Modelling energy production in a stellar core

Ava Ramirez
(Dated: July 13, 2023)

I. INTRODUCTION

At a distance of 150 million km from us, in the center of our solar system, lies the Sun. Being our closest star, the Sun provide us on Earth with energy that is vital for our existence. But how can a star produce this energy that we so desperately need? [3]

Stars are concentrations of gas in different sizes and in the case of our Sun it consists of mainly Hydrogen (91%) and Helium (8.9%). In the core the temperature is high enough for thermonuclear fusion of lighter elements into heavier elements, which is how the stars stay alive as it produces energy as well as creating a pressure outwards [3] to support the hydrostatic equilibrium (B.V. Gudiksen, 2021 [1]). In the Sun it is mostly fusion of hydrogen into helium. The fusion process can happen in different steps and reactions depending on the temperature in the core. How efficient a process is depends mainly on the temperature and pressure, which in our Sun is $T_{core} \approx 1.57 \cdot 10^7$ K and $\rho_{core} = 1.62 \cdot 10^5$ kg m^{-3}. (B.V. Gudiksen, 2021 [1])

In this report we are going to look into the energy production in the center of a star, using known values of density and temperature of our Sun as references. We will also look at the energies produced and lost in each reaction. Even though energy production is something that happens over time, we will only look at a "snapshot" of what is being produced per second, and not look into the over-time-development. In section II we will explain the steps for making our energy-production function. In sections III and IV we show and explain our results and discuss them using what we know of the fusion processes.

II. METHOD

This section is split up into subsections explaining the different steps that is needed in order to calculate the energy production and other values we are interested in. The results from our calculations are presented in section III.

A. Reaction chains

In order to calculate the energy production in the center of a star, we need to know what types of reactions are happening in the core. In our case, we have based our model on a star with core temperatures T_{core} and mass density ρ_{core} not far from our own Sun. Based on what we know about stars of this type, the reaction chains most likely to occur is the PP chain and the dominant CNO cycle (B.V. Gudiksen, 2021 [1]).

1. *PP chain*

In the PP chain, we have three different branches, all resulting in the production of ${}^4_2He$. Before the branching of the chain we have two reactions, which we will refer to as the *common steps* (equations 1 and 2):

$$ {}^1_1H + {}^1_1H \rightarrow {}^2_1D + e^+ + \nu_e + 0.155\text{Mev} \quad (1)$$

$$ {}^2_1D + {}^1_1H \rightarrow {}^3_2He + 5.494\text{MeV} \quad (2)$$

Assuming the ${}^2_1D$ is consumed right away in equation 2, we will combine the common steps into a single step. The energies with units of MeV are the energies resulting from the reaction not including the annihilation energy from the positrons e^+. This is further explained in section II B.

After the two first steps, the PP chains branches out into PPI, PPII and PPIII.

PPI is just one reaction where two ${}^3_2He$ atoms, which we get from two rounds of the reactions in 1 and 2, reacts and produces one ${}^4_2He$ and two ${}^1_1H$. This means that all in all PPI spend $4 \cdot {}^1_1H$ for each ${}^4_2He$. This is also the case for PPII and PPIII, though they also include heavier elements in their reactions [1].

The PPI branch is shown in equation 3

$$ {}^3_2He + {}^3_2He \rightarrow {}^4_2He + 2{}^1_1H + 12.860 \text{ MeV} \quad (3)$$

With the helium produced in equations 1, 2 and 3, the second PP branch, PPII, can start. The PPII branch is shown in equations 4, 5 and 6:

$$^3_2He +^4_2 He \rightarrow^7_4 Be + 1.586 \text{ MeV} \quad (4)$$
$$^7_4Be + e^- \rightarrow^7_3 Li + \nu_e + \delta E \quad (5)$$
$$^7_3Li +^1_1 H \rightarrow 2^4_2He + 17.346 \text{ MeV} \quad (6)$$

In the case of our Sun the final branch in the PP chain, PPIII, is only responsible for a small part of the energy production. This is because it needs a temperature above $20 \cdot 10^6$ K to produce a lot of energy. The first reaction of the PPIII branch is the same as the first reaction in PPII [1]. The four reactions are shown in equations 7, 8, 9 and 10

$$^3_2He +^4_2 He \rightarrow^7_4 Be + 1.586 \text{ MeV} \quad (7)$$
$$^7_4Be +^1_1 H \rightarrow^8_5 B + 0.137 \text{ MeV} \quad (8)$$
$$^8_5B \rightarrow^8_4 Be + e^+ + \nu_e + 7.345 \text{ MeV} \quad (9)$$
$$^8_4Be \rightarrow 2^4_2He + 2.995 \text{ MeV} \quad (10)$$

In the third reaction of PPIII, the ^8_5B goes through a β-decay, which is resulting in a ^8_4Be particle, a positron and a neutrino ν_e.

