3.3 Compact Sets

Definition 3.3.1 (Compactness): A set $K \subseteq \mathbb{R}$ is compact if every sequence in K has a subsequence that converges to a limit also in K.

Ex 3.3.2

- A closed interval is the most basic example of a compact set
  - if (a_n) is contained in $[c,d]$, by the Bolzano-Weierstrass Theorem we can find a convergent subsequence (a_{n_k}). And because a closed interval is a closed set, then $\lim a_{n_k} \in [c,d]$

Definition 3.3.3: A set $A \subseteq \mathbb{R}$ is bounded if $\exists M > 0$ s.t. $|a| \leq M$ $\forall a \in A$.

Theorem 3.3.4 (Characterization of Compactness in $\mathbb{R}$):
A set $K \subseteq \mathbb{R}$ is compact iff it is closed and bounded.

Proof:

Assume K is compact. If K was unbounded, then we could define a strictly increasing unbounded sequence contained in K, which has no convergent subsequences. This violates compactness, so K must be bounded. Now we must show that K is closed. Consider a limit point x of K. There exists an (a_n) contained in K with $(a_n) \to x$. Because K is compact, there exists some $(a_{n_k}) \to l$ s.t. $l \in K$. But because all subsequences of a convergent sequence converge to the same limit, $l = x$, and so $x \in K$. Hence, K is closed and bounded.

Assume K is closed and bounded. Since K is bounded, every sequence of K is bounded, and by the Bolzano-Weierstrass Theorem every sequence of K has a convergent subsequence. Because K is closed, all these convergent subsequences converge to a limit in K, and therefore K is compact $\square$

Ex 3.3.3

- important to think of compact sets as generalizations of closed intervals → whenever a fact involving closed intervals is true, we can replace "closed interval" with "compact set" (usually)

Theorem 3.3.5 (Nested Compact Set Property)

If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ is a nested sequence of nonempty <u>compact</u> sets, then $\bigcap_{n=1}^{\infty} K_n$ is not empty.

Proof:

We want to construct a sequence that is eventually in each one of these sets. For each $n \in \mathbb{N}$, pick $x_n \in K_n$. Because the sets are nested, (x_n) is contained in K_1. Then (x_n) has an $(x_{n_k}) \to x \in K_1$. Now we show that x belongs to every K_n. Given $n_0 \in \mathbb{N}$, the terms of (x_n) are contained in K_{n_0} as long as $n \geq n_0$. Thus (x_{n_k}) is eventually contained in K_{n_0}, and so $x \in K_{n_0}$. Since n_0 is arbitrary, $x \in \bigcap_{n=1}^{\infty} K_n$ □

Open Covers

- like completeness, there are different ways to define/characterize compactness in $\mathbb{R}$. Could have defined compact sets as closed and bounded, and then shown that convergent subsequences in compact sets have limits in the set.
- there is a 3rd characterization in terms of open covers and finite subcovers.

Definition 3.3.6 (Open Cover and Finite Subcover)

Let $A \subseteq \mathbb{R}$. An open cover for A is a possibly infinite collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains A ($A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$).
Given an open cover for A, a finite subcover is a finite sub-collection of open sets from the original open cover whose union still manages to completely contain A.

Ex 3.3.7

Consider $(0,1)$. For each point $x \in (0,1)$, let O_x be $(x/2, 1)$. Then $\{O_x : x \in (0,1)\}$ forms an open cover. It is impossible, however, to form a finite subcover of $(0,1)$ given this open cover. Any proposed finite subcollection $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$ can be disqualified as a finite subcover if we set $x' = \min\{x_1, x_2, \ldots, x_n\}$ and observe a real number y s.t. $0 < y \leq x'/2 \Rightarrow y \in (0,1)$ but $y \notin \bigcup_{i=1}^{n} O_{x_i}$

Now consider a similar open cover for $[0,1]$. Let $O_x = (x/2, 1)$ for $x \in (0,1)$, and then let $O_0 = (-\varepsilon, \varepsilon)$ and $O_1 = (1-\varepsilon, 1+\varepsilon)$ to cover the endpoints. $\{O_0, O_1, O_x : x \in (0,1)\}$ is an open cover. For a finite subcover, we could be boring and have defined $\varepsilon = 0.6 \Rightarrow \{O_0, O_1\}$ is a finite subcover, or in general choose x' s.t. $x'/2 < \varepsilon \Rightarrow \{O_0, O_1, O_{x'}\}$ is a finite subcover.

