2.4 Monotone Convergence Theorem and Infinite Series

- We know that convergent sequences are bounded, but bounded sequences do not necessarily converge

* but if a bounded sequence is monotone, then it does converge

Definition 2.4.1 (Monotonicity)
A sequence (a_n) is increasing if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$, and decreasing if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$. A sequence is monotone if it is either increasing or decreasing.

Theorem 2.4.2 (Monotone Convergence Theorem)
If a sequence is monotone and bounded, then it converges

Proof:
Suppose (a_n) is monotone and bounded, and let $a = \sup\{a_n\}$. By definition of supremum, there exists an $a_N \in \{a_n\}$ such that $a - \varepsilon < a_N \Rightarrow a - a_N < \varepsilon \Rightarrow |a_N - a| < \varepsilon$. Without loss of generality, suppose a_n is monotonically increasing (if it was decreasing, then use $\inf\{a_n\} = a$). Then we know $\forall n \geq N$, $|a_n - a| \leq |a_N - a| < \varepsilon$. Because $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$, (a_n) is convergent $\square$

Definition 2.4.3 (Convergence of a Series)
Let (b_n) be a sequence. An infinite series is a formal expression of the form $\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \ldots$

We define the corresponding sequence of partial sums (s_m) by $s_m = b_1 + b_2 + \ldots + b_m$, and we say that $\sum_{n=1}^{\infty} b_n$ converges to B if (s_m) converges to B. Written as $\sum_{n=1}^{\infty} b_n = B$.

Ex 2.4.4. Convergent Series by Monotone Convergence Theorem

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The sequence of partial sums is $S_m = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{m^2}$.

Because all the terms in the series are positive, then (S_m) is increasing. Can we find an upper bound on (S_m)? If so, then (S_m) converges and so does the series.

$$S_m = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{m^2} < 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \dots + \frac{1}{m(m-1)}$$

$$= 1 + (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \dots + \left(\frac{1}{m(m-1)} - \frac{1}{m}\right)$$

$$= 1 + 1 - \frac{1}{m} < 2.$$

Thus 2 is an upper bound on (S_m), and by the MCT, $\sum_{n=1}^{\infty} 1/n^2$ converges

Ex 2.4.5 Harmonic Series

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$ with $S_m = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$. Again, because all the terms in the series are positive, then (S_m) is increasing.

Notice that

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(1/8 + 1/8 + 1/8 + 1/8\right) = \frac{5}{2}$$

$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + 1/8\right) + \dots + \left(\frac{1}{2^{k-1}+1} + \dots + \frac{1}{2^k}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^k} + \dots + \frac{1}{2^k}\right)$$

$$= 1 + \frac{1}{2} + 2(1/4) + 4(1/8) + \dots + 2^{k-1}(1/2^k)$$

$$= 1 + 1/2 + 1/2 + 1/2 + \dots + 1/2$$

$= 1 + k(1/2)$, which is unbounded, so the harmonic series diverges.

Theorem 2.4.6 (Cauchy Condensation Test)
Suppose (b_n) is a decreasing sequence that satisfies $b_n \geq 0$ $\forall n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} b_n$ converges iff $\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \ldots$ converges.

Proof:
Assume $\sum_{n=1}^{\infty} 2^n b_{2^n}$ converges. We know the partial sums are bounded because they are convergent: $t_k = b_1 + 2b_2 + 4b_4 + \ldots + 2^k b_{2^k}$. Now we want to prove that $\sum_{n=1}^{\infty} b_n$ converges. Because $b_n \geq 0$, we know partial sums are increasing, so we only need to show that $S_m = b_1 + b_2 + b_3 + \ldots + b_m$ is bounded. Fix m and let k be large enough to ensure $m \leq 2^{k+1} - 1$. Then, $S_m \leq S_{2^{k+1}-1}$ and

$$S_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \ldots + (b_{2^k} + \ldots + b_{2^{k+1}-1})$$
$$\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \ldots + (b_{2^k} + \ldots + b_{2^k})$$
$$= b_1 + 2b_2 + 4b_4 + \ldots + 2^k b_{2^k} = t_k$$

Thus, $S_m \leq t_k \leq M$, and so S_m is bounded and $\sum_{n=1}^{\infty} b_n$ converges.

