

1

i $\vec{u} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Ser først på divergensen

$$\nabla \cdot \vec{u} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z$$

$$\nabla \cdot \vec{u} = 1 + 1 + 1$$

$$\nabla \cdot \vec{u} = 3$$

Ser så på virvlingen

$$\nabla \times \vec{u} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$\nabla \times \vec{u} = \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y \right) \hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z \right) \hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x \right) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = \vec{0}$$

ii $\vec{u} = r \cos(\theta)\hat{\mathbf{i}}_r + r \sin(\theta)\hat{\mathbf{i}}_\theta + z\hat{\mathbf{k}}$ Ser først på divergensen

$$\nabla \cdot \vec{u} = \frac{1}{r} \left(\frac{\partial}{\partial r} (r (r \cos(\theta))) + \frac{\partial}{\partial \theta} (r \sin(\theta)) \right) + \frac{\partial}{\partial z}z$$

$$\nabla \cdot \vec{u} = \frac{1}{r} \left(\cos(\theta) \frac{\partial}{\partial r} r^2 + r \frac{\partial}{\partial \theta} \sin(\theta) \right) + 1$$

$$\nabla \cdot \vec{u} = \frac{1}{r} (2r \cos(\theta) + r \cos(\theta)) + 1$$

$$\nabla \cdot \vec{u} = 2 \cos(\theta) + \cos(\theta) + 1$$

$$\nabla \cdot \vec{u} = 3 \cos(\theta) + 1$$

Ser så på virvlingen

$$\nabla \times \vec{u} = 0\hat{\mathbf{i}}_r + 0\hat{\mathbf{i}}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r (r \sin(\theta))) - \frac{\partial}{\partial \theta} (r \cos(\theta)) \right) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = \frac{1}{r} \left(\sin(\theta) \frac{\partial}{\partial r} (r^2) - r \frac{\partial}{\partial \theta} (\cos(\theta)) \right) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = \frac{1}{r} (2r \sin(\theta) + r \sin(\theta)) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = (2 \sin(\theta) + \sin(\theta)) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = 3 \sin(\theta) \hat{\mathbf{k}}$$

iii $\vec{u} = \hat{\mathbf{i}}_r + \hat{\mathbf{i}}$

Konverterer først over til rene cylinder koordinater

$$\hat{\mathbf{i}} = \cos(\theta)\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta$$

$$\vec{u} = \hat{\mathbf{i}}_r + \cos(\theta)\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta$$

$$\vec{u} = (1 + \cos(\theta))\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta$$

Ser først på divergensen

$$\nabla \cdot \vec{u} = \frac{1}{r} \left(\frac{\partial}{\partial r} (r(1 + \cos(\theta))) \frac{\partial}{\partial \theta} (-\sin(\theta)) \right)$$

$$\nabla \cdot \vec{u} = \frac{1}{r} (1 + \cos(\theta) - \cos(\theta))$$

$$\nabla \cdot \vec{u} = \frac{1}{r}$$

Ser så på virvlingen

$$\nabla \times \vec{u} = 0\hat{\mathbf{i}}_r + 0\hat{\mathbf{i}}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r(-\sin(\theta))) - \frac{\partial}{\partial \theta} (1 + \cos(\theta)) \right) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = \frac{1}{r} (-\sin(\theta) + \sin(\theta)) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = \vec{0}$$

2

a

For å finne enhetsvektorene så må man først finne skaleringsfaktorene. Starter med skaleringsfaktoren til u

