In this note we will look at representations of finite groups, and how the character isomorphism lets us understand them.

So first, recall that a representation of a finite group G is a vector space V with a group homomorphism $\rho : G \to Aut(V)$, and a morphism of representations is a linear map that commutes with the action of G on both vector spaces. These naturally form a category, $Rep(G)$.

Definition 1. *The group algebra $k[G]$ of a group G over the field k is the algebra with basis e_g for $g \in G$, with $e_g e_h = e_{gh}$.*

Proposition 1. *We have a canonical equivalence of categories between left $k[G]$ modules and $Rep(G)$.*

Proof. For any representation of G, we extend the map $\rho : G \to Aut(V)$ to an algebra map $\tilde{\rho} : k[G] \to End(V)$ by linearity, and any algebra map $k[G] \to End(V)$ by restriction induces a group homomorphism $G \to Aut(V)$. One may check that the notions of morphisms in the respective categories correspond also. □

So from here we will identify representations of G and $k[G]$ modules, and simple modules correspond to irreducible representations.

Since we will exclusively be dealing with finite groups, and finite dimensional vector spaces, we will abuse our previous notation slightly and let $Rep(G)$ and $k[G] - Mod$ denote the categories of finite dimensional representations, and finitely generated $k[G]$ modules. We will sometimes refer to irreducible representations as irreps, and simple modules as simples.

Theorem 1 (Maschke)**.** *For G a finite group, k an field of characteristic coprime to $|G|$, we have that $k[G]$ is a reducible $k[G]$ module, and is thus semisimple if k algebraically closed.*

Proof. To show that any submodule $V \subset k[G]$ is G invariant, then we will show the inclusion is split as a map of $k[G]$ modules, so giving a morphism of $k[G]$ modules $\tilde{\pi} : k[G] \to V$ which is the identity on V. The kernel of this projection will then be our complementary $k[G]$ module, and since $k[G]$ is finite dimensional, this process will end up expressing $k[G]$ as a direct sum of simple submodules. So in order to come up with a G invariant projection onto V, pick any k vector space projection $\pi : k[G] \to V$, and construct $\tilde{\pi}$ as:

$$\tilde{\pi} = \sum_{g \in G} g\pi(g^{-1}).$$

So $\tilde{\pi}(v) = \sum_{g \in G} g.\pi(g^{-1}v)$. This is a morphism of $k[G]$ modules, since

$$\tilde{\pi}(h.v) = \sum_{g \in G} g.\pi(g^{-1}h.v) = \sum_{g \in G} hh^{-1}g.\pi(g^{-1}h.v) = h \sum_{k \in G} k.\pi(k^{-1}v) = h.\tilde{\pi}(v).$$

□

From here, we will assume that k is algebraically closed of characteristic coprime to $|G|$. In this setting, our previous theory tells us that $k[G]$ is a sum of matrix algebras, and we have one irreducible representation for each matrix factor.

Example 1. *As a small application of this, we can tell how many irreducible representations there are of S_3, and what their dimensions are. We know that there are finitely many irreducibles, and the sum $\sum_i dim(V_i)^2 = 6$. Since S_3 is not abelian, $k[S_3]$ is not abelian, so one of our matrix algebras must be of dimension > 1. This forces the numbers to be $1, 1, 2$, so from this we read that S_3 has three isomorphism classes of irreps, and has a normal subgroup, since it has a nonzero abelian quotient (the image of S_3 in the nontrivial one dimensional rep).*

We can now identify the characters of $k[G]$ as k functions on G, that are constant on conjugacy classes.

Proposition 2. *Characters on $k[G]$ can be identified with k functions on G, that are constant on conjugacy classes.*

Proof. The dual of $k[G]$ is naturally identified with k functions on G, and the condition that $\chi(ab) = \chi(ba)$ can be verified on the basis e_g, where it becomes $\chi(aba^{-1}) = \chi(b)$. □

Corollary 1. *We have a natural basis of $Char(G)$, that of the indicator functions $\mathbf{1}_C$ on conjugacy classes, so since χ is an isomorphism, we have the number of irreps of G is equal to the number of conjugacy classes.*

So we are in the unusual sitation of having a vector space $R[G]\otimes k$, with two, completely distinct, natural bases, that of irreducible characters, and that of conjugacy class indicators. To a large extent, the comparison between these two bases is responsible for the usefulness of character theory in studying properties of G. Many phenomena that are opaque in one basis can be understood through the other, and the map χ respects a great deal of structure on $Char(G)$.