2. *CNO cycle*

The second chain of reactions we used was the (dominant) CNO cycle. In addition to hydrogen particles, carbon, nitrogen and oxygen are used to produce helium. In the core of our Sun, the CNO cycle is responsible for only $\approx 1.7\%$ of the fusion processes, due to its high temperature requirement. A star needs a temperature above 10^7 K for the cycle to be effective. The dominant cycle is shown in equations 11-16:

$$^{12}_6C +^1_1 H \rightarrow^{13}_7 N + 1.944 \text{ MeV} \quad (11)$$
$$^{13}_7N \rightarrow^{13}_6 C\ e^+ + \nu_e + 0.491 \text{ MeV} \quad (12)$$
$$^{13}_6C +^1_1 H \rightarrow^{14}_7 N + 7.551 \text{ MeV} \quad (13)$$
$$^{14}_7N\ ^1_1H \rightarrow^{15}_8 O + 7.297 \text{ MeV} \quad (14)$$
$$^{15}_8O \rightarrow^{15}_7 N + e^+\nu_e + 0.735 \text{ MeV} \quad (15)$$
$$^{15}_7N +^1_1 H \rightarrow^{12}_6 C +^4_2 He + 4.966 \text{ MeV} \quad (16)$$

In the PP chain, no carbon is being produced, and in the CNO cycle it is only used to spark the reactions as a catalyst. So where does the $^{12}_6C$ in the CNO cycle come from?
One way to produce carbon is through the "Triple-alpha process", where three ^4_2He nuclei are transformed into $^{12}_6C$. This process needs a core temperature of at least 10^8K, and happens only when the star has consumed all its hydrogen. This means, the carbon in the stars that are yet to fuse all their hydrogen, which is the case of our stellar core, comes from other stars that has already produced carbon. For instance, if a star forms in the remnants of a supernova, it can contain some of the heavier elements produced in the explosion [4].

Some of the reactions in the PP chain and CNO cycle produces a neutrino through a β-decay. In Table I we have listed the energies of the neutrinos for the respective reaction, in units of MeV

Reaction	$\mathbf{Q}_\nu$**[MeV]**
$^1_1H +^1_1 H \rightarrow\ ^2_1D + e^+ + \nu_e$	0.265
$^7_4Be + e^- \rightarrow^7_3 Li + \nu_e$	0.81
$^8_5B \rightarrow^8_4 Be + e^+ + \nu_e$	6.711
$^{13}_7N \rightarrow^{13}_6 C\ e^+ + \nu_e$	0.707
$^{15}_8O \rightarrow^{15}_7 N + e^+\nu_e$	0.997

Table I. Table listing the energies of the neutrinos produced in the PP chain and CNO cycle

The neutrino energy for reaction $^7_4Be + e^- \rightarrow^7_3 Li + \nu_e$ is based on a weighted average of the two neutrino energies that it can produce. The two different energies are a result of the energy level of the produced ^7_3Li particle. (B.V. Gudiksen, 2021 [1])

B. Energy output

The first step in calculating the energy production, is to look at the energy that each reaction produces. In both the PP chain and the CNO cycle, 26.732 Mev is produced per ^4_2He nucleus, which is the mass loss between four ^1_1H and one ^4_2He nucleus. This is the total energy being produced, not accounting for energy loss. When calculating the energy output in each reaction, we use Einsteins mass-energy principle in equation 17:

$$\delta E = \delta m c^2 \quad (17)$$

where δE is the energy we get or lose from the mass difference δm and $c = 299792458$ m/s is the speed of light. Here, δm is the difference in mass that goes in and comes out of the reaction.(B.V. Gudiksen, 2021 [1])

Now we need the masses of the particles that are involved in the reactions. Equation 17 has units [kg m^2/s^2],

which means we needed the masses in kg. For consistency, we calculated the masses of all particles involved using the atomic weight. To convert the atomic weight into units of [kg] we used the atomic mass unit $m_u = 1.6605 \cdot 10{-}27$ kg (B.V. Gudiksen, 2021 [1]) In table II we have listed the masses used in units of atomic weight [5].

Element	**Atomic weight [u]**
${}^{1}_{1}H$	1.00782503207
${}^{2}_{1}D$	2.01410177785
${}^{3}_{2}He$	3.01602931914
${}^{4}_{2}He$	4.00260325415
${}^{7}_{4}Be$	7.016929828
${}^{7}_{3}Li$	7.016004548
${}^{8}_{4}Be$	8.005305103
${}^{8}_{5}B$	8.024607233
${}^{12}_{6}C$	12
${}^{13}_{7}N$	13.005738609
${}^{13}_{6}C$	13.00335483778
${}^{14}_{7}N$	14.00307400478
${}^{15}_{8}O$	15.99491461956
${}^{15}_{7}N$	15.00010889823

Table II. Table listing the masses of elements in the PP chain and CNO cycles. The masses have units u, which is the atomic weight.

