2.5 Subsequences and Bolzano-Weierstrass Theorem

Definition 2.5.1 (Subsequence)

Let (a_n) be a sequence of reals, and let $n_1 < n_2 < n_3 < \dots$ be an increasing sequence of natural numbers. Then the sequence $(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ is a subsequence of (a_n), and is denoted by (a_{n_k}) where $k \in \mathbb{N}$ indexes the subsequence

- Notice that order of terms in a subsequence is the same as the original, and no repeats are allowed.

Theorem 2.5.2 Subsequences of a convergent sequence converge to the same limit as the original sequence

Proof:

Suppose $(a_n) \to a$ and let (a_{n_k}) be any subsequence. Then $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$. Since $n_k \geq k$ $\forall k \in \mathbb{N}$, if we let $k \geq N$, then $n_k \geq N$, and so $|a_{n_k} - a| < \varepsilon$. This implies $\lim a_{n_k} = a = \lim a_n$ $\square$

Ex 2.5.3 (Applying Thm 2.5.2 to find a limit)

Let $0 < b < 1$. Then $(b^n) = b, b^2, b^3, b^4, \dots$ is decreasing and bounded below by 0, so $\lim b^n = l$ exists. Consider the subsequence (b^{2n}). By Thm 2.5.2, $(b^{2n}) \to l$. But $b^{2n} = b^n \cdot b^n$, and by ALT $(b^{2n}) = l \cdot l = l^2$. So if $l = l^2$, then $l = 0$.

Ex 2.5.4 (Applying Thm 2.5.2 to prove divergence)

Consider $(1, -1/2, 1/3, -1/4, 1/5, -1/5, 1/5, -1/5, \dots)$. Pretty sure it diverges... Consider subsequence $(1/5, 1/5, 1/5, \dots)$ that converges to $1/5$. Now consider the subsequence $(-1/5, -1/5, -1/5, \dots)$ that converges to $-1/5$. Because we have 2 subsequences that converge to different limits, the original sequence must diverge.

• It was pretty easy for us to pick out convergent subsequences in the last example. That's because it's always possible for any bounded sequence

Theorem 2.5.5 (Bolzano–Weierstrass Theorem):
Every bounded sequence contains a convergent subsequence.

Proof:
Let (a_n) be a bounded sequence so that $\exists M > 0$ s.t. $|a_n| < M$ $\forall n \in \mathbb{N}$. Bisect $[-M, M]$ into $[-M, 0]$ and $[0, M]$. At least one of these has an infinite number of terms of (a_n). Select a half for which this is the case and label as I_1. Then let a_{n_1} be some term of a_n satisfying $a_{n_1} \in I_1$. Bisect I_1 into closed intervals of even length, and let I_2 be a half that contains an infinite number of terms of the original sequence. Because I_2 has infinitely many terms, we can select a_{n_2} with $n_2 > n_1$ and $a_{n_2} \in I_2$. Continue by constructing the closed interval I_k by taking a half of I_{k-1} with infinitely many terms and select $n_k > n_{k-1} > \ldots > n_1$ so that $a_{n_k} \in I_k$. Notice the sets $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ form a nested sequence of closed intervals, and by Nested Interval Property $\exists x \in \mathbb{R}$ s.t. $x \in I_k$ $\forall k$. Let this x be the candidate limit for our emerging subsequence.
Let $\varepsilon > 0$. The length of I_k is $M(1/2)^{k-1}$, which converges to 0. Choose N so that $k \geq N$ implies the length of $I_k < \varepsilon$. Because a_{n_k} and x are in I_k, $|a_{n_k} - x| < \varepsilon$, and so $(a_{n_k}) \to x$ $\square$

2.5 Exercises

1) Give an example, or argue that it is impossible.

impossible. If (a_{n_k}) is bounded subsequence of (a_n), then (a_{n_k}) has a convergent subsequence $(a_{n_{k_r}})$ that is a subsequence of (a_n).

a) Sequence that has a subsequence that is bounded but contains no subsequence that converges

Consider $a_n = \begin{cases} -1 & \text{if } n \text{ is a multiple of } 5, \text{ but not } 10 \\ -2 & \text{if } n \text{ is a multiple of } 10 \\ n & \text{otherwise} \end{cases}$

$a_n = (1,2,3,4,-1,6,7,8,9,-2,11,12,13,14,-1,\ldots)$ and a bounded sequence that does not converge is given by $a_{n_k} = (a_{n_5}, a_{n_{10}}, a_{n_{15}}, \ldots)$. a_{n_k} is bounded by $[-2,-1]$, and it does not converge because we can find two subsequences that converge to different limits: $(a_{n_5}, a_{n_{15}}, a_{n_{25}}, \ldots) \to -1$ and $(a_{n_{10}}, a_{n_{20}}, a_{n_{30}}, \ldots) \to -2$

b) A sequence that does not contain 0 or 1 as a term, but contains subsequences converging to each of these values.

