6.5 Power Series

- A power series is a function of the form
$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Theorem 6.5.1: If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x_0 \in \mathbb{R}$, then it converges absolutely for $|x| < |x_0|$.

Proof:
If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then $|a_n x_0^n| \leq M$ for $M \geq 0$, $n \in \mathbb{N}$. If $|x| < |x_0|$, then $|a_n x^n| < |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n$.
But since $\sum_{n=0}^{\infty} M \left|\frac{x}{x_0}\right|^n$ is a geometric series with $|r| < 1$, then by the comparison test $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely □

- main implication of this theorem is that the set of points for which a given power series converges is $\{0\}$, $\mathbb{R}$, or some bounded interval centered at 0: $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$
  - value of R is the <u>radius of convergence</u>

Theorem 6.5.2: If a power series converges absolutely at a point x_0, then it converges uniformly on $[-c, c]$, where $c = |x_0|$

Proof: (Exercise 6.5.3)
Define $M_n = |a_n x_0^n|$. Since for any $x \in [-c, c]$, $|a_n x^n| \leq M_n$, and $\sum_{n=0}^{\infty} |a_n x_0^n|$ converges, then by the M-test $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly for $x \in [-c, c]$ □

- Some nice results follow from Thm 6.5.2
  - power series that converges on open interval is necessarily continuous on that interval
- don't really know what to say about endpoint behavior
- Note that if power series converges conditionally at $x=R$, then it is possible for it to diverge at $-R$.
  - example $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ with $R=1$

Lemma 6.5.3 (Abel's Lemma)

Let b_n satisfy $b_1 \geq b_2 \geq \dots \geq 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded, ie assume $\exists A > 0$ s.t. $|a_1 + a_2 + \dots + a_n| \leq A \; \forall n \in \mathbb{N}$. Then $\forall n \in \mathbb{N}$, $|a_1 b_1 + a_2 b_2 + \dots + a_n b_n| \leq A b_1$

Proof:

$$\left|\sum_{k=1}^{n} a_k b_k\right| = \left|s_n b_{n+1} + \sum_{k=1}^{n} s_k (b_k - b_{k+1})\right| \text{ by Exercise 2.7.12}$$

$$\leq A b_{n+1} + \sum_{k=1}^{n} A (b_k - b_{k+1}) = A b_{n+1} + A b_1 - A b_{n+1} = A b_1 \; \square$$

Theorem 6.5.4 (Abel's Theorem)

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges at $x = R > 0$. Then the series converges uniformly on $[0, R]$. A similar result holds if the series converges at $x = -R$.

Proof:

Rewrite $g(x) = \sum_{n=0}^{\infty} a_n x_n = \sum_{n=0}^{\infty} (a_n R^n) \left(\frac{x}{R}\right)^n$. Because $\sum_{n=0}^{\infty} a_n R^n$ converges, $\exists N$ s.t. $|a_{m+1} R^{m+1} + \dots + a_n R^n| < \varepsilon/2$ for $n > m \geq N$. Use this $\varepsilon/2$ as an upper bound for the partial sums of $a_n R^n$ (omitting first m terms), and then realizing $\left(\frac{x}{R}\right)^{m+j}$ is monotone decreasing, apply Abel's Lemma to see that

$$\left|a_{m+1} R^{m+1} \left(\frac{x}{R}\right)^{m+1} + \dots + a_n R^n \left(\frac{x}{R}\right)^n\right| \leq \varepsilon/2 \left(\frac{x}{R}\right)^{m+1} < \varepsilon.$$ So $g(x)$ converges uniformly by the Cauchy Criterion for uniform convergence $\square$

• summarize Thm 6.5.2 and Abel's Theorem with the following

Theorem 6.5.5: If a power series converges pointwise on $A \subseteq \mathbb{R}$, then it converges uniformly on any compact set $K \subseteq A$.

