6.4 Series of Functions

Definition 6.4.1

For each $n \in \mathbb{N}$, let f_n and f be functions on $A \subseteq \mathbb{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \dots$$

<u>converges pointwise</u> on A to $f(x)$ if the sequence of partial sums

$$S_k(x) = f_1(x) + f_2(x) + \dots + f_k(x)$$

converges pointwise ~~to~~ $f(x)$. The series converges uniformly on A to f if $S_k(x)$ converges uniformly on A to $f(x)$.

Theorem 6.4.2 (Term-by-term Continuity)

Let f_n be continuous functions on $A \leq \mathbb{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to f. Then f is continuous on A.

Proof:

Assume $f_n : A \to \mathbb{R}$ are continuous on A, and $\sum_{n=1}^{\infty} f_n = f$ uniformly. Since each f_n is continuous, by the Algebraic Continuity Theorem the sequence of partial sums $S_k(x)$ is also continuous. Since $\sum_{n=1}^{\infty} f_n = f$ implies $S_k(x) \to f(x)$ uniformly, then by the continuous limit theorem $f(x)$ is continuous on A $\square$

Theorem 6.4.3 (Term-by-term Differentiability)

Let f_n be differentiable functions on an interval A, and assume $\sum_{n=1}^{\infty} f_n'(x) = g(x)$ uniformly. If $\exists x_0 \in [a,b]$ where $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$. In other words,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{and} \quad f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

Proof:

Because Thm 5.2.4 asserts that the sum of differentiable functions is differentiable, and its derivative equals the sum of the derivatives of its differentiable functions, $S_k'(x) = f_1'(x) + f_2'(x) + \dots + f_k'(x)$. This means $S_k'(x)$ is the sequence of partial sums for $\sum_{n=1}^{\infty} f_n'(x)$, and so $(S_k') \to g$ uniformly. Because $\sum_{n=1}^{\infty} f_n(x_0)$ converges, $S_k(x_0)$ converges by definition of infinite series. Then by Theorem 6.3.3, $(S_k) \to f$ uniformly where $f' = g$. This implies that $\sum_{n=1}^{\infty} f_n(x) = f(x)$, and $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ $\square$

Theorem 6.4.4 (Cauchy Criterion for Uniform Convergence of Series)
A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbb{R}$ iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|f_{m+1}(x) + f_{m+2}(x) + \dots + f_n(x)| < \varepsilon$ whenever $n > m \geq N$ and $x \in A$

- Because uniform convergence is so useful, we'd like a way to determine when a series converges uniformly ↓

Corollary 6.4.5 (Weierstrass M-test)
For each $n \in \mathbb{N}$, let f_n be a function on $A \subseteq \mathbb{R}$, and let $M_n > 0$ satisfy $|f_n(x)| \leq M_n$ $\forall x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Proof: Exercise 6.4.1

6.4 Exercises

1) Supply proof of Weierstrass M-test

Proof:
Suppose f_n is on $A \subseteq \mathbb{R}$ and $M_n > 0$ s.t. $|f_n| \leq M_n$ $\forall x \in A$. Assume $\sum_{n=1}^{\infty} M_n$ converges. By the Cauchy Criterion for infinite series, $\exists N$ s.t. whenever $n > r \geq N$, $|M_{r+1} + M_{r+2} + \dots + M_n| < \varepsilon$. This implies that $|f_{r+1} + f_{r+2} + \dots + f_n| \leq |M_{r+1} + M_{r+2} + \dots + M_n| < \varepsilon$ whenever $n > r \geq N$, and so by the Cauchy Criterion, $\sum_{n=1}^{\infty} f_n$ converges uniformly $\square$

2) True or False - prove or counterexample

a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to 0

True. If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\exists N \in \mathbb{N}$ s.t. whenever $n > m \geq N$, $|S_n - S_m| < \varepsilon$ (S_n being sequence of partial sums). Consider $n = m+1$, then whenever $m \geq N$, $|g_{m+1}| < \varepsilon$, and so $(g_n) \to 0$ uniformly because N works $\forall x \in A$.

b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

True. Similar to proof for 1. The N that works for g_n suffices for f_n

c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then $\exists M_n$ constants such that $|f_n(x)| \leq M_n$ $\forall x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

False. Let $f_n = \dfrac{(-1)^{n+1} \lceil x \rceil}{n}$ on $[0,1]$. Then (f_n) is identical to the alternating harmonic series, which converges and thus follows the Cauchy Criterion, so (f_n) converges uniformly. But $|f_n|$ is the harmonic series, which diverges.

3.

a) Show that $g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$ is continuous on $\mathbb{R}$

$\cos(2^n x)$ is continuous, so $\frac{\cos(2^n x)}{2^n} = g_n$ is continuous $\forall n \in \mathbb{N}$.
It is also apparent that $\sum_{n=0}^{\infty} g_n$ converges uniformly, so by Theorem 6.4.2 $g(x)$ is continuous on $\mathbb{R}$.

b) g is continuous but nowhere differentiable. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable.
$g'_n = -\sin(2^n x)$, so $\sum_{n=0}^{\infty} g'_n(x)$ does not converge uniformly and Theorem 6.4.3 does not apply.

4) Define $g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(1+x^{2n})}$. Find the values of x where this series converges and show we get a continuous function on this set.

Since $\frac{x^{2n}}{1+x^{2n}} < x^{2n}$ on $(-1,1)$ and $\sum_{n=0}^{\infty} x^{2n}$ converges geometrically, the series converges on $(-1,1)$.

No idea how to show continuity.

6) Let $f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \frac{1}{x+4} + \dots$
Show f is defined $\forall x > 0$. Is f continuous on $(0,\infty)$? Differentiable?

$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{x+n}$. Since regardless of x, the sequence decreases and converges to 0, the series converges by the AST and so f is defined for $x>0$. Each f_n is differentiable: $f_n'(x) = (-1)^{n+1}\left(\frac{1}{(x+n)^2}\right)$ and by the M-test $|f_n'| \leq 1/n^2$, so $\sum f_n'$ converges uniformly. Because $\sum f_n(x_0)$ converges $\forall x_0 \in (0,\infty)$, $\sum f_n$ converges uniformly. Thus f is differentiable and continuous.

9) Let $h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$.

a) Show h is continuous on $\mathbb{R}$.

Each h_n is continuous on $\mathbb{R}$, and by the M-test $|h_n| < \frac{1}{n^2}$, so $\sum h_n$ is uniformly convergent. Thus, h is continuous on $\mathbb{R}$.

b) Is h differentiable? If so, is h' continuous?

Each h_n is differentiable. $h'_n = \frac{-1}{(x^2+n^2)^2}$. By the M-test, $|h'_n| \leq 1/n^2$, so $\sum_{n=1}^{\infty} h'_n$ converges uniformly. Because $\sum_{n=1}^{\infty} f_n(x_0)$ converges $\forall x_0 \in \mathbb{R}$, by Thm 6.4.3 h is differentiable, and because each h'_n is continuous, h' is also continuous.