$$\begin{aligned}
 h_u &= \left| \frac{\partial \vec{r}}{\partial u} \right| \\
 h_u &= \left| a \cos(v) \frac{\partial}{\partial u} \cosh(u) \hat{\mathbf{i}} + a \sin(v) \frac{\partial}{\partial v} \sinh(u) \hat{\mathbf{j}} \right| \\
 \boxed{a = 1} \\
 h_u &= \left| \cos(v) \sinh(u) \hat{\mathbf{i}} + \sin(v) \cosh(u) \hat{\mathbf{j}} \right| \\
 h_u &= \sqrt{(\cos(v) \sinh(u))^2 + (\sin(v) \cosh(u))^2} \\
 h_u &= \sqrt{\cos^2(v) \sinh^2(u) + \sin^2(v) \cosh^2(u)} \\
 \boxed{\cosh^2(u) = 1 + \sinh^2(u)} \\
 h_u &= \sqrt{\cos^2(v) \sinh^2(u) + \sin^2(v) (1 + \sinh^2(u))} \\
 h_u &= \sqrt{\cos^2(v) \sinh^2(u) + \sinh^2(u) \sin^2(v) + \sin^2(v)} \\
 h_u &= \sqrt{\sinh^2(u) (\cos^2(v) + \sin^2(v)) + \sin^2(v)} \\
 h_u &= \sqrt{\sinh^2(u) + \sin^2(v)}
 \end{aligned}$$

Finner så skaleringsfaktoren til v

$$\begin{aligned}
 h_v &= \left| \frac{\partial \vec{r}}{\partial v} \right| \\
 h_v &= \left| \cosh(u) \frac{\partial}{\partial v} \cos(v) \hat{\mathbf{i}} + \sinh(u) \frac{\partial}{\partial v} \sin(v) \hat{\mathbf{j}} \right| \\
 h_v &= \left| -\cosh(u) \sin(v) \hat{\mathbf{i}} + \sinh(u) \cos(v) \hat{\mathbf{j}} \right| \\
 h_v &= \sqrt{(-\cosh(u) \sin(v))^2 + (\sinh(u) \cos(v))^2} \\
 h_v &= \sqrt{\cosh^2(u) \sin^2(v) + \sinh^2(u) \cos^2(v)} \\
 h_v &= \sqrt{(1 + \sinh^2(u)) \sin^2(v) + \sinh^2(u) \cos^2(v)} \\
 h_v &= \sqrt{\sin^2(v) + \sinh^2(u) \sin^2(v) + \sinh^2(u) \cos^2(v)} \\
 h_v &= \sqrt{\sin^2(v) + \sinh^2(u) (\sin^2(v) + \cos^2(v))} \\
 h_v &= \sqrt{\sinh^2(u) + \sin^2(v)}
 \end{aligned}$$

Ser da at $h_u = h_v$ Enhetsvektorer er da gitt ved

$$\mathbf{e}_u = \frac{1}{h_u} \frac{\partial \vec{r}}{\partial u}$$

$$\mathbf{e}_u = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}} \left(\cos(v) \sinh(u) \hat{\mathbf{i}} + \sin(v) \cosh(u) \hat{\mathbf{j}} \right)$$

Og

$$\mathbf{e}_v = \frac{1}{h_v} \frac{\partial \vec{r}}{\partial v}$$

$$\mathbf{e}_v = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}} \left(-\cosh(u) \sin(v) \hat{\mathbf{i}} + \sinh(u) \cos(v) \hat{\mathbf{j}} \right)$$

Hvis de er ortogonale så må $\mathbf{e}_u \cdot \mathbf{e}_v = 0$

$$\mathbf{e}_u \cdot \mathbf{e}_v = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}} \left(\frac{\cos(v) \sinh(u)}{\sin(v) \cosh(u)} \right) \cdot \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}} \left(\frac{-\cosh(u) \sin(v)}{\sinh(u) \cos(v)} \right)$$

$$\mathbf{e}_u \cdot \mathbf{e}_v = \frac{1}{\sinh^2(u) \sin^2(v)} \left(-\cos(v) \sinh(u) \cosh(u) \sin(v) + \sin(v) \cosh(u) \sinh(u) \cos(v) \right)$$

$$\mathbf{e}_u \cdot \mathbf{e}_v = \left(-\sin(v) \cos(v) \sinh(u) \cosh(u) + \sin(v) \cos(v) \sinh(u) \cosh(u) \right)$$

$$\mathbf{e}_u \cdot \mathbf{e}_v = 0$$

Med dette så observeres det at enhetsvektorene er ortogonale

b

Siden ingen f har blitt oppgitt så tar jeg utgangspunkt i en helt generell f . Gradienten blir da

$$\nabla f = \frac{1}{h_u} \frac{\partial f}{\partial u} \mathbf{e}_u + \frac{1}{h_v} \frac{\partial f}{\partial v} \mathbf{e}_v$$

$$\boxed{h_u = h_v = h}$$

$$\nabla f = \frac{1}{h} \left(\frac{\partial f}{\partial u} \mathbf{e}_u + \frac{\partial f}{\partial v} \mathbf{e}_v \right)$$

