Derivations for PINNs

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1 Introduction

1.1 Incompressible Navier-Stokes Equations, and Expansion

The incompressible Navier-Stokes equations in vector form read:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0 \tag{1}$$

with the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

If we consider a two dimensional case, and expand:

$$\frac{\partial u_x}{\partial t} + \left(\frac{\partial u_x u_x}{\partial x} + \frac{\partial u_y u_x}{\partial y} \right) + \frac{\partial p}{\partial x} - \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) = 0 \tag{3}$$

$$\frac{\partial u_y}{\partial t} + \left(\frac{\partial u_x u_y}{\partial x} + \frac{\partial u_y u_y}{\partial y} \right) + \frac{\partial p}{\partial y} - \nu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) = 0 \tag{4}$$

with the continuity equation:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{5}$$

1.2 Steady NS equations

In the case that the flow is completely steady, the time derivative term can be negleted, giving the form:

$$(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0 \tag{6}$$

with the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{7}$$

Or expanded:

$$\left(\frac{\partial u_x u_x}{\partial x} + \frac{\partial u_y u_x}{\partial y} \right) + \frac{\partial p}{\partial x} - \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) = 0 \tag{8}$$

$$\left(\frac{\partial u_x u_y}{\partial x} + \frac{\partial u_y u_y}{\partial y} \right) + \frac{\partial p}{\partial y} - \nu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) = 0 \tag{9}$$

with the continuity equation:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{10}$$

1.3 Reynolds Decomposition

We consider the decomposition into a mean and fluctuating part according to the following equation:

$$\mathbf{u}(\mathbf{x},t) = \overline{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x},t) \tag{11}$$

with the properties that:

$$\overline{\overline{\mathbf{u}}} = \overline{\mathbf{u}} \tag{12}$$

$$\overline{\mathbf{u}'} = 0 \tag{13}$$

1.4 Derivation of the RANS Equations

We can first begin by substituting the the Reynolds decomposition into the continuity equation:

$$\frac{\partial(\overline{u_x} + u'_x)}{\partial x} + \frac{\partial(\overline{u_y} + u'_y)}{\partial y} = 0 \tag{14}$$

$$\frac{\partial \overline{u_x}}{\partial x} + \frac{\partial u'_x}{\partial x} + \frac{\partial \overline{u_y}}{\partial y} + \frac{\partial u'_y}{\partial y} = 0 \tag{15}$$

We can then take the time average of the equations:

$$\overline{\frac{\partial \overline{u_x}}{\partial x}} + \overline{\frac{\partial u'_x}{\partial x}} + \overline{\frac{\partial \overline{u_y}}{\partial y}} + \overline{\frac{\partial u'_y}{\partial y}} = 0 \tag{16}$$

Since the time average of the velocity field is defined as the mean velocity field, it's derivative is the derivative of the mean velocity field. Since the time average of the fluctuating component is defined as zero, it's spatial derivatives are zero. Thus Reynolds averaged Continuity equation is:

$$\frac{\partial \overline{u_x}}{\partial x} + \frac{\partial \overline{u_y}}{\partial y} = 0 \tag{17}$$

We can then consider the momentum equations. We begin by substituting the Reynolds decomposition into the x-momentum equation:

$$\begin{aligned}\frac{\partial(\overline{u_x} + u'_x)}{\partial t} + \left(\frac{\partial(\overline{u_x} + u'_x)(\overline{u_x} + u'_x)}{\partial x} + \frac{\partial(\overline{u_x} + u'_x)(\overline{u_y} + u'_y)}{\partial y}\right) \\ + \frac{\partial(\overline{p} + p')}{\partial x} - \nu\left(\frac{\partial^2(\overline{u_x} + u'_x)}{\partial x^2} + \frac{\partial^2(\overline{u_x} + u'_x)}{\partial y^2}\right) = 0\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial \overline{u_x}}{\partial t} + \frac{\partial u'_x}{\partial t} + \left(\frac{\partial \overline{u_x}\,\overline{u_x}}{\partial x} + 2\frac{\partial \overline{u_x} u'_x}{\partial x} + \frac{\partial u'_x u'_x}{\partial x} + \frac{\partial \overline{u_x}\,\overline{u_y}}{\partial y} + \frac{\partial \overline{u_x} u'_y}{\partial y} + \frac{\partial u'_x u'_y}{\partial y}\right) \\ + \frac{\partial \overline{p}}{\partial x} + \frac{\partial p'}{\partial x} - \nu\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2} + \frac{\partial^2 u'_x}{\partial x^2} + \frac{\partial^2 u'_x}{\partial y^2}\right) = 0\end{aligned} \tag{19}$$

If we take the time average of the decomposed NS equation, we get:

$$\begin{aligned}\overline{\frac{\partial \overline{u_x}}{\partial t}} + \overline{\frac{\partial u'_x}{\partial t}} + \left(\overline{\frac{\partial \overline{u_x}\,\overline{u_x}}{\partial x}} + 2\overline{\frac{\partial \overline{u_x} u'_x}{\partial x}} + \overline{\frac{\partial u'_x u'_x}{\partial x}} + \overline{\frac{\partial \overline{u_x}\,\overline{u_y}}{\partial y}} + \overline{\frac{\partial \overline{u_x} u'_y}{\partial y}} + \overline{\frac{\partial u'_x u'_y}{\partial y}}\right) \\ + \overline{\frac{\partial \overline{p}}{\partial x}} + \overline{\frac{\partial p'}{\partial x}} - \nu\left(\overline{\frac{\partial^2 \overline{u_x}}{\partial x^2}} + \overline{\frac{\partial^2 \overline{u_x}}{\partial y^2}} + \overline{\frac{\partial^2 u'_x}{\partial x^2}} + \overline{\frac{\partial^2 u'_x}{\partial y^2}}\right) = 0\end{aligned} \tag{20}$$

Since by definition $\mathbf{u}'$ is a vector field with mean of zero at each point, it's derivative must also have a zero mean, thus the time average of $\frac{\partial \mathbf{u}'}{\partial t} = 0$. $\frac{\partial \mathbf{u}}{\partial t} = 0$ for steady flow. This leaves the terms:

$$\left(\frac{\partial \overline{u_x}\,\overline{u_x}}{\partial x} + \frac{\partial \overline{u_x}\,\overline{u_y}}{\partial y} + \overline{\frac{\partial u'_x u'_x}{\partial x}} + \overline{\frac{\partial u'_x u'_y}{\partial y}}\right) + \frac{\partial \overline{p}}{\partial x} - \nu\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2}\right) = 0 \tag{21}$$

Applying the chain rule to the first two terms:

$$\left(\overline{u_x}\frac{\partial\overline{u_x}}{\partial x}+\overline{u_x}\frac{\partial\overline{u_x}}{\partial x}+\overline{u_x}\frac{\partial\overline{u_y}}{\partial y}+\overline{u_y}\frac{\partial\overline{u_x}}{\partial y}+\frac{\partial\overline{u'_x u'_x}}{\partial x}+\frac{\partial\overline{u'_x u'_y}}{\partial y}\right)+\frac{\partial\overline{p}}{\partial x}-\nu\left(\frac{\partial^2\overline{u_x}}{\partial x^2}+\frac{\partial^2\overline{u_x}}{\partial y^2}\right)=0 \quad (22)$$

$$\left(\overline{u_x}\frac{\partial\overline{u_x}}{\partial x}+\overline{u_x}\left[\frac{\partial\overline{u_x}}{\partial x}+\frac{\partial\overline{u_y}}{\partial y}\right]+\overline{u_y}\frac{\partial\overline{u_x}}{\partial y}+\frac{\partial\overline{u'_x u'_x}}{\partial x}+\frac{\partial\overline{u'_x u'_y}}{\partial y}\right)+\frac{\partial\overline{p}}{\partial x}-\nu\left(\frac{\partial^2\overline{u_x}}{\partial x^2}+\frac{\partial^2\overline{u_x}}{\partial y^2}\right)=0 \quad (23)$$

The term in the square brackets is the continuity equation and thus zero. This gives the final form of the x-momentum RANS equation:

$$\left(\overline{u_x}\frac{\partial\overline{u_x}}{\partial x}+\overline{u_y}\frac{\partial\overline{u_x}}{\partial y}+\frac{\partial\overline{u'_x u'_x}}{\partial x}+\frac{\partial\overline{u'_x u'_y}}{\partial y}\right)+\frac{\partial\overline{p}}{\partial x}-\nu\left(\frac{\partial^2\overline{u_x}}{\partial x^2}+\frac{\partial^2\overline{u_x}}{\partial y^2}\right)=0 \quad (24)$$

We can isolate the terms corresponding to the Reynolds stress tensor:

$$\left(\overline{u_x}\frac{\partial\overline{u_x}}{\partial x}+\overline{u_y}\frac{\partial\overline{u_x}}{\partial y}\right)+\frac{\partial\overline{p}}{\partial x}-\nu\left(\frac{\partial^2\overline{u_x}}{\partial x^2}+\frac{\partial^2\overline{u_x}}{\partial y^2}\right)+\frac{\partial\overline{u'_x u'_x}}{\partial x}+\frac{\partial\overline{u'_x u'_y}}{\partial y}=0 \quad (25)$$

We can use a similar procedure to arrive at the y-momentum equation.

1.5 Summary of 2D RANS equations

The 2D momentum equations are:

$$\left(\overline{u_x}\frac{\partial\overline{u_x}}{\partial x}+\overline{u_y}\frac{\partial\overline{u_x}}{\partial y}\right)+\frac{\partial\overline{p}}{\partial x}-\nu\left(\frac{\partial^2\overline{u_x}}{\partial x^2}+\frac{\partial^2\overline{u_x}}{\partial y^2}\right)+\frac{\partial\overline{u'_x u'_x}}{\partial x}+\frac{\partial\overline{u'_x u'_y}}{\partial y}=0 \quad (26)$$

$$\left(\overline{u_x}\frac{\partial\overline{u_y}}{\partial x}+\overline{u_y}\frac{\partial\overline{u_y}}{\partial y}\right)+\frac{\partial\overline{p}}{\partial y}-\nu\left(\frac{\partial^2\overline{u_y}}{\partial x^2}+\frac{\partial^2\overline{u_y}}{\partial y^2}\right)+\frac{\partial\overline{u'_x u'_y}}{\partial x}+\frac{\partial\overline{u'_y u'_y}}{\partial y}=0 \quad (27)$$

with the continuity equation:

$$\frac{\partial\overline{u_x}}{\partial x}+\frac{\partial\overline{u_y}}{\partial y}=0 \quad (28)$$

In vector form:

$$\left(\overline{\mathbf{u}}\cdot\nabla\right)\overline{\mathbf{u}}+\nabla\overline{p}-\nu\nabla^2\overline{\mathbf{u}}+\nabla\cdot\overline{\mathbf{u'u'}}=0 \quad (29)$$

with the continuity equation:

$$\nabla\cdot\overline{\mathbf{u}}=0 \quad (30)$$

This gives a neural network with the equation:

$$\left(\overline{u_x},\overline{u_y},\overline{u'_x u'_x},\overline{u'_x u'_y},\overline{u'_y u'_y}\right)[p]=f(x,y) \quad (31)$$

with the loss functions:

$$L_R=\frac{1}{N_c}\sum_{1}^{N_c}\left(\left(\overline{\mathbf{u}}\cdot\nabla\right)\overline{\mathbf{u}}+\nabla\overline{p}-\nu\nabla^2\overline{\mathbf{u}}+\nabla\cdot\overline{\mathbf{u'u'}}\right) \quad (32)$$

$$L_C=\frac{1}{N_c}\sum_{1}^{N_c}\left(\nabla\cdot\overline{\mathbf{u}}\right) \quad (33)$$

$$L_D=\frac{1}{N_D}\sum_{1}^{N_D}(\overline{u_x}_{PINN}-\overline{u_x}_D)^2+(\overline{u_y}_{PINN}-\overline{u_y}_D)^2+(\overline{u'_x u'_x}_{PINN}-\overline{u'_x u'_x}_D)^2 \quad (34)$$

$$+(\overline{u'_x u'_y}_{PINN}-\overline{u'_x u'_y}_D)^2+(\overline{u'_y u'_y}_{PINN}-\overline{u'_y u'_y}_D)^2 \quad (35)$$

$$L=\gamma\left(L_R+L_C\right)+L_D \quad (36)$$

2 Triple Decomposition

There was an interesting paper Baj 2015 that applies OMD to a multi-scale problem. Instead of decomposing the flow using the Reynolds decomposition, we can instead decompose into a mean, coherent, and incoherent part:

$$\mathbf{u}(\mathbf{x},t) = \overline{\mathbf{u}}(\mathbf{x}) + \widetilde{\mathbf{u}}(\mathbf{x},t) + \mathbf{u}''(\mathbf{x},t) \tag{37}$$

If we substitute this into the x-momentum equation:

$$\begin{aligned} &\frac{\partial(\overline{u_x} + \widetilde{u}_x + u''_x)}{\partial t} + \left(\frac{\partial(\overline{u_x} + \widetilde{u}_x + u''_x)(\overline{u_x} + \widetilde{u}_x + u''_x)}{\partial x} + \frac{\partial(\overline{u_x} + \widetilde{u}_x + u''_x)(\overline{u_y} + \widetilde{u}_y + u''_y)}{\partial y} \right) \\ &\qquad + \frac{\partial(\overline{p} + \widetilde{p} + p'')}{\partial x} - \nu \left(\frac{\partial^2(\overline{u_x} + \widetilde{u}_x + u''_x)}{\partial x^2} + \frac{\partial^2(\overline{u_x} + \widetilde{u}_x + u''_x)}{\partial y^2} \right) = 0 \end{aligned} \tag{38}$$

$$\begin{aligned} &\left(\frac{\partial \overline{u_x}}{\partial t} + \frac{\partial \widetilde{u}_x}{\partial t} + \frac{\partial u''_x}{\partial t} \right) + \left(\frac{\partial \overline{u_x}\,\overline{u_x}}{\partial x} + 2\frac{\partial \overline{u_x}\widetilde{u}_x}{\partial x} + 2\frac{\partial \overline{u_x} u''_x}{\partial x} + \frac{\partial \widetilde{u}_x \widetilde{u}_x}{\partial x} + 2\frac{\partial \widetilde{u}_x u''_x}{\partial x} + \frac{\partial u''_x u''_x}{\partial x} + \right. \\ &\quad \left. \frac{\partial \overline{u_x}\,\overline{u_y}}{\partial y} + \frac{\partial \overline{u_x}\widetilde{u}_y}{\partial y} + \frac{\partial \overline{u_x} u''_y}{\partial y} + \frac{\partial \widetilde{u}_x \overline{u_y}}{\partial y} + \frac{\partial \widetilde{u}_x \widetilde{u}_y}{\partial y} + \frac{\partial \widetilde{u}_x u''_y}{\partial y} + \frac{\partial u''_x \overline{u_y}}{\partial y} + \frac{\partial u''_x \widetilde{u}_y}{\partial y} + \frac{\partial u''_x u''_y}{\partial y} \right) \\ &\qquad + \left(\frac{\partial \overline{p}}{\partial x} + \frac{\partial \widetilde{p}}{\partial x} + \frac{\partial p''}{\partial x} \right) - \nu \left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \widetilde{u}_x}{\partial x^2} + \frac{\partial^2 u''_x}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2} + \frac{\partial^2 \widetilde{u}_x}{\partial y^2} + \frac{\partial^2 u''_x}{\partial y^2} \right) = 0 \end{aligned} \tag{39}$$

Now the y-momentum equation:

$$\begin{aligned} &\frac{\partial(\overline{u_y} + \widetilde{u}_y + u''_y)}{\partial t} + \left(\frac{\partial(\overline{u_x} + \widetilde{u}_x + u''_x)(\overline{u_y} + \widetilde{u}_y + u''_y)}{\partial x} + \frac{\partial(\overline{u_y} + \widetilde{u}_y + u''_y)(\overline{u_y} + \widetilde{u}_y + u''_y)}{\partial y} \right) \\ &\qquad + \frac{\partial(\overline{p} + \widetilde{p} + p'')}{\partial y} - \nu \left(\frac{\partial^2(\overline{u_y} + \widetilde{u}_y + u''_y)}{\partial x^2} + \frac{\partial^2(\overline{u_y} + \widetilde{u}_y + u''_y)}{\partial y^2} \right) = 0 \end{aligned} \tag{40}$$

$$\begin{aligned} &\left(\frac{\partial \overline{u_y}}{\partial t} + \frac{\partial \widetilde{u}_y}{\partial t} + \frac{\partial u''_y}{\partial t} \right) + \\ &\left(\frac{\partial \overline{u_x}\,\overline{u_y}}{\partial x} + \frac{\partial \overline{u_x}\widetilde{u}_y}{\partial x} + \frac{\partial \overline{u_x} u''_y}{\partial x} + \frac{\partial \widetilde{u}_x \overline{u_y}}{\partial x} + \frac{\partial \widetilde{u}_x \widetilde{u}_y}{\partial x} + \frac{\partial \widetilde{u}_x u''_y}{\partial x} + \frac{\partial u''_x \overline{u_y}}{\partial x} + \frac{\partial u''_x \widetilde{u}_y}{\partial x} + \frac{\partial u''_x u''_y}{\partial x} \right. \\ &\quad \left. + \frac{\partial \overline{u_y}\,\overline{u_y}}{\partial y} + 2\frac{\partial \overline{u_y}\widetilde{u}_y}{\partial y} + 2\frac{\partial \overline{u_y} u''_y}{\partial y} + \frac{\partial \widetilde{u}_y \widetilde{u}_y}{\partial y} + 2\frac{\partial \widetilde{u}_y u''_y}{\partial y} + \frac{\partial u''_y u''_y}{\partial y} \right) \\ &\qquad + \left(\frac{\partial \overline{p}}{\partial y} + \frac{\partial \widetilde{p}}{\partial y} + \frac{\partial p''}{\partial y} \right) - \nu \left(\frac{\partial^2 \overline{u_y}}{\partial x^2} + \frac{\partial^2 \widetilde{u}_y}{\partial x^2} + \frac{\partial^2 u''_y}{\partial x^2} + \frac{\partial^2 \overline{u_y}}{\partial y^2} + \frac{\partial^2 \widetilde{u}_y}{\partial y^2} + \frac{\partial^2 u''_y}{\partial y^2} \right) = 0 \end{aligned} \tag{41}$$

