6.6 Taylor Series

- On its interval of convergence, a power series is continuous and infinitely differentiable
- We know that some infinitely differentiable functions like $\arctan(x)$ and $\sqrt{1+x}$ can be represented by power series

* Do all infinitely differentiable functions of calculus have representations as power series?

* How can we find/construct a power series representation for a given function (assuming such a representation exists)?

Example 6.6.1

Given $\frac{1}{1-x} = 1+x+x^2+x^3+\ldots \Rightarrow \frac{1}{1+x^2} = 1-x^2+x^4-x^6+\ldots$ for $|x|<1$
and $(\arctan(x))' = \frac{1}{1+x^2}$, we can use term-by-term antidifferentiation
and get $\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \quad \forall x \in (-1,1)$ (also $x = \pm 1$)
Similar methods result in power series for $\log(1+x)$ and $x/(1+x^2)^2$

- Manipulating existing series to produce new ones is fine, but can we build one from scratch if given a function?

Theorem 6.6.2 (Taylor's Formula)

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ be defined on some nontrivial interval centered at 0. Then $a_n = \frac{f^{(n)}(0)}{n!}$

Proof Exercise 6.3.3

- Let's use Taylor's formula for $f(x) = \sin(x)$.

$a_0 = \sin(0) = 0 \quad a_1 = \cos(0)/1 = 1 \quad a_2 = -\sin(0)/2 = 0 \quad a_3 = -\cos(0)/3! = -1/3!$
$a_4 = \sin(0)/4! = 0 \quad a_5 = \cos(0)/5! \ldots \quad \sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \ldots$

← can only use "equals" bc we assumed $f(x) = \sum_{n=0}^{\infty} a_n x^n$ in the 1st place.

- The question we want to answer is the converse
  - If f is infinitely differentiable in a neighborhood of 0 and we let $a_n = \frac{f^{(n)}(0)}{n!}$, does $\sum_{n=0}^{\infty} a_n x^n = f(x)$?
    ↳ Is it possible for $\sum a_n x_n$ to converge to the wrong thing?
- Let $S_N(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N$. We want to know whether $\lim_{N\to\infty} S_N(x) = f(x)$ for $x \neq 0$.

Theorem 6.6.3 (Lagrange's Remainder Theorem)
Let f be differentiable $N+1$ times on $(-R,R)$, define $a_n = \frac{f^{(n)}(0)}{n!}$ for $n = 0,1,2,3,\dots,N$ and let $S_N(x) = a_0 + a_1 x + \dots + a_N x^N$. Given $x \neq 0$ in $(-R,R)$, $\exists c$ s.t. $|c| < |x|$ where the error function $E_N(x) = f(x) - S_N(x)$ satisfies
$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

Example 6.6.4
- How well does $S_5(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ approximate $\sin(x)$ on $[-2,2]$?
Theorem 6.6.3 asserts that $E_5(x) = \frac{-\sin(c)}{6!} x^6$ for some $c \in (-|x|, |x|)$.
Noticing that $|\sin(c)| \leq 1$, and that we're dealing with $[-2,2]$,
$$|E_5(x)| \leq \frac{2^6}{6!} \approx 0.089$$

To prove that $S_N(x) \to \sin(x)$ uniformly on $[-2,2]$, observe $|f^{(N+1)}(c)| \leq 1$.
So $|E_N(x)| \leq 2^{N+1}/(N+1)!$, and thus by the M-test $E_N(x) \to 0$ uniformly

Proof of Theorem 6.6.3:
The nature of Taylor coefficients is such that $f^{(n)}(0) = S_N^{(n)}(0)$ for $0 \leq n \leq N$. This implies that $E_N^{(n)}(0) = 0$ for $n = 0,1,2,\dots$ Assume $x > 0$ and apply the Generalized MVT to $E_N(x)$ and x^{N+1} on $[0,x]$. $\exists x_1 \in (0,x)$ s.t. $\frac{E_N'(x_1)}{(N+1)x_1^N} = \frac{E_N(x)}{x^{N+1}}$. Repeating this process on $E_N'(x)$ and $(N+1)x^N$ on $[0,x_1]$ gets us $\frac{E_N''(x_2)}{(N+1)N x_2^{N-1}} = \frac{E_N'(x_1)}{(N+1)x_1^N} = \frac{E_N(x)}{x^{N+1}}$ for $x_2 \in (0, x_1)$

Continuing in this manner we arrive at

$$\frac{E_N(x)}{x^{N+1}} = \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!} \quad \text{for } x_{N+1} \in (0, x_N) \subseteq \dots \subseteq (0, x). \text{ Set } c = x_{N+1}.$$

Because $S_N^{(N+1)}(x) = 0$, we have $E_N^{(N+1)}(x) = f^{(N+1)}(x)$, and it follows that $E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$ □

- If we don't want to center the Taylor series at 0, easy fix
  - Taylor series around a is
    $$\sum_{n=0}^{\infty} c_n (x-a)^n \text{ where } c_n = