2.6 The Cauchy Criterion

Definition 2.6.1 (Cauchy Sequence)
A sequence (a_n) is called a Cauchy sequence if, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. whenever $m, n \geq N$, it follows that $|a_n - a_m| < \epsilon$

- Very, very similar to definition for convergence
  - convergence talks about terms in sequence getting arbitrarily close to some number
  - Cauchy sequence talks about terms in sequence getting arbitrarily close to each other
- Turns out the definitions are equivalent
  ↳ We want to prove that a sequence converges, if and only if, it is a Cauchy sequence.

Theorem 2.6.2: Every convergent sequence is a Cauchy Sequence.
Proof given in Ex 1

- Proving the converse is difficult bc to prove something converges directly, we need a candidate for the limit.

Lemma 2.6.3 Cauchy sequences are bounded
Proof: Let $\epsilon = 1$. There exists N such that $|x_m - x_n| < 1 \;\forall m, n \geq N$. Thus, we must have $|x_n| < |x_N| + 1 \;\forall n \geq N$. It follows that $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N + 1|\}$ is a bound for (x_n)

Because $||x_n| - |x_N|| \leq |x_n - x_N| < 1$, ← reverse triangle inequality: See problem 1.2.6d
so $|x_n| - |x_N| < 1$
$\Rightarrow |x_n| < 1 + |x_N|$

Theorem 2.6.4 (Cauchy Criterion)

A sequence converges iff it is a Cauchy sequence

Proof:

($\Rightarrow$) Use Thm 2.6.2

($\Leftarrow$) Start with a Cauchy sequence (x_n) that is bounded by lemma 2.6.3. By Bolzano-Weierstrass, we produce a convergent subsequence (x_{n_K}). Set $x = \lim x_{n_K}$. The idea is to show that $(x_n) \to x$. Let $\varepsilon > 0$. Because (x_n) is Cauchy, $\exists N$ such that $|x_n - x_m| < \varepsilon/2$ whenever $m, n \geq N$. We also know that $(x_{n_K}) \to x$, so choose a term in this subsequence x_{n_K}, with $K \geq N$, and $|x_{n_K} - x| < \varepsilon/2$. Observe that if $n \geq N$, then

$$|x_n - x| = |x_n - x_{n_K} + x_{n_K} - x|$$
$$\leq |x_n - x_{n_K}| + |x_{n_K} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \square$$

2.6 Exercises

1) Prove Theorem 2.6.2: Every convergent sequence is a Cauchy sequence. (Hint: Use triangle inequality)

Proof: Assume $(x_n) \to x$. To prove (x_n) is Cauchy, we must find a point in the sequence after which we have $|x_n - x_m| < \varepsilon$. Since $(x_n) \to x$, then $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$, $|x_n - x| < \varepsilon/2$ and $|x_m - x| < \varepsilon/2$. Now consider the expression $|x_n - x_m|$. Expand as $|x_n - x + x - x_m| = |(x_n - x) + (x - x_m)|$. Then by the triangle inequality, $|(x_n - x) + (x - x_m)| \leq |x_n - x| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ □

2) Give an example, or argue that such a request is impossible

a) A Cauchy sequence that is not monotone.

Simply any convergent sequence that is not monotone: $(1, 2, 1, 1/2, 1/4, 1/8, \ldots)$

b) A Cauchy sequence with an unbounded subsequence

Impossible. By lemma 2.6.3, a Cauchy sequence (x_n) is bounded so $\exists M \in \mathbb{R}$ s.t. $|x_n| < M$ $\forall n \in \mathbb{N}$. Now consider an unbounded subsequence (x_{n_k}). Then $\forall B \in \mathbb{R}$, $\exists k \in \mathbb{N}$ s.t. $|x_{n_k}| \geq B$. This means that there is some term x_{n_k} with $|x_{n_k}| \geq M$. This contradicts the finding that (x_n) is bounded, so the request is impossible □

c) A divergent monotone sequence with a Cauchy subsequence.