And the continuity equation:

$$\frac{\partial(\overline{u_x} + \widetilde{u}_x + u''_x)}{\partial x} + \frac{\partial(\overline{u_y} + \widetilde{u}_y + u''_y)}{\partial y} = 0 \tag{42}$$

$$\frac{\partial \overline{u_x}}{\partial x} + \frac{\partial \widetilde{u}_x}{\partial x} + \frac{\partial u''_x}{\partial x} + \frac{\partial \overline{u_y}}{\partial y} + \frac{\partial \widetilde{u}_y}{\partial y} + \frac{\partial u''_y}{\partial y} = 0 \tag{43}$$

2.1 Reynolds Averaged Triple Decomposition

If we take the time average of the terms, for a statistically stationary flow:

$$\begin{aligned} &\left(\frac{\partial \overline{\overline{u_x}}}{\partial t} + \frac{\partial \overline{\widetilde{u}_x}}{\partial t} + \frac{\partial \overline{u''_x}}{\partial t} \right) + \left(\frac{\partial \overline{\overline{u_x}\,\overline{u_x}}}{\partial x} + 2\frac{\partial \overline{\overline{u_x}\widetilde{u}_x}}{\partial x} + 2\frac{\partial \overline{\overline{u_x} u''_x}}{\partial x} + \frac{\partial \overline{\widetilde{u}_x u'_x}}{\partial x} + 2\frac{\partial \overline{\widetilde{u}_x u''_x}}{\partial x} + \frac{\partial \overline{u''_x u''_x}}{\partial x} + \right. \\ &\quad \left. \frac{\partial \overline{\overline{u_x}\,\overline{u_y}}}{\partial y} + \frac{\partial \overline{\overline{u_x}\widetilde{u}_y}}{\partial y} + \frac{\partial \overline{\overline{u_x} u''_y}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x \overline{u_y}}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x u''_y}}{\partial y} + \frac{\partial \overline{u''_x \overline{u_y}}}{\partial y} + \frac{\partial \overline{u''_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{u''_x u''_y}}{\partial y} \right) \\ &\qquad + \left(\frac{\partial \overline{\overline{p}}}{\partial x} + \frac{\partial \overline{\widetilde{p}}}{\partial x} + \frac{\partial \overline{p''}}{\partial x} \right) - \nu \left(\frac{\partial^2 \overline{\overline{u_x}}}{\partial x^2} + \frac{\partial^2 \overline{\widetilde{u}_x}}{\partial x^2} + \frac{\partial^2 \overline{u''_x}}{\partial x^2} + \frac{\partial^2 \overline{\overline{u_x}}}{\partial y^2} + \frac{\partial^2 \overline{u'_x}}{\partial y^2} + \frac{\partial^2 \overline{u''_x}}{\partial y^2} \right) = 0 \end{aligned} \tag{44}$$

$$\left(\frac{\partial \overline{u_x}\,\overline{u_x}}{\partial x} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_x}}{\partial x} + 2\frac{\partial \overline{\widetilde{u}_x u''_x}}{\partial x} + \frac{\partial \overline{u''_x u''_x}}{\partial x} + \frac{\partial \overline{u_x}\,\overline{u_y}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x u''_y}}{\partial y} + \frac{\partial \overline{u''_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{u''_x u''_y}}{\partial y}\right) + \left(\frac{\partial \overline{p}}{\partial x}\right) - \nu\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2}\right) = 0 \tag{45}$$

Applying product rule to the mean field terms:

$$\left(2\overline{u_x}\frac{\partial \overline{u_x}}{\partial x} + \overline{u_x}\frac{\partial \overline{u_y}}{\partial y} + \overline{u_y}\frac{\partial \overline{u_x}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_x}}{\partial x} + 2\frac{\partial \overline{\widetilde{u}_x u''_x}}{\partial x} + \frac{\partial \overline{u''_x u''_x}}{\partial x} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x u''_y}}{\partial y} + \frac{\partial \overline{u''_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{u''_x u''_y}}{\partial y}\right) + \left(\frac{\partial \overline{p}}{\partial x}\right) - \nu\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2}\right) = 0 \tag{46}$$

$$\left(\overline{u_x}\frac{\partial \overline{u_x}}{\partial x} + \overline{u_x}\left[\frac{\partial \overline{u_x}}{\partial x} + \frac{\partial \overline{u_y}}{\partial y}\right] + \overline{u_y}\frac{\partial \overline{u_x}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_x}}{\partial x} + 2\frac{\partial \overline{\widetilde{u}_x u''_x}}{\partial x} + \frac{\partial \overline{u''_x u''_x}}{\partial x} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x u''_y}}{\partial y} + \frac{\partial \overline{u''_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{u''_x u''_y}}{\partial y}\right) + \left(\frac{\partial \overline{p}}{\partial x}\right) - \nu\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2}\right) = 0 \tag{47}$$

By continuity the square brackets are zero:

$$\left(\overline{u_x}\frac{\partial \overline{u_x}}{\partial x} + \overline{u_y}\frac{\partial \overline{u_x}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_x}}{\partial x} + 2\frac{\partial \overline{\widetilde{u}_x u''_x}}{\partial x} + \frac{\partial \overline{u''_x u''_x}}{\partial x} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x u''_y}}{\partial y} + \frac{\partial \overline{u''_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{u''_x u''_y}}{\partial y}\right) + \left(\frac{\partial \overline{p}}{\partial x}\right) - \nu\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2}\right) = 0 \tag{48}$$

A numerical investigation of the DNS data for a fixed cylinder at Re=150 showed that the cross terms have zero mean. Is this a general result? Simplifying:

$$\left(\overline{u_x}\frac{\partial \overline{u_x}}{\partial x} + \overline{u_y}\frac{\partial \overline{u_x}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_x}}{\partial x} + \frac{\partial \overline{u''_x u''_x}}{\partial x} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{u''_x u''_y}}{\partial y}\right) + \left(\frac{\partial \overline{p}}{\partial x}\right) - \nu\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2}\right) = 0 \tag{49}$$

And for the y-momentum equation we would have:

$$\left(\overline{u_x}\frac{\partial \overline{u_y}}{\partial x} + \overline{u_y}\frac{\partial \overline{u_y}}{\partial y} + \frac{\partial \overline{\widetilde{u}_x \widetilde{u}_y}}{\partial x} + \frac{\partial \overline{u''_x u''_y}}{\partial x} + \frac{\partial \overline{\widetilde{u}_y \widetilde{u}_y}}{\partial y} + \frac{\partial \overline{u''_y u''_y}}{\partial y}\right) + \frac{\partial \overline{p}}{\partial y} - \nu\left(\frac{\partial^2 \overline{u_y}}{\partial x^2} + \frac{\partial^2 \overline{u_y}}{\partial y^2}\right) = 0 \tag{50}$$

2.1.1 A note on the coherent fluctuating term applied to POD modes

I am not entirely sure that the following is correct, but here goes. If we substitute a set of POD modes in for $\widetilde{u}_x$:

$$\begin{aligned}
\frac{\partial \overline{\widetilde{u}_x \widetilde{u}_x}}{\partial x} &= \frac{\partial}{\partial x}\overline{(\phi_{1,x}a_1 + \phi_{2,x}a_2)(\phi_{1,x}a_1 + \phi_{2,x}a_2)} \\
&= \frac{\partial}{\partial x}\overline{(\phi_{1,x}^2 a_1^2 + 2\phi_{1,x}a_1\phi_{2,x}a_2 + \phi_{2,x}^2 a_2^2)} \\
&= \frac{\partial}{\partial x}(\overline{\phi_{1,x}^2 a_1^2} + \overline{2\phi_{1,x}a_1\phi_{2,x}a_2} + \overline{\phi_{2,x}^2 a_2^2})
\end{aligned} \tag{51}$$

but a_1 and a_2 are orthogonal so the middle term is zero. Since the spatial modes don't depend on time, we can just apply the time average to a_i^2, which is just a scalar value. If we define:

$$A_i = \overline{a_i^2} \tag{52}$$

Thus the term becomes:

$$
\begin{aligned}
&= \frac{\partial}{\partial x}(\overline{\phi_{1,x}^2 a_1^2} + \overline{\phi_{2,x}^2 a_2^2}) \\
&= \frac{\partial}{\partial x}(\phi_{1,x}^2 A_1 + \phi_{2,x}^2 A_2) \\
&= 2A_1\phi_{1,x}\frac{\partial \phi_{1,x}}{\partial x} + 2A_2\phi_{2,x}\frac{\partial \phi_{2,x}}{\partial x}
\end{aligned} \tag{53}
$$

If we examine the $\widetilde{u}_x\widetilde{u}_y$ term:

$$
\begin{aligned}
\frac{\partial \overline{\widetilde{u}_x\widetilde{u}_y}}{\partial y} &= \frac{\partial}{\partial x}\overline{(\phi_{1,x}a_1 + \phi_{2,x}a_2)(\phi_{1,y}a_1 + \phi_{2,y}a_2)} \\
&= \frac{\partial}{\partial y}\overline{(\phi_{1,x}\phi_{1,y}a_1^2 + \phi_{1,x}a_1\phi_{2,y}a_2 + \phi_{1,y}a_1\phi_{2,x}a_2 + \phi_{2,x}\phi_{2,y}a_2^2)} \\
&= \frac{\partial}{\partial y}(\overline{\phi_{1,x}\phi_{1,y}a_1^2} + \overline{\phi_{1,x}a_1\phi_{2,y}a_2} + \overline{\phi_{1,y}a_1\phi_{2,x}a_2} + \overline{\phi_{2,x}\phi_{2,y}a_2^2}) \\
&= A_1\frac{\partial \phi_{1,x}\phi_{1,y}}{\partial y} + A_2\frac{\partial \phi_{2,x}\phi_{2,y}}{\partial y} \\
&= A_1\left(\phi_{1,x}\frac{\partial \phi_{1,y}}{\partial y} + \phi_{1,y}\frac{\partial \phi_{1,x}}{\partial y}\right) + A_2\left(\phi_{2,x}\frac{\partial \phi_{2,y}}{\partial y} + \phi_{2,y}\frac{\partial \phi_{2,x}}{\partial y}\right)
\end{aligned} \tag{54}
$$

This is valid for all of the $''$ terms, so the 4 terms in full are, for some arbitrary truncation of POD modes:

$$
\begin{aligned}
\frac{\partial \overline{\widetilde{u}_x\widetilde{u}_x}}{\partial x} &= \sum_i^{n_t} 2A_i\phi_{i,x}\frac{\partial \phi_{i,x}}{\partial x} \\
\frac{\partial \overline{\widetilde{u}_x\widetilde{u}_y}}{\partial x} &= \sum_i^{n_t} A_i\left(\phi_{i,x}\frac{\partial \phi_{i,y}}{\partial x} + \phi_{i,y}\frac{\partial \phi_{i,x}}{\partial x}\right) \\
\frac{\partial \overline{\widetilde{u}_x\widetilde{u}_y}}{\partial y} &= \sum_i^{n_t} A_i\left(\phi_{i,x}\frac{\partial \phi_{i,y}}{\partial y} + \phi_{i,y}\frac{\partial \phi_{i,x}}{\partial y}\right) \\
\frac{\partial \overline{\widetilde{u}_y\widetilde{u}_y}}{\partial y} &= \sum_i^{n_t} 2A_i\phi_{i,y}\frac{\partial \phi_{i,y}}{\partial y}
\end{aligned} \tag{55}
$$

2.1.2 Neural network for the RANS-triple decomposition

Using the above, we can train our network on the POD modes. First I would try the following with 2 modes:

$$
\begin{aligned}
(\overline{u_x}, \overline{u_y}, \phi_1, \phi_2, \overline{u_x''u_x''}, \overline{u_x''u_y''}, \overline{u_y''u_y''})[\overline{p}] &= f(x,y) \\
(A_1, A_2) &= f(1)
\end{aligned} \tag{56}
$$

With 4 modes, we capture about 98% of the energy, so this is probably good enough to continue to the next step:

$$
\begin{aligned}
(\overline{u_x}, \overline{u_y}, \phi_1, \phi_2, \phi_3, \phi_4, \overline{u_x''u_x''}, \overline{u_x''u_y''}, \overline{u_y''u_y''})[\overline{p}] &= f(x,y) \\
(A_1, A_2, A_3, A_4) &= f(1)
\end{aligned} \tag{57}
$$

If this works well also for the 2 cylinder flow, we can try it on the real data. We can also build a model of the Re=150 fixed cylinder using a galerkin projection.

2.2 Expansion of the time-varying triple decomposition

Lets expand the x-momentum equation using the product rule, so we can evaluate the terms:

$$\begin{aligned}
&\left(\frac{\partial \overline{u_x}}{\partial t}+\frac{\partial \widetilde{u}_x}{\partial t}+\frac{\partial u''_x}{\partial t}\right)\\
&+\left(\frac{\partial \overline{u_x}\,\overline{u_x}}{\partial x}+2\frac{\partial \overline{u_x}\widetilde{u}_x}{\partial x}+2\frac{\partial \overline{u_x}u''_x}{\partial x}\right.\\
&+\frac{\partial \widetilde{u}_x\widetilde{u}_x}{\partial x}+2\frac{\partial \widetilde{u}_x u''_x}{\partial x}+\frac{\partial u''_x u''_x}{\partial x}\\
&+\frac{\partial \overline{u_x}\,\overline{u_y}}{\partial y}+\frac{\partial \overline{u_x}\widetilde{u}_y}{\partial y}+\frac{\partial \overline{u_x}u''_y}{\partial y}\\
&+\frac{\partial \widetilde{u}_x\overline{u_y}}{\partial y}+\frac{\partial \widetilde{u}_x\widetilde{u}_y}{\partial y}+\frac{\partial \widetilde{u}_x u''_y}{\partial y}\\
&\left.+\frac{\partial u''_x\overline{u_y}}{\partial y}+\frac{\partial u''_x\widetilde{u}_y}{\partial y}+\frac{\partial u''_x u''_y}{\partial y}\right)\\
&+\left(\frac{\partial \overline{p}}{\partial x}+\frac{\partial \widetilde{p}}{\partial x}+\frac{\partial p''}{\partial x}\right)\\
-\nu&\left(\frac{\partial^2 \overline{u_x}}{\partial x^2}+\frac{\partial^2 \widetilde{u}_x}{\partial x^2}+\frac{\partial^2 u''_x}{\partial x^2}+\frac{\partial^2 \overline{u_x}}{\partial y^2}+\frac{\partial^2 \widetilde{u}_x}{\partial y^2}+\frac{\partial^2 u''_x}{\partial y^2}\right)=0
\end{aligned} \tag{58}$$

$$\begin{aligned}
&\left(\frac{\partial \overline{u_x}}{\partial t}+\frac{\partial \widetilde{u}_x}{\partial t}+\frac{\partial u''_x}{\partial t}\right)\\
&+\left(2\overline{u_x}\frac{\partial \overline{u_x}}{\partial x}+2\overline{u_x}\frac{\partial \widetilde{u}_x}{\partial x}+2\widetilde{u}_x\frac{\partial \overline{u_x}}{\partial x}+2\overline{u_x}\frac{\partial u''_x}{\partial x}+2u''_x\frac{\partial \overline{u_x}}{\partial x}\right.\\
&+\widetilde{u}_x\frac{\partial \widetilde{u}_x}{\partial x}+\widetilde{u}_x\frac{\partial \widetilde{u}_x}{\partial x}+2\widetilde{u}_x\frac{\partial u''_x}{\partial x}+2u''_x\frac{\partial \widetilde{u}_x}{\partial x}+2u''_x\frac{\partial u''_x}{\partial x}\\
&+\overline{u_x}\frac{\partial \overline{u_y}}{\partial y}+\overline{u_y}\frac{\partial \overline{u_x}}{\partial y}+\overline{u_x}\frac{\partial \widetilde{u}_y}{\partial y}+\widetilde{u}_y\frac{\partial \overline{u_x}}{\partial y}+\overline{u_x}\frac{\partial u''_y}{\partial y}+u''_y\frac{\partial \overline{u_x}}{\partial y}\\
&+\widetilde{u}_x\frac{\partial \overline{u_y}}{\partial y}+\overline{u_y}\frac{\partial \widetilde{u}_x}{\partial y}+\widetilde{u}_x\frac{\partial \widetilde{u}_y}{\partial y}+\widetilde{u}_y\frac{\partial \widetilde{u}_x}{\partial y}+\widetilde{u}_x\frac{\partial u''_y}{\partial y}+u''_y\frac{\partial \widetilde{u}_x}{\partial y}\\
&\left.+u''_x\frac{\partial \overline{u_y}}{\partial y}+\overline{u_y}\frac{\partial u''_x}{\partial y}+u''_x\frac{\partial \widetilde{u}_y}{\partial y}+\widetilde{u}_y\frac{\partial u''_x}{\partial y}+u''_x\frac{\partial u''_y}{\partial y}+u''_y\frac{\partial u''_x}{\partial y}\right)\\
&+\left(\frac{\partial \overline{p}}{\partial x}+\frac{\partial \widetilde{p}}{\partial x}+\frac{\partial p''}{\partial x}\right)\\
-\nu&\left(\frac{\partial^2 \overline{u_x}}{\partial x^2}+\frac{\partial^2 \widetilde{u}_x}{\partial x^2}+\frac{\partial^2 u''_x}{\partial x^2}+\frac{\partial^2 \overline{u_x}}{\partial y^2}+\frac{\partial^2 \widetilde{u}_x}{\partial y^2}+\frac{\partial^2 u''_x}{\partial y^2}\right)=0
\end{aligned} \tag{59}$$

