1

i $\vec{u} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
Ser først på divergensen

$$\begin{aligned}
\nabla \cdot \vec{u} &= \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z \\
\nabla \cdot \vec{u} &= 1 + 1 + 1 \\
\nabla \cdot \vec{u} &= 3
\end{aligned}$$

Ser så på virvlingen

$$\begin{aligned}
\nabla \times \vec{u} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
\nabla \times \vec{u} &= \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right)\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= \vec{0}
\end{aligned}$$

ii $\vec{u} = r\cos(\theta)\hat{\mathbf{i}}_r + r\sin(\theta)\hat{\mathbf{i}}_\theta + z\hat{\mathbf{k}}$ Ser først på divergensen

$$\begin{aligned}
\nabla \cdot \vec{u} &= \frac{1}{r}\left(\frac{\partial}{\partial r}\Big(r\big(r\cos(\theta)\big)\Big) + \frac{\partial}{\partial \theta}\big(r\sin(\theta)\big)\right) + \frac{\partial}{\partial z}z \\
\nabla \cdot \vec{u} &= \frac{1}{r}\left(\cos(\theta)\frac{\partial}{\partial r}r^2 + r\frac{\partial}{\partial \theta}\sin(\theta)\right) + 1 \\
\nabla \cdot \vec{u} &= \frac{1}{r}\big(2r\cos(\theta) + r\cos(\theta)\big) + 1 \\
\nabla \cdot \vec{u} &= 2\cos(\theta) + \cos(\theta) + 1 \\
\nabla \cdot \vec{u} &= 3\cos(\theta) + 1
\end{aligned}$$

Ser så på virvlingen

$$\begin{aligned}
\nabla \times \vec{u} &= 0\hat{\mathbf{i}}_r + 0\hat{\mathbf{i}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\Big(r\big(r\sin(\theta)\big)\Big) - \frac{\partial}{\partial \theta}\big(r\cos(\theta)\big)\right)\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= \frac{1}{r}\left(\sin(\theta)\frac{\partial}{\partial r}\Big(r^2\Big) - r\frac{\partial}{\partial \theta}\big(\cos(\theta)\big)\right)\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= \frac{1}{r}\big(2r\sin(\theta) + r\sin(\theta)\big)\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= \big(2\sin(\theta) + \sin(\theta)\big)\hat{\mathbf{k}} \\
\nabla \times \vec{u} &= 3\sin(\theta)\hat{\mathbf{k}}
\end{aligned}$$

iii $\vec{u} = \hat{\mathbf{i}}_r + \hat{\mathbf{i}}$

Konverterer først over til rene sylinder koordinater

$$\boxed{\hat{\mathbf{i}} = \cos(\theta)\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta}$$

$$\begin{aligned}\vec{u} &= \hat{\mathbf{i}}_r + \cos(\theta)\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta\\ \vec{u} &= \big(1+\cos(\theta)\big)\,\hat{\mathbf{i}}_r - \sin(\theta)\hat{\mathbf{i}}_\theta\end{aligned}$$

Ser først på divergensen

$$\begin{aligned}\nabla\cdot\vec{u} &= \frac{1}{r}\left(\frac{\partial}{\partial r}\Big(r\left(1+\cos(\theta)\right)\Big)\frac{\partial}{\partial\theta}\left(-\sin(\theta)\right)\right)\\ \nabla\cdot\vec{u} &= \frac{1}{r}\left(1+\cos(\theta)-\cos(\theta)\right)\\ \nabla\cdot\vec{u} &= \frac{1}{r}\end{aligned}$$

Ser så på virvlingen

$$\begin{aligned}\nabla\times\vec{u} &= 0\hat{\mathbf{i}}_r + 0\hat{\mathbf{i}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\Big(r\left(-\sin(\theta)\right)\Big) - \frac{\partial}{\partial\theta}\left(1+\cos(\theta)\right)\right)\hat{\mathbf{k}}\\ \nabla\times\vec{u} &= \frac{1}{r}\left(-\sin(\theta)+\sin(\theta)\right)\hat{\mathbf{k}}\\ \nabla\times\vec{u} &= \vec{0}\end{aligned}$$

2

a

For å finne enhetsvektorene så må man først finne skaleringsfaktorene. Starter med skaleringsfaktoren til u

