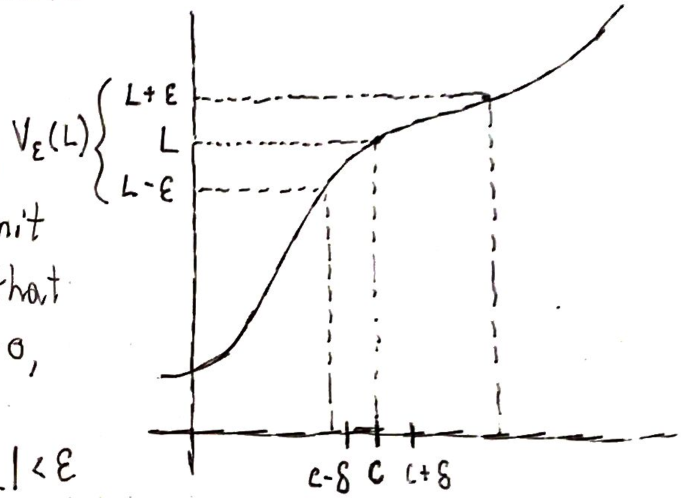
4.2 Functional Limits



Definition 4.2.1 (Functional Limit)
Let $f: A \supseteq \mathbb{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = L$ provided that, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. whenever $0 < |x-c| < \delta$ and $x \in A$, it follows that $|f(x)-L| < \varepsilon$

- Known as the "ε-δ" version of the definition for functional limits
  - challenge/response: challenge mode w/ ε, response with δ
- Recall that $|f(x)-L| < \varepsilon \iff f(x) \in V_\varepsilon(L)$ and $|x-c| < \delta \iff x \in V_\delta(c)$
- The additional restriction $0 < |x-c|$ just means $x \neq c$ - remember that c is a limit point

Definition 4.2.1B (Functional Limit - Topological Version)
Let c be a limit point of the domain of $f: A \to \mathbb{R}$. We say $\lim_{x\to c} f(x) = L$ provided that, for every $V_\varepsilon(L)$, there exists a $V_\delta(c)$ with the property that $\forall\, x \in V_\delta(c)$ $x \neq c$, $x \in A$, it follows that $f(x) \in V_\varepsilon(L)$

Ex. 4.2.2

(i) Prove that if $f(x) = 3x+1$, then $\lim_{x\to 2} f(x) = 7$.
$|f(x)-7| < \varepsilon \iff |3x+1-7| < \varepsilon \iff 3|x-2| < \varepsilon \iff |x-2| < \varepsilon/3$. So choose $\delta = \varepsilon/3$, and then $|x-2| < \delta$ implies $|f(x)-7| < \varepsilon$, so $\lim_{x\to 2} f(x) = 7$

(ii) Prove $\lim_{x\to 2} g(x) = 4$ when $g(x) = x^2$.
$|g(x)-4| = |x^2-4| = |x+2||x-2|$. If we agree that $\delta \leq 1$, then x can be no larger than 3. This gives an upper bound $|x+2| \leq 5$, so $|x+2||x-2| \leq 5|x-2|$. Now choose $\delta = \min\{1, \varepsilon/5\}$ and let $|x-2| < \delta$. Then
$|g(x)-4| = |x+2||x-2| < 5(\varepsilon/5) = \varepsilon$.

Theorem 4.2.3 (Sequential Criterion for Functional Limits)

Given a function $f: A \to \mathbb{R}$ and a limit point c of A, the following 2 statements are equivalent:

(i) $\lim_{x \to c} f(x) = L$

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$

Proof:

($\Rightarrow$) Assume $\lim_{x \to c} f(x) = L$, and let $(x_n) \subseteq A$ with $x_n \neq c$ and $(x_n) \to c$. By the limit of a sequence, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - c| < \delta$. And since $x_n \in A$, then by the definition of functional limits $\forall n \geq N$: $|f(x_n) - L| < \varepsilon$. This implies that $f(x_n) \to L$

($\Leftarrow$). Proof by contrapositive. Suppose $\lim_{x \to c} f(x) \neq L$ and assume for contradiction that (ii) holds. This means $\exists \varepsilon_0$ s.t. $\forall \delta$, $\exists x \in V_\delta(c)$ with $x \neq c$ for which $f(x) \neq V_{\varepsilon_0}(L)$. Now let $\delta_n = 1/n$, and then for each $n \in \mathbb{N}$ pick $x_n \in V_{\delta_n}(c)$ with $x_n \neq c$ s.t. $f(x_n) \notin V_{\varepsilon_0}(L)$. This yields $(x_n) \subseteq A$, $x_n \neq c$, $(x_n) \to c$ with $\lim f(x_n) \neq L$, contradicting (ii). Thus, (ii) $\Rightarrow$ (i) $\square$

Corollary 4.2.4 (Algebraic Limit Theorem for Functional Limits)

Let f and g be functions on $A \subseteq \mathbb{R}$, and assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. Then,

(i) $\lim_{x \to c} kf(x) = kL \quad \forall k \in \mathbb{R}$

(ii) $\lim_{x \to c} [f(x) + g(x)] = L + M$

(iii) $\lim_{x \to c} [f(x) g(x)] = LM$

(iv) $\lim_{x \to c} f(x)/g(x) = L/M$, provided $M \neq 0$

Proof: Exercise 4.2.1

Corollary 4.2.5 (Divergence Criterion for Functional Limits)

Let f be a function defined on A, and let c be a limit point of A. If there exist 2 sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and $\lim x_n = \lim y_n = c$ but $\lim f(x_n) \neq \lim f(y_n)$, then we conclude that the functional limit $\lim_{x \to c} f(x)$ does not exist.

