We will be looking at the representation theory of finite groups, though a large amount of the basic theory follows from more general module/representation theoretic principles. So we will develop these first, and later see what special properties groups give us.

For simplicity, throughout, all our fields k will be algebraically closed, and our algebras finite dimensional over such fields. The results in this section hold with nigh identical proofs slightly more generally, and it is a good exercise to think about what happens when we relax the algebraically closed hypothesis, we may have more exotic coefficients of our matrix algebras, but this is the only real change.

Throughout, A will denote a k algebra, and we will use the terms module and representation interchangably, that is, a representation of A is just an A-module.

Definition 1. *An A-module V is simple if it contains no proper nonzero A submodules.*

We have the following critical result, which is also easy to prove.

Lemma 1 (Schur)**.** *If V, W are simple A-modules, then any A-module morphism between them is either an isomorphism, or zero.*

Proof. If $f : V \to W$, then the kernel of f must be zero or V, and the image must be zero or W by simplicity. □

Corollary 1. *For A finite dimensional over algebraically closed k, for any simple A module V, we have $End_A(V) \cong k$.*

Proof. Schurs lemma gives that this ring of endomorphisms is a division ring, and since V is simple, it is a quotient of A as a left A module, so is finite dimensional over k. Thus, $\mathrm{End}_A(V)$ is finite dimensional, hence algebraic over k, so is exactly k by algebraic closedness. □

Now we reach our most important definition, that of a semisimple algebra.

Definition 2. *An algebra A is semisimple if A, viewed as a left A module decomposes into a direct sum of simple A modules.*

From here, all of our algebras are finite dimensional over algebraically closed fields k, but the theory works analogously for artinian rings with the same notion of semisimplicity.

Example 1. *For V a finite dimensional k vector space, the ring of endomorphisms of V is semisimple. For this algebra A, V is a simple A module, and identifying A with $n \times n$ matrices, we see that the left multiplication action is the direct sum of n copies of this simple module. (draw the picture of matrices acting on columns to see this)*

Example 2. *Similarly, products of matrix algebras are also semisimple, since we have an explicit description of their simple summands via their matrix algebra factors.*

It turns out that with our hyptheses, these are the only examples of semisimple algebras.

Theorem 1. *If A is finite dimensional over k algebraically closed, and semisimple, then A is a finite product of matrix algebras over k.*

To prove this, we need another definition, that of reducibility of an A module.

Definition 3. *An A module V is reducible if V is expressible as a finite direct sum of simple modules.*

So in particular, A is semisimple if and only if A is reducible, viewed as a left A module.

Note that trivially, direct sums of reducible modules remain reducible.

Lemma 2. *If V is a reducible A module, and $\phi : V \to W$ is surjective map of A modules, then W is a reducible A module.*

Proof. Induct on the minimal number of simple summands of V, noting that the claim is clear when this is one. Pick a decomposition of V into $V' \oplus V''$, where V' is simple, so V'' is in our inductive range. If $\phi|_{V''} : V'' \to W$ is surjective, then we are done by induction, so we may assume the image of V'' under ϕ is proper. Then we have the two submodules $\phi(V')$ and $\phi(V'')$ that generate W, and since $\phi(V'')$ is proper, $\phi(V')$ is nonzero, and therefore simple. So since $\phi(V') \cap \phi(V'')$ is a submodule of $\phi(V')$, and $\phi(V')$ is not a subset of $\phi(V'')$, this intersection must be zero by simplicity of $\phi(V')$. So W is the direct sum of $\phi(V')$ and $\phi(V'')$, and this first factor is simple, and the second is reducible by induction. $\square$

Corollary 2. *Any finitely generated module over a semisimple algebra is reducible.*

Proof. Being finitely generated is to say admits a surjective map from a free module, which are reducible by semisimplicity. $\square$

So as a corollary, going back to our friend the matrix algebra, we have that all finite dimensional representations of $M_n(k)$ are direct sums of the obvious representation on k^n.

Exercise: Show that more generally, the categories of R modules and $M_n(R)$ modules are equivalent.

Now we need two basic lemmas, whose proofs we will leave to the reader. Let R be any ring, and R^{op} denote its opposite ring, the ring with the same elements but $x * y = yx$.

Lemma 3. *For a left R module M, we have $End_R(M^n) \cong M_n(End_R(M))$, the ring of $n \times n$ matrices with coefficients in $End_R(M)$.*

Hint: Use the universal property of direct sums, which are both products and coproducts in the category of R modules, until you see why this is true.

Lemma 4. *Viewing R as a left R module, we have $End_R(R) \cong R^{op}$.*

Hint: Multiply on the right.

