Math 320 Final Notes - Week 3

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2.2

Definition 1 (sequence)**.** *A sequence is a function whose domain is* $\mathbb{N}$*.*

Definition 2 (convergence of a sequence)**.** *A sequence* (a_n) *converges to a real number* a *if, for every positive number* ϵ*, there exists an* $N \in \mathbb{N}$ *such that whenever* $n \geq N$ *it follows that* $|a_n - a| < \epsilon$*.*

Definition 3 (ϵ-neighborhood)**.** *Given a real number* $a \in \mathbb{R}$ *and a positive number* $\epsilon > 0$*, the set*

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the ϵ*-neighborhood of* a*.*

Definition 4 (convergence of a sequence, topologically)**.** *A sequence* (a_n) *converges to* a *if, given any* ϵ*-neighborhood* $V_\epsilon(a)$ *of* a*, there exists a point in the sequence after which all of the terms are in* $V_\epsilon(a)$*. In other words, every* ϵ*-neighborhood contains all but a finite number of the terms of* (a_n)*.*

Theorem 1 (Uniqueness of Limits)**.** *The limit of a sequence, when it exists, must be unique.*

Definition 5 (divergence)**.** *A sequence that does not converge is said to diverge.*

2.3

Definition 6 (bounded)**.** *A sequence* (x_n) *is bounded if there exists a number* $M > 0$ *such that* $|x_n| \leq M$ *for all* $n \in \mathbb{N}$*.*

Theorem 2 (2.3.2)**.** *Every convergent sequence is bounded.*

Theorem 3 (Algebraic Limit Theorem)**.** *Let* $\lim a_n = a$*, and* $\lim b_n = b$*. Then,*

(i) $\lim(ca_n) = ca$*, for all* $c \in \mathbb{R}$*;*

(ii) $\lim(a_n + b_n) = a + b$*;*

(iii) $\lim(a_n b_n) = ab$*;*

(iv) $\lim(a_n/b_n) = a/b$*, provided* $b \neq 0$*.*

Theorem 4 (Order Limit Theorem)**.** *Assume* $\lim a_n = a$ *and* $\lim b_n = b$*.*

(i) If $a_n \geq 0$ *for all* $n \in \mathbb{N}$*, then* $a \geq 0$*.*

(ii) If $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*, then* $a \leq b$*.*

(iii) If there exists $c \in \mathbb{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbb{N}$*, then* $c \leq b$*. Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$*, then* $a \leq c$*.*