In this note we will look at functors between representation categories of different groups. For instance, this gives us a way to understand the representation theory and characters of a group by understanding its subgroups.

We will start abstract and get progressively more concrete. Those only interested in characters, restriction, and induction should skip ahead a bit and prove the adjunction on the character level, or in the special case of group algebras.

So to start, we let $\phi : R \to S$ be a map of (not necessarily commutative) rings. Using ϕ, we can define a pair of adjoint functors between the categories $R - Mod$ and $S - Mod$.

Definition 1. *For $\phi : R \to S$, we define the functor $\phi_* : S - Mod \to R - mod$ on objects by $\phi_* M = M$ as an abelian group, with R action given by $r.m = \phi(r).m$ for $r \in R$, $m \in M$. Morphisms of S modules are naturally morphisms of R modules with this R module structure, so giving a functor.*

This functor is known as restriction of scalars when $R \to S$ is an inclusion, and in general corresponds intuitively to "viewing M as an R module via ϕ". For example, we can view any R module as a $\mathbb{Z}$ module via ϕ_* for the canonical map $\phi : \mathbb{Z} \to R$. This functor clearly preserves direct sums of modules, and when R and S are commutative, also preserves tensor products. The following two exercises can help give some familiarity with ϕ_*.
Exercise: For an inclusion of fields $i : K \to L$ of finite degree, and a finite dimensional L module V of dimension n, what is the K dimension of $i_* V$?
Exercise: Show that if M is a finitely generated S module, then $\phi_* M$ need not be finitely generated as an R module.
This ϕ_* has a left adjoint, denoted ϕ^*, a functor $R - Mod \to S - Mod$. By uniqueness of adjoints, this determines ϕ^* uniquely up to unique isomorphism if it exists, but we still need to construct it.

Definition 2. *For an R module N, we define $\phi^* N$ an S module first as an abelian group as $S \otimes_{\mathbb{Z}} N$ modded out by the subgroup generated by elements of the form $s\phi(r) \otimes n - s \otimes r.n$. The S action is given on the first coordinate, so $s.(s' \otimes n) = ss' \otimes n$.*

Working a bit less formally, elements of $\phi^* N$ are formal sums of $s \otimes n$, but where elements of R are allows to "cross the bridge" of the tensor symbol, so $s\phi(r) \otimes n = s \otimes r.n$ in this module. The following exercises can give some familiarity with ϕ^*.
Exercise: Show that if N is a free R module, $\phi^* N$ is also free, on the same basis.
Exercise: Show that if N is finitely generated as an R module, then $\phi^* N$ is finitely generated as an S module.
Exercise: Show that for the projection map $\pi : \mathbb{Z} \to \mathbb{F}_p$, we have $\pi^* \mathbb{Z}/q\mathbb{Z} = 0$ if q is coprime to p.

Proposition 1. *For a map of rings $\phi : R \to S$, the functors ϕ^* and ϕ_* are an* *adjoint pair:*

$$R - Mod \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} S - Mod \ .$$

That is, we have an isomorphism, natural in both variables:

$$\hom_{S-Mod}(\phi^* N, M) \cong \hom_{R-Mod}(N, \phi_* M).$$

Proof. Let $f : \phi^* N \to M$ be an S module morphism. We will obtain the mate of f by first applying ϕ_* to f, that is, viewing it as a morphism of R modules via ϕ, then precomposing with the map $N \to \phi_* \phi^* N$ given as $n \to 1 \otimes n$. That is, the composition $N \to \phi_* \phi^* N \to \phi_* M$ is the desired map $N \to \phi_* M$.

To give the mate of $g : N \to \phi_* M$, we define an S module morphism $\phi^* N \to M$ by $s \otimes n \to s.g(n)$. We leave it to the reader to verify that these are inverse bijections, natural in N and M. $\square$

This is the general case, so now we will specialise to $\phi : k[H] \to k[G]$ for ϕ induced from a group homomorphism $\phi : H \to G$. Looking first at ϕ_*, we see that $\phi_* V$ for V a representation of G is just given by composing $H \xrightarrow{\phi} G \to Aut(V)$. So for characters, this is just pulling back a class function on G to a class function on H.

