

1

- i $\vec{u} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
Ser først på divergensen

$$\begin{aligned}\nabla \cdot \vec{u} &= \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z \\ \nabla \cdot \vec{u} &= 1 + 1 + 1 \\ \nabla \cdot \vec{u} &= 3\end{aligned}$$

Ser så på virvlingen

$$\begin{aligned}\nabla \times \vec{u} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ \nabla \times \vec{u} &= \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y \right) \hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z \right) \hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x \right) \hat{\mathbf{k}} \\ \nabla \times \vec{u} &= \vec{0}\end{aligned}$$

- ii $\vec{u} = r \cos(\theta)\hat{\mathbf{i}}_r + r \sin(\theta)\hat{\mathbf{i}}_\theta + z\hat{\mathbf{k}}$ Ser først på divergensen

$$\begin{aligned}\nabla \cdot \vec{u} &= \frac{1}{r} \left(\frac{\partial}{\partial r} (r (r \cos(\theta))) + \frac{\partial}{\partial \theta} (r \sin(\theta)) \right) + \frac{\partial}{\partial z}z \\ \nabla \cdot \vec{u} &= \frac{1}{r} \left(\cos(\theta) \frac{\partial}{\partial r} r^2 + r \frac{\partial}{\partial \theta} \sin(\theta) \right) + 1 \\ \nabla \cdot \vec{u} &= \frac{1}{r} (2r \cos(\theta) + r \cos(\theta)) + 1 \\ \nabla \cdot \vec{u} &= 2 \cos(\theta) + \cos(\theta) + 1 \\ \nabla \cdot \vec{u} &= 3 \cos(\theta) + 1\end{aligned}$$

Ser så på virvlingen

$$\begin{aligned}\nabla \times \vec{u} &= 0\hat{\mathbf{i}}_r + 0\hat{\mathbf{i}}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r (r \sin(\theta))) - \frac{\partial}{\partial \theta} (r \cos(\theta)) \right) \hat{\mathbf{k}} \\ \nabla \times \vec{u} &= \frac{1}{r} \left(\sin(\theta) \frac{\partial}{\partial r} (r^2) - r \frac{\partial}{\partial \theta} (\cos(\theta)) \right) \hat{\mathbf{k}} \\ \nabla \times \vec{u} &= \frac{1}{r} (2r \sin(\theta) + r \sin(\theta)) \hat{\mathbf{k}} \\ \nabla \times \vec{u} &= (2 \sin(\theta) + \sin(\theta)) \hat{\mathbf{k}} \\ \nabla \times \vec{u} &= 3 \sin(\theta) \hat{\mathbf{k}}\end{aligned}$$

iii $\vec{u} = \hat{\mathbf{i}}_r + \hat{\mathbf{i}}$

Konverterer først over til rene cylinder koordinater

$$\boxed{\hat{\mathbf{i}} = \cos(\theta)\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta}$$

$$\vec{u} = \hat{\mathbf{i}}_r + \cos(\theta)\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta$$

$$\vec{u} = (1 + \cos(\theta))\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta$$

Ser først på divergensen

$$\nabla \cdot \vec{u} = \frac{1}{r} \left(\frac{\partial}{\partial r} (r(1 + \cos(\theta))) \frac{\partial}{\partial \theta} (-\sin(\theta)) \right)$$

$$\nabla \cdot \vec{u} = \frac{1}{r} (1 + \cos(\theta) - \cos(\theta))$$

$$\nabla \cdot \vec{u} = \frac{1}{r}$$

Ser så på virvlingen

$$\nabla \times \vec{u} = 0\hat{\mathbf{i}}_r + 0\hat{\mathbf{i}}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r(-\sin(\theta))) - \frac{\partial}{\partial \theta} (1 + \cos(\theta)) \right) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = \frac{1}{r} (-\sin(\theta) + \sin(\theta)) \hat{\mathbf{k}}$$

$$\nabla \times \vec{u} = \vec{0}$$

2

a

For å finne enhetsvektorene så må man først finne skaleringsfaktorene. Starter med skaleringsfaktoren til u

$$\begin{aligned}
 h_u &= \left| \frac{\partial \vec{r}}{\partial u} \right| \\
 h_u &= \left| a \cos(v) \frac{\partial}{\partial u} \cosh(u) \hat{\mathbf{i}} + a \sin(v) \frac{\partial}{\partial v} \sinh(u) \hat{\mathbf{j}} \right| \\
 \boxed{a = 1} \\
 h_u &= \left| \cos(v) \sinh(u) \hat{\mathbf{i}} + \sin(v) \cosh(u) \hat{\mathbf{j}} \right| \\
 h_u &= \sqrt{(\cos(v) \sinh(u))^2 + (\sin(v) \cosh(u))^2} \\
 h_u &= \sqrt{\cos^2(v) \sinh^2(u) + \sin^2(v) \cosh^2(u)} \\
 \boxed{\cosh^2(u) = 1 + \sinh^2(u)} \\
 h_u &= \sqrt{\cos^2(v) \sinh^2(u) + \sin^2(v) (1 + \sinh^2(u))} \\
 h_u &= \sqrt{\cos^2(v) \sinh^2(u) + \sinh^2(u) \sin^2(v) + \sin^2(v)} \\
 h_u &= \sqrt{\sinh^2(u) (\cos^2(v) + \sin^2(v)) + \sin^2(v)} \\
 h_u &= \sqrt{\sinh^2(u) + \sin^2(v)}
 \end{aligned}$$

Finner så skaleringsfaktoren til v

$$\begin{aligned}
 h_v &= \left| \frac{\partial \vec{r}}{\partial v} \right| \\
 h_v &= \left| \cosh(u) \frac{\partial}{\partial v} \cos(v) \hat{\mathbf{i}} + \sinh(u) \frac{\partial}{\partial v} \sin(v) \hat{\mathbf{j}} \right| \\
 h_v &= \left| -\cosh(u) \sin(v) \hat{\mathbf{i}} + \sinh(u) \cos(v) \hat{\mathbf{j}} \right| \\
 h_v &= \sqrt{(-\cosh(u) \sin(v))^2 + (\sinh(u) \cos(v))^2} \\
 h_v &= \sqrt{\cosh^2(u) \sin^2(v) + \sinh^2(u) \cos^2(v)} \\
 h_v &= \sqrt{(1 + \sinh^2(u)) \sin^2(v) + \sinh^2(u) \cos^2(v)} \\
 h_v &= \sqrt{\sin^2(v) + \sinh^2(u) \sin^2(v) + \sinh^2(u) \cos^2(v)} \\
 h_v &= \sqrt{\sin^2(v) + \sinh^2(u) (\sin^2(v) + \cos^2(v))} \\
 h_v &= \sqrt{\sinh^2(u) + \sin^2(v)}
 \end{aligned}$$

