5.3 The Mean Value Theorems

- Mean value theorem comes in varying forms of generality
  - most specific is Rolle's Theorem

Theorem 5.3.1 (Rolle's Theorem)
Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then $\exists c \in (a,b)$ where $f'(c) = 0$

Proof:
Let f be differentiable on (a,b) and continuous on $[a,b]$, and let $f(a) = f(b)$. By the extreme value theorem, f attains a maximum and minimum on $[a,b]$. If $f(a)$ and $f(b)$ are the maximum and minimum, then f is constant, and so $\forall c \in (a,b)$ $f'(c) = 0$. If, WLOG, $f(a)$ is not the maximum, then neither is $f(b)$, and so $\exists c \in (a,b)$ s.t. $f'(c) = 0$ $\square$

Theorem 5.3.2 (Mean Value Theorem)
If $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ where $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

Proof:
Define $g(x) = (b-a)f(x) - (f(b) - f(a))x$. g is still continuous on $[a,b]$ and differentiable on (a,b). Notice $g(a) = g(b)$. So by Rolle's Theorem, $\exists c \in (a,b)$ s.t. $g'(\;) = 0$. $g'(c) = (b-a)f'(c) - (f(b) - f(a)) = 0$
$\Rightarrow f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $\square$

★ The MVT is very important - it makes its way into nearly every proof related to the geometric nature of the derivative

Corollary 5.3.3: If $g: A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ $\forall x \in A$, then $g(x) = k$ for some $k \in \mathbb{R}$

Proof:
Let $a, b \in A$ s.t. $a < b$ WLOG. By the MVT, $\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a} = 0 \Rightarrow f(b) = f(a) = k$. Because we showed for any two arbitrary points $a, b \in A$, $f(a) = f(b) = k$, then $f(x) = k$ $\forall x \in K$.
(Just fix a, and then $f(x) = f(a) = k$ $\forall x \in A$) $\square$

Corollary 5.3.4: If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ $\forall x \in A$, then $f(x) = g(x) + k$ for some $k \in \mathbb{R}$.

Proof:
Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$. So by corollary 5.3.3, $h(x) = f(x) - g(x) = k \Rightarrow f(x) = g(x) + k$ $\square$

- Even more generalized form of MVT exists that is needed for L'Hospital's rules and Lagrange's Remainder Theorem.

Theorem 5.3.5 (Generalized Mean Value Theorem)
If f and g are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), $\exists c \in (a,b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a,b) then the conclusion can be stated as: $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Proof: Exercise 5.3.5

Theorem 5.3.6 (L'Hospital's Rule: 0/0 Case)

Let f, g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ $\forall x \neq a$, then $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L$

Proof: Exercise 5.3.11

Definition 5.3.7: Given $g: A \to \mathbb{R}$ and a limit point c of A, we say $\lim_{x \to c} g(x) = \infty$ if, $\forall M > 0$, $\exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow g(x) \geq M$.

Theorem 5.3.8 (L'Hospital's Rule: ∞/∞ Case)

Assume f and g are differentiable on (a, b) and $g'(x) \neq 0$ $\forall x \in (a, b)$. If $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)}$.

Proof:

Let $\varepsilon > 0$. $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ implies $\exists \delta_1 > 0$ s.t. $\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon/2$ $\forall a < x < a + \delta_1$. Let $t = a + \delta_1$. For $x \in (a, t)$, apply the Generalized MVT on $[x, t]$ to get $\frac{f(x) - f(t)}{g(x) - g(t)} = \frac{f'(c)}{g'(c)}$ for some $c \in (x, t)$. Because $c < t$, we have