Note that from this perspective, we see that ϕ_* is not only additive, but also multiplicative on tensor products, inducing a ring homomorphism $\phi_* : Char(G) \to Char(H)$. One can also see directly from the construction of tensor products that $\phi_*(V \otimes W) \cong \phi_* V \otimes \phi_* W$, as well as duals.

In this setting, we have two extremes, when our map ϕ is either injective, or surjective. The injective situation is more familiar, and more important, so we will only briefly treat the case of ϕ surjective. In the case of ϕ_*, this takes representations of G/N and treats them as representations of G.

Exercise: Show that if $\pi : G \to G/N$ is sujective, then $\pi_* W$ is an irreducible representation of G if and only if W is an irreducible representation of G/N.

In the more usual setting of an inclusion $H \to G$, we denote this functor by Res^G_H, the restriction of a representation of G to a representation of H. We have seen that the induced map $Res^G_H : R[G] \to R[H]$ is a ring homomorphism, so lets work out what this is for the inclusion $e \to G$ of the trivial group. In this case, $R[H] = R[e] = \mathbb{Z}$, generated by the one dimensional trivial representation. So we have a canonical map $R[G] \to \mathbb{Z}$. This map is given on representations by taking the dimension. To see this, note that Res^G_H preserves dimension for all G, H, and representations of e are determined by their dimension.

Now we turn to our other functor ϕ^*, which is somewhat more involved to define. We leave the surjective case as an exercise, dealing with the more important subgroup inclusion $H < G$.

Exercise: Show that for $\pi : G \to G/N$ the projection, V a representation of G, we have $\pi^* V \cong V_N := V/n.v - v$, the coinvariants of N on V.

So now lets construct $\phi^* V$ for $H < G$ a subgroup inclusion, which we denote $Ind^G_H V$. Note that our construction may look somewhat arbitrary, but once we show it is isomorphic to $\phi^* V$, we can rest easy by the uniqueness of adjoints.

Definition 3. *For a subgroup inclusion $H < G$, and W a representation of H, a_i a fixed set of left coset representatives, we define the induced representation $Ind_H^G W$ first as a vector space:*

$$Ind_H^G = \bigoplus_i (a_i, W)$$

And we define the action of $g \in G$ on (a_i, w) by first noting that $ga_i = a_j h$ for a unique a_j, and $h \in H$. So then we define $g.(a_i, w) = (a_j, h.w)$. We then extend by linearity.

To see that this construction is isomorphic to the $\phi^* W$ defined earlier, note that a_i is a basis of $k[G]$ as a right $k[H]$ module, so we have a natural isomorphism $\phi^* W \to Ind_H^G W$ mapping $a_i h \otimes w \to (a_i, h.w)$.
Exercise: Show that Ind_H^G is not a ring homomorphism unless $H = G$.
Important exercise: Show that for $\mathbf{1}$ the trivial representation of e, we have $Ind_e^G(\mathbf{1}) \cong k[G]$.
Lets determine the character of $Ind_H^G V$ in terms of the character of V. Note that on $Ind_H^G V$, each element $g \in G$ permutes the blocks (a_i, W), so viewing this as a matrix, we see the diagonal entries will only come from blocks that are fixed. What does it mean for g to fix the block (a_i, W)? This is to say that $ga_i = a_i h$ for some $h \in H$. So we see that the character of $Ind_H^G V$ on g is the sum of $\chi_V(a_i^{-1} g a_i)$ over a choice of coset representatives a_i, where we set $\chi_V(k) = 0$ if $k \notin H$.
So we define induction of class functions by precisely this formula, for ψ a class function on H, we have:

