Math 320 Final Notes - Week 10

Christopher Kapic

December 2020

5.3

Definition 1 (differentiability). *Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the derivative of g at c is defined by*

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case we say g is differentiable at c. If g' exists for all points $c \in A$, we say that g is differentiable on A.

Theorem 1 (5.2.3). *If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Proof on page 149.

Theorem 2 (Algebraic Differentiability Theorem - 5.2.4). *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f + g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, *provided that* $g(c) \neq 0$.

Proof on page 149.

Theorem 3 (Chain Rule - 5.2.5). *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Proof on page 150.

Theorem 4 (Interior Extremum Theorem - 5.2.6). *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$ (i.e., $f(c) \geq f(x)$ for all $x \in (a, b)$), then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Proof on page 151.

Theorem 5 (Darboux's Theorem - 5.2.7). *If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.*

Proof on page 152.

7.2 - The Definition of the Riemann Integral

Definition 2 (partition, lower sum, upper sum). *A partition P of $[a, b]$ is a finite set of points from $[a, b]$ that includes both a and b. The notational convention is to always list the points of a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ in increasing order; thus,*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For each subinterval $[x_{k-1}, x_k]$ of P, let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \text{ and } M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The lower sum of f with respect to P is given by

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Likewise, we define the upper sum of f with respect to P by

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

For a particular partion P, it is clear that $U(f, P) \geq L(f, P)$. The fact that this same inequality holds if the upper and lower sums are computed with respect to different partitions is the content of the next two lemmas.

Definition 3 (refinement - 7.2.2)**.** *A partition Q is a refinement of a partition P if Q contains all of the points of P; that is, if $P \subseteq Q$.*

Lemma 1 (7.2.3)**.** *If $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.*
Proof on page 218.

Lemma 2 (7.2.4)**.** *If P_1 and P_2 are any two partitions of $[a, b]$, then $L(f, P_1) \leq U(f, P_2)$.*
Proof on page 219.

Definition 4 (upper integral, lower integral)**.** *Let $\mathcal{P}$ be the collection of all possible partitions of the interval $[a, b]$: The upper integral of f is defined to be*

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}.$$

In a similar way, define the lower integral of f by

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}.$$

Lemma 3 (7.2.6)**.** *For any bounded function f on $[a, b]$, it is always the case that $U(f) \geq L(f)$.*

Definition 5 (Riemann Integrability)**.** *A bounded function f defined on the interval $[a, b]$ is Riemann-integrable if $U(f) = L(f)$. In this case, we define $\int_a^b f$ or $\int_a^b f(x)dx$ to be this common value; namely,*

$$\sum_{a}^{b} f = U(f) = L(f).$$

Theorem 6 (Integrability Criterion - 7.2.8)**.** *A bounded function f is integrable on $[a, b]$ if and only if, for every $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that*

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Proof on page 221.

Theorem 7 (7.2.9)**.** *If f is continuous on $[a, b]$, then it is integrable.*
Proof on page 222.

7.3 - Integrating Functions with Discontinuities

Theorem 8 (7.3.2)**.** *If $f : [a, b] \to \mathbb{R}$ is bounded, and f is integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$. An analgous result holds at the other endpoint.*
Proof on page 224.

7.4 - Properties of the Integral

Theorem 9 (7.4.1)**.** *Assume $f : [a,b] \to \mathbb{R}$ is bounded, and let $c \in (a,b)$. Then, f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and $[c,b]$. In this case, we have*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof on page 228.

Theorem 10 (7.4.2)**.** *Assume f and g are integrable functions on the interval $[a,b]$.*

(i) The function $f+g$ is integrable on $[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(ii) For $k \in \mathbb{R}$, the function kf is integrable with $\int_a^b kf = k\int_a^b f$.

(iii) If $m \le f(x) \le M$ on $[a,b]$, then $m(b-a) \le \int_a^b f \le M(b-a)$.

(iv) If $f(x) \le g(x)$ on $[a,b]$, then $\int_a^b f \le \int_a^b g$.

(v) The function $|f|$ is integrable and $|\int_a^b f| \le \int_a^b |f|$.

Proof on page 230.

Definition 6 (7.4.3)**.** *If f is integrable on the interval $[a,b]$, define*

$$\int_b^a f = -\int_a^b f.$$

Also, for $c \in [a,b]$ define

$$\int_c^c f = 0.$$

Theorem 11 (Integrable Limit Theorem - 7.4.4)**.** *Assume that $f_n \to f$ uniformly on $[a,b]$ and that each f_n is integrable. Then, f is integrable and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof on page 232.