity[n] + (l1[n] + 2.*l2[n] + 2.*l3[n] + l4[n]) /
   6.;
```

Where $F(n, t)$ is the force on the n particle at time t .

4 Results and Discussion

We simulate the motion of a particle in the Penning trap for $50\mu s$ with initial conditions

$$(x_0, y_0, z_0) = (20, 0, 20)\mu m, \quad (45)$$

$$(u_0, v_0, w_0) = (0, 25, 0)\mu m/\mu s. \quad (46)$$

We show the motion of the particle in the z -direction over time in [Figure 2](#). The solution is a harmonic oscillator, as seen from [Equation 8](#). The dashed lines in the figure indicate the anticipated period of $T = \frac{2\pi}{\omega_z} \approx 9 \mu s$.

We simulate decreasing time steps to test the convergence of FE and RK4. We divide our domain into 4000 and double the number of steps consecutively three times, resulting in four simulations with decreasing time steps. The results are in [Figure 3](#). We see from this plot that the relative error in the RK4 solution is roughly two orders of magnitude lower than that of the FE solution. However, the rate of convergence is higher for FE than for RK4. This is also confirmed by the convergence rate, which is $r_{RK4} \approx 1$ and $r_{FE} \approx 1.4$. We expected FE to have a convergence rate of about $r_{FE} \approx 1$, since it is an $\mathcal{O}(h)$ accurate method. RK4 should have an accuracy of $\mathcal{O}(h^4)$, and a convergence rate of around $r_{RK4} \approx 4$, there is probably a bug in our code. We are not sure if this is a bug in our error calculation or in RK4. Since only RK4 is very off, it seems the bug is in our RK4 implementation. The formula used to calculate the convergence rate is shown in [section 5](#). We expect to see the RK4 algorithm converge faster as the time step decreases. However, this is not the case, it might be that the algorithm quickly reaches a threshold, but it seems more likely that there is a minor bug in the code. The spiky wiggles are also unexpected, from a working implementation we would normally have expected a steady increase in the relative error over time. The relative error is of around 1%. The bug is not devastating and we can still use our simulations to get a qualitative understanding of the motion of the particles in the Penning trap. The results discussed below, concerning one or two particles were obtained using this code.

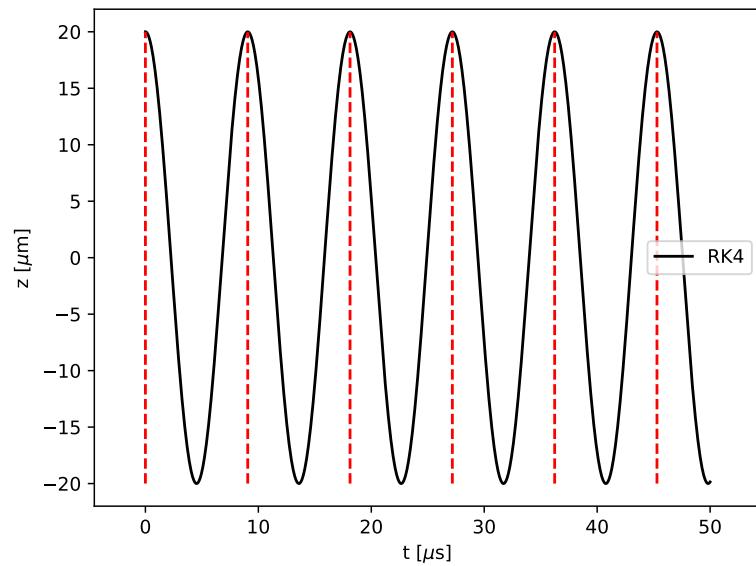


Figure 2: The motion of a single particle in the z -direction simulated for $50 \mu\text{s}$. The dashed lines indicate the expected period of $T \approx 9 \mu\text{s}$.