If we assume the flow is statistically stationary we can eliminate the time derivative of the mean field, and we can also use RANS continuity to eliminate a term for the mean field:

$$
\begin{aligned}
&\left(\frac{\partial \widetilde{u}_x}{\partial t} + \frac{\partial u''_x}{\partial t}\right) \\
&+\left(\overline{u_x}\frac{\partial \overline{u_x}}{\partial x} + 2\overline{u_x}\frac{\partial \widetilde{u}_x}{\partial x} + 2\widetilde{u}_x\frac{\partial \overline{u_x}}{\partial x} + 2\overline{u_x}\frac{\partial u''_x}{\partial x} + 2u''_x\frac{\partial \overline{u_x}}{\partial x}\right. \\
&+\widetilde{u}_x\frac{\partial \widetilde{u}_x}{\partial x} + \widetilde{u}_x\frac{\partial \widetilde{u}_x}{\partial x} + 2\widetilde{u}_x\frac{\partial u''_x}{\partial x} + 2u''_x\frac{\partial \widetilde{u}_x}{\partial x} + 2u''_x\frac{\partial u''_x}{\partial x} \\
&+\overline{u_y}\frac{\partial \overline{u_x}}{\partial y} + \overline{u_x}\frac{\partial \widetilde{u}_y}{\partial y} + \widetilde{u}_y\frac{\partial \overline{u_x}}{\partial y} + \overline{u_x}\frac{\partial u''_y}{\partial y} + u''_y\frac{\partial \overline{u_x}}{\partial y} \\
&+\widetilde{u}_x\frac{\partial \overline{u_y}}{\partial y} + \overline{u_y}\frac{\partial \widetilde{u}_x}{\partial y} + \widetilde{u}_x\frac{\partial \widetilde{u}_y}{\partial y} + \widetilde{u}_y\frac{\partial \widetilde{u}_x}{\partial y} + \widetilde{u}_x\frac{\partial u''_y}{\partial y} + u''_y\frac{\partial \widetilde{u}_x}{\partial y} \\
&\left.+u''_x\frac{\partial \overline{u_y}}{\partial y} + \overline{u_y}\frac{\partial u''_x}{\partial y} + u''_x\frac{\partial \widetilde{u}_y}{\partial y} + \widetilde{u}_y\frac{\partial u''_x}{\partial y} + u''_x\frac{\partial u''_y}{\partial y} + u''_y\frac{\partial u''_x}{\partial y}\right) \\
&+\left(\frac{\partial \overline{p}}{\partial x} + \frac{\partial \widetilde{p}}{\partial x} + \frac{\partial p''}{\partial x}\right) \\
&-\nu\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \widetilde{u}_x}{\partial x^2} + \frac{\partial^2 u''_x}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2} + \frac{\partial^2 \widetilde{u}_x}{\partial y^2} + \frac{\partial^2 u''_x}{\partial y^2}\right) = 0
\end{aligned}
\tag{60}
$$

For the y-momentum equation:

$$
\begin{aligned}
&\left(\frac{\partial \overline{u_y}}{\partial t} + \frac{\partial \widetilde{u}_y}{\partial t} + \frac{\partial u''_y}{\partial t}\right) + \\
&\left(\frac{\partial \overline{u_x u_y}}{\partial x} + \frac{\partial \overline{u_x}\widetilde{u}_y}{\partial x} + \frac{\partial \overline{u_x}u''_y}{\partial x}\right. \\
&+\frac{\partial \widetilde{u}_x\overline{u_y}}{\partial x} + \frac{\partial \widetilde{u}_x\widetilde{u}_y}{\partial x} + \frac{\partial \widetilde{u}_x u''_y}{\partial x} \\
&+\frac{\partial u''_x\overline{u_y}}{\partial x} + \frac{\partial u''_x\widetilde{u}_y}{\partial x} + \frac{\partial u''_x u''_y}{\partial x} \\
&+\frac{\partial \overline{u_y u_y}}{\partial y} + 2\frac{\partial \overline{u_y}\widetilde{u}_y}{\partial y} + 2\frac{\partial \overline{u_y}u''_y}{\partial y} \\
&\left.+\frac{\partial \widetilde{u}_y\widetilde{u}_y}{\partial y} + 2\frac{\partial \widetilde{u}_y u''_y}{\partial y} + \frac{\partial u''_y u''_y}{\partial y}\right) \\
&+\left(\frac{\partial \overline{p}}{\partial y} + \frac{\partial \widetilde{p}}{\partial y} + \frac{\partial p''}{\partial y}\right) \\
&-\nu\left(\frac{\partial^2 \overline{u_y}}{\partial x^2} + \frac{\partial^2 \widetilde{u}_y}{\partial x^2} + \frac{\partial^2 u''_y}{\partial x^2} + \frac{\partial^2 \overline{u_y}}{\partial y^2} + \frac{\partial^2 \widetilde{u}_y}{\partial y^2} + \frac{\partial^2 u''_y}{\partial y^2}\right) = 0
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&\left(\frac{\partial \overline{u_y}}{\partial t} + \frac{\partial \widetilde{u}_y}{\partial t} + \frac{\partial u''_y}{\partial t}\right) + \\
&\left(\overline{u_x}\frac{\partial \overline{u_y}}{\partial x} + \overline{u_y}\frac{\partial \overline{u_x}}{\partial x} + \overline{u_x}\frac{\partial \widetilde{u}_y}{\partial x} + \widetilde{u}_y\frac{\partial \overline{u_x}}{\partial x} + \overline{u_x}\frac{\partial u''_y}{\partial x} + u''_y\frac{\partial \overline{u_x}}{\partial x}\right. \\
&+\widetilde{u}_x\frac{\partial \overline{u_y}}{\partial x} + \overline{u_y}\frac{\partial \widetilde{u}_x}{\partial x} + \widetilde{u}_x\frac{\partial \widetilde{u}_y}{\partial x} + \widetilde{u}_y\frac{\partial \widetilde{u}_x}{\partial x} + \widetilde{u}_x\frac{\partial u''_y}{\partial x} + u''_y\frac{\partial \widetilde{u}_x}{\partial x} \\
&+u''_x\frac{\partial \overline{u_y}}{\partial x} + \overline{u_y}\frac{\partial u''_x}{\partial x} + u''_x\frac{\partial \widetilde{u}_y}{\partial x} + \widetilde{u}_y\frac{\partial u''_x}{\partial x} + u''_x\frac{\partial u''_y}{\partial x} + u''_y\frac{\partial u''_x}{\partial x} \\
&+2\overline{u_y}\frac{\partial \overline{u_y}}{\partial y} + 2\overline{u_y}\frac{\partial \widetilde{u}_y}{\partial y} + 2\widetilde{u}_y\frac{\partial \overline{u_y}}{\partial y} + 2\overline{u_y}\frac{\partial u''_y}{\partial y} + 2u''_y\frac{\partial \overline{u_y}}{\partial y} \\
&\left.+\widetilde{u}_y\frac{\partial \widetilde{u}_y}{\partial y} + \widetilde{u}_y\frac{\partial \widetilde{u}_y}{\partial y} + 2\widetilde{u}_y\frac{\partial u''_y}{\partial y} + 2u''_y\frac{\partial \widetilde{u}_y}{\partial y} + 2u''_y\frac{\partial u''_y}{\partial y}\right) \\
&+\left(\frac{\partial \overline{p}}{\partial y} + \frac{\partial \widetilde{p}}{\partial y} + \frac{\partial p''}{\partial y}\right) \\
&-\nu\left(\frac{\partial^2 \overline{u_y}}{\partial x^2} + \frac{\partial^2 \widetilde{u}_y}{\partial x^2} + \frac{\partial^2 u''_y}{\partial x^2} + \frac{\partial^2 \overline{u_y}}{\partial y^2} + \frac{\partial^2 \widetilde{u}_y}{\partial y^2} + \frac{\partial^2 u''_y}{\partial y^2}\right) = 0
\end{aligned}
\tag{62}
$$

Similar to above, we can simplify:

$$\begin{aligned}
\left(\frac{\partial \overline{u_y}}{\partial t}+\frac{\partial \widetilde{u}_y}{\partial t}+\frac{\partial u_y''}{\partial t}\right)&+\\
\left(\overline{u_x}\frac{\partial \overline{u_y}}{\partial x}+\overline{u_x}\frac{\partial \widetilde{u}_y}{\partial x}+\widetilde{u}_y\frac{\partial \overline{u_x}}{\partial x}+\overline{u_x}\frac{\partial u_y''}{\partial x}+u_y''\frac{\partial \overline{u_x}}{\partial x}\right.&\\
+\widetilde{u}_x\frac{\partial \overline{u_y}}{\partial x}+\overline{u_y}\frac{\partial \widetilde{u}_x}{\partial x}+\widetilde{u}_x\frac{\partial \widetilde{u}_y}{\partial x}+\widetilde{u}_y\frac{\partial \widetilde{u}_x}{\partial x}+\widetilde{u}_x\frac{\partial u_y''}{\partial x}+u_y''\frac{\partial \widetilde{u}_x}{\partial x}&\\
+u_x''\frac{\partial \overline{u_y}}{\partial x}+\overline{u_y}\frac{\partial u_x''}{\partial x}+u_x''\frac{\partial \widetilde{u}_y}{\partial x}+\widetilde{u}_y\frac{\partial u_x''}{\partial x}+u_x''\frac{\partial u_y''}{\partial x}+u_y''\frac{\partial u_x''}{\partial x}&\\
+\overline{u_y}\frac{\partial \overline{u_y}}{\partial y}+2\overline{u_y}\frac{\partial \widetilde{u}_y}{\partial y}+2\widetilde{u}_y\frac{\partial \overline{u_y}}{\partial y}+2\overline{u_y}\frac{\partial u_y''}{\partial y}+2u_y''\frac{\partial \overline{u_y}}{\partial y}&\\
\left.+\widetilde{u}_y\frac{\partial \widetilde{u}_y}{\partial y}+\widetilde{u}_y\frac{\partial \widetilde{u}_y}{\partial y}+2\widetilde{u}_y\frac{\partial u_y''}{\partial y}+2u_y''\frac{\partial \widetilde{u}_y}{\partial y}+2u_y''\frac{\partial u_y''}{\partial y}\right)&\\
+\left(\frac{\partial \overline{p}}{\partial y}+\frac{\partial \widetilde{p}}{\partial y}+\frac{\partial p''}{\partial y}\right)&\\
-\nu\left(\frac{\partial^2 \overline{u_y}}{\partial x^2}+\frac{\partial^2 \widetilde{u}_y}{\partial x^2}+\frac{\partial^2 u_y''}{\partial x^2}+\frac{\partial^2 \overline{u_y}}{\partial y^2}+\frac{\partial^2 \widetilde{u}_y}{\partial y^2}+\frac{\partial^2 u_y''}{\partial y^2}\right)&=0
\end{aligned} \tag{63}$$

In the case of our PINNs, as long as we can evaluate each of the 3 parts of the expression ($\overline{u}, u', u''$) and their derivatives, we can solve the above equations. $\overline{u}$ is assimilated using the RANS equation. In the case of the computational data, u'' is known since we can compute it based on $u''=u-\overline{u}-u'$. For experimental data, we can only estimate u'' using this method because u does not necessarily satisfy the NS equations; but perhaps it makes a good initial guess for the true u''. Since this is a quantity we will solve using the PINN, this should be OK. Alternatively we can investigate approaches to model u'' somehow. Perhaps there is a connection to $\overline{u''}$. Finally we just need an expression for the coherent part u'.

2.3 Linear Combination of Fourier/POD Modes

If we take the triple decomposition and decompose the coherent part into a linear combination of n_t velocities:

$$\mathbf{u}'=\sum_{i=1}^{n_t}\mathbf{u}_\mathbf{i}' \tag{64}$$

Let's let our modes be POD or Fourier modes which have the form:

$$\mathbf{u}_\mathbf{i}'(\mathbf{x},t)=\phi_i(\mathbf{x})a_i(t)=\phi_i a_i \tag{65}$$

Also noting that we get pressure modes from the extended POD, which have the form:

$$\mathbf{p}_\mathbf{i}'(\mathbf{x},t)=\psi_i(\mathbf{x})a_i(t)=\psi_i a_i \tag{66}$$

Then for some arbitrary order, we can construct our low order model and compute the derivative terms. For the fixed cylinder, 3 mode pairs should be sufficient to capture 95% of the energy of our flow. Thus we get:

$$\begin{aligned}
\widetilde{u}_x&=\phi_{1,x}a_1+\phi_{2,x}a_2+\phi_{3,x}a_3+\phi_{4,x}a_4+\phi_{5,x}a_5+\phi_{6,x}a_6\\
\widetilde{u}_y&=\phi_{1,y}a_1+\phi_{2,y}a_2+\phi_{3,y}a_3+\phi_{4,y}a_4+\phi_{5,y}a_5+\phi_{6,y}a_6\\
\widetilde{p}&=\psi_1a_1+\psi_2a_2+\psi_3a_3+\psi_4a_4+\psi_5a_5+\psi_6a_6
\end{aligned} \tag{67}$$

Which gives the derivative terms:

$$\begin{aligned}
\frac{\partial \widetilde{u}_x}{\partial x} &= a_1\frac{\partial \phi_{1,x}}{\partial x} + a_2\frac{\partial \phi_{2,x}}{\partial x} + a_3\frac{\partial \phi_{3,x}}{\partial x} + a_4\frac{\partial \phi_{4,x}}{\partial x} + a_5\frac{\partial \phi_{5,x}}{\partial x} + a_6\frac{\partial \phi_{6,x}}{\partial x}\\
\frac{\partial \widetilde{u}_x}{\partial y} &= a_1\frac{\partial \phi_{1,x}}{\partial y} + a_2\frac{\partial \phi_{2,x}}{\partial y} + a_3\frac{\partial \phi_{3,x}}{\partial y} + a_4\frac{\partial \phi_{4,x}}{\partial y} + a_5\frac{\partial \phi_{5,x}}{\partial y} + a_6\frac{\partial \phi_{6,x}}{\partial y}\\
\frac{\partial \widetilde{u}_y}{\partial x} &= a_1\frac{\partial \phi_{1,y}}{\partial x} + a_2\frac{\partial \phi_{2,y}}{\partial x} + a_3\frac{\partial \phi_{3,y}}{\partial x} + a_4\frac{\partial \phi_{4,y}}{\partial x} + a_5\frac{\partial \phi_{5,y}}{\partial x} + a_6\frac{\partial \phi_{6,y}}{\partial x}\\
\frac{\partial \widetilde{u}_y}{\partial y} &= a_1\frac{\partial \phi_{1,y}}{\partial y} + a_2\frac{\partial \phi_{2,y}}{\partial y} + a_3\frac{\partial \phi_{3,y}}{\partial y} + a_4\frac{\partial \phi_{4,y}}{\partial y} + a_5\frac{\partial \phi_{5,y}}{\partial y} + a_6\frac{\partial \phi_{6,y}}{\partial y}
\end{aligned}\tag{68}$$

$$\begin{aligned}
\frac{\partial^2 \widetilde{u}_x}{\partial x^2} &= a_1\frac{\partial^2 \phi_{1,x}}{\partial x^2} + a_2\frac{\partial^2 \phi_{2,x}}{\partial x^2} + a_3\frac{\partial^2 \phi_{3,x}}{\partial x^2} + a_4\frac{\partial^2 \phi_{4,x}}{\partial x^2} + a_5\frac{\partial^2 \phi_{5,x}}{\partial x^2} + a_6\frac{\partial^2 \phi_{6,x}}{\partial x^2}\\
\frac{\partial^2 \widetilde{u}_x}{\partial y^2} &= a_1\frac{\partial^2 \phi_{1,x}}{\partial y^2} + a_2\frac{\partial^2 \phi_{2,x}}{\partial y^2} + a_3\frac{\partial^2 \phi_{3,x}}{\partial y^2} + a_4\frac{\partial^2 \phi_{4,x}}{\partial y^2} + a_5\frac{\partial^2 \phi_{5,x}}{\partial y^2} + a_6\frac{\partial^2 \phi_{6,x}}{\partial y^2}\\
\frac{\partial^2 \widetilde{u}_y}{\partial x^2} &= a_1\frac{\partial^2 \phi_{1,y}}{\partial x^2} + a_2\frac{\partial^2 \phi_{2,y}}{\partial x^2} + a_3\frac{\partial^2 \phi_{3,y}}{\partial x^2} + a_4\frac{\partial^2 \phi_{4,y}}{\partial x^2} + a_5\frac{\partial^2 \phi_{5,y}}{\partial x^2} + a_6\frac{\partial^2 \phi_{6,y}}{\partial x^2}\\
\frac{\partial^2 \widetilde{u}_y}{\partial y^2} &= a_1\frac{\partial^2 \phi_{1,y}}{\partial y^2} + a_2\frac{\partial^2 \phi_{2,y}}{\partial y^2} + a_3\frac{\partial^2 \phi_{3,y}}{\partial y^2} + a_4\frac{\partial^2 \phi_{4,y}}{\partial y^2} + a_5\frac{\partial^2 \phi_{5,y}}{\partial y^2} + a_6\frac{\partial^2 \phi_{6,y}}{\partial y^2}
\end{aligned}\tag{69}$$

$$\begin{aligned}
\frac{\partial \widetilde{u}_x}{\partial t} &= \phi_{1,x}\frac{\partial a_1}{\partial t} + \phi_{2,x}\frac{\partial a_2}{\partial t} + \phi_{3,x}\frac{\partial a_3}{\partial t} + \phi_{4,x}\frac{\partial a_4}{\partial t} + \phi_{5,x}\frac{\partial a_5}{\partial t} + \phi_{6,x}\frac{\partial a_6}{\partial t}\\
\frac{\partial \widetilde{u}_y}{\partial t} &= \phi_{1,y}\frac{\partial a_1}{\partial t} + \phi_{2,y}\frac{\partial a_2}{\partial t} + \phi_{3,y}\frac{\partial a_3}{\partial t} + \phi_{4,y}\frac{\partial a_4}{\partial t} + \phi_{5,y}\frac{\partial a_5}{\partial t} + \phi_{6,y}\frac{\partial a_6}{\partial t}
\end{aligned}\tag{70}$$

$$\begin{aligned}
\frac{\partial \widetilde{p}}{\partial x} &= a_1\frac{\partial \psi_1}{\partial x} + a_2\frac{\partial \phi_2}{\partial x} + a_3\frac{\partial \psi_3}{\partial x} + a_4\frac{\partial \psi_4}{\partial x} + a_5\frac{\partial \psi_5}{\partial x} + a_6\frac{\partial \psi_6}{\partial x}\\
\frac{\partial \widetilde{p}}{\partial y} &= a_1\frac{\partial \psi_1}{\partial y} + a_2\frac{\partial \phi_2}{\partial y} + a_3\frac{\partial \psi_3}{\partial y} + a_4\frac{\partial \psi_4}{\partial y} + a_5\frac{\partial \psi_5}{\partial y} + a_6\frac{\partial \psi_6}{\partial y}
\end{aligned}\tag{71}$$