Veit ikke hva w_u og w_v inneholder så blir generell divergens

$$\nabla \cdot \vec{w} = \frac{1}{h_u h_v} \left(\frac{\partial}{\partial u} (w_u \mathbf{e}_u h_v) + \frac{\partial}{\partial v} (w_v \mathbf{e}_v h_u) \right)$$

Laplace operatoren blir

$$\nabla^2 = \frac{1}{h_u h_v} \left[\frac{\partial}{\partial u} \left(\frac{h_u}{h_v} \frac{\partial}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_v}{h_u} \frac{\partial}{\partial v} \right) \right]$$

$$\boxed{h_v = h_u = h}$$

$$\nabla^2 = \frac{1}{h^2} \left[\frac{\partial}{\partial u} \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \frac{\partial}{\partial v} \right]$$

$$\nabla^2 = \frac{1}{h^2} \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right]$$

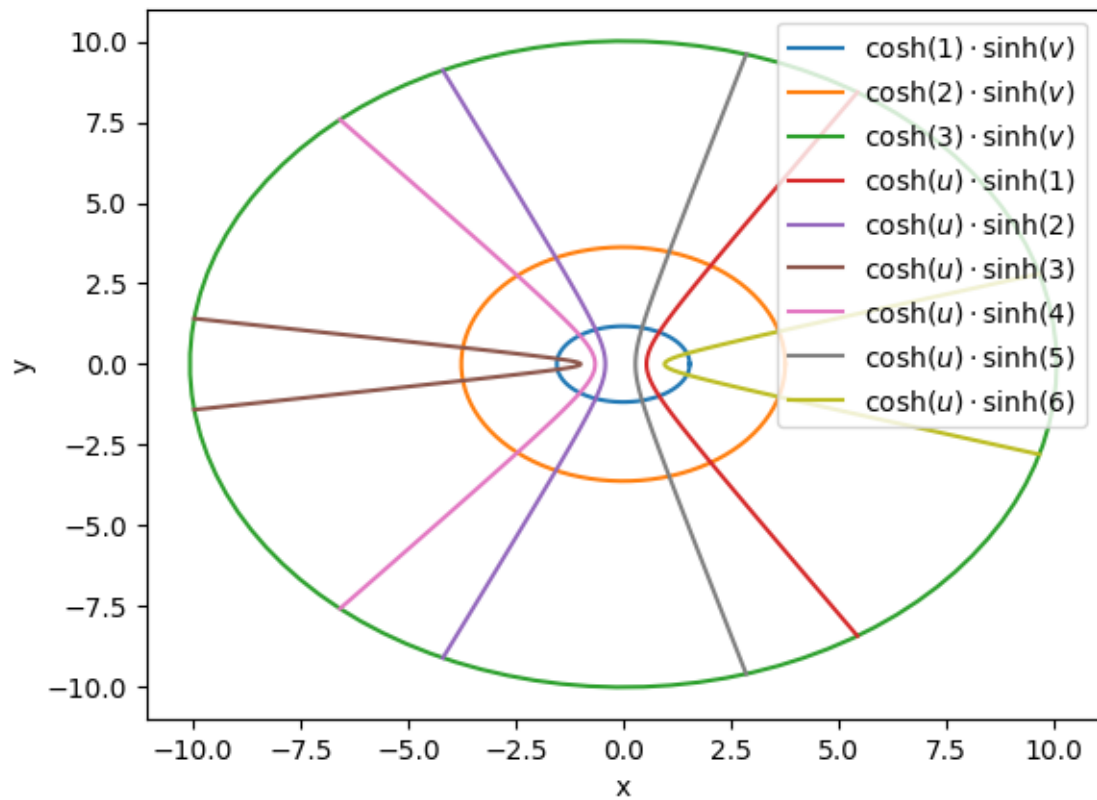
c

Python kode til skissen er

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 N = 100
5 u = np.linspace(-3, 3, N)
6 v = np.linspace(0, 2*np.pi, N)
7
8 for i in range(1, 4):
9     x = np.cosh(i)*np.cos(v)
10    y = np.sinh(i)*np.sin(v)
11    plt.plot(x, y, label=f"$\cosh({i}) \cdot \sinh(v)$")
12
13 for i in range(1, 7):
14     x = np.cosh(u)*np.cos(i)
15     y = np.sinh(u)*np.sin(i)
16     plt.plot(x, y, label=f"$\cosh(u) \cdot \sinh({i})$")
17
18 plt.xlabel("x")
19 plt.ylabel("y")
20 plt.legend()
21 plt.savefig("2c.png")
22 plt.show()
```

Som produserer skissen

Figur 1: Skissen til 2c



d

Python kode til dette er

```

1 import numpy as np
2 import sympy as sp
3 import matplotlib.pyplot as plt
4
5 u, v = psi = sp.symbols("u, v", real=True)
6 r = (sp.cosh(u)*sp.cos(v), sp.sinh(u)*sp.sin(v))
7
8 def basisvektor(psi, r):
9     b = np.zeros((len(psi), len(r)), dtype=object)
10    for i, ui in enumerate(psi):
11        for j, rj in enumerate(r):
12            b[i, j] = sp.simplify(rj.diff(ui, 1))
13    return b
14
15 def skaleringsfaktor(b):
16     h = np.zeros(b.shape[0], dtype=object)
17     for i, s in enumerate(np.sum(b**2, axis=1)):
18         h[i] = sp.simplify(sp.sqrt(s))
19     return h