$$Ind_H^G \psi(g) = \sum_i \psi(a^{-1} g a).$$

Exercise: Show directly (without representations) that this formula turns class functions on H into class functions on G.
Now we will use our abstract nonsense, the adjunction of proposition 1. In our setting of $H \xrightarrow{i} G$, since $Ind_H^G = i^*$, and $Res_H^G = i_*$, this asserts the existence of a canonical isomorphism:

$$hom_G(Ind_H^G V, W) \cong hom_H(V, Res_H^G W).$$

In view of our inner product on $Char(G) \cong R[G] \otimes k$, we have the following theorem, known as Frobenius reciprocity.

Theorem 1. *For V a rep of H, W a rep of G, we have:*

$$\langle Ind_H^G V, W \rangle = \langle V, Res_H^G W \rangle$$

Exercise for those who don't like abstract nonsense: Show that this holds from the definitions of restriction and induction of characters.

To be more at peace with this result, we'll sketch what the adjunction isomorphism looks like in terms of representations. So given a G morphism φ from

$Ind_H^G V$ to W, we may restrict it to get a linear map from the summand (e, W) to W, and this map is H linear (since φ was G linear). For the other direction, given a map $\gamma : W \to Res_H^G V$, we define a map from the induced representation by sending (a_i, w) to $a_i.w$. Note that even though $Res_H^G V$ is just a representaion of H, we needed to remember it came from G in order to define this map.
As a (somewhat circular) application of Frobenius reciprocity, lets combine it with the fact that $Ind_e^G \mathbf{1} = k[G]$. We have, for each irrep W of G:

$$\langle k[G], W\rangle = \langle Ind_e^G \mathbf{1}, W\rangle = \langle \mathbf{1}, Res_e^G W\rangle = \dim_k W$$

So since our inner product is symmetric, we see the number of copies of W in $k[G]$ is equal to the dimension of W.
Induction gives us a way of constructing representations and characters of G from subgroups we hopefully understand better, it is a source of characters. We have another source of characters, and representations, given from the connection between G sets and representations of G.

Let $G - Set$ be the category of finite G sets, which we note has a coproduct given by disjoint union, and a product given by the setwise product of G sets with diagonal action. We observe that every finite G set breaks up uniquely into orbits, and each orbit is a transitive G set (these are the analogue of our irreps). So each G set X is uniquely determined up to isomorphism by its multiplicities of its isomorphism classes of orbits.

So in view of this, we can define the Burnside ring $B[G]$ of G to be the ring generated by isomorphism classes of finite G sets, with addition given by disjoint union, and multiplication given by product. Note we want an actual ring, so we have to include negatives, but they don't really make much of a difference, and shouldn't be worried over.

Definition 4. *For a finite G set X, we define the representation V_X of G to have basis e_x, with action $g.e_x = e_{g.x}$. This induces a functor $G-Set \to Rep(G)$, and this functor preserves the coproduct and monoidal structures on these categories (it turns disjoint unions into direct sums, and products into tensor products), so induces a ring morphism $B[G] \to R[G]$.*

This gives us another method of finding representations and characters of a finite group, by identifying actions of G on sets. However, this map comes with a small downside, for any G set X that has more than one element, V_X is not irreducible. One way to see this is that $\sum_{x\in X} e_x$ is invariant, but lets also give a character theoretic proof of this fact.
First, observe that the character of V_X is given by $\chi_{V_X}(g) = |X^g|$, the number of points of X fixed by G. So we see that $\sum_{g\in G} \chi_{V_X}(g) = \sum_{g\in G} |X^g| > 0$. So we see that $\langle \mathbf{1}, \chi_{V_X}\rangle > 0$, so since this is an integer, if X is not a singleton, V_X has at least one copy of the trivial representation within it.
A useful exercise to get used to induction and this G set construction is to show that for the action of G on the cosets G/H, we have a canonical isomorphism of representations $V_{G/H} \cong Ind_H^G \mathbf{1}$.