2.2 The Limit of a Sequence

Definition 2.2.1 (Sequence)
A sequence is a function whose domain is $\mathbb{N}$

- can easily see how a sequence is then depicted as an ordered list of real numbers
- Given $f: \mathbb{N} \to \mathbb{R}$, $f(n)$ is the n^{th} term in the sequence

Ex. 2.2.2 Sequence Notation Examples

(i) $(1, 1/2, 1/3, 1/4, \ldots)$

(ii) $\left(\frac{1+n}{n}\right)_{n=1}^{\infty} = \left(2, \frac{3}{2}, \frac{4}{3}, \ldots\right)$

(iii) (a_n) where $a_n = 2^n \ \forall n \in \mathbb{N}$

(iv) (x_n) where $x_1 = 2$ and $x_{n+1} < \frac{x_n + 1}{2}$

- sequences can sometimes be indexed at $n=0$ or $n=n_0$ instead of 1.

* essential for sequence to be <u>infinite</u>.

Definition 2.2.3 (Convergence of a Sequence)
A sequence (a_n) converges to a real number a if, for every positive number ε, there exists an $N \in \mathbb{N}$ s.t. whenever $n \geq N$, it follows that $|a_n - a| < \varepsilon$.

- To indicate (a_n) converges to a, write $\lim a_n = a$, or $(a_n) \to a$

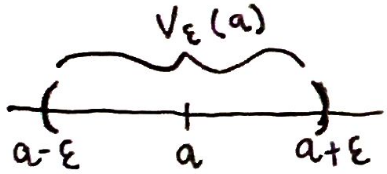
Personal Example: Prove that $(a_n) \to 1$ for $a_n = \frac{1+n}{n}$

Proof:
We seek $N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon$ whenever $n \geq N$. In our case we have $|a_n - 1| < \varepsilon$. Rewrite this as $-\varepsilon < a_n - 1 < \varepsilon \iff 1 - \varepsilon < a_n < 1 + \varepsilon$ $\iff 1 - \varepsilon < \frac{1+n}{n} < 1 + \varepsilon \iff n - n\varepsilon < 1 + n < n + n\varepsilon \iff -n\varepsilon < 1 < n\varepsilon$. Now choose $n > \frac{1}{\varepsilon}$. This leads to $n\varepsilon > 1$ and $-n\varepsilon < 1$, so the inequality $-n\varepsilon < 1 < n\varepsilon$ $\iff |a_n - 1| < \varepsilon$ is true for $n \geq N > 1/\varepsilon$ and thus a_n converges to 1 $\square$

Definition 2.2.4 (ε-neighborhood):

Given $a \in \mathbb{R}$ and $\varepsilon > 0$, the set $V_\varepsilon(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$ is called the ε-neighborhood of a.

- in other words, $V_\varepsilon(a)$ is an interval centered at a w/ radius ε.



- Recasting Def 2.2.3 in terms of ε-neighborhoods gives a better geometric impression

Definition 2.2.3B (Topological Convergence of a sequence):

A sequence (a_n) converges to a if, given any $V_\varepsilon(a)$ of a, there exists a point in the sequence after which all the terms are in $V_\varepsilon(a)$.

- in other words, every ε-neighborhood contains all but a finite number of the terms (a_n)

Ex 2.2.5 Convergence of $a_n = \frac{1}{\sqrt{n}}$

interesting bc short & sweet, but does not follow problem-solving process

Claim that $\lim(1/\sqrt{n}) = 0$.

Proof:

Let $\varepsilon > 0$. Choose N satisfying $N > 1/\varepsilon^2$. Now verify that this choice of N has the desired property. Let $n \geq N$, then $n > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon$ and hence $|a_n - 0| < \varepsilon$ $\square$

Template for Convergence Proof:

1) Let $\varepsilon > 0$ be arbitrary
2) Demonstrate choice of $N \in \mathbb{N}$. Requires work to find - done before formal proof.
3) Assume $n \geq N$
4) Derive $|x_n - x| < \varepsilon$

Ex 2.2.6 Revisiting $\lim\left(\frac{1+n}{n}\right)=1$

Proof:

Let $\varepsilon>0$ be arbitrary. Choose $N>1/\varepsilon$. Now assume $n\geq N$, then $n>\frac{1}{\varepsilon} \Rightarrow \frac{1}{n}<\varepsilon$. Since $\frac{1}{n}=\frac{n+1}{n}-1$, then $\frac{n+1}{n}-1<\varepsilon$. Similarly, $n>\frac{1}{\varepsilon} \Rightarrow \frac{1}{n}<\varepsilon \Rightarrow \frac{1}{n}>-\varepsilon \Rightarrow \frac{n+1}{n}-1>-\varepsilon$. So $\left|\frac{n+1}{n}-1\right|<\varepsilon$ □

Theorem 2.2.7 (Uniqueness of Limits)

The limit of a sequence, when it exists, must be unique.

