6.3 Uniform Convergence and Differentiation

Theorem 6.3.1 (Differentiable Limit Theorem)
Let $f_n \to f$ pointwise on $[a,b]$ and assume each f_n is differentiable. If (f_n') converges uniformly on $[a,b]$ to a function g, then f is differentiable and $f' = g$.

Proof:
Fix $c \in [a,b]$ and let $\varepsilon > 0$. We wish to show that $f'(c) = g(c)$. That is, $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = g(c)$. To do this, we need a δ so that

$|x - c| < \delta$ implies $\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \varepsilon$. Summon some terms and

apply the triangle inequality to get

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} + \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) + f_n'(c) - g(c) \right|$$

$$\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| + \left| f_n'(c) - g(c) \right|. \quad \forall x \in [a,b]$$

We want to get each of these terms smaller than $\varepsilon/3$. First, because $f_n'(c) \to g(c)$, then $\exists N_1$ s.t. $n \geq N_1 \Rightarrow |f_n'(c) - g(c)| < \varepsilon/3$. By uniform convergence and Thm 6.2.5, $\exists N_2$ s.t. $|f_n'(c) - f_m'(c)| < \varepsilon/3 \ \forall m, n \geq N_2$.
Choose $N = \max\{N_1, N_2\}$. From the definition of the derivative, $\exists \delta$ s.t. $|x - c| < \delta \Rightarrow \left| \frac{f_N(x) - f_N(c)}{x - c} - f_N'(c) \right| < \varepsilon/3$. This is the desired δ, but

we still have the first term left. Fix x so $|x - c| < \delta$, let $m \geq N$, and the MVT says for $[c, x]$ WLOG that $\exists \alpha \in (c, x)$ s.t.

$$f_m'(\alpha) - f_N'(\alpha) = \frac{(f_m(x) - f_N(x)) - (f_m(c) - f_N(c))}{x - c}.$$ Since $|f_m'(\alpha) - f_N'(\alpha)| < \varepsilon/3$,

it follows $\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| < \varepsilon/3$, and by OLT, taking $\lim_{m \to \infty}$ yields

$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| < \varepsilon/3$. Put them together and voila $\square$

- Thm 6.3.1 hypothesis unnecessarily strong, don't need to assume $f_n(x) \to f(x)$ because assumption that (f'_n) converges uniformly is enough to prove that $(f_n) \to f$ <u>uniformly</u>

Theorem 6.3.2
Let (f_n) be a sequence of differentiable functions on $[a,b]$ and assume (f'_n) converges uniformly on $[a,b]$. If $\exists x_0 \in [a,b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a,b]$.

Proof: Exercise 6.3.7

- Combine 6.3.1 and 6.3.2 to get a stronger version

Theorem 6.3.3
Let (f_n) be a sequence of differentiable functions defined on $[a,b]$ and assume (f'_n) converges uniformly to g on $[a,b]$. If $\exists x_0 \in [a,b]$ for which $f_n(x_0)$ is convergent, then (f_n) converges uniformly. Moreover, $f = \lim f_n$ is differentiable and $f' = g$.

6.3 Exercises

1) Consider $g_n(x) = \frac{x^n}{n}$

a) Show (g_n) converges uniformly on $[0,1]$ and find $g = \lim g_n$.
Show g is differentiable and compute $g'(x)$ $\forall x \in [0,1]$.

$\lim g_n = 0$. Choose $N > 1/\varepsilon$, and then $\forall n \geq N$, $\frac{1}{n} < \varepsilon$, and $\frac{x^n}{n} \leq \frac{1}{n} < \varepsilon$, so $|x^n/n| < \varepsilon$. Since g is constant, it's differentiable, and so $g'(x) = 0$ $\forall x \in [0,1]$

* ⇝ needed help disproving uniform convergence

b) Now show (g'_n) converges on $[0,1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h and g'. Are they the same?

