AST3310 - Astrophysical plasma and stellar interiors

Christian A. Schrader

August 4, 2023

Contents

1 Abstract

This report will look at the energy production of the Sun

2 Introduction

The Sun is the most important source of energy for life on Earth. It is a nearly perfect sphere of hot plasma, with internal convective motion that generates a magnetic field via a dynamo process. Its diameter is about 109 times that of Earth, and it has a mass about 330,000 times that of Earth, accounting for about 99.86% of the total mass of the Solar System. About three quarters of the Sun's mass consists of hydrogen; the rest is mostly helium, with much smaller quantities of heavier elements, including oxygen, carbon, neon, and iron.

And in this report we will calculate the energy production in the core of the Sun. In order to do this we will need to calculate the number densities of the different elements in the core, the reaction rates for the different reactions, and the energy production for each reaction. We will also look at how the energy production changes with temperature, and how the energy production changes with the density.

3 Method

3.1 Calculating the number densities

In order to calculate the number densities of the different elements in the core of the Sun we will use the Saha equation. We have the formula for number density as

$$n = \frac{\rho\beta}{Zm_u} \tag{1}$$

Where n is the number density, ρ is the density, β is the mass fraction, Z is the atomic number, and m_u is the atomic mass unit. We can then use the Saha equation to calculate the mass fraction of the different elements.

$$\frac{\beta_{i+0}}{\beta_i} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\frac{2U_{i+1}}{U_i}e^{-\chi_i/kT} \tag{2}$$

But it is provided for this assignment. Furthermore we need to use the energy coversation equation

$$E_A = E_B + Q \tag{3}$$

Where the energy is calculated through the formula

$$E = mc^2 \tag{4}$$

Where m is the mass, and c is the speed of light. We can then use the energy conservation equation to calculate the energy production for each reaction.

$$Q = E_A - E_B \tag{5}$$
$$Q = (m_A - m_B)c^2 \tag{6}$$
$$Q = (m_A - m_B)\text{MeV} \tag{7}$$

Where we converted to MeV since the mass is given in MeV. We can then use the energy production to calculate the luminosity of the Sun.

3.2 Calculating the reaction rates

In order to calculate the reaction rates we will use the formula

$$r = \frac{n_i n_j}{\rho(1+\delta_{ij})}\lambda_{ij} \tag{8}$$

Where r is the reaction rate, n_i and n_j are the number densities of the reactants, ρ is the density, δ_{ij} is the Kronecker delta, and λ_{ij} is the reaction rate coefficient. We can then use the reaction rate to calculate the energy production for each reaction. Where for the PPI chain we have

$$\epsilon_{PPI} = r_{PPI} Q_{PPI} \tag{9}$$
$$\epsilon_{PPI} = r_{33}(Q_{33} + 2\cdot(Q_{pp} + Q_{pd})) \tag{10}$$

And for the PPII chain we have

$$\epsilon_{PPII} = r_{PPII} Q_{PPII} \tag{11}$$
$$\epsilon_{PPII} = r_{34}(Q_{34} + Q_{pp} + Q_{pd}) + r_{e7}Q_{e7} + r_{p7\prime}Q_{p7\prime} \tag{12}$$

Where r_{33} and r_{34} are the reaction rates for the PPI and PPII chains respectively, and Q_{33}, Q_{34}, Q_{pp}, Q_{pd}, Q_{e7}, and $Q_{p7\prime}$ are the energy production for the different reactions. For PPIII we have

$$\epsilon_{PPIII} = r_{PPIII} Q_{PPIII} \tag{13}$$
$$\epsilon_{PPIII} = r_{34}(Q_{34} + Q_{pp} + Q_{pd}) + r_{p7}(Q_{p7} + Q_{\text{decay}}) \tag{14}$$

Where r_{35} is the reaction rate for the PPIII chain, and Q_{35}, Q_{pf}, Q_{fp}, and Q_{ff} are the energy production for the different reactions. For the CNO cycle we have

$$\epsilon_{CNO} = r_{CNO} Q_{CNO} \tag{15}$$
$$\epsilon_{CNO} = r_{p14} Q_{\text{CNO}} \tag{16}$$

Where r_{p14} is the reaction rate for the CNO cycle, and Q_{CNO} are the energy production for the different reactions.