The reactions listed in Table I produces a neutrino ν_e which, in the case of our Sun, does not contribute to the "thermal bath" where the reactions occur [1]. This is due to the mean free path of the neutrinos in the core; $l_{mfp} \approx 10^{15}$ m [6], is larger than the radius of the Sun; $R_{sun} = 6.96 \cdot 10^{8}$ m [1]. Since our model is based on stars with temperature and mass densities close to that of our Sun, we then have to subtract the energy from the neutrinos in order to get the energy output.

The positron which is produced in some of the equation, annihilates immediately with an electron after the reaction. The annihilation produces two photons γ, each with an energy of 0.511 MeV. Given enough energy, electrons can overcome the electrostatic binding force between itself and the atomic nucleus. Due to the immense pressure and temperature in the core of a star, we therefore assumed that all the elements are fully ionized. The energy from the annihilation is already a part of the δ_m in equation 17, which means we do not have to add this energy to our energy output. If we want to compare the energy output to the energies in equations 1-16, we then have to add the annihilation energy from the positron and electron to the reactions that produces a positron. This way we can compare our calculations to the actual values.

We know the energies of the neutrinos and annihilation in units of MeV, so to convert it into the SI units J we use that $1\text{eV} = 1.6022 \cdot 10^{-19}$ J.

Further we will call the total energy output, which does not account for the lost neutrino-energy, Q_{ik}. For our energy production we need to remove the neutrino energy, and we will call the energy output which accounts for the lost neutrino Q'_{ik}. It is Q'_{ik} that we will use in the energy production.

Our general equation for calculating the energy output Q_{ik} is shown in equation 18

$$Q'_{ik} = \delta m_{ik} \cdot c^2 - Q_{\nu_{ik}} \tag{18}$$

where δm is the difference in mass and Q_ν is the energy of the neutrino that is lost. The subscripts i, k indicates the particles that goes in to the reaction, for instance Q'_{33} for the PPI reaction between two ${}^{3}_{2}He$ particles.

In general we calculate the energy output for all the reactions the same way: subtract the mass on the right side of the reaction (except positrons as they annihilate) from the mass on the left side of the reaction, convert this to energy and subtract any energy from a neutrino. But in the case of reaction 5 where we have the ${}^{7}_{4}Be$ electron capture, we do not add the mass of the electron on the left side. Here the ${}^{7}_{4}Be$ core absorbs one of its own electrons in the electron capture, meaning that we do not add an extra electron mass in δm.

In the PPIII branch, the last two reactions are both decays. To make the calculations easier we combined the output energy of both decays into one Q'_{decay}.

In section III we have a table IV listing our calculated energy output for the PP chain and CNO cycle, using the masses in table II. They are listed in both units of J and MeV, which makes it easier to calculate the energy output in SI units as well as comparing the calculated values to the ones produced in the reaction chains, as they are most often presented in units of MeV.

1. *Energy lost to neutrinos*

We wanted to know how much of the produced energy is lost to the neutrinos, which carries the energies listed in Table I.
To calculate this we find the ratio between the total neutrino energy Q_ν in the branches and the total energy output Q_{tot} in PPI, PPII, PPIII and CNO cycle. Equation 19 show how we calculated this:

$$Q_{\text{lost}} = \frac{Q_{\nu,tot}}{Q_{tot}} \tag{19}$$

where $Q_{\nu,tot}$ is the total neutrino energy lost in a branch. Since the common steps happen in all of the PP branches, we need to include the neutrino in equation 1 in all of then as many times as the common steps occur.

In table V in section III we have added the calculated percentages for all the chains of reactions.

C. Reaction rates

Now that we know how much energy is produced in each reaction, we need to know at what rate they occur. To calculate the rate for each reaction we used equation 20:

$$r_{ik} = \frac{n_i n_k}{\rho(1+\delta_{ik})}\lambda_{ik} \tag{20}$$

The units of r_{ik} is $[\text{kg}^{-1}\text{s}^{-1}]$.Here ρ is the mass density and the δ_{ik} takes into consideration reactions between same types of particles. The λ_{ik} is a proportionality function which in our case only depends on temperature. [1]. In this equation, we need λ_{ik} to get the rate itself, and we can find the equations for each reaction in table 3.1, page 32 in B.V Gudiksen (2021) [1]. The table 3.1 lists $N_A\lambda_{ik}$, where N_A is Avogrado's number: $N_A = 6.0221 \cdot 10^{23}$. We express N_A in moles. which means the reaction rates in table 3.1 has the units of cm^{-3} s mol -1. To get the λ_{ik} we therefore divide by N_A and also multiply it with a factor of $1/100^3$ to convert it into SI units.

