Math 320 Final Notes - Week 6

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3.2 - Open and Closed Sets

Definition 1 (open). *A set $O \subseteq \mathbb{R}$ is open if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.*

Theorem 1 (3.2.3). *(i) The union of an arbitrary collection of open sets is open.*

(ii) The intersection of a finite collection of open sets is open.

Definition 2 (limit point). *A point x is a limit point of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.*

Theorem 2 (3.2.5). *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Definition 3 (isolated point). *A point $a \in A$ is an isolated point of A if it is not a limit point of A.*

Definition 4 (closed). *A set $F \subseteq \mathbb{R}$ is closed if it contains its limit points.*

Theorem 3 (3.2.8). *A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Theorem 4 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.*

Definition 5 (closure). *Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\overline{A} = A \cup L$.*

Theorem 5 (3.2.12). *For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.*

Theorem 6 (3.2.13). *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

Theorem 7 (3.2.14). *(i) The union of a finite collection of closed sets is closed.*

(ii) The intersection of an arbitrary collection of closed sets is closed.

3.3 - Compact Sets

Definition 6 (compactness). *A set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a subsequence that converges to a limit that is also in K.*

Definition 7 (bounded). *A set $A \subseteq \mathbb{R}$ is bounded if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.*

Theorem 8 (Characterization of Compactness in $\mathbb{R}$ - 3.3.4). *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof on page 96.

Theorem 9 (Nested Compact Set Property - 3.3.5). *If*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is not empty.

Proof on page 97.

Definition 8 (open cover, finite subcover). *Let $A \subseteq \mathbb{R}$. An open cover for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A; that is, $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$. Given an open cover for A, a finite subcover is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.*

Theorem 10 (Heine-Borel Theorem - 3.3.8). *Let K be a subset of $\mathbb{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others:*

(i) K is compact.

(ii) K is closed and bounded.

(iii) Every open cover for K has a finite subcover.

Proof on page 98.

3.4 - Perfect Sets and Connected Sets

Definition 9 (perfect). *A set $P \subseteq \mathbb{R}$ is perfect if it is closed and contains no isolated points.*

Theorem 11 (3.4.3). *A nonempty perfect set is uncountable.*

Proof on page 102.

Definition 10 (separated, disconnected, connected). *Two nonempty sets $A, B \subseteq \mathbb{R}$ are separated if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty. A set $E \subseteq \mathbb{R}$ is disconnected if it can be written as $E = A \cup B$, where A and B are nonempty separated sets.*

A set that is not disconnected is called a connected set.

Theorem 12 (3.4.6). *A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.*

Proof on page 104.

Theorem 13 (3.4.7). *A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well.*

Proof on page 105.