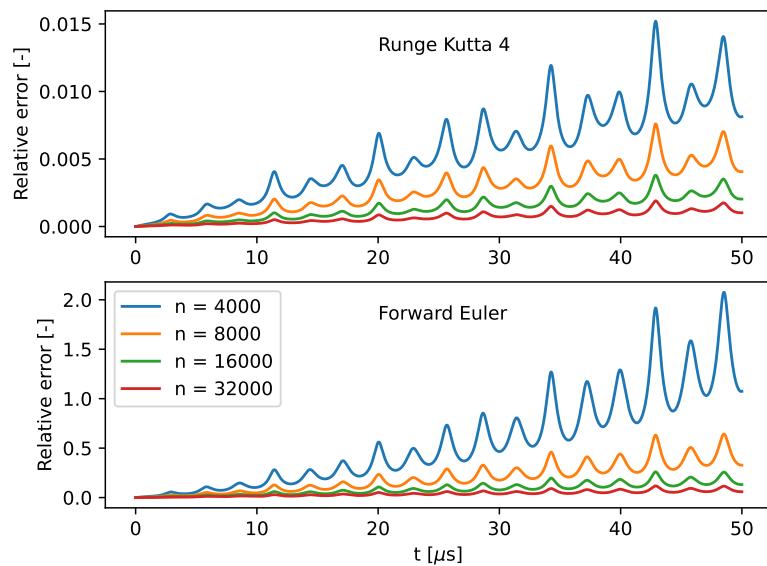


Figure 3: The relative error at time t for number of steps n

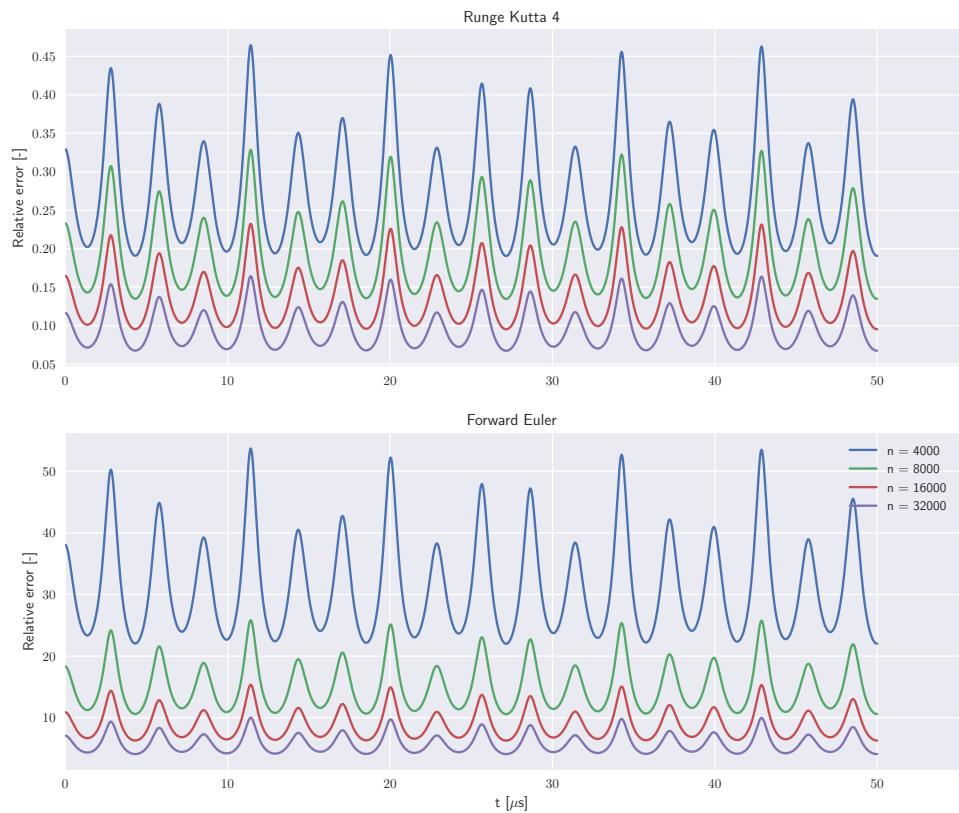


Figure 4: The relative error of our fixed code. RK4 has a relative error of around 10% for our finest resolution, and FE has an error of around 100%. Ouch.

We also ran simulations for 100 particles for different frequencies to investigate resonance phenomena. Before we did this we debugged our implementations of RK4 and FE. Their performance is shown in [Figure 4](#). After fixing the bug, and creating many new ones, this is the code we used for finding the resonance frequencies. We used $N = 32000$ time steps for simulating the particle motion for $500\mu\text{s}$. This is a coarser grid than any the coarsest grid in [Figure 4](#) and should give a relative error of well over 45% by the end of the simulation. Surprisingly enough, our results weren't utterly horrendus. We suspect there might be a bug in our calculation of the error, since the results we found for the resonance frequencies seem remarkably sensible, given our error estimation above. In [Figure 10](#) we see that we find resonance frequencies roughly in the middle between two frequencies found in the analytical solution, indicated by the black dots.

To look at the effect of the Coulomb interaction, we add another particle to the mix with the initial conditions

$$(x_0, y_0, z_0) = (25, 25, 0)\mu\text{m}, \quad (47)$$

$$(u_0, v_0, w_0) = (0, 40, 5)\mu\text{m}/\mu\text{s}. \quad (48)$$

The trajectories of the particles are shown in [Figure 5](#), both calculating with interactions and neglecting interactions. To better understand the effects of the interactions. We look at two different projections of the trajectories, onto the xz -plane and the xy -plane.

