In this note we will look at some of the examples we covered in class, computing the characters associated to representations of some finite groups G. There is a succint way to organise this data, known as a character table. This is a table with rows indexed by irreducible representations of G, columns by conjugacy classes, with the (χ, C) entry the value of $\chi(g)$ for any g in the conjugacy class C. Lets see our first example, the cyclic group on three elements, e, g, g^2. For simplicity, throughout we will be working over $\mathbb{C}$. I'll apologise upfront for the quality of the tables, using latex is not a strength of mine.

The character table is given as:

Character table of C_3			
	e	g	g^2
1	1	1	1
ζ	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
ζ^2	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$

This is a concrete way of expressing the characters of C_3 in terms of the basis of conjugacy class indicators, which we label by their conjugacy class.

Now lets see how we actually obtained these representations, now that we understand the format. We know immeadiately that we have the trivial representation, where every element has trace 1, which gives us our first row. To find another representation, we note that C_3 naturally acts on a one dimensional complex vector space as the root of unity $e^{\frac{2\pi i}{3}}$, and this gives us our second row, the representation we call ζ. From these two, we see that the dual of ζ is another, distinct irreducible representation, which we could have constructed directly by choosing the action of $g \in C_3$ on $\mathbb{C}$ to be multiplication by the "other" cube root of unity. Since $\zeta \otimes \zeta$ is isomorphic to ζ, we denote this representation by ζ^2.

This example was almost too simple, so lets look at a more complicated group, to see how some of our general properties of the character table can help us. Before moving on, lets prove the following basic linear algebraic lemma.

Lemma 1. *If we have A an $n \times n$ matrix, D an invertible diagonal matrix, with $ADA^T = Id$, then the columns of A are orthogonal.*

Proof. Inverting the equation gives us $(A^T)^{-1}D^{-1}A^{-1} = \mathrm{Id}$, so since $(A^T)^{-1} = (A^{-1})^T$, multiplying on either side gives $A^T A = D^{-1}$, which is exactly the claim that the columns are orthogonal. □

So in our case, lets check that for the diagonal matrix with entries $(\frac{|C_e|}{|G|}, \frac{|C_1|}{|G|}, \dots \frac{|C_n|}{|G|})$, the conditions of the lemma hold, where C_i are our conjugacy classes. So letting $\chi_V(C)$ denote the value of χ_V on the conjugacy class C we need to check that the sum:

$$\sum_{C_i} \frac{|C_i|}{|G|} \chi_V(C_i) \chi_W(C_i)$$

is equal to 1 if the irreps are isomorphic, and 0 else. But this is exactly the expression that the characters are orthonormal, which we showed in the previous note. So the lemma gives us the following column orthogonality relation on the character table.

Corollary 1. *For any two elements $g, h \in G$, we have that the sum*

$$\sum_{\chi} \chi(g)\chi(h)$$

over the set of irreducible characters χ is zero unless $g = h$, where it equals $\frac{|G|}{|C_g|}$, where C_g is the conjugacy class containing g.

Here we are identifying our irreps and characters, and calling a character irreducible if it corresponds to an irreducible representation.

Now lets look at the group S_3, the symmetric group on $\{1, 2, 3\}$. We know that in S_n, the conjugacy classes are determined by their cycle type, so we write representatives at the top to indicate the cycle type. At the moment, we know that S_3 has three conjugacy classes, hence three irreps, and the sum of their dimensions squared equals 6, so they must be $1, 1$ and 2 dimensional. We also know about the trivial representation, so we have the following partially filled in character table.

Character table of S_3			
	e	(12)	(123)
1	1	1	1
??	1		
??	2		

Now we don't have much to work with here, so lets try come up with another representation. We saw in the last note that some methods of finding representations "externally" are:

- Pulling back from irreps of quotients.
- Inducing from subgroups.
- Looking for group actions of G on finite sets.

All three of these will actually work in this case, but lets use the first strategy, noting the surjection $S_3 \to \{1, -1\}$ given by taking the sign of a permutation. We've written the cyclic group of order 2 in this way to show that it naturally acts, nontrivially, on a one dimensional vector space. This is therefore a one dimensional irrep, which we call σ, whose character we can add to our table.