Consider $a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ n/(n+1) & \text{if } n \text{ is odd} \end{cases}$. $a_n = (1/2, 1/2, 3/4, 1/4, 4/5, 1/5, \ldots)$ never contains 0 or 1. ($n/(n+1) = 1 \Rightarrow n = n+1$ FALSE. $n/(n+1) = 0 \Rightarrow n = 0$ FALSE. $1/n = 0 \Rightarrow 1 = 0$ FALSE, $\frac{1}{n} = 1 \Rightarrow n = 1$ FALSE bc n even). However, its subsequences $a_{n_k} = (a_{n_1}, a_{n_3}, a_{n_5}, \ldots) = (1/2, 3/4, 4/5, \ldots)$ and $a_{n_r} = (a_{n_2}, a_{n_4}, \ldots) = (1/2, 1/4, \ldots)$ converge to 1 and 0, respectively.

c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

Consider $(a_n) = (1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, \ldots)$

* can construct as something similar $a_n = \begin{cases} 1 & \text{if prime} \\ \dfrac{1}{\frac{1}{a_{n-1}} + 1} & \text{otherwise} \end{cases}$

d) Sequence with subsequences converging to every point in $\{1, 1/2, 1/3, 1/4, \dots\}$, and no subsequences converging to points outside the set.

Impossible. Proof:

Suppose a_n is a sequence with subsequences converging to every point in $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, and no subsequences converging to points outside the set. Let (a_{n_k}) be a subsequence of (a_n) containing only the subsequences of (a_n) that converge. Now we show that (a_{n_k}) is eventually in $\{x \in \mathbb{R} : x \geq 0\}$. Assume for contradiction that (a_{n_k}) is not eventually in $\{x \in \mathbb{R} : x \geq 0\}$. That is, $\forall K \in \mathbb{N}, \exists k \geq K$ such that $a_{n_k} < 0$. Then we can construct a subsequence purely of negative numbers, which cannot converge to a point in S by the Order Limit Theorem, and so (a_{n_k}) must eventually be in $\{x \in \mathbb{R} : x \geq 0\}$. This means $\exists K_1 \in \mathbb{N}$ such that $\forall k \geq K_1$, $a_{n_k} \geq 0$. Now let $\varepsilon > 0$ and find $m \in \mathbb{N}$ such that $1/m < \varepsilon$. We can split $1/m$ into $1/p + 1/q$ for $p, q \in \mathbb{N}$. Then $\exists K_2 \geq K_1$ such that $\forall k \geq K_2$, $|a_{n_k} - \frac{1}{p}| < \frac{1}{q}$ $\Rightarrow a_{n_k} < 1/q + 1/p = \frac{1}{m} < \varepsilon \Rightarrow a_{n_k} \geq \varepsilon$. And because $k \geq K_1$, $a_{n_k} \leq \varepsilon \Rightarrow |a_{n_k}| < \varepsilon$, so $(a_{n_k}) \to 0$. But this is a contradiction since $0 \notin S$, and (a_{n_k}) is a subsequence of (a_n). Thus, the request is impossible □

2) Decide whether the following are true or false, and justify.

a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

True. Consider the proper subsequence $(x_2, x_3, x_4, \dots) = x_{n+1}$. $(x_{n+1}) \to x$, so $\exists N_1$ s.t $\forall n+1 \geq N_1$, $|x_{n+1} - x| < \varepsilon$. Now let $N = N_1 - 1$. Then $\forall n \geq N$, $|x_n - x| < \varepsilon$, so (x_n) converges □

↳ would have been easier to say $\lim x_{n+1} = x$, and $\lim x_{n+1} = \lim x_n$.

b) If (x_n) contains a divergent subsequence, then (x_n) diverges

True. Let (x_{n_K}) be a divergent subsequence of (x_n). Then $\forall K \in \mathbb{N}, \exists k \geq K$ s.t. $|x_{n_K} - x| \geq \varepsilon$ $\forall x \in \mathbb{R}$. Because this $x_{n_K} \in \{x_n\}$, we can always find a term of (x_n) such that $|x_n - x| \geq \varepsilon$ $\forall x \in \mathbb{R}$. Thus (x_n) diverges $\square$

c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

True. If (x_n) is bounded, then there exists a subsequence $(x_{n_K}) \to x$. If all other subsequences converged to x, then x_n would converge. Thus, at least one other subsequence of x_n must converge to something other than $\lim(x_{n_K})$ $\square$

d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

True. Without loss of generality assume (x_n) increases - so $x_m \geq x_n$ for $m \geq n$. Let (x_{n_K}) be a convergent subsequence. Then (x_{n_K}) is bounded: $|x_{n_K}| \leq M$. Now because $n \leq n_K$, then $x_n \leq x_{n_K} \leq M$, and so (x_n) is bounded as well. Thus (x_n) converges by MCT $\square$

3)

a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \dots$ converges to a limit L (ie $(S_n) \to L$). Then show that any regrouping of the terms $(a_1 + a_2 + \dots + a_{n_1}) + (a_{n_1+1} + \dots + a_{n_2}) + (a_{n_2+1} + \dots + a_{n_3}) + \dots$ leads to a series that converges to L.