Ser da at $h_u = h_v$ Enhetsvektorer er da gitt ved

$$\mathbf{e}_u = \frac{1}{h_u} \frac{\partial \vec{r}}{\partial u}$$

$$\mathbf{e}_u = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}} \left(\cos(v) \sinh(u) \hat{\mathbf{i}} + \sin(v) \cosh(u) \hat{\mathbf{j}} \right)$$

Og

$$\mathbf{e}_v = \frac{1}{h_v} \frac{\partial \vec{r}}{\partial v}$$

$$\mathbf{e}_v = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}} \left(-\cosh(u) \sin(v) \hat{\mathbf{i}} + \sinh(u) \cos(v) \hat{\mathbf{j}} \right)$$

Hvis de er ortogonale så må $\mathbf{e}_u \cdot \mathbf{e}_v = 0$

$$\mathbf{e}_u \cdot \mathbf{e}_v = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}} \left(\frac{\cos(v) \sinh(u)}{\sin(v) \cosh(u)} \right) \cdot \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}} \left(\frac{-\cosh(u) \sin(v)}{\sinh(u) \cos(v)} \right)$$

$$\mathbf{e}_u \cdot \mathbf{e}_v = \frac{1}{\sinh^2(u) \sin^2(v)} \left(-\cos(v) \sinh(u) \cosh(u) \sin(v) + \sin(v) \cosh(u) \sinh(u) \cos(v) \right)$$

$$\mathbf{e}_u \cdot \mathbf{e}_v = \left(-\sin(v) \cos(v) \sinh(u) \cosh(u) + \sin(v) \cos(v) \sinh(u) \cosh(u) \right)$$

$$\mathbf{e}_u \cdot \mathbf{e}_v = 0$$

Med dette så observeres det at enhetsvektorene er ortogonale

b

Siden ingen f har blitt oppgitt så tar jeg utgangspunkt i en helt generell f . Gradienten blir da

$$\nabla f = \frac{1}{h_u} \frac{\partial f}{\partial u} \mathbf{e}_u + \frac{1}{h_v} \frac{\partial f}{\partial v} \mathbf{e}_v$$

$$\boxed{h_u = h_v = h}$$

$$\nabla f = \frac{1}{h} \left(\frac{\partial f}{\partial u} \mathbf{e}_u + \frac{\partial f}{\partial v} \mathbf{e}_v \right)$$

Veit ikke hva w_u og w_v inneholder så blir generell divergens

$$\nabla \cdot \vec{w} = \frac{1}{h_u h_v} \left(\frac{\partial}{\partial u} (w_u \mathbf{e}_u h_v) + \frac{\partial}{\partial v} (w_v \mathbf{e}_v h_u) \right)$$

Laplace operatoren blir

$$\nabla^2 = \frac{1}{h_u h_v} \left[\frac{\partial}{\partial u} \left(\frac{h_u}{h_v} \frac{\partial}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_v}{h_u} \frac{\partial}{\partial v} \right) \right]$$

$$\boxed{h_v = h_u = h}$$

$$\nabla^2 = \frac{1}{h^2} \left[\frac{\partial}{\partial u} \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \frac{\partial}{\partial v} \right]$$

$$\nabla^2 = \frac{1}{h^2} \left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right]$$

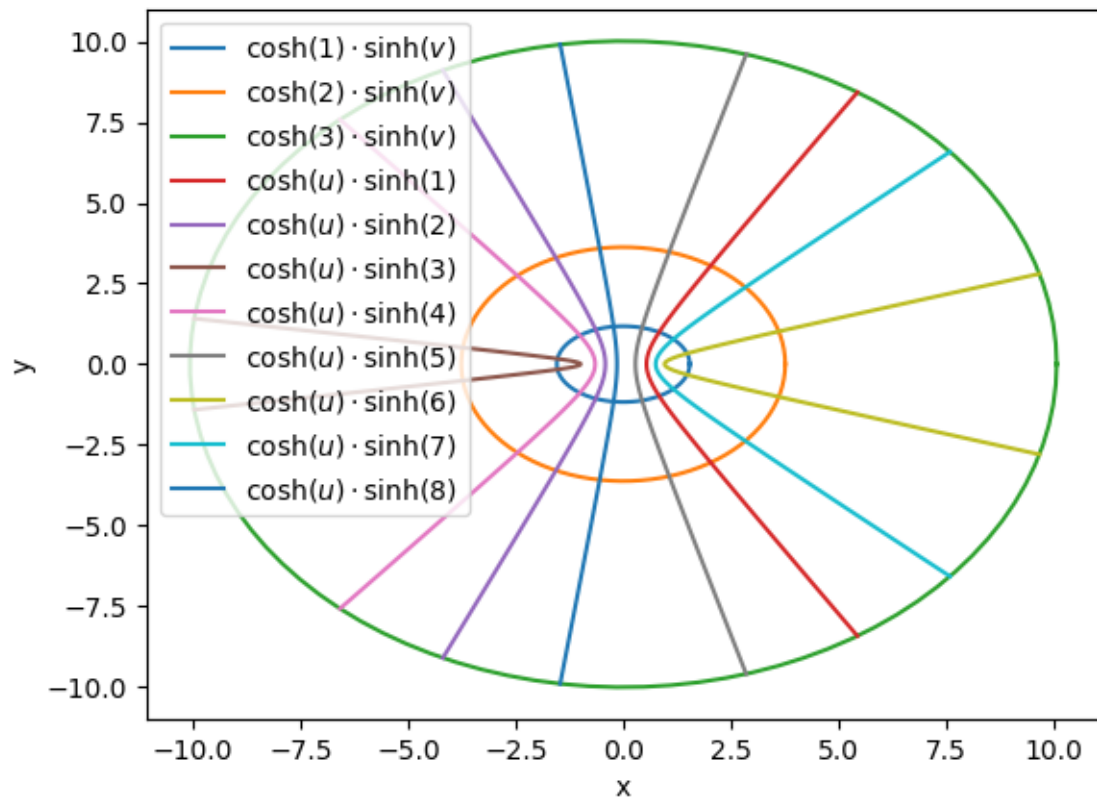
c

Python kode til skissen er

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 N = 100
5 u = np.linspace(-3, 3, N)
6 v = np.linspace(0, 2*np.pi, N)
7
8 for i in range(1, 4):
9     x = np.cosh(i)*np.cos(v)
10    y = np.sinh(i)*np.sin(v)
11    plt.plot(x, y, label=f"$\cosh({i}) \cdot \sinh(v)$")
12
13 for i in range(1, 9):
14     x = np.cosh(u)*np.cos(i)
15     y = np.sinh(u)*np.sin(i)
16     plt.plot(x, y, label=f"$\cosh(u) \cdot \sinh({i})$")
17
18 plt.xlabel("x")
19 plt.ylabel("y")
20 plt.legend()
21 plt.savefig("2c.png")
22 plt.show()
```

Som produserer skissen

Figur 1: Skissen til 2c



d

Python kode til dette er