Therefore, we need to train the PINN as follows:

$$\begin{aligned}
(a_1, a_2, a_3, a_4, a_5, a_6) &= f(t)\\
(u'')[p''] &= f(x,t)\\
(\phi_1, \psi_1, \phi_2, \psi_2, \phi_3, \psi_3, \phi_4, \psi_4, \phi_5, \psi_5, \phi_6, \psi_6) &= f(x)
\end{aligned}\tag{72}$$

with the pretrained network from RANS for:

$$(\overline{u})[\overline{p}] = f(x) \tag{73}$$

2.4 Fourier-averaged Navier Stokes

Let's define the kth Fourier mode of the velocity, with i as the imaginary unit:

$$\hat{\phi}_k(\mathbf{x}) = \frac{1}{2L}\int_{t=-L}^{L} \mathbf{u}(\mathbf{x},t)e^{-i\pi kt/L}dt \tag{74}$$

$$\hat{\phi}_k(\mathbf{x}) = \frac{1}{2L}\int_{t=-L}^{L} \mathbf{u}(\mathbf{x},t)e^{-i\omega_k t}dt \tag{75}$$

$$\hat{\psi}_k(\mathbf{x}) = \frac{1}{2L}\int_{t=-L}^{L} p(\mathbf{x},t)e^{-i\pi kt/L}dt \tag{76}$$

$$\hat{\psi}_k(\mathbf{x}) = \frac{1}{2L}\int_{t=-L}^{L} p(\mathbf{x},t)e^{-i\omega_k t}dt \tag{77}$$

Noting Euler's formula, with $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$:

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t) \tag{78}$$

Thus we can reconstruct our velocity field according to the inverse transform:

$$\mathbf{u}(\mathbf{x},t) = \sum_{k=-\infty}^{\infty} \hat{\phi}_k(\mathbf{x})e^{i\pi kt/L} \tag{79}$$

$$\omega_k = \pi k/L$$

$$e^{i\omega_k t} = \cos(\omega_k t) + i\sin(\omega_k t)$$

$$\mathbf{u}(\mathbf{x}, t) = \sum_{k=-\infty}^{\infty} \hat{\phi}_k(\mathbf{x}) e^{i\pi kt/L} \tag{80}$$

$$p(\mathbf{x}, t) = \sum_{k=-\infty}^{\infty} \hat{\psi}_k(\mathbf{x}) e^{i\pi kt/L} \tag{81}$$

Let's also note that $e^{i\omega_k t}$ forms an orthogonal basis. Consider two frequencies ω_1 and ω_2:

$$e^{i\omega_1 t} = \cos(\omega_1 t) + i\sin(\omega_1 t)$$
$$e^{i\omega_2 t} = \cos(\omega_2 t) + i\sin(\omega_2 t)$$

If we take the product of the two:

$$\sum_{-\infty}^{\infty} e^{i\omega_1 t} e^{i\omega_2 t} = \sum_{-\infty}^{\infty} \cos(\omega_1 t)\cos(\omega_2 t)$$
$$-\sum_{-\infty}^{\infty} \sin(\omega_1 t)\sin(\omega_2 t) dt + \sum_{-\infty}^{\infty} i\sin(\omega_1 t)\cos(\omega_2 t) + i\sum_{-\infty}^{\infty} \cos(\omega_1 t)\sin(\omega_2 t) \tag{82}$$

It can be shown that the imaginary terms of the previous equation are always orthogonal, and that the real-valued terms are orthogonal for $\omega_1 \neq \omega_2$. In order for the reconstructed flow field to be completely real valued, the modes must obey the following property:

$$\phi_{-k} = \phi_k^* \tag{83}$$

Where * is the complex conjugate. For example if:

$$\phi_k = a + ib$$
$$\phi_{-k} = a - ib$$
$$\phi_k + \phi_{-k} = a + ib + a - ib = 2a \tag{84}$$

Thus the reconstruction will be real valued.

2.5 Derivation

Let's begin with the continuity equation:

$$(\nabla \cdot \mathbf{u}) = 0$$
$$\frac{\partial u_q}{\partial q} = 0$$
$$\sum_{k=-\infty}^{\infty} \frac{\hat{\phi}_{k,q}}{\partial q} e^{i\omega_k t} = 0 \tag{85}$$

We can then get the part corresponding to lth frequency mode by multiplying by $e^{-i\pi lt/L} = e^{-i\omega_l t}$ and integrating:

$$\frac{1}{2L}\int_{-L}^{L} e^{-i\omega_l t} \sum_{k=-\infty}^{\infty} \frac{\hat{\phi}_{k,q}}{\partial q} e^{i\omega_k t} dt = 0$$
$$\frac{1}{2L}\int_{-L}^{L} \sum_{k=-\infty}^{\infty} \frac{\hat{\phi}_{k,q}}{\partial j} e^{it - i\omega_l t} dt = 0$$
$$\frac{1}{2L}\int_{-L}^{L} \sum_{k=-\infty}^{\infty} \frac{\hat{\phi}_{k,q}}{\partial j} e^{i(\omega_k - \omega_l)t} dt = 0 \tag{86}$$

For $k \neq l$ the exponential term is zero due to orthogonality, thus we can consider the nonzero element of the sum. For k=l, we can see that the exponential function becomes:

$$\omega_k - \omega_l = \omega_l - \omega_l = 0$$
$$\frac{1}{2L}\int_{-L}^{L} \frac{\hat{\phi}_{l,q}}{\partial q} e^0 dt = 0$$

Since the Fourier mode doesn't depend on time, we can take it out of the integral:

$$\frac{\hat{\phi}_{l,q}}{\partial q}\frac{1}{2L}\int_{-L}^{L} 1 dt = 0$$

Evaluating the integral:

$$\frac{\hat{\phi}_{l,q}}{\partial q}\frac{1}{2L}\left(t|_{-L}^{L}\right) = \frac{\hat{\phi}_{l,q}}{\partial q}\frac{1}{2L}\left(\text{Ł} - (-L)\right) = 0$$

Giving the result:

$$\frac{\hat{\phi}_{l,q}}{\partial q} = 0 \tag{87}$$

$$\frac{\hat{\phi}_{l,x}}{\partial x} + \frac{\hat{\phi}_{l,y}}{\partial y} = 0 \tag{88}$$

$$\nabla \cdot \hat{\phi} = 0 \tag{89}$$

Interestingly, we get a continuity equation for both the real and imaginary parts of a complex mode. Let's now consider the momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = 0 \tag{90}$$

Let's first define the notation:

$$\omega_k = \pi k / L \tag{91}$$

$$\sum_{k=-\infty}^{\infty} = \sum_{k} \tag{92}$$

First we can consider the derivative term:

$$\frac{\partial \mathbf{u}}{\partial t} = \sum_k \frac{\partial \hat{\phi}_k e^{i\pi k t/L}}{\partial t} = \sum_k \frac{\partial \hat{\phi}_k e^{i\omega_k t}}{\partial t} \tag{93}$$

Since the Fourier mode is independent of time:

$$\begin{aligned} \sum_k \frac{\partial \hat{\phi}_k e^{i\omega_k t}}{\partial t} &= \sum_k \hat{\phi}_k \frac{\partial e^{i\omega_k t}}{\partial t} = \sum_k i\omega_k \hat{\phi}_k e^{i\omega_k t} \\ \frac{\partial \mathbf{u}}{\partial t} &= \sum_k i\omega_k e^{i\omega_k t} \begin{bmatrix} \hat{\phi}_{k,x} \\ \hat{\phi}_{k,y} \end{bmatrix} \end{aligned} \tag{94}$$

If we extract the mth component, using orthogonality:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= \frac{1}{2L}\int_{-L}^{L} e^{-i\omega_m t} \sum_k i\omega_k e^{i\omega_k t} \begin{bmatrix} \hat{\phi}_{k,x} \\ \hat{\phi}_{k,y} \end{bmatrix} dt \\ &= \left(i\omega_m \begin{bmatrix} \hat{\phi}_{m,x} \\ \hat{\phi}_{m,y} \end{bmatrix} \right) \frac{1}{2L}\int_{-L}^{L} e^{i(\omega_m - \omega_m)t} dt \\ &= i\omega_m \begin{bmatrix} \hat{\phi}_{m,x} \\ \hat{\phi}_{m,y} \end{bmatrix} \end{aligned} \tag{95}$$

Now considering the non-linear term:

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u} = \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)\begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$= \sum_k \begin{bmatrix} u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} \\ u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} \end{bmatrix} = \sum_k\sum_l \begin{bmatrix} \hat{\phi}_{k,x}e^{i\omega_k t}\frac{\partial \hat{\phi}_{l,x}e^{i\omega_l t}}{\partial x} + \hat{\phi}_{k,y}e^{i\omega_k t}\frac{\partial \hat{\phi}_{l,x}e^{i\omega_l t}}{\partial y} \\ \hat{\phi}_{k,x}e^{i\omega_k t}\frac{\partial \hat{\phi}_{l,y}e^{i\omega_l t}}{\partial x} + \hat{\phi}_{k,y}e^{i\omega_k t}\frac{\partial \hat{\phi}_{l,y}e^{i\omega_l t}}{\partial y} \end{bmatrix}$$

$$= \sum_k\sum_l \begin{bmatrix} \hat{\phi}_{k,x}e^{i\omega_k t}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{k,y}e^{i\omega_k t}e^{i\omega_l t}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{k,x}e^{i\omega_k t}e^{it}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{k,y}e^{i\omega_k t}e^{i\omega_l t}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix}$$

$$= \sum_k\sum_l \begin{bmatrix} \hat{\phi}_{k,x}e^{i(\omega_k+\omega_l)t}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{k,y}e^{i(\omega_k+\omega_l)t}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{k,x}e^{i(\omega_k+\omega_l)t}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{k,y}e^{i(\omega_k+\omega_l)t}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix}$$

$$= \sum_k\sum_l e^{i(\omega_k+\omega_l)t}\begin{bmatrix} \hat{\phi}_{k,x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{k,x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix} \tag{96}$$

We should specifically extract the special mean-field case where $l = 0$, first noting that:

$$\begin{aligned} \hat{\phi}_{0,x} &= \overline{u_x} \\ \hat{\phi}_{0,y} &= \overline{u_y} \end{aligned} \tag{97}$$

Thus the nonlinear term for $l = 0$ case becomes:

$$= \sum_k e^{i\omega_k t}\begin{bmatrix} \hat{\phi}_{k,x}\frac{\partial \overline{u_x}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \overline{u_x}}{\partial y} \\ \hat{\phi}_{k,x}\frac{\partial \overline{u_y}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \overline{u_y}}{\partial y} \end{bmatrix} \tag{98}$$

Which is the convection of the mode due to the mean field. If we rewrite the nonlinear term, as a whole, we get:

$$\begin{aligned} &= \sum_k e^{i\omega_k t}\begin{bmatrix} \hat{\phi}_{k,x}\frac{\partial \overline{u_x}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \overline{u_x}}{\partial y} \\ \hat{\phi}_{k,x}\frac{\partial \overline{u_y}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \overline{u_y}}{\partial y} \end{bmatrix} \\ &+ \sum_k\sum_{l\neq 0} e^{i(\omega_k+\omega_l)t}\begin{bmatrix} \hat{\phi}_{k,x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{k,x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix} \end{aligned} \tag{99}$$

If we extract the mth component:

$$\begin{aligned} &= \frac{1}{2L}\int_{t=-L}^{L} e^{-i\omega_m t}\sum_k e^{i\omega_k t}\begin{bmatrix} \hat{\phi}_{k,x}\frac{\partial \overline{u_x}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \overline{u_x}}{\partial y} \\ \hat{\phi}_{k,x}\frac{\partial \overline{u_y}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \overline{u_y}}{\partial y} \end{bmatrix} dt \\ &+ \frac{1}{2L}\int_{t=-L}^{L} e^{-i\omega_m t}\sum_k\sum_{l\neq 0} e^{i(\omega_k+\omega_l)t}\begin{bmatrix} \hat{\phi}_{k,x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{k,x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix} dt \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2L}\int_{t=-L}^{L} \sum_k e^{i(\omega_k-\omega_m)t}\begin{bmatrix} \hat{\phi}_{k,x}\frac{\partial \overline{u_x}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \overline{u_x}}{\partial y} \\ \hat{\phi}_{k,x}\frac{\partial \overline{u_y}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \overline{u_y}}{\partial y} \end{bmatrix} dt \\ &+ \frac{1}{2L}\int_{t=-L}^{L} \sum_k\sum_{l\neq 0} e^{i(\omega_k+\omega_l-\omega_m)t}\begin{bmatrix} \hat{\phi}_{k,x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{k,x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{k,y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix} dt \end{aligned}$$

We can see that the sinusoidal part will be non-orthogonal for $k = m - l$. In the case $l = m$, we would have $k = m - m = 0$ thus we get another mean field term:

$$\begin{aligned} &= \sum_{l=m}\begin{bmatrix} \hat{\phi}_{(m-m),x}\frac{\partial \hat{\phi}_{m,x}}{\partial x} + \hat{\phi}_{(m-m),y}\frac{\partial \hat{\phi}_{m,x}}{\partial y} \\ \hat{\phi}_{(m-m),x}\frac{\partial \hat{\phi}_{m,y}}{\partial x} + \hat{\phi}_{(m-m),y}\frac{\partial \hat{\phi}_{m,y}}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \overline{u}_x\frac{\partial \hat{\phi}_{m,x}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,x}}{\partial y} \\ \overline{u}_x\frac{\partial \hat{\phi}_{m,y}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,y}}{\partial y} \end{bmatrix} \end{aligned}$$

Otherwise we can just substitute for k:

$$= \begin{bmatrix} \hat{\phi}_{m,x}\frac{\partial \overline{u_x}}{\partial x} + \hat{\phi}_{m,y}\frac{\partial \overline{u_x}}{\partial y} \\ \hat{\phi}_{m,x}\frac{\partial \overline{u_y}}{\partial x} + \hat{\phi}_{m,y}\frac{\partial \overline{u_y}}{\partial y} \end{bmatrix} + \begin{bmatrix} \overline{u}_x\frac{\partial \hat{\phi}_{m,x}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,x}}{\partial y} \\ \overline{u}_x\frac{\partial \hat{\phi}_{m,y}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,y}}{\partial y} \end{bmatrix}$$

$$+ \sum_{l\neq 0,m} \begin{bmatrix} \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix}$$

Now considering the pressure term:

$$\nabla p = \nabla \sum_k \left(\psi_k e^{i\omega_k t}\right) = e^{i\omega_k t}\frac{\partial \psi_{k,q}}{\partial q} = \sum_k e^{i\omega_k t}\begin{bmatrix} \frac{\partial \psi_{k,x}}{\partial x} \\ \frac{\partial \psi_{k,y}}{\partial y} \end{bmatrix} \tag{100}$$

We can extract the mth component:

$$= \frac{1}{2L}\int_{-L}^{L} e^{-i\omega_m t}\sum_k e^{i\omega_k t}\begin{bmatrix} \frac{\partial \psi_{k,x}}{\partial x} \\ \frac{\partial \psi_{k,y}}{\partial y} \end{bmatrix} dt$$

$$= \sum_k \begin{bmatrix} \frac{\partial \psi_{k,x}}{\partial x} \\ \frac{\partial \psi_{k,y}}{\partial y} \end{bmatrix} \frac{1}{2L}\int_{-L}^{L} e^{i(\omega_k-\omega_m)t}dt$$

$$= \begin{bmatrix} \frac{\partial \psi_{m,x}}{\partial x} \\ \frac{\partial \psi_{m,y}}{\partial y} \end{bmatrix} \tag{101}$$

Considering the viscous term:

$$-\nu\nabla^2\mathbf{u} = -\nu\sum_k\left(\frac{\partial^2 \hat{\phi}_k e^{i\omega_k t}}{\partial q^2}\right)$$

$$= -\nu\sum_k e^{i\omega_k t}\left(\frac{\partial^2\hat{\phi}_k}{\partial q^2}\right) = -\nu\sum_k e^{i\omega_k t}\begin{bmatrix} \frac{\partial^2\hat{\phi}_{k,x}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{k,x}}{\partial y^2} \\ \frac{\partial^2\hat{\phi}_{k,y}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{k,y}}{\partial y^2} \end{bmatrix} \tag{102}$$

Again extracting the mth component:

$$= -\nu\frac{1}{2L}\int_{-L}^{L} e^{-i\omega_m t}\sum_k e^{i\omega_k t}\begin{bmatrix} \frac{\partial^2\hat{\phi}_{k,x}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{k,x}}{\partial y^2} \\ \frac{\partial^2\hat{\phi}_{k,y}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{k,y}}{\partial y^2} \end{bmatrix} dt$$

$$= -\nu\sum_k \begin{bmatrix} \frac{\partial^2\hat{\phi}_{k,x}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{k,x}}{\partial y^2} \\ \frac{\partial^2\hat{\phi}_{k,y}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{k,y}}{\partial y^2} \end{bmatrix} \frac{1}{2L}\int_{-L}^{L} e^{i(\omega_k-\omega_m)t}dt$$

$$= -\nu\begin{bmatrix} \frac{\partial^2\hat{\phi}_{m,x}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{m,x}}{\partial y^2} \\ \frac{\partial^2\hat{\phi}_{m,y}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{m,y}}{\partial y^2} \end{bmatrix} \tag{103}$$