Proof:

Assume the infinite series $\sum_{n=1}^{\infty} a_n$ converges to L, so the sequence of partial sums (S_n) also converges to L. Now group the terms in the series as follows: $(a_1 + a_2 + \dots + a_{n_1}) + (a_{n_1+1} + \dots + a_{n_2}) + (a_{n_2+1} + \dots + a_{n_3}) + \dots$
We can rewrite as $S_{n_1} + (S_{n_2} - S_{n_1}) + (S_{n_3} - S_{n_2}) + \dots$ We can write this as the infinite series $\sum_{i=1}^{\infty} (S_{n_i} - S_{n_{i-1}})$, where $S_{n_0} = 0$. Then the sequence of partial sums for this series $Z_m = \sum_{i=1}^{m} (S_{n_i} - S_{n_{i-1}}) = S_{n_m}$.
Because $Z_m = S_{n_m}$, then $(Z_m) \to L \Rightarrow \sum_{i=1}^{\infty} (S_{n_i} - S_{n_{i-1}}) \to L \Rightarrow \sum_{n=1}^{\infty} a_n \to L$ $\square$

b) Why doesn't the proof in (a) apply to $\sum_{n=1}^{\infty} (-1)^n$?

The sequence of partial sums for this series does not converge, so we cannot take the step "Because $Z_m = S_{n_m}$, then $(Z_m) \to L$".

8) Another way to prove Bolzano-Weierstrass is to show every sequence has a monotone subsequence. "Peak term" is useful: Given (x_n), term x_m is a peak term if $x_m \geq x_n \ \forall n \geq m$

a) Find examples of sequences w/ 0, 1, 2 peak terms. Find an example of a sequence with infinitely many peak terms not monotone

0 peak terms: $(1, 2, 3, \ldots)$
1 peak term: $(1, 1/2, 2/3, 3/4, \ldots)$
2 peak terms: $(2, 1, 1/2, 2/3, 3/4, \ldots)$
infinitely many - not monotone: $(1, 0, 1, 0, 1, 0, \ldots)$

b) Show that every sequence has a monotone subsequence, and show how this leads to a proof of Bolzano-Weierstrass.

Proof:
A sequence (x_n) can have no peak terms, a finite number of peak terms, or an infinite number of peak terms. If (x_n) has no peak terms, then $x_m \leq x_n \ \forall n \geq m$, and so it has a monotone subsequence (equal to itself, or really any subsequence). If (x_n) has a finite number of peak terms, then $\exists m$ s.t. x_m is the last peak term. Then define a subsequence $(x_{m+1}, x_{m+2}, \ldots)$ and see that it is monotone. If there are infinitely many peak terms, then construct a subsequence out of all the peak terms, and see that it is also monotone. Therefore, every sequence has a monotone subsequence. If a sequence (x_n) is bounded, then it has a monotone subsequence that is necessarily bounded, and therefore convergent. Thus, every bounded sequence has a convergent subsequence $\square$

9) Let (a_n) be a bounded sequence, and define the set $S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many terms } a_n\}$. Show there exists a subsequence (a_{n_k}) converging to $s = \sup S$.

Proof:

First, note that S has no maximum, because for any $x < a_n$, we can find $x' = x + \varepsilon < a_n$. This means $s > x \ \forall x \in S$. Now define A to be the set of all terms of (a_n) such that $a_n \geq s$. Since $\forall x \in S$, $\forall a \in A$, $x < a$, then by definition of S conclude A is an infinite set. Because A is infinite, we can pick some $a_{n_1} \in A$, and always find a_{n_2} where $n_2 > n_1$. Continuing this process yields a subsequence (a_{n_k}). Now assume for contradiction that (a_{n_k}) does not converge to s. Then $\forall \varepsilon > 0$, $\forall K \in \mathbb{N}$, $\exists k \geq K$ s.t. $|a_{n_k} - s| \geq \varepsilon$. This implies that $a_{n_k} \geq s + \varepsilon$, and we can find a real number $x = s + \varepsilon/2$ so that $s < x < a_{n_k}$. Because we can do this for all natural numbers, then there exists $x \in \mathbb{R}$ s.t. $s < x < a_{n_k}$ for infinitely many a_{n_k}, which means $x \in S$. This is a contradiction because $x \in S$ and $x > s$, so we conclude that (a_{n_k}) must converge $\square$