Math 320 Final Notes - Week 1

Christopher Kapic

December 2020

1.2

Theorem 1. *The triangle inequality states that*

$$|a - b| \leq |a - c| + |c - b|.$$

Theorem 2. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

1.3 - The Axiom of Completeness

Theorem 3 (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded has a least upper bound.*

Definition 1 (bounded)**.** *A set $A \subseteq \mathbb{R}$ is bounded above if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an upper bound for A.*

Similarly, the set A is bounded below if there exists a lower bound $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition 2 (least upper bound, supremum)**.** *A real number s is the least upper bound for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:*

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

The least upper bound is also known as the supremum of the set A.

Definition 3 (maximum)**.** *A real number a_0 is a maximum of the set A is a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a minimum of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.*

Lemma 1. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

1.4 - Consequences of Completeness

Theorem 4 (Nested Interval Property)**.** *For each $n \in \mathbb{R}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem 5 (Archimedean Property)**.**

(i) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.

(ii) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

Theorem 6 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Furthermore, given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.

Theorem 7 ($\sqrt{2}$ exists). *There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.*

Math 320 Final Notes - Week 2

Christopher Kapic

December 2020

1.5 - Cardinality

Definition 1 (one-to-one, onto)**.** *A function $f : A \to B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is onto if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.*

Definition 2 (bijectivity)**.** *The set A has the same cardinality as B if there exists $f : A \to B$ that is 1-1 and onto. In this case, we write $A \sim B$.*

Definition 3 (countable)**.** *A set A is countable if $\mathbb{N} \sim A$. An infinite set that is not countable is called an uncountable set.*

Theorem 1 (1.5.6)**.** *(i) The set $\mathbb{Q}$ is countable.*

(ii) The set $\mathbb{R}$ is uncountable.

Theorem 2 (1.5.7)**.** *If $A \subseteq B$ and B is countable, then A is either countable or finite.*

Theorem 3 (1.5.8)**.** *(i) If $A_1, A_2, \ldots A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.*

(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Math 320 Final Notes - Week 3

Christopher Kapic

December 2020

2.2

Definition 1 (sequence)**.** *A sequence is a function whose domain is* $\mathbb{N}$.

Definition 2 (convergence of a sequence)**.** *A sequence* (a_n) *converges to a real number* a *if, for every positive number* ϵ*, there exists an* $N \in \mathbb{N}$ *such that whenever* $n \geq N$ *it follows that* $|a_n - a| < \epsilon$.

Definition 3 (ϵ-neighborhood)**.** *Given a real number* $a \in \mathbb{R}$ *and a positive number* $\epsilon > 0$*, the set*

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the ϵ*-neighborhood of* a.

Definition 4 (convergence of a sequence, topologically)**.** *A sequence* (a_n) *converges to* a *if, given any* ϵ*-neighborhood* $V_\epsilon(a)$ *of* a*, there exists a point in the sequence after which all of the terms are in* $V_\epsilon(a)$*. In other words, every* ϵ*-neighborhood contains all but a finite number of the terms of* (a_n).

Theorem 1 (Uniqueness of Limits)**.** *The limit of a sequence, when it exists, must be unique.*

Definition 5 (divergence)**.** *A sequence that does not converge is said to diverge.*

2.3

Definition 6 (bounded)**.** *A sequence* (x_n) *is bounded if there exists a number* $M > 0$ *such that* $|x_n| \leq M$ *for all* $n \in \mathbb{N}$.

Theorem 2 (2.3.2)**.** *Every convergent sequence is bounded.*

Theorem 3 (Algebraic Limit Theorem)**.** *Let* $\lim a_n = a$*, and* $\lim b_n = b$*. Then,*

(i) $\lim(ca_n) = ca$*, for all* $c \in \mathbb{R}$*;*

(ii) $\lim(a_n + b_n) = a + b$*;*

(iii) $\lim(a_n b_n) = ab$*;*

(iv) $\lim(a_n/b_n) = a/b$*, provided* $b \neq 0$.

