6.2 Uniform Convergence of a Sequence of Functions

Definition 6.2.1 (Pointwise Convergence)
For each $n \in \mathbb{N}$, let f_n be a function defined on $A \subseteq \mathbb{R}$. The sequence of functions (f_n) converges pointwise on A to a function f if, $\forall x \in A$, $f_n(x)$ converges to $f(x)$

- Written as $f_n \to f$, $\lim f_n = f$, or $\lim_{n\to\infty} f_n(x) = f(x)$

Example 6.2.2 (i)
Consider $f_n(x) = (x^2 + nx)/n$ on $\mathbb{R}$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2+nx}{n} = \lim_{n\to\infty} \frac{x^2}{n} + x = x$$

So $(f_n) \to f(x) = x$ pointwise on $\mathbb{R}$.

(ii) Let $g_n(x) = x^n$ on $[0,1]$
If $0 \leq x < 1$, then $x^n \to 0$, but when $x = 1$, $x^n = 1$. So $g_n \to g$ where

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

(iii) Consider $h_n(x) = x^{1 + \frac{1}{2n-1}}$ on $[-1,1]$
Since $2n-1$ is odd, no function in the series will ever take the square root of a negative number, and $h_n(x) \geq 0$ $\forall x \in [-1,1]$.
At the same time, $(1/2n-1) \to 0$, so $h_n(x) \to |x|$

$$\lim_{n\to\infty} x^{1+\frac{1}{2n-1}} = x \lim_{n\to\infty} x^{\frac{1}{2n-1}} = |x|$$

- pointwise convergence not strong enough to justify properties we'd like to have. For ex, the pointwise limit of continuous functions need not be continuous - like Example 6.2.2(ii)

Definition 6.2.3 (Uniform Convergence)

Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$. Then (f_n) converges uniformly on A to a limit function f defined on A if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$ and $x \in A$.

• restate pointwise definition to emphasize the difference

Definition 6.2.1B (Pointwise Convergence v2)

Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$. Then (f_n) converges pointwise on A to a limit f defined on A if, $\forall \varepsilon > 0$ and $x \in A$, $\exists N \in \mathbb{N}$ (perhaps dependent on x) such that $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$.

Example 6.2.4 (i)

Let $g_n(x) = \dfrac{1}{n(1+x^2)}$. It's immediately obvious that $g_n(x) \to 0$ for any fixed $x \in \mathbb{R}$, so $g(x) = 0$ is the pointwise limit of g_n.

Does $g_n(x)$ converge uniformly to 0? Need to find N s.t. $|f_n(x)| < \varepsilon \ \forall n \geq N$.

Notice $1/(1+x^2) \leq 1 \ \forall x \in \mathbb{R}$. Choose $N > 1/\varepsilon$, then $1/N < \varepsilon$. Thus $\frac{1}{N}\left(1/(1+x^2)\right) < \varepsilon(1) = \varepsilon$ and the convergence is uniform.

(ii) From Example 6.2.2 (i), $f_n(x) = (x^2 + nx)/n$ converges pointwise on $\mathbb{R}$ to $f(x) = x$. This convergence is not uniform.

$|f_n(x) - f(x)| = \left| \dfrac{x^2 + nx}{n} - x \right| = \dfrac{x^2}{n}$. For $\varepsilon > 0$, to make $|f_n(x) - f(x)| < \varepsilon$,

$N > x^2/\varepsilon$ is necessary. Because the choice of N depends on x, the convergence is not uniform.

* If we restrict domain to $[-b, b]$, we see $x^2/n \leq b^2/n$, so $N > b^2/\varepsilon$ assures uniform convergence.

Theorem 6.2.5 (Cauchy Criterion for Uniform Convergence)
(f_n) on $A \subseteq \mathbb{R}$ converges uniformly on A iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|f_n(x) - f_m(x)| < \varepsilon$ whenever $m, n \geq N$ and $x \in A$.

Proof: Exercise 6.2.5

Theorem 6.2.6 (Continuous Limit Theorem)
Let (f_n) on $A \subseteq \mathbb{R}$ converge uniformly on A to f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof:
Fix $c \in A$, $\varepsilon > 0$. Pick N so that $|f_N(x) - f(x)| < \varepsilon/3$ $\forall x \in A$. Because f_N is continuous, $\exists \delta > 0$ s.t. $|f_N(x) - f_N(c)| < \varepsilon/3$ whenever $|x - c| < \delta$.
This implies
$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \ \square \end{aligned}$$

6.2 Exercises

1) Let $f_n(x) = \frac{nx}{1+nx^2}$

a) Find the pointwise limit of (f_n) $\forall x \in (0,\infty)$

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \frac{1}{x}$$

b) Is the convergence uniform on $(0,\infty)$

No. $|f_n(x) - f(x)| = \left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \left|\frac{1}{x(1+nx^2)}\right|$. So if $\frac{1}{x(1+nx^2)} < \varepsilon$, then must choose $N > \frac{1}{x^3\varepsilon} - \frac{1}{x^2}$. This depends on x, so the convergence is not uniform.

c) Is the convergence uniform on $(0,1)$

No, still choose $N > \frac{1}{x^3\varepsilon} - \frac{1}{x^2}$, which is unbounded as $x \to 0$.

d) Is the convergence uniform on $(1,\infty)$

Yes. On $(1,\infty)$, $1/x^3\varepsilon - 1/x^2$ is bounded above by 0 and below by $-4/27\varepsilon^2$ (with calculus). So if $N > \frac{4}{27\varepsilon^2}$, then $|f_n(x) - f(x)| < \varepsilon$ $\forall n > N$

2. Define a sequence of functions on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, 1/2, 1/3, \ldots, 1/n \\ 0 & \text{otherwise} \end{cases}$$ and let f be the pointwise limit of f_n.

a) Is each f_n continuous at 0? Does $f_n \to f$ uniformly on $\mathbb{R}$? Is f continuous at 0?

