Now we are going to work through a more difficult example, in fact an infinite family of examples. The group we will be considering is the 3×3 heisenberg group H over $\mathbb{F}_q$, so matrices of the following form:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Where a, b and c are in $\mathbb{F}_q$. Before finding the representations of this group, we will need to fully understand it, so lets write down the multiplication law, and see what properties we can deduce.

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & c+ab'+c' \\ 0 & 1 & b+b' \\ 0 & 0 & 1 \end{pmatrix}$$

First, we see that when $a = b = 0$, we have a central element, so we call the subgroup of elements of this form Z, which is abstractly isomorphic to $\mathbb{F}_q$ under addition. It turns out that this is the whole centre of H, though we won't be using that fact.

$$Z = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We also see that mapping an element onto the pair (a, b) is a surjective group homomorphism onto $\mathbb{F}_q^2$. So our group H fits into a short exact sequence

$$0 \to Z \to H \to \mathbb{F}_q^2 \to 0.$$

Now we are going to count the number of conjugacy classes in H. Lets now look at the conjugation equation:

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & b-a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a' & b-a-b(a+a')+ab'+c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}$$

From this, we see that for fixed $(a, b) \neq (0, 0)$, all values of c give rise to the same conjugacy class. So counting the number of conjugacy classes, we see there are $q^2 - 1$ corresponding to nonzero (a, b) values, and q corresponding to the elements of Z, since Z is abelian and central. So our total number of conjuagcy classes is $q^2 + q - 1$.

Now lets find $q^2 + q - 1$ irreducibles. We see from the surjection to $\mathbb{F}_q^2$, we have q^2 linear (one dimensional) irreps, by pulling these back to H. (Recall what irreps of abelian groups look like, hopefully you've done that assignment question by now).Furthermore, we see that for each of these linear characters λ, every element z in Z acts as the identity (since its in the kernel of the map to $GL(V)$, by construction). This gives that

$$\langle \mathbf{1}, Res_Z^H(\lambda) \rangle = 1$$

For each of our q^2 linear characters pulled back from $\mathbb{F}_q^2$. But Frobenius reciprocity, this yields

$$\langle Ind_Z^H \mathbf{1}, \lambda \rangle = 1.$$

So now note that the index of Z in H is q^2, so the representation $Ind_Z^H \mathbf{1}$ is q^2 dimensional. So now we've found q^2 distinct one dimensional representations of H lying inside a q^2 dimensional representation, so we've found every irrep inside $Ind_Z^H \mathbf{1}$, $Ind_Z^H \mathbf{1}$ is isomorphic to the sum of the p^2 λ representations we defined earlier.

Given that inducing the linear $\mathbf{1}$ from Z to H had such a nice structure, lets check what happens when we induce other linear representations of Z. There will be $q - 1$ linear, nontrivial representations to choose from. We claim that for each γ_1, γ_2, irreducible, nontrivial, distinct representations, we have

$$\langle Ind_Z^H \gamma_1, Ind_Z^H \gamma_2 \rangle = 0.$$

By Frobenius reciprocity, we need to check that

$$\langle \gamma_1, Res_Z^H Ind_Z^H \gamma_2 \rangle = 0.$$

So lets compute what the character of $Res_Z^H Ind_Z^H \gamma_2$ is, from our formula for induced characters:

$$Res_Z^H Ind_Z^H \gamma_2(z) = Ind_Z^H \gamma_2(z) = \sum_{a_i} \gamma_2(a_i z a_i^{-1}).$$

Recall that the a_i are a set of left coset representatives of Z in H, and in the last sum we define $\gamma_2(a_i z a_i^{-1}) = 0$ if $a_i z a_i^{-1} \notin Z$. But in our case, z is central, so commutes with every element, and so this character is just $q^2\gamma_2$, since the number of coset representatives equals the index of Z in H, q^2. Since γ_1 and γ_2 were assume distinct, their inner product is zero, so we obtain our result.

So from this, we know we have q^2 linear irreps, and we have at least one irrep contained in each $Ind_Z^H(\gamma)$ for $\gamma \neq \mathbf{1}$. What the previous result gives us is that each irreducible character of H is contained in $Ind_Z^H(\gamma)$ for a *unique* γ. So in view of our conjugacy class count, we see that each $Ind_Z^H(\gamma)$ must be some multiple of a single irrep, which we will call V_γ. What is the multiplicity of V_γ in $Ind_Z^H(\gamma)$? We have

$$\langle Ind_Z^H(\gamma), V_\gamma \rangle = \langle \gamma, Res_Z^H(V_\gamma) \rangle = \dim_k(V_\gamma)$$

Where this last equality follows from our observation that $Res_Z^H Ind_Z^H \gamma = q^2\gamma$. So counting dimensions, we have $q^2 = \dim_k(V_\gamma)^2$, so V_γ has dimension q.

So we've found all our irreps, we have q^2 linear irreps, pulled back from $\mathbf{F}_q^2$, and we have $q-1$ irreps of dimension q, each of which is the only constituent V_γ of $Ind_Z^H(\gamma)$, where it occurs with multiplicity q.