Theorem 4 (Order Limit Theorem)**.** *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

(i) If $a_n \geq 0$ *for all* $n \in \mathbb{N}$*, then* $a \geq 0$.

(ii) If $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*, then* $a \leq b$.

(iii) If there exists $c \in \mathbb{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbb{N}$*, then* $c \leq b$*. Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$*, then* $a \leq c$.

Math 320 Final Notes - Week 4

Christopher Kapic

December 2020

2.4 The Monotone Convergence Theorem, Infinite Series

Definition 1 (increasing, decreasing, monotone). *A sequence (a_n) is increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is monotone if it is either increasing or decreasing.*

Theorem 1 (Monotone Convergence Theorem). *If a sequence is monotone and bounded, then it converges.*

Definition 2 (convergence of a series). *Let (b_n) be a sequence. An infinite series is a formal expression of the form*

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \dots.$$

We define the corresponding sequence of partial sums (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ converges to B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Theorem 2 (Cauchy Condensation Test). *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \dots$$

converges.

Theorem 3 (Corrollary 2.4.7). *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

2.5 - Subsequences and the Bolzano-Weierstrass Theorem

Definition 3 (subsequence). *Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \dots$ be an increasing sequence of natural numbers. Then the sequence*

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \dots)$$

is called a subsequence of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 4 (2.5.2). *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Theorem 5 (Bolzano-Weierstrass Theorem). *Every bounded sequences contains a convergent subsequence.*

2.7 - Properties of Infinite Series

Theorem 6 (Algebraic Limit Theorem for Series)**.** *If* $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$*, then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$ *and*

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

Theorem 7 (Cauchy Criterion for Series)**.** *The series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if, given* $\epsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that whenever* $n > m \geq N$ *it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Theorem 8 (2.7.3)**.** *If the series* $\sum_{k=1}^{\infty} a_k$ *converges, then* $(a_k) \to 0$.

Theorem 9 (Comparison Test)**.** *Assume* (a_k) *and* (b_k) *are sequences satisfying* $0 \leq a_k \leq b_k$ *for all* $k \in \mathbb{N}$.

(i) If $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(ii) If $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Theorem 10 (Absolute Convergence Test)**.** *If the series* $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges as well.*

Theorem 11 (Alternating Series Test)**.** *Let* (a_n) *be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \ldots$ *and*

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges.*

Definition 4 (absolute, conditional convergence)**.** *If* $\sum_{n=1}^{\infty} |a_n|$ *converges, then we say that the original series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely. If, on the other hand, the series* $\sum_{n=1}^{\infty} a_n$ *converges but the series of absolute values* $\sum_{n=1}^{\infty} |a_n|$ *does not converge, then we say that the original series* $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

Definition 5 (rearrangement)**.** *Let* $\sum_{k=1}^{\infty} a_k$ *be a series. A series* $\sum_{k=1}^{\infty} b_k$ *is called a rearrangement of* $\sum_{k=1}^{\infty} a_k$ *if there exists a one-to-one, onto function* $f : \mathbb{N} \to \mathbb{N}$ *such that* $b_{f(k)} = a_k$ *for all* $k \in \mathbb{N}$.

Theorem 12 (2.7.10)**.** *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*

Math 320 Final Notes - Week 5

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2.8 - Double Summations and Products of Infinite Series

Theorem 1 (2.8.1). *Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover,

$$\lim_{n \to \infty} s_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$.

3.2 - Open and Closed Sets

Definition 1 (open). *A set $O \subseteq \mathbb{R}$ is open if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.*

Theorem 2 (3.2.3). *(i) The union of an arbitrary collection of open sets is open.*

(ii) The intersection of a finite collection of open sets is open.

