1.4 Consequences of Completeness

another way to think of $\mathbb{R}$ as gapless

Theorem 1.4.1 (Nested Interval Property):
For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume I_n contains I_{n+1}. Then the resulting sequence of nested intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ has a nonempty intersection: $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

Proof:
We need to show $\exists x \in \mathbb{R}$ s.t. $x \in I_n \ \forall n \in \mathbb{N}$. Consider the set of all left endpoints $A = \{a_n : n \in \mathbb{N}\}$. A is nonempty and bounded above by $B = \{b_n : n \in \mathbb{N}\}$, the set of all right endpoints. Let $x = \sup A$. Then $x \geq a_n \ \forall a_n \in A$, and $x \leq b_n \ \forall b_n \in B$. That is, $a_n \leq x \leq b_n$, so $x \in I_n \ \forall n \in \mathbb{N}$ $\square$

Density of $\mathbb{Q}$ in $\mathbb{R}$

- $\mathbb{Q}$ extends $\mathbb{N}$ and $\mathbb{R}$ extends $\mathbb{Q}$ - how do $\mathbb{Q}$ and $\mathbb{N}$ sit inside $\mathbb{R}$?

Theorem 1.4.2 (Archimedean Property)
(i) Given any $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $n > x$ ($\mathbb{N}$ is unbounded)
(ii) Given real number $y > 0$, $\exists n \in \mathbb{N}$ s.t. $1/n < y$

Proof of (i):
Suppose for contradiction $\mathbb{N}$ is bounded. That is, $a = \sup \mathbb{N}$. Then by the Axiom of Completeness, $\exists n \in \mathbb{N}$ s.t. $a - 1 < n$. But this implies that $a < n+1$. Because $n+1 \in \mathbb{N}$, this contradicts the fact that $a = \sup \mathbb{N}$, so our assumption that $\mathbb{N}$ is bounded is false. Thus, $\mathbb{N}$ is unbounded $\square$

Proof of (ii):

Aim to show $\exists n \in \mathbb{N}$ such that $1/n < y$ for $y \in \mathbb{R}^+$. Rewrite the inequality as $n > 1/y$. From (i) we have that given any real number x, $\exists n \in \mathbb{N}$ s.t. $n > x$. Let $x = 1/y$. Then $\exists n \in \mathbb{N}$ s.t. $n > 1/y$ $\square$

Theorem 1.4.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$):

$\forall a, b \in \mathbb{R}$ with $a < b$, $\exists r \in \mathbb{Q}$ s.t. $a < r < b$

"$\mathbb{Q}$ is dense in $\mathbb{R}$"

Proof:

Since $r \in \mathbb{Q}$, $r = p/q$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. Rewrite the inequality as $a < p/q < b$. The proof now relies on careful choices of p and q to make the inequality true. First, choose denominator q to be large enough so that increments of size $1/q$ are smaller than the interval $b-a$: $1/q < b-a$. We can do this by the Archimedean Property. Now rewrite the inequality to be proved as $qa < p < qb$, and focus on p. Want to pick p as the smallest integer greater than qa: $p - 1 \leq qa < p$. From this we see that $a < p/q$. All that's left is to show $p/q < b \iff p < qb$. Note that $a < b - 1/q$. Use this in the manipulation of $p - 1 \leq qa$.

$$
\begin{aligned}
& p - 1 \leq qa \\
\iff & p \leq qa + 1 < q(b - 1/q) + 1 \\
\iff & p < qb - 1 + 1 \\
\iff & p < qb \;\square
\end{aligned}
$$

Corollary 1.4.4 (Density of $\mathbb{I}$ in $\mathbb{R}$):
Given $a, b \in \mathbb{R}$, $\exists t \in \mathbb{I}$ s.t. $a < t < b$
* Proof given in exercise 1.4.5

Existence of Square Roots

Theorem 1.4.5 (Existence of $\sqrt{2}$ in $\mathbb{R}$):
There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$