Yes, pick $\delta = 1/n$, then if $|x| < \delta$, $|f_n(x) - f| = 0 < \varepsilon$.
No, for any choice of N, if $n \geq N$ then for $x = \frac{1}{n+1}$, $|f_n(x) - f| = 1$
No, f is not continuous at 0 becouse because we can find x of the form $1/n$ arbitrarily close to 0 so that $f(x) = 1$
This does not contradict Thm 6.2.6. f need not be continuous at 0 since f does not converge uniformly.

3) For $n \in \mathbb{N}$ and $x \in [0, \infty)$, let $g_n(x) = \frac{x}{1+x^n}$ and $h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n \end{cases}$

a) Find pointwise limit on $[0, \infty)$ for (g_n) and (h_n)

$$(g_n) \to \begin{cases} x & \text{if } x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases} \qquad (h_n) \to \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

* b) How do we know convergence cannot be uniform on $[0, \infty)$
Each g_n is continuous at $x = 1$, and each h_n is continuous at $x = 0$. If convergence was uniform, then g would be continuous at 1 and h would be continuous at 0. However, this is not the case.

c) Choose a smaller set over which convergence is uniform.
$[0, 1)$ for g and $[a, \infty)$ for h where $a > 0$
$x^n/(1+x^n) < x^n < \varepsilon$ and $N \geq 1/a$, and then $|h_n(x) - h| = 0 < \varepsilon$

*5) Using the Cauchy Criterion (Thm 2.6.4), prove Thm 6.2.5. First define a candidate for $f(x)$, and then argue $f_n \to f$ uniformly

Proof:

$\Rightarrow$ Assume (f_n) converges uniformly on $A \subseteq \mathbb{R}$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon/2$. For $m > N$, $|f_m(x) - f(x)| < \varepsilon/2$. By the triangle inequality, $|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| < |f_n(x) - f(x)| + |f_m(x) - f(x)|$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\Leftarrow$ Assume for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|f_n(x) - f_m(x)| < \varepsilon$ $\forall m, n \geq N$, and $x \in A$. For any fixed point $x \in A$, by the Cauchy criterion $f_n(x) \to y$. The mapping of each x to its respective y produces the function f that $f_n(x)$ is pointwise convergent to. By the ALT we know $\lim_{m \to \infty} f_n(x) - f_m(x) = f_n(x) - f(x)$. From this we can use the OLT to assert that $|f_n(x) - f(x)| < \varepsilon$ □

9) Assume (f_n) and (g_n) are uniformly convergent sequences of functions.

a) Show $(f_n + g_n)$ is a uniformly convergent sequence of functions

We know $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $|f_n(x) - f(x)| < \varepsilon/2$. Similarly, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$, $|g_n(x) - g(x)| < \varepsilon/2$. Choose $N = \max\{N_1, N_2\}$, and by triangle inequality $|f_n(x) + g_n(x) - (f(x) + g(x))| < |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

b) Give an example to show $(f_n g_n)$ may not converge uniformly

$f_n(x) = x + 1/n$ $g_n(x) = x + 1/n$. $(f_n g_n) = (x + 1/n)^2 \to x^2$ pointwise.
Want to find N that makes $|2x/n + 1/n^2| < \varepsilon$ $\forall n \geq N$. Choice of N clearly depends on x, so $f_n g_n$ does not converge uniformly.

c) Prove that if $\exists M > 0$ s.t. $|f_n| \le M$, $|g_n| \le M$, then $(f_n g_n)$ does converge uniformly

Proof:

$|f_n g_n - fg| = |f_n g_n - f g_n + f g_n - fg| \le |f_n g_n - f g_n| + |f g_n - fg|$
$= |g_n||f_n - f| + |f||g_n - g| \le M(|f_n - f| + |g_n - g|)$ by the OLT. Choose N so that $|f_n - f| < \varepsilon/2M$ and $|g_n - g| < \varepsilon/2M$, and then we have $M(|f_n - f| + |g_n - g|) = \varepsilon$ D

13) Bolzano-Weierstrass for bounded sequences of functions.
Let $A = \{x_1, x_2, \dots\}$ be a countable set. For each $n \in \mathbb{N}$, let f_n be defined on A and assume $\exists M > 0$ s.t. $|f_n(x)| \le M$ $\forall n \in \mathbb{N}$ and $x \in A$. Use a-c to show there exists a subsequence of (f_n) that converges pointwise on A.

a) Why does the sequence $f_n(x_1)$ necessarily contain a convergent subsequence (f_{n_k})? To indicate the subsequence (f_{n_k}) is generated by considering values of functions at x_1, use notation $f_{n_k} = f_{1,k}$

By Bolzano-Weierstrass because $f_n(x_1)$ is bounded.

b) Explain why $f_{1,k}(x_2)$ contains a convergent subsequence

Same as (a), $f_{1,k}(x_2)$ is bounded.

→ Just don't understand this at all

c) Construct a nested family of subsequences $(f_{m,k})$, and produce a single subsequence of (f_n) that converges to every point of A.

← seriously what is going on?

Let $f_{2,k} = f_{1,k}(x_2)$ and so on. Let (g_k) be the sequence of all functions that appear in $(f_{m,k})$. $g_k(x_m)$ converges for $x_m \in A$????