# 1

i $\vec{u} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
Ser først på divergensen

$$\nabla \cdot \vec{u} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z$$
$$\nabla \cdot \vec{u} = 1 + 1 + 1$$
$$\nabla \cdot \vec{u} = 3$$

Ser så på virvlingen

$$\nabla \times \vec{u} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$
$$\nabla \times \vec{u} = \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right)\hat{\mathbf{k}}$$
$$\nabla \times \vec{u} = \vec{0}$$

ii $\vec{u} = r\cos(\theta)\hat{\mathbf{i}}_r + r\sin(\theta)\hat{\mathbf{i}}_\theta + z\hat{\mathbf{k}}$ Ser først på divergensen

$$\nabla \cdot \vec{u} = \frac{1}{r}\left(\frac{\partial}{\partial r}\Big(r\left(r\cos(\theta)\right)\Big) + \frac{\partial}{\partial \theta}\left(r\sin(\theta)\right)\right) + \frac{\partial}{\partial z}z$$
$$\nabla \cdot \vec{u} = \frac{1}{r}\left(\cos(\theta)\frac{\partial}{\partial r}r^2 + r\frac{\partial}{\partial \theta}\sin(\theta)\right) + 1$$
$$\nabla \cdot \vec{u} = \frac{1}{r}\big(2r\cos(\theta) + r\cos(\theta)\big) + 1$$
$$\nabla \cdot \vec{u} = 2\cos(\theta) + \cos(\theta) + 1$$
$$\nabla \cdot \vec{u} = 3\cos(\theta) + 1$$

Ser så på virvlingen

$$\nabla \times \vec{u} = 0\hat{\mathbf{i}}_r + 0\hat{\mathbf{i}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\Big(r\left(r\sin(\theta)\right)\Big) - \frac{\partial}{\partial \theta}\left(r\cos(\theta)\right)\right)\hat{\mathbf{k}}$$
$$\nabla \times \vec{u} = \frac{1}{r}\left(\sin(\theta)\frac{\partial}{\partial r}\left(r^2\right) - r\frac{\partial}{\partial \theta}\left(\cos(\theta)\right)\right)\hat{\mathbf{k}}$$
$$\nabla \times \vec{u} = \frac{1}{r}\big(2r\sin(\theta) + r\sin(\theta)\big)\hat{\mathbf{k}}$$
$$\nabla \times \vec{u} = \big(2\sin(\theta) + \sin(\theta)\big)\hat{\mathbf{k}}$$
$$\nabla \times \vec{u} = 3\sin(\theta)\hat{\mathbf{k}}$$

iii $\vec{u} = \hat{\mathbf{i}}_r + \hat{\mathbf{i}}$

Konverterer først over til rene sylinder koordinater

$$\boxed{\hat{\mathbf{i}} = \cos(\theta)\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta}$$

$$\begin{aligned}
\vec{u} &= \hat{\mathbf{i}}_r + \cos(\theta)\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta \\
\vec{u} &= \big(1 + \cos(\theta)\big)\,\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta
\end{aligned}$$

Ser først på divergensen

$$\begin{aligned}
\nabla \cdot \vec{u} &= \frac{1}{r}\left(\frac{\partial}{\partial r}\Big(r\,\big(1 + \cos(\theta)\big)\Big)\,\frac{\partial}{\partial \theta}\big(-\sin(\theta)\big)\right) \\
\nabla \cdot \vec{u} &= \frac{1}{r}\big(1 + \cos(\theta) - \cos(\theta)\big) \\
\nabla \cdot \vec{u} &= \frac{1}{r}
\end{aligned}$$

Ser så på virvlingen

$$\begin{aligned}
\nabla \times \vec{u} &= 0\hat{\mathbf{i}}_r + 0\hat{\mathbf{i}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\Big(r\,\big(-\sin(\theta)\big)\Big) - \frac{\partial}{\partial \theta}\big(1 + \cos(\theta)\big)\right)\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= \frac{1}{r}\big(-\sin(\theta) + \sin(\theta)\big)\,\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= \vec{0}
\end{aligned}$$

# 2

## a

For å finne enhetsvektorene så må man først finne skaleringsfaktorene. Starter med skaleringsfaktoren til $u$

$$h_u = \left|\frac{\partial \vec{r}}{\partial u}\right|$$
$$h_u = \left|a\cos(v)\frac{\partial}{\partial u}\cosh(u)\hat{\mathbf{i}} + a\sin(v)\frac{\partial}{\partial v}\sinh(u)\hat{\mathbf{j}}\right|$$
$$\boxed{a = 1}$$
$$h_u = \left|\cos(v)\sinh(u)\,\hat{\mathbf{i}} + \sin(v)\cosh(u)\right|$$
$$h_u = \sqrt{\big(\cos(v)\sinh(u)\big)^2 + \big(\sin(v)\cosh(u)\big)^2}$$
$$h_u = \sqrt{\cos^2(v)\sinh^2(u) + \sin^2(v)\cosh^2(u)}$$
$$\boxed{\cosh^2(u) = 1 + \sinh^2(u)}$$
$$h_u = \sqrt{\cos^2(v)\sinh^2(u) + \sin^2(v)\Big(1 + \sinh^2(u)\Big)}$$
$$h_u = \sqrt{\cos^2(v)\sinh^2(u) + \sinh^2(u)\sin^2(v) + \sin^2(v)}$$
$$h_u = \sqrt{\sinh^2(u)\Big(\cos^2(v) + \sin^2(v)\Big) + \sin^2(v)}$$
$$h_u = \sqrt{\sinh^2(u) + \sin^2(v)}$$

Finner så skaleringsfaktoren til $v$

$$h_v = \left|\frac{\partial \vec{r}}{\partial v}\right|$$
$$h_v = \left|\cosh(u)\frac{\partial}{\partial v}\cos(v)\hat{\mathbf{i}} + \sinh(u)\frac{\partial}{\partial v}\sin(v)\hat{\mathbf{j}}\right|$$
$$h_v = \left|-\cosh(u)\sin(v)\hat{\mathbf{i}} + \sinh(u)\cos(v)\hat{\mathbf{j}}\right|$$
$$h_v = \sqrt{\big(-\cosh(u)\sin(v)\big)^2 + \big(\sinh(u)\cos(v)\big)^2}$$
$$h_v = \sqrt{\cosh^2(u)\sin^2(v) + \sinh^2(u)\cos^2(v)}$$
$$h_v = \sqrt{\Big(1 + \sinh^2(u)\Big)\sin^2(v) + \sinh^2(u)\cos^2(v)}$$
$$h_v = \sqrt{\sin^2(v) + \sinh^2(u)\sin^2(v) + \sinh^2(u)\cosh^2(v)}$$
$$h_v = \sqrt{\sin^2(v) + \sinh^2(u)\Big(\sin^2(v) + \cos^2(v)\Big)}$$
$$h_v = \sqrt{\sinh^2(u) + \sin^2(v)}$$

Ser da at $h_u = h_v$ Enhetsvektorer er da gitt ved

$$\mathbf{e}_u = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u}$$
$$\mathbf{e}_u = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}}\left(\cos(v)\sinh(u)\,\hat{\mathbf{i}} + \sin(v)\cosh(u)\,\hat{\mathbf{j}}\right)$$

Og

$$\mathbf{e}_v = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v}$$
$$\mathbf{e}_v = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}}\left(-\cosh(u)\sin(v)\hat{\mathbf{i}} + \sinh(u)\cos(v)\hat{\mathbf{j}}\right)$$