$L - \varepsilon/2 < \frac{f(x) - f(t)}{g(x) - g(t)} < L + \varepsilon/2$ $\forall x \in (a, t)$. To isolate $f(x)/g(x)$, we can multiply by $(g(x) - g(t))/g(x)$, but we have to make sure this quantity is positive. In other words, $1 - g(t)/g(x) \geq 0$. Because $\lim_{x \to a} g(x) = \infty$, $\exists \delta_2 > 0$ s.t. $g(x) \geq g(t)$ $\forall a < x < a + \delta_2$. Multiplication yields

$$L - \varepsilon/2 + \frac{-Lg(t) + \varepsilon/2\, g(t) + f(t)}{g(x)} < \frac{f(x)}{g(x)} < L + \varepsilon/2 + \frac{-Lg(t) + \varepsilon/2\, g(t) + f(t)}{g(x)}$$

We choose a δ_3 so that $a < x < a + \delta_3$ implies $\left| \frac{-Lg(t) + \varepsilon/2\, g(t) + f(t)}{g(x)} \right| < \varepsilon/2$, and then $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ ensures $\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$ $\forall a < x < a + \delta$ $\square$

5.3 Exercises

1. Recall $f: A \to \mathbb{R}$ is Lipschitz on A if $\exists M > 0$ such that
$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq M \quad \forall x \neq y \in A$$

a) Show that if f is differentiable on $[a,b]$ and if f' is continuous on $[a,b]$, then f is Lipschitz on $[a,b]$

Proof:
Because f' is continuous on the compact set $[a,b]$, it attains a maximum and minimum value at x_0 and $x_1 \in [a,b]$. Let $M = \max\{|f'(x_0)|, |f'(x_1)|\}$. It follows that $|f'(x)| \leq M$ for all $x \in [a,b]$. Now consider any $x, y \in [a,b]$. By the MVT, $\exists c \in (y,x)$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$ (assuming WLOG $x > y$).
But $|f'(c)| \leq M \ \forall c \in [a,b]$, so $\left|\frac{f(x)-f(y)}{x-y}\right| \leq M$ and f is Lipschitz $\square$

2. Let f be differentiable on A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show the converse need not be true.

Proof by Contrapositive:
Suppose f is not one-to-one on A. Then we can find some $x \neq y \in A$ ($x < y$ WLOG) such that $f(x) = f(y)$. Then by Rolle's Theorem $\exists c \in (x,y)$ such that $f'(c) = 0$. This shows that $f'(x) = 0$ for at least one $x \in A$. Because the contrapositive is true, the original statement must also be true.

Converse example: $f(x) = x^2$ on $A = [0,1]$. $f'(0) = 0$.

3. Let h be differentiable and defined on $[0,3]$, and assume $h(0)=1$, $h(1)=2$, $h(3)=2$.

a) Argue $\exists d \in [0,3]$ where $h(d)=d$

Let $g(x)=h(x)-x$ on $[0,3]$. g is continuous and $g(1)=1$, $g(3)=-1$, so by IVT $\exists d \in (1,3)$ s.t. $g(d)=0 \Rightarrow h(d)=d$.

b) Argue $\exists c \in [0,3]$ s.t. $h'(c)=1/3$.

Because $\frac{h(3)-h(0)}{3-0}=\frac{1}{3}$, by the MVT $\exists c \in (0,3)$ s.t. $h'(c)=1/3$

c) Argue $h'(x)=1/4$ at some point in the domain

Because $\frac{h(3)-h(1)}{2}=0$, by MVT $\exists a \in (1,3)$ s.t. $h'(a)=0$. From (b) we have $b \in (0,3)$ s.t. $h'(b)=1/3$. Assume WLOG $a<b$. Because h is differentiable on $[a,b]$, and $h'(a)<1/4<h'(b)$, $\exists c \in (a,b)$ s.t. $h'(c)=1/4$.

4. Let f be differentiable on interval A containing 0, and assume $(x_n) \subseteq A$ with $(x_n) \to 0$ and $x_n \neq 0$

a) If $f(x_n)=0$ $\forall n \in \mathbb{N}$, show $f(0)=0$ and $f'(0)=0$

Because $f(x_n)=0$ $\forall n \in \mathbb{N}$, it is trivial that $f(x_n) \to 0$. And because f is continuous at 0, $f(x_n) \to f(0)$, so $f(0)=0$. $f'(0)=\lim_{x\to 0} f(x)/x = L$, and by the sequential criterion for functional limits, because $(x_n) \subseteq A$, $x_n \neq 0$ and $(x_n) \to 0$, $(f(x_n)/x_n) \to L$. But $f(x_n)/x_n = 0$ $\forall n \in \mathbb{N}$, so $\lim_{x\to 0} f(x)/x = 0$ and $f'(0)=0$.