Proof:

Suppose (a_n) is an arbitrary sequence, and $\lim(a_n)=a$ and $\lim(a_n)=b$. Then we have $V_\varepsilon(a)$ and $V_\varepsilon(b)$. If we can show that $V_\varepsilon(a)=V_\varepsilon(b)$, then $a=b$ and we are done. Suppose $V_\varepsilon(a)\nsubseteq V_\varepsilon(b)$. Then there exists a set $A=V_\varepsilon(a)\setminus V_\varepsilon(b)\neq\emptyset$. Because $V_\varepsilon(a)$ is infinite, A is also potentially infinite. If this the case, then an infinite number of terms of (a_n) exist outside of $V_\varepsilon(b)$, which violates the definition of $V_\varepsilon(b)$. Thus, $V_\varepsilon(a)\subseteq V_\varepsilon(b)$. Through identical reasoning we conclude that $V_\varepsilon(b)\subseteq V_\varepsilon(a)$, and therefore $V_\varepsilon(a)=V_\varepsilon(b)$. This means the two ε-neighborhoods must be centered at the same point $a=\lim(a_n)=b$, and so the limit of a sequence is unique □

Divergence

Definition 2.2.9 (Divergence)

A sequence that does not converge is said to diverge

- details on how to argue divergence left for 2.5

2.2 Exercises

1) Definition: A sequence (x_n) vercongés to x if $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \varepsilon$.
Give, an example of a vercongent sequence. Is there a divergent vercongent sequence? Can a sequence vercongé to 2 dif values? What is being described here?

$a_n = \frac{1}{2^n} = (1/2, 1/4, 1/8, \ldots)$ is vercongent to $1/2$. Let $\varepsilon = 1/2$. It is true that $|x_n - 1/2| < 1/2$ $\forall N \in \mathbb{N}$ and thus $\forall n \geq N$.

The sequence $(1, -1/2, 1/3, -1/4, 1/4, -1/5, 1/5, -1/5, 1/5, \ldots)$ is a divergent sequence vercongent to $x = 1$. Let $\varepsilon = 1.6$. It is true that $|x_n - 1| \leq 1.5 < 1.6$ $\forall N \in \mathbb{N}$ and thus $\forall n \geq N$.

Yes - any vercongent sequence vercongés to more than 1 value. This is because vercongent sequences describe bounded sequences.

2) Verify that the following sequences converge to the proposed limit

a) $\lim \left(\frac{2n+1}{5n+4}\right) = \frac{2}{5}$ * caused a lot of headache, so going to show whole process.

Need to show $|a_n - 2/5| < \varepsilon \Rightarrow \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \varepsilon$. $\frac{2n+1}{5n+4} - \frac{2}{5} = \frac{-3}{25n+20} = \frac{-3/5}{5n+4}$.

So $\left|\frac{-3/5}{5n+4}\right| < \varepsilon \Rightarrow \left|\frac{3/5}{5n+4}\right| < \varepsilon \Rightarrow \frac{3}{5\varepsilon} < 5n+4$ choose N to make this true.

← proof starts here

Then $\forall n > N$, $3/5\varepsilon < 5n+4 \Rightarrow \left|\frac{3/5}{5n+4}\right| < \varepsilon \Rightarrow \left|\frac{2}{5} - \frac{3/5}{5n+4} - \frac{2}{5}\right| < \varepsilon$

$\Rightarrow \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \varepsilon$ $\square$

b) $\lim\left(\frac{2n^2}{n^3+3}\right)=0$

Proof:

Choose $N > 2/\varepsilon$, so $\forall n \geq N$, $n > 2/\varepsilon \Rightarrow \frac{2}{n} < \varepsilon$. Notice that $\frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} < \frac{2}{n}$, so $\left|\frac{2n^2}{n^3+3}\right| < \left|\frac{2}{n}\right| < \varepsilon$ and therefore $\left|\frac{2n^2}{n^3+3}\right| < \varepsilon$ $\square$

c) $\lim\left(\frac{\sin(n^2)}{\sqrt[3]{n}}\right)=0$

Proof:

Choose $N > 1/\varepsilon^3$ for some arbitrary ε. Then $\forall n \geq N$, $n > 1/\varepsilon^3$. This implies $n^{1/3} > 1/\varepsilon \Rightarrow 1/n^{1/3} < \varepsilon$. Notice that $\left|\frac{\sin(n^2)}{n^{1/3}}\right| \leq 1/n^{1/3} < \varepsilon$, so $\left|\frac{\sin(n^2)}{n^{1/3}}\right| < \varepsilon$ $\square$

4) Give example or show why impossible

a) Sequence with infinite number of ones that does not converge to one

Let $a_n = (-1)^n = (-1, 1, -1, 1, -1, 1, \ldots)$

b) Sequence with infinite number of ones that does converge to a limit not equal to 1.

Impossible. Suppose we have a sequence a_n with an infinite number of ones that converges to $a \neq 1$. Then $|a_n - a| < \varepsilon$ must be true for all a_n. Since a_n could equal 1 for any n, we must always be able to satisfy $|1-a| < \varepsilon$. But $|1-a| > 0$, so if we let $\varepsilon = |1-a|$, then $|1-a| \not< \varepsilon$, and the request is impossible $\square$

c) A divergent sequence such that $\forall n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Let $a_n = \begin{cases} n & \text{if } \sqrt{n} \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$

this works with primes too bc there are arbitrarily large gaps b/w primes

This works because the gap between two consecutive squares n^2 and $(n+1)^2$ is $2n+1$. So if we want m consecutive ones, find $n \geq \frac{m-1}{2}$ and there you will have at least m consecutive ones.