The n_i and n_k are the number density of the elements in the left side of the reaction, which can be calculated using 21:

$$n_i = \frac{\rho X_i}{m_i}, n_i = \frac{\rho Y_i}{m_i}, n_i = \frac{\rho Z_i}{m_i} \tag{21}$$

where X, Y, Z are the mass fractions of hydrogen, helium and metals [8], ρ is the mass density and m_i are the mass of the corresponding to the particle. We assumed that the mass fraction of each element used was independent of radius. In table III we have listed the mass fractions used in our calculations.

In order for a reaction to happen, there must be a sufficient amount of the particles that reacts with each other. This means, if the reaction in PPI and the first reaction in PPII and PPIII happens much faster than the common steps, there will not be enough helium for the PP branches. To control that a reaction does not consume more than what the previous reaction produces, we will do a scaling of the reaction rates. For the first part of the PP chain, this means that if the combined reaction rates of equations 3, 4 and 7 (here 4 and 7 have the same reaction rate and we count this one time, as this is one way the particles can react) are larger than the reaction rate we used for the common steps, r_{pp}, we scale them so they consume only what equations 1 and 2 produce. In equation 22 we show how we do this:

Mass fraction	**Value**
X	0.7
$Y_{^3_2He}$	10^{-10}
Y	0.29
$Z_{^7_3Li}$	10^{-7}
$Z_{^7_4Be}$	10^{-7}
$Z_{^{14}_7N}$	10^{-11}

Table III. Table listing the mass fractions of hydrogen, helium and metals used for the energy production in the different fusion reactions.

$$\text{R1}_{\text{scale}} = \frac{r_{pp}}{2 \cdot r_{33} + r_{34}} \tag{22}$$

When we then use the reaction rates further in the energy production, the rates r_{33}, r_{34} will be scaled as such: $r_{33} \cdot \text{R1}_{\text{scale}}$ and $r_{34} \cdot \text{R1}_{\text{scale}}$.

The PPII and PPIII branch have the same first reaction: ${}^3_2He + {}^4_2He \rightarrow {}^7_4Be + 1.586$ MeV. To make sure they do not consume too much of the helium, we make a scale similar to 22. In this scale we make sure that the reactions rates of 5 and 8 do not get too large compared to the reaction rate r_{34}. In equation 23 we have set up the scale factor R2_{scale}:

$$\text{R2}_{\text{scale}} = \frac{r_{34}}{r_{e7} + r_{17}} \tag{23}$$

We then write the new reaction rates as: $r_{e7} \cdot \text{R2}_{\text{scale}}$ and $r_{17} \cdot \text{R2}_{\text{scale}}$

The final scaling we need to do is for the last reaction in the PPII branch. Here we have to make sure that the rate of equation 6; r_{e7}, is not larger than the rate $\text{r}_{17'}$ for reaction 6. We then only need to scale r_{e7} by $\text{r}_{17'}$ to make sure the final reaction in PPII consumes no more than what is being produced in the previous reaction. In equation 24 we show this scaling factor:

$$\mathrm{R3_{scale}} = \frac{r_{e7}}{r_{17'}} \tag{24}$$

The new rate of $\mathrm{r_{17'}}$ then becomes: $r_{17'} \cdot R3_{scale}$.

D. Energy production

We now know how much energy is being produced in the reactions, their reaction rates and have also made sure we do not consume more particles than what is produced. We have all we need in order to calculate the energy production. To calculate the total energy production, we used equation 25. In this equation, $\mathrm{Q'_{ik}}$ is the energy output after we have accounted for the lost neutrinos using equation 18, and $\mathrm{r_{ik}}$ is the reaction rate, with the subscripts indicating which reaction we are using. In section II C we showed how we calculated the reaction rates as well as how we scaled them in order to prevent over consuming the particles needed. In order to calculate the energy production, we now need to combine the energy outputs and reaction rates in a way that accounts for how different reactions depends on the previous ones as well as the assumptions we made.

$$\varepsilon = \sum \mathrm{Q'_{ik} r_{ik}} \tag{25}$$

The PPI branch requires two ${}^3_2He$ to produce one ${}^4_2He$, which means we need the common steps in equation 1 and 2 to happen twice. In equation 22 we make sure we do not consume more of the ${}^3_2He$ than what is produced in the common steps. Since we also only look at one reaction at a time and we only need two rounds of the common steps in order fore PPI to begin, we do not include the reaction rate r_{pp}. We instead combine $\mathrm{Q'_{pp}}$ and $\mathrm{Q'_{pd}}$ which are energy outputs of the common steps, with the energy output in equation 3; $\mathrm{Q'_{33}}$, and use only the limiting reaction rate r_{33}.