```

1 import numpy as np
2 import sympy as sp
3 import matplotlib.pyplot as plt
4
5 u, v = psi = sp.symbols("u, v", real=True)
6 r = (sp.cosh(u)*sp.cos(v), sp.sinh(u)*sp.sin(v))
7
8 def basisvektor(psi, r):
9     b = np.zeros((len(psi), len(r)), dtype=object)
10    for i, ui in enumerate(psi):
11        for j, rj in enumerate(r):
12            b[i, j] = sp.simplify(rj.diff(ui, 1))
13    return b
14
15 def skaleringsfaktor(b):
16     h = np.zeros(b.shape[0], dtype=object)
17     for i, s in enumerate(np.sum(b**2, axis=1)):
18         h[i] = sp.simplify(sp.sqrt(s))
19     return h

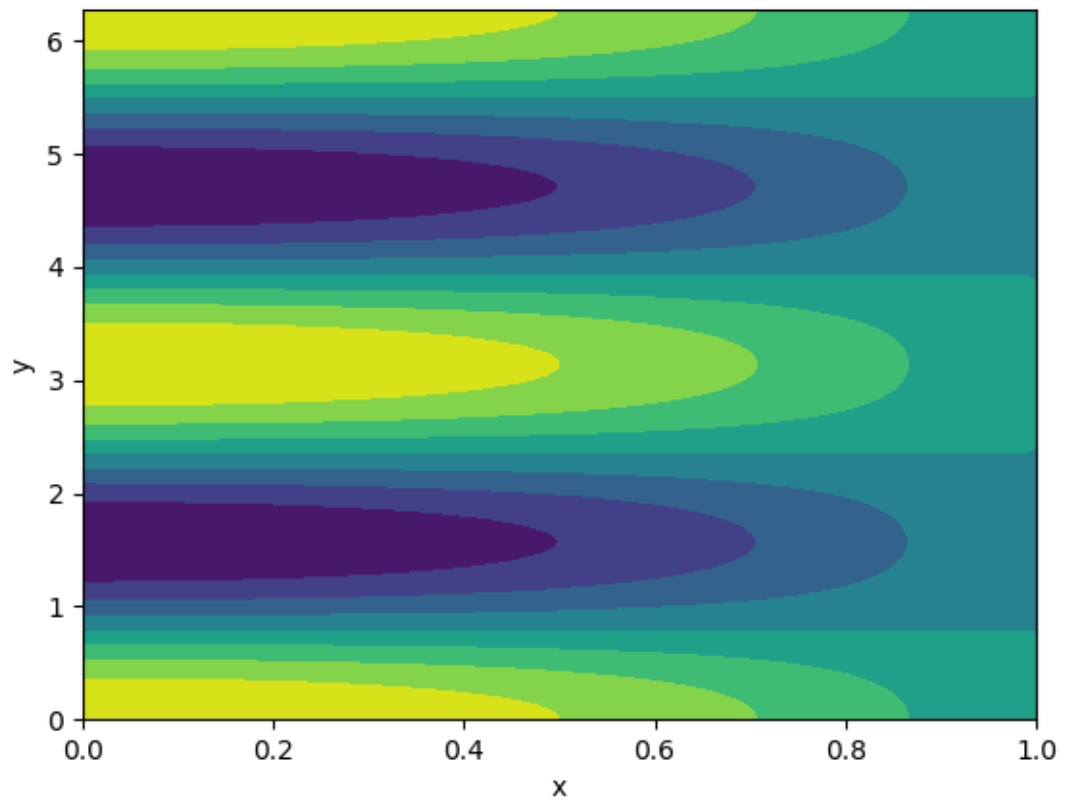
```

```

20
21 def enhetsvektor(psi, r):
22     b = basisvektor(psi, r)
23     hi = skaleringsfaktor(b)
24     return b/ hi[None, :], hi
25
26 e, h = enhetsvektor(psi, r)
27
28 f = (1 - u**2)*sp.cos(2*v)
29
30 N = 100
31 ui = np.broadcast_to(np.linspace(0, 1, N)[: , None], (N, N))
32 vi = np.broadcast_to(np.linspace(0, 2*np.pi, N)[None, :], (N, N))
33 fj = sp.lambdify((u, v), f)(ui, vi)
34
35 mesh = []
36 for rj in r:
37     mesh.append(sp.lambdify((u, v), rj)(ui, vi))
38 x, y = mesh
39
40 plt.contourf(x, y, fj)
41
42 df = np.array((1/h[0]*f.diff(u, 1), 1/h[1]*f.diff(v, 1)))
43
44 gradf = e[0]*df[0] + e[1]*df[1]
45
46 dfdxi = sp.lambdify((u, v), gradf[0])(ui, vi)
47 dfdyi = sp.lambdify((u, v), gradf[1])(ui, vi)
48 plt.contourf(x, y, fj)
49 plt.quiver(x[:,50], y[:,50], dfdxi[:,50], dfdyi[:,50], scale=15, pivot="middle")
50
51 plt.xlabel("x")
52 plt.ylabel("y")
53
54 plt.savefig("2d.png")
55 plt.show()
56 plt.close()
57
58 plt.contourf(ui, vi, fj)
59 plt.xlabel("x")
60 plt.ylabel("y")
61 plt.savefig("2d-elliptical.png")
62 plt.show()