$g'_n(x) = x^{n-1}$. $\lim_{n \to \infty} g'_n = h = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1 \end{cases}$ $g'(x) = 0$ and they are not the same. $g'_n(x)$ does not converge uniformly. Consider the sequence $x_n = 2^{-1/n-1}$. Then $|g'_n(x_n) - h(x_n)| \geq 1/2$

3) Consider $f_n(x) = \frac{x}{1 + nx^2}$

a) Find points on $\mathbb{R}$ where each $f_n(x)$ attains max and min value. Prove (f_n) converges uniformly on $\mathbb{R}$. What is the limit function?

$f'_n(x) = \frac{1 - nx^2}{(1 + nx^2)^2} = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$. $f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2\sqrt{n}}$.
So $|f_n(x)| \leq 1/2\sqrt{n}$ $\forall n \in \mathbb{N}$. Choose $N > 1/4\varepsilon^2$, and it follows that $|f_n(x)| \leq 1/2\sqrt{n} < \varepsilon$ $\forall n \geq N$. Thus $f_n(x)$ converges uniformly to 0.

b) Let $f = \lim f_n$. Compute $f'_n(x)$ and find all values of x for which $f'(x) = \lim f'_n(x)$.

$f = \lim f_n = 0$. $f'_n(x) = \frac{1-nx^2}{(1+nx^2)^2}$. $\lim f'_n(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

$f'(x) = \lim f'_n(x)$ when $x \neq 0$.

4) Let $h_n(x) = \frac{\sin(nx)}{\sqrt{n}}$. Show $h_n \to 0$ uniformly on $\mathbb{R}$ but (h'_n) diverges $\forall x \in \mathbb{R}$.

Choose $N > 1/\varepsilon^2$. Then $\forall n \geq N$ $\frac{1}{\sqrt{n}} < \varepsilon$, and $\left|\frac{\sin(nx)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} < \varepsilon$, so h_n converges uniformly to 0.

$h'_n = \sqrt{n}\cos(nx)$, which is unbounded as $n \to \infty$.

6) Provide an example or explain why it's impossible domain of all is $\mathbb{R}$

a) (f_n) of nowhere differentiable functions with $f_n \to f$ uniformly and f everywhere differentiable.

$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$ $f_n(x) = \frac{g(x)}{n}$. $f_n \to f$ uniformly, and since $f = 0$ it is everywhere differentiable.

b) (f_n) of differentiable functions s.t. (f'_n) converges uniformly but (f_n) does not converge $\forall x \in \mathbb{R}$

$f_n(x) = \frac{\sin(x)}{n} + n$ does not converge

$f'_n(x) = \frac{\cos(x)}{n}$ does.

c) (f_n) differentiable, (f_n), (f'_n) converge uniformly but $f = \lim f_n$ is not differentiable at any point.

Impossible, violates differentiable limit theorem

7) Use the MVT to prove Thm 6.3.2. Observe that the triangle inequality implies for any $x \in [a,b]$ and $m,n \in \mathbb{N}$,

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$$

Proof:

Suppose (f_n) is a sequence of differentiable functions on $[a,b]$, and that (f'_n) converges uniformly on $[a,b]$. If $\exists x_0 \in [a,b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a,b]$. We want to show $\exists N \in \mathbb{N}$ s.t. $\forall m,n \geq N$, $|f_n(x) - f_m(x)| < \varepsilon$, which would make (f_n) uniformly convergent by Cauchy Criterion. Observe $|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$. Because $f_n(x_0)$ is convergent, $\exists N_1$ s.t. $\forall n,m \geq N_1$ $|f_n(x_0) - f_m(x_0)| < \varepsilon/2$. Now by the MVT we know $\exists c \in (x_0, x)$ WLOG s.t. $f'_n(c) = (f_n(x) - f_n(x_0))/(x - x_0)$ and $f'_m(c) = (f_m(x) - f_m(x_0))/(x - x_0)$. Because f'_n converges uniformly, $\exists N_2$ s.t. $\forall m,n \geq N_2$, $|f'_n(c) - f'_m(c)| < \varepsilon/2(x - x_0)$. This expands to

$$\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} \right| < \frac{\varepsilon}{2(x - x_0)} \Rightarrow |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| < \frac{\varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$, and we have $\forall n,m \geq N$, $|f_n(x) - f_m(x)| < \varepsilon$ $\square$