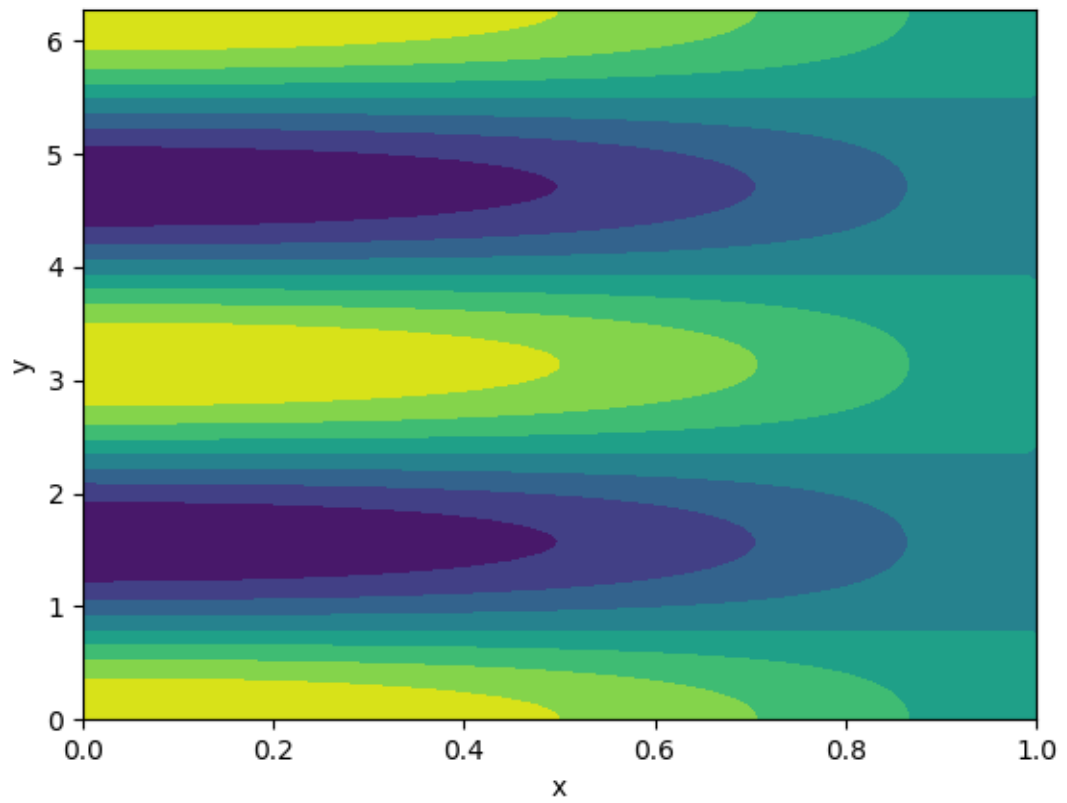
```

```

20
21 def enhetsvektor(psi, r):
22     b = basisvektor(psi, r)
23     hi = skaleringsfaktor(b)
24     return b/ hi[None, :], hi
25
26 e, h = enhetsvektor(psi, r)
27
28 f = (1 - u**2)*sp.cos(2*v)
29
30 N = 100
31 ui = np.broadcast_to(np.linspace(0, 1, N)[: , None], (N, N))
32 vi = np.broadcast_to(np.linspace(0, 2*np.pi, N)[None, :], (N, N))
33 fj = sp.lambdify((u, v), f)(ui, vi)
34
35 mesh = []
36 for rj in r:
37     mesh.append(sp.lambdify((u, v), rj)(ui, vi))
38 x, y = mesh
39
40 plt.contourf(x, y, fj)
41
42 df = np.array((1/h[0]*f.diff(u, 1), 1/h[1]*f.diff(v, 1)))
43
44 gradf = e[0]*df[0] + e[1]*df[1]
45
46 dfdxi = sp.lambdify((u, v), gradf[0])(ui, vi)
47 dfdyi = sp.lambdify((u, v), gradf[1])(ui, vi)
48 plt.contourf(x, y, fj)
49 plt.quiver(x[:,50], y[:,50], dfdxi[:,50], dfdyi[:,50], scale=15, pivot="middle")
50
51 plt.xlabel("x")
52 plt.ylabel("y")
53
54 plt.savefig("2d.png")
55 plt.show()
56 plt.close()
57
58 plt.contourf(ui, vi, fj)
59 plt.xlabel("x")
60 plt.ylabel("y")
61 plt.savefig("2d-elliptical.png")
62 plt.show()