Definition 2 (limit point). *A point x is a limit point of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.*

Theorem 3 (3.2.5). *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Definition 3 (isolated point). *A point $a \in A$ is an isolated point of A if it is not a limit point of A.*

Definition 4 (closed). *A set $F \subseteq \mathbb{R}$ is closed if it contains its limit points.*

Theorem 4 (3.2.8). *A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Theorem 5 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.*

Definition 5 (closure). *Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\overline{A} = A \cup L$.*

Theorem 6 (3.2.12). *For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.*

Theorem 7 (3.2.13). *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

Theorem 8 (3.2.14). *(i) The union of a finite collection of closed sets is closed.*

(ii) The intersection of an arbitrary collection of closed sets is closed.

Math 320 Final Notes - Week 6

Christopher Kapic

December 2020

3.2 - Open and Closed Sets

Definition 1 (open)**.** *A set $O \subseteq \mathbb{R}$ is open if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.*

Theorem 1 (3.2.3)**.** *(i) The union of an arbitrary collection of open sets is open.*

(ii) The intersection of a finite collection of open sets is open.

Definition 2 (limit point)**.** *A point x is a limit point of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.*

Theorem 2 (3.2.5)**.** *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Definition 3 (isolated point)**.** *A point $a \in A$ is an isolated point of A if it is not a limit point of A.*

Definition 4 (closed)**.** *A set $F \subseteq \mathbb{R}$ is closed if it contains its limit points.*

Theorem 3 (3.2.8)**.** *A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Theorem 4 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.*

Definition 5 (closure)**.** *Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\overline{A} = A \cup L$.*

Theorem 5 (3.2.12)**.** *For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.*

Theorem 6 (3.2.13)**.** *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

Theorem 7 (3.2.14)**.** *(i) The union of a finite collection of closed sets is closed.*

(ii) The intersection of an arbitrary collection of closed sets is closed.

3.3 - Compact Sets

Definition 6 (compactness)**.** *A set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a subsequence that converges to a limit that is also in K.*

Definition 7 (bounded)**.** *A set $A \subseteq \mathbb{R}$ is bounded if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.*

Theorem 8 (Characterization of Compactness in $\mathbb{R}$ - 3.3.4)**.** *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof on page 96.

Theorem 9 (Nested Compact Set Property - 3.3.5). *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ *is not empty.*

Proof on page 97.

Definition 8 (open cover, finite subcover). *Let* $A \subseteq \mathbb{R}$. *An open cover for* A *is a (possibly infinite) collection of open sets* $\{O_\lambda : \lambda \in \Lambda\}$ *whose union contains the set* A*; that is,* $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$. *Given an open cover for* A*, a finite subcover is a finite subcollection of open sets from the original open cover whose union still manages to completely contain* A.

Theorem 10 (Heine-Borel Theorem - 3.3.8). *Let* K *be a subset of* $\mathbb{R}$. *All of the following statements are equivalent in the sense that any one of them implies the two others:*

(i) K *is compact.*

(ii) K *is closed and bounded.*

(iii) Every open cover for K *has a finite subcover.*

Proof on page 98.

3.4 - Perfect Sets and Connected Sets

Definition 9 (perfect). *A set* $P \subseteq \mathbb{R}$ *is perfect if it is closed and contains no isolated points.*

Theorem 11 (3.4.3). *A nonempty perfect set is uncountable.*

Proof on page 102.

Definition 10 (separated, disconnected, connected). *Two nonempty sets* $A, B \subseteq \mathbb{R}$ *are separated if* $\overline{A} \cap B$ *and* $A \cap \overline{B}$ *are both empty. A set* $E \subseteq \mathbb{R}$ *is disconnected if it can be written as* $E = A \cup B$*, where* A *and* B *are nonempty separated sets.*

A set that is not disconnected is called a connected set.

Theorem 12 (3.4.6). *A set* $E \subseteq \mathbb{R}$ *is connected if and only if, for all nonempty disjoint sets* A *and* B *satisfying* $E = A \cup B$*, there always exists a convergent sequence* $(x_n) \to x$ *with* (x_n) *contained in one of* A *or* B*, and* x *an element of the other.*

Proof on page 104.