Hvis de er ortogonale så må $\mathbf{e}_u \cdot \mathbf{e}_v = 0$

$$\mathbf{e}_u \cdot \mathbf{e}_v = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}}\begin{pmatrix}\cos(v)\sinh(u)\\ \sin(v)\cosh(u)\end{pmatrix} \cdot \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}}\begin{pmatrix}-\cosh(u)\sin(v)\\ \sinh(u)\cos(v)\end{pmatrix}$$
$$\mathbf{e}_u \cdot \mathbf{e}_v = \frac{1}{\sinh^2(u)\sin^2(v)}\left(-\cos(v)\sinh(u)\cosh(u)\sin(v) + \sin(v)\cosh(u)\sinh(u)\cos(v)\right)$$
$$\mathbf{e}_u \cdot \mathbf{e}_v = \left(-\sin(v)\cos(v)\sinh(u)\cosh(u) + \sin(v)\cos(v)\sinh(u)\cosh(u)\right)$$
$$\mathbf{e}_u \cdot \mathbf{e}_v = 0$$

Med dette så observeres det at enhetsvektorene er ortogonale

## b

Siden ingen $f$ har blitt oppgitt så tar jeg utgangspunkt i en helt generell $f$. Gradienten blir da

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v$$
$$\boxed{h_u = h_v = h}$$
$$\nabla f = \frac{1}{h}\left(\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{\partial f}{\partial v}\mathbf{e}_v\right)$$

Veit ikke hva $w_u$ og $w_v$ inneholder så blir generell divergens

$$\nabla \cdot \vec{w} = \frac{1}{h_u h_v}\left(\frac{\partial}{\partial u}\left(w_u \mathbf{e}_u h_v\right) + \frac{\partial}{\partial v}\left(w_v \mathbf{e}_v h_u\right)\right)$$

Laplace operatoren blir

$$\nabla^2 = \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}\left(\frac{h_u}{h_v}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_v}{h_u}\frac{\partial}{\partial v}\right)\right]$$

$$\boxed{h_v = h_u = h}$$

$$\nabla^2 = \frac{1}{h^2}\left[\frac{\partial}{\partial u}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\frac{\partial}{\partial v}\right]$$

$$\nabla^2 = \frac{1}{h^2}\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]$$

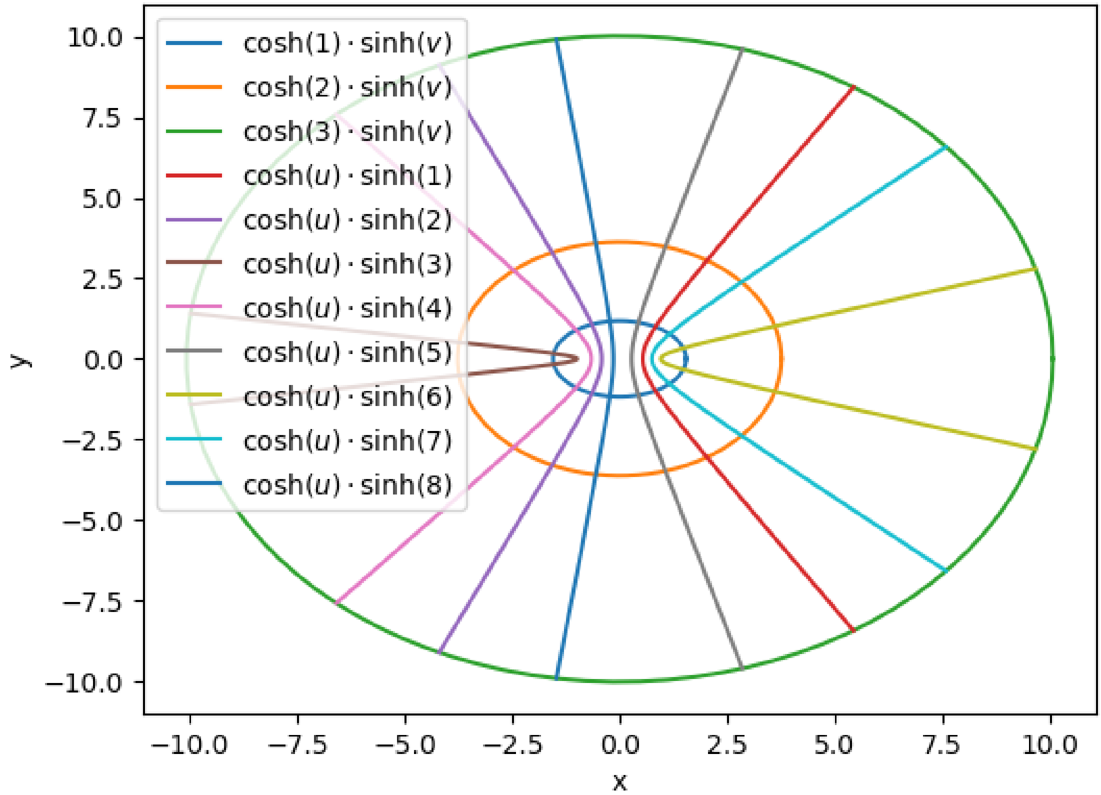
## c

Python kode til skissen er

```python
import numpy as np
import matplotlib.pyplot as plt

N = 100
u = np.linspace(-3, 3, N)
v = np.linspace(0, 2*np.pi, N)

for i in range(1, 4):
    x = np.cosh(i)*np.cos(v)
    y = np.sinh(i)*np.sin(v)
    plt.plot(x, y, label=f"$\cosh({i}) \cdot \sinh(v)$")

for i in range(1, 9):
    x = np.cosh(u)*np.cos(i)
    y = np.sinh(u)*np.sin(i)
    plt.plot(x, y, label=f"$\cosh(u) \cdot \sinh({i})$")

plt.xlabel("x")
plt.ylabel("y")
plt.legend()
plt.savefig("2c.png")
plt.show()
```