Proof:
Consider $T = \{t \in \mathbb{R} : t^2 < 2\}$ and set $\alpha = \sup T$. Assume for contradiction $\alpha^2 < 2$. We want to show that $\alpha + \varepsilon \in T$ for $\varepsilon > 0$ to contradict that $\alpha^2 < 2$. Let $\varepsilon = 1/n$ for $n \in \mathbb{N}$. Then

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha + 1}{n}.$$

Now choose $n \in \mathbb{N}$ so that $(2\alpha+1)/n < 2 - \alpha^2$. Then $\alpha^2 + (2\alpha+1)/n < \alpha^2 + 2 - \alpha^2 \Rightarrow (\alpha + 1/n)^2 < 2 \Rightarrow (\alpha + 1/n) \in T$. Thus α^2 cannot be less than 2 for α to be a supremum. Now assume for contradiction that $\alpha^2 > 2$. We want to show that $2 < (\alpha - \varepsilon)^2 < \alpha^2$ to contradict that $\alpha^2 > 2$ (and α is a supremum). Consider $\varepsilon = 1/n$. Then:

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}.$$

Now choose $n \in \mathbb{N}$ such that $-2\alpha/n > 2 - \alpha^2$. Then $\alpha^2 - 2\alpha/n > \alpha^2 + 2 - \alpha^2 \Rightarrow \alpha^2 - 2\alpha/n > 2$. So $2 < \alpha^2 - 2\alpha/n < \alpha^2$, contradicting our assumption that $\alpha^2 > 2$.
Since $\alpha^2 \not< 2$ and $\alpha^2 \not> 2$, then $\alpha^2 = 2$, and thus $\sqrt{2} \in \mathbb{R}$.

1.4 Exercises

1) I is the set of irrationals

a) Show if $a,b \in \mathbb{Q}$, then ab and $a+b \in \mathbb{Q}$

Proof:

Since $a,b \in \mathbb{Q}$, we can rewrite them as $a = \frac{m}{n}$, $b = \frac{p}{q}$ for $m,n,p,q \in \mathbb{Z}$ with $n,q \neq 0$. ab becomes $(m/n)(p/q) = mp/nq$. Both mp and nq are integers because $\mathbb{Z}$ is closed under multiplication, so ab can be written as the division of two integers with a nonzero denominator. Thus $ab \in \mathbb{Q}$.

Similarly, $a+b = m/n + p/q = (mq+pn)/nq$. As before, $mq, pn, nq \in \mathbb{Z}$, and $mq+pn \in \mathbb{Z}$ because of closure of $\mathbb{Z}$ under addition and multiplication. Since we wrote $a+b$ as the division of two integers with a non-zero denominator, $a+b \in \mathbb{Q}$ □

b) Show that if $a \in \mathbb{Q}$ and $t \in I$, then $a+t \in I$ and $at \in I$ as long as $a \neq 0$.

Proof:

Suppose for contradiction that $a+t \notin I \Rightarrow a+t \in \mathbb{Q}$. Then $a+t = p/q$ for $p,q \in \mathbb{Z}$ and $q \neq 0$. Solving for t yields $t = p/q - a \in \mathbb{Q}$ because the sum of two rationals is rational. This contradicts $t \in I$, so our assumption is false and $a+t \in I$.

Again suppose for contradiction that $at \in \mathbb{Q}$. Then $at = p/q$ for $p,q \in \mathbb{Z}$ and $q \neq 0$. Solving for t yields $t = p/qa \in \mathbb{Q}$ (because $a \neq 0$). This contradicts $t \in I$, so our assumption is false and $at \in I$ □

c) Is I closed under addition and multiplication $(s+t, st)$?
No. Consider $t = -s$. Then $s+t = s-s = 0 \in \mathbb{Q}$.
Consider $t = 1/s$. Then $st = s(1/s) = 1 \in \mathbb{Q}$.