Theorem 13 (3.4.7). *A set* $E \subseteq \mathbb{R}$ *is connected if and only if whenever* $a < c < b$ *with* $a, b \in E$*, it follows that* $c \in E$ *as well.*

Proof on page 105.

Math 320 Final Notes - Week 7

Christopher Kapic

December 2020

4.2 - Functional Limits

Definition 1 (functional limit)**.** *Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L < \epsilon$.*

Definition 2 (functional limit: topological version)**.** *Let c be a limit point of the domain of $f : A \to \mathbb{R}$. We say $\lim_{x\to c} f(x) = L$ provided that, for every ϵ-neighborhood $V_\epsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.*

Theorem 1 (Sequential Criterion for Functional Limits - 4.2.3)**.** *Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:*

(i) $\lim_{x\to c} f(x) = L$.

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Proof on page 118.

Theorem 2 (Corollary 4.2.4 - Algebraic Limit Theorem for Functional Limits)**.** *Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then,*

(i) $\lim_{x\to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$,

(ii) $\lim_{x\to c}[f(x) + g(x)] = L + M$,

(iii) $\lim_{x\to c}[f(x)g(x)] = LM$, *and*

(iv) $\lim_{x\to c} f(x)/g(x) = L/M$, *provided* $M \neq 0$.

Proof on page 119.

4.3 - Continuous Functions

Definition 3 (continuity)**.** *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.*

If f is continuous at every point in the domain A, then we say that f is continuous on A.

Theorem 3 (Characterizationf of Continuity - 4.3.2)**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

(i) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;

(ii) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;

(iii) *For all* $(x_n) \to c$ *(with* $x_n \in A$*), it follows that* $f(x_n) \to f(c)$*.*

If c *is a limit point of* A*, then the above conditions are equivalent to...*

(iv) $\lim_{x \to c} f(x) = f(c)$*.*

Proof on page 123.

Theorem 4 (Corollary 4.3.3 - Criterion for Discontinuity)**.** *Let* $f : A \to \mathbb{R}$*, and let* $c \in A$ *be a limit point of* A*. If there exists a sequence* $(x_n) \subseteq A$ *where* $(x_n) \to c$ *but such that* $f(x_n)$ *does not converge to* $f(c)$*, we may conclude that* f *is not continuous at* c*.*

Theorem 5 (Algebraic Continuity Theorem - 4.3.4)**.** *Assume* $f : A \to \mathbb{R}$ *and* $g : A \to \mathbb{R}$ *are continuous at a point* $c \in A$*. Then,*

(a) $kf(x)$ *is continuous at* c *for all* $k \in \mathbb{R}$*;*

(b) $f(x) + g(x)$ *is continuous at* c*;*

(c) $f(x)g(x)$ *is continuous at* c*; and*

(d) $f(x)/g(x)$ *is continuous at* c*, provided the quotient is defined.*

Proof on page 124.

Theorem 6 (Composition of Continuous Functions - 4.3.9)**.** *Given* $f : A \to \mathbb{R}$ *and* $g : B \to \mathbb{R}$*, assume that the range* $f(A) = \{f(x) : x \in A\}$ *is contained in the domain* B *so that the composition* $g \circ f(x) = g(f(x))$ *is defined on* A*.*

If f *is continuous at* $c \in A$*, and if* g *is continuous at* $f(c) \in B$*, then* $g \circ f$ *is continuous at* c*.*

Math 320 Final Notes - Week 8

Christopher Kapic

December 2020

4.3 - Continuous Functions

Definition 1 (continuity). *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.*

If f is continuous at every point in the domain A, then we say that f is continuous on A.

Theorem 1 (Characterizationf of Continuity - 4.3.2). *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

(i) *For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;*

(ii) *For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;*

(iii) *For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.*

If c is a limit point of A, then the above conditions are equivalent to...

(iv) $\lim_{x \to c} f(x) = f(c)$.

Proof on page 123.