Character table of S_3			
	e	(12)	(123)
1	1	1	1
σ	1	-1	1
??	2		

So lets now see how can we work "internally" to the data we already have, so find the character of the last irrep. We will use out orthogonality relations, comparing our mystery columns to our known first column, noting that their dot product must be zero. If we call this last irrep S, this gives $\chi_S((12)) = 0$, and $\chi_S((123)) = -1$, so we arrive at the completed table:

Character table of S_3			
	e	(12)	(123)
1	1	1	1
σ	1	-1	1
χ_S	2	0	-1

It is a good exercise to show how to obtain the mystery two dimensional representation using the other two external methods, induction, and group actions.

Lets now look at a trickier group, the symmetric group on 4 symbols, $\{1, 2, 3, 4\}$. Again, we know the conjugacy classes, so we know the number of irreps, and we can identify the trivial and sign irrep, as before, giving our partial table:

Character table of S_4					
	e	(12)	(123)	(12)(34)	(1234)
1	1	1	1	1	1
σ	1	-1	1	1	-1
??					
??					
??					

Now we will use method three of our extrinsic irrep finding methods, we will look at the representation V_X for a natural S_4 set X. We have a natural choice, given that S_4 is given to us as the full permutation group on $\{1, 2, 3, 4\}$, so the associated character ψ is given on g by the number of fixed points of g on $\{1, 2, 3, 4\}$:

	e	(12)	(123)	(12)(34)	(1234)
ψ	4	2	1	0	0

So lets now check the norm of ψ:

$$\langle \psi, \psi \rangle = \frac{1}{24} = \sum_{g \in S_4} |\psi(g)|^2 = \frac{1}{24}(4^2 * 1 + 2^2 * 6 + 1^2 * 8) = 2$$

Remember when we compute the norm of an irrep from the character table, we need to include the size of each conguacy class, in our case, these are 1, 6, and 8. If one wants to do serious computation with a character table, it is useful to have these numbers on hand.

So from this computation, we see that ψ is the sum of precisely two non-isomorphic irreps, by Schurs lemma and orthonormality of irreps. But we know one of these irreducible constituents already, its a trivial representation, since the element $e_1 + e_2 + e_3 + e_4$ is invariant in V_X. (Alternatively, compute $\langle \mathbf{1}, \psi \rangle = 1$).

So the character $\chi_T := \psi - \mathbf{1}$ is irreducible, and so is $\chi_T \otimes \sigma$ (by the assignment problem). So we have two more irreps, and our table looks like:

Character table of S_4					
	e	(12)	(123)	(12)(34)	(1234)
$\mathbf{1}$	1	1	1	1	1
σ	1	-1	1	1	-1
??					
χ_T	3	1	0	-1	-1
$\chi_T \otimes \sigma$	3	-1	0	-1	1

From this, we can deduce the final character from orthogonality relations, its dimension is 2, and the other entries follow from column orthogonality compared to the first column. So we finally arrive at the completed character table:

Character table of S_4					
	e	(12)	(123)	(12)(34)	(1234)
$\mathbf{1}$	1	1	1	1	1
σ	1	-1	1	1	-1
χ_S	2	0	-1	2	0
χ_T	3	1	0	-1	-1
$\chi_T \otimes \sigma$	3	-1	0	-1	1

Lets observe some things. First, lets note that the subset $\{e, (12)(34), (13)(24), (14)(23)\}$ can be described as those elements of S_4 which have trace 2 on the representation S. This representation is two dimensional however, and since the trace is the sum of the eigenvalues, which are roots of unity, we see that all these elements must act as the identity on S. That is, these elements form the kernel of the associated map $S_4 \to Aut(S)$, so in particular, they are a normal subgroup of S. So we have determined the *existence* of a normal subgroup of S_4 purely by analysing characters. As a much deeper application of this idea, we have the following theorem, originally due to Frobenius.

Theorem 1. *Let G be a finite group acting on a set X, such that each non-identity element g of G fixes at most 1 element of X. Then the elements of G that don't fix any element of X, along with $\{e\}$ form a normal subgroup K.*

The difficulty of this theorem lies entirely in proving that the the set K is a subgroup, of which the only known proofs use character theory, by building a character which has "kernel" K.