1.3 Axiom of Completeness

Initial Definition of $\mathbb{R}$

- informally, fills the gaps of $\mathbb{Q}$
- this "filling of gaps" given by axiom of completeness

Axiom of Completeness

- Every nonempty set of real numbers that is bounded above has a least upper bound

Least Upper Bound (LUB) and Greatest Lower Bound (GLB)

Definition: "bounded above/below" "upper/lower bound"

- A set $A \subseteq \mathbb{R}$ is bounded above if $\exists b \in \mathbb{R}$ s.t. $a \leq b \ \forall a \in A$.
  - this b is called an upper bound
- A set $A \subseteq \mathbb{R}$ is bounded below if $\exists b \in \mathbb{R}$ s.t. $b \leq a \ \forall a \in A$

Definition: "Least Upper Bound" = "supremum"

- $s \in \mathbb{R}$ is the least upper bound of $A \subseteq \mathbb{R}$ if
  (i) s is an upper bound for A
  (ii) if b is any upper bound for A, $s \leq b$

Definition: "Greatest Lower Bound" = "infimum"

- $s \in \mathbb{R}$ is greatest lower bound of $A \subseteq \mathbb{R}$ if
  (i) s is lower bound for A
  (ii) if b is any lower bound for A, $b \leq s$

- The sup and inf for a set are unique
  Proof: Let s_1 and s_2 be $\sup A$ for some $A \subseteq \mathbb{R}$. Then $s_1 \leq s_2$ and $s_2 \leq s_1$, so $s_1 = s_2$ $\square$

Ex 1.3.3 - Finding a supremum

Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$

Some upper bounds include, $3, 2, 3/2$. But what is $\sup A$?
$\sup A = 1$. Proof: 1 is an upper bound because $1 \geq 1/n \ \forall n \in \mathbb{N}$.
Let b be an upper bound. Then $1 \leq b$ by definition of upper bound ($1 \in A$). So because 1 is an upper bound and $1 \leq b$ for all upper bounds b, $1 = \sup A$ $\square$

* Note that $\sup A$ and $\inf A$ may or may not be elements of A.

Definition: Maximum of a set / Minimum of a set
- $a_0 \in \mathbb{R}$ is a maximum of the set A if $a_0 \in A$ and $a_0 \geq a \ \forall a \in A$.
- $a_1 \in \mathbb{R}$ is a minimum of the set A if $a_1 \in A$ and $a_1 \leq a \ \forall a \in A$

Ex 1.3.5 Max/Min vs sup/inf

Consider $(0,2) = \{x \in \mathbb{R} : 0 < x < 2\}$ and $[0,2] = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$.
Both sets bounded above and below ($\sup = 2$, $\inf = 0$). However, $(0,2)$ does not have a maximum or a minimum - cannot find any elements in the set that satisfy the requirements.
* Supremum can exist and not be a maximum, but when a maximum exists, it is the supremum

Why the Axiom of Completeness does not apply to $\mathbb{Q}$

Ex 1.3.6

Consider $S = \{r \in \mathbb{Q} : r^2 < 2\}$. Nonempty set that is bounded above, but what is LUB (world consists only of rationals)?

$b = 1.4$ is upper bound, but not LUB, bc $b = 1.41$ is better, and $b = 1.414$ is better, and $b = 1.4142$ even better... so no LUB exists bc whenever one is proposed, a better one can be found.

Ex 1.3.7 An algebraic property of supremums

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define $c + A = \{c + a : a \in A\}$.

Then $\sup(c+A) = c + \sup A$.

Proof: Let $s = \sup A$. Then $a \leq s \ \forall a \in A$, and $a + c \leq s + c$. So $s + c$ is an upper bound on $c + A$. Let b be an upper bound on $c + A$. Then $a + c \leq b \ \forall a \in A$, and $a \leq b - c$. Because $s = \sup A$, $s \leq b - c$, and $s + c \leq b$, so because we have shown that $s + c$ is an upper bound on $c + A$, and $s + c \leq b$ for all upper bounds b on $c + A$, by definition of supremum $\sup(c+A) = c + \sup A$ $\square$

- Part (ii) of the definition of LUB ($s \leq b \ \forall$ upper bounds b) can be restated in a potentially more useful way

Lemma 1.3.8: Assume $s \in \mathbb{R}$ is upper bound for $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if, $\forall \varepsilon > 0$, $\exists\, a \in A$ satisfying $s - \varepsilon < a$

My Proof:

($\Rightarrow$): Suppose $s = \sup A$ and $\exists\, \varepsilon > 0$ s.t $\forall a \in A$, $s - \varepsilon \geq a$. Then $s - \varepsilon$ is an uppor bound on A, but $s - \varepsilon < s$, so s cannot be a supremum. This contradicts the original assumption that s is a supremum.

($\Leftarrow$): Suppose $\forall \varepsilon > 0$, $\exists\, a \in A$ s.t. $s - \varepsilon < a$, and $s \neq \sup A$. If $s \neq \sup A$, then $s > b$ for some upper bound b. Choose $\varepsilon_0 = s - b$. Then $s - \varepsilon_0 = b \Rightarrow s - \varepsilon_0 \geq a$ by definition of upper bound. This contradicts the initial assumption that $\forall \varepsilon > 0$, $\exists\, a \in A$ s.t. $s - \varepsilon < a$ $\square$

1.3 Exercises

2) Give an example or show request is impossible

a) a set B with $\inf B \geq \sup B$

$B = \{1\}$ $\inf B = 1$, $\sup B = 1$ $\inf B = \sup B$

b) finite set that contains infimum but not supremum

Impossible

Key is finite. If the set is finite, then we can find a minimum and a maximum. Since min/max exist, then $\inf = \min$ and $\sup = \max$, and both inf and sup must then be in the set.