Theorem 3.3.8 (Heine-Borel Theorem)

Let $K \subseteq \mathbb{R}$. All of the following are equivalent in that any one of them implies the two others:

(i) K is compact

(ii) K is closed and bounded

(iii) Every open cover for K has a finite subcover

Proof:

(i) and (ii) are already equivalent, so we only need to show that (i)=(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i)=(ii), so we start with (iii) and try to imply (ii). Assume every open cover for $K \subseteq \mathbb{R}$ has a finite subcover $\{O_1, O_2, \ldots, O_n\}$. Because this subcover is bounded and K is contained in the subcover, K is bounded. Assume for contradiction that K is not closed, so $\exists (a_n) \to a$ s.t. (a_n) is contained in K and $a \notin K$. Construct an open cover by taking O_x to be an interval of radius $|k-a|/2$ for every $k \in K$. We can also assume a finite subcover $\{O_{k_1}, O_{k_2}, \ldots, O_{k_n}\}$. Let $\varepsilon_0 = \min\left\{\frac{|k_i - a|}{2} : 1 \leq i \leq n\right\}$. Since $(a_n) \to a$, $\exists a_N$ s.t. $|a_N - a| < \varepsilon_0$. But if this is the case, a_N cannot belong to any O_{k_i}, so K is closed.

Now we prove the converse that (ii) implies (iii). Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, assume no finite subcover exists. Let I_0 be a closed interval containing K. Specifically, let $I_0 = [\inf K, \sup K]$. $I_0 \cap K$ cannot be finitely covered because $I_0 \cap K = K$. Split I_0 into halves $[\inf I_0, \frac{\inf I_0 + \sup I_0}{2}]$, $[\frac{\inf I_0 + \sup I_0}{2}, \sup I_0]$, and choose I_1 as the half ~~such~~ that ~~$I_1 \cap K$~~ cannot be finitely covered. Inductively define I_n as the half of I_{n-1} ~~such~~ that ~~$I_n \cap K$~~ cannot be finitely covered. The existence of this half is guaranteed because if both halves could be finitely covered, then I_{n-1} can be finitely covered, and so can $I_{n-1} \cap K$. Because $|I_n| = (\frac{1}{2})^n I_0$, $\lim |I_n| = 0$. Each $I_n \cap K \neq \emptyset$, because if it were empty then it could be finitely covered. Therefore, $\forall n \; \exists x$ s.t. $x \in K$ and $x \in I_n$. For each I_n we can call this element x_n. Since $\lim |I_n| = 0$, let $|I_N| = \varepsilon$. Because $\forall n, m \geq N$, $x_n \in I_N$ and $x_m \in I_N$, it follows that $|x_n - x_m| < \varepsilon$. By the Cauchy Criterion, we have constructed a sequence $(x_n) \to x$. Because (x_n) is contained in K and K is closed, $x \in K$. x must also be in $I_n \; \forall n \in \mathbb{N}$, because if it were not, we could find a term in the sequence also outside of all I_n, which violates our construction. Now because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. This means $\exists V_\varepsilon(x) \subseteq O_{\lambda_0}$. But because $\lim |I_n| = 0$, $\exists N$ s.t. $I_N \cap K$ cannot be finitely covered, but $I_N \subseteq V_\varepsilon(x) \subseteq O_{\lambda_0}$, implying that O_{λ_0} is a finite subcover of $I_N \cap K$. Thus, a finite subcover must exist $\square$

$\leftarrow$ Ex 3.3.9

X = incorrect
O = no idea, looked it up
* = looked for hints

3.3 Exercises

1. Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

Proof:
Because K is compact, it is bounded, so by Axiom of Completeness both $\inf K$ and $\sup K$ exist. Then since K is closed, $K = \bar{K}$, and by Ex 3.2.4 both $\sup K$ and $\inf K$ are elements of K. □

2/11. Decide if the set is compact. If it is not, provide a sequence contained in the set that does not possess a subsequence converging to a limit in the set. (Ex 11) Also provide an open cover for which there is no finite subcover.

a) $\mathbb{N}$

Not compact - the sequence $a_n = n$ is contained in $\mathbb{N}$ and has no subsequence converging to anything. For each $n \in \mathbb{N}$, let $O_n = (n-\varepsilon, n+\varepsilon)$ for $\varepsilon > 0$. Then $\{O_n : n \in \mathbb{N}\}$ is an open cover of $\mathbb{N}$ with no finite subcover.

b) $\mathbb{Q} \cap [0,1]$ had to look up the open cover

Not compact. Let x be an irrational number $0 < x < 1$. Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, we can find a sequence (a_n) contained in $\mathbb{Q} \cap [0,1]$ that converges to x. Because $(a_n) \to x$, all subsequences of (a_n) also converge to x, and since $x \notin \mathbb{Q} \cap [0,1]$, the set is not compact. Let an open cover for the set be $\left\{(-2, 1/\sqrt{2} - 1/n) \cup (1/\sqrt{2} + 1/n, 2) : n \in \mathbb{N}\right\}$. No finite subcover for the set.

c) The Cantor set

Compact. It is bounded below by 0 and above by 1, and it is closed because it is the intersection of an arbitrary collection of closed sets:
$C = \bigcap_{n=0}^{\infty} C_n$, and each C_n is the union of a finite collection of closed intervals.

d) $\{1 + 1/2^2 + 1/3^2 + \ldots + 1/n^2 : n \in \mathbb{N}\}$

Not compact. The sequence $(1, 1+1/4, 1+1/4+1/9, 1+1/4+1/9+1/16, \ldots)$ converges to $\pi^2/6$, which is not in the set. Because all subsequences converge to the same value, the set is not compact. There is no finite subcover for the open cover given by $\{(0, \pi^2/6 - 1/n) : n \in \mathbb{N}\}$.

e) $\{1, 1/2, 2/3, 3/4, \ldots\}$

Compact. The set is bounded below by $1/2$ and above by 1. All sequences and therefore subsequences of the set converge to 1, which is in the set.

