Math 320 Final Notes - Week 5

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2.8 - Double Summations and Products of Infinite Series

Theorem 1 (2.8.1). *Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array of real numbers. If*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover,

$$\lim_{n \to \infty} s_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij},$$

where $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$.

3.2 - Open and Closed Sets

Definition 1 (open). *A set $O \subseteq \mathbb{R}$ is open if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.*

Theorem 2 (3.2.3). *(i) The union of an arbitrary collection of open sets is open.*

(ii) The intersection of a finite collection of open sets is open.

Definition 2 (limit point). *A point x is a limit point of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.*

Theorem 3 (3.2.5). *A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Definition 3 (isolated point). *A point $a \in A$ is an isolated point of A if it is not a limit point of A.*

Definition 4 (closed). *A set $F \subseteq \mathbb{R}$ is closed if it contains its limit points.*

Theorem 4 (3.2.8). *A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.*

Theorem 5 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For every $y \in \mathbb{R}$, there exists a sequence of rational numbers that converges to y.*

Definition 5 (closure). *Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\overline{A} = A \cup L$.*

Theorem 6 (3.2.12). *For any $A \subseteq \mathbb{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.*

Theorem 7 (3.2.13). *A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.*

Theorem 8 (3.2.14). *(i) The union of a finite collection of closed sets is closed.*

(ii) The intersection of an arbitrary collection of closed sets is closed.