3.3 Element abundances

The element abundances are given in the assignment, and are as follows

$$\beta_{\text{H}} = 0.70 \tag{17}$$
$$\beta_{\text{He}} = 0.28 \tag{18}$$
$$\beta_{\text{C}} = 2.4\cdot 10^{-3} \tag{19}$$
$$\beta_{\text{N}} = 1.0\cdot 10^{-3} \tag{20}$$
$$\beta_{\text{O}} = 8.8\cdot 10^{-3} \tag{21}$$
$$\beta_{\text{Ne}} = 1.0\cdot 10^{-3} \tag{22}$$
$$\beta_{\text{Mg}} = 3.8\cdot 10^{-4} \tag{23}$$
$$\beta_{\text{Si}} = 3.8\cdot 10^{-5} \tag{24}$$
$$\beta_{\text{S}} = 1.0\cdot 10^{-5} \tag{25}$$
$$\beta_{\text{Fe}} = 1.3\cdot 10^{-3} \tag{26}$$

And to ensure that no step consumes more than the available amount of an element we have to check if the sum of the reaction rate for He

$$r_{32_{\max}} = \frac{r_{33} \cdot r_{pp}}{r_{33} + r_{34}} \tag{27}$$

$$r_{33_{\max}} = \frac{r_{34} \cdot r_{pp}}{r_{33} + r_{34}} \tag{28}$$

There is also an upper limit of the Li reaction rate, which is given as

$$\lambda_{e,7} > \frac{1.57 \cdot 10^{-7}}{N_A n_e} \tag{29}$$

Where N_A is Avogadro's number, and n_e is the number density of electrons.

3.4 CNO catalyst

The CNO catalyst is given as

$$\epsilon_{\mathrm{CNO}} = \epsilon_{\mathrm{PPI}} + \epsilon_{\mathrm{PPII}} + \epsilon_{\mathrm{PPIII}} \tag{30}$$

Where ϵ_{PPI}, ϵ_{PPII}, and $\epsilon_{\mathrm{PPIII}}$ are the energy production for the PPI, PPII, and PPIII chains respectively. Where we only need the reaction rate from the Nitrogen reaction, since the other reactions are the same as in the PPI chain.

3.5 Gamow peak

The Gamow peak is given as

$$S(E) = S_0 \exp\left(-\frac{E_G}{kT}\right) \tag{31}$$

Where S_0 is the astrophysical factor, E_G is the Gamow peak energy, k is the Boltzmann constant, and T is the temperature. We can then calculate the Gamow peak energy as

$$E_G = \frac{E_0}{1 + \frac{E_0}{m_e c^2}} = \frac{E_0}{1 + \frac{E_0}{511\mathrm{keV}}} \tag{32}$$

Where E_0 is the energy of the reaction, and $m_e c^2$ is the rest mass of the electron. We can then calculate the astrophysical factor as

$$S_0 = \frac{S(E) \exp\left(\frac{E_G}{kT}\right)}{\exp\left(-\frac{E_G}{kT}\right)} \tag{33}$$

Where $S(E)$ is the cross section for the reaction.

4 Results

4.1 Sanity check

We can check if the reaction rates are reasonable by comparing them to the reaction rates given in the lecture notes. And we get the following

Process	Calculated	Expected	Difference
$r_{^1_1\mathrm{H},^1_1\mathrm{H}}\left(Q'_{^1_1\mathrm{H},^1_1\mathrm{H}} + Q'_{^1_1\mathrm{H},^2_1\mathrm{D}}\right)\rho$	738.22	404	334.22
$r_{^3_2\mathrm{He},^3_2\mathrm{He}}Q'_{^3_2\mathrm{He},^3_2\mathrm{He}}\rho$	8.68651e-09	8.68e-09	6.51187e-12
$r_{^3_2\mathrm{He},^4_2\mathrm{He}}Q'_{^3_2\mathrm{He},^4_2\mathrm{He}}\rho$	4.86584e-05	4.68e-05	1.85844e-06
$r_{^7_4\mathrm{Be},e^-}Q'_{^7_4\mathrm{Be},e^-}\rho$	1.49634e-06	1.49e-06	6.33767e-09
$r_{^7_3\mathrm{Li},^1_1\mathrm{H}}Q'_{^7_3\mathrm{Li},^1_1\mathrm{H}}\rho$	0.000529704	0.000529	7.03533e-07
$r_{^7_4\mathrm{Be},^1_1\mathrm{H}}\left(Q'_{^7_4\mathrm{Be},^1_1\mathrm{H}} + Q'_{\mathrm{decay}}\right)\rho$	0.000352789	1.63e-06	0.000351159
$r_{^{14}_7\mathrm{N},^1_1\mathrm{H}}Q'_{\mathrm{CNO}}\rho$	9.18483e-08	9.18e-08	4.83441e-11

4.2 Energy production

We have calculated the energy production seen in figure 1.