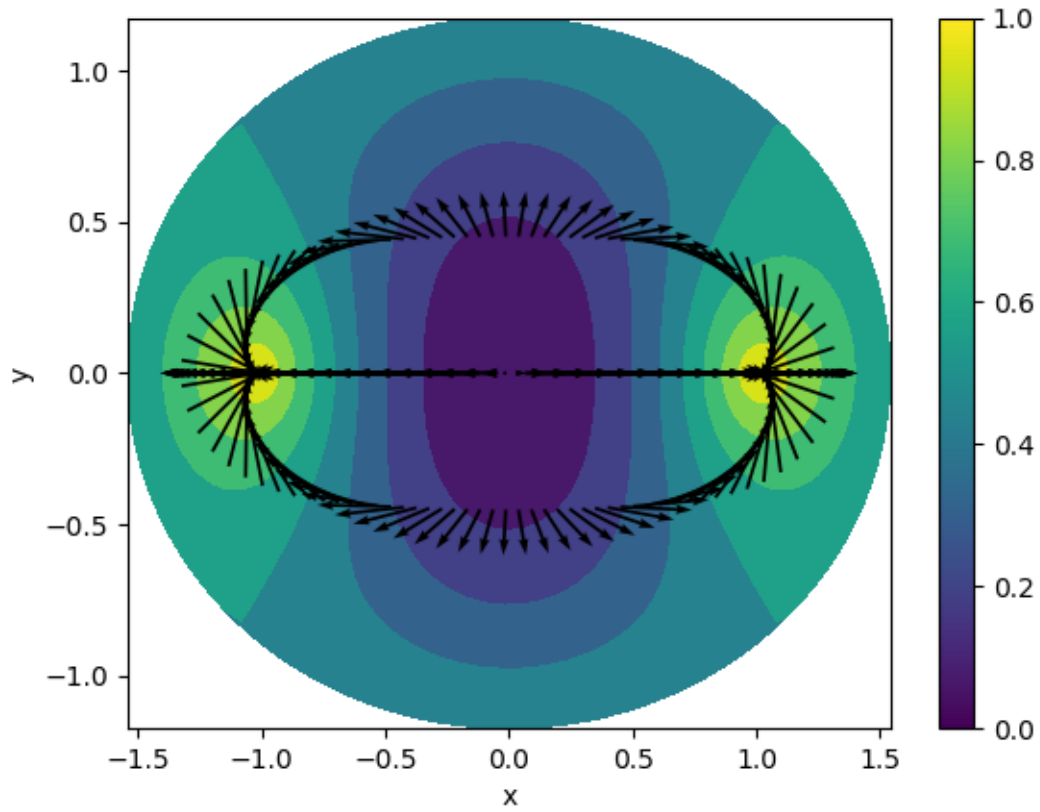
```

Figur 2: Konturplott i elliptiske koordinater oppgave 2d



Observerer da at vektorene peker ut i fra det lilla og blir slukt ned i det gule

Figur 3: Pilplott til oppgave 2d



3

a

Regner ut $\nabla \cdot \vec{u}$

$$\nabla \cdot \vec{u} = \frac{\partial}{\partial x} (\cos(x) \sin(y)) + \frac{\partial}{\partial y} (-\sin(x) \cos(y))$$

$$\nabla \cdot \vec{u} = -\sin(x) \sin(y) + \sin(x) \sin(y)$$

$$\nabla \cdot \vec{u} = 0$$

Divergensen lik 0 derfor finnes det en Strømfunksjon

Strømfunksjon

$$u_x = \frac{\partial \psi}{\partial y} \wedge u_y = \frac{\partial \psi}{\partial x}$$

$$\begin{array}{lll}
\frac{\partial \psi}{\partial y} = -u_x & \wedge & \frac{\partial \psi}{\partial x} = u_y \\
\psi(x, y) = - \int \cos(x) \sin(y) e^{-2vt} dy & \wedge & \psi(x, y) = \int -\sin(x) \cos(y) e^{-2vt} dx \\
\psi(x, y) = -\cos(x) \cos(y) e^{-2vt} & \wedge & \psi(x, y) = -\cos(x) \cos(y) e^{-2vt}
\end{array}$$

Ser så om den har ett skalarpotensial. Hvis det har det så må $\nabla \times \vec{u} = \vec{0}$

$$\begin{aligned}
\nabla \times \vec{u} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(x) \sin(y) e^{-2vt} & -\sin(x) \cos(y) e^{-2vt} & 0 \end{vmatrix} \\
\nabla \times \vec{u} &= e^{-2vt} \left(\frac{\partial}{\partial x} (-\sin(x) \cos(y)) - \frac{\partial}{\partial y} (\cos(x) \sin(y)) \right) \hat{\mathbf{k}} \\
\nabla \times \vec{u} &= e^{-2vt} (-\cos(x) \cos(y) - \cos(x) \cos(y)) \hat{\mathbf{k}} \\
\nabla \times \vec{u} &= -2e^{-2vt} \cos(x) \cos(y) \hat{\mathbf{k}}
\end{aligned}$$

Dette er ikke lik 0 så da finnes det ikke noe skalarpotensial.

b

Python kode til pilplott med strømlinjer

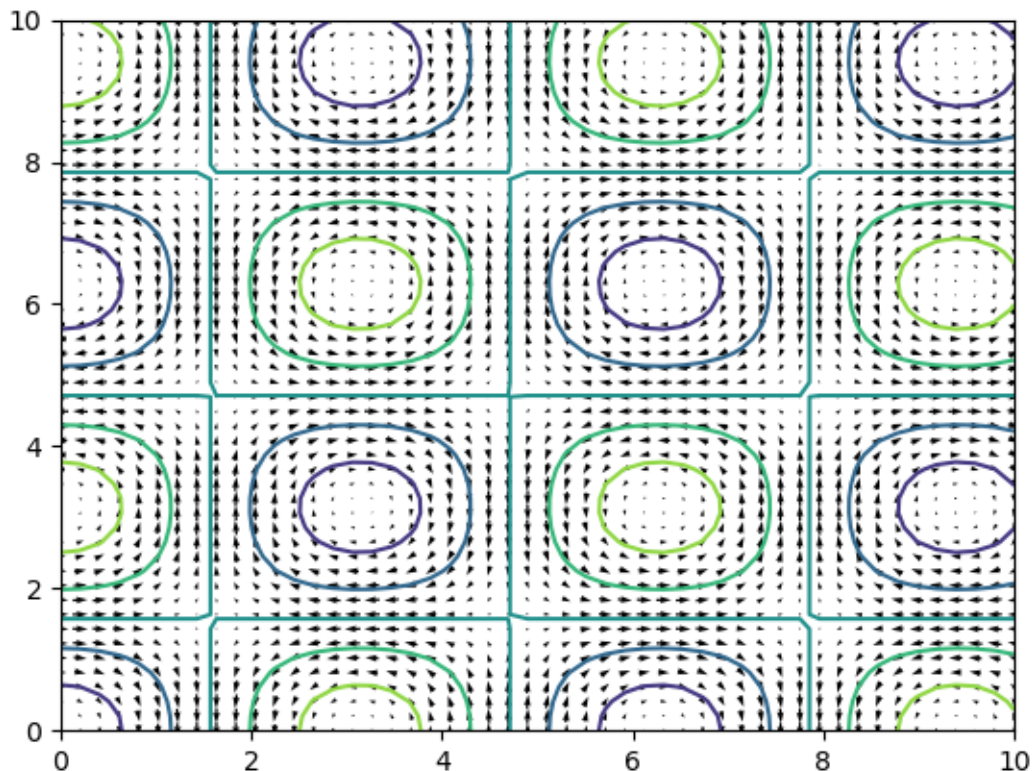
```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 t = np.linspace(0, 10, 50)
5 x, y = np.meshgrid(t, t, indexing="ij")
6
7 plt.contour(x, y, -np.cos(x)*np.cos(y), 4)
8 plt.quiver(x, y, np.cos(x)*np.sin(y), -np.sin(x)*np.cos(y), pivot="middle")
9
10 plt.savefig("3b.png")
11 plt.show()

```

Som produserer følgende graf

Figur 4: Pilplott med strømlinjer til 3b



c

Hvis $\vec{F}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$ er et vektorfelt i planet er

$$\nabla \cdot \vec{F}(x, y) = \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y)$$

Dersom C er en enkel, parametrisert kurve i planet som omslutter et område R . Så må det finnes en to-dimensjonal versjon av Gauss' sats som sier

$$\iint_R \nabla \cdot \vec{F} \, dx \, dy = \int_C \vec{F} \cdot \vec{n} \, ds$$

der \vec{n} er enhetsnormalvektoren til C som peker ut av området R . Skal nå vise at dette bare er en omformulering av Greens teorem. Starter med å velge en positiv orientert parametrisering $\vec{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$, $t \in [a, b]$, som gjennomløper C med fart konstant lik 1. Da er $\vec{r}'(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}}$ en tangentvektor med lengde 1. Det betyr at $\vec{n}(t) = y'(t)\hat{\mathbf{i}} - x'(t)\hat{\mathbf{j}}$ er en enhetsnormalvektor som

peker ut av området R .

$$\int_C \vec{F} \cdot \vec{n} \, ds = \int_a^b \left(P(\vec{r}(t)) y'(t) + Q(\vec{r}(t)) (-x'(t)) \right) dt = \int_C -Q \, dx + P \, dy$$

Bruker så Greens teorem

$$\int_C -Q \, dx + P \, dy = \iint_R \left(\frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y} \right) dx \, dy = \iint_R \nabla \cdot \vec{F} \, dx \, dy$$

Dermed er

$$\iint_R \nabla \cdot \vec{F} \, dx \, dy = \int_C \vec{F} \cdot \vec{n} \, ds$$