The energy production for the branch can then be written as:

$$\varepsilon_{PPI} = \mathrm{r_{33}}(\mathrm{Q'_{33}} + 2(\mathrm{Q'_{pp}} + \mathrm{Q_{pd}})) \tag{26}$$

In the PPII branch, we use the same argument for the rate of the common steps as we did for, only now the common steps need to have happened only once. We add the energy output of the common steps to the energy output $\mathrm{Q'_{34}}$ of the first reaction, and use the reaction rate in equation 27.

$$\mathrm{r_{34}} = \frac{n_{{}^3_2He} n_{{}^4_2He}}{(\rho_{core})} \lambda_{34} \tag{27}$$

For the other two reactions in the PPII branch, we used the energy outputs $\mathrm{Q'_{e7}}$ and $\mathrm{Q'_{17'}}$ which corresponded to the two reaction rates:

$$\mathrm{r_{e7}} = \frac{n_e n_{{}^7_4Be}}{(\rho_{core})} \lambda_{e7} \tag{28}$$

$$\mathrm{r_{17'}} = \frac{n n_{{}^4_2He}}{(\rho_{core})} \lambda_{34} \tag{29}$$

For temperatures below $T = 10^6$ K, the electron capture in reaction 9 has an upper limit $N_A \lambda_{e7} \leq 1.57 \cdot 10^{-7}/n_e$. We implemented this using the same method for converting to SI units as for the other reactions in table 3.1, page 32 in B.V Gudiksen (2021).

The energy production for the PPII chain then becomes:

$$\varepsilon_{PPII} = \mathrm{r_{34}}(\mathrm{Q'_{34}} + (\mathrm{Q'_{pp}} + \mathrm{Q'_{pd}})) + \mathrm{r_{e7}} \mathrm{Q'_{e7}} + \mathrm{r_{17'}} \mathrm{Q'_{17'}} \tag{30}$$

For the last branch in the PP chain, PPIII, we add in the energy output from the common steps to the energy output of the first reaction, as we did in PPII. For the decay energy output $\mathrm{Q'_{decay}}$ in PPIII, we assume the previous reaction in equation 8 happens slower and is more limiting due to the lower temperatures as the T_{core} for our Sun and becaus the decays are not dependent on the proton-capture. We therefore combine $\mathrm{Q'_{decay}}$ with $\mathrm{Q'_{17}}$ and use the reaction rate $\mathrm{r_{17}}$.

The energy production from the PPIII branch becomes:

$$\varepsilon_{PPIII} = \mathrm{r_{34}}(\mathrm{Q'_{34}} + (\mathrm{Q'_{pp}} + \mathrm{Q'_{pd}})) + \mathrm{r_{17}}(\mathrm{Q'_{17}} + \mathrm{Q'_{decay}}) \tag{31}$$

The CNO cycle we have used is a part of the "Cold CNO" cycles. This means that the energy production depends on the reaction rates of the proton capture processes. The reaction in equation 14 is the slowest one, which is why we will use its reaction rate $\mathrm{r_{p14}}$ for the whole CNO cycle (M. Wiescher et. al, 2010 [2]). We combine the energy output of the whole CNO cycle into one $\mathrm{Q'_{CNO}}$, and get the equation for energy production in 32.

$$\varepsilon_{CNO} = \mathrm{r_{p14}} \mathrm{Q'_{CNO}} \tag{32}$$

In section III we have listed the resulting energy productions in table VI, using the core temperature of the

Sun and its mass density. We have also plotted the relative energy production from each PP branch and the CNO cycle with $T = [10^4, 10^9]$ K and using the mass density of the Sun. This is shown in Figure 1.

1. Gamow peak

In equation II C, we have the proportionality constant λ_{ik}, which can be formulated as in equation 33.

$$\lambda_{ik} = \sqrt{\frac{8}{m\pi(k_B T)^3}} \int_0^\infty \exp\left\{(-\frac{E}{k_B T})\right\} E\sigma(E)dE \quad (33)$$

Here $k_B = 1.3806\ 10^{-23}\ m^2\ kg\ s^{-2}K^{-4}$ is the Boltzmann's constant, T is the core temperature, and $m = m_i m_k/(m_i + m_k)$ is the reduced mass. $E = 1/2mv^2$ is the kinetic energy in the center of mass system and the cross section σ can be written on the form:

$$\sigma(E) = E^{-1}S(E)\exp\left\{-\sqrt{\frac{m}{2E}}\frac{Z_i Z_k e^2 \pi}{\epsilon : 0h}\right\} \quad (34)$$

where $S(E)$ is a function that varies slowly and and we assume to be of order one with units J m^2, Z_i, Z_k are the atomic numbers of the particles i, k, e is the elementary charge, h is Planck's constant and ϵ_0 is the permittivity of free space.

Based on equation 33, λ_{ik} is proportional to the integral of the two exponential functions $\exp\left\{(-\frac{E}{k_B T})\right\}$ and $\sigma(E)$. If we multiply these two functions together, and plot it against a range of energies E, we can analyze at what kinetic energies the different reaction rates are most effective. In Figure 2 we show the plot of the relative probability of the gamow peak for some of the reaction rates from PPI, PPII, PPIII and CNO with an energy range of $E = [10^{-17}, 10^{-13}]$ J. Since $\sigma(E)$ includes the atomic numbers, we do not include reactions which are decays or electron captures. In Table VII we have listed the energies E of the Gamow peaks of the same reactions.