The particle trajectories in the xz -plane are shown in [Figure 7](#). When we include interaction, the particles span a larger region in the z -direction, as expected. Since the particles are repulsive, they tend to be far apart. A similar effect can be seen in the xy -plane, [Figure 6](#). Here we can tell that the presence of Particle 2 (P2) pushes Particle (P1) beyond the bounds of its analytical solution. At first, it looks as if P1 is being squashed in towards the centre. Then they swap places, and P1 is on the outside and the P2 seems to have a compressed trajectory with denser precessions.

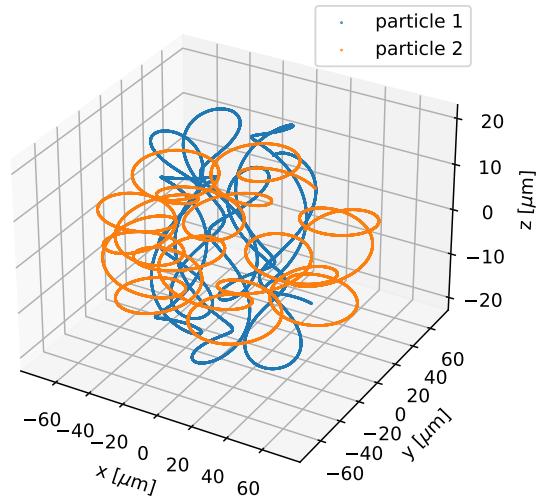
We see a plot of the particles' motion in phase space along the z -axis in [Figure 8](#). Without particle interactions, the phase space trajectories are elliptic. The energy of each particle is conserved and transitions back and forth between kinetic and potential energy.

When the Coulomb interactions are included, the particles can exchange energy. This makes it possible for both particles to lose or gain energy, making a larger region of phase space available for them. The potential energy of one particle, due to the presence of the other particle, is indicated by the colour of the trajectory. Red indicates high energy, and blue indicates low energy. Notice that at a portion of the trajectories where the particles are close, P1 is generally closer to the origin, implying it has lower energy. In contrast, P2 is further from the origin, implying it has higher potential or kinetic energy. The interactions between the particles allow them to swap places in phase space.

In the phase space projected onto the x -direction, [Figure 9](#), the phase space plots look more complicated. We, for example, recognize the precessions also seen in [Figure 6](#), the particle frequently changes direction, that is crosses $\dot{x} = 0$ in the figure. The equation of motion for a particle in the x -direction [Equation 6](#) has three variables, x , \dot{x} and y . As in the z -direction, the particle's energy is conserved. Furthermore, we see a transition of energy between potential energy. Depending on the position through both x and y and the kinetic energy. If we were to simulate this for longer, we expect the trajectory to close in on itself after one orbit about the trap's centre, encircling a torus in 3D phase space. This is because the particle's energy is conserved, and three initial conditions, x_0 , y_0 and \dot{x}_0 , would then uniquely determine its path through phase space. Leinaas The top figure in [Figure 9](#) is what the torus would look like from the side, whereas a trajectory like those seen in [Figure 6](#) is what it would look like top down, that is, looking through the doughnut. In [section 5](#) we see a part of the torus.

Again if we include interactions, things get more disordered. The interactions distort the trajectories in phase space.

Particle trajectory without interaction



Particle trajectory including interaction

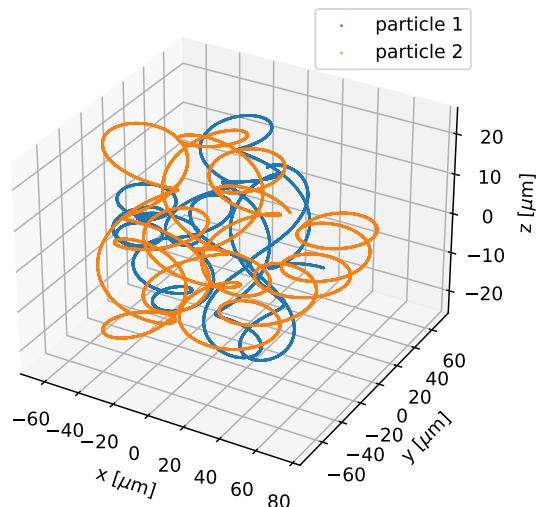


Figure 5: 3D plots of the particle trajectories with and without interactions. The trajectory of Particle 1 is shown in blue, whereas the trajectory of Particle 2 is orange.