Lets look at some of the structures preserved by χ. Since we know that χ is an isomorphism, we should explain what we mean by this. We can show that $R[G] \otimes k$ and $Char(G)$ have a bounty of natural structures idependent of one another, and the map χ respects them. It is interesting that most of these structures are present on the free abelian groups $R[G]$ and $\mathbb{Z}[\mathbf{1}_C]$, but there is no map between these in general (over $\mathbb{Z}$).

The easiest, that we have already implicitly dealt with, is addition. Characters can be added, and representations can be "added" by direct sum, and the map χ respects this.

Next, we may endow both $Char(G)$ and $R[G]$ with a canonical linear involution, which is preserved by χ. On the representation side, this is given by taking the dual of a representation V.

Definition 2. *For a representation V of G, we define the dual representation as V^*, with the action of G on functionals given by $g.\phi(v) = \phi(g^{-1}.v)$. This* *yields a contravariant functor* $* : Rep(G) \to Rep(G)$*, and* $** \cong Id_{Rep(G)}$*, just like the usual vector space case.*

On the character side, this is taking $\chi^*(g) = \chi(g^{-1})$. Note that in the case of $k = \mathbb{C}$, this is just taking the complex conjugate of χ, since the eigenvalues of the inverse matrix $\rho(g)^{-1}$ are the inverses of the eigenvalues of $\rho(g)$, and all these eigenvalues lie on the unit circle, since g has finite order.

Proposition 3. *The map* χ *respects this, in that* $\chi_{V^*} = \chi_V^*$*.*

Proof. There are many ways to see this, one method is to note that we only need to verify this as a representation of the cyclic group $< g >$, and since this is abelian, all irreps are one dimensional (think of matrix algebras), and in the case of a one dimensional irrep, the formula is clear. (alternatively, pick bases and compute) □

Next, we endow both $R[G]$ and $Char(G)$ with natural ring structures. Lets look at $Char(G)$ first, as the ring structure is simpler. Here, we just take pointwise multiplication of functions, $(\chi.\psi)(g) = \chi(g)\psi(g)$. This clearly results in another character on G, and we see that $\mathbb{1}_C$ are a very natural basis of idempotents in this ring.

We will define the ring structure on $R[G]$ as the decategorification of a tensor product on $Rep(G)$.

Definition 3. *For* G *representations* V, W*, we define the structure of a representation on* $V \otimes_k W$ *by*

$$g.(v \otimes w) = g.v \otimes g.w$$

We will write the representation as $V \otimes W$*, and note that this constructs a billinear functor* $\otimes : Rep(G) \times Rep(G) \to Rep(G)$*.*

Be aware, this formula is how we define the action of elements of g on basis elements of $V \otimes W$, and we extend by linearity in $k[G]$, so we do *not* have $x.(v \otimes w) = x.v \otimes x.w$ for arbitrary $x \in k[G]$.

So now, since this functor is bilinear, on isomorphism classes of representations, we can define $[V].[W] := [V \otimes W]$, and this gives the structure of a ring to $R[G]$, with unit the trivial representation.

Proposition 4. *The map* χ *identifies these ring structures, in that* $\chi_{V \otimes W} = \chi_V . \chi_W$*.*

Proof. This follows at once from the fact that $\rho_{V \otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$, coupled with the fact that $Tr(\phi \otimes \psi) = Tr(\phi).Tr(\psi)$ for any linear maps ϕ, ψ. If you haven't seen this linear algebra fact before, you should prove it yourself, eg, by basis computations or eigenvalue density arguments. □

Our last piece of structure is that of a nondegenerate sesquilinear form on the vector space $Char(G)$ and $R[G]$ (here we interpret sesquilinear with respect to the canonical involution defined earlier). This all works over arbitrary fields, but for simplicity, we will be working over $\mathbb{C}$, and our form will become a hermitian inner product.

Lets first define this for characters. In this case, each character is a function on the discrete space $|G|$, with values in $\mathbb{C}$, so we take the (normalised) L^2 inner product of functions:

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g)\overline{\chi(g)}.$$

We see that with respect to this, our basis of conjugacy classes are orthogonal, though not orthonormal if G is not abelian.

Now for representations, we will again work on the categorical level, and decategorify to see what we get in $R[G]$. For this, we introduce another structure on represenations, an *internal* hom functor.

