Math 320 Final Notes - Week 1

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December 2020

1.2

Theorem 1. *The triangle inequality states that*

$$|a - b| \leq |a - c| + |c - b|.$$

Theorem 2. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

1.3 - The Axiom of Completeness

Theorem 3 (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded has a least upper bound.*

Definition 1 (bounded)**.** *A set $A \subseteq \mathbb{R}$ is bounded above if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an upper bound for A.*

Similarly, the set A is bounded below if there exists a lower bound $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Definition 2 (least upper bound, supremum)**.** *A real number s is the least upper bound for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:*

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

The least upper bound is also known as the supremum of the set A.

Definition 3 (maximum)**.** *A real number a_0 is a maximum of the set A is a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a minimum of A if $a_1 \in A$ and $a_1 \leq a$ for every $a \in A$.*

Lemma 1. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

1.4 - Consequences of Completeness

Theorem 4 (Nested Interval Property)**.** *For each $n \in \mathbb{R}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ldots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem 5 (Archimedean Property)**.** *(i) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.*

(ii) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

Theorem 6 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Furthermore, given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.

Theorem 7 ($\sqrt{2}$ exists)**.** *There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.*