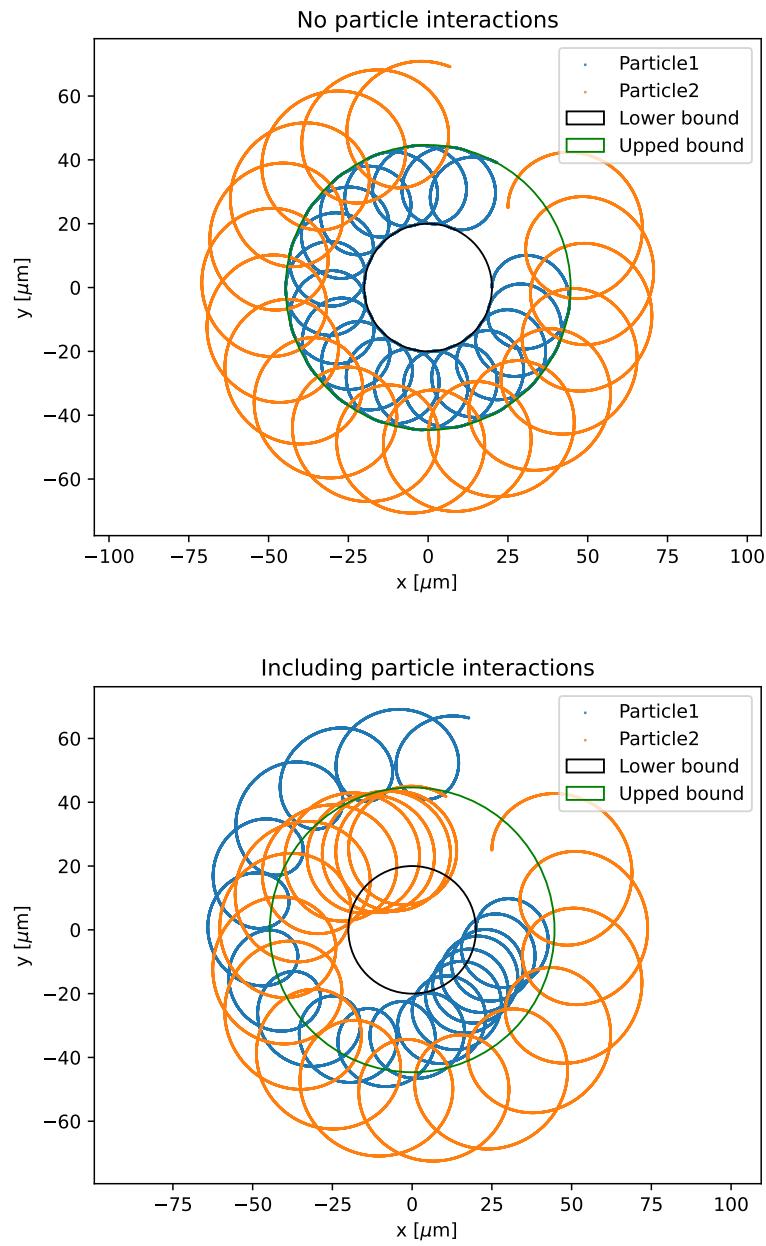


Figure 6: Plot of Particle 1 (blue) and Particle 2 (orange) with and without the effects from the Coulomb interactions. The simulations start in the right half of the figure, and the particles 'orbit' clockwise about the trap's centre. The upper and lower bounds on Particle 1's analytical solutions are displayed as green and black circles.

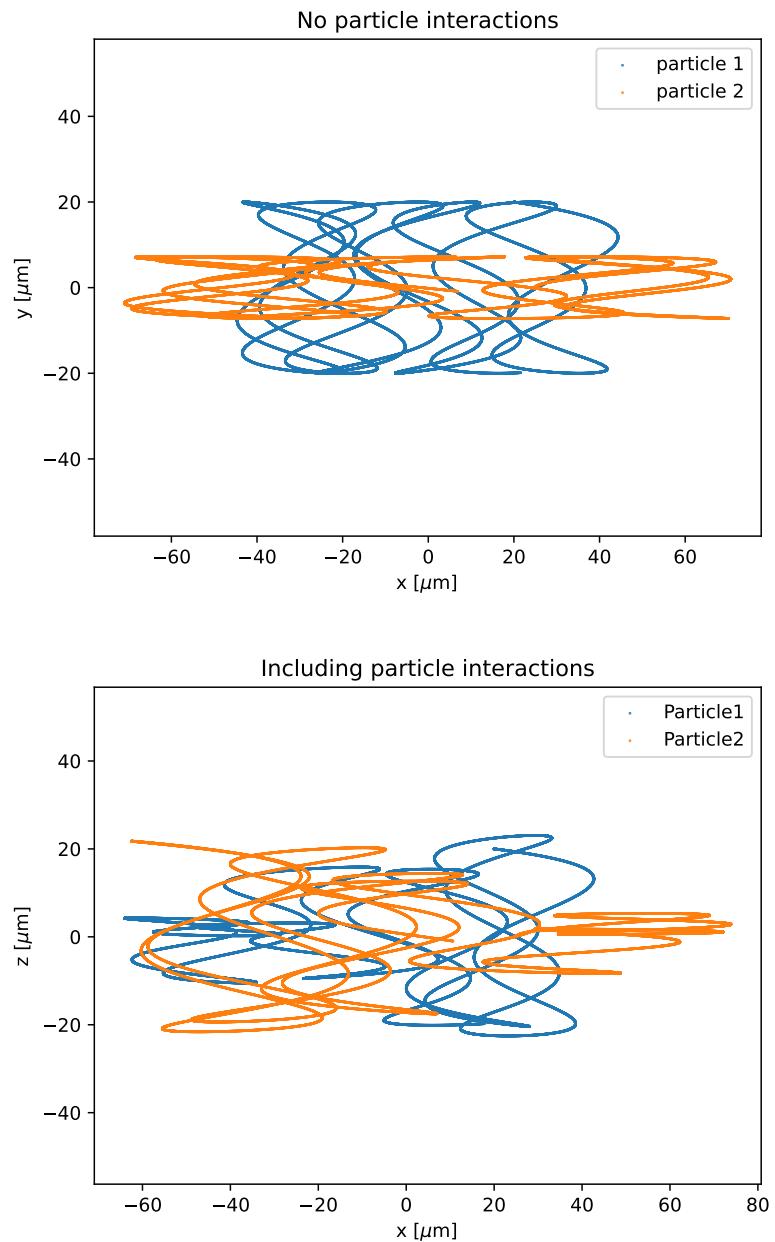


Figure 7: Particle 1 (blue) and Particle 2 (orange) with and without taking into account Coulomb interactions.