Ex 4.2.6

Show $\lim_{x \to 0} \sin(1/x)$ does not exist.

If $x_n = 1/2\pi n$ and $y_n = 1/(2\pi n + \pi/2)$, then $\lim(x_n) = \lim(y_n) = 0$.

But $\sin(1/x_n) = 0$ and $\sin(1/y_n) = 1$ $\forall n \in \mathbb{N}$. So $\lim_{x \to 0} \sin(1/x)$ DNE.

4.2 Exercises

X = incorrect
O = no idea, looked it up
* = looked for hints

1.

a) Show how 4.2.4(ii) follows from Thm 4.2.3 and the ALT for sequences

Proof:
Let f, g be functions on $A \subseteq \mathbb{R}$ s.t. $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, for some limit point c of A. By theorem 4.2.3, suppose (a_n) and (b_n) are arbitrary sequences contained in A with $\lim a_n = \lim b_n = c$ such that $\lim f(a_n) = L$ and $\lim g(b_n) = M$. Then by the ALT for sequences it follows that $\lim [f(a_n) + g(b_n)] = L + M$. Since (a_n) and (b_n) are arbitrary, we can conclude that $\lim_{x \to c} [f(x) + g(x)] = L + M$ $\square$

b) Prove (ii) again directly from Def 4.2.1 without Thm 4.2.3

Proof:
Let $f, g: A \subseteq \mathbb{R}$ s.t. $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some lim pt. c of A. By definition 4.2.1, we can say that whenever $0 < |x - c| < \delta$, it follows that $|f(x) - L| < \varepsilon/2$ and $|g(x) - M| < \varepsilon/2$. By the triangle inequality, $|(f(x) + g(x)) - (L + M)| < |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, so again by definition 4.2.1, $\lim_{x \to c} [f(x) + g(x)] = L + M$ $\square$

2) For each stated limit, find the largest possible δ-neighborhood that is a proper response to the ε challenge.

a) $\lim_{x \to 3} (5x - 6) = 9$; $\varepsilon = 1$

$|5x - 6 - 9| < 1 \Rightarrow |5x - 15| < 1 \Rightarrow 5|x - 3| < 1 \Rightarrow |x - 3| < 1/5$

$\delta = 1/5$

b) $\lim_{x \to 4} \sqrt{x} = 2$, $\varepsilon = 1$

$|\sqrt{x} - 2| < 1 \Rightarrow 1 < \sqrt{x} < 3 \Rightarrow 1 < x < 9$. We want to find δ s.t. $|x-4| < \delta$ implies $1 < x < 9$. Now $|x-4| < \delta \Rightarrow 4-\delta < x < 4+\delta$, so $\delta = \min\{3, 5\} = 3$.

5. Use Definition 4.2.1 to prove the following

a) $\lim_{x \to 2} (3x+4) = 10$

Let $\delta = \varepsilon/3$. Then whenever $0 < |x-2| < \varepsilon/3$, it follows that $3|x-2| < \varepsilon \Rightarrow |3x-6| < \varepsilon \Rightarrow |(3x+4)-10| < \varepsilon$ □

b) $\lim_{x \to 0} x^3 = 0$

Choose $\delta = \min\{1, \epsilon\}$. If $\epsilon > 1$, then $\delta = 1$, and so whenever $|x| < 1 < \epsilon$, it follows that $|x^3| < |x| < \epsilon \Rightarrow |x^3| < \epsilon$. If $\epsilon \le 1$, then $\delta = \epsilon$. Whenever $|x| < \epsilon < 1$, it follows that $|x^3| < |x| < \epsilon \Rightarrow |x^3| < \epsilon$ □

c) $\lim_{x \to 2} (x^2 + x - 1) = 5$

Choose $\delta = \min\{1, \epsilon/6\}$. This means that whenever $|x-2| < \delta$, $\Rightarrow 1 < x < 3$, and so $|x+3| \le 6$. This is useful in just a second. If $\epsilon/6 > 1$, then $\delta = 1$, and $|x+3| \le 6$ and $|x-2| < 1$. Thus $|x+3||x-2| = |(x+3)(x-2)| = |x^2 + x - 1 - 5| < \epsilon$ because $\epsilon > 6$ and $|x+3||x-2| < 6$. If $\epsilon/6 \le 1$, then $\delta = \epsilon/6$. Then $|x+3||x-2| < 6(\epsilon/6) = \epsilon$ □