c) Bounded subset of $\mathbb{Q}$ that contains supremum but not infimum

$A = \{x \in \mathbb{Q} : 0 < x \leq 1\}$ $\inf A = 0 \notin A$, $\sup A = 1 \in A$

3)

a) Let A be nonempty & bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is lower bound for } A\}$. Show $\sup B = \inf A$

By definition of infimum part (i), inf A is a lower bound on A, so $\inf A \in B$. By definition of infimum (ii), $\inf A \geq b \ \forall b \in B$. Because $\inf A \in B$ and $\inf A \geq b \ \forall b \in B$, inf A is a maximum of B. The maximum of a set equals the supremum of a set, so $\inf A = \sup B$ $\square$

b) Use (a) to explain why there is no need to assert that infimums exist as part of the Axiom of Completeness.

The infimum of a set A is equivalent to the supremum of a set B of all lower bounds on A.

10) The Cut Property of $\mathbb{R}$ is the following:
If A and B are nonempty, disjoint sets w/ $A \cup B = \mathbb{R}$ and $a < b$ $\forall a \in A$ and $\forall b \in B$, then $\exists c \in \mathbb{R}$ s.t $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

a) Use axiom of completeness to prove the cut property
Proof:
If $A \cup B = \mathbb{R}$ and $a < b$ $\forall a \in A, b \in B$, then A is bounded above and B is bounded below, and both are nonempty, so $\sup A$ and $\inf B$ exist. Let $t = \sup A$ and $s = \inf B$. Aim to show that $s = t$.
Suppose $s > t$. Then $s - t = \varepsilon_0 \Rightarrow s - \varepsilon_0 = t \Rightarrow t < s - \varepsilon_0/2 < s$.
This implies that $s - \varepsilon_0/2 \in \mathbb{R}$ that is greater than $\sup A$ and less than $\inf B$, which means $s - \varepsilon_0/2 \notin A$ and $s - \varepsilon_0/2 \notin B$.
This is not possible because $A \cup B = \mathbb{R}$. So our original assumption that $s > t$ is wrong. Now suppose $s < t$. By identical reasoning, we arrive at a contradiction when we derive a number $t - \varepsilon_0/2 \notin A$ and $t - \varepsilon_0/2 \notin B$. So s, t exist but $s \not< t$ and $s \not> t$, so $\sup A$ must equal $\inf B$. That is, we have one number $c = \sup A = \inf B$ that satisfies $x \leq c$ when $x \in A$ and $x \geq c$ when $x \in B$ $\square$

b) Now assume $\mathbb{R}$ possesses cut property and let E be a nonempty set bounded above. Prove sup E exists.

Proof:

Since E is bounded above, let F be a nonempty set such that $e < f$ $\forall e \in E$ and $f \in F$, and $E \cup F = \mathbb{R}$. Then by the Cut property $\exists c \in \mathbb{R}$ s.t. $c \geq e$ $\forall e \in E$ and $c \leq f$ $\forall f \in F$. Since $c \geq e$ $\forall e \in E$, c is an upper bound on E. $\{c\} \cup F$ represents all the upper bounds on E, since $e \leq x$ $\forall x \in \{c\} \cup F$ and $\forall e \in E$. $c \leq x$ $\forall x \in \{c\} \cup F$, that is, c is less than or equal to all the upper bounds on E. Therefore, $c = \sup E$ and $\sup E$ exists $\square$

c) Cut Property could be used in place of Axiom of Completeness as axiom that defines $\mathbb{R}$ from $\mathbb{Q}$. Give an example showing that the Cut Property is not valid when $\mathbb{R}$ is replaced w/ $\mathbb{Q}$.

Let $A = \{a \in \mathbb{Q} : a^2 < 2\}$ and $B = \{b \in \mathbb{Q} : b^2 > 2\}$. Then A and B are nonempty, disjoint sets such that $A \cup B = \mathbb{Q}$. By the Cut Property for rational numbers, $\exists c \in \mathbb{Q}$ such that $c \geq a$ for $a \in A$ and $c \leq b$ for $b \in B$. Because A is open, it has no maximum, so $c \notin A$. Because B is open, it has no minimum, so $c \notin B$. This is a contradiction: c cannot be in $\mathbb{Q}$ but neither in A nor B, because $A \cup B = \mathbb{Q}$. Thus, no such c can exist and the Cut Property does not hold for rational numbers.

11) Decide True/False. Prove if true, give counterexample if False.

a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

True. Proof:
Let $s = \sup A$ and $t = \sup B$, and suppose $s > t$. Let $\varepsilon_0 = s - t$. By the definition of supremum, $\exists a \in A$ s.t. $a > s - \varepsilon_0/2$. But because $t < s - \varepsilon_0/2 < a < s$, $a \notin B$. This is a contradiction of the fact that $A \subseteq B$, so our supposition that $s > t$ is false and therefore $\sup A \leq \sup B$ $\square$

b) If $\sup A < \inf B$ for sets A and B, then $\exists c \in \mathbb{R}$ s.t. $a < c < b$ for all $a \in A$, $b \in B$.

True. Proof:
If $\sup A = s$ is less that $\inf B = t$, then $a \leq s < t \leq b$ for all $a \in A$, $b \in B$. Let $\varepsilon = t - s$. Then $a \leq s < t - \varepsilon/2 < t \leq b$, which implies $a < c < b$ where $c = t - \varepsilon/2$ $\square$

c) If $\exists c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$, $b \in B$, then $\sup A < \inf B$.

False. Let $A = (0,1)$ $B = (1,2)$. Then $a < 1 < b$ for all $a \in A$ and $b \in B$, but $\sup A = \inf B = 1$.