Corollary 2.4.7: The series $1/n^p$ converges iff $p > 1$

2.4 Exercises

1) Consider $x_1 = 3$, $x_{n+1} = \frac{1}{4-x_n}$

a) Prove the sequence converges

Proof:
First we must show the sequence is monotonically decreasing. That is, $x_{n+1} \leq x_n \ \forall n \in \mathbb{N}$. The base case is simple: $x_1 = 3$ and $x_2 = 1$ so $x_1 \geq x_2$. Now suppose that $x_{k+1} \leq x_k \ \forall k \in \mathbb{N}$. We aim to show that $x_{k+2} \leq x_{k+1}$ is true. Rewrite as $\frac{1}{4-x_{k+1}} \leq \frac{1}{4-x_k}$

$4 - x_k \leq 4 - x_{k+1} \Rightarrow x_k \geq x_{k+1}$, which is true by the induction hypothesis. Thus the sequence is decreasing.
Now we need to show the sequence is bounded. Since $x_1 = 3$ and the sequence decreases, 3 is an upper bound. Since $x_n < 4 \ \forall n \in \mathbb{N}$, all terms in the sequence are positive, so x_n is bounded below by 0. Because the sequence is decreasing and bounded, by the MCT it converges □

b) $\lim x_n$ exists. Show why $\lim x_{n+1}$ must also exist and equal the same value.

x_{n+1} is still decreasing and bounded, so $\lim x_{n+1}$ exists. And because $\{x_{n+1}\} \subseteq \{x_n\}$, x_{n+1} has a lower supremum but the same infimum as $\{x_n\}$. We also know that $\lim x_n = \inf\{x_n\}$ because x_n is decreasing and bounded. Therefore,
$\lim x_{n+1} = \inf\{x_{n+1}\} = \inf\{x_n\} = \lim x_n$.

c) Take the limit of each side of recursive equation to compute $\lim x_n$.

$\lim x_{n+1} = \lim\left(\frac{1}{4-x_n}\right) \Rightarrow \lim x_{n+1} = \frac{1}{4-\lim x_n}$, and since $\lim x_{n+1} = \lim x_n = x$, then $x = \frac{1}{4-x} \Rightarrow -x^2+4x-1=0 \Rightarrow$ $x = \boxed{2-\sqrt{3} = \lim x_n}$

3) Show that $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ converges & find the limit.

Proof of Convergence:

First show the sequence is increasing. The recursive definition of the sequence is $a_{n+1} = \sqrt{2+a_n}$. Base case is $\sqrt{2} \le \sqrt{2+\sqrt{2}} \Rightarrow a_1 \le a_2$. Now suppose $a_{n+1} \ge a_n$ $\forall n \in \mathbb{N}$. We show that $a_{n+2} \ge a_{n+1}$. Rewrite as $\sqrt{2+a_{n+1}} \overset{?}{\ge} \sqrt{2+a_n} \Rightarrow 2+a_{n+1} \ge 2+a_n \Rightarrow a_{n+1} \ge a_n$ which is true by induction hypothesis, so (a_n) is increasing and therefore monotone. Now we show the sequence is bounded. Consider 10 as a potential upper bound. That is, we want to show $|a_n| < 10$ $\forall n \in \mathbb{N}$. The base case $\sqrt{2} < 10$ holds. Now suppose $|a_n| < 10$ $\forall n \in \mathbb{N}$. We aim to show $|a_{n+1}| < 10$. Rewrite as $\sqrt{2+a_n} < 10 \Rightarrow 2+a_n < 100 \Rightarrow a_n < 98 \Rightarrow |a_n| < 98$, which is true by induction hypothesis. Then (a_n) is monotone and bounded, so $\lim a_n$ exists.

Limit computation:

because $\lim a_{n+1} = \lim a_n$

$\lim a_{n+1} = \lim\sqrt{2+a_n} \Rightarrow \lim a_{n+1} = \sqrt{2+\lim a_n} \Rightarrow a = \sqrt{2+a}$

$\Rightarrow a^2 - a - 2 = 0 \Rightarrow a = 2$. So $\lim a_n = 2$.

b) Does the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converge? If so, find limit

Yes. The sequence is increasing. $\sqrt{2} < \sqrt{2\sqrt{2}}$. Suppose $a_{n+1} \geq a_n$, then $a_{n+2} \geq a_{n+1}$ because $\sqrt{2a_{n+1}} \geq \sqrt{2a_n} \Rightarrow a_{n+1} \geq a_n$. The sequence is also bounded by considering upper bound of 2: $\sqrt{2} \leq 2$. Suppose $|a_n| \leq 2$, then $|a_{n+1}| \leq 2 \Rightarrow \sqrt{2a_n} \leq 2 \Rightarrow 2a_n \leq 4 \Rightarrow a_n \leq 2$. Therefore, the sequence converges.