Theorem 2 (Corollary 4.3.3 - Criterion for Discontinuity). *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.*

Theorem 3 (Algebraic Continuity Theorem - 4.3.4). *Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at a point $c \in A$. Then,*

(a) *$kf(x)$ is continuous at c for all $k \in \mathbb{R}$;*

(b) *$f(x) + g(x)$ is continuous at c;*

(c) *$f(x)g(x)$ is continuous at c; and*

(d) *$f(x)/g(x)$ is continuous at c, provided the quotient is defined.*

Proof on page 124.

Theorem 4 (Composition of Continuous Functions - 4.3.9). *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A.*

If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

4.4 - Continuous Functions on Compact Sets

Theorem 5 (Preservation of Compact Sets - 4.4.1)**.** *Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.*

Proof on page 130.

Theorem 6 (Exterme Value Theorem - 4.4.2)**.** *If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

Proof on page 130.

Definition 2 (uniform continuity)**.** *A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.*

Theorem 7 (Sequential Criterion for Absence of Uniform Continuity - 4.4.5)**.** *A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying*

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Proof on page 132.

Theorem 8 (Uniform Continuity on Compact Sets - 4.4.7)**.** *A function that is continuous on a compact set K is uniformly continuous on K.*

Proof on page 133.

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4.5 - The Intermediate Value Theorem

Theorem 1 (Preservation of Connected Sets - 4.5.2). *Let $f : G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.*

Proof on page 137.

Definition 1 (intermediate value property). *A function f has the intermediate value property on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L$.*

Another way to summarize the Intermediate Value Theorem is to say that every continuous function on $[a, b]$ has the intermediate value property.

5.2 - Derivatives and the Intermediate Value Property

Definition 2 (differentiability). *Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the derivative of g at c is defined by*

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case we say g is differentiable at c. If g' exists for all points $c \in A$, we say that g is differentiable on A.

Theorem 2 (5.2.3). *If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Proof on page 149.

Theorem 3 (Algebraic Differentiability Theorem - 5.2.4). *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f + g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, *provided that* $g(c) \neq 0$.

Proof on page 149.

Theorem 4 (Chain Rule - 5.2.5). *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Proof on page 150.

Theorem 5 (Interior Extremum Theorem - 5.2.6). *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$ (i.e., $f(c) \geq f(x)$ for all $x \in (a, b)$), then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Proof on page 151.

Theorem 6 (Darboux's Theorem - 5.2.7)**.** *If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.*
Proof on page 152.

5.3 - The Mean Value Theorems

Theorem 7 (Rolle's Theorem - 5.3.1)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*
Proof on page 156.

Theorem 8 (Mean Value Theorem - 5.3.2)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof on page 156.

Theorem 9 (Corollary 5.3.3)**.** *If $g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.*
Proof on page 157.

Theorem 10 (Corollary 5.3.4)**.** *If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$.*
Proof on page 158.

Theorem 11 (Generalized Mean Value Theorem - 5.3.5)**.** *If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof on page 158.

Theorem 12 (L'Hospital's Rule: 0/0 case)**.** *Let f and g be continuous on an interval conataining a, and assume f and g are differentiable on this interavl with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \textit{ implies } \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof on page 159.

Definition 3 (5.3.7)**.** *Given $g : A \to \mathbb{R}$ and a limit point c of A, we say that $\lim_{x \to c} g(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) \geq M$.*

Theorem 13 (L'Hospital's Rule: ∞/∞ case)**.** *Assume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \textit{ implies } \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof on page 159.

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5.3

Definition 1 (differentiability)**.** *Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the derivative of g at c is defined by*

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case we say g is differentiable at c. If g' exists for all points $c \in A$, we say that g is differentiable on A.

Theorem 1 (5.2.3)**.** *If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Proof on page 149.

Theorem 2 (Algebraic Differentiability Theorem - 5.2.4)**.** *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f + g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, *provided that* $g(c) \neq 0$.

Proof on page 149.

Theorem 3 (Chain Rule - 5.2.5)**.** *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Proof on page 150.