Har med det vist at den to-dimensjonale versjonen av Gauss' sats følger fra Green
Regner så ut fluksen med Gauss' sats

$$\int_C \vec{u} \cdot \vec{n} \, ds = \iint_R \nabla \cdot \vec{u} \, dx \, dy$$

Fra **a** veit man at divergensen er 0. Følgelig er fluksen da 0
Sirkulasjonen blir da

$$\oint \vec{u} \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \vec{u} \cdot d\vec{r}_{x,0} + \int_0^{\frac{\pi}{2}} \vec{u} \cdot d\vec{r}_{y,0} + \int_{\frac{\pi}{2}}^0 \vec{u} \cdot d\vec{r}_{x,1} + \int_{\frac{\pi}{2}}^0 \vec{u} \cdot d\vec{r}_{y,1}$$

$\vec{r}_{x,0}(t') = t' \hat{\mathbf{i}} + 0 \hat{\mathbf{j}} \quad \wedge \quad \vec{r}_{y,0}(t') = \frac{\pi}{2} \hat{\mathbf{i}} + t' \hat{\mathbf{j}} \quad \wedge \quad \vec{r}_{x,1}(t') = t' \hat{\mathbf{i}} + \frac{\pi}{2} \hat{\mathbf{j}} \quad \wedge \quad \vec{r}_{y,1}(t') = 0 \hat{\mathbf{i}} + t' \hat{\mathbf{j}}$
$\vec{r}'_{x,0}(t') = \hat{\mathbf{i}} \quad \wedge \quad \vec{r}'_{y,0}(t') = \hat{\mathbf{j}} \quad \wedge \quad \vec{r}'_{x,1}(t') = \hat{\mathbf{i}} \quad \wedge \quad \vec{r}'_{y,1}(t') = \hat{\mathbf{j}}$

$$\begin{aligned} \oint \vec{u} \cdot d\vec{r} = & \int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') \, dt' + \int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') \, dt' \\ & + \int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1}(t') \, dt' + \int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1}(t') \, dt' \end{aligned}$$

For å få det litt mer oversiktlig så regner jeg ett integral av gangen. Starter med

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') dt' &= \int_0^{\frac{\pi}{2}} (\cos(t') \sin(0) e^{-2vt}) \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') dt' &= \int_0^{\frac{\pi}{2}} (\cos(t') \sin(0) e^{-2vt}) dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') dt' &= \int_0^{\frac{\pi}{2}} 0 dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{x,0}(t')) \cdot \vec{r}'_{x,0}(t') dt' &= 0
\end{aligned}$$

Ser så på

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= \int_0^{\frac{\pi}{2}} \left(-\sin\left(\frac{\pi}{2}\right) \cos(t') e^{-2vt} \right) \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= \int_0^{\frac{\pi}{2}} \left(-\sin\left(\frac{\pi}{2}\right) \cos(t') e^{-2vt} \right) dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= -e^{-2vt} \int_0^{\frac{\pi}{2}} \cos(t') dt' \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= -e^{-2vt} \left[\sin(t') \right]_0^{\frac{\pi}{2}} \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= -e^{-2vt} (1 - 0) \\
\int_0^{\frac{\pi}{2}} \vec{u}(\vec{r}_{y,0}(t')) \cdot \vec{r}'_{y,0}(t') dt' &= -e^{-2vt}
\end{aligned}$$

Ser så på

$$\begin{aligned}
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1} dt' &= \int_{\frac{\pi}{2}}^0 \left(\cos(t') \sin\left(\frac{\pi}{2}\right) \right) \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} dt' \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1} dt' &= e^{-2vt} \int_{\frac{\pi}{2}}^0 \cos(t') dt' \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1} dt' &= e^{-2vt} \left[\sin(t') \right]_{\frac{\pi}{2}}^0 \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1} dt' &= e^{-2vt} (0 - 1) \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{x,1}(t')) \cdot \vec{r}'_{x,1} dt' &= -e^{-2vt}
\end{aligned}$$

Ser så på

$$\begin{aligned}