3) Prove that $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$. Notice this demonstrates that intervals must be closed for Nested Interval Property to be true.
Proof:
Let $x \in \bigcap_{n=1}^{\infty} (0, 1/n)$. Then $x > 0$ and $x < 1/n \ \forall n \in \mathbb{N}$. This contradicts part (ii) of Archimedean property, which states for all $x \in \mathbb{R}: x > 0, \exists n \in \mathbb{N}$ s.t. $1/n < x$. Therefore, no such x exists and $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$ $\square$

5) Prove Corollary 1.4.4: Given any 2 real numbers a, b, there exists an irrational number t satisfying $a < t < b$. Consider the numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.
Proof:
Let $a, b \in \mathbb{R}$ and $a < b$. Then $a - \sqrt{2} < b - \sqrt{2}$. By the rational density theorem, $a - \sqrt{2} < r < b - \sqrt{2}$. Add $\sqrt{2}$ to get $a < r + \sqrt{2} < b$. $r + \sqrt{2} \in I$, so we are done $\square$

6) A set B is dense in $\mathbb{R}$ if $\forall a, b \in \mathbb{R}$ $a < b$, $\exists x \in B$ s.t. $a < x < b$.
Let $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Which are dense in $\mathbb{R}$?

a) Set of all rationals p/q with $q \leq 10$.
Not dense - consider $a = .01$, $b = .02$. There is no element x in the given set such that $.01 < x < .02$. The closest we get is $x = 0$ and $x = .1$.

b) Set of all rationals p/q where q is a power of 2

Dense. Proof: Let $q = 2^n$ for $n = 0, 1, 2, \ldots$ - this ensures $q \in \mathbb{N}$ and q is a power of 2. Now choose q so that $1/q < b-a$, for $a, b \in \mathbb{R}$, $a < b$. We can do this by finding n s.t. $1/n < b-a$ $n \in \mathbb{N}$ by Archimedean property, and then $1/2^n < 1/n < b-a$, let $q = 2^n$. The proof proceeds identically to that of the rational density theorem.

c) Set of rationals p/q where $10|p| \geq q$.

Not dense - this means that $\frac{|p|}{q} \geq \frac{1}{10}$, so elements in the set have a magnitude greater than $1/10$. This means no such p/q exists between $a = -1/10$ and $b = 1/10$.

8) Give example or prove impossible

a) Sets A and B with $A \cap B = \emptyset$, $\sup A = \sup B$, $\sup A \notin A$ and $\sup B \notin B$.

$$A = \{x \in \mathbb{Q} : x < 1\} \quad B = \{x \in \mathbb{I} : x < 1\}$$

$A \cap B = \emptyset$ because $\mathbb{Q} = \overline{\mathbb{I}}$. $\sup A = \sup B = 1$. $1 \notin A$ and $1 \notin B$.

b) A sequence of nested unbounded closed intervals $J_1 \supseteq J_2 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but finite

Let $J_n = [-1/n, 1/n]$. Then $\bigcap_{n=1}^{\infty} J_n = \{0\}$

c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq \ldots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. ($L_n$ of the form $[a, \infty)$)

Let $L_n = [n, \infty)$. Proof $\bigcap_{n=1}^{\infty} L_n$ is empty:
Suppose $x \in \bigcap_{n=1}^{\infty} L_n$. Then $x \in [n, \infty)$ $\forall n \in \mathbb{N}$, which means $x \geq n$. But by archimedean principle, we can always find $n > x$, so no such x exists and $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

* d) A sequence of closed, bounded (not necessarily nested) intervals $I_1, I_2, \dots$ with $\bigcap_{n=1}^{N} I_n \neq \emptyset$ $\forall N \in \mathbb{N}$ but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

?

Does not exist. Solution sketchy, but if $\bigcap_{n=1}^{N} I_n \neq \emptyset$ then this necessitates nested intervals. Since they're closed and bounded, then by Nested Interval Theorem $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.