Proof:
A compact set K has a maximum x_1 and a minimum x_0. Since $x_0, x_1 \in A$, Abel's theorem implies that the power series converges uniformly on $[x_0, x_1]$, which must contain K $\square$

• leads to desirable conclusion that power series are continuous on every point at which it converges.

Theorem 6.5.6: If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets in $(-R, R)$

Proof: Exercise 6.5.5

Theorem 6.5.7: Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on interval $A \subseteq \mathbb{R}$. f is continuous on A and differentiable on any $(-R, R) \subseteq A$. The derivative is given by $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Moreover, f is infinitely differentiable on $(-R, R)$, and successive derivatives are obtained by term-by-term differentiation.

Proof: Have already shown why f is continuous. Thm 6.5.6 establishes f' converges uniformly, so Thm 6.4.3 claims f is differentiable. f' is still a power series, and Thm 6.5.6 shows the radius of convergence doesn't change, so it is infinitely differentiable $\square$

6.5 Exercises

1) Consider $g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$

a) Is g defined on $(-1,1)$? Is it continuous on this set? Is g defined on $(-1,1]$? Is it continuous on this set? What happens on $[-1,1]$? Can the power series possibly converge for any $|x|>1$?

Since $g(1)$ converges, then g is defined on $(-1,1)$ by Theorem 6.5.1. It is also defined on $(-1,1]$. By theorem 6.5.?, g is continuous on both of these intervals. $g(-1)$ diverges, so g is not well defined or continuous on $[-1,1]$. The series cannot converge for $|x|>1$. By Alembert's Rule, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$, then a sequence diverges. Consider the subsequence x^{2n}/n. Applying Alembert's rule, we get $\lim_{n\to\infty}\left|\frac{x^{2n+2}\,2n}{2(n+1)\,x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^2 n}{n+1}\right| = \lim_{n\to\infty} x^2 > 1$ since $|x|>1$. Because this subsequence diverges, so does the sequence $\frac{(-1)^n x^n}{n}$, and so the series cannot converge.

b) For what values of x is $g'(x)$ defined? Find a formula for $g'(x)$.

By Thm 6.5.?, $g'(x)$ is defined on $(-1,1)$. $g'(x) = 1 - x + (-x)^2 + (-x)^3 + \ldots$ so by sum of geometric series, this equals $1/1-(-x) = 1/1+x$.

2) Find suitable coefficients (a_n) so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why impossible

a) Converges $\forall x \in \mathbb{R}$.

$a_n = 0$.

Non-trivially, $\frac{1}{n!}$ works $\sum \frac{x^n}{n!} = e^x$

b) Diverges $\forall x \in \mathbb{R}$.

Impossible. $\sum a_n x^n$ will always converge when $x=0$.

c) Converges absolutely $\forall x \in [-1,1]$, and diverges off this set.

$a_n = \frac{1}{n^2}$

d) Converges conditionally at $x=-1$ and absolutely at $x=1$

Impossible. If it converges absolutely at $x=1$, this implies the series also must converge absolutely at $x=-1$, which contradicts the given statement.

e) Converges conditionally at both $x=-1$ and $x=1$

$$a_n = \begin{cases} \frac{(-1)^{n/2}}{n} & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

5)

a) If s satisfies $0<s<1$, show ns^{n-1} is bounded $\forall n \geq 1$

Again use Alembert's rule. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)s^n}{ns^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)s}{n}\right|$
$= \lim_{n\to\infty} s < 1$. Thus $(ns^{n-1}) \to 0$ and therefore must be bounded.

* b) Given $x \in (-R,R)$, pick t to satisfy $|x|<t<R$. Use this to construct a proof for Thm 6.5.6.