Theorem 4 (Interior Extremum Theorem - 5.2.6)**.** *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$ (i.e., $f(c) \geq f(x)$ for all $x \in (a, b)$), then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Proof on page 151.

Theorem 5 (Darboux's Theorem - 5.2.7)**.** *If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.*

Proof on page 152.

7.2 - The Definition of the Riemann Integral

Definition 2 (partition, lower sum, upper sum)**.** *A partition P of $[a, b]$ is a finite set of points from $[a, b]$ that includes both a and b. The notational convention is to always list the points of a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ in increasing order; thus,*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For each subinterval $[x_{k-1}, x_k]$ of P, let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \text{ and } M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The lower sum of f with respect to P is given by

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Likewise, we define the upper sum of f with respect to P by

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

For a particular partion P, it is clear that $U(f, P) \geq L(f, P)$. The fact that this same inequality holds if the upper and lower sums are computed with respect to different partitions is the content of the next two lemmas.

Definition 3 (refinement - 7.2.2)**.** *A partition Q is a refinement of a partition P if Q contains all of the points of P; that is, if $P \subseteq Q$.*

Lemma 1 (7.2.3)**.** *If $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.*
Proof on page 218.

Lemma 2 (7.2.4)**.** *If P_1 and P_2 are any two partitions of $[a, b]$, then $L(f, P_1) \leq U(f, P_2)$.*
Proof on page 219.

Definition 4 (upper integral, lower integral)**.** *Let $\mathcal{P}$ be the collection of all possible partitions of the interval $[a, b]$: The upper integral of f is defined to be*

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}.$$

In a similar way, define the lower integral of f by

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}.$$

Lemma 3 (7.2.6)**.** *For any bounded function f on $[a, b]$, it is always the case that $U(f) \geq L(f)$.*

Definition 5 (Riemann Integrability)**.** *A bounded function f defined on the interval $[a, b]$ is Riemann-integrable if $U(f) = L(f)$. In this case, we define $\int_a^b f$ or $\int_a^b f(x)dx$ to be this common value; namely,*

$$\sum_{a}^{b} f = U(f) = L(f).$$

Theorem 6 (Integrability Criterion - 7.2.8)**.** *A bounded function f is integrable on $[a, b]$ if and only if, for every $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that*

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Proof on page 221.

Theorem 7 (7.2.9)**.** *If f is continuous on $[a, b]$, then it is integrable.*
Proof on page 222.

7.3 - Integrating Functions with Discontinuities

Theorem 8 (7.3.2)**.** *If $f : [a, b] \to \mathbb{R}$ is bounded, and f is integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$. An analgous result holds at the other endpoint.*
Proof on page 224.

7.4 - Properties of the Integral

Theorem 9 (7.4.1)**.** *Assume $f : [a, b] \to \mathbb{R}$ is bounded, and let $c \in (a, b)$. Then, f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$. In this case, we have*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof on page 228.

Theorem 10 (7.4.2)**.** *Assume f and g are integrable functions on the interval $[a, b]$.*

(i) The function $f + g$ is integrable on $[a, b]$ with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

(ii) For $k \in \mathbb{R}$, the function kf is integrable with $\int_a^b kf = k \int_a^b f$.

(iii) If $m \leq f(x) \leq M$ on $[a, b]$, then $m(b - a) \leq \int_a^b f \leq M(b - a)$.

(iv) If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f \leq \int_a^b g$.

(v) The function $|f|$ is integrable and $|\int_a^b f| \leq \int_a^b |f|$.

Proof on page 230.

Definition 6 (7.4.3)**.** *If f is integrable on the interval $[a, b]$, define*

$$\int_b^a f = -\int_a^b f.$$

Also, for $c \in [a, b]$ define

$$\int_c^c f = 0.$$

Theorem 11 (Integrable Limit Theorem - 7.4.4)**.** *Assume that $f_n \to f$ uniformly on $[a, b]$ and that each f_n is integrable. Then, f is integrable and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof on page 232.