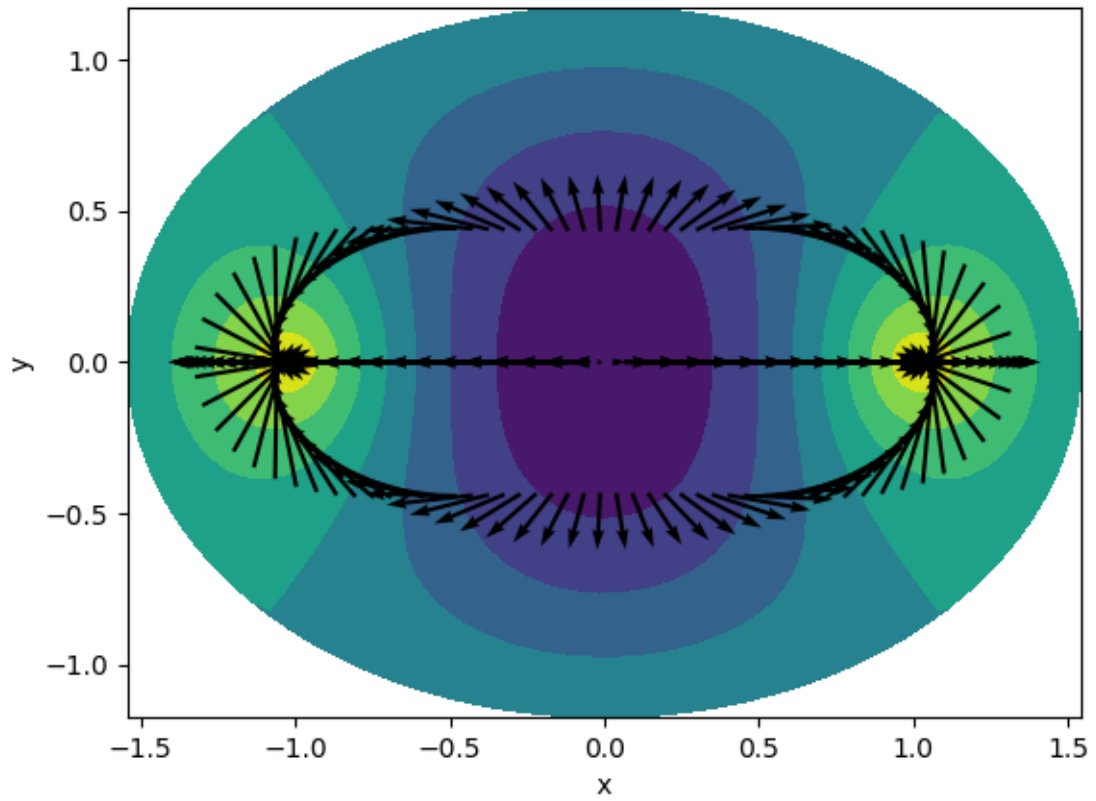
```

Figur 2: Konturplott i elliptiske koordinater oppgave 2d



Observerer da at vektorene peker ut i fra origo og blir slukt ned i brennpunktene $(-1, 0)$ og $(1, 0)$

Figur 3: Pilplott til oppgave 2d



3

a

Strømfunksjon

$$v_x = \frac{\partial \psi}{\partial y} \wedge v_y = \frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial y} = -v_x \quad \wedge \quad \frac{\partial \psi}{\partial x} = v_y$$

$$\psi(x, y) = - \int \cos(x) \sin(y) e^{-2vt} dy \quad \wedge \quad \psi(x, y) = \int -\sin(x) \cos(y) e^{-2vt} dx$$

$$\psi(x, y) = -\cos(x) \cos(y) e^{-2vt} + C \quad \wedge \quad \psi(x, y) = -\cos(x) \cos(y) e^{-2vt} + C$$

Ser så om den har ett skalarpotensial. Hvis det har det så må

$$\begin{array}{lll} \frac{\partial \phi}{\partial x} = v_x & \wedge & \frac{\partial \phi}{\partial y} = v_y \\ \phi(x, y) = \int \cos(x) \sin(y) e^{-2vt} dx & \wedge & \phi(x, y) = \int \left(-\sin(x) \cos(y) e^{-2vt} \right) dy \\ \phi = \sin(x) \sin(y) e^{-2vt} + f(y) & \wedge & \phi = -\sin(x) \sin(y) e^{-2vt} + f(x) \end{array}$$

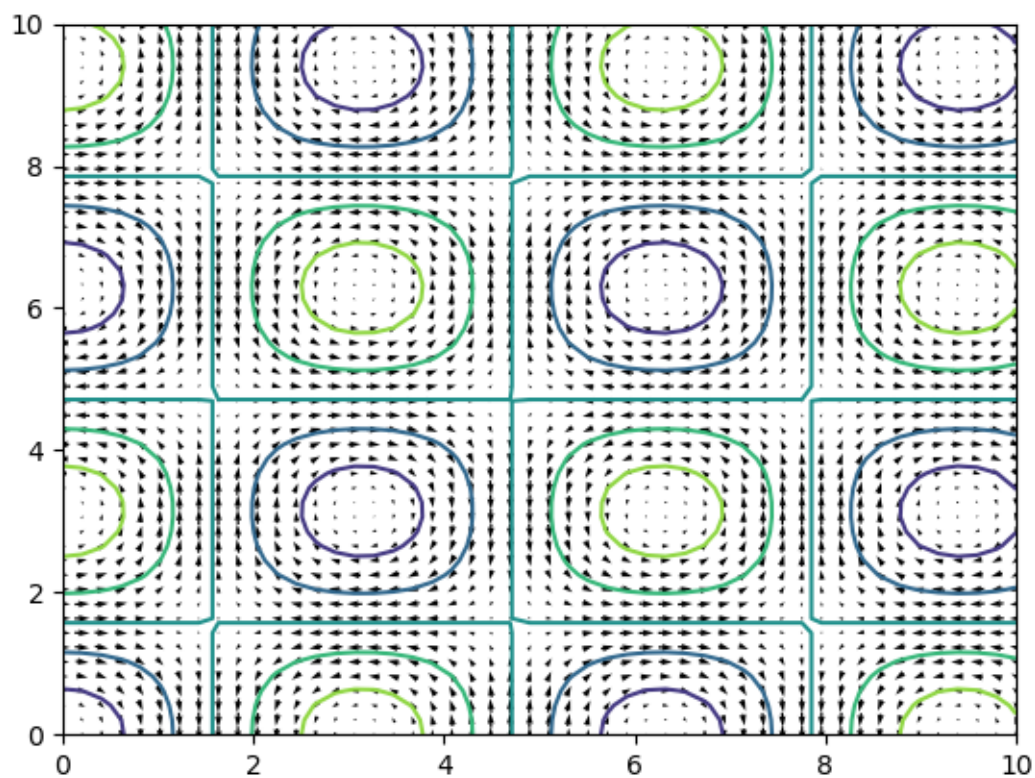
Disse kan ikke være like så det finnes ikke noe skalarpotensial

b

Python kode til pilplott med strømlinjer

Som produserer følgende graf

Figur 4: Pilplott med strømlinjer til 3b



c

Kan bruke Gauss' divergensteorem siden det er en generalisering av Green Sett inn mer her siden

$$\int_C \vec{u} \cdot \vec{n} \, ds = \iiint_R \nabla \cdot \vec{u} \, dx \, dy$$

Regner ut $\nabla \cdot \vec{u}$

$$\nabla \cdot \vec{u} = \frac{\partial}{\partial x} (\cos(x) \sin(y)) + \frac{\partial}{\partial y} (-\sin(x) \cos(y))$$

$$\nabla \cdot \vec{u} = -\sin(x) \sin(y) + \sin(x) \sin(y)$$

$$\nabla \cdot \vec{u} = 0$$

Følgelig er fluksen da 0