So now lets combine what we have thus far, and let A be a semisimple algebra over k. We have a decomposition of A as

$$A \cong \bigoplus_{i=1}^{n} V_i^{n_i}.$$

So taking the endomorphisms of A as a left A module, on one hand we obtain A^{op}, and by Schur's lemma, and the previous proposition, we obtain:

$$A^{op} \cong \prod_{i=1}^{n} M_{n_i}(k).$$

So we see that A^{op} is a direct sum of matrix algebras, but matrix algebras over a field are isomorphic to their own opposites, by taking the transpose, so we have that A is a finite product of matrix algebras.

Now recall that modules M over a product ring $R \times S$ are just given by pairs (M', M''). That is, modules over a product ring are exactly what you would hope for, just collections of modules over each piece.

Exercise: Prove this.

So we see that a semisimple algebra has finitely many isomorphism classes of simple modules, one for each algebra factor, and if V_i is simple, then the number of copies of V_i in A is exactly $\dim_k(V_i)$.

Corollary 3. *For a finite dimensional semisimple algebra A over algebraically closed k, we have:*

$$\dim_k(A) = \sum \dim_k(V_i)^2.$$

Where this sum runs over the isomorphism classes of simple A modules.

This lets us upgrade our theorem 1, we dont just know A is isomorphic to a product of matrix algebras, we have that A is actually the product of the algebras of k endomorphisms of its simple modules.

Theorem 2. *For each simple module V_i over A semisimple, we have the associated map $A \to End_k(V_i)$. The product of these maps*

$$A \to \prod_i End_k(V_i)$$

is an isomorphism of k algebras.

Proof. Since A is semisimple, each nonzero element of A acts nontrivially on A, so acts nontrivially on some simple A module, so this map is injective. Corollary 3 then gives surjectivity by counting dimensions. $\square$

We have that for semisimple A, all modules are direct sums of simple modules, and we have finitely many isomorphism classes of simple modules. Lets now see that any two decompositions of V into simple submodules have the same multiplicities of each simple module V_i.

To prove this, for any V, denote the maximal number of summands of any V_i occurring in a decomposition of V as $n_i = \dim_k(Hom_A(V_i, V))$. We have the following isomorphisms of k vector spaces:

$$V \cong Hom_A(A, V) \cong \bigoplus_i Hom_A(V_i^{\dim_k V_i}, V)$$

So we see that $\dim_k(V) = \sum_i n_i \dim_k(V_i)$, so any direct sum decomposition must have n_i copies of V_i by maximality.

These multiplicities let us map the set of isomorphism classes of A modules injectively into a free abelian group, with basis corresponding to our simple modules. That is, we can think of a module V over semisimple A as a collection of multiplicities n_i of each irrep V_i in V. We will denote this abelian group $R(A)$, the representation group of A, and note that it is K_0 of the category of finitely generated A modules.

We are now going to use our structure theorem to identify $R(A)$ with another group, that of the characters on A.

Definition 4. *A character of a k-algebra A is a k-linear map $\chi : A \to k$ such that $\chi(ab) = \chi(ba)$.*

We denote the k-vector space of characters on A by $Char(A)$. We have the following natural map χ from $R(A)$ to $Char(A)$:

$$[V] \mapsto \begin{tikzcd} A \arrow[r, "\rho_V"] \arrow[dr, "\chi_V"] & \mathrm{End}(V) \arrow[d, "Tr"] \\ & k \end{tikzcd}$$

Extending by linearity. In other words, for each representation V, we set the character χ_V to be $\chi_V(x) = Tr(\rho_V(x))$, where ρ_V is the representation map.

This map is well defined on isomorphism classes, and yields a character since $Tr(ab) = Tr(ba)$, and it respects the natural abelian group structure on either side. The map χ yields the following isomorphism.

Theorem 3. *For a semisimple algebra A, the map χ induces an isomorphism*

$$R(A) \otimes_{\mathbb{Z}} k \to Char(A).$$

Proof. First, this reduces to the matrix algebra case. Both sides of this map respect the product decomposition of A into matrix algebras, so it suffices to prove the statement for $A = M_n(k)$. This then boils down to the statement that the subspace of matrices spanned by $AB - BA$ is codimension 1 is the space of all matrices. In other words, a matrix of trace zero is a sum of matrix commutators. This subspace is conjugation invariant, so we can verify the following:

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

Embedding these in the top 2×2 part of an $n \times n$ matrix, we see that these can be conjugated to a basis of the $n \times n$ traceless matrices, giving the result. □