Som produserer skissen

Figur 1: Skissen til 2c



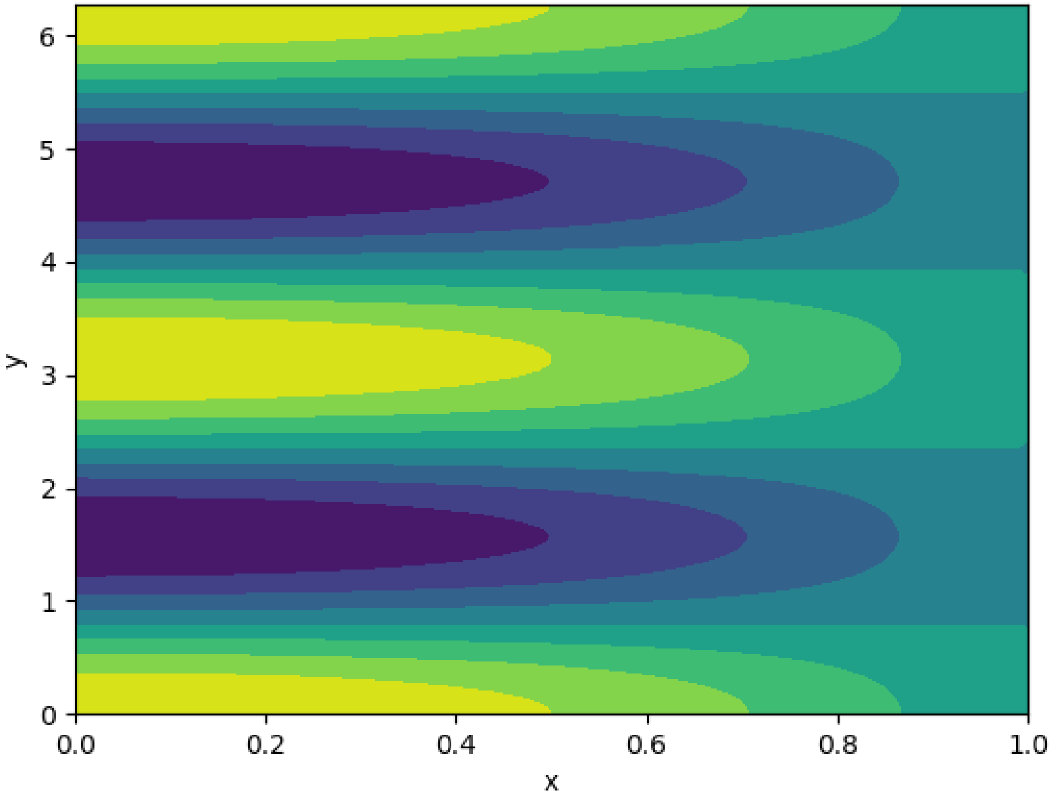

### d

Python kode til dette er

```
import numpy as np
import sympy as sp
import matplotlib.pyplot as plt

u, v = psi = sp.symbols("u, v", real=True)
r = (sp.cosh(u)*sp.cos(v), sp.sinh(u)*sp.sin(v))

def basisvektor(psi, r):
    b = np.zeros((len(psi), len(r)), dtype=object)
    for i, ui, in enumerate(psi):
        for j, rj in enumerate(r):
            b[i, j] = sp.simplify(rj.diff(ui, 1))
    return b

def skaleringsfaktor(b):
    h = np.zeros(b.shape[0], dtype=object)
    for i, s in enumerate(np.sum(b**2, axis=1)):
        h[i] = sp.simplify(sp.sqrt(s))
    return h
```

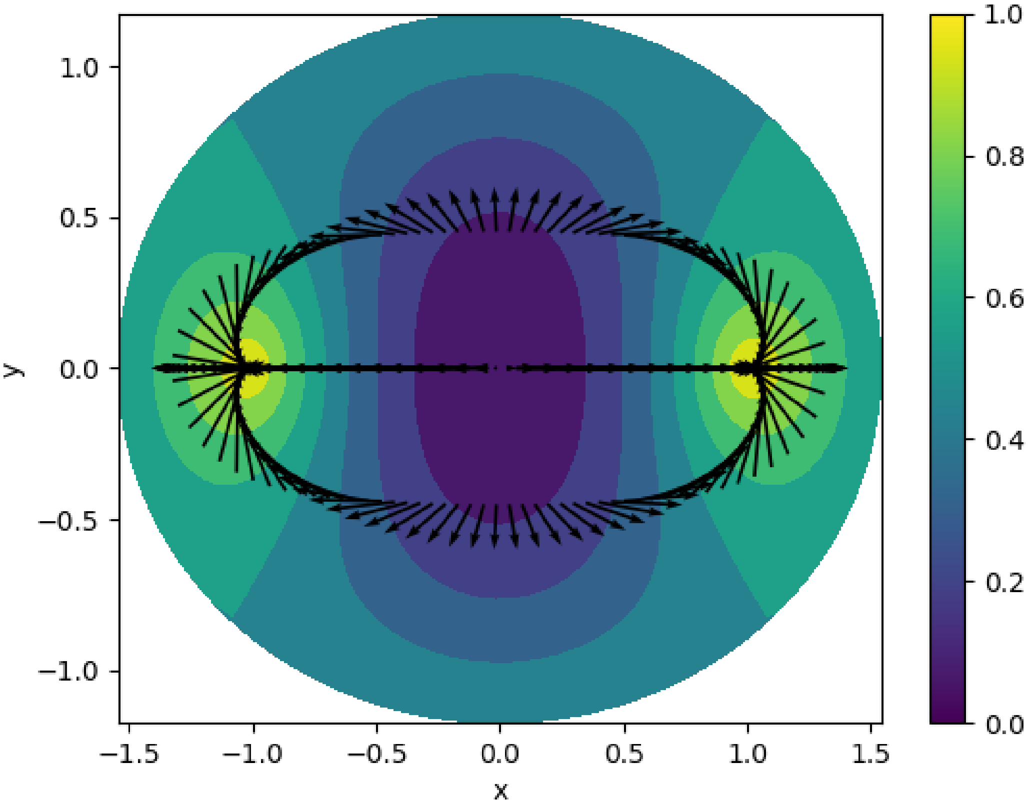
```python

def enhetsvektor(psi, r):
    b = basisvektor(psi, r)
    hi = skaleringsfaktor(b)
    return b/ hi[None, :], hi

e, h = enhetsvektor(psi, r)

f = (1 - u**2)*sp.cos(2*v)

N = 100
ui = np.broadcast_to(np.linspace(0, 1, N)[:, None], (N, N))
vi = np.broadcast_to(np.linspace(0, 2*np.pi, N)[None, :], (N, N))
fj = sp.lambdify((u, v), f)(ui, vi)

mesh = []
for rj in r:
    mesh.append(sp.lambdify((u, v), rj)(ui, vi))
x, y = mesh

plt.contourf(x, y, fj)

df = np.array((1/h[0]*f.diff(u, 1), 1/h[1]*f.diff(v, 1)))

gradf = e[0]*df[0] + e[1]*df[1]

dfdxi = sp.lambdify((u, v), gradf[0])(ui, vi)
dfdyi = sp.lambdify((u, v), gradf[1])(ui, vi)
plt.contourf(x, y, fj)
plt.quiver(x[::50], y[::50], dfdxi[::50], dfdyi[::50], scale=15, pivot="middle")

plt.xlabel("x")
plt.ylabel("y")

plt.savefig("2d.png")
plt.show()
plt.close()

plt.contourf(ui, vi, fj)
plt.xlabel("x")
plt.ylabel("y")
plt.savefig("2d_elliptical.png")
plt.show()
```

Figur 2: Konturplott i elliptiske koordinater oppgave 2d




Observerer da at vektorene peker ut i fra det lilla og blir slukt ned i det gule

Figur 3: Pilplott til oppgave 2d




# 3

## a

Regner ut $\nabla \cdot \vec{u}$

$$\begin{aligned}
\nabla \cdot \vec{u} &= \frac{\partial}{\partial x}\big(\cos(x)\sin(y)\big) + \frac{\partial}{\partial y}\big(-\sin(x)\cos(y)\big) \\
\nabla \cdot \vec{u} &= -\sin(x)\sin(y) + \sin(x)\sin(y) \\
\nabla \cdot \vec{u} &= 0
\end{aligned}$$

Divergensen lik 0 derfor finnes det en Strømfunksjon
Strømfunksjon

$$u_x = \frac{\partial \psi}{\partial y} \wedge u_y = \frac{\partial \psi}{\partial x}$$

$$
\begin{aligned}
\frac{\partial \psi}{\partial y} &= -u_x && \wedge && \frac{\partial \psi}{\partial x} = u_y \\
\psi(x,y) &= -\int \cos(x)\sin(y)\mathrm{e}^{-2vt}\,\mathrm{d}y && \wedge && \psi(x,y) = \int -\sin(x)\cos(y)\mathrm{e}^{-2vt}\,\mathrm{d}x \\
\psi(x,y) &= -\cos(x)\cos(y)\mathrm{e}^{-2vt} && \wedge && \psi(x,y) = -\cos(x)\cos(y)\mathrm{e}^{-2vt}
\end{aligned}
$$

Ser så om den har ett skalarpotensial. Hvis det har det så må $\nabla \times \vec{u} = \vec{0}$

$$
\begin{aligned}
\nabla \times \vec{u} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(x)\sin(y)\mathrm{e}^{-2vt} & -\sin(x)\cos(y)\mathrm{e}^{-2vt} & 0 \end{vmatrix} \\
\nabla \times \vec{u} &= \mathrm{e}^{-2vt}\left(\frac{\partial}{\partial x}\left(-\sin(x)\cos(y)\right) - \frac{\partial}{\partial y}\left(\cos(x)\sin(y)\right)\right)\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= \mathrm{e}^{-2vt}\left(-\cos(x)\cos(y) - \cos(x)\cos(y)\right)\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= -2\mathrm{e}^{-2vt}\cos(x)\cos(y)\hat{\mathbf{k}}
\end{aligned}
$$

Dette er ikke lik 0 så da finnes det ikke noe skalarpotensial.