Sirkulasjonen blir da

$$\oint \vec{u} \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \vec{u} \cdot d\vec{r}_{x,0} + \int_0^{\frac{\pi}{2}} \vec{u} \cdot d\vec{r}_{y,0} + \int_{\frac{\pi}{2}}^0 \vec{u} \cdot d\vec{r}_{x,1} + \int_{\frac{\pi}{2}}^0 \vec{u} \cdot d\vec{r}_{y,1}$$

$\vec{r}_{x,0}(t') = t'\hat{\mathbf{i}} + 0\hat{\mathbf{j}}$	\wedge	$\vec{r}_{y,0}(t') = \frac{\pi}{2}\hat{\mathbf{i}} + t'\hat{\mathbf{j}}$	\wedge	$\vec{r}_{x,1}(t') = t'\hat{\mathbf{i}} + \frac{\pi}{2}\hat{\mathbf{j}}$	\wedge	$\vec{r}_{y,1}(t') = 0\hat{\mathbf{i}} + t'\hat{\mathbf{j}}$
$\vec{r}'_{x,0}(t') = \hat{\mathbf{i}}$	\wedge	$\vec{r}'_{y,0}(t') = \hat{\mathbf{j}}$	\wedge	$\vec{r}'_{x,1}(t') = \hat{\mathbf{i}}$	\wedge	$\vec{r}'_{y,1}(t') = \hat{\mathbf{j}}$

$$\begin{aligned} \oint \vec{u} \cdot d\vec{r} = & \int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') \, dt' + \int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') \, dt' \\ & + \int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1}(t') \, dt' + \int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1}(t') \, dt' \end{aligned}$$

For å få det litt mer oversiktlig så regner jeg ett integral av gangen. Starter med

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') \, dt' &= \int_0^{\frac{\pi}{2}} (\cos(t') \sin(0) e^{-2vt}) \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} \, dt' \\ \int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') \, dt' &= \int_0^{\frac{\pi}{2}} (\cos(t') \sin(0) e^{-2vt}) \, dt' \\ \int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') \, dt' &= \int_0^{\frac{\pi}{2}} 0 \, dt' \\ \int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') \, dt' &= 0 \end{aligned}$$

Ser så på