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1} dt' &= \int_{\frac{\pi}{2}}^0 \left(-\sin(0) \cos(t') e^{-2vt} \right) \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} dt' \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1} dt' &= \int_{\frac{\pi}{2}}^0 \left(-\sin(0) \cos(t') e^{-2vt} \right) dt' \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1} dt' &= \int_{\frac{\pi}{2}}^0 0 dt' \\
\int_{\frac{\pi}{2}}^0 \vec{u}(\vec{r}_{y,1}(t')) \cdot \vec{r}'_{y,1} dt' &= 0
\end{aligned}$$

Kan så gå tilbake til sirkulasjonen

$$\oint \vec{u} \cdot d\vec{r} = 0 - e^{-2vt} - e^{-2vt} + 0 = -2e^{-2vt}$$

Kan også bruke Stokes sats som sier

$$\oint_C \vec{u} \cdot d\vec{r} = \int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} d\sigma$$

Fra **a** veit man at

$$\nabla \times \vec{u} = -2e^{-2vt} \cos(x) \cos(y) \hat{\mathbf{k}}$$

Og $\vec{n} = \hat{\mathbf{k}}$ siden den har positiv orientering om z . Da blir Stokes

$$\begin{aligned}
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= \int_{\sigma} \left(-2e^{-2vt} \cos(x) \cos(y) \right) \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \, d\sigma \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_{\sigma} (\cos(x) \cos(y)) \, d\sigma \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(x) \cos(y) \, dx \, dy \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_0^{\frac{\pi}{2}} \cos(y) [\sin(x)]_0^{\frac{\pi}{2}} \, dy \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_0^{\frac{\pi}{2}} \cos(y) (1 - 0) \, dy \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} \int_0^{\frac{\pi}{2}} \cos(y) \, dy \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} [\sin(y)]_0^{\frac{\pi}{2}} \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt} (1 - 0) \\
\int_{\sigma} (\nabla \times \vec{u}) \cdot \vec{n} \, d\sigma &= -2e^{-2vt}
\end{aligned}$$

d

Finner flatenormalen med formelen

$$\vec{n} = \frac{\nabla \beta}{|\nabla \beta|}$$

Der

$$\begin{aligned}
\nabla \beta &= -\frac{\partial \psi}{\partial x} \hat{\mathbf{i}} - \frac{\partial \psi}{\partial y} \hat{\mathbf{j}} + \frac{\partial z}{\partial z} \hat{\mathbf{k}} \\
\nabla \beta &= -\sin(x) \cos(y) \hat{\mathbf{i}} - \sin(y) \cos(x) \hat{\mathbf{j}} + \hat{\mathbf{k}}
\end{aligned}$$

og

$$|\nabla \beta| = \sqrt{(\sin(x) \cos(y))^2 + (\sin(y) \cos(x))^2 + 1}$$

Da blir flatenormalen

$$\vec{n} = \frac{\nabla\beta}{|\nabla\beta|}$$

$$\vec{n} = \frac{-\sin(x)\cos(y)\hat{\mathbf{i}} - \sin(y)\cos(x)\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{(\sin(x)\cos(y))^2 + (\sin(y)\cos(x))^2 + 1}}$$

Setter $x(t) = \cos(t)$ og $y(t) = \sin(t)$ da blir

$$z(t) = \cos(\cos(t))\cos(\sin(t))$$

Deriverer vi x, y med hensyn på t får vi da

$$x'(t) = -\sin(t) \quad \wedge \quad y'(t) = \cos(t)$$

og z

$$z'(t) = \frac{d}{dt} (\cos(\cos(t))\cos(\sin(t)))$$

$$z'(t) = \cos(\sin(t)) \frac{d}{dt} (\cos(\cos(t))) + \cos(\cos(t)) \frac{d}{dt} (\cos(\sin(t)))$$

$u = \cos(t) \wedge v = \sin(t)$

$$z'(t) = \cos(\sin(t)) \frac{d}{du} (\cos(u)) \frac{d}{dt} (\cos(t)) + \cos(\cos(t)) \frac{d}{dv} (\cos(v)) \frac{d}{dt} \sin(t)$$

$$z'(t) = \cos(\sin(t)) (-\sin(u)) (-\sin(t)) + \cos(\cos(t)) (-\sin(v)) \cos(t)$$

$$z'(t) = \cos(\sin(t)) \sin(\cos(t)) \sin(t) - \cos(\cos(t)) \sin(\sin(t)) \cos(t)$$

For å finne buelengden så kan man bruke formelen

$$L(a, b) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

Der $a = 0$ og $b = 2\pi$ så blir integrallet

$$\int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (\cos(\sin(t)) \sin(\cos(t)) \sin(t) - \cos(\cos(t)) \sin(\sin(t)) \cos(t))^2} dt$$

Bruker Python til å regne ut det integrallet og koden ser slik ut

