Math 320 Final Notes - Week 9

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December 2020

4.5 - The Intermediate Value Theorem

Theorem 1 (Preservation of Connected Sets - 4.5.2). *Let $f : G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.*

Proof on page 137.

Definition 1 (intermediate value property). *A function f has the intermediate value property on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L$.*

Another way to summarize the Intermediate Value Theorem is to say that every continuous function on $[a, b]$ has the intermediate value property.

5.2 - Derivatives and the Intermediate Value Property

Definition 2 (differentiability). *Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the derivative of g at c is defined by*

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

provided this limit exists. In this case we say g is differentiable at c. If g' exists for all points $c \in A$, we say that g is differentiable on A.

Theorem 2 (5.2.3). *If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Proof on page 149.

Theorem 3 (Algebraic Differentiability Theorem - 5.2.4). *Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f + g)'(c) = f'(c) + g'(c)$,

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$,

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, *provided that* $g(c) \neq 0$.

Proof on page 149.

Theorem 4 (Chain Rule - 5.2.5). *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Proof on page 150.

Theorem 5 (Interior Extremum Theorem - 5.2.6). *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$ (i.e., $f(c) \geq f(x)$ for all $x \in (a, b)$), then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Proof on page 151.

Theorem 6 (Darboux's Theorem - 5.2.7)**.** *If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(a) > \alpha > f'(b)$), then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.*

Proof on page 152.

5.3 - The Mean Value Theorems

Theorem 7 (Rolle's Theorem - 5.3.1)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Proof on page 156.

Theorem 8 (Mean Value Theorem - 5.3.2)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof on page 156.

Theorem 9 (Corollary 5.3.3)**.** *If $g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.*

Proof on page 157.

Theorem 10 (Corollary 5.3.4)**.** *If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$.*

Proof on page 158.

Theorem 11 (Generalized Mean Value Theorem - 5.3.5)**.** *If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof on page 158.

Theorem 12 (L'Hospital's Rule: 0/0 case)**.** *Let f and g be continuous on an interval conataining a, and assume f and g are differentiable on this interavl with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \textit{ implies } \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof on page 159.

Definition 3 (5.3.7)**.** *Given $g : A \to \mathbb{R}$ and a limit point c of A, we say that $\lim_{x \to c} g(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) \geq M$.*

Theorem 13 (L'Hospital's Rule: ∞/∞ case)**.** *Assume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \textit{ implies } \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof on page 159.