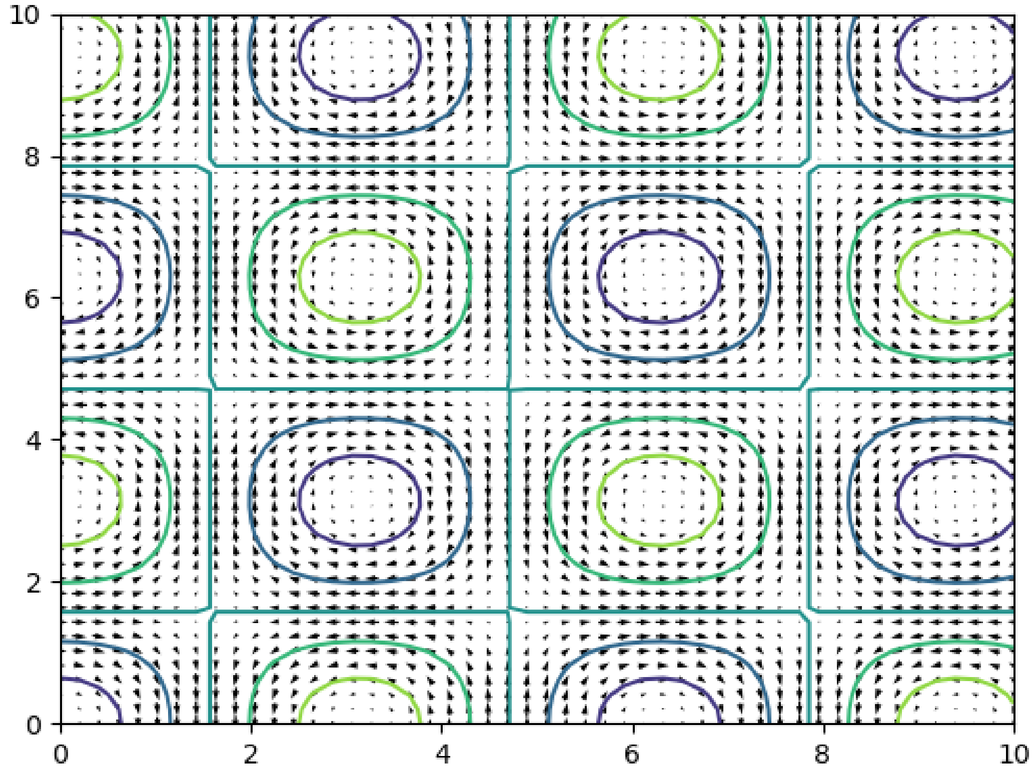
## b

Python kode til pilplott med strømlinjer

```python
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(0, 10, 50)
x, y = np.meshgrid(t, t, indexing="ij")

plt.contour(x, y, -np.cos(x)*np.cos(y), 4)
plt.quiver(x, y, np.cos(x)*np.sin(y), -np.sin(x)*np.cos(y), pivot="middle")

plt.savefig("3b.png")
plt.show()
```

Som produserer følgende graf

Figur 4: Pilplott med strømlinjer til 3b




**c**

Hvis $\vec{F}(x,y) = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}}$ er et vektorfelt i planet er

$$\nabla \cdot \vec{F}(x,y) = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y)$$

Dersom $C$ er en enkel, parametrisert kurve i planet som omslutter et område $R$. Så må det finnes en to-dimensjonal versjon av Gauss' sats som sier

$$\iint_R \nabla \cdot \vec{F} \,\mathrm{d}x\,\mathrm{d}y = \int_C \vec{F} \cdot \vec{n} \,\mathrm{d}s$$

der $\vec{n}$ er enhetsnormalvektoren til $C$ som peker ut av området $R$. Skal nå vise at dette bare er en omfurmelering av Greens teorem. Starter med å velge en positiv orientert parametrisering $\vec{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}, t \in [a,b]$, som gjennomløper $C$ med fart konstant lik 1. Da er $\overrightarrow{r}'(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}}$ en tangentvektor med lengde 1. Det betyr at $\vec{n}(t) = y'(t)\hat{\mathbf{i}} - x'(t)\hat{\mathbf{j}}$ er en enhetsnormalvektor som peker ut av av området $R$.

$$\int_C \vec{F} \cdot \vec{n} \, \mathrm{d}s = \int_a^b \Big(P\left(\vec{r}(t)\right) y'(t) + Q\left(\vec{r}(t)\right)\left(-x'(t)\right)\Big) \, \mathrm{d}t = \int_C -Q \, \mathrm{d}x + P \, \mathrm{d}y$$

Bruker så Greens teorem

$$\int_C -Q \, \mathrm{d}x + P \, \mathrm{d}y = \iint_R \left(\frac{\partial p}{\partial x} - \frac{\partial(-Q)}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y = \iint_R \nabla \cdot \vec{F} \, \mathrm{d}x \, \mathrm{d}y$$

Dermed er

$$\iint_R \nabla \cdot \vec{F} \, \mathrm{d}x \, \mathrm{d}y = \int_C \vec{F} \cdot \vec{n} \, \mathrm{d}s$$

Har med det vist at den to-dimensjonale versjonen av Gauss' sats følger fra Green
Regner så ut fluksen med Gauss' sats

$$\int_C \vec{u} \cdot \vec{n} \, \mathrm{d}s = \iint_R \nabla \cdot \vec{u} \, \mathrm{d}x \, \mathrm{d}y$$

Fra **a** veit man at divergensen er 0. Følgelig er fluksen da 0
Sirkulasjonen blir da

$$\oint \vec{u} \cdot \mathrm{d}\vec{r} = \int_0^{\frac{\pi}{2}} \vec{u} \cdot \mathrm{d}\vec{r}_{x,0} + \int_0^{\frac{\pi}{2}} \vec{u} \cdot \mathrm{d}\vec{r}_{y,0} + \int_{\frac{\pi}{2}}^0 \vec{u} \cdot \mathrm{d}\vec{r}_{x,1} + \int_{\frac{\pi}{2}}^0 \vec{u} \cdot \mathrm{d}\vec{r}_{y,1}$$

$$\boxed{\begin{array}{llll} \vec{r}_{x,0}(t') = t'\hat{\mathbf{i}} + 0\hat{\mathbf{j}} \quad \wedge & \vec{r}_{y,0}(t') = \frac{\pi}{2}\hat{\mathbf{i}} + t'\hat{\mathbf{j}} \wedge & \vec{r}_{x,1}(t') = t'\hat{\mathbf{i}} + \frac{\pi}{2}\hat{\mathbf{j}} \quad \wedge & \vec{r}_{y,1}(t') = 0\hat{\mathbf{i}} + t'\hat{\mathbf{j}} \\ \overrightarrow{r}'_{x,0}(t') = \hat{\mathbf{i}} \quad \wedge & \overrightarrow{r}'_{y,0}(t') = \hat{\mathbf{j}} \wedge & \overrightarrow{r}'_{x,1}(t') = \hat{\mathbf{i}} \quad \wedge & \overrightarrow{r}'_{y,1}(t') = \hat{\mathbf{j}} \end{array}}$$