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= \int_0^{\frac{\pi}{2}} \left(-\sin\left(\frac{\pi}{2}\right) \cos(t') e^{-2vt} \right) \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= \int_0^{\frac{\pi}{2}} \left(-\sin\left(\frac{\pi}{2}\right) \cos(t') e^{-2vt} \right) dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= -e^{-2vt} \int_0^{\frac{\pi}{2}} \cos(t') dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= -e^{-2vt} \left[\sin(t') \right]_0^{\frac{\pi}{2}} \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= -e^{-2vt} (1 - 0) \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= -e^{-2vt}
\end{aligned}$$

Ser så på

$$\begin{aligned}
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1}(t') dt' &= \int_{\frac{\pi}{2}}^0 \left(\cos(t') \sin\left(\frac{\pi}{2}\right) \right) \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} dt' \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1}(t') dt' &= e^{-2vt} \int_{\frac{\pi}{2}}^0 \cos(t') dt' \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1}(t') dt' &= e^{-2vt} \left[\sin(t') \right]_{\frac{\pi}{2}}^0 \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1}(t') dt' &= e^{-2vt} (0 - 1) \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1}(t') dt' &= -e^{-2vt}
\end{aligned}$$

Ser så på

$$\begin{aligned}\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1} dt' &= \int_{\frac{\pi}{2}}^0 \left(-\sin(0) \cos(t') e^{-2vt} \right) \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} dt' \\ \int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1} dt' &= \int_{\frac{\pi}{2}}^0 \left(-\sin(0) \cos(t') e^{-2vt} \right) dt' \\ \int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1} dt' &= \int_{\frac{\pi}{2}}^0 0 dt' \\ \int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1} dt' &= 0\end{aligned}$$