d) $\lim_{x \to 3} 1/x = 1/3$

Choose $\delta = \min\{1, 12\epsilon\}$. If $12\epsilon > 1$, then $\delta = 1$ and $\epsilon > 1/12$ and so $|x-3| < 1$. $|x-3| < 1 \Rightarrow x < 4$, so we also can say $|3x| < 12$. Now rewrite $|1/x - 1/3|$ as $|x-3|/|3x|$, and we can conclude that $|x-3|/|3x| < 1/12 < \epsilon$. If $12\epsilon < 1$, then $\delta = 12\epsilon$ and $|x-3| < 12\epsilon$. From before, we still have $|3x| < 12$, so $|x-3|/|3x| < 12\epsilon/12 = \epsilon$ □

6) Decide if true or false, and give justifications

a) If a particular δ has been constructed as a response to a particular ϵ challenge, then any smaller positive δ will also suffice.

True. Suppose we have some $V_\epsilon(L)$ for which $V_\delta(c)$ is a suitable response, and let $0 < \delta' < \delta$. Because $V_{\delta'}(c) \subseteq V_\delta(c)$, then it is always true that $x \in V_{\delta'}(c) \Rightarrow f(x) \in V_\epsilon(L)$.

b) If $\lim_{x \to a} f(x) = L$ and a is in the domain of f, then $L = f(a)$.

False. Consider $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 10 & \text{if } x = 0 \end{cases}$. $\lim_{x \to 0} f(x) = 0$, and 0 is in the domain of $f(x)$, but $f(0) = 10 \neq 0$.

c) If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} 3[f(x) - 2]^2 = 3(L-2)^2$

True. Expand $3[f(x)-2]^2 \Rightarrow 3[f(x)^2 - 4f(x) + 4]$ and apply ALT to get $3[L^2 - 4L + 4] = 3(L-2)^2$

d) If $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x)g(x) = 0$ for any function g with domain equal to that of f.

False. Let $f(x) = x$ and $g(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$. f and g have the same domain and $\lim_{x \to 0} f(x) = 0$. But $f(x)g(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$, and so $\lim_{x \to 0} f(x)g(x) = 1$

9) $\lim_{x \to 0} \frac{1}{x^2} = \infty$ makes sense. To write a rigorous definition for an infinite limit like this, the $\epsilon > 0$ challenge is replaced with an arbitrarily large $M > 0$ challenge. Definition: $\lim_{x \to c} f(x) = \infty$ means that $\forall M > 0$ $\exists \delta > 0$ s.t. whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$

a) Show $\lim_{x \to 0} \frac{1}{x^2} = \infty$ with the above definition.

Choose $\delta = \frac{1}{\sqrt{M}}$. Then if $|x| < \frac{1}{\sqrt{M}}$, $\Rightarrow x^2 < \frac{1}{M} \Rightarrow \frac{1}{x^2} > M$

b) Construct a definition for the statement $\lim_{x \to \infty} f(x) = L$. Show $\lim_{x \to \infty} 1/x = 0$

$\lim_{x \to \infty} f(x) = L$ means $\forall \epsilon > 0$, $\exists M > 0$ such that whenever $x > M$, it follows that $|f(x) - L| < \epsilon$.

Choose $M = \frac{1}{\epsilon}$. Then $x > 1/\epsilon \Rightarrow |x| > 1/\epsilon \Rightarrow |1/x| < \epsilon$.

c) Write a rigorous definition for $\lim_{x \to \infty} f(x) = \infty$. Give an example

$\lim_{x \to \infty} f(x) = \infty$ means $\forall N > 0$, $\exists M > 0$ such that whenever $x > M \Rightarrow f(x) > N$.

$\lim_{x \to \infty} \sqrt{x} = \infty$ because we can choose $M = N^2$, so $x > N^2 \Rightarrow \sqrt{x} > N$.

10) The "right-hand limit" of a function is informally defined as the limit obtained by "letting x approach a from the right-hand side"

a) Give a proper definition in the style of Definition 4.2.1 for the right/left-hand limits: $\lim_{x\to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = M$

$\lim_{x\to a^+} f(x) = L$ means $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < x - c < \delta$, it follows that $|f(x) - L| < \epsilon$.

$\lim_{x\to a^-} f(x) = L$ means $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < c - x < \delta$, it follows that $|f(x) - L| < \epsilon$

b) Prove that $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$.

Proof:

$(\Rightarrow)$. Suppose $\lim_{x\to a} f(x) = L$. Then $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$. If we restrict $x > a$, then $|x - a| = x - a$, and $0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$, so $\lim_{x\to a^+} f(x) = L$. Similarly, if we restrict $x < a$, then $|x - a| = a - x$ and $0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$ so $\lim_{x\to a^-} f(x) = L$.

$(\Leftarrow)$. Suppose $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$. Then $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < x - c < \delta$, or $0 < c - x < \delta$, it follows that $|f(x) - L| < \epsilon$. This implies that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$, so $\lim_{x\to a} f(x) = L$.