$$\oint \vec{u} \cdot \mathrm{d}\vec{r} = \begin{array}{l} \displaystyle\int_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{x,0}(t')\right) \cdot \overrightarrow{r}'_{x,0}(t') \, \mathrm{d}t' + \int_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{y,0}(t')\right) \cdot \overrightarrow{r}'_{y,0}(t') \, \mathrm{d}t' \\ \displaystyle + \int_{\frac{\pi}{2}}^0 \vec{u}\left(\vec{r}_{x,1}(t')\right) \cdot \overrightarrow{r}'_{x,1} \, \mathrm{d}t' + \int_{\frac{\pi}{2}}^0 \vec{u}\left(\vec{r}_{y,1}(t')\right) \cdot \overrightarrow{r}'_{y,1} \, \mathrm{d}t' \end{array}$$

For å få det litt mer oversiktelig så regner jeg ett integral av gangen. Starter med

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{x,0}(t')\right) \cdot \overrightarrow{r}'_{x,0}(t')\,\mathrm{d}t' = \int\limits_0^{\frac{\pi}{2}} \Big(\cos\big(t'\big)\sin(0)\mathrm{e}^{-2vt}\Big)\,\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}\,\mathrm{d}t'$$

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{x,0}(t')\right) \cdot \overrightarrow{r}'_{x,0}(t')\,\mathrm{d}t' = \int\limits_0^{\frac{\pi}{2}} \Big(\cos\big(t'\big)\sin(0)\mathrm{e}^{-2vt}\Big)\,\mathrm{d}t'$$

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{x,0}(t')\right) \cdot \overrightarrow{r}'_{x,0}(t')\,\mathrm{d}t' = \int_0^{\frac{\pi}{2}} 0\,\mathrm{d}t'$$

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{x,0}(t')\right) \cdot \overrightarrow{r}'_{x,0}(t')\,\mathrm{d}t' = 0$$

Ser så på

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{y,0}(t')\right) \cdot \overrightarrow{r}'_{y,0}(t')\,\mathrm{d}t' = \int\limits_0^{\frac{\pi}{2}} \left(-\sin\left(\frac{\pi}{2}\right)\cos\big(t'\big)\mathrm{e}^{-2vt}\right)\hat{\mathbf{j}}\cdot\hat{\mathbf{j}}\,\mathrm{d}t'$$

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{y,0}(t')\right) \cdot \overrightarrow{r}'_{y,0}(t')\,\mathrm{d}t' = \int\limits_0^{\frac{\pi}{2}} \left(-\sin\left(\frac{\pi}{2}\right)\cos\big(t'\big)\mathrm{e}^{-2vt}\right)\mathrm{d}t'$$

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{y,0}(t')\right) \cdot \overrightarrow{r}'_{y,0}(t')\,\mathrm{d}t' = -\mathrm{e}^{-2vt}\int\limits_0^{\frac{\pi}{2}} \cos\big(t'\big)\,\mathrm{d}t'$$

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{y,0}(t')\right) \cdot \overrightarrow{r}'_{y,0}(t')\,\mathrm{d}t' = -\mathrm{e}^{-2vt}\Big[\sin\big(t'\big)\Big]_0^{\frac{\pi}{2}}$$

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{y,0}(t')\right) \cdot \overrightarrow{r}'_{y,0}(t')\,\mathrm{d}t' = -\mathrm{e}^{-2vt}\,(1-0)$$

$$\int\limits_0^{\frac{\pi}{2}} \vec{u}\left(\vec{r}_{y,0}(t')\right) \cdot \overrightarrow{r}'_{y,0}(t')\,\mathrm{d}t' = -\mathrm{e}^{-2vt}$$

Ser så på

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{x,1}(t')\right) \cdot \overrightarrow{r}'_{x,1}\,\mathrm{d}t' = \int_{\frac{\pi}{2}}^{0} \left(\cos(t')\sin\left(\frac{\pi}{2}\right)\right)\hat{\mathbf{i}}\cdot\hat{\mathbf{i}}\,\mathrm{d}t'$$

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{x,1}(t')\right) \cdot \overrightarrow{r}'_{x,1}\,\mathrm{d}t' = \mathrm{e}^{-2vt}\int_{\frac{\pi}{2}}^{0} \cos(t')\,\mathrm{d}t'$$

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{x,1}(t')\right) \cdot \overrightarrow{r}'_{x,1}\,\mathrm{d}t' = \mathrm{e}^{-2vt}\left[\sin(t')\right]_{\frac{\pi}{2}}^{0}$$

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{x,1}(t')\right) \cdot \overrightarrow{r}'_{x,1}\,\mathrm{d}t' = \mathrm{e}^{-2vt}\,(0-1)$$

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{x,1}(t')\right) \cdot \overrightarrow{r}'_{x,1}\,\mathrm{d}t' = -\mathrm{e}^{-2vt}$$

Ser så på

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{y,1}(t')\right) \cdot \overrightarrow{r}'_{y,1}\,\mathrm{d}t' = \int_{\frac{\pi}{2}}^{0} \left(-\sin(0)\cos(t')\mathrm{e}^{-2vt}\right)\hat{\mathbf{j}}\cdot\hat{\mathbf{j}}\,\mathrm{d}t'$$

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{y,1}(t')\right) \cdot \overrightarrow{r}'_{y,1}\,\mathrm{d}t' = \int_{\frac{\pi}{2}}^{0} \left(-\sin(0)\cos(t')\mathrm{e}^{-2vt}\right)\mathrm{d}t'$$

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{y,1}(t')\right) \cdot \overrightarrow{r}'_{y,1}\,\mathrm{d}t' = \int_{\frac{\pi}{2}}^{0} 0\,\mathrm{d}t'$$

$$\int_{\frac{\pi}{2}}^{0} \vec{u}\left(\vec{r}_{y,1}(t')\right) \cdot \overrightarrow{r}'_{y,1}\,\mathrm{d}t' = 0$$

Kan så gå tilbake til sirkulasjonen

$$\oint \vec{u}\cdot\mathrm{d}\vec{r} = 0 - \mathrm{e}^{-2vt} - \mathrm{e}^{-2vt} + 0 = -2\mathrm{e}^{-2vt}$$

Kan også bruke Stokes sats som sier

$$\oint_C \vec{u}\cdot\mathrm{d}\vec{r} = \int_\sigma (\nabla\times\vec{u})\cdot\vec{n}\,\mathrm{d}\sigma$$