Limit Computation:

$\lim a_{n+1} = \lim(\sqrt{2a_n}) \Rightarrow \lim a_{n+1} = \sqrt{2 \lim a_n} \Rightarrow a = \sqrt{2a}$

$\Rightarrow a^2 = 2a \Rightarrow a = 2$, so $\lim a_n = 2$.

6) Arithmetic vs. Geometric Mean

a) Explain why $\sqrt{xy} \leq \frac{x+y}{2}$ $\forall x, y \in \mathbb{R}^+$.

We know that $(x-y)^2 \geq 0$. $\Rightarrow (x-y)^2 = (x+y)^2 - 4xy \geq 0 \Rightarrow$
$(x+y)^2 \geq 4xy \Rightarrow x+y \geq 2\sqrt{xy} \Rightarrow (x+y)/2 \geq \sqrt{xy}$

b) Let $0 \leq x_1 \leq y_1$ and define $x_{n+1} = \sqrt{x_n y_n}$ $y_{n+1} = \frac{x_n + y_n}{2}$
Show $\lim x_n$ and $\lim y_n$ exist and are equal.

First we show that (x_n) is increasing and (y_n) is decreasing. From part (a), we know that $y_n \geq x_n$ $\forall n \in \mathbb{N}$. Then $x_n y_n \geq x_n^2 \Rightarrow \sqrt{x_n y_n} \geq x_n \Rightarrow x_{n+1} \geq x_n$, and so (x_n) increases. Similarly, $x_n \leq y_n \Rightarrow x_n + y_n \leq 2y_n \Rightarrow (x_n + y_n)/2 \leq y_n \Rightarrow y_{n+1} \leq y_n$, and so (y_n) decreases. Next we need to show that (x_n) and (y_n) are bounded. Since $x_1 \leq x_{n>1}$, (x_n) is bounded below by x_1. We propose y_1 as an upper bound for (x_n) and prove by induction. Our base case $x_1 \leq y_1$ is given.

Now suppose $x_n \le y_1 \ \forall n \in \mathbb{N}$. Then $x_n y_1 \le y_1^2$, and $x_n y_n \le x_n y_1 \le y_1^2$. So $x_n y_n \le y_1^2 \Rightarrow \sqrt{x_n y_n} \le y_1 \Rightarrow x_{n+1} \le y_1$. So (x_n) is bounded above by y_1. Since (y_n) is decreasing, y_1 is an upper bound. We propose 0 as a lower bound for (y_n) and prove by induction. $0 \le y_1$ is the given base case. Now suppose $0 \le y_n \ \forall n \in \mathbb{N}$. Then $0 \le x_n + y_n$ because $x_n \ge x_1 \ge 0$. Dividing by 2 yields $0 \le (x_n + y_n)/2 \Rightarrow 0 \le y_{n+1}$, and so (y_n) is bounded below by 0. By the MCT, $\lim x_n$ and $\lim y_n$ exist $\square$

Now we show that $\lim y_n = \lim x_n$:

$$y_{n+1} = \frac{x_n + y_n}{2} \Rightarrow \lim y_{n+1} = \lim\left(\frac{x_n + y_n}{2}\right) \Rightarrow \lim y_{n+1} = \frac{\lim x_n + \lim y_n}{2}$$

$$\Rightarrow 2\lim y_n = \lim x_n + \lim y_n \Rightarrow \lim y_n = \lim x_n.$$

7) Let (a_n) be a bounded sequence

a) Prove that the sequence defined by $y_n = \sup\{a_k : k \ge n\}$ converges

Proof:
Because (a_n) is bounded, so is (y_n). Because $\{a_k : k \ge n+1\} \subseteq \{a_k : k \ge n\}$, then $\sup\{a_k : k \ge n+1\} \le \sup\{a_k : k \ge n\}$, and therefore $y_{n+1} \le y_n$, so (y_n) is decreasing. Thus, (y_n) converges $\square$

b) The limit superior of (a_n), or $\lim\sup a_n$, is defined by $\lim\sup a_n = \lim y_n$, where $y_n = \sup\{a_k : k \ge n\}$. Provide a definition for $\lim\inf a_n$, and briefly explain why it always exists for any bounded sequence.