We can then construct the full set of equations:

$$i\omega_m\begin{bmatrix} \hat{\phi}_{m,x} \\ \hat{\phi}_{m,y} \end{bmatrix} + \begin{bmatrix} \hat{\phi}_{m,x}\frac{\partial \overline{u_x}}{\partial x} + \hat{\phi}_{m,y}\frac{\partial \overline{u_x}}{\partial y} \\ \hat{\phi}_{m,x}\frac{\partial \overline{u_y}}{\partial x} + \hat{\phi}_{m,y}\frac{\partial \overline{u_y}}{\partial y} \end{bmatrix} + \begin{bmatrix} \overline{u}_x\frac{\partial \hat{\phi}_{m,x}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,x}}{\partial y} \\ \overline{u}_x\frac{\partial \hat{\phi}_{m,y}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,y}}{\partial y} \end{bmatrix}$$

$$+ \sum_{l\neq 0,m} \begin{bmatrix} \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix} + \begin{bmatrix} \frac{\partial \psi_{m,x}}{\partial x} \\ \frac{\partial \psi_{m,y}}{\partial y} \end{bmatrix} - \nu\begin{bmatrix} \frac{\partial^2\hat{\phi}_{m,x}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{m,x}}{\partial y^2} \\ \frac{\partial^2\hat{\phi}_{m,y}}{\partial x^2} + \frac{\partial^2\hat{\phi}_{m,y}}{\partial y^2} \end{bmatrix} = 0 \tag{104}$$

Or in vector form, which as noted in the derivation by ??? is quite similar to the RANS equation, with the addition of a 'Fourier stress' term:

$$i\omega_m\hat{\phi}_m + (\hat{\phi}_m\cdot\nabla)\overline{u} + (\overline{u}\cdot\nabla)\hat{\phi}_m + \sum_{l\neq 0,m}(\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l + \nabla\psi_m - \nu\nabla^2\hat{\phi}_m = 0 \tag{105}$$

2.5.1 Alternate expansion of the non-linear term

Considering the non-linear term, we have:

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u} = \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)\begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$= \begin{bmatrix} u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} \\ u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} \end{bmatrix} \tag{106}$$

Alternatively we can write:

$$\nabla \cdot \mathbf{uu} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} u_x u_x + u_x u_y \\ u_y u_x + u_y u_y \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x u_x}{\partial x} + \frac{\partial u_x u_y}{\partial y} \\ \frac{\partial u_y u_x}{\partial x} + \frac{\partial u_y u_y}{\partial y} \end{bmatrix} \tag{107}$$

Applying the chain rule:

$$= \begin{bmatrix} u_x \frac{\partial u_x}{\partial x} + u_x \frac{\partial u_x}{\partial x} + u_x \frac{\partial u_y}{\partial y} + u_y \frac{\partial u_x}{\partial y} \\ u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_y \frac{\partial u_y}{\partial y} \end{bmatrix} \tag{108}$$

Factoring:

$$= \begin{bmatrix} u_x \frac{\partial u_x}{\partial x} + u_x \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + u_y \frac{\partial u_x}{\partial y} \\ u_x \frac{\partial u_y}{\partial x} + u_y \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + u_y \frac{\partial u_y}{\partial y} \end{bmatrix} \tag{109}$$

We can see that the term in the brackets is the continuity equation for incompressible flow, and thus is zero, giving:

$$\nabla \cdot \mathbf{uu} = \begin{bmatrix} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \\ u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \end{bmatrix} = (\mathbf{u} \cdot \nabla)\mathbf{u} \tag{110}$$

Consider the non-linear term of the FANS:

$$\nabla \cdot \sum_k \sum_l \hat{\phi}_k e^{i\omega_k t} \hat{\phi}_l e^{i\omega_l t} = \nabla \cdot \sum_k \sum_l \hat{\phi}_k \hat{\phi}_l e^{i(\omega_l + \omega_k)t} \tag{111}$$

We can put the derivative onto the spatial part only, since it doesn't depend on time:

$$= \sum_k \sum_l \left(\nabla \cdot \hat{\phi}_k \hat{\phi}_l \right) e^{i(\omega_l + \omega_k)t} \tag{112}$$

$$= \sum_k \sum_l \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \phi_{k,x}\phi_{l,x} + \phi_{k,x}\phi_{l,y} \\ \phi_{k,y}\phi_{l,x} + \phi_{k,y}\phi_{l,y} \end{bmatrix} e^{i(\omega_l + \omega_k)t} = \sum_k \sum_l \begin{bmatrix} \frac{\partial \phi_{k,x}\phi_{l,x}}{\partial x} + \frac{\partial \phi_{k,x}\phi_{l,y}}{\partial y} \\ \frac{\partial \phi_{k,y}\phi_{l,x}}{\partial x} + \frac{\partial \phi_{k,y}\phi_{l,y}}{\partial y} \end{bmatrix} e^{i(\omega_l + \omega_k)t} \tag{113}$$

Applying the chain rule:

$$= \sum_k \sum_l \begin{bmatrix} \phi_{k,x} \frac{\partial \phi_{l,x}}{\partial x} + \phi_{l,x} \frac{\partial \phi_{k,x}}{\partial x} + \phi_{k,x} \frac{\partial \phi_{l,y}}{\partial y} + \phi_{l,y} \frac{\partial \phi_{k,x}}{\partial y} \\ \phi_{k,y} \frac{\partial \phi_{l,x}}{\partial x} + \phi_{l,x} \frac{\partial \phi_{k,y}}{\partial x} + \phi_{k,y} \frac{\partial \phi_{l,y}}{\partial y} + \phi_{l,y} \frac{\partial \phi_{k,y}}{\partial y} \end{bmatrix} e^{i(\omega_l + \omega_k)t} \tag{114}$$

If we factor:

$$= \sum_k \sum_l \begin{bmatrix} \phi_{k,x} \left(\frac{\partial \phi_{l,x}}{\partial x} + \frac{\partial \phi_{l,y}}{\partial y} \right) + \phi_{l,x} \frac{\partial \phi_{k,x}}{\partial x} + \phi_{l,y} \frac{\partial \phi_{k,x}}{\partial y} \\ \phi_{k,y} \left(\frac{\partial \phi_{l,x}}{\partial x} + \frac{\partial \phi_{l,y}}{\partial y} \right) + \phi_{l,x} \frac{\partial \phi_{k,y}}{\partial x} + +\phi_{l,y} \frac{\partial \phi_{k,y}}{\partial y} \end{bmatrix} e^{i(\omega_l + \omega_k)t} \tag{115}$$

If we recall the continuity equation in Fourier domain:

$$\frac{\hat{\phi}_{l,x}}{\partial x} + \frac{\hat{\phi}_{l,y}}{\partial y} = 0 \tag{116}$$

Thus the form becomes:

$$= \sum_k \sum_l \begin{bmatrix} \phi_{l,x} \frac{\partial \phi_{k,x}}{\partial x} + \phi_{l,y} \frac{\partial \phi_{k,x}}{\partial y} \\ \phi_{l,x} \frac{\partial \phi_{k,y}}{\partial x} + \phi_{l,y} \frac{\partial \phi_{k,y}}{\partial y} \end{bmatrix} e^{i(\omega_l + \omega_k)t} \tag{117}$$

If we extract the mth coefficient:

$$= \int_{t=-L}^{L} e^{-i\omega_m t} \sum_k \sum_l \begin{bmatrix} \phi_{l,x}\frac{\partial \phi_{k,x}}{\partial x} + \phi_{l,y}\frac{\partial \phi_{k,x}}{\partial y} \\ \phi_{l,x}\frac{\partial \phi_{k,y}}{\partial x} + \phi_{l,y}\frac{\partial \phi_{k,y}}{\partial y} \end{bmatrix} e^{i(\omega_l+\omega_k)t} dt$$

$$= \int_{t=-L}^{L} \sum_k \sum_l \begin{bmatrix} \phi_{l,x}\frac{\partial \phi_{k,x}}{\partial x} + \phi_{l,y}\frac{\partial \phi_{k,x}}{\partial y} \\ \phi_{l,x}\frac{\partial \phi_{k,y}}{\partial x} + \phi_{l,y}\frac{\partial \phi_{k,y}}{\partial y} \end{bmatrix} e^{i(\omega_l+\omega_k-\omega_m)t} dt$$

Which as before is not orthogonal for $k = m - l$, so we can simplify as before to:

$$\sum_l \begin{bmatrix} \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix}$$

Where we can specifically break out the two mean field terms for $l = 0, l = m$:

$$= \begin{bmatrix} \hat{\phi}_{m,x}\frac{\partial \overline{u_x}}{\partial x} + \hat{\phi}_{m,y}\frac{\partial \overline{u_x}}{\partial y} \\ \hat{\phi}_{m,x}\frac{\partial \overline{u_y}}{\partial x} + \hat{\phi}_{m,y}\frac{\partial \overline{u_y}}{\partial y} \end{bmatrix} + \begin{bmatrix} \overline{u}_x\frac{\partial \hat{\phi}_{m,x}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,x}}{\partial y} \\ \overline{u}_x\frac{\partial \hat{\phi}_{m,y}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,y}}{\partial y} \end{bmatrix}$$

$$+ \sum_{l\neq 0,m} \begin{bmatrix} \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix}$$

Or alternatively, we can state:

$$\left(\phi_{(m-l)} \cdot \nabla\right) \phi_l = \nabla \cdot \left(\phi_{(m-l)}\phi_l\right) \tag{118}$$

Thus we get the alternative form of the FANS equation, for incompressible flow:

$$i\omega_m\hat{\phi}_m + (\hat{\phi}_m \cdot \nabla)\overline{u} + (\overline{u} \cdot \nabla)\hat{\phi}_m + \sum_{l\neq 0,m} \nabla \cdot \left(\hat{\phi}_{(m-l)}\hat{\phi}_l\right) + \nabla\psi_m - \nu\nabla^2\hat{\phi}_m = 0 \tag{119}$$

2.6 FANS: Implementation in real-valued equations

Since the numerical frameworks for the PINNs accept real valued inputs only, we need to convert these complex valued equations into real valued ones. Starting with 104:

$$i\omega_m \begin{bmatrix} \hat{\phi}_{m,x} \\ \hat{\phi}_{m,y} \end{bmatrix} + \begin{bmatrix} \hat{\phi}_{m,x}\frac{\partial \overline{u_x}}{\partial x} + \hat{\phi}_{m,y}\frac{\partial \overline{u_x}}{\partial y} \\ \hat{\phi}_{m,x}\frac{\partial \overline{u_y}}{\partial x} + \hat{\phi}_{m,y}\frac{\partial \overline{u_y}}{\partial y} \end{bmatrix} + \begin{bmatrix} \overline{u}_x\frac{\partial \hat{\phi}_{m,x}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,x}}{\partial y} \\ \overline{u}_x\frac{\partial \hat{\phi}_{m,y}}{\partial x} + \overline{u}_y\frac{\partial \hat{\phi}_{m,y}}{\partial y} \end{bmatrix}$$

$$+ \sum_{l\neq 0,m} \begin{bmatrix} \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,x}}{\partial y} \\ \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,y}}{\partial y} \end{bmatrix} + \begin{bmatrix} \frac{\partial \psi_{m,x}}{\partial x} \\ \frac{\partial \psi_{m,y}}{\partial y} \end{bmatrix} - \nu \begin{bmatrix} \frac{\partial^2 \hat{\phi}_{m,x}}{\partial x^2} + \frac{\partial^2 \hat{\phi}_{m,x}}{\partial y^2} \\ \frac{\partial^2 \hat{\phi}_{m,y}}{\partial x^2} + \frac{\partial^2 \hat{\phi}_{m,y}}{\partial y^2} \end{bmatrix} = 0$$

Let's first denote the real and complex parts of the modes:

$$\Re(\hat{\phi}_{m,q}) = \phi_{m,q,r} \tag{120}$$

$$\Im(\hat{\phi}_{m,q}) = \phi_{m,q,i} \tag{121}$$

Taking the real part of the x-momentum equation:

$$\begin{aligned} -\omega_m\phi_{m,x,i} + \left(\phi_{m,x,r}\frac{\partial \overline{u_x}}{\partial x} + \phi_{m,y,r}\frac{\partial \overline{u_x}}{\partial y}\right) + \left(\overline{u_x}\frac{\partial \phi_{m,x,r}}{\partial x} + \overline{u_y}\frac{\partial \phi_{m,x,r}}{\partial y}\right) \\ + \sum_{l\neq 0,m}\left(\phi_{(m-l),x,r}\frac{\partial \phi_{l,x,r}}{\partial x} + \phi_{(m-l),y,r}\frac{\partial \phi_{l,x,r}}{\partial y} - \phi_{(m-l),x,i}\frac{\partial \phi_{l,x,i}}{\partial x} - \phi_{(m-l),y,i}\frac{\partial \phi_{l,x,i}}{\partial y}\right) \\ +\psi_{m,x,r} - \nu\left(\frac{\partial^2 \phi_{m,x,r}}{\partial x^2} + \frac{\partial^2 \phi_{m,x,r}}{\partial y^2}\right) = 0 \end{aligned} \tag{122}$$

The complex part of the x-momentum equation:

$$\begin{aligned} \omega_m\phi_{m,x,r} + \left(\phi_{m,x,i}\frac{\partial \overline{u_x}}{\partial x} + \phi_{m,y,i}\frac{\partial \overline{u_x}}{\partial y}\right) + \left(\overline{u_x}\frac{\partial \phi_{m,x,i}}{\partial x} + \overline{u_y}\frac{\partial \phi_{m,x,i}}{\partial y}\right) \\ + \sum_{l\neq 0,m}\left(\phi_{(m-l),x,r}\frac{\partial \phi_{l,x,i}}{\partial x} + \phi_{(m-l),y,r}\frac{\partial \phi_{l,x,i}}{\partial y} + \phi_{(m-l),x,i}\frac{\partial \phi_{l,x,r}}{\partial x} + \phi_{(m-l),y,i}\frac{\partial \phi_{l,x,r}}{\partial y}\right) \\ +\psi_{m,x,r} - \nu\left(\frac{\partial^2 \phi_{m,x,r}}{\partial x^2} + \frac{\partial^2 \phi_{m,x,r}}{\partial y^2}\right) = 0 \end{aligned} \tag{123}$$

The real part of the y-momentum equation:

$$-\omega_m \phi_{m,y,i} + \left(\phi_{m,x,r}\frac{\partial \overline{u_y}}{\partial x} + \phi_{m,y,r} + \frac{\partial \overline{u_y}}{\partial y}\right) + \left(\overline{u_x}\frac{\partial \phi_{m,y,r}}{\partial x} + \overline{u_y}\frac{\partial \phi_{m,y,r}}{\partial y}\right)$$
$$+ \sum_{l\neq 0,m}\left(\phi_{(m-l),x,r}\frac{\partial \phi_{l,y,r}}{\partial x} + \phi_{(m-l),y,r}\frac{\partial \phi_{l,y,r}}{\partial y} - \phi_{(m-l),x,i}\frac{\partial \phi_{l,y,i}}{\partial x} - \phi_{(m-l),y,i}\frac{\partial \phi_{l,y,i}}{\partial y}\right)$$
$$+\psi_{m,y,r} - \nu\left(\frac{\partial^2 \phi_{m,y,r}}{\partial x^2} + \frac{\partial^2 \phi_{m,y,r}}{\partial y^2}\right) = 0 \qquad (124)$$

The complex part of the y-momentum equation:

$$\omega_m \phi_{m,y,r} + \left(\phi_{m,x,i}\frac{\partial \overline{u_y}}{\partial x} + \phi_{m,y,i}\frac{\partial \overline{u_y}}{\partial y}\right) + \left(\overline{u_x}\frac{\partial \phi_{m,y,i}}{\partial x} + \overline{u_y}\frac{\partial \phi_{m,y,i}}{\partial y}\right)$$
$$+ \sum_{l\neq 0,m}\left(\phi_{(m-l),x,r}\frac{\partial \phi_{l,y,i}}{\partial x} + \phi_{(m-l),y,r}\frac{\partial \phi_{l,y,i}}{\partial y} + \phi_{(m-l),x,i}\frac{\partial \phi_{l,y,r}}{\partial x} + \phi_{(m-l),y,i}\frac{\partial \phi_{l,y,r}}{\partial y}\right)$$
$$+\psi_{m,y,i} - \nu\left(\frac{\partial^2 \phi_{m,y,i}}{\partial x^2} + \frac{\partial^2 \phi_{m,y,i}}{\partial y^2}\right) = 0 \qquad (125)$$

2.7 FANS: Dealing with the Fourier-stress term

Interesting paper found by Maziyar, need to try to understand it:

An exact representation of the nonlinear triad interaction terms in spectral space. J. Fluid Mech. (2014), vol 748, pp. 175-188. doi:10.1017/jfm.2014.179

There are some questions here:

1. These modal interactions occur at defined frequencies, which is connected to the turbulent energy cascade. For a small set of modes, we can write out the interactions in our model. What do the Fourier stresses look like in space and time?
2. Can we identify a technique that works to model the unresolved scales of the flow? Eddy viscosity model?
3. When we have modulated frequency behavior (such as captured by a POD mode) can we capture this well with just Fourier modes? Can we use the spatial information to help us identify groups of modes corresponding to modulated behavior?
4. What is the connection here with the Bi-spectral modal decomposition?