Fra **a** veit man at

$$\nabla\times\vec{u} = -2\mathrm{e}^{-2vt}\cos(x)\cos(y)\hat{\mathbf{k}}$$

Og $\vec{n} = \hat{\mathbf{k}}$ siden den har positiv orientering om $z$. Da blir Stokes

$$\begin{aligned}
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= \int\limits_{\sigma} \Big( -2\mathrm{e}^{-2vt} \cos(x)\cos(y) \Big) \, \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \, \mathrm{d}\sigma \\
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= -2\mathrm{e}^{-2vt} \int\limits_{\sigma} \big( \cos(x)\cos(y) \big) \, \mathrm{d}\sigma \\
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= -2\mathrm{e}^{-2vt} \int\limits_{0}^{\frac{\pi}{2}} \int\limits_{0}^{\frac{\pi}{2}} \cos(x)\cos(y) \, \mathrm{d}x \, \mathrm{d}y \\
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= -2\mathrm{e}^{-2vt} \int\limits_{0}^{\frac{\pi}{2}} \cos(y) \, \big[ \sin(x) \big]_{0}^{\frac{\pi}{2}} \, \mathrm{d}y \\
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= -2\mathrm{e}^{-2vt} \int\limits_{0}^{\frac{\pi}{2}} \cos(y) \, (1 - 0) \, \mathrm{d}y \\
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= -2\mathrm{e}^{-2vt} \int\limits_{0}^{\frac{\pi}{2}} \cos(y) \, \mathrm{d}y \\
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= -2\mathrm{e}^{-2vt} \, \big[ \sin(y) \big]_{0}^{\frac{\pi}{2}} \\
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= -2\mathrm{e}^{-2vt} \, (1 - 0) \\
\int\limits_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, \mathrm{d}\sigma &= -2\mathrm{e}^{-2vt}
\end{aligned}$$

## d

Finner flatenormalen med formelen

$$\vec{n} = \frac{\nabla\beta}{|\nabla\beta|}$$

Der

$$\begin{aligned}
\nabla\beta &= -\frac{\partial\psi}{\partial x}\hat{\mathbf{i}} - \frac{\partial\psi}{\partial y}\hat{\mathbf{j}} + \frac{\partial z}{\partial z}\hat{\mathbf{k}} \\
\nabla\beta &= -\sin(x)\cos(y)\hat{\mathbf{i}} - \sin(y)\cos(x)\hat{\mathbf{j}} + \hat{\mathbf{k}}
\end{aligned}$$

og

$$|\nabla\beta| = \sqrt{\big(\sin(x)\cos(y)\big)^2 + \big(\sin(y)\cos(x)\big)^2 + 1}$$

Da blir flatenormalen

$$\vec{n} = \frac{\nabla\beta}{|\nabla\beta|}$$
$$\vec{n} = \frac{-\sin(x)\cos(y)\hat{\mathbf{i}} - \sin(y)\cos(x)\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{\big(\sin(x)\cos(y)\big)^2 + \big(\sin(y)\cos(x)\big)^2 + 1}}$$

Setter $x(t) = \cos(t)$ og $y(t) = \sin(t)$ da blir

$$z(t) = \cos\big(\cos(t)\big)\cos\big(\sin(t)\big)$$

Deriverer vi $x, y$ med hensyn på $t$ får vi da

$$x'(t) = -\sin(t) \qquad \wedge \qquad y'(t) = \cos(t)$$

og $z$

$$z'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\cos\big(\cos(t)\big)\cos\big(\sin(t)\big)\Big)$$
$$z'(t) = \cos\big(\sin(t)\big)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\cos\big(\cos(t)\big)\Big) + \cos\big(\cos(t)\big)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\cos\big(\sin(t)\big)\Big)$$
$$\boxed{u = \cos(t) \wedge v = \sin(t)}$$
$$z'(t) = \cos\big(\sin(t)\big)\frac{\mathrm{d}}{\mathrm{d}u}\big(\cos(u)\big)\frac{\mathrm{d}}{\mathrm{d}t}\big(\cos(t)\big) + \cos\big(\cos(t)\big)\frac{\mathrm{d}}{\mathrm{d}v}\big(\cos(v)\big)\frac{\mathrm{d}}{\mathrm{d}t}\sin(t)$$
$$z'(t) = \cos\big(\sin(t)\big)\big(-\sin(u)\big)\big(-\sin(t)\big) + \cos\big(\cos(t)\big)\big(-\sin(v)\big)\cos(t)$$
$$z'(t) = \cos\big(\sin(t)\big)\sin\big(\cos(t)\big)\sin(t) - \cos\big(\cos(t)\big)\sin\big(\sin(t)\big)\cos(t)$$

For å finne buelengden så kan man bruke formelen

$$L(a,b) = \int_a^b \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}\,\mathrm{d}t$$

Der $a = 0$ og $b = 2\pi$ så blir integrallet

$$\int_0^{2\pi} \sqrt{\big(-\sin(t)\big)^2 + \big(\cos(t)\big)^2 + \Big(\cos\big(\sin(t)\big)\sin\big(\cos(t)\big)\sin(t) - \cos\big(\cos(t)\big)\sin\big(\sin(t)\big)\cos(t)\Big)^2}\,\mathrm{d}t$$

Bruker Python til å regne ut det integrallet og koden ser slik ut

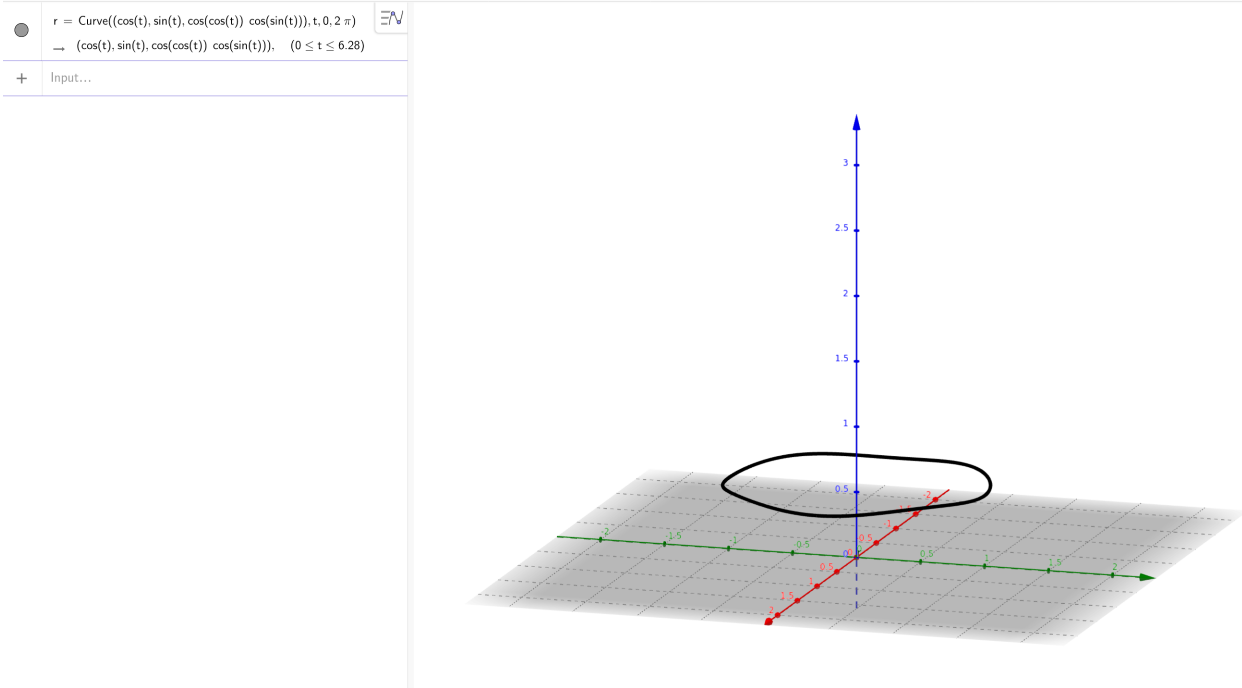
```python
import numpy as np

t = np.linspace(0, 2*np.pi, 1000)

dx = -np.sin(t)
dy = np.cos(t)
dz = np.cos(np.cos(t))*np.sin(np.cos(t))*np.sin(t) - np.cos(np.cos(t))*np.sin(np.sin(↩
    t))*np.cos(t)

L = np.trapz(np.sqrt(dx**2 + dy**2 + dz**2), t)

print(f"{L:f}")
```

