Math 320 Final Notes - Week 4

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2.4 The Monotone Convergence Theorem, Infinite Series

Definition 1 (increasing, decreasing, monotone)**.** *A sequence (a_n) is increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is monotone if it is either increasing or decreasing.*

Theorem 1 (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it converges.*

Definition 2 (convergence of a series)**.** *Let (b_n) be a sequence. An infinite series is a formal expression of the form*

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \dots.$$

We define the corresponding sequence of partial sums (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m,$$

and say that the series $\sum_{n=1}^{\infty} b_n$ converges to B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Theorem 2 (Cauchy Condensation Test)**.** *Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series*

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \dots$$

converges.

Theorem 3 (Corrollary 2.4.7)**.** *The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.*

2.5 - Subsequences and the Bolzano-Weierstrass Theorem

Definition 3 (subsequence)**.** *Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \dots$ be an increasing sequence of natural numbers. Then the sequence*

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \dots)$$

is called a subsequence of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 4 (2.5.2)**.** *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Theorem 5 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequences contains a convergent subsequence.*

2.7 - Properties of Infinite Series

Theorem 6 (Algebraic Limit Theorem for Series). *If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then*

(i) $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$ *and*

(ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

Theorem 7 (Cauchy Criterion for Series). *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Theorem 8 (2.7.3). *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.*

Theorem 9 (Comparison Test). *Assume (a_k) and (b_k) are sequences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.*

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Theorem 10 (Absolute Convergence Test). *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.*

Theorem 11 (Alternating Series Test). *Let (a_n) be a sequence satisfying,*

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \ldots$ *and*

(ii) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Definition 4 (absolute, conditional convergence). *If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ converges conditionally.*

Definition 5 (rearrangement). *Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a rearrangement of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one, onto function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.*

Theorem 12 (2.7.10). *If a series converges absolutely, then any rearrangement of this series converges to the same limit.*