```

1 import numpy as np
2
3 t = np.linspace(0, 2*np.pi, 1000)
4
5 dx = -np.sin(t)
6 dy = np.cos(t)
7 dz = np.cos(np.cos(t))*np.sin(np.cos(t))*np.sin(t) - np.cos(np.cos(t))*np.sin(np.sin(t))*np.cos(t)
8
9 L = np.trapz(np.sqrt(dx**2 + dy**2 + dz**2), t)
10
11 print(f"{L:f}")

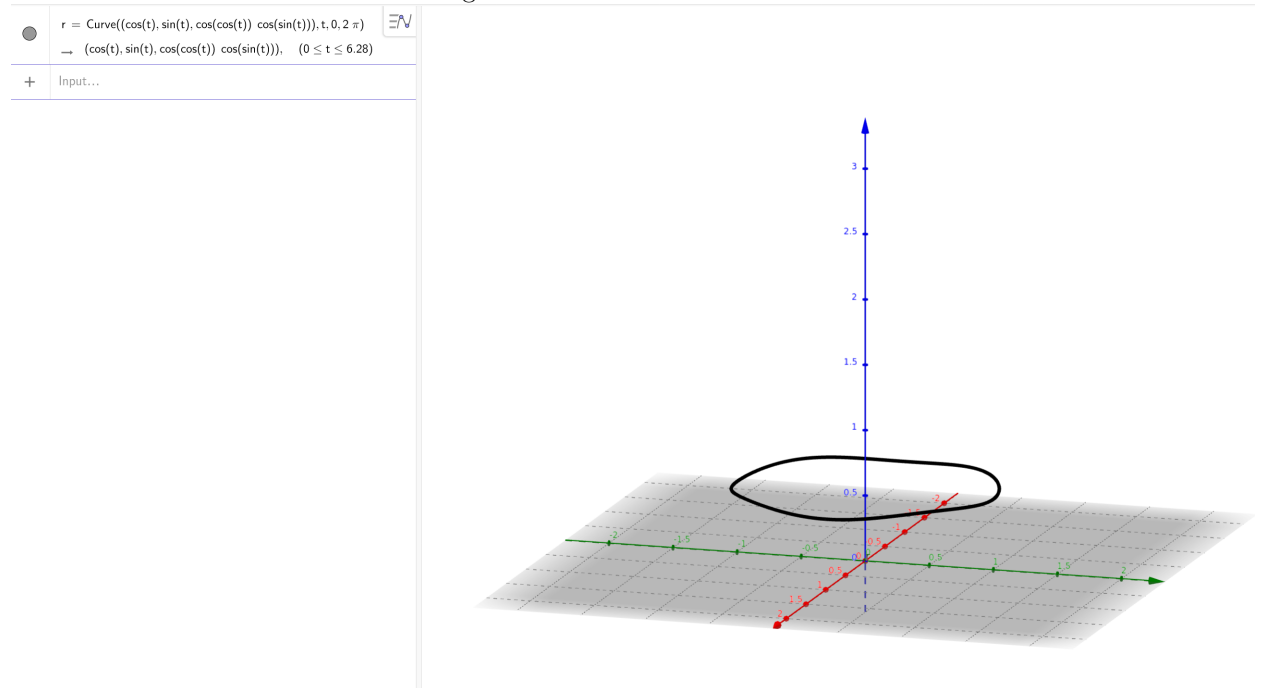
```

Som gir meg at

$$L = 6.284694$$

Som er litt større enn 2π som gir mening siden kurven ser noe slikt ut i Geogebra

Figur 5: Kurven til 3d



e

Bruker formelen

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$$

Der $\vec{v} = \vec{u} = (\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}}) e^{-2vt}$

Finner først lokalakselerasjonen

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} &= \frac{\partial}{\partial t} (\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}}) e^{-2vt} \\ \frac{\partial \vec{u}}{\partial t} &= (\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}}) \frac{\partial}{\partial t} e^{-2vt} \\ \frac{\partial \vec{u}}{\partial t} &= -2v (\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}}) e^{-2vt} \\ \frac{\partial \vec{u}}{\partial t} &= -2v \vec{u} \end{aligned}$$

Finner så den konvekktive akselerasjonen

$$\vec{u} \cdot \nabla \vec{u} = (\vec{u} \cdot \nabla u_x) \hat{\mathbf{i}} + (\vec{u} \cdot \nabla u_y) \hat{\mathbf{j}}$$

$$\vec{u} \cdot \nabla \vec{u} = u_x \frac{\partial \vec{u}}{\partial x} + u_y \frac{\partial \vec{u}}{\partial y}$$

$$\vec{u} \cdot \nabla \vec{u} = u_x \frac{\partial}{\partial x} \left((\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}}) e^{-2vt} \right) + u_y \frac{\partial}{\partial y} \left((\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}}) e^{-2vt} \right)$$

$$\vec{u} \cdot \nabla \vec{u} = e^{-2vt} \left(u_x \left(-\sin(x) \sin(y) \hat{\mathbf{i}} - \cos(x) \cos(y) \hat{\mathbf{j}} \right) + u_y \left(\cos(x) \cos(y) \hat{\mathbf{i}} + \sin(x) \sin(y) \hat{\mathbf{j}} \right) \right)$$

$$\vec{u} \cdot \nabla \vec{u} = e^{-2vt} \left(\cos(x) \sin(y) e^{-2vt} \left(-\sin(x) \sin(y) \hat{\mathbf{i}} - \cos(x) \cos(y) \hat{\mathbf{j}} \right) - \sin(x) \cos(y) e^{-2vt} \left(\cos(x) \cos(y) \hat{\mathbf{i}} + \sin(x) \sin(y) \hat{\mathbf{j}} \right) \right)$$

$$\vec{u} \cdot \nabla \vec{u} = e^{-4vt} \left(\cos(x) \sin(y) \left(-\sin(x) \sin(y) \hat{\mathbf{i}} - \cos(x) \cos(y) \hat{\mathbf{j}} \right) - \sin(x) \cos(y) \left(\cos(x) \cos(y) \hat{\mathbf{i}} + \sin(x) \sin(y) \hat{\mathbf{j}} \right) \right)$$

$$\vec{u} \cdot \nabla \vec{u} = e^{-4vt} \left(-\sin^2(y) \cos(x) \sin(x) \hat{\mathbf{i}} - \cos^2(x) \cos(y) \sin(y) \hat{\mathbf{j}} - \cos^2(y) \cos(x) \sin(x) \hat{\mathbf{i}} - \sin^2(x) \cos(y) \sin(y) \hat{\mathbf{j}} \right)$$

$$\vec{u} \cdot \nabla \vec{u} = e^{-4vt} \left(- \left(\left(\sin^2(y) \cos(x) \sin(x) + \cos^2(y) \cos(x) \sin(x) \right) \hat{\mathbf{i}} + \left(\cos^2(x) \cos(y) \sin(y) + \sin^2(x) \cos(y) \sin(y) \right) \hat{\mathbf{j}} \right) \right)$$

$$\vec{u} \cdot \nabla \vec{u} = -e^{-4vt} \left(\cos(x) \sin(x) \left(\sin^2(y) + \cos^2(y) \right) \hat{\mathbf{i}} + \cos(y) \sin(y) \left(\sin^2(x) + \cos^2(x) \right) \hat{\mathbf{j}} \right)$$

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

$$\vec{u} \cdot \nabla \vec{u} = -e^{-4vt} \left(\cos(x) \sin(x) \hat{\mathbf{i}} + \cos(y) \sin(y) \hat{\mathbf{j}} \right)$$

$$\boxed{2 \cos(x) \sin(x) = \sin(2x)}$$

$$\vec{u} \cdot \nabla \vec{u} = -e^{-4vt} \left(\frac{1}{2} \sin(2x) \hat{\mathbf{i}} + \frac{1}{2} \sin(2y) \hat{\mathbf{j}} \right)$$

$$\vec{u} \cdot \nabla \vec{u} = -\frac{1}{2} e^{-4vt} \left(\sin(2x) \hat{\mathbf{i}} + \sin(2y) \hat{\mathbf{j}} \right)$$

Da blir

$$\frac{D\vec{u}}{Dt} = -2v\vec{u} - \frac{1}{2} \left(\sin(2x) \hat{\mathbf{i}} + \sin(2y) \hat{\mathbf{j}} \right) e^{-4vt}$$

Skal så finne trykket p

$$\frac{D\vec{u}}{Dt} = -\nabla p + v \nabla^2 \vec{u}$$

$$\nabla p = v \nabla^2 \vec{u} - \frac{D\vec{u}}{Dt}$$

$$\nabla p = v \nabla^2 \vec{u} + 2v\vec{u} + \frac{1}{2} \left(\sin(2x) \hat{\mathbf{i}} + \sin(2y) \hat{\mathbf{j}} \right) e^{-4vt}$$

Må først da regne ut