Som gir meg at

$$L = 6.284694$$

Som er litt større enn $2\pi$ som gir mening siden kurven ser noe slikt ut i Geogebra

Figur 5: Kurven til 3d




### e

Bruker formelen

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$$

Der $\vec{v} = \vec{u} = \left(\cos(x)\sin(y)\hat{\mathbf{i}} - \sin(x)\cos(y)\hat{\mathbf{j}}\right) \mathrm{e}^{-2vt}$
Finner først lokalakselerasjonen

$$\begin{aligned}
\frac{\partial \vec{u}}{\partial t} &= \frac{\partial}{\partial t}\left(\cos(x)\sin(y)\hat{\mathbf{i}} - \sin(x)\cos(y)\hat{\mathbf{j}}\right) \mathrm{e}^{-2vt} \\
\frac{\partial \vec{u}}{\partial t} &= \left(\cos(x)\sin(y)\hat{\mathbf{i}} - \sin(x)\cos(y)\hat{\mathbf{j}}\right) \frac{\partial}{\partial t}\mathrm{e}^{-2vt} \\
\frac{\partial \vec{u}}{\partial t} &= -2v\left(\cos(x)\sin(y)\hat{\mathbf{i}} - \sin(x)\cos(y)\hat{\mathbf{j}}\right) \mathrm{e}^{-2vt} \\
\frac{\partial \vec{u}}{\partial t} &= -2v\vec{u}
\end{aligned}$$

Finner så den konvektive akselerasjonen

$$\vec{u}\cdot\nabla\vec{u} = (\vec{u}\cdot\nabla u_x)\,\hat{\mathbf{i}} + \left(\vec{u}\cdot\nabla u_y\right)\hat{\mathbf{j}}$$

$$\vec{u}\cdot\nabla\vec{u} = u_x\frac{\partial\vec{u}}{\partial x} + u_y\frac{\partial\vec{u}}{\partial y}$$

$$\vec{u}\cdot\nabla\vec{u} = u_x\frac{\partial}{\partial x}\left(\left(\cos(x)\sin(y)\hat{\mathbf{i}} - \sin(x)\cos(y)\hat{\mathbf{j}}\right)\mathrm{e}^{-2vt}\right) + u_y\frac{\partial}{\partial y}\left(\left(\cos(x)\sin(y)\hat{\mathbf{i}} - \sin(x)\cos(y)\hat{\mathbf{j}}\right)\mathrm{e}^{-2vt}\right)$$

$$\vec{u}\cdot\nabla\vec{u} = \mathrm{e}^{-2vt}\left(u_x\left(-\sin(x)\sin(y)\hat{\mathbf{i}} - \cos(x)\cos(y)\hat{\mathbf{j}}\right) + u_y\left(\cos(x)\cos(y)\hat{\mathbf{i}} + \sin(x)\sin(y)\hat{\mathbf{j}}\right)\right)$$

$$\vec{u}\cdot\nabla\vec{u} = \mathrm{e}^{-2vt}\left(\begin{array}{l}\cos(x)\sin(y)\mathrm{e}^{-2vt}\left(-\sin(x)\sin(y)\hat{\mathbf{i}} - \cos(x)\cos(y)\hat{\mathbf{j}}\right)\\ -\sin(x)\cos(y)\mathrm{e}^{-2vt}\left(\cos(x)\cos(y)\hat{\mathbf{i}} + \sin(x)\sin(y)\hat{\mathbf{j}}\right)\end{array}\right)$$

$$\vec{u}\cdot\nabla\vec{u} = \mathrm{e}^{-4vt}\left(\cos(x)\sin(y)\left(-\sin(x)\sin(y)\hat{\mathbf{i}} - \cos(x)\cos(y)\hat{\mathbf{j}}\right) - \sin(x)\cos(y)\left(\cos(x)\cos(y)\hat{\mathbf{i}} - \sin(x)\sin(y)\hat{\mathbf{j}}\right)\right)$$

$$\vec{u}\cdot\nabla\vec{u} = \mathrm{e}^{-4vt}\left(-\sin^2(y)\cos(x)\sin(x)\hat{\mathbf{i}} - \cos^2(x)\cos(y)\sin(y)\hat{\mathbf{j}} - \cos^2(y)\cos(x)\sin(x)\hat{\mathbf{i}} - \sin^2(x)\cos(y)\sin(y)\hat{\mathbf{j}}\right)$$

$$\vec{u}\cdot\nabla\vec{u} = \mathrm{e}^{-4vt}\left(\begin{array}{l}-\left(\left(\sin^2(y)\cos(x)\sin(x) + \cos^2(y)\cos(x)\sin(x)\right)\hat{\mathbf{i}}\right.\\ \left.+\left(\cos^2(x)\cos(y)\sin(y) + \sin^2(x)\cos(y)\sin(y)\right)\hat{\mathbf{j}}\right)\end{array}\right)$$

$$\vec{u}\cdot\nabla\vec{u} = -\mathrm{e}^{-4vt}\left(\cos(x)\sin(x)\left(\sin^2(y) + \cos^2(y)\right)\hat{\mathbf{i}} + \cos(y)\sin(y)\left(\sin^2(x) + \cos^2(x)\right)\hat{\mathbf{j}}\right)$$

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

$$\vec{u}\cdot\nabla\vec{u} = -\mathrm{e}^{-4vt}\left(\cos(x)\sin(x)\hat{\mathbf{i}} + \cos(y)\sin(y)\hat{\mathbf{j}}\right)$$

$$\boxed{2\cos(x)\sin(x) = \sin(2x)}$$

$$\vec{u}\cdot\nabla\vec{u} = -\mathrm{e}^{-4vt}\left(\frac{1}{2}\sin(2x)\hat{\mathbf{i}} + \frac{1}{2}\sin(2y)\hat{\mathbf{j}}\right)$$

$$\vec{u}\cdot\nabla\vec{u} = -\frac{1}{2}\mathrm{e}^{-4vt}\left(\sin(2x)\hat{\mathbf{i}} + \sin(2y)\hat{\mathbf{j}}\right)$$

Da blir

$$\frac{D\vec{u}}{Dt} = -2v\vec{u} - \frac{1}{2}\left(\sin(2x)\hat{\mathbf{i}} + \sin(2y)\hat{\mathbf{j}}\right)\mathrm{e}^{-4vt}\ \blacksquare$$

Skal så finne trykket $p$

$$\frac{D\vec{u}}{Dt} = -\nabla p + v\nabla^2\vec{u}$$

$$\nabla p = v\nabla^2\vec{u} - \frac{D\vec{u}}{Dt}$$

$$\nabla p = v\nabla^2\vec{u} + 2v\vec{u} + \frac{1}{2}\left(\sin(2x)\hat{\mathbf{i}} + \sin(2y)\hat{\mathbf{j}}\right)\mathrm{e}^{-4vt}$$

