Please submit answers to at least ten of the questions, if you submit more, the best ten will be counted.

Throughout these questions, G is a finite group. All representations are over a field algebraically closed of characteristic zero, which you may (and sometimes will need to) take to be $\mathbb{C}$.

1. Compute the character table of A_4, the alternating group on a four element set.

2. Prove that a group G is abelian if and only if all irreps are one dimensional.

3. For λ a linear character of G, V is irreducible if and only if $V \otimes \lambda$ is.

4. Show that $V \otimes V^*$ is never irreducible if $\dim_k(V) \geq 2$.

5. Let G be a non-abelian group of order 8. Compute the character table of G without using the classification of nonabelian groups of order 8, showing that the character table does not determine the group. (Hint: Can $G/Z(G)$ be cyclic?)

6. For N a normal subgroup of G, G acts naturally acts on the category $RepN$, by $F_g(\rho)(n) = \rho(gng^{-1})$, where we identify a representation with the map $\rho : N \to Aut(V)$. This descends to an action of G on the set of isomorphism classes of representations of N. Show that:

   - This action factors through G/N.
   - If V is an irrep of G, then the restriction of V to a representation of N decomposes into a single G/N orbit of N irreducibles, with equal multiplicity. That is:

   $$Res_N^G \cong (\bigoplus_i V_i)^e$$

   where the V_i are a single orbit under the action of G/N.

7. Compute the character table of D_n, the dihedral group of order $2n$. That is, describe its irreps and character values on those irreps. (Hint: Think geometrically about this group).

8. Explain how to answer the following questions about G from its character table:

   - What is the order of G?
   - Is G abelian?
   - Is the centre of G trivial?
   - Is G simple?
   - Is G perfect?

9. Show that for a finite abelian group A, the group of linear characters, $\hat{A}$ under $\otimes$ is (noncanonically) isomorphic to A, but we have a canonical isomorphism $A \cong \hat{\hat{A}}$.

10. * For a G set X, show that

$$V_X \cong \mathbf{1} \oplus U$$

with U irreducible iff G acts 2 transitively on X.

11. * Compute the character table of the finite group $Aff_1(\mathbb{F}_q)$, the group of automorphisms of the affine line over $\mathbb{F}_q$. Explicitly, this is maps of the form $x \mapsto ax + b$ for $a \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$.(Hint: The irreps are either very small or very large.)

12. * Show that is V is a faithful irrep of G, then every irrep can be found within $V^{\otimes n}$ for some n.

13. * Let A be an abelian subgroup of G. Show that $\dim_k(V) \leq [G : A]$ for all irreps V of G. (Hint: Induction)

14. ** Let A be an abelian subgroup of G, and V an irrep of G with $\dim_k(V) = [G : A]$. Show that G contains a normal abelian subgroup. (Hint: Determine $|\chi_V(g)|$ for $g \in A$, and $g \notin A$).