III. RESULTS

A. Results: Energy output

In section II B we showed how we calculated the energy output for the reactions in the PP chain and CNO cycle. In Table IV we have listed our calculated values, using the masses in Table II and the neutrino energies in Table I. We have also added the Percent Error [9] showing the error between our calculations and the actual values of each Q'_{ik}. The actual values are listed in table 3.2 and 3.3 in B.V. Gudiksen, page 33, 2021 [1], and to convert the values into units of J we used that 1eV $= 1.6022 \cdot 10^{-19}$J. We can also compare the values using the energies in equations 1-16 and adding potential annihilation energy. This is then in units of MeV.

$\mathbf{Q_{ik}}$	**Calc. [J]**	**Calc. [MeV]**	**Error %**
$Q'_{^1_1\mathrm{H},^1_1\mathrm{H}}$	1.886 10^{-13}	1.177	0.016
$Q'_{^1_1\mathrm{H},^2_1\mathrm{D}}$	8.801 10^{-13}	5.493	0.012
$Q'_{^3_2\mathrm{He},^3_2\mathrm{He}}$	2.06 10^{-12}	12.86	0.0057
$Q'_{^3_2\mathrm{H},^1_3 4\mathrm{H}}$	2.541 10^{-13}	1.586	0.0038
$Q'_{^7_4\mathrm{Be},_e}$	8.311 10^{-15}	0.05187	5.9
$Q'_{^7_3\mathrm{Li},^1_1\mathrm{H}}$	2.779 10^{-12}	17.35	0.005
$Q'_{^7_4\mathrm{Be},^1_1\mathrm{H}}$	2.203 10^{-14}	0.1375	0.37
$Q'_{^8_5\mathrm{B}}$	1.805 10^{-12}	11.27	35
$Q'_{^8_4\mathrm{Be}}$	1.471 10^{-14}	0.09184	97
Q'_{decay}	1.82 10^{-12}	11.36	0.016
$Q'_{^{12}_6\mathrm{C},^1_1\mathrm{H}}$	3.114 10^{-13}	1.943	0.029
$Q'_{^{13}_7\mathrm{N}}$	2.425 10^{-13}	1.513	0.028
$Q'_{^{13}_6\mathrm{C},^1_1\mathrm{H}}$	1.21 10^{-12}	7.55	0.0081
$Q'_{^{14}_7\mathrm{N},^1_1\mathrm{H}}$	1.169 10^{-12}	7.297	0.0053
$Q'_{^{15}_8\mathrm{O}}$	2.815 10^{-13}	1.757	0.0058
$Q'_{^{15}_7\mathrm{N},^1_1\mathrm{H}}$	7.955 10^{-13}	4.965	0.013

Table IV. Table listing the calculated values for Q'_{ik} in units of J and MeV as well as the percent error compared to the actual values.

In Table V we have listed the energy lost to neutrinos for each of the reaction branches, including their total energy output Q_{tot}.

Reaction branch	$\mathbf{Q_{tot}}$ **[MeV]**	$\mathbf{Q_\nu}$**[MeV]**	**Percentage lost**
PPI	26.7303	0.53	1.98%
PPII	26.7303	1.075	4.02%
PPIII	26.7303	6.976	26.10%
CNO	26.7303	1.704	6.37%

Table V. Table listing the percentage of energy lost to neutrinos in the PP chain and CNO cycle, as well as the total energy output in each of the branches. The Q_ν is the total energy from neutrinos in each branch.

B. Result: Energy production

In section II D we showed how we calculated the total energy production in the PP chain and the CNO cycle. In table VI we have listed the calculated values for

the energy production in each reaction branch as well as the total energy produced by all the reactions together.

Reaction branch	ε [J kg^{-1} s^{-1}]
PPI	1.08 10^{-13}
PPII	4.81 10^{-9}
PPIII	1.56 10^{-9}
CNO	5.62 10^{-13}
Total	6.38 10^{-9}

Table VI. Table listing the energy production in the PP chain and CNO cycle, as well as the total energy production, using the core temperature and mass density of the Sun.

In Figure 1 we show the plot of the relative energy production of PPI, PPII, PPIII and CNO cycle. The y-axis shows the relative probability based on the scaling using the total energy ε_{tot}. The x-axis is in logarithmic scale, and the values along the axis is the powers of the temperature scaled by 10^9.

