Math 320 Final Notes - Week 7

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4.2 - Functional Limits

Definition 1 (functional limit). *Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L < \epsilon$.*

Definition 2 (functional limit: topological version). *Let c be a limit point of the domain of $f : A \to \mathbb{R}$. We say $\lim_{x\to c} f(x) = L$ provided that, for every ϵ-neighborhood $V_\epsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.*

Theorem 1 (Sequential Criterion for Functional Limits - 4.2.3). *Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:*

(i) $\lim_{x\to c} f(x) = L$.

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Proof on page 118.

Theorem 2 (Corollary 4.2.4 - Algebraic Limit Theorem for Functional Limits). *Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then,*

(i) $\lim_{x\to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$,

(ii) $\lim_{x\to c}[f(x) + g(x)] = L + M$,

(iii) $\lim_{x\to c}[f(x)g(x)] = LM$, *and*

(iv) $\lim_{x\to c} f(x)/g(x) = L/M$, *provided* $M \neq 0$.

Proof on page 119.

4.3 - Continuous Functions

Definition 3 (continuity). *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$.*

If f is continuous at every point in the domain A, then we say that f is continuous on A.

Theorem 3 (Characterizationf of Continuity - 4.3.2). *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

(i) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;

(ii) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;

(iii) *For all* $(x_n) \to c$ *(with* $x_n \in A$*), it follows that* $f(x_n) \to f(c)$.

If c *is a limit point of* A*, then the above conditions are equivalent to...*

(iv) $\lim_{x \to c} f(x) = f(c)$.

Proof on page 123.

Theorem 4 (Corollary 4.3.3 - Criterion for Discontinuity)**.** *Let* $f : A \to \mathbb{R}$*, and let* $c \in A$ *be a limit point of* A*. If there exists a sequence* $(x_n) \subseteq A$ *where* $(x_n) \to c$ *but such that* $f(x_n)$ *does not converge to* $f(c)$*, we may conclude that* f *is not continuous at* c*.*

Theorem 5 (Algebraic Continuity Theorem - 4.3.4)**.** *Assume* $f : A \to \mathbb{R}$ *and* $g : A \to \mathbb{R}$ *are continuous at a point* $c \in A$*. Then,*

(a) $kf(x)$ *is continuous at* c *for all* $k \in \mathbb{R}$*;*

(b) $f(x) + g(x)$ *is continuous at* c*;*

(c) $f(x)g(x)$ *is continuous at* c*; and*

(d) $f(x)/g(x)$ *is continuous at* c*, provided the quotient is defined.*

Proof on page 124.

Theorem 6 (Composition of Continuous Functions - 4.3.9)**.** *Given* $f : A \to \mathbb{R}$ *and* $g : B \to \mathbb{R}$*, assume that the range* $f(A) = \{f(x) : x \in A\}$ *is contained in the domain* B *so that the composition* $g \circ f(x) = g(f(x))$ *is defined on* A*.*

If f *is continuous at* $c \in A$*, and if* g *is continuous at* $f(c) \in B$*, then* $g \circ f$ *is continuous at* c*.*