Kan så gå tilbake til sirkulasjonen

$$\oint \vec{u} \cdot d\vec{r} = 0 - e^{-2vt} - e^{-2vt} + 0 = -2e^{-2vt}$$

Kan også bruke Stokes sats som sier

$$\oint_C \vec{u} \cdot d\vec{r} = \int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} d\sigma$$

Der

$$\begin{aligned}\nabla \times \vec{u} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(x) \sin(y) e^{-2vt} & -\sin(x) \cos(y) e^{-2vt} & 0 \end{vmatrix} \\ \nabla \times \vec{u} &= e^{-2vt} \left(\frac{\partial}{\partial x} (-\sin(x) \cos(y)) - \frac{\partial}{\partial y} (\cos(x) \sin(y)) \right) \hat{\mathbf{k}} \\ \nabla \times \vec{u} &= e^{-2vt} (-\cos(x) \cos(y) - \cos(x) \cos(y)) \hat{\mathbf{k}} \\ \nabla \times \vec{u} &= -2e^{-2vt} \cos(x) \cos(y) \hat{\mathbf{k}}\end{aligned}$$

Og $\vec{n} = \hat{\mathbf{k}}$ siden den har positiv orientering om z . Da blir Stokes

$$\begin{aligned}
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= \int_{\sigma} \left(-2e^{-2vt} \cos(x) \cos(y) \right) \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \, d\sigma \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_{\sigma} (\cos(x) \cos(y)) \, d\sigma \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(x) \cos(y) \, dx \, dy \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_0^{\frac{\pi}{2}} \cos(y) [\sin(x)]_0^{\frac{\pi}{2}} \, dy \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_0^{\frac{\pi}{2}} \cos(y) (1 - 0) \, dy \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_0^{\frac{\pi}{2}} \cos(y) \, dy \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} [\sin(y)]_0^{\frac{\pi}{2}} \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} (1 - 0) \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt}
\end{aligned}$$