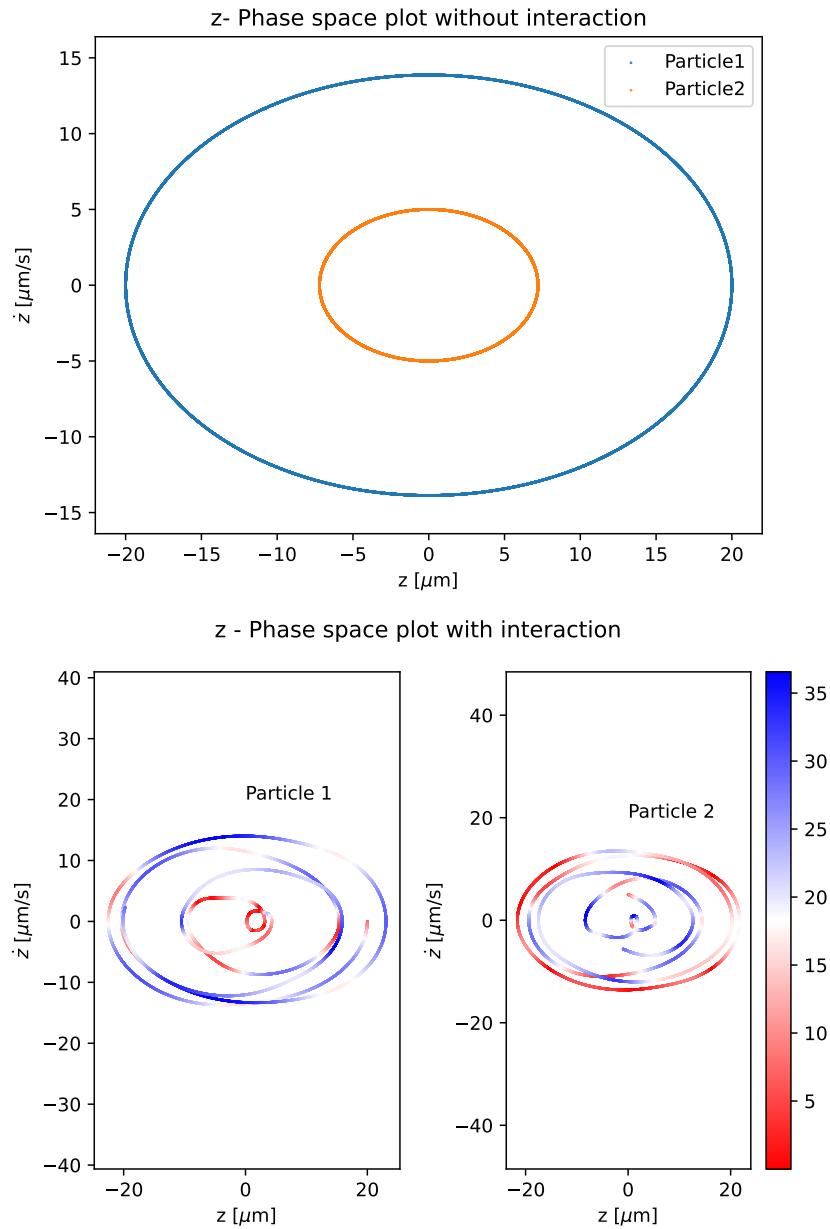


Figure 8: Phase space trajectories (z, \dot{z}) of P1 and P2. Above, P1 is blue, and P2 is orange. The simulations for the topmost plot were run neglecting Coulomb interactions. The bottommost simulation includes Coulomb interactions. In the bottom figure, we draw the particle trajectories in separate plots for clarity, the colour indicates the distance between the particles. The colour bar scale is μm .

