4.5 The Intermediate Value Theorem

Theorem 4.5.1 (Intermediate Value Theorem)
Let $f:[a,b] \to \mathbb{R}$ be continuous. If $L \in \mathbb{R}$ s.t. $f(a) < L < f(b)$, or $f(a) > L > f(b)$, then $\exists\, c \in (a,b)$ where $f(c) = L$.

- One way we can prove this is by classifying it as a special case of another theorem...

Theorem 4.5.2 (Preservation of Connected Sets)
Let $f: G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.

> ☆ I skipped 3.4 and missed important definitions and theorems on connected sets. Here they are:
>
> closure
>
> Definition 3.4.4: Two nonempty sets $A, B \subseteq \mathbb{R}$ are <u>separated</u> if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty. A set $E \subseteq \mathbb{R}$ is <u>disconnected</u> if it can be written as $E = A \cup B$, where A and B are separated. A set that is not disconnected is <u>connected</u>.
>
> Thm 3.4.6: A set $E \subseteq \mathbb{R}$ is connected iff $\forall$ nonempty disjoint sets A and B satisfying $E = A \cup B$, $\exists\, (x_n) \to x$ with $(x_n) \subseteq A$ or $(x_n) \subseteq B$, and x an element of the other.
>
> Thm 3.4.7: A set $E \subseteq \mathbb{R}$ is connected iff $a < c < b$ with $a, b \in E$ implies that $c \in E$.

Proof

Proof (Thm 4.5.2):

Let $f: G \to \mathbb{R}$ be continuous and $E \subseteq G$ be connected. We aim to show $f(E)$ is connected. Consider all nonempty disjoint subsets A and B such that $f(E) = A \cup B$, and let $f^{-1}(A) = \{x \in E : f(x) \in A\}$, $f^{-1}(B) = \{x \in E : f(x) \in B\}$. Because A and B are nonempty, we are assured that $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint, because if $x \in f^{-1}(A) \cap f^{-1}(B)$, then $f(x) \in f(A) \cap f(B)$, which cannot be true. Now since $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ and both sets are nonempty, and $f^{-1}(A) \cup f^{-1}(B) = E$, because E is connected by Theorem 3.4.6 $\exists (x_n) \to x$ such that $(x_n) \subseteq f^{-1}(A)$ and $x \in f^{-1}(B)$ WLOG. f is continuous, so $f(x_n) \to f(x)$, but $f(x_n) \subseteq A$ and $f(x) \in B$. Thus again by Theorem 3.4.6, $f(E)$ is connected $\square$

- In $\mathbb{R}$, a set is connected iff it is a (possibly unbounded) interval, we can use this with Theorem 4.5.2 to prove the IVT.

Proof (Exercise 4.5.1):

Let $f: [a,b] \to \mathbb{R}$ be continuous, and WLOG $L \in \mathbb{R}$ s.t. $f(a) < L < f(b)$. $[a,b]$ is connected, so by Theorem 4.5.2, $f([a,b])$ is connected. Because $f(a), f(b) \in f([a,b])$ and $f(a) < L < f(b)$, we know by Theorem 3.4.7 that $L \in f([a,b])$. This implies $\exists\, c \in [a,b]$ s.t. $f(c) = L$ $\square$

- We can use IVT to prove $\sqrt{2}$ exists. Let $f(x) = x^2 - 2$. We see $f(1) = -1$ and $f(2) = 2$. So by IVT there exists a point $c \in (1,2)$ s.t. $f(c) = 0$, and solving algebraically we get $c = \sqrt{2}$. This implies some connection between the IVT and completeness
- Turns out, we can prove the IVT both with the AoC and the Nested Interval Property. Proofs in Exercise 4.5.5

Definition 4.5.3 (Intermediate Value Property)

A function f has the intermediate value property on $[a,b]$ if $\forall\, x < y \in [a,b]$ and $\forall\, L$ between $f(x)$ and $f(y)$, $\exists\, c \in (x,y)$ s.t. $f(c) = L$.