Impossible. Proof:

Assume for contradiction that (x_n) is a divergent monotone sequence with a Cauchy subsequence (x_{n_k}). We know (x_n) must be unbounded, because otherwise by the MCT (x_n) would converge. By lemma 2.6.3 we also know (x_{n_k}) is bounded. The question is now reduced to whether an unbounded monotone sequence (x_n) can have a bounded subsequence (x_{n_k}). Since (x_{n_k}) is bounded, then $\exists M \in \mathbb{R}$ such that $\forall k \in \mathbb{N}$, $|x_{n_k}| < M$. Because (x_n) is unbounded, then we can find $n \in \mathbb{N}$ such that $|x_n| \geq M$. Now find k so that $n_k \geq n$, and then because (x_n) is monotone (assume increasing WLOG), then $x_{n_k} \geq x_n \geq M$. This contradicts the finding that $|x_{n_k}| < M$, so the request is impossible $\square$

d) An unbounded sequence containing a subsequence that is Cauchy.

Consider $x_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ n & \text{if } n \text{ even} \end{cases} = (1,2,1,4,1,6,\ldots)$. (x_n) is unbounded, and $(x_{n_k}) = (1,1,1,1,\ldots)$ is a Cauchy subsequence.

3) If (x_n) and (y_n) are Cauchy sequences, then by the Cauchy Criterion (x_n) and (y_n) converge, and then by ALT (x_n+y_n) converges and it's hence Cauchy.

a) Give a direct argument that (x_n+y_n) is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

for cauchy sequences (x_n) and (y_n)

Proof:

We want to show there $\exists N \in \mathbb{N}$ such that $\forall m,n \geq N$, $|(x_n+y_n)-(x_m+y_m)| < \varepsilon$. Let $N_1 \in \mathbb{N}$ such that $\forall m,n \geq N_1$, $|x_n - x_m| < \varepsilon/2$, and let $N_2 \in \mathbb{N}$ such that $\forall m,n \geq N_2$, $|y_n-y_m| < \varepsilon/2$. Then choose $N = \max\{N_1, N_2\}$, and both expressions still hold $\forall m,n \geq N$. Now we know that $\varepsilon = \varepsilon/2 + \varepsilon/2 > |x_n - x_m| + |y_n - y_m|$ $\geq |(x_n-x_m)+(y_n-y_m)| = |(x_n+y_n)-(x_m+y_m)|$, and so (x_n+y_n) is a Cauchy sequence $\square$

b) Do the same for the product $(x_n y_n)$

Proof:

We want to show there $\exists N \in \mathbb{N}$ s.t. $\forall m,n \geq N$, $|x_n y_n - x_m y_m| < \varepsilon$, assuming (x_n) and (y_n) are Cauchy sequences. Because (x_n) and (y_n) are Cauchy, they are bounded. So $\exists X \in \mathbb{R}$ s.t. $|x_n| < X$ $\forall n \in \mathbb{N}$, and $\exists Y \in \mathbb{R}$ s.t. $|y_n| < Y$ $\forall n \in \mathbb{N}$. Set $N_1 \in \mathbb{N}$ s.t. $\forall m,n \geq N_1$, $|x_n - x_m| < \frac{\varepsilon}{2|Y|}$ and set $N_2 \in \mathbb{N}$ s.t. $\forall m,n \geq N_2$, $|y_n - y_m| < \frac{\varepsilon}{2|X|}$. Then we can show that

$$\varepsilon > |X|\frac{\varepsilon}{2|X|} + |Y|\frac{\varepsilon}{2|Y|} > |X||y_n - y_m| + |Y||x_n - x_m| > |x_n||y_n-y_m| + |y_n||x_n - x_m|$$

$$> |x_n(y_n-y_m) - y_m(x_m - x_n)| > |x_n y_n + x_n y_m - x_n y_m - x_m y_m|$$

$= |x_n y_n - x_m y_m|$. So the product $(x_n y_n)$ is Cauchy $\square$