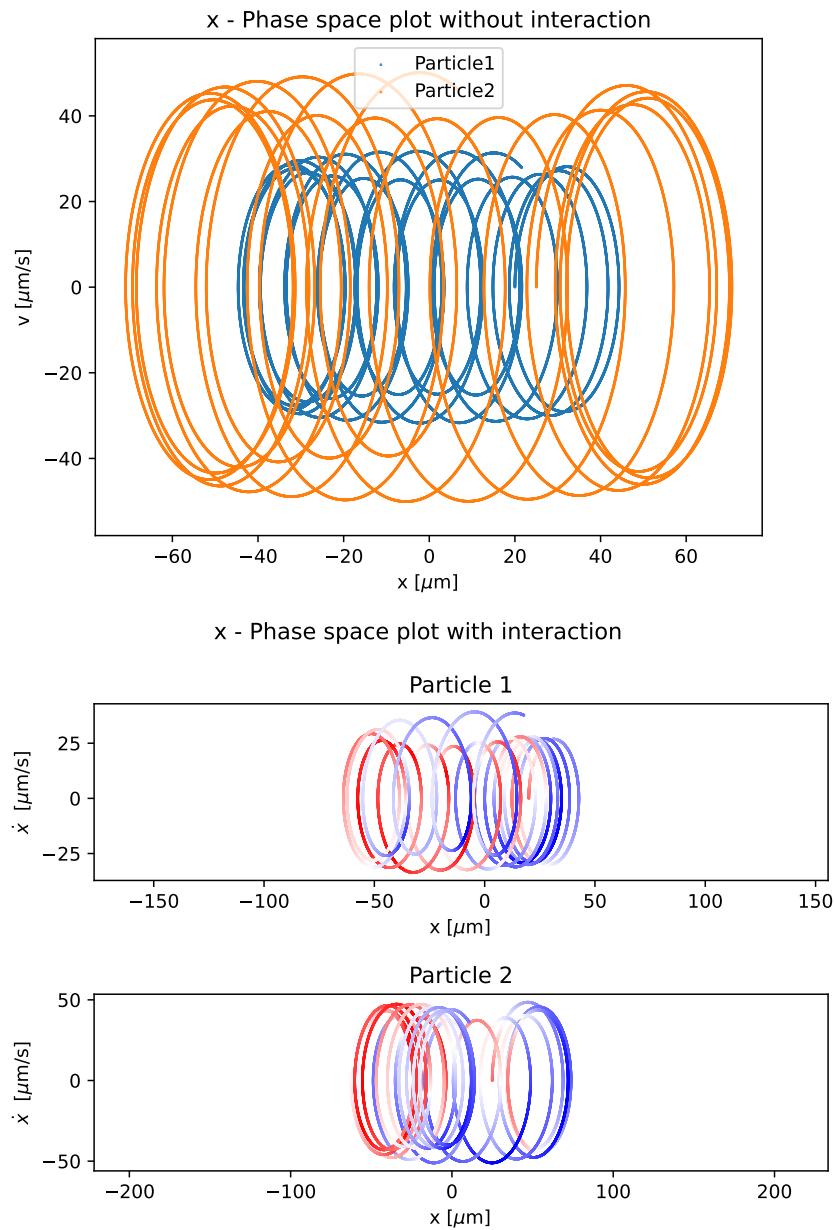


Figure 9: Phase space plots of P1 (blue) and P2 (orange), the trajectories look a bit like cylinders, we expect them to encircle a torus in 3D. When we include Coulomb interactions, the trajectories get distorted compared to the idealized case.