Definition 4. *For G representations V, W, we define the structure of a representation on $Hom_k(V, W)$ by*

$$g.\phi(v) = g.\phi(g^{-1}.v)$$

This representation is the internal hom of V and W, denoted $Hom(V, W)$.

We have an important distinction to make here, we have two distinct functors which we are calling hom, we have $\hom_G(V, W)$, the morphisms between V and W in the category $Rep(G)$, which is *not* a representation of G, and the internal hom functor $\mathrm{Hom}(V, W)$ which is a G representation. This is somewhat confusing, but there is a simple relationship between these vector spaces.

Proposition 5. *The G invariants of $Hom(V, W)$ are exactly the G morphisms between V and W. That is:*

$$Hom(V, W)^G \cong \hom_G(V, W)$$

Proof. A k linear map ϕ between representations (as k vector spaces only) is a morphism of representations if and only if $g.\phi(g^{-1.v}) = \phi(v)$ for all $g \in G$, which is to say a G invariant of $\mathrm{Hom}(V, W)$. $\square$

Before going on, lets note that this Hom functor is actually not anything new, in that we have a canonical isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$ of G representations. To see this, simply note that natural map of vector spaces

$$V^* \otimes W \to \mathrm{Hom}(V, W)$$

$$\phi \otimes w \mapsto v \to \phi(v)w$$

Is an isomorphism of vector spaces by counting dimensions, and respects our G action.

Just like vector spaces, we have a tensor hom adjunction, now with G actions present everywhere.

Proposition 6. *For any V, W, U representations of G, we have a canonical isomorphism of representations:*

$$Hom(V \otimes W, U) \cong Hom(V, Hom(W, U))$$

And by taking invariants, we get the usual tensor hom adjunction:

$$\hom_G(V \otimes W, U) \cong \hom_G(V, Hom(W, U))$$

Proof. We leave this to the reader, as it is a simple verification that the usual tensor hom adjunction respects the G action everywhere. $\square$

So now we define our inner product on $R[G]$ as

$$\langle [V], [W] \rangle = \dim_k \hom_G(V, W).$$

By Schur's lemma, this is equivalent to setting the irreps as an orthonormal basis of $R[G]$. Lets show now that the map χ identifies these inner products, it is an isometry.

Proposition 7. *For all G representations V, W, we have that:*

$$\langle V, W \rangle = \langle \chi_V, \chi_W \rangle.$$

Proof. Lets first show that this holds when V is the trivial representation $\mathbf{1}$. In this case, we want to identify the number $\frac{1}{|G|}\sum_{g \in G} \chi_W(g)$ with the number of irreducible summands of W isomorphic to $\mathbf{1}$, that is, the dimension of the G invariants of W. So note that the element $T = \frac{1}{|G|}\sum_{g \in G} e_g$ in $k[G]$ is an idempotent element, so its image in $\mathrm{End}(W)$ is also idempotent. Also observe that any element of W in the image of $\rho_W(T)$ is invariant, and $\rho_W(T)$ fixes these. So we may identify $\rho_W(T)$ with the projection morphism onto the G invariants of W. Now by picking a suitable basis, we see for any idempotent linear map $T : V \to V$, we have that the trace of T is equal to the dimension of $im(T)$.
So putting this all together, we have:

$$\langle \mathbf{1}, W \rangle = \dim_k W^G = Tr(\rho(T)) = Tr(\frac{1}{|G|}\sum_{g \in G} \rho(g)) = \frac{1}{|G|}\sum_{g \in G} \chi_W(g) = \langle \chi_{\mathbf{1}}, \chi_W \rangle$$

Now for the general case, we will use our adjunctions. We have from our adjunction that:

$$\hom_G(V, W) \cong \hom_G(\mathbf{1}, \mathrm{Hom}(V, W))$$

Aswell as

$$\mathrm{Hom}(V, W) \cong V^* \otimes W$$

Since we already established that χ was a ring morphism, we have that:

$$\langle V, W \rangle = \langle \mathbf{1}, V^* \otimes W \rangle = \langle \chi_{\mathbf{1}}, \chi_{V^* \otimes W} \rangle = \langle \chi_{\mathbf{1}}, \chi_V^* \cdot \chi_W \rangle = \langle \chi_V, \chi_W \rangle$$

Where this last equality is immeadiate from the definition of the inner product on characters.

$\square$