$\lim\inf a_n = \lim y_n$ where $y_n = \inf\{a_k : k \ge n\}$.
Since $\{a_k : k \ge n+1\} \subseteq \{a_k : k \ge n\}$, then $\inf\{a_k : k \ge n+1\} \ge \inf\{a_k : k \ge n\}$, so $y_{n+1} \ge y_n$. (y_n) is monotone & bounded, so $\lim y_n = \lim\inf a_n$ exists.

c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example where the inequality is strict.

Proof:
Let (a_n) be a bounded sequence, let $x_n = \inf\{a_k : k \geq n\}$, and let $y_n = \sup\{a_k : k \geq n\}$. By definition of infimum and supremum, $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$, so $x_n \leq y_n \ \forall n \in \mathbb{N}$. From parts (a) and (b), we know $\limsup a_n = \lim y_n$, and $\liminf a_n = \lim x_n$, and by the Order Limit Theorem, $\lim x_n \leq \lim y_n$, so $\liminf a_n \leq \limsup a_n$ $\square$

Example: Let $a_n = (-1)^n$. Then $\liminf a_n = -1 < 1 = \limsup a_n$

d) Show that $\liminf a_n = \limsup a_n$ iff $\lim a_n$ exists. In this case, all 3 share the same value.

Proof:
We know $\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}$, and so by OLT, $\liminf a_n \leq \lim a_n \leq \limsup a_n$, and the limit exists. By the Squeeze Theorem, $\lim a_n = \liminf a_n = \limsup a_n$.
If $\lim a_n = a$ exists, then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$. And for any $k \geq n$, $k \geq N$, so $|\sup\{a_k : k \geq n\} - a| < \varepsilon \Rightarrow \limsup a_n = a$ exists. Similarly $\liminf a_n = a$ exists, and $\lim a_n = \liminf a_n = \limsup a_n$ $\square$

8) For each series, find an explicit formula for sequence of partial sums and determine convergence

a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ $\quad S_m = \frac{2^m - 1}{2^{m-1}}$

The series converges. Choose $N > 1/\varepsilon$. Then $\forall n \geq N$, $n > 1/\varepsilon \Rightarrow \frac{1}{n} < \varepsilon$

$\Rightarrow \frac{1}{2^{n-1}} \leq \frac{1}{n} < \varepsilon \Rightarrow \frac{1}{2^{n-1}} < \varepsilon \Rightarrow \left|\frac{-1}{2^{n-1}}\right| < \varepsilon \Rightarrow \left|\frac{2^n - 1 - 2^n}{2^{n-1}}\right| < \varepsilon$

$\Rightarrow \left|\frac{2^n - 1}{2^{n-1}} - 2\right| < \varepsilon$, so $\lim S_m$ exists and the series converges $\square$

b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ $\quad S_m = \frac{m}{m+1}$

The series converges. Choose $N > 1/\varepsilon$, then $\forall n \geq N$, $\frac{1}{n} < \varepsilon \Rightarrow \frac{1}{n+1} < \frac{1}{n} < \frac{1}{\varepsilon}$

$\Rightarrow \left|\frac{-1}{n+1}\right| < \frac{1}{\varepsilon} \Rightarrow \left|\frac{n-(n+1)}{n+1}\right| < \varepsilon \Rightarrow \left|\frac{n}{n+1} - 1\right| < \varepsilon$, so $\lim S_m$ exists & series converges.

c) $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$ $\quad S_m = \log(n+1)$

The series diverges. Proof:

Suppose the series converges. Then S_m is bounded $\Rightarrow |\log(n+1)| < M$.

Choose $n > e^M - 1$, Then $|\log(n+1)| > \log(e^M) = M$. This contradicts our assumption that S_m converges, so the series diverges $\square$

9) Complete the proof of Thm 2.4.6 by showing that if $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$

Proof:

Let $S_{2^{k+1}-1} = b_1 + b_2 + b_3 + \dots + b_{2^{k+1}-1}$ be partial sums of (b_n)

Since (b_n) decreases,

$$\begin{aligned} 2S_{2^{k+1}-1} &= 2b_1 + 2(b_2 + b_3) + \dots + 2(b_{2^k} + \dots + b_{2^{k+1}-1}) \\ &\geq 2b_2 + 2(2b_4) + \dots + 2(2^k b_{2^{k+1}-1}) \\ &= 2b_2 + 4b_4 + 8b_8 + \dots + 2^{k+1} b_{2^{k+1}} \\ &= -b_1 + \sum_{n=0}^{k+1} 2^n b_{2^n} \end{aligned}$$

, which we already know diverges,

so $\sum_{n=1}^{\infty} b_n$ diverges $\square$