2.8 Expansion of flow with single frequency and harmonics

Let's consider an example flow with 3 only frequencies (ω_a,$2\omega_a$,$3\omega_a$), plus the negative frequencies ($-\omega_a$,$-2\omega_a$,$-3\omega_a$). Let's write the non-linear term for each of the cases. Let's start with $\omega_m = \omega_a$:

$$\sum_{l\neq 0,m}\begin{bmatrix}\hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,x}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,x}}{\partial y}\\ \hat{\phi}_{(m-l),x}\frac{\partial \hat{\phi}_{l,y}}{\partial x} + \hat{\phi}_{(m-l),y}\frac{\partial \hat{\phi}_{l,y}}{\partial y}\end{bmatrix}$$
$$= \begin{bmatrix}\hat{\phi}_{(a-2a),x}\frac{\partial \hat{\phi}_{2a,x}}{\partial x} + \hat{\phi}_{(a-2a),y}\frac{\partial \hat{\phi}_{2a,x}}{\partial y}\\ \hat{\phi}_{(a-2a),x}\frac{\partial \hat{\phi}_{2a,y}}{\partial x} + \hat{\phi}_{(a-2a),y}\frac{\partial \hat{\phi}_{2a,y}}{\partial y}\end{bmatrix} + \begin{bmatrix}\hat{\phi}_{(a-3a),x}\frac{\partial \hat{\phi}_{3a,x}}{\partial x} + \hat{\phi}_{(a-3a),y}\frac{\partial \hat{\phi}_{3a,x}}{\partial y}\\ \hat{\phi}_{(a-3a),x}\frac{\partial \hat{\phi}_{3a,y}}{\partial x} + \hat{\phi}_{(a-3a),y}\frac{\partial \hat{\phi}_{3a,y}}{\partial y}\end{bmatrix}$$
$$+ \begin{bmatrix}\hat{\phi}_{(a--a),x}\frac{\partial \hat{\phi}_{-a,x}}{\partial x} + \hat{\phi}_{(a--a),y}\frac{\partial \hat{\phi}_{-a,x}}{\partial y}\\ \hat{\phi}_{(a--a),x}\frac{\partial \hat{\phi}_{-a,y}}{\partial x} + \hat{\phi}_{(a--a),y}\frac{\partial \hat{\phi}_{-a,y}}{\partial y}\end{bmatrix} + \begin{bmatrix}\hat{\phi}_{(a--2a),x}\frac{\partial \hat{\phi}_{-2a,x}}{\partial x} + \hat{\phi}_{(a--2a),y}\frac{\partial \hat{\phi}_{-2a,x}}{\partial y}\\ \hat{\phi}_{(a--2a),x}\frac{\partial \hat{\phi}_{-2a,y}}{\partial x} + \hat{\phi}_{(a--2a),y}\frac{\partial \hat{\phi}_{-2a,y}}{\partial y}\end{bmatrix}$$
$$+ \begin{bmatrix}\hat{\phi}_{(a--3a),x}\frac{\partial \hat{\phi}_{-3a,x}}{\partial x} + \hat{\phi}_{(a--3a),y}\frac{\partial \hat{\phi}_{-3a,x}}{\partial y}\\ \hat{\phi}_{(a--3a),x}\frac{\partial \hat{\phi}_{-3a,y}}{\partial x} + \hat{\phi}_{(a--3a),y}\frac{\partial \hat{\phi}_{-3a,y}}{\partial y}\end{bmatrix} \qquad (126)$$

Simplifying the frequencies, and since the 4th harmonic is neglected we can remove the last term:

$$\begin{aligned}
&\sum_{l\neq 0,m}\begin{bmatrix}\hat{\phi}_{(m-l),x}\frac{\partial\hat{\phi}_{l,x}}{\partial x}+\hat{\phi}_{(m-l),y}\frac{\partial\hat{\phi}_{l,x}}{\partial y}\\ \hat{\phi}_{(m-l),x}\frac{\partial\hat{\phi}_{l,y}}{\partial x}+\hat{\phi}_{(m-l),y}\frac{\partial\hat{\phi}_{l,y}}{\partial y}\end{bmatrix}\\
&=\begin{bmatrix}\hat{\phi}_{(-a),x}\frac{\partial\hat{\phi}_{2a,x}}{\partial x}+\hat{\phi}_{(-a),y}\frac{\partial\hat{\phi}_{2a,x}}{\partial y}\\ \hat{\phi}_{(-a),x}\frac{\partial\hat{\phi}_{2a,y}}{\partial x}+\hat{\phi}_{(-a),y}\frac{\partial\hat{\phi}_{2a,y}}{\partial y}\end{bmatrix}+\begin{bmatrix}\hat{\phi}_{(-2a),x}\frac{\partial\hat{\phi}_{3a,x}}{\partial x}+\hat{\phi}_{(-2a),y}\frac{\partial\hat{\phi}_{3a,x}}{\partial y}\\ \hat{\phi}_{(-2a),x}\frac{\partial\hat{\phi}_{3a,y}}{\partial x}+\hat{\phi}_{(-2a),y}\frac{\partial\hat{\phi}_{3a,y}}{\partial y}\end{bmatrix}\\
&+\begin{bmatrix}\hat{\phi}_{(2a),x}\frac{\partial\hat{\phi}_{-a,x}}{\partial x}+\hat{\phi}_{(2a),y}\frac{\partial\hat{\phi}_{-a,x}}{\partial y}\\ \hat{\phi}_{(2a),x}\frac{\partial\hat{\phi}_{-a,y}}{\partial x}+\hat{\phi}_{(2a),y}\frac{\partial\hat{\phi}_{-a,y}}{\partial y}\end{bmatrix}+\begin{bmatrix}\hat{\phi}_{(3a),x}\frac{\partial\hat{\phi}_{-2a,x}}{\partial x}+\hat{\phi}_{(3a),y}\frac{\partial\hat{\phi}_{-2a,x}}{\partial y}\\ \hat{\phi}_{(3a),x}\frac{\partial\hat{\phi}_{-2a,y}}{\partial x}+\hat{\phi}_{(3a),y}\frac{\partial\hat{\phi}_{-2a,y}}{\partial y}\end{bmatrix}
\end{aligned}\tag{127}$$

If we apply the property from equation 83, we can rewrite the above example:

$$\begin{aligned}
&\sum_{l\neq 0,m}\begin{bmatrix}\hat{\phi}_{(m-l),x}\frac{\partial\hat{\phi}_{l,x}}{\partial x}+\hat{\phi}_{(m-l),y}\frac{\partial\hat{\phi}_{l,x}}{\partial y}\\ \hat{\phi}_{(m-l),x}\frac{\partial\hat{\phi}_{l,y}}{\partial x}+\hat{\phi}_{(m-l),y}\frac{\partial\hat{\phi}_{l,y}}{\partial y}\end{bmatrix}\\
&=\begin{bmatrix}\hat{\phi}^*_{a,x}\frac{\partial\hat{\phi}_{2a,x}}{\partial x}+\hat{\phi}^*_{(a),y}\frac{\partial\hat{\phi}_{2a,x}}{\partial y}\\ \hat{\phi}^*_{a,x}\frac{\partial\hat{\phi}_{2a,y}}{\partial x}+\hat{\phi}^*_{a,y}\frac{\partial\hat{\phi}_{2a,y}}{\partial y}\end{bmatrix}+\begin{bmatrix}\hat{\phi}^*_{(2a),x}\frac{\partial\hat{\phi}_{3a,x}}{\partial x}+\hat{\phi}^*_{(2a),y}\frac{\partial\hat{\phi}_{3a,x}}{\partial y}\\ \hat{\phi}^*_{(2a),x}\frac{\partial\hat{\phi}_{3a,y}}{\partial x}+\hat{\phi}^*_{(2a),y}\frac{\partial\hat{\phi}^*_{3a,y}}{\partial y}\end{bmatrix}\\
&+\begin{bmatrix}\hat{\phi}_{(2a),x}\frac{\partial\hat{\phi}^*_{a,x}}{\partial x}+\hat{\phi}_{(2a),y}\frac{\partial\hat{\phi}^*_{a,x}}{\partial y}\\ \hat{\phi}_{(2a),x}\frac{\partial\hat{\phi}^*_{a,y}}{\partial x}+\hat{\phi}_{(2a),y}\frac{\partial\hat{\phi}^*_{a,y}}{\partial y}\end{bmatrix}+\begin{bmatrix}\hat{\phi}_{(3a),x}\frac{\partial\hat{\phi}^*_{2a,x}}{\partial x}+\hat{\phi}_{(3a),y}\frac{\partial\hat{\phi}^*_{2a,x}}{\partial y}\\ \hat{\phi}_{(3a),x}\frac{\partial\hat{\phi}^*_{2a,y}}{\partial x}+\hat{\phi}_{(3a),y}\frac{\partial\hat{\phi}^*_{2a,y}}{\partial y}\end{bmatrix}\\
&=(\hat{\phi}^*_a\cdot\nabla)\hat{\phi}_{2a}+(\hat{\phi}^*_{2a}\cdot\nabla)\hat{\phi}_{3a}+(\hat{\phi}_{2a}\cdot\nabla)\hat{\phi}^*_a+(\hat{\phi}_{3a}\cdot\nabla)\hat{\phi}_{2a}
\end{aligned}\tag{128}$$

Now consider $\omega_m = 2\omega_a$:

$$\begin{aligned}
&\sum_{l\neq 0,m}(\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l\\
&=(\hat{\phi}_{2a-a}\cdot\nabla)\hat{\phi}_a+(\hat{\phi}_{2a-3a}\cdot\nabla)\hat{\phi}_{3a}+(\hat{\phi}_{2a--a}\cdot\nabla)\hat{\phi}_{-a}\\
&+(\hat{\phi}_{2a--2a}\cdot\nabla)\hat{\phi}_{-2a}+(\hat{\phi}_{2a--3a}\cdot\nabla)\hat{\phi_{-3a}}\\
&=(\hat{\phi}_a\cdot\nabla)\hat{\phi}_a+(\hat{\phi}^*_a\cdot\nabla)\hat{\phi}_{3a}+(\hat{\phi}_{3a}\cdot\nabla)\hat{\phi}^*_a
\end{aligned}\tag{129}$$

Now consider $\omega_m = 3\omega_a$:

$$\begin{aligned}
&\sum_{l\neq 0,m}(\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l\\
&=(\hat{\phi}_{3a--3a}\cdot\nabla)\hat{\phi_{-3a}}+(\hat{\phi}_{3a--2a}\cdot\nabla)\hat{\phi_{-2a}}\\
&+(\hat{\phi}_{3a--a}\cdot\nabla)\hat{\phi_{-a}}+(\hat{\phi}_{3a-a}\cdot\nabla)\hat{\phi_a}+(\hat{\phi}_{3a-2a}\cdot\nabla)\hat{\phi_{2a}}\\
&=(\hat{\phi}_{2a}\cdot\nabla)\hat{\phi_a}+(\hat{\phi}_a\cdot\nabla)\hat{\phi_{2a}}
\end{aligned}\tag{130}$$

2.9 Expansion of flow with two dynamical frequencies

This would be applicable to the modulated periodic 2 cylinder case. Consider the flow is sparse except for the frequencies $\omega_a, 2\omega_a, 3\omega_a, \omega_b, 2\omega_b, 3\omega_b$, with $\omega_b > \omega_a$. Begin with $\omega_m = \omega_b$:

$$\begin{aligned}
&\sum_{l\neq 0,m}(\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l\\
&=(\hat{\phi}_{b--3b}\cdot\nabla)\hat{\phi}_{-3b}+(\hat{\phi}_{b--2b}\cdot\nabla)\hat{\phi}_{-2b}+(\hat{\phi}_{b--b}\cdot\nabla)\hat{\phi}_{-b}\\
&+(\hat{\phi}_{b-2b}\cdot\nabla)\hat{\phi}_{2b}+(\hat{\phi}_{b-3b}\cdot\nabla)\hat{\phi}_{3b}+(\hat{\phi}_{b--2a}\cdot\nabla)\hat{\phi}_{-2a}\\
&+(\hat{\phi}_{b--a}\cdot\nabla)\hat{\phi}_{-a}+(\hat{\phi}_{b-a}\cdot\nabla)\hat{\phi}_a+(\hat{\phi}_{b-2a}\cdot\nabla)\hat{\phi}_{-2a}+(\hat{\phi}_{b-3a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned}\tag{131}$$

$$\begin{aligned}
&=(\hat{\phi}_{3b}\cdot\nabla)\hat{\phi}^*_{2b}+(\hat{\phi}_{2b}\cdot\nabla)\hat{\phi}^*_b+(\hat{\phi}^*_b\cdot\nabla)\hat{\phi}_{2b}+(\hat{\phi}^*_{2b}\cdot\nabla)\hat{\phi}_{3b}+(\hat{\phi}_{b+2a}\cdot\nabla)\hat{\phi}^*_{2a}\\
&+(\hat{\phi}_{b+a}\cdot\nabla)\hat{\phi}^*_a+(\hat{\phi}_{b-a}\cdot\nabla)\hat{\phi}_a+(\hat{\phi}_{b-2a}\cdot\nabla)\hat{\phi}^*_{2a}+(\hat{\phi}_{b-3a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned}\tag{132}$$

Now continue with $\omega_m = 2\omega_b$:

$$\begin{aligned}
&\sum_{l\neq 0,m} (\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l \\
&= (\hat{\phi}_{2b--3b}\cdot\nabla)\hat{\phi}_{-3b} + (\hat{\phi}_{2b--2b}\cdot\nabla)\hat{\phi}_{-2b} + (\hat{\phi}_{2b--b}\cdot\nabla)\hat{\phi}_{-b} \\
&+(\hat{\phi}_{2b-b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{2b-3b}\cdot\nabla)\hat{\phi}_{3b} + (\hat{\phi}_{2b--3a}\cdot\nabla)\hat{\phi}_{-3a} \\
&+(\hat{\phi}_{2b--2a}\cdot\nabla)\hat{\phi}_{-2a} + (\hat{\phi}_{2b--a}\cdot\nabla)\hat{\phi}_{-a} + (\hat{\phi}_{2b-a}\cdot\nabla)\hat{\phi}_{a} \\
&+(\hat{\phi}_{2b-2a}\cdot\nabla)\hat{\phi}_{2a} + (\hat{\phi}_{2b-3a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned} \tag{133}$$

$$\begin{aligned}
&= (\hat{\phi}_{3b}\cdot\nabla)\hat{\phi}^*_{b} + (\hat{\phi}_{b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}^*_{b}\cdot\nabla)\hat{\phi}_{3b} \\
&+(\hat{\phi}_{2b+3a}\cdot\nabla)\hat{\phi}^*_{3a} + (\hat{\phi}_{2b+2a}\cdot\nabla)\hat{\phi}^*_{2a} + (\hat{\phi}_{2b+a}\cdot\nabla)\hat{\phi}^*_{a} \\
&+(\hat{\phi}_{2b-a}\cdot\nabla)\hat{\phi}_{a} + (\hat{\phi}_{2b-2a}\cdot\nabla)\hat{\phi}_{2a} + (\hat{\phi}_{2b-3a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned} \tag{134}$$

Now continue with $\omega_m = 3\omega_b$:

$$\begin{aligned}
&\sum_{l\neq 0,m} (\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l \\
&= (\hat{\phi}_{3b--3b}\cdot\nabla)\hat{\phi}_{-3b} + (\hat{\phi}_{3b--2b}\cdot\nabla)\hat{\phi}_{-2b} + (\hat{\phi}_{3b--b}\cdot\nabla)\hat{\phi}_{-b} \\
&+(\hat{\phi}_{3b-b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{3b-2b}\cdot\nabla)\hat{\phi}_{2b} + (\hat{\phi}_{3b--3a}\cdot\nabla)\hat{\phi}_{-3a} \\
&+(\hat{\phi}_{3b--2a}\cdot\nabla)\hat{\phi}_{-2a} + (\hat{\phi}_{3b--a}\cdot\nabla)\hat{\phi}_{-a} + (\hat{\phi}_{3b-a}\cdot\nabla)\hat{\phi}_{a} \\
&+(\hat{\phi}_{3b-2a}\cdot\nabla)\hat{\phi}_{2a} + (\hat{\phi}_{3b-3a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned} \tag{135}$$

$$\begin{aligned}
&= (\hat{\phi}_{2b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{b}\cdot\nabla)\hat{\phi}_{2b} + (\hat{\phi}_{3b+3a}\cdot\nabla)\hat{\phi}^*_{3a} \\
&+(\hat{\phi}_{3b-a}\cdot\nabla)\hat{\phi}_{a} + (\hat{\phi}_{3b-2a}\cdot\nabla)\hat{\phi}_{2a} + (\hat{\phi}_{3b-3a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned} \tag{136}$$

Now continue with $\omega_m = \omega_a$:

$$\begin{aligned}
&\sum_{l\neq 0,m} (\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l \\
&(\hat{\phi}_{a--3b}\cdot\nabla)\hat{\phi}_{-3b} + (\hat{\phi}_{a--2b}\cdot\nabla)\hat{\phi}_{-2b} + (\hat{\phi}_{a--b}\cdot\nabla)\hat{\phi}_{-b} \\
&+(\hat{\phi}_{a-b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{a-2b}\cdot\nabla)\hat{\phi}_{2b} + (\hat{\phi}_{-3b}\cdot\nabla)\hat{\phi}_{3b} + (\hat{\phi}_{a--3a}\cdot\nabla)\hat{\phi}_{-3a} \\
&+(\hat{\phi}_{a--2a}\cdot\nabla)\hat{\phi}_{-2a} + (\hat{\phi}_{a--a}\cdot\nabla)\hat{\phi}_{-a} \\
&+(\hat{\phi}_{a-2a}\cdot\nabla)\hat{\phi}_{2a} + (\hat{\phi}_{a-3a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned} \tag{137}$$

$$\begin{aligned}
&= (\hat{\phi}_{a+2b}\cdot\nabla)\hat{\phi}^*_{2b} + (\hat{\phi}_{a+b}\cdot\nabla)\hat{\phi}^*_{b} \\
&+(\hat{\phi}_{a-b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{a-2b}\cdot\nabla)\hat{\phi}_{2b} + (\hat{\phi}_{a-3b}\cdot\nabla)\hat{\phi}_{3b} \\
&+(\hat{\phi}_{3a}\cdot\nabla)\hat{\phi}^*_{2a} + (\hat{\phi}_{2a}\cdot\nabla)\hat{\phi}^*_{a} + (\hat{\phi}^*_{a}\cdot\nabla)\hat{\phi}_{2a} + (\hat{\phi}^*_{2a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned} \tag{138}$$