Proof:

We want to show that if $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x \in (-R,R)$, then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges $\forall x \in (-R,R)$. Given any $x \in (-R,R)$, pick t so that $|x|<t<R$. It is apparent that $\sum_{n=1}^{\infty} a_n t^{n-1}$ converges, and that $n\left(\frac{|x|}{t}\right)^{n-1} \leq M$ $\forall n \in \mathbb{N}$ from (a). These facts will be used shortly.

Notice $\sum_{n=1}^{\infty} n a_n x^{n-1} \leq \sum_{n=1}^{\infty} n a_n |x|^{n-1} = \sum_{n=1}^{\infty} n a_n \left(\frac{|x|}{t}\right)^{n-1} t^{n-1}$

$\leq M \sum_{n=1}^{\infty} a_n t^{n-1}$, which converges $\square$

* $\to$ didn't get $\sum_{n=1}^{\infty} n/2^n$ strategy

6) The geometric series $1+x+x^2+x^3+\ldots = \frac{1}{1-x}$ $\forall |x|<1$.

Use the results about power series proved in this section to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$.

$d/dx\left(1/1-x\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$

$1/2\left(1/(1-x)^2\right) = 1/2 + x + 3/2\, x^2 + \ldots$ Now use $x = 1/2$

$= 1(1/2)^1 + 2(1/2)^2 + 3(1/2)^3 = \sum n/2^n$

So $1/2\left(\frac{1}{(1-1/2)^2}\right) = 2$.

Take $d/dx\left(1/1-x\right) = \frac{1}{(1-x)^2} = 1+2x+3x^2+\ldots$. Multiply by x to get $x/(1-x)^2 = x + 2x^2 + 3x^3 + 4x^4 + \ldots$. Differentiate again to get $1/(1-x)^2 + 2x/(1-x)^3 = 1 + 4x + 9x^2 + 16x^3 + \ldots$ Then multiply by x to get $x/(1-x)^2 + 2x^2/(1-x)^3 = x + 4x^2 + 9x^3 + 16x^4 + \ldots$ Plug in $x = 1/2$ to get $\sum n^2/2^n = 6$.

8)

a) Show power series representations are unique. If we have $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \quad \forall x \in (-R, R)$, prove that $a_n = b_n \ \forall n = 0, 1, 2, \ldots$

Plug in $x=0$ and see that $a_0 = b_0$. Now differentiate both sides of $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ to get $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n b_n x^{n-1}$. Again plugging in 0 yields $a_1 = b_1$. Since power series are infinitely differentiable, we can write the k^{th} derivatives as $\sum_{n=k}^{\infty} a_n \frac{n!}{(n-k)!} x^{n-k} = \sum_{n=k}^{\infty} b_n \frac{n!}{(n-k)!} x^{n-k}$. Plugging in $x=0$ will give $a_k = b_k$ for all $k = 0, 1, 2, \ldots$

b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$ and assume $f'(x) = f(x)$ $\forall x \in (-R, R)$ and $f(0) = 1$. Deduce values of a_n

Since $a_0 = 1$, and $f'(x) = f(x)$, then $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and from part (a) this gives us (by plugging in $x=0$) that $a_0 = a_1 = 1$. Doing this again gives $\sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} \Rightarrow a_1 = 2a_2 \Rightarrow a_2 = 1/2$. Once more: $\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} = \sum_{n=3}^{\infty} a_n (n)(n-1)(n-2) x^{n-3}$ $\Rightarrow 2a_2 = 3(2)a_3 \Rightarrow a_3 = 1/6$. In general, we have $\sum_{n=k}^{\infty} a_n \frac{n!}{(n-k)!} x^{n-k} = \sum_{n=k+1}^{\infty} a_n \frac{n!}{(n-(k+1))!} x^{n-(k+1)}$. Plugging in $x=0$ yields $a_k k! = a_{k+1} (k+1)! \Rightarrow a_{k+1} = \frac{a_k k!}{(k+1)!} \Rightarrow a_{k+1} = \frac{a_k}{k}$. This recurrence relation has the explicit formula $a_n = 1/n!$.