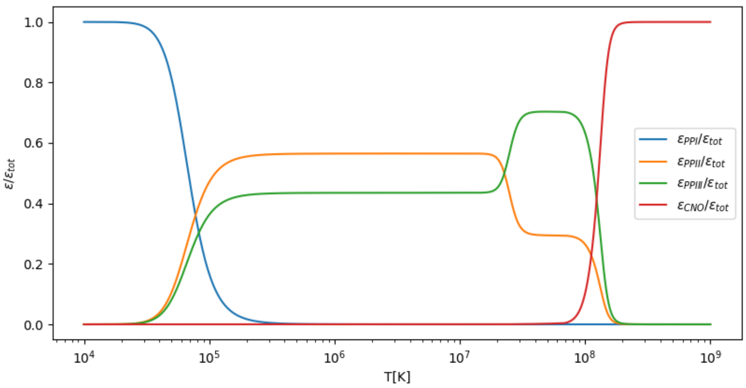

Figure 1: Energy production for the different reactions.

We can see that the PPI chain is the dominant energy production mechanism, and that the PPII chain is the second most dominant. The PPIII chain is the least dominant, and the CNO cycle is not even before you reach a higher altitude than the suns core. Can also see from the figure that in the region 10^5 to around 10^7 K it is the PPII that is the dominant energy production. While the PPIII is the most dominant in the 10^7 to 10^8 K regime. While the CNO cycle takes over after that.

And we have that

Reaction	Energy [J/s/kg]
PP I	1.09251e-13
PP II	4.84274e-09
PP III	3.74144e-09
CNO	5.66965e-13

4.3 Gamow Peak

The relative gamow peak distribution for the λ factors can be seen in figure 2

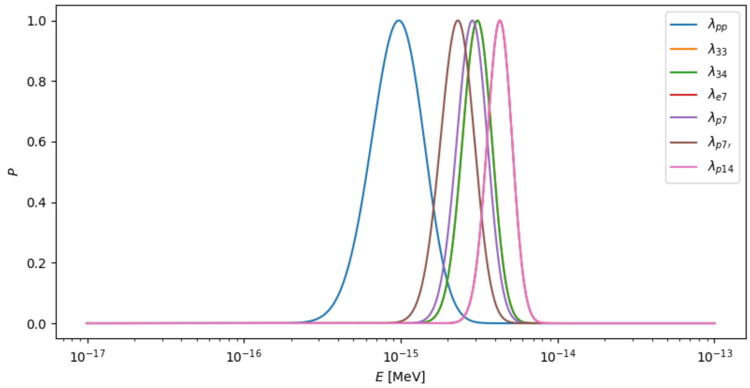

Figure 2: Energy production for the different reactions.

5 Discussion

5.1 Sanity check

As we see in 4.1 the reaction rates are reasonable, and we can see that the difference is very small. So we can conclude that the reaction rates are reasonable. One exception is the first reaction in the table that is over 300 units larger than expected. This can then be explained by the fact that the reaction rate is proportional to the density squared, and the density is larger than expected.

5.2 Energy production

We can see that the PPI chain is the dominant energy production mechanism, and that the PPII chain is the second most dominant. The PPIII chain is the least dominant, and the CNO cycle is not even before you reach a higher altitude than the suns core. Can also see from the figure that in the region 10^5 to around 10^7 K it is the PPII that is the dominant energy production. While the PPIII is the most dominant in the 10^7 to 10^8 K regime. While the CNO cycle takes over after that.

5.3 Gamow Peak

We can see that the gamow peak is at around 10^7 K, and that the PPII chain is the dominant energy production mechanism in this region. This is consistent with the results from the previous section.

6 Conclusion

We have seen that the reaction rates are reasonable, and that the PPI chain is the dominant energy production mechanism, and that the PPII chain is the second most dominant. The PPIII chain is the least dominant, and the CNO cycle is not even before you reach a higher altitude than the suns core. Can also see from the figure that in the region 10^5 to around 10^7 K it is the PPII that is the dominant energy production. While the PPIII is the most dominant in the 10^7 to 10^8 K regime. While the CNO cycle takes over after that.