$$v \cdot \nabla^2 \vec{u} = v \cdot \left(\frac{\partial^2}{\partial x^2} \vec{u} + \frac{\partial^2}{\partial y^2} \vec{u} \right)$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \vec{u} &= \frac{\partial^2}{\partial x^2} \left(\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}} \right) e^{-2vt} \\ \frac{\partial^2}{\partial x^2} \vec{u} &= \frac{\partial}{\partial x} \left(-\sin(x) \sin(y) \hat{\mathbf{i}} - \cos(x) \cos(y) \hat{\mathbf{j}} \right) e^{-2vt} \\ \frac{\partial^2}{\partial x^2} \vec{u} &= \left(-\cos(x) \sin(y) \hat{\mathbf{i}} + \sin(x) \cos(y) \hat{\mathbf{j}} \right) e^{-2vt} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2} \vec{u} &= \frac{\partial^2}{\partial y^2} \left(\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}} \right) e^{-2vt} \\ \frac{\partial^2}{\partial y^2} \vec{u} &= \frac{\partial}{\partial y} \left(\cos(x) \cos(y) \hat{\mathbf{i}} + \sin(x) \sin(y) \hat{\mathbf{j}} \right) e^{-2vt} \\ \frac{\partial^2}{\partial y^2} \vec{u} &= \left(-\cos(x) \sin(y) \hat{\mathbf{i}} + \sin(x) \cos(y) \hat{\mathbf{j}} \right) e^{-2vt} \end{aligned}$$

$$v \cdot \nabla^2 \vec{u} = v \cdot \left(\left(-\cos(x) \sin(y) \hat{\mathbf{i}} + \sin(x) \cos(y) \hat{\mathbf{j}} \right) e^{-2vt} + \left(-\cos(x) \sin(y) \hat{\mathbf{i}} + \sin(x) \cos(y) \hat{\mathbf{j}} \right) e^{-2vt} \right)$$

$$v \cdot \nabla^2 \vec{u} = v e^{-2vt} \cdot \left(-2 \cos(x) \sin(y) \hat{\mathbf{i}} + 2 \sin(x) \cos(y) \hat{\mathbf{j}} \right)$$

$$v \cdot \nabla^2 \vec{u} = -2v \left(\cos(x) \sin(y) \hat{\mathbf{i}} - \sin(x) \cos(y) \hat{\mathbf{j}} \right) e^{-2vt}$$

$$v \cdot \nabla^2 \vec{u} = -2v \vec{u}$$

Da blir

$$\nabla p = 2v \vec{u} + \frac{1}{2} \left(\sin(2x) \hat{\mathbf{i}} + \sin(2y) \hat{\mathbf{j}} \right) e^{-4vt} - 2v \vec{u}$$

Ser så på integrallet med hensyn på x

$$p(x, y, t) = \int \left(2v \vec{u} + \frac{1}{2} \left(\sin(2x) + \sin(2y) \right) e^{-4vt} - 2v \vec{u} \right) dx$$

Tar integrallene hver for seg

$$\begin{aligned} \int 2v \vec{u} dx &= 2v \int \left(\cos(x) \sin(y) - \sin(x) \cos(y) \right) e^{-2vt} dx \\ \int 2v \vec{u} dx &= 2v e^{-2vt} \left(-\sin(x) \sin(y) - \cos(x) \cos(y) \right) \end{aligned}$$

Ser så på

$$\begin{aligned} \int \frac{1}{2} \left(\sin(2x) + \sin(2y) \right) e^{-4vt} dx &= \frac{1}{2} e^{-4vt} \int \left(\sin(2x) + \sin(2y) \right) dx \\ \int \frac{1}{2} \left(\sin(2x) + \sin(2y) \right) e^{-4vt} dx &= \frac{1}{2} e^{-4vt} \cdot \left(-\frac{1}{2} \cos(2x) \right) \\ \int \frac{1}{2} \left(\sin(2x) + \sin(2y) \right) e^{-4vt} dx &= -\frac{1}{4} e^{-4vt} \cos(2x) \end{aligned}$$

Observerer at det første og siste integrallet er likt bare med motsatt fortegn

$$\int -2v\vec{u} \, dx = -2ve^{-2vt} (-\sin(x)\sin(y) - \cos(x)\cos(y))$$

Da blir

$$\begin{aligned} p(x, y, t) &= 2ve^{-2vt} (-\sin(x)\sin(y) - \cos(x)\cos(y)) - \frac{1}{4}e^{-4vt} \cos(2x) \\ &\quad - 2ve^{-2vt} (-\sin(x)\sin(y) - \cos(x)\cos(y)) + f(y) \\ p(x, y, t) &= -\frac{1}{4}e^{-4vt} \cos(2x) + f(y) \end{aligned}$$

Ser så med hensyn på y

$$p(x, y, t) = \int \left(2v\vec{u} + \frac{1}{2} (\sin(2x) + \sin(2y)) e^{-4vt} - 2v\vec{u} \right) dy$$

Tar integrallene hver for seg

$$\begin{aligned} \int 2v\vec{u} \, dy &= 2v \int (\cos(x)\sin(y) - \sin(x)\cos(y)) e^{-2vt} \, dy \\ \int 2v\vec{u} \, dy &= 2ve^{-2vt} (\cos(x)\cos(y) + \sin(x)\sin(y)) \end{aligned}$$

Ser så på

$$\begin{aligned} \int \frac{1}{2} (\sin(2x) + \sin(2y)) e^{-4vt} \, dy &= \frac{1}{2}e^{-4vt} \int (\sin(2x) + \sin(2y)) \, dy \\ \int \frac{1}{2} (\sin(2x) + \sin(2y)) e^{-4vt} \, dy &= \frac{1}{2}e^{-4vt} \cdot \left(-\frac{1}{2} \cos(2y) \right) \\ \int \frac{1}{2} (\sin(2x) + \sin(2y)) e^{-4vt} \, dy &= -\frac{1}{4}e^{-4vt} \cos(2y) \end{aligned}$$

Observerer at det første og siste integrallet er likt bare med motsatt fortegn

$$\int -2v\vec{u} \, dy = -2ve^{-2vt} (\cos(x)\cos(y) + \sin(x)\sin(y))$$

Da blir

$$\begin{aligned} p(x, y, t) &= 2ve^{-2vt} (-\sin(x)\sin(y) - \cos(x)\cos(y)) - \frac{1}{4}e^{-4vt} \cos(2y) \\ &\quad - 2ve^{-2vt} (-\sin(x)\sin(y) - \cos(x)\cos(y)) + g(x) \\ p(x, y, t) &= -\frac{1}{4}e^{-4vt} \cos(2y) + g(x) \end{aligned}$$

Sammenligner jeg disse 2 p -ene så ser jeg at

$$g(x) = -\frac{1}{4}e^{-4vt} \cos(2x) \wedge f(y) = -\frac{1}{4}e^{-4vt} \cos(2y)$$

Følgelig blir

$$p(x, y, t) = -\frac{1}{4}e^{-4vt} \cos(2x) - \frac{1}{4}e^{-4vt} \cos(2y) = -\frac{1}{4}e^{-2vt} (\cos(2x) + \cos(2y)) + p_0$$