Continuing with $\omega_m = 2\omega_a$:

$$\begin{aligned}
&\sum_{l\neq 0,m} (\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l \\
&(\hat{\phi}_{2a--3b}\cdot\nabla)\hat{\phi}_{-3b} + (\hat{\phi}_{2a--2b}\cdot\nabla)\hat{\phi}_{-2b} + (\hat{\phi}_{2a--b}\cdot\nabla)\hat{\phi}_{-b} \\
&+(\hat{\phi}_{2a-b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{2a-2b}\cdot\nabla)\hat{\phi}_{2b} + (\hat{\phi}_{2a-3b}\cdot\nabla)\hat{\phi}_{3b} + (\hat{\phi}_{2a--3a}\cdot\nabla)\hat{\phi}_{-3a} \\
&+(\hat{\phi}_{2a--2a}\cdot\nabla)\hat{\phi}_{-2a} + (\hat{\phi}_{2a--a}\cdot\nabla)\hat{\phi}_{-a} \\
&+(\hat{\phi}_{2a-a}\cdot\nabla)\hat{\phi}_{a} + (\hat{\phi}_{2a-3a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned} \tag{139}$$

$$\begin{aligned}
&= (\hat{\phi}_{2a+2b}\cdot\nabla)\hat{\phi}^*_{2b} + (\hat{\phi}_{2a+b}\cdot\nabla)\hat{\phi}^*_{b} \\
&+(\hat{\phi}_{2a-b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{2a-2b}\cdot\nabla)\hat{\phi}_{2b} + (\hat{\phi}_{2a-3b}\cdot\nabla)\hat{\phi}_{3b} \\
&+(\hat{\phi}_{3a}\cdot\nabla)\hat{\phi}^*_{a} + (\hat{\phi}_{a}\cdot\nabla)\hat{\phi}_{a} + (\hat{\phi}^*_{a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned} \tag{140}$$

Continuing with $\omega_m = 3\omega_a$:

$$
\begin{aligned}
&\sum_{l\neq 0,m} (\hat{\phi}_{m-l}\cdot\nabla)\hat{\phi}_l \\
&(\hat{\phi}_{3a--3b}\cdot\nabla)\hat{\phi}_{-3b} + (\hat{\phi}_{3a--2b}\cdot\nabla)\hat{\phi}_{-2b} + (\hat{\phi}_{3a--b}\cdot\nabla)\hat{\phi}_{-b} \\
&+(\hat{\phi}_{3a-b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{3a-2b}\cdot\nabla)\hat{\phi}_{2b} + (\hat{\phi}_{3a-3b}\cdot\nabla)\hat{\phi}_{3b} + (\hat{\phi}_{3a--3a}\cdot\nabla)\hat{\phi}_{-3a} \\
&+(\hat{\phi}_{3a--2a}\cdot\nabla)\hat{\phi}_{-2a} + (\hat{\phi}_{3a--a}\cdot\nabla)\hat{\phi}_{-a} \\
&+(\hat{\phi}_{3a-a}\cdot\nabla)\hat{\phi}_{a} + (\hat{\phi}_{3a-2a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned}
\tag{141}
$$

$$
\begin{aligned}
&= (\hat{\phi}_{3a+2b}\cdot\nabla)\hat{\phi}^*_{2b} + (\hat{\phi}_{3a+b}\cdot\nabla)\hat{\phi}^*_{b} \\
&+(\hat{\phi}_{3a-b}\cdot\nabla)\hat{\phi}_{b} + (\hat{\phi}_{3a-2b}\cdot\nabla)\hat{\phi}_{2b} + (\hat{\phi}_{3a-3b}\cdot\nabla)\hat{\phi}_{3b} \\
&+(\hat{\phi}_{2a}\cdot\nabla)\hat{\phi}_{a} + (\hat{\phi}_{a}\cdot\nabla)\hat{\phi}_{3a}
\end{aligned}
\tag{142}
$$

We notice that we get a series of frequency triads ().

2.9.1 RANS of Fourier Modes

We can also evaluate the Reynolds stress contributions of the modes, with the RANS equation:

$$
\begin{aligned}
&\left(\overline{u_x}\frac{\partial\overline{u_x}}{\partial x} + \overline{u_y}\frac{\partial\overline{u_x}}{\partial y} + \frac{\partial\overline{\widetilde{u}_x\widetilde{u}_x}}{\partial x} + \frac{\partial\overline{u''_x u''_x}}{\partial x} + \frac{\partial\overline{\widetilde{u}_x\widetilde{u}_y}}{\partial y} + \frac{\partial\overline{u''_x u''_y}}{\partial y}\right) \\
&\qquad + \left(\frac{\partial\overline{p}}{\partial x}\right) - \nu\left(\frac{\partial^2\overline{u_x}}{\partial x^2} + \frac{\partial^2\overline{u_x}}{\partial y^2}\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\left(\overline{u_x}\frac{\partial\overline{u_y}}{\partial x} + \overline{u_y}\frac{\partial\overline{u_y}}{\partial y} + \frac{\partial\overline{\widetilde{u}_x\widetilde{u}_y}}{\partial x} + \frac{\partial\overline{u''_x u''_y}}{\partial x} + \frac{\partial\overline{\widetilde{u}_y\widetilde{u}_y}}{\partial y} + \frac{\partial\overline{u''_y u''_y}}{\partial y}\right) \\
&\qquad + \frac{\partial\overline{p}}{\partial y} - \nu\left(\frac{\partial^2\overline{u_y}}{\partial x^2} + \frac{\partial^2\overline{u_y}}{\partial y^2}\right) = 0
\end{aligned}
$$

with:

$$
\begin{aligned}
\frac{\partial\overline{\widetilde{u}_x\widetilde{u}_x}}{\partial x} &= \sum_i^{n_t} 2A_i\phi_{i,x}\frac{\partial\phi_{i,x}}{\partial x} \\
\frac{\partial\overline{\widetilde{u}_x\widetilde{u}_y}}{\partial x} &= \sum_i^{n_t} A_i\left(\phi_{i,x}\frac{\partial\phi_{i,y}}{\partial x} + \phi_{i,y}\frac{\partial\phi_{i,x}}{\partial x}\right) \\
\frac{\partial\overline{\widetilde{u}_x\widetilde{u}_y}}{\partial y} &= \sum_i^{n_t} A_i\left(\phi_{i,x}\frac{\partial\phi_{i,y}}{\partial y} + \phi_{i,y}\frac{\partial\phi_{i,x}}{\partial y}\right) \\
\frac{\partial\overline{\widetilde{u}_y\widetilde{u}_y}}{\partial y} &= \sum_i^{n_t} 2A_i\phi_{i,y}\frac{\partial\phi_{i,y}}{\partial y}
\end{aligned}
$$

For Fourier modes, $A_i = 0.5$. Probably we need to consider the negative frequencies here as well, TBD.

2.10 Fourier transform of the non-linear term

If we consider the vector form of the FANS equation:

$$
i\omega_m\hat{\phi}_m + (\hat{\phi}_m\cdot\nabla)\overline{u} + (\overline{u}\cdot\nabla)\hat{\phi}_m + \sum_{l\neq 0,m}\nabla\cdot\left(\hat{\phi}_{(m-l)}\hat{\phi}_l\right) + \nabla\psi_m - \nu\nabla^2\hat{\phi}_m = 0 \tag{143}
$$

Instead of computing the non-linear term directly, we can compute the fourier transform of the product $\mathbf{u}'\mathbf{u}'$ which we will show is equivalent. Let's define the kth Fourier mode of the fluctuating product $\mathbf{u}'\mathbf{u}'$:

$$
\hat{\tau}_k = \frac{1}{2L}\int_{t=-L}^{L}(\mathbf{u}'\mathbf{u}')e^{-i\omega_k t}dt \tag{144}
$$

with the components:

$$\begin{aligned}
\hat{\tau}_{k,xx} &= \frac{1}{2L}\int_{t=-L}^{L}(u'_x u'_x)e^{-i\omega_k t}dt \\
\hat{\tau}_{k,xy} &= \frac{1}{2L}\int_{t=-L}^{L}(u'_x u'_y)e^{-i\omega_k t}dt \\
\hat{\tau}_{k,yx} = \hat{\tau}_{xy} &= \frac{1}{2L}\int_{t=-L}^{L}(u'_y u'_x)e^{-i\omega_k t}dt \\
\hat{\tau}_{k,yy} &= \frac{1}{2L}\int_{t=-L}^{L}(u'_y u'_y)e^{-i\omega_k t}dt
\end{aligned} \tag{145}$$

We can reconstruct $\mathbf{u'u'}$ using the following sums:

$$\mathbf{u'u'} = \sum_k \hat{\tau}_k e^{i\omega_k t} \tag{146}$$

or the components:

$$\begin{aligned}
u'_x u'_x &= \sum_k \hat{\tau}_{k,xx} e^{i\omega_k t} \\
u'_x u'_y &= \sum_k \tau_{k,xy} e^{i\omega_k t} \\
u'_y u'_x = u'_x u'_y &= \sum_k \hat{\tau}_{k,yx} e^{i\omega_k t} \\
u'_y u'_y &= \sum_k \hat{\tau}_{k,yy} e^{i\omega_k t}
\end{aligned} \tag{147}$$

However, we can also define the reconstruction of $\mathbf{u'u'}$ based on the Fourier transform of $\mathbf{u}$. If we recall the inverse transform for our velocity field:

$$\mathbf{u}(\mathbf{x}, t) = \sum_k \hat{\phi}_k e^{i\omega_k t} \tag{148}$$

We can consider the fluctuating part only by considering:

$$\mathbf{u}' = \sum_{k\neq 0} \hat{\phi}_k e^{i\omega_k t} \tag{149}$$

Thus we can write $\mathbf{u'u'}$ as

$$\begin{aligned}
\mathbf{u'u'} &= \sum_{k\neq 0} \hat{\phi}_k e^{i\omega_k t} \sum_{l\neq 0} \hat{\phi}_l e^{i\omega_l t} \\
&= \sum_{k\neq 0}\sum_{l\neq 0} \hat{\phi}_k \hat{\phi}_l e^{i(\omega_k+\omega_l)t}
\end{aligned} \tag{150}$$

Thus we get the equality based on the Fourier transform of $\mathbf{u'u'}$:

$$\mathbf{u'u'} = \sum_k \hat{\tau}_k e^{i\omega_k t} = \sum_{k\neq 0}\sum_{l\neq 0} \hat{\phi}_k \hat{\phi}_l e^{i(\omega_k+\omega_l)t} \tag{151}$$

Take the part corresponding to the frequency m:

$$\begin{aligned}
= \frac{1}{2L}\int_{t=-L}^{L} e^{-i\omega_m t}\sum_k \hat{\tau}_k e^{i\omega_k t}dt &= \frac{1}{2L}\int_{t=-L}^{L} e^{-i\omega_m t}\sum_{k\neq 0}\sum_{l\neq 0}\hat{\phi}_k\hat{\phi}_l e^{i(\omega_k+\omega_l)t}dt \\
= \hat{\tau}_m &= \frac{1}{2L}\int_{t=-L}^{L}\sum_{k\neq 0}\sum_{l\neq 0}\hat{\phi}_k\hat{\phi}_l e^{i(\omega_k+\omega_l-\omega_m)t}dt
\end{aligned} \tag{152}$$

The right hand side is recognizable as the Fourier stress term. Using the orthogonality argument from before, the sum is nonzero for only $k + l - m = 0$. We can thus let $k = m - l$:

$$= \nabla \cdot \hat{\tau}_m = \sum_{l\neq 0,m} \nabla \cdot \left(\hat{\phi}_{(m-l)}\hat{\phi}_l\right) \tag{153}$$

Thus we can write our equation using the Fourier coefficients of $\mathbf{u}'\mathbf{u}'$ to replace the Fourier stress term:

$$i\omega_m\hat{\phi}_m + (\hat{\phi}_m \cdot \nabla)\overline{u} + (\overline{u} \cdot \nabla)\hat{\phi}_m + (\nabla \cdot \hat{\tau}_m) + \nabla\psi_m - \nu\nabla^2\hat{\phi}_m = 0 \tag{154}$$

If we write the network notation for these we get:

$$\begin{aligned} \left[\hat{\phi}_m, \hat{\tau}_m\right]\left(\hat{\psi}_m\right) &= f(\mathbf{x}) \\ \text{given: } \left[\overline{u}, \overline{\mathbf{u}'\mathbf{u}'}\right](\overline{p}) &= f(\mathbf{x}) \end{aligned} \tag{155}$$

with the loss function:

$$L_F = \frac{1}{N_c}\sum_{1}^{N_c}\left(i\omega_m\hat{\phi}_m + (\hat{\phi}_m \cdot \nabla)\overline{u} + (\overline{u} \cdot \nabla)\hat{\phi}_m + (\nabla \cdot \hat{\tau}_m) + \nabla\psi_m - \nu\nabla^2\hat{\phi}_m\right) \tag{156}$$

$$L_C = \frac{1}{N_c}\sum_{1}^{N_c}\left(\nabla \cdot \hat{\phi}\right) \tag{157}$$

$$L_D = \frac{1}{N_D}\sum_{1}^{N_D}\left((\hat{\phi}_{PINN} - \hat{\phi}_D)^2 + (\hat{\tau}_{PINN} - \hat{\tau}_D)^2\right) \tag{158}$$

$$L = \gamma\left(L_F + L_C\right) + L_D \tag{159}$$

Revisiting the previous expansion into real valued equations, with the new form for the fourier stress term:

$$\begin{aligned} -\omega_m\phi_{m,x,i} &+ \left(\phi_{m,x,r}\frac{\partial\overline{u_x}}{\partial x} + \phi_{m,y,r}\frac{\partial\overline{u_x}}{\partial y}\right) + \left(\overline{u_x}\frac{\partial\phi_{m,x,r}}{\partial x} + \overline{u_y}\frac{\partial\phi_{m,x,r}}{\partial y}\right) \\ &+ \left(\frac{\partial\tau_{m,xx,r}}{\partial x} + \frac{\partial\tau_{m,xy,r}}{\partial y}\right) + \psi_{m,x,r} - \nu\left(\frac{\partial^2\phi_{m,x,r}}{\partial x^2} + \frac{\partial^2\phi_{m,x,r}}{\partial y^2}\right) = 0 \end{aligned} \tag{160}$$

The complex part of the x-momentum equation:

$$\begin{aligned} \omega_m\phi_{m,x,r} &+ \left(\phi_{m,x,i}\frac{\partial\overline{u_x}}{\partial x} + \phi_{m,y,i}\frac{\partial\overline{u_x}}{\partial y}\right) + \left(\overline{u_x}\frac{\partial\phi_{m,x,i}}{\partial x} + \overline{u_y}\frac{\partial\phi_{m,x,i}}{\partial y}\right) \\ &+ \left(\frac{\partial\tau_{m,xx,i}}{\partial x} + \frac{\partial\tau_{m,xy,i}}{\partial y}\right) + \psi_{m,x,i} - \nu\left(\frac{\partial^2\phi_{m,x,i}}{\partial x^2} + \frac{\partial^2\phi_{m,x,i}}{\partial y^2}\right) = 0 \end{aligned} \tag{161}$$

The real part of the y-momentum equation:

$$\begin{aligned} -\omega_m\phi_{m,y,i} &+ \left(\phi_{m,x,r}\frac{\partial\overline{u_y}}{\partial x} + \phi_{m,y,r}\frac{\partial\overline{u_y}}{\partial y}\right) + \left(\overline{u_x}\frac{\partial\phi_{m,y,r}}{\partial x} + \overline{u_y}\frac{\partial\phi_{m,y,r}}{\partial y}\right) \\ &+ \left(\frac{\partial\tau_{m,xy,r}}{\partial x} + \frac{\partial\tau_{m,yy,r}}{\partial y}\right) + \psi_{m,y,r} - \nu\left(\frac{\partial^2\phi_{m,y,r}}{\partial x^2} + \frac{\partial^2\phi_{m,y,r}}{\partial y^2}\right) = 0 \end{aligned} \tag{162}$$

The complex part of the y-momentum equation:

$$\begin{aligned} \omega_m\phi_{m,y,r} &+ \left(\phi_{m,x,i}\frac{\partial\overline{u_y}}{\partial x} + \phi_{m,y,i}\frac{\partial\overline{u_y}}{\partial y}\right) + \left(\overline{u_x}\frac{\partial\phi_{m,y,i}}{\partial x} + \overline{u_y}\frac{\partial\phi_{m,y,i}}{\partial y}\right) \\ &+ \left(\frac{\partial\tau_{m,xy,i}}{\partial x} + \frac{\partial\tau_{m,yy,i}}{\partial y}\right) + \psi_{m,y,i} - \nu\left(\frac{\partial^2\phi_{m,y,i}}{\partial x^2} + \frac{\partial^2\phi_{m,y,i}}{\partial y^2}\right) = 0 \end{aligned} \tag{163}$$

2.11 Energy equation

Considering the

3 Galerkin Projection on NS

The approach above produces an orthogonal basis that can be used for a Galerkin projection of the NSE. Assume that the modes provide a representation of the flow as

$$\mathbf{u}(x, y, t) = \sum_k a_k(t)\widetilde{u}_k(x, y) \tag{164}$$

When this approach is inserted into the NSE this leads to a linear quadratic system of ODEs given by

$$\frac{\mathrm{d}a_k}{\mathrm{d}t} = F_k + L_{kl}a_l + Q_{klm}a_l a_m \tag{165}$$

with constant, linear, and quadratic coefficients given by

$$F_k = \left\langle \widetilde{u}_k, -(\overline{u}\cdot\nabla)\overline{u} - \nabla\overline{p} + Re^{-1}\nabla^2\overline{u} \right\rangle \tag{166}$$
$$L_{kl} = \left\langle \widetilde{u}_k, -(\overline{u}\cdot\nabla)\overline{u} - (\overline{u}\cdot\nabla)\widetilde{u}_k - \nabla\widetilde{p} + Re^{-1}\nabla^2\widetilde{u}_l \right\rangle \tag{167}$$
$$Q_{klm} = \langle \widetilde{u}_k, -(\overline{u}\cdot\nabla)\widetilde{u}_l \rangle \tag{168}$$
$$\tag{169}$$

These coefficient can be computed from the fields assimilated wit the PINNs. The brackets $\langle , \rangle$ indicate the inner product. To stabilise this system this might require further introduction of a shift mode as proposed by Noack et al. 2003, or the use of the multiscale model reduction approach of Callaham et al. 2022.