7) Here are 2 useful definitions:

(i) A sequence (a_n) is eventually in a set $A \subseteq \mathbb{R}$ if $\exists N \in \mathbb{N}$ s.t. $a_n \in A$ $\forall n \geq N$

(ii) A sequence (a_n) is frequently in a set $A \subseteq \mathbb{R}$ if, $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. $a_n \in A$.

a) Is $a_n = (-1)^n$ eventually or frequently in the set $\{1\}$?

Frequently. For any $N \in \mathbb{N}$, if $a_n \notin A$ for $n = N$, then it is guaranteed that $a_{n+1} \in A$. However, for any $N \in \mathbb{N}$ I can do the opposite, and find an $a_n \notin A$ for $n \geq N$. So a_n is not eventually in A.

b) Which definition is stronger?

Eventually implies frequently.

c) Give alternate phrasing to Definition 2.2.3B w/ one of these terms

Def: A sequence a_n converges to a if, given any $V_\varepsilon(a)$, a_n is eventually in $V_\varepsilon(a)$.

d) Suppose an infinite number of terms of sequence x_n equal to 2. Is (x_n) necessarily eventually in $(1.9, 2.1)$ or frequently in $(1.9, 2.1)$?

Frequently, just like (a), let $x_n = (-1)^n(2) = (-2, 2, -2, 2, \ldots)$.

8) Definition: A sequence (x_n) is zero-heavy if $\exists M \in \mathbb{N}$ s.t $\forall N \in \mathbb{N}$ there exists n satisfying $N \leq n \leq N+M$ where $x_n = 0$

a) Is $(0,1,0,1,0,1,0,\ldots)$ zero-heavy?

Yes, let $M=1$. Then in the interval $[a_n, a_{n+1}]$ we can always find a zero. Thus the sequence is zero-heavy.

b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide counterexample.

Yes. Proof:

Let (a_n) be a zero-heavy sequence. We can show that (a_n) has an infinite number of zeros by partitioning it into an infinite number of disjoint intervals, each with at least one zero. This is done by induction. Consider the intervals $[a_1, a_{1+M}]$ and $[a_{2+M}, a_{2+2M}]$, where $M \in \mathbb{N}$ s.t. the criterion of zero-heavy is satisfied for (a_n). It is obvious that the two intervals are disjoint because the lowest index of the second interval is greater than the highest index of the first. $[a_1, a_{1+M}]$ also contains a term equaling 0 by definition of zero-heavy, and so does $[a_{2+M}, a_{2+2M}]$ if we let $N = 2+M$. This is the base case. Now suppose $[a_{N+KM+K}, a_{N+(K+1)M+K}]$ is an interval of terms of (a_n) such that the zero term it contains is distinct from all prior intervals (ie those with indices less than $N+KM+K$), and $N, K, M \in \mathbb{N}$ where M lets the interval contain a zero by the zero-heavy property. We aim to show that $[a_{N+(K+1)M+K+1}, a_{N+(K+2)M+K+1}]$ contains its own distinct zero. Because $N+(K+1)M+K+1 > N+KM+K$, this interval is disjoint from $[a_{N+KM+K}, a_{N+(K+1)M+K}]$. If we rearrange the indices as follows: $[a_{\underbrace{(N+(K+1)M+K+1)}_{\text{"}N\text{"}}}, a_{\underbrace{(N+(K+1)M+K+1)}_{\text{"}N\text{"}}+M}]$, it becomes apparent that the interval contains a zero. Thus, there are infinitely many disjoint intervals that contain zeros, so the sequence has infinitely many zeros $\square$

This proof is really messy but the idea is SOLID. Better if by contradiction

c) If a sequence has infinite zeros, is it necessarily zero-heavy. If not, provide counterexample.

Consider $x_n = \begin{cases} 0 & \text{if } \sqrt{n} \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$

(x_n) has infinite zeros because there are infinite squares. However, (x_n) is not zero-heavy. Suppose (x_n) was zero-heavy. Then $\exists M \in \mathbb{N}$ s.t. $\forall N \in \mathbb{N}$ $\exists n$ in $N \leq n \leq N+M$ s.t. $x_n = 0$. Because our sequence only has $x_n = 0$ when n is a square, M must be at least the size of a gap between squares. However, the gap between squares can be arbitrarily large ($2n+1$ for the gap between n^2 and $(n+1)^2$), so if $M = 2k+1$, then $M < 2(k+1)+1$, and we can find two zeros in the sequence that are at least $2(k+1)+1$ terms apart with no zeros in between. Thus, (x_n) cannot be zero-heavy $\square$

d) Logically negate the definition of zero-heavy.

A sequence is not zero-heavy if for all $M \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all n satisfying $N \leq n \leq N+M$, $x_n \neq 0$.