$$
\begin{aligned}
&h_u = \left|\frac{\partial \vec{r}}{\partial u}\right| \\
&h_u = \left|a\cos(v)\frac{\partial}{\partial u}\cosh(u)\hat{\mathbf{i}} + a\sin(v)\frac{\partial}{\partial v}\sinh(u)\hat{\mathbf{j}}\right| \\
&\boxed{a = 1} \\
&h_u = \left|\cos(v)\sinh(u)\,\hat{\mathbf{i}} + \sin(v)\cosh(u)\right| \\
&h_u = \sqrt{\left(\cos(v)\sinh(u)\right)^2 + \left(\sin(v)\cosh(u)\right)^2} \\
&h_u = \sqrt{\cos^2(v)\sinh^2(u) + \sin^2(v)\cosh^2(u)} \\
&\boxed{\cosh^2(u) = 1 + \sinh^2(u)} \\
&h_u = \sqrt{\cos^2(v)\sinh^2(u) + \sin^2(v)\left(1 + \sinh^2(u)\right)} \\
&h_u = \sqrt{\cos^2(v)\sinh^2(u) + \sinh^2(u)\sin^2(v) + \sin^2(v)} \\
&h_u = \sqrt{\sinh^2(u)\left(\cos^2(v) + \sin^2(v)\right) + \sin^2(v)} \\
&h_u = \sqrt{\sinh^2(u) + \sin^2(v)}
\end{aligned}
$$

Finner så skaleringsfaktoren til v

$$
\begin{aligned}
&h_v = \left|\frac{\partial \vec{r}}{\partial v}\right| \\
&h_v = \left|\cosh(u)\frac{\partial}{\partial v}\cos(v)\hat{\mathbf{i}} + \sinh(u)\frac{\partial}{\partial v}\sin(v)\hat{\mathbf{j}}\right| \\
&h_v = \left|-\cosh(u)\sin(v)\hat{\mathbf{i}} + \sinh(u)\cos(v)\hat{\mathbf{j}}\right| \\
&h_v = \sqrt{\left(-\cosh(u)\sin(v)\right)^2 + \left(\sinh(u)\cos(v)\right)^2} \\
&h_v = \sqrt{\cosh^2(u)\sin^2(v) + \sinh^2(u)\cos^2(v)} \\
&h_v = \sqrt{\left(1 + \sinh^2(u)\right)\sin^2(v) + \sinh^2(u)\cos^2(v)} \\
&h_v = \sqrt{\sin^2(v) + \sinh^2(u)\sin^2(v) + \sinh^2(u)\cosh^2(v)} \\
&h_v = \sqrt{\sin^2(v) + \sinh^2(u)\left(\sin^2(v) + \cos^2(v)\right)} \\
&h_v = \sqrt{\sinh^2(u) + \sin^2(v)}
\end{aligned}
$$

Ser da at $h_u = h_v$ Enhetsvektorer er da gitt ved

$$\mathbf{e}_u = \frac{1}{h_u}\frac{\partial \vec{r}}{\partial u}$$
$$\mathbf{e}_u = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}}\left(\cos(v)\sinh(u)\,\hat{\mathbf{i}} + \sin(v)\cosh(u)\,\hat{\mathbf{j}}\right)$$

Og

$$\mathbf{e}_v = \frac{1}{h_v}\frac{\partial \vec{r}}{\partial v}$$
$$\mathbf{e}_v = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}}\left(-\cosh(u)\sin(v)\hat{\mathbf{i}} + \sinh(u)\cos(v)\hat{\mathbf{j}}\right)$$

Hvis de er ortogonale så må $\mathbf{e}_u \cdot \mathbf{e}_v = 0$

$$\mathbf{e}_u \cdot \mathbf{e}_v = \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}}\begin{pmatrix}\cos(v)\sinh(u)\\ \sin(v)\cosh(u)\end{pmatrix} \cdot \frac{1}{\sqrt{\sinh^2(u) + \sin^2(v)}}\begin{pmatrix}-\cosh(u)\sin(v)\\ \sinh(u)\cos(v)\end{pmatrix}$$
$$\mathbf{e}_u \cdot \mathbf{e}_v = \frac{1}{\sinh^2(u)\sin^2(v)}\left(-\cos(v)\sinh(u)\cosh(u)\sin(v) + \sin(v)\cosh(u)\sinh(u)\cos(v)\right)$$
$$\mathbf{e}_u \cdot \mathbf{e}_v = \left(-\sin(v)\cos(v)\sinh(u)\cosh(u) + \sin(v)\cos(v)\sinh(u)\cosh(u)\right)$$
$$\mathbf{e}_u \cdot \mathbf{e}_v = 0$$

Med dette så observeres det at enhetsvektorene er ortogonale

3

a

Strømfunksjon

$$v_x = \frac{\partial \psi}{\partial y} \wedge v_y = \frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial y} = -v_x \qquad \wedge \qquad \frac{\partial \psi}{\partial x} = v_y$$
$$\psi(x,y) = -\int \cos(x)\sin(y)\mathrm{e}^{-2vt}\,\mathrm{d}y \qquad \wedge \qquad \psi(x,y) = \int -\sin(x)\cos(y)\mathrm{e}^{-2vt}\,\mathrm{d}x$$
$$\psi(x,y) = -\cos(x)\cos(y)\mathrm{e}^{-2vt} + C \qquad \wedge \qquad \psi(x,y) = -\cos(x)\cos(y)\mathrm{e}^{-2vt} + C$$