CSE 250B: Section 3 - Sharad Vikram

1. Warmup: Find the eigenvalues and eigenvectors for the following two matrices and write down their spectral decomposition:

   (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

   Solution: Solving $|A - \lambda I| = 0$:

   $$\begin{aligned} (1-\lambda)(1-\lambda) - 4 &= 0 \\ \lambda^2 - 2\lambda + 1 - 4 &= 0 \\ \lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda+1)(\lambda-3) &= 0 \end{aligned}$$

   We get two eigenvalues, $\lambda = -1, \lambda = 3$. Now we solve for the eigenvectors with the characteristic equation.

   $$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   For $\lambda = -1$, we get an eigenvector of the form $x_1 = -x_2$. For $\lambda = 3$, we get an eigenvector of the form $x_1 = x_2$.
   Thus the spectral decomposition is:

   $$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

   (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

   Solution: Solving $|A - \lambda I| = 0$:

   $$\begin{aligned} (1-\lambda)(1-\lambda) - 1 &= 0 \\ \lambda^2 - 2\lambda + 1 - 1 &= 0 \\ \lambda^2 - 2\lambda &= 0 \\ \lambda(\lambda-2) &= 0 \end{aligned}$$

   We get two eigenvalues, $\lambda = 0, \lambda = 2$. Now we solve for the eigenvectors with the characteristic equation.

   $$\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   For $\lambda = 0$, we get an eigenvector of the form $x_1 = x_2$. For $\lambda = 2$, we get an eigenvector of the form $x_1 = -x_2$.

Thus the spectral decomposition is:

$$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

2. Warmup: Prove that the following matrices are PSD or not PSD.

(a) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

Solution: This is not PSD because it is not symmetric.

(b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Solution: Solution 1: Look at the eigenvalues from the previous problem. None are negative, so this matrix is PSD. Solution 2: Recall the definition of PSD. $\forall x\ x^T Ax \geq 0$.

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \tag{1}$$

$$x_1^2 - 2x_1x_2 + x_2^2 \geq 0 \tag{2}$$

$$(x_1 - x_2)^2 \geq 0 \tag{3}$$

This is a squared term so this matrix is PSD.

(c) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

Solution: To show a matrix is not PSD, we can simply show an x such that $x^T Ax < 0$. For example, $x = \begin{bmatrix} 1 & 1 \end{bmatrix}$ works.

3. Let $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$, and let $A \in \mathbb{R}^{n \times n}$ be the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

(a) Give an explicit formula for $x^\top Ax$. Write your answer as a sum involving the elements of A and x.

(b) Show that if A is positive definite, then the entries on the diagonal of A are positive (that is, $a_{ii} > 0$ for all $1 \leq i \leq n$).

Solution:

(a)
$$x^\top Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

(b) Let $i \in [1, n]$, and let e_i be the i^{th} standard basis vector (that is, the vector of all zeros except for a single 1 in the i^{th} position). Then, by the positive definiteness of A, we have $e_i^\top Ae_i = a_{ii} > 0$.

4. Let B be a positive semidefinite matrix. Show that $B + \gamma I$ is positive definite for any $\gamma > 0$.

Solution: Let $x \neq 0$. Then

$$\begin{aligned} x^\top(B+\gamma I)x &= x^\top Bx + x^\top \gamma Ix \\ &= x^\top Bx + \gamma\|x\|^2 \\ &> 0 \end{aligned}$$

because $x^\top Bx \geq 0$ (since B is positive semidefinite) and $\|x\|^2 > 0$ (because $x \neq 0$). Hence $B + \gamma I$ is positive definite.

5. The square root of a matrix is defined as follows: matrix B is said to be a square root of A if the matrix product $BB = A$. For a real symmetric positive semidefinite matrix A, find its square root B.

Solution:

- For real symmetric matrix A, there exists a real orthogonal matrix Q such that $A = Q\Lambda Q^T$. This is according to spectral theorem. Λ is the diagonal matrix having all real eigenvalues and the columns of Q are eigenvectors of A.
- All eigenvalues of positive semidefinite matrix are non-negative.

If A is a real symmetric matrix, then there exists an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$. $\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$ is the diagonal matrix composed with all eigenvalues $\lambda_i \geq 0$ and $QQ^T = I$.

Let $B = Q\begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}Q^T = Q\Sigma Q^T$

where $\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{pmatrix}$

Then we have

$$BB = (Q\Sigma Q^T)(Q\Sigma Q^T) = Q\Sigma(Q^T Q)\Sigma Q^T = Q\Sigma I\Sigma Q^T = Q\Lambda Q^T = A$$

So B is the square root of A.

6. Multivariate Gaussian

(a) **True/False** If X_1 and X_2 are both normally distributed and independent, then (X_1, X_2) must have multivariate normal distribution.

Solution: True. (X_1, X_2) will be bivariate-normally distributed with a diagonal covariance matrix.

(b) **True/False** If (X_1, X_2) has multivariate normal distribution, then X_1 and X_2 are independent.

Solution: False. If the off diagonal elements of the covariance matrix Σ are not zeros, it means $cov(X_1, X_2) \neq 0$. Then they are not independent.

c) **Challenge:** Transforming a Standard Normal Multivariate Gaussian

We are given a 2 dimensional Multivariate Gaussian random variable Z, with mean 0 and covariance I. We want to transform this Gaussian into something cooler. Find the covariance matrix of a Multivariate Gaussian such that the axes x_1 and x_2 of the isocontours of the density are elliptically shaped with major/minor axis lengths in a 4:3 ratio, and the axes are rotated 45 degrees counterclockwise.

Solution:

Recall that any symmetric matrix Σ can be decomposed as $U\Lambda U^T$, where U is an orthogonal matrix of eigenvectors and Λ is a diagonal matrix of corresponding eigenvalues. Also recall that the columns of U are the directions of the ellipsoid axes and the values of $\Lambda^{\frac{1}{2}}$ correspond to the length of those axes.

(a) First, we find Λ. Recall that multiplying a diagonal matrix D to Z will scale the variances by the squares of the diagonal (the new covariance matrix of DZ is $DID^T = D^2$). The lengths of the axes of the ellipsoid are proportional to the **standard deviation** of each individual component. So, in order to scale by 3 and 4, we simply create the matrix:

$$\Lambda = \begin{pmatrix} 3^2 & 0 \\ 0 & 4^2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}$$

Note that we are finding the spectral decomposition. In order to achieve this type of scaling we would multiply Z by $\Lambda^{\frac{1}{2}}$.

(b) Next we need to find a rotation matrix U such that it rotates the standard cartesian coordinate system 45 degrees counter clockwise. There are 2 ways to do this:

(1) Remember that a rotation matrix has the form:

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Plugging in $\theta = \frac{\pi}{4}$ gives us

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(2) Another way to do this is to realize that we want e_1 to be rotate 45 degrees counterclockwise. Writing that out mathematically, we have

$$U * \binom{1}{0} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \binom{1}{0} = \binom{u_{11}}{u_{21}} = \frac{1}{\sqrt{2}} \binom{1}{1}$$

where the far right hand side is the coordinates of rotating e_1 45 degrees counterclockwise on the unit circle. Doing the same for e_2 gives us the same result as the first method.

(c) Finally, we simply multiply out to find the new covariance matrix.

$$\Sigma = U\Lambda U^T = \frac{1}{2} \begin{pmatrix} 25 & -7 \\ -7 & 25 \end{pmatrix}$$