Math 320 Final Notes - Week 8

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4.3 - Continuous Functions

Definition 1 (continuity)**.** *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.*

If f is continuous at every point in the domain A, then we say that f is continuous on A.

Theorem 1 (Characterizationf of Continuity - 4.3.2)**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

(i) *For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;*

(ii) *For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;*

(iii) *For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.*

If c is a limit point of A, then the above conditions are equivalent to...

(iv) $\lim_{x\to c} f(x) = f(c)$.

Proof on page 123.

Theorem 2 (Corollary 4.3.3 - Criterion for Discontinuity)**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.*

Theorem 3 (Algebraic Continuity Theorem - 4.3.4)**.** *Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at a point $c \in A$. Then,*

(a) *$kf(x)$ is continuous at c for all $k \in \mathbb{R}$;*

(b) *$f(x) + g(x)$ is continuous at c;*

(c) *$f(x)g(x)$ is continuous at c; and*

(d) *$f(x)/g(x)$ is continuous at c, provided the quotient is defined.*

Proof on page 124.

Theorem 4 (Composition of Continuous Functions - 4.3.9)**.** *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A.*

If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

4.4 - Continuous Functions on Compact Sets

Theorem 5 (Preservation of Compact Sets - 4.4.1)**.** *Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.*

Proof on page 130.

Theorem 6 (Exterme Value Theorem - 4.4.2)**.** *If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

Proof on page 130.

Definition 2 (uniform continuity)**.** *A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.*

Theorem 7 (Sequential Criterion for Absence of Uniform Continuity - 4.4.5)**.** *A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying*

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Proof on page 132.

Theorem 8 (Uniform Continuity on Compact Sets - 4.4.7)**.** *A function that is continuous on a compact set K is uniformly continuous on K.*

Proof on page 133.