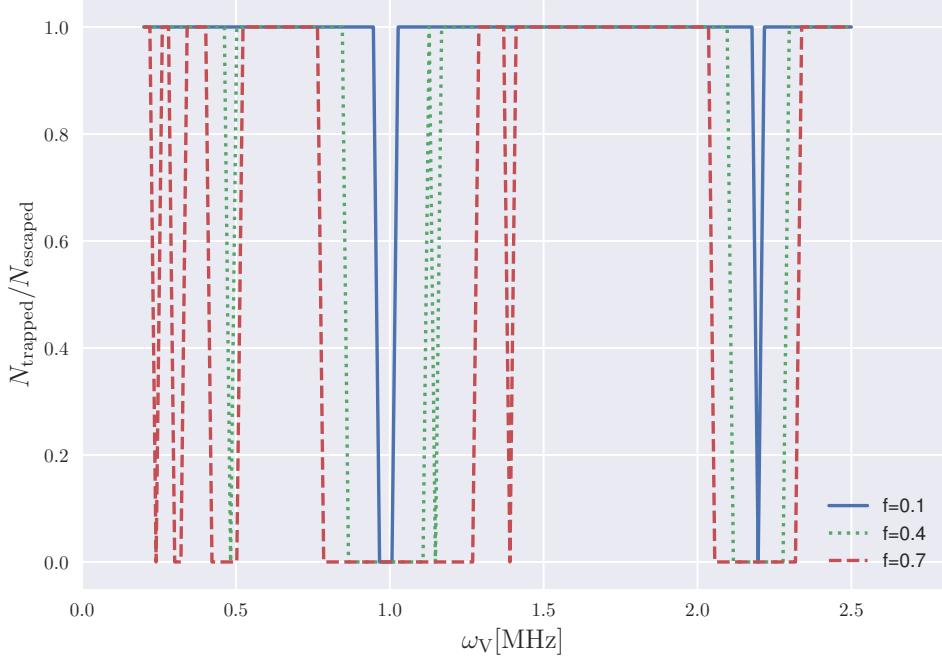


Figure 10: The fraction of the initial 100 particles left after $500 \mu\text{s}$ in the Penning trap with an oscillating \mathbf{E} -field over the oscillation frequency. The simulations were run with three different amplitudes f . The angular frequencies of the analytical solution for the motion of a single particle is marked by black dots. $\omega_+ \approx 2.3 \text{ MHz}$ and $2\omega_- \approx 2.1 \text{ MHz}$.

To isolate only a few particles, one practical strategy might be to fill the trap with many particles and then try to kick out most of them. To do this we can add a periodically oscillating perturbation to the electric potential strength

$$V_0 \rightarrow V_0(1 + f \cos(\omega_V t)), \quad (49)$$

where f is a dimensionless amplitude, and ω_V is the angular frequency of the perturbation. We expect the particle motion in the trap to display some degree of periodicity. We can exploit this by aligning the oscillation electric potential with the inherent oscillations of the particles. This can amplify the translation of the particles to make them leave the trap altogether.

We simulate 100 particles randomly initialized in the Penning trap for $500\mu\text{s}$ for different angular frequencies in the domain $\omega_V \in [0.2, 2.5] \text{ MHz}$. In this simulation, we ignore the Coulomb interactions, which correspond to simulating 100 particles individually.

The fraction of particles left in the trap after $500\mu\text{s}$ as a function of frequency is shown in [Figure 10](#).

The fraction of particles left in the trap is drastically reduced for some frequencies. These correspond to the natural frequencies of the system. Since the fraction of particles left quite abruptly reaches zero using this method to isolating particles might need some fine tuning. The potential should probably not be oscillating at the resonance frequencies for the full $500\mu\text{s}$ if our goal is to have some particles left in the end. Let us say a particle is in the left

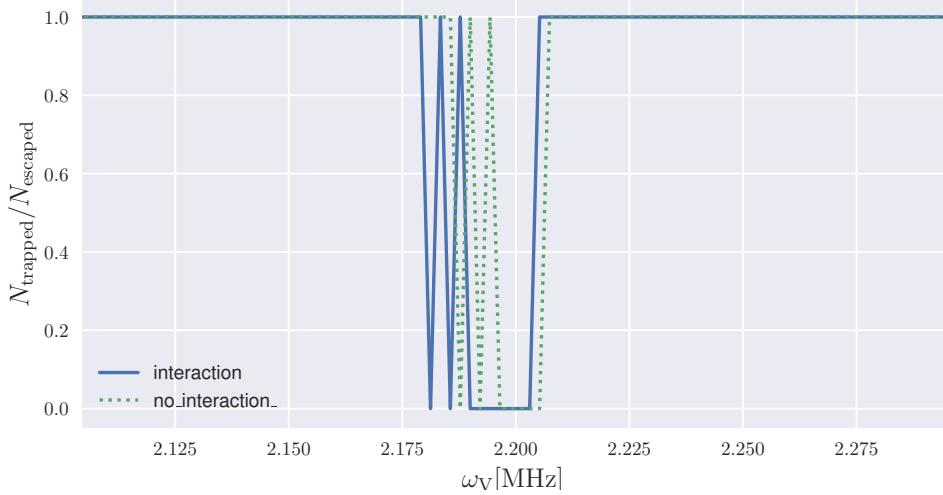


Figure 11: Fraction of particles left of the initial 100 particles, the blue line is for interacting particles, the green dotted line is for non-interacting particles. The perturbation amplitude is $f = 0.1$.