(a)
5) Prove the Generalized Mean Value Theorem

Proof:

Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b). Define $h(x) = [f(b)-f(a)]g(x) - [g(b)-g(a)]f(x)$. Notice that
$h'(x) = [f(b)-f(a)]g'(x) - [g(b)-g(a)]f'(x)$,
$h(a) = f(b)g(a) - f(a)g(b)$, and $h(b) = f(b)g(a) - f(a)g(b)$. Then $[h(b)-h(a)]/b-a = 0$, and so by the MVT, $\exists c \in (a,b)$ such that
$h'(c) = [f(b)-f(a)]g'(c) - [g(b)-g(a)]f'(c) = 0$
$\Rightarrow [f(b)-f(a)]g'(c) = [g(b)-g(a)]f'(c)$ $\square$

8) Assume f is continuous on an interval containing 0 and differentiable $\forall x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show $f'(0)$ exists & equals L

Proof:
$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x}$. Notice that because f is continuous, $\lim_{x\to 0} f(x) = f(0)$, so the limit has indeterminate form $0/0$. Apply L'Hospital's Rule to get $f'(0) = \lim_{x\to 0} f'(x) = L$ $\square$

9) Assume f and g are as described in Theorem 5.3.6, but now f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Prove L'Hospital's $0/0$ rule under this stronger hypothesis.

Proof:
Suppose f and g are continuous on an interval $A \ni a$ and differentiable at a. Also, f' and g' are continuous at a, $g'(a) \neq 0$, $f(a) = g(a) = 0$, and $g'(x) \neq 0$ $\forall x \in A$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then $L = f'(a)/g'(a)$ by continuity. By definition of derivative, $f'(a)/g'(a)$ simplifies to $\lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$. Thus, $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ implies $\lim_{x\to a} f(x)/g(x) = L$ $\square$

10) Let $f(x) = x\sin(1/x^4)e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. Compute $\lim_{x\to 0}$ of $f(x)$, $g(x)$, $f(x)/g(x)$, $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with L'Hospital's Rule (Thm 5.3.6)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}(x\sin(1/x^4)) \cdot \lim_{x\to 0} e^{-1/x^2} = 0\cdot 0 = 0$$

$\lim_{x\to 0} g(x) = 0$ $\quad \lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

$f'(x) = 2x^{-3}e^{-x^{-2}}\left[x\sin(1/x^4) + \sin(1/x^4) - 2x^{-1}\cos(1/x^4)\right]$

$g'(x) = 2x^{-3}e^{-x^{-2}}$

$\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist. This does not contradict Thm 5.3.6 because $\lim_{x\to 0} f(x)/g(x)$ can be vacuously true. That is, $\lim_{x\to 0} f'(x)/g'(x)$ can be false, yet $\lim_{x\to 0} f(x)/g(x) = L$ can still be true.

11. a) Use the Generalized MVT to prove Theorem 5.3.6

Proof:

Let f and g be continuous on an interval $A \ni a$, and assume f and g are differentiable on A but not necessarily at a. Suppose $f(a) = g(a) = 0$ and $g'(x) \neq 0$ $\forall x \neq a$, and $\lim_{x\to a} f'(x)/g'(x) = L$. By definition of functional limits, for $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f'(x)/g'(x) - L| < \varepsilon$. By the Generalized MVT, $\exists c \in (a,x)$ (or $c \in (x,a)$) s.t. $f'(c)/g'(c) = f(x)/g(x)$. Then because $|c-a| < \delta$, it follows that $|f(x)/g(x) - L| < \varepsilon$. And thus $\lim_{x\to a} f(x)/g(x) = L$ $\square$