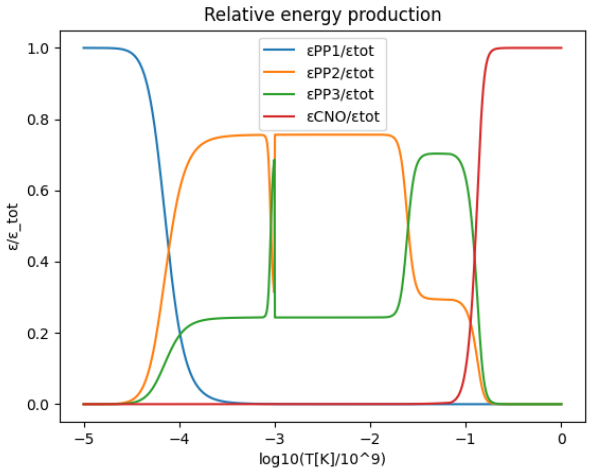

Figure 1. Relative energy production for $T = [10^4, 10^9]K$

In Table VII we have listed the energies at the different Gamow peaks for our reactions.

Figure 2 shows the relative probability of the Gamow peak, using our method in section II D 1. The y-axis is the relative probability, which is found by dividing the Gamow peak for each reaction that varies with energy E, with the sum of all the Gamow peaks for the reaction. The x-axis logarithmic, showing the power of the energies E.

Reaction rate	E [J]
λ_{pp}	9.759 10^{-16}
λ_{pd}	1.074 10^{-15}
λ_{33}	3.544 10^{-15}
λ_{34}	3.7 10^{-15}
$\lambda\prime_{17}$	2.445 10^{-15}
λ_{17}	2.961 10^{-15}
λ_{p12}	3.951 10^{-15}
λ_{p13}	3.958 10^{-15}
λ_{p14}	4.396 10^{-15}
λ_{p15}	4.4 10^{-15}

Table VII. Table listing the energies of the Gamow peaks in the reactions.

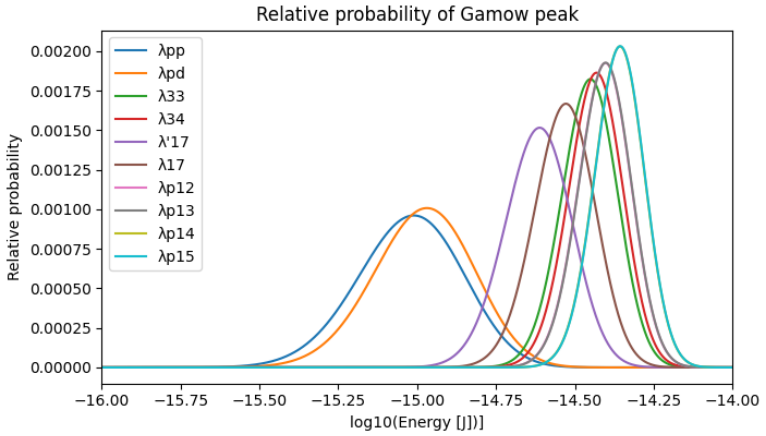

Figure 2. Relative probability of Gamow peak for $E = [10^{-17}, 10^{-13}]J$

IV. DISCUSSION

In Table IV we have listed our results from section II B. We calculated the percent errors to easily spot larger deviations in our results. In all of the reactions we got an error smaller than 0.5% except for three reactions: $Q'_{^7_4\mathrm{Be,e}}$, $Q'_{^8_5\mathrm{B}}$ and $Q'_{^8_4\mathrm{Be}}$.

For the reaction $Q'_{^7_4\mathrm{Be,e}}$, one reason for the larger error could be due to the fact we use an averaged value for the neutrino energy. Since the neutrino can have two different energies, depending on the energy level of the produced ^7_3Li particle, the energy output cannot be calculated directly [1].

For the other two reactions it is more tricky. We have used atomic weight from the same source as the other particles, and followed the same method for calculating the energy output. However, even though they individually have a higher error compared to the actual values, their combined energy output Q'_{decay} has

an error of only 0.016%. Since Q'_{decay} is the one we used in our calculations, we can neglect the large errors of the decays in terms of the use in further calculations.

In Table V we notice that the Q_{tot} for all the branches is the same; 26.7303 MeV. In section II B we mentioned that all of the reaction branches has a total energy output: 26.732 MeV, which is not far from our calculated values. Since we should expect the same values for the total energy output, this indicates that our total energy output in each branch sums up to be somewhat correct, event though we have some larger errors for some of them individually.

As for the percentage of energy lost to neutrinos, we have compared them to known values for the Sun from [7]. Here they list the following neutrino energy loss for the PP chain: PPI : 2%, $PPII$: 4% and $PPIII$: 28.3%. Our values have an error of less than 10%, which we would consider a good result due to our assumptions and other errors from the previous calculation.

Our main goal of the project was to model the energy production in a stellar core. In Table VI we have listed our results for the energy production using the core temperature and density of the Sun. For our perticular model, PPII is the reaction chain with the highest energy production. Our value for PPIII is slightly smaller, while CNO is about a factor of 10^4 smaller and PPI even more so. The low value for the CNO cycle is expected, as we know it is dependent on higher temperatures and do not contribute much for our Sun. The reason PPI has such small values could be due to the reaction rates and how we implemented them. We expect PPIII to have lower values due to the low temperature, but it seems that with our assumptions and reaction rates PPII and PPIII produces the most energy per unit mass.
Something that could be a reason for our values is the mass fractions we assumed to be constant of radius. Also, we have only now looked at the reactions in the core, which is where the energy and density is the highest. This could be why PPII and PPIII are the dominant reaction chains in our model.

Our calculated values in Table VI seems to fit well with our relative energy production plot in Figure 1. Here, the relative energy production for PPI is dominant in the beginning where we have the lowest temperatures, but it quickly drops as the temperature rises closer to the core temperature of the Sun. In the Figure, this temperature is around $log10(T[K]/10^9) = -2$. We also see that the CNO cycle is also close to zero for this temperature, but clearly dominates for $log10(T[K]/10^9) = -0.5$.
In the figure we can see the effect of the upper limit of the electron capture at temperatures below $T = 10^6\ K$, which is at $log10(T[K]/10^9) \approx -3$. Before this point it was a constant reaction rate, but then it "switched over" to the reaction rate that varied more with the temperature. This is why we have a drop in the relative energy production for the PPII branch. And because of this drop, the total energy production gets smaller and therefor the PPIII branch drastically rises in a peak at the same temperature.

PPII, which had the highest energy production seems to be on average higher in the relative energy production in Figure 1, with PPIII being below but not changing as drastically as PPI and CNO. Based on our calculation, or plot seems to work, especially since it features the drop due to the upper limit of the electron capture.

Our final result was the energies and relative probability of the Gamow peak, which we have listed in Table VII and plotted in Figure 2. As mentioned in section I, how effective the reactions are depends on the temperature and density in the core. In the case of our Gamow peaks, they are based on and kinetic energy range $E = [10^{-17}, 10^{-13}]$ which also depends on the temperature and density.
In both the table and the plot we see that the reaction rates of the common steps are most effective at kinetic energies close to 10^{-15}. This makes sense, as the PP chain is responsible for the highest energy production in Sun-like stars, while in contrast the CNO cycle is more effective for higher temperatures and energies. Comparing our plot and table, we see that the reactions rate from the CNO cycle almost overlaps in pairs for the higher energies. λ_{p12} and λ_{p13} have very close Gamow Peaks, and you have to zoom in on the plot in order to spot it. The same goes for λ_{p14} and λ_{p15}. Of the reactions in the PP chain, the rate λ_{34} has the highest relative probability for its Gamow peak. This could maybe explain why PPII and PPIII had such high energy production compared to PPI and CNO, but then the reaction rate λ_{33} in PPI is very close in relative probability.

We have discussed some possible errors in our calculations, and in general we think that the main source of errors could be converting units and finding values for masses and other constants. Comparing with results found in other literary works might not have had the same assumptions, nor used the same values for the masses or constant mass fractions. Our calculations seems to be consistent with our various results, with the results in the tables agreeing with the figures.

As a part of our project we implemented a "sanity check" to check our calculations. Our values in the check is very close to what they should be, and it indicates that our calculations might be overall correct, with some errors along the way.

V. CONCLUSION

Using the the reactions from the PP chain and the CNO cycle, we made a model for the energy production in a stellar core, based on the parameters of our Sun. Our model calculated the energy output of all the reactions, giving us values with mostly low percent errors, as well as percentages of energy lost to neutrinos that fit well with other sources. Our model calculated the highest energy production for PPII and PPIII, and the lowest for PPI and CNO cycle.
From this project I have learned how to combine the different steps in order to calculate energy production, especially that you have to think about the consummation of particles. The most troublesome part was to get the energy output right, with understanding what was included and not included of the energies in the different reactions.

VI. SOFTWARES

- Python 3: programming language. Used for calculating our results and visualizing plots.

VII. REFERENCES

[1] B.V. Gudiksen, 2021, "AST3310: Astrophysical plasma and stellar interiors"
[2] M. Wiescher et. al, 2010, "The Cold and Hot CNO Cycles"
[3] https://solarsystem.nasa.gov/solar-system/sun/overview/
[4] https://astronomy.swin.edu.au/cosmos/t/Triple+Alpha+Process
[5] https://periodictable.com/index.html
[6] Solution to exercise 3.6 in B.V. Gudiksen, 2021, "AST3310: Astrophysical plasma and stellar interiors", page 16. Exercise solved as a part of group session.
[7] https://en.wikipedia.org/wiki/Proton%E2%80%93proton$_chain$
[8] In astronomy it is common to refer to elements heavier than hydrogen and helium as 'metals'.
[9] Percent error $= 100 \cdot |\text{Calculated values} - \text{Actual values}|/\text{Actual values}$