- every continuous function on $[a,b]$ has the intermediate value property on $[a,b]$ - this is the IVT
- Just because a function has the intermediate value property does not imply continuity

  $g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases}$ is not continuous at 0, but it has the IVP on $[0,1]$
- Exercise 4.5.3 shows that IVP $\Rightarrow$ continuity if the function is monotone.

4.5 Exercises

X = incorrect
O = no idea, looked it up
* = looked for hints

2) Give an example or prove impossible

a) Continuous function defined on an open interval with range equal to a closed interval

$f(x) = 1 \quad x \in (0,1)$

b) Continuous function defined on a closed interval with range equal to an open interval

$f(x) = x \quad f(\mathbb{R}) = \mathbb{R} = (-\infty, \infty)$

c) Continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.

$$f(x) = \begin{cases} \ln(x) & x \geq 1 \\ 0 & 0 < x < 1 \end{cases} \qquad f: (0,\infty) \to [0,\infty)$$

→ didn't think about connectedness :n

d) Continuous function defined on $\mathbb{R}$ with range equal to $\mathbb{Q}$

Impossible. $\mathbb{R}$ is connected so $f(\mathbb{R})$ should be connected, but $\mathbb{Q}$ is disconnected.

3) A function is increasing on A if $f(x) \leq f(y)$ $\forall x, y \in A$ $x < y$.
Show that if f is increasing on $[a,b]$ and satisfies the intermediate value property, then f is continuous on $[a,b]$

Proof:

Let $f: A \to \mathbb{R}$ be a function that is increasing and satisfies the IVP on $[a,b] \subseteq A$. Let $c \in [a,b]$ and $\varepsilon > 0$. We know $f(a) \leq f(c) \leq f(b)$, and by the IVP we can find $y \in (a,b)$ s.t. $|f(y) - f(c)| < \varepsilon$. Let $\delta = |y - c|$. Whenever $|x - c| < \delta$ for $x \in [a,b]$, it follows that (WLOG) $y < x < c \Rightarrow f(y) < f(x) < f(c)$, and so $|f(x) - f(c)| < |f(y) - f(c)| < \varepsilon$. Thus, f is continuous $\square$

5.

a) Prove the IVT with the AoC

Proof:

Let f be continuous on $[a,b]$, and $L \in \mathbb{R}$ s.t. $f(a) < L < f(b)$.
Now let $K = \{x \in [a,b] : f(x) \leq L\}$. $a \in K$ and K is bounded above by b, so $c = \sup K$ exists. There are 3 cases to consider: $f(c) > L$, $f(c) < L$, or $f(c) = L$.
If $f(c) > L$, then let $\varepsilon = |f(c) - L|$. Because f is continuous, $\exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$. By definition of supremum, we also know $c - \delta < x$ for $x \in K \Rightarrow |c - x| < \delta \Rightarrow |f(c) - f(x)| < |f(c) - L|$. This means that either $L < f(c) < f(x)$ or $L < f(x) < f(c)$. Both imply $x \notin K$, so this case cannot be true.
If $f(c) < L$, then let $\varepsilon = |f(c) - L|$ and consider $x > c$ so $|x - c| < \delta$. This leads to $|f(x) - f(c)| < |f(c) - L|$, and so either $f(c) < f(x) < L$ or $f(x) < f(c) < L$. Both imply $x \in K$, so this case cannot be true.
That leaves $f(c) = L$. We have produced $c \in (a,b)$ ($c \neq a$ and $c \neq b$ because $f(a) \neq f(c) \neq f(b)$) such that $f(c) = L$, and so we are done $\square$

b) Prove the IVT with the nested interval property

Proof:

Let $f:[a,b]\to\mathbb{R}$ be continuous, and $L\in\mathbb{R}$ s.t. $f(a)<L<f(b)$. Let $I_0=[a_0,b_0]$ and $z_1=(a+b)/2$. If $f(z)\geq L$, then $a_1=a_0$ $b_1=z_1$. If $f(z)<L$ then $a_1=z_1$ and $b_1=b_0$. Repeat and inductively define $I_n=[a_n,b_n]$ where $a_n=a_{n-1}$ and $b_n=z_n$, or vice versa, where $z_n=(a_{n-1}+b_{n-1})/2$. We now have a sequence of nested intervals I_n with the property that $f(a_n)<L<f(b_n)$ $\forall n\in\mathbb{N}$. By the NIP, $\exists\, c\in\bigcap_{n=1}^{\infty} I_n$. This means $a_n\leq c\leq b_n$ $\forall n\in\mathbb{N}$. We know that $|I_n|=b_n-a_n=(1/2)^n(b-a)\to 0$, so $|b_n-a_n|<\varepsilon$ $\forall n\geq N$ for some $N\in\mathbb{N}$. It follows that $|b_n-c|<|b_n-a_n|<\varepsilon$, so $(b_n)\to c$. Now suppose $f(c)<L$. Let $\varepsilon=|f(c)-L|$. By continuity, $\exists\,\delta$ s.t. $|x-c|<\delta\Rightarrow|f(x)-f(c)|<\varepsilon$. Because $(b_n)\to c$ we can find $|b_n-c|<\delta\Rightarrow|f(b_n)-f(c)|<|f(c)-L|=\varepsilon$. This implies that either $f(b_n)<f(c)<L$ or $f(c)<f(b_n)<L$. This violates $f(a_n)<L<f(b_n)$, so $f(c)\geq L$. A symmetric argument with a_n shows $f(c)>L$ cannot be true, which leaves $f(c)=L$. Thus, $c\in(a,b)$ and $f(c)=L$ □

6. Let $f:[0,1]\to\mathbb{R}$ be continuous with $f(0)=f(1)$.

* ⇝ struggled to put together formal argument

a) Show $\exists\, x,y\in[0,1]$ s.t. $|x-y|=1/2$ and $f(x)=f(y)$

WLOG, assume $f(1/2)>f(0)$ and define $g(x)=f(1/2+x)-f(x)$. g is continuous because it is the sum of 2 continuous functions, and $g(0)=f(1/2)-f(0)>0$ and $g(1/2)=f(1)-f(1/2)<0$. Then by the IVT $\exists\, c\in(0,1/2)$ s.t. $g(c)=0$. That is, $f(1/2+c)-f(c)=0\Rightarrow f(1/2+c)=f(c)$. Let $x=1/2+c$ and $y=c$. Then $|x-y|=1/2$ and $f(x)=f(y)$ □

~ solution used a trick I was not going to think of.

(b) Show $\forall n \in \mathbb{N}$ $\exists x_n, y_n \in [0,1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

WLOG, assume $f(1/n) > f(0)$ and let $g(x) = f(1/n + x) - f(x)$.
$g(0) = f(1/n) - f(0) > 0$. By expanding it is apparent that
$g(0) + g(1/n) + \dots + g(\frac{n-1}{n}) = 0$. And because $g(0) > 0$, at least one of
the $g(k/n)$ must be negative. This is enough to appeal to the IVT:
$\exists\, c \in (0, k/n)$ s.t. $g(c) = 0 \Rightarrow f(1/n + c) = f(c)$ $\square$

~ why was this so hard?

(c) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x-y| = h$ satisfying $f(x) = f(y)$. Provide an example with $h = 2/5$.

$f(x) = \cos(5\pi x) + 2x$ $\quad f(0) = f(1)$ ✓

$f(x + 2/5) - f(x) = 4/5$ so it is never the case that $f(x + 2/5) = f(x)$.

7) Let f be continuous on $[0,1]$ with range contained in $[0,1]$.
Prove f must have a fixed point: $f(x) = x$ for some $x \in [0,1]$

Proof:
Assume $f(0) \neq 0$ and $f(1) \neq 1$ (otherwise we would be done).
Define $g(x) = f(x) - x$. Then $g(0) = f(0) - 0 > 0$ and
$g(1) = f(1) - 1 < 0$. Thus by IVT $\exists\, c \in (0,1)$ s.t. $g(c) = 0$ and
therefore $f(c) = c$ $\square$