3.1 Expansion based on Moritz's derivation

Since the quantities in both sides of the inner products are vectors in R^2 or R^3, the inner product here becomes a dot product summed over the spatial domain:

$$\begin{aligned} F_k = \sum_\Omega (F_{k,x} + F_{k,y}) = \sum_\Omega \phi_{k,x} & \left(-\left(\frac{\partial \overline{u}_x\overline{u}_x}{\partial x} + \frac{\partial \overline{u}_y\overline{u}_x}{\partial y} \right) - \frac{\partial \overline{p}}{\partial x} + \nu \left(\frac{\partial^2 \overline{u}_x}{\partial x^2} + \frac{\partial^2 \overline{u}_x}{\partial y^2} \right) \right) \\ + \phi_{k,y} & \left(-\left(\frac{\partial \overline{u}_x\overline{u}_y}{\partial x} + \frac{\partial \overline{u}_y\overline{u}_y}{\partial y} \right) - \frac{\partial \overline{p}}{\partial y} + \nu \left(\frac{\partial^2 \overline{u}_y}{\partial x^2} + \frac{\partial^2 \overline{u}_y}{\partial y^2} \right) \right) \end{aligned} \tag{170}$$

$$\begin{aligned} L_{kl} = L_{kl,x} + L_{kl,y} = \sum_\Omega \Phi_{k,x} & \left(-\left(\frac{\partial \overline{u}_x\widetilde{u}_x}{\partial x} + \frac{\partial \overline{u}_y\widetilde{u}_x}{\partial y} \right) - \right) \\ + \Phi_{k,y} & \left(-\left(\frac{\partial \overline{u}_x\widetilde{u}_y}{\partial x} + \frac{\partial \overline{u}_y\widetilde{u}_y}{\partial y} \right) - \right) \end{aligned} \tag{171}$$

3.2 Derivation

We can construct our Galerkin model by constructing a weak form of the NS equations using the POD modes as test functions.

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = 0 \\ \frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} \end{aligned} \tag{172}$$

Tried some examples with different random vectors, and it seems possible to distribute the inner product in R^2 and R^3. Take the inner product with the mode:

$$\left\langle \phi_k, \frac{\partial \mathbf{u}}{\partial t} \right\rangle = -\langle \phi_k, (\mathbf{u}\cdot\nabla)\,\mathbf{u} \rangle - \langle \phi_k, \nabla p \rangle + \langle \phi_k, \nu\nabla^2\mathbf{u} \rangle \tag{173}$$

Let $\mathbf{u}$ and p take the form:

$$\mathbf{u} = \sum_{k=0}^{N} u_k = \overline{u} + \sum_{k=1}^{N} \phi_k a_k \tag{174}$$

$$p = \sum_{k=0}^{N} p_k = \overline{p} + \sum_{k=1}^{N} \psi_k a_k \tag{175}$$

Noting that $k = 0$ is the mean field. Looking at the time derivative term:

$$\left\langle \phi_k, \frac{\partial \mathbf{u}}{\partial t} \right\rangle = \sum_{i=0}^{N} \phi_k \cdot \frac{d\mathbf{u}_k}{dt} = 0 + \sum_{i=1}^{N} \phi_k \cdot \phi_l \frac{da_l}{dt} = \delta_{kl}\frac{da_l}{dt} \tag{176}$$

Since the basis functions are orthogonal, we arrive at:

$$\left\langle \phi_k, \frac{\partial \mathbf{u}}{\partial t} \right\rangle = \frac{da_k}{dt} \tag{177}$$

Lets next consider the convective term:

$$\left\langle \phi_k, ([\overline{u} + \sum_{l=1}^{N} \phi_l a_l] \cdot \nabla)(\overline{u} + \sum_{m=1}^{N} \phi_m a_m) \right\rangle$$

Expanding the right hand component of the inner product:

$$\begin{aligned} ([\overline{u} + \sum_{l=1}^{N} \phi_l a_l] \cdot \nabla)(\overline{u} + \sum_{m=1}^{N} \phi_m a_m) = \\ (\overline{u} \cdot \nabla)\overline{u} + (\sum_{l=1}^{N} \phi_l a_l \cdot \nabla)\overline{u} + (\overline{u} \cdot \nabla) \sum_{m=1}^{N} \phi_m a_m + (\sum_{l=1}^{N} \phi_l a_l \cdot \nabla) \sum_{m=1}^{N} \phi_m a_m \end{aligned} \tag{178}$$

We can see that we get a constant term, two linear terms, and one quadratic term. If we factor out the temporal coefficients and reindex the linear terms to a single dummy index:

$$(\overline{u} \cdot \nabla)\overline{u} + a_l(\phi_l \cdot \nabla)\overline{u} + a_l(\overline{u} \cdot \nabla)\phi_l + a_l a_m(\phi_l \cdot \nabla)\phi_m \tag{179}$$

We can rewrite our system of equations into constant, linear and quadratic terms:

$$\begin{aligned} \frac{da_k}{dt} &= F_k + L_{kl} a_l + Q_{klm} a_l a_m \\ F_k &= -\langle \phi_k, (\overline{u} \cdot \nabla)\overline{u} \rangle + ... \\ L_{kl} &= -\langle \phi_k, (\phi_l \cdot \nabla)\overline{u} \rangle - \langle \phi_k, (\overline{u} \cdot \nabla)\phi_l \rangle + ... \\ Q_{klm} &= -\langle \phi_k, (\phi_l \cdot \nabla)\phi_m \rangle + ... \end{aligned} \tag{180}$$

Let's complete the remaining terms, starting with the pressure term:

$$\langle \phi_k, -\nabla p \rangle = \langle \phi_k, -\nabla \overline{p} \rangle + \left\langle \phi_k, \sum_{k=1}^{N} \nabla \psi_l a_l \right\rangle = \langle \phi_k, -\nabla \overline{p} \rangle + a_l \langle \phi_k, \nabla \psi_l \rangle \tag{181}$$

Noticing that the left term is constant, and the right term is linear, we can add them to our system of equations:

$$\begin{aligned} \frac{da_k}{dt} &= F_k + L_{kl} a_l + Q_{klm} a_l a_m \\ F_k &= -\langle \phi_k, (\overline{u} \cdot \nabla)\overline{u} \rangle + \langle \phi_k, \nabla \overline{p} \rangle + ... \\ L_{kl} &= -\langle \phi_k, (\phi_l \cdot \nabla)\overline{u} \rangle - \langle \phi_k, (\overline{u} \cdot \nabla)\phi_l \rangle - \langle \phi_k, \nabla \psi_l \rangle + ... \\ Q_{klm} &= -\langle \phi_k, (\phi_l \cdot \nabla)\phi_m \rangle + ... \end{aligned} \tag{182}$$

Now let's consider the viscous term:

$$\begin{aligned} \langle \phi_k, \nu \nabla^2 \mathbf{u} \rangle &= \nu \langle \phi_k, \nabla^2 \overline{u} \rangle + \langle \phi_k, \nabla^2 \phi_l a_l \rangle \\ &= \nu \langle \phi_k, \nabla^2 \overline{u} \rangle + \nu a_l \langle \phi_k, \nabla^2 \phi_l \rangle \end{aligned} \tag{183}$$

Which gives the final form for the Galerkin system:

$$\begin{aligned} \frac{da_k}{dt} &= F_k + L_{kl} a_l + Q_{klm} a_l a_m \\ F_k &= -\langle \phi_k, (\overline{u} \cdot \nabla)\overline{u} \rangle - \langle \phi_k, \nabla \overline{p} \rangle + \nu \langle \phi_k, \nabla^2 \overline{u} \rangle \\ L_{kl} &= -\langle \phi_k, (\phi_l \cdot \nabla)\overline{u} \rangle - \langle \phi_k, (\overline{u} \cdot \nabla)\phi_l \rangle - \langle \phi_k, \nabla \psi_l \rangle + \nu \langle \phi_k, \nabla^2 \phi_l \rangle \\ Q_{klm} &= -\langle \phi_k, (\phi_l \cdot \nabla)\phi_m \rangle \end{aligned} \tag{184}$$

3.3 Expansion based on second derivation

Looking at the constant term:

$$
\begin{aligned}
F_k = -\left\langle \phi_k, (\overline{u}\cdot\nabla)\overline{u}\right\rangle - \left\langle \phi_k, \nabla\overline{p}\right\rangle + \nu\left\langle \phi_k, \nabla^2\overline{u}\right\rangle \\
F_k = -\Phi_{k,x}\left(\frac{\partial\overline{u}_x\overline{u}_x}{\partial x} + \frac{\partial\overline{u}_y\overline{u}_x}{\partial y}\right) - \phi_{k,y}\left(\frac{\partial\overline{u}_x\overline{u}_y}{\partial x} + \frac{\partial\overline{u}_y\overline{u}_y}{\partial y}\right) - \Phi_{k,x}\frac{\partial\overline{p}}{\partial x} - \Phi_{k,y}\frac{\partial\overline{p}}{\partial y} \\
+\nu\left[\phi_{k,x}\left(\frac{\partial^2\overline{u}_x}{\partial^2 x} + \frac{\partial^2\overline{u}_x}{\partial^2 y}\right) + \phi_{k,y}\left(\frac{\partial^2\overline{u}_y}{\partial^2 x} + \frac{\partial^2\overline{u}_y}{\partial^2 y}\right)\right]
\end{aligned}
\tag{185}
$$

Then the linear term:

$$
\begin{aligned}
L_{kl} &= -\left\langle \phi_k, (\phi_l\cdot\nabla)\overline{u}\right\rangle - \left\langle \phi_k, (\overline{u}\cdot\nabla)\phi_l\right\rangle - \left\langle \phi_k, \nabla\psi_l\right\rangle + \nu\left\langle \phi_k, \nabla^2\phi_l\right\rangle \\
&= -\left\langle \phi_k, \left(\phi_{l,x}\frac{\partial}{\partial x} + \phi_{l,y}\frac{\partial}{\partial y}\right)\overline{u}\right\rangle - \left\langle \phi_k, \left(\overline{u}_x\frac{\partial}{\partial x} + \overline{u}_y\frac{\partial}{\partial y}\right)\phi_l\right\rangle \\
&\quad - \left\langle \phi_k, \begin{bmatrix} \frac{\partial\psi_l}{\partial x} \\ \frac{\partial\psi_l}{\partial y} \end{bmatrix}\right\rangle + \nu\left\langle \phi_k, \begin{bmatrix} \frac{\partial^2\phi_{l,x}}{\partial x^2} + \frac{\partial^2\phi_{l,x}}{\partial y^2} \\ \frac{\partial^2\psi_{l,y}}{\partial x^2} + \frac{\partial^2\phi_{l,y}}{\partial y^2} \end{bmatrix}\right\rangle
\end{aligned}
\tag{186}
$$

$$
\begin{aligned}
= -\left\langle \phi_k, \begin{bmatrix} \phi_{l,x}\frac{\partial\overline{u}_x}{\partial x} + \phi_{l,y}\frac{\partial\overline{u}_x}{\partial y} \\ \phi_{l,x}\frac{\partial\overline{u}_y}{\partial x} + \phi_{l,y}\frac{\partial\overline{u}_y}{\partial y} \end{bmatrix}\right\rangle - \left\langle \phi_k, \begin{bmatrix} \overline{u}_x\frac{\partial\phi_{l,x}}{\partial x} + \overline{u}_y\frac{\partial\phi_{l,x}}{\partial y} \\ \overline{u}_x\frac{\partial\phi_{l,y}}{\partial x} + \overline{u}_y\frac{\partial\phi_{l,y}}{\partial y} \end{bmatrix}\right\rangle \\
-\phi_{k,x}\frac{\partial\psi_l}{\partial x} - \phi_{k,y}\frac{\partial\psi_l}{\partial y} + \nu\left[\phi_{k,x}\left(\frac{\partial^2\phi_{l,x}}{\partial x^2} + \frac{\partial^2\phi_{l,x}}{\partial y^2}\right) + \phi_{k,y}\left(\frac{\partial^2\psi_{l,y}}{\partial x^2} + \frac{\partial^2\phi_{l,y}}{\partial y^2}\right)\right]
\end{aligned}
\tag{187}
$$

$$
\begin{aligned}
L_{kl} = -\phi_{k,x}\left(\phi_{l,x}\frac{\partial\overline{u}_x}{\partial x} + \phi_{l,y}\frac{\partial\overline{u}_x}{\partial y}\right) - \phi_{k,y}\left(\phi_{l,x}\frac{\partial\overline{u}_y}{\partial x} + \phi_{l,y}\frac{\partial\overline{u}_y}{\partial y}\right) \\
-\phi_{k,x}\left(\overline{u}_x\frac{\partial\phi_{l,x}}{\partial x} + \overline{u}_y\frac{\partial\phi_{l,x}}{\partial y}\right) - \phi_{k,y}\left(\overline{u}_x\frac{\partial\phi_{l,y}}{\partial x} + \overline{u}_y\frac{\partial\phi_{l,y}}{\partial y}\right) \\
-\phi_{k,x}\frac{\partial\psi_l}{\partial x} - \phi_{k,y}\frac{\partial\psi_l}{\partial y} + \nu\left[\phi_{k,x}\left(\frac{\partial^2\phi_{l,x}}{\partial x^2} + \frac{\partial^2\phi_{l,x}}{\partial y^2}\right) + \phi_{k,y}\left(\frac{\partial^2\phi_{l,y}}{\partial x^2} + \frac{\partial^2\phi_{l,y}}{\partial y^2}\right)\right]
\end{aligned}
\tag{188}
$$

Finally the quadratic term, summing over space, for all combinations l and m:

$$
\begin{aligned}
Q_{klm} = -\left\langle \phi_k, (\phi_l\cdot\nabla)\phi_m\right\rangle = -\left\langle \phi_k, \left(\phi_{l,x}\frac{\partial}{\partial x} + \phi_{l,y}\frac{\partial}{\partial y}\right)\phi_m\right\rangle \\
= -\left\langle \phi_k, \begin{bmatrix} \left(\phi_{l,x}\frac{\partial\phi_{m,x}}{\partial x} + \phi_{l,y}\frac{\partial\phi_{m,x}}{\partial y}\right) \\ \left(\phi_{l,x}\frac{\partial\phi_{m,y}}{\partial x} + \phi_{l,y}\frac{\partial\phi_{m,y}}{\partial y}\right) \end{bmatrix}\right\rangle \\
Q_{klm} = -\phi_{k,x}\left(\phi_{l,x}\frac{\partial\phi_{m,x}}{\partial x} + \phi_{l,y}\frac{\partial\phi_{m,x}}{\partial y}\right) - \phi_{k,y}\left(\phi_{l,x}\frac{\partial\phi_{m,y}}{\partial x} + \phi_{l,y}\frac{\partial\phi_{m,y}}{\partial y}\right)
\end{aligned}
\tag{189}
$$

3.4 Calibrating the Pressure Term

Where the pressure field is available, it is possible to calibrate an alternative linear representation for the pressure term, following equation C1 of Noack 2005:

$$
\langle u_i, -\nabla p\rangle_\Omega = \sum_{j=0}^{N} l_{ij}^{\pi} a_j \tag{190}
$$

If we recast the notation of the spatial modes to be consistent with the rest of the derivations:

$$
\langle \phi_i, -\nabla p\rangle_\Omega = \sum_{j=0}^{N} l_{ij}^{\pi} a_j \tag{191}
$$

If we then multiply the equation by a_i, we can use the orthogonality of the temporal coefficients to reduce the right hand side:

$$\langle \phi_i a_i, -\nabla p \rangle_\Omega = \sum_{j=0}^{N} l^\pi_{ij} a_j a_i \tag{192}$$

$$\langle \phi_i a_i, -\nabla p \rangle_\Omega = \sum_{j=0}^{N} l^\pi_{ij} \lambda_i \delta_{ij} \tag{193}$$

$$\langle \phi_i a_i, -\nabla p \rangle_\Omega = l^\pi_{ii} \lambda_i \tag{194}$$

$$\tag{195}$$

Alternatively we can compute the terms using a linear system:

$$\langle \phi_i, -\nabla p \rangle_\Omega = \sum_{j=0}^{N} l^\pi_{ij} a_j \tag{196}$$

If we construct a matrix A of all the temporal coefficients:

$$A = \begin{bmatrix} a_1 \\ a_2 \\ .. \\ a_{N_m} \end{bmatrix} \tag{197}$$

And matrix L_i of the pressure coefficients to be computed for the ith mode:

$$L^\pi_i = \begin{bmatrix} l_{i,1} & l_{i,2} & .. & l_{i,N_m} \end{bmatrix} \tag{198}$$

we can thus rewrite the pressure expression as:

$$\langle \phi_i, -\nabla p \rangle_\Omega = L^\pi_i A \tag{199}$$

If B^* is the Moore-Penrose pseudoinverse of B, we can find L^π_i in a least squares sense:

$$\langle \phi_i, -\nabla p \rangle_\Omega A^* = L^\pi_i \tag{200}$$

$$\tag{201}$$

$$-\phi_{i,x} \frac{\partial p^m}{\partial x} - \phi_{i,y} \frac{\partial p^m}{\partial y} = \sum_{j=0}^{N} l^\pi_{ij} a^m_j \tag{202}$$