Må først da regne ut

$$v \cdot \nabla^2 \vec{u} = v \cdot \left( \frac{\partial^2}{\partial x^2} \vec{u} + \frac{\partial^2}{\partial y^2} \vec{u} \right)$$

$$\boxed{\begin{aligned} \frac{\partial^2}{\partial x^2} \vec{u} &= \frac{\partial^2}{\partial x^2} \left( \cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt} \\ \frac{\partial^2}{\partial x^2} \vec{u} &= \frac{\partial}{\partial x} \left( - \sin(x) \sin(y) \hat{\mathbf{i}} - \cos(x) \cos(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt} \\ \frac{\partial^2}{\partial x^2} \vec{u} &= \left( - \cos(x) \sin(y) \hat{\mathbf{i}} + \sin(x) \cos(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt} \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{\partial^2}{\partial y^2} \vec{u} &= \frac{\partial^2}{\partial y^2} \left( \cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt} \\ \frac{\partial^2}{\partial y^2} \vec{u} &= \frac{\partial}{\partial y} \left( \cos(x) \cos(y) \hat{\mathbf{i}} + \sin(x) \sin(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt} \\ \frac{\partial^2}{\partial y^2} \vec{u} &= \left( - \cos(x) \sin(y) \hat{\mathbf{i}} + \sin(x) \cos(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt} \end{aligned}}$$

$$v \cdot \nabla^2 \vec{u} = v \cdot \left( \left( - \cos(x) \sin(y) \hat{\mathbf{i}} + \sin(x) \cos(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt} \cdot \left( - \cos(x) \sin(y) \hat{\mathbf{i}} + \sin(x) \cos(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt} \right)$$

$$v \cdot \nabla^2 \vec{u} = v \mathrm{e}^{-2vt} \cdot \left( -2 \cos(x) \sin(y) \hat{\mathbf{i}} + 2 \sin(x) \cos(y) \hat{\mathbf{j}} \right)$$

$$v \cdot \nabla^2 \vec{u} = -2v \left( \cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}} \right) \mathrm{e}^{-2vt}$$

$$v \cdot \nabla^2 \vec{u} = -2v \vec{u}$$

Da blir

$$\nabla p = 2v\vec{u} + \frac{1}{2} \left( \sin(2x) \hat{\mathbf{i}} + \sin(2y) \hat{\mathbf{j}} \right) \mathrm{e}^{-4vt} - 2v\vec{u}$$

Ser så på integrallet med hensyn på $x$

$$p(x, y, t) = \int \left( 2v\vec{u} + \frac{1}{2} \left( \sin(2x) + \sin(2y) \right) \mathrm{e}^{-4vt} - 2v\vec{u} \right) \mathrm{d}x$$

Tar integrallene hver for seg

$$\int 2v\vec{u} \, \mathrm{d}x = 2v \int \left( \cos(x) \sin(y) - \sin(x) \cos(y) \right) \mathrm{e}^{-2vt} \, \mathrm{d}x$$

$$\int 2v\vec{u} \, \mathrm{d}x = 2v \mathrm{e}^{-2vt} \left( - \sin(x) \sin(y) - \cos(x) \cos(y) \right)$$

Ser så på

$$\int \frac{1}{2} \left( \sin(2x) + \sin(2y) \right) \mathrm{e}^{-4vt} \, \mathrm{d}x = \frac{1}{2} \mathrm{e}^{-4vt} \int \left( \sin(2x) + \sin(2y) \right) \mathrm{d}x$$

$$\int \frac{1}{2} \left( \sin(2x) + \sin(2y) \right) \mathrm{e}^{-4vt} \, \mathrm{d}x = \frac{1}{2} \mathrm{e}^{-4vt} \cdot \left( - \frac{1}{2} \cos(2x) \right)$$

$$\int \frac{1}{2} \left( \sin(2x) + \sin(2y) \right) \mathrm{e}^{-4vt} \, \mathrm{d}x = - \frac{1}{4} \mathrm{e}^{-4vt} \cos(2x)$$

Observerer at det første og siste integrallet er likt bare med motsatt fortegn

$$\int -2v\vec{u}\,\mathrm{d}x = -2v\mathrm{e}^{-2vt}\big(-\sin(x)\sin(y) - \cos(x)\cos(y)\big)$$

Da blir

$$\begin{aligned} p(x,y,t) &= \begin{aligned} &2v\mathrm{e}^{-2vt}\big(-\sin(x)\sin(y) - \cos(x)\cos(y)\big) - \frac{1}{4}\mathrm{e}^{-4vt}\cos(2x) \\ &- 2v\mathrm{e}^{-2vt}\big(-\sin(x)\sin(y) - \cos(x)\cos(y)\big) + f(y) \end{aligned} \\ p(x,y,t) &= -\frac{1}{4}\mathrm{e}^{-4vt}\cos(2x) + f(y) \end{aligned}$$

Ser så med hensyn på $y$

$$p(x,y,t) = \int \left(2v\vec{u} + \frac{1}{2}\big(\sin(2x) + \sin(2y)\big)\mathrm{e}^{-4vt} - 2v\vec{u}\right)\mathrm{d}y$$

Tar integrallene hver for seg

$$\begin{aligned} \int 2v\vec{u}\,\mathrm{d}y &= 2v\int \big(\cos(x)\sin(y) - \sin(x)\cos(y)\big)\mathrm{e}^{-2vt}\,\mathrm{d}y \\ \int 2v\vec{u}\,\mathrm{d}y &= 2v\mathrm{e}^{-2vt}\big(\cos(x)\cos(y) + \sin(x)\sin(y)\big) \end{aligned}$$

Ser så på

$$\begin{aligned} \int \frac{1}{2}\big(\sin(2x) + \sin(2y)\big)\mathrm{e}^{-4vt}\,\mathrm{d}y &= \frac{1}{2}\mathrm{e}^{-4vt}\int \big(\sin(2x) + \sin(2y)\big)\,\mathrm{d}y \\ \int \frac{1}{2}\big(\sin(2x) + \sin(2y)\big)\mathrm{e}^{-4vt}\,\mathrm{d}y &= \frac{1}{2}\mathrm{e}^{-4vt}\cdot\left(-\frac{1}{2}\cos(2y)\right) \\ \int \frac{1}{2}\big(\sin(2x) + \sin(2y)\big)\mathrm{e}^{-4vt}\,\mathrm{d}y &= -\frac{1}{4}\mathrm{e}^{-4vt}\cos(2y) \end{aligned}$$

Observerer at det første og siste integrallet er likt bare med motsatt fortegn

$$\int -2v\vec{u}\,\mathrm{d}y = -2v\mathrm{e}^{-2vt}\big(\cos(x)\cos(y) + \sin(x)\sin(y)\big)$$

Da blir

$$\begin{aligned} p(x,y,t) &= \begin{aligned} &2v\mathrm{e}^{-2vt}\big(-\sin(x)\sin(y) - \cos(x)\cos(y)\big) - \frac{1}{4}\mathrm{e}^{-4vt}\cos(2y) \\ &- 2v\mathrm{e}^{-2vt}\big(-\sin(x)\sin(y) - \cos(x)\cos(y)\big) + g(x) \end{aligned} \\ p(x,y,t) &= -\frac{1}{4}\mathrm{e}^{-4vt}\cos(2y) + g(x) \end{aligned}$$

Sammenligner jeg disse 2 $p$-ene så ser jeg at

$$g(x) = -\frac{1}{4}\mathrm{e}^{-4vt}\cos(2x) \wedge f(y) = -\frac{1}{4}\mathrm{e}^{-4vt}\cos(2y)$$

Følgelig blir

$$p(x,y,t) = -\frac{1}{4}\mathrm{e}^{-4vt}\cos(2x) - \frac{1}{4}\mathrm{e}^{-4vt}\cos(2y) = -\frac{1}{4}\mathrm{e}^{-2vt}\big(\cos(2x) + \cos(2y)\big) + p_0$$