half of the trap as shown in [Figure 1](#) and travelling to the left. If the field is now strong, the particle's velocity will increase. The magnetic field accelerates the particle to turn and travel towards the right in the figure. Suppose the potential oscillations align with the particle's motion. In that case, the potential is weak when the particle is in the left half travelling towards the right and strong again when the particle is entering the right half. The potential will add a substantially higher amount of energy to the particle than it detracts. Eventually, the particle will move too fast for the magnetic field to confine it within the trap.

We see that the resonance frequencies are roughly halved. This corresponds to 'kicking' the particle only once per period or every second per period. This can explain why the low amplitude oscillations do not manage to kick out particles at lower frequencies. It is unable to provide enough energy for the particle to escape. It is also worth noting that the resolution is sharper for the lower amplitude oscillations need to sync up better with the resonance frequency to deliver energy to the system. Intuitively this makes much sense. The force is applied to the particles over a specific period. If the oscillations are not quite in sync, the **E**-field will decrease the kinetic energy of the particles for some part of this time, as it accelerates the particle in the direction opposite of its motion. If the applied force is strong, the particle will still have a net gain in kinetic energy. If the applied force is weak, this effect might stall the particles. The low amplitude oscillations give a better resolution because it is more dependent on syncing up with the natural frequency of the particles to provide a net gain in energy.

The angular frequencies of the analytical solution for the motion of a single particle [18](#) is $\omega_+ \approx 2.30$ and $\omega_- \approx 1.05$. These are marked in the figure as black dots. Notice that the highest resonance frequency appears dead in the centre of these.

To see the effect of Coulomb interactions on the resonance of the trap we perform simulations for the same duration and the same amount of particles in the region around the resonance frequencies we found. We chose to only simulate the amplitude $f = 0.1$ as this gave the sharpest troughs in the former simulation.

The results are shown in [Figure 11](#). Higher resolution shows that more frequencies

are close together, not just one. This is true both in the case of single particles, ignoring interactions and with many particles. This might be due to the precessions of the particles in the xy -plane. Different frequencies might align up with different 'subcircles', as seen in [Figure 6](#). It might also be the case that it will align up with one such precession for the first kick and then shift by one for the next kick. This is plausible if the particles have a steady time period for one orbit about the centre. This is somewhat speculative. We need to find out if the orbits are steady, even in the case where we ignore the interactions. Probably the particle periods are only partially constant.

In comparing the interacting particles to the non-interacting particles, we notice a slight shift in all the resonance frequencies. It appears to be constant, but this might be an effect of too low resolution. Alas, we do not know.

A possible explanation of why the resonance frequencies are lower in the case of interacting particles might be because the particles can distribute the energy among themselves. Let us imagine one particle is being kicked, pushing the next particle 'in line', which again pushes the next particle and so forth. This highly idealized illustration can explain how a compression wave propagates through the particles in the trap. If some of the energy is bound to compression waves, we expect that the mean velocity of the particles is somewhat slower, making the orbital period longer. This is one reasonable explanation which we do not intend to verify.

5 Appendix

The largest absolute error in any single timestep of our simulation is given by

$$\Delta_{\max,k} = \max_i |\mathbf{r}_{i,\text{exact}} - \mathbf{r}_i| \quad (50)$$

We estimate the convergence rate by evaluating the change in maximum absolute error relative to the change in step size. We write this as

$$r_{\text{err}} = \frac{1}{3} \sum_{k=2}^4 \frac{\log(\Delta_{\max,k}) - \log(\Delta_{\max,k-1})}{\log(h_k) - \log(h_{k-1})} \quad (51)$$

The proposed torus is mentioned in the [??](#). We have not stored data from a simulation that would verify that the trajectory closes in on itself after encircling the trap's centre.

Particle trajectory without interaction

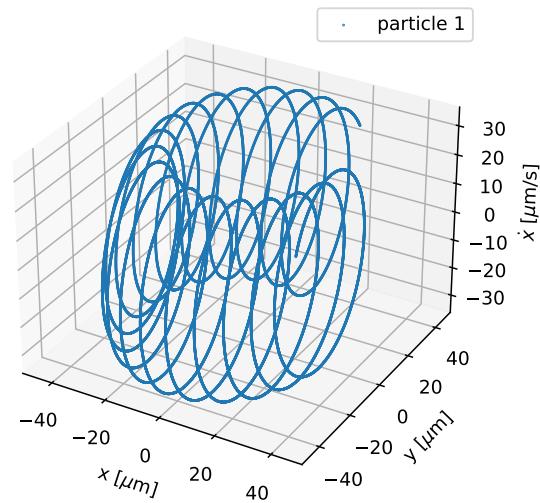


Figure 12: The phase space trajectory of a particle with no interactions. It seems to encircle a torus.

References

Leinaas, Jon Magne. *Classical Mechanics and Electrodynamics*. World Scientific, 2019.