4) Assume K is compact and F is closed. Decide if definitely compact, definitely closed, both, or neither

a) $K \cap F$

Intersection of 2 closed sets is closed, and because K is bounded $K \cap F$ is also bounded, so definitely compact

b) $\overline{F^c \cup K^c}$

K^c is unbounded, but $\overline{F^c \cup K^c}$ is closed, so definitely closed.

c) $K \setminus F = \{x \in K : x \notin F\}$

Neither. Let $K = [0,2]$, $F = [1, \infty)$. $K \setminus F = [0,1)$, which is not closed

d) $\overline{K \cap F^c}$

definitely compact. $K \cap F^c$ is bounded, so $\overline{K \cap F^c}$ is bounded and closed

5) Prove if true, counterexample if false.

a) The arbitrary intersection of compact sets is compact

True. Proof:
Since compact sets are closed, $\bigcap_{n=1}^{\infty} A_n$ for compact sets A_n is closed. And if any set in an intersection is bounded, then the intersection is also bounded. Thus $\bigcap_{n=1}^{\infty} A_n$ is compact $\square$

b) The arbitrary union of compact sets is compact.

False. Define $A_n = [0, n]$. Then $\bigcup_{n=1}^{\infty} A_n$ is unbounded and not compact

c) Let A be arbitrary, and let K be compact. Then $A \cap K$ is compact

False. Let $K = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and $A = \{1/n : n \in \mathbb{N}\}$. Then $A \cap K = \{1/n : n \in \mathbb{N}\}$, which does not contain its limit point, so not compact.

d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ is a nested sequence of nonempty closed sets, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$

False. Let $F_n = [n, \infty)$. $\bigcap_{n=1}^{\infty} F_n = \emptyset$

10) Alternate proof to final implication of Heine-Borel Theorem.
Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a,b]$, and define S to be the set of all $x \in [a,b]$ s.t. $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$

a) Argue S is nonempty and bounded so $s = \sup S$ exists

There must exist some O_{λ_0} from the open cover s.t. $a \in O_{\lambda_0}$. Because O_{λ_0} is open, $\exists N_\varepsilon(a) \subseteq O_{\lambda_0}$, and therefore $[a, a+\varepsilon/2] \subseteq O_{\lambda_0}$. Let $x = a + \varepsilon/2$. Then $[a,x]$ has finite subcover O_{λ_0} and so S is nonempty. S is clearly bounded since $x \in [a,b]$.

b) Now show $s = b$, which implies $[a,b]$ has a finite subcover

First we must show that $s \in S$. Assume S is infinite (finite trivial). Then we can find $x \in S$ arbitrarily close to s: $|s - x| < \varepsilon$. There exists some O_{λ_x} from the original open cover that contains this x. Then $\exists \varepsilon$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq O_{\lambda_x}$. Because $|s-x| < \varepsilon \Rightarrow x \leq s < x+\varepsilon$, then $[x, s] \subseteq (x-\varepsilon, x+\varepsilon) \subseteq O_{\lambda_x}$ and so $F \cup O_{\lambda_x}$ is a finite subcover for $[a,x] \cup [x,s] = [a,s]$, implying $s \in S$.
Because $s \in S$, $s \leq b$. Assume for contradiction $s < b$. Then $\exists \delta$ s.t. $s + \delta = b$. There also exists an O_{λ_s} s.t. $s \in O_{\lambda_s}$, implying $(s-\varepsilon, s+\varepsilon) \subseteq O_{\lambda_s}$. Choose δ' s.t. $\delta' < \delta$ and $\delta' < \varepsilon$. Then $[s, s+\delta'] \subseteq (s-\varepsilon, s+\varepsilon) \subseteq O_{\lambda_s}$. Again because $s \in S$, $[a, s]$ has a finite subcover G. Hence $G \cup O_{\lambda_s}$ is a finite subcover for $[a,s] \cup [s, s+\delta'] = [a, s+\delta']$, implying $s+\delta' \in S$. But $s + \delta' > s$, which is a contradiction, so $s = b$. And since $s = b$, and $s \in S$, $[a,b]$ has a finite subcover. $\square$

c) Prove the theorem for an arbitrary closed and bounded set K.

If K is an arbitrary closed and bounded set with $a = \inf K$ and $b = \sup K$, then $K \subseteq [a,b]$. Since any open cover or finite subcover of $[a,b]$ also applies to K, then with (a) and (b) we are done $\square$