CSE 250B: Section 3 - Sharad Vikram

1. Warmup: Find the eigenvalues and eigenvectors for the following two matrices and write down their spectral decomposition:

   (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

2. Warmup: Prove that the following matrices are PSD or not PSD.

   (a) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

3. Let $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$, and let $A \in \mathbb{R}^{n \times n}$ be the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

   (a) Give an explicit formula for $x^\top Ax$. Write your answer as a sum involving the elements of A and x.

   (b) Show that if A is positive definite, then the entries on the diagonal of A are positive (that is, $a_{ii} > 0$ for all $1 \leq i \leq n$).

4. Let B be a positive semidefinite matrix. Show that $B + \gamma I$ is positive definite for any $\gamma > 0$.

5. The square root of a matrix is defined as follows: matrix B is said to be a square root of A if the matrix product $BB = A$. For a real symmetric positive semidefinite matrix A, find its square root B.

6. Multivariate Gaussian

   (a) **True/False** If X_1 and X_2 are both normally distributed and independent, then (X_1, X_2) must have multivariate normal distribution.

   (b) **True/False** If (X_1, X_2) has multivariate normal distribution, then X_1 and X_2 are independent.

   c) **Challenge:** Transforming a Standard Normal Multivariate Gaussian
   We are given a 2 dimensional Multivariate Gaussian random variable Z, with mean 0 and covariance I. We want to transform this Gaussian into something cooler. Find the covariance matrix of a Multivariate Gaussian such that the axes x_1 and x_2 of the isocontours of the density are elliptically shaped with major/minor axis lengths in a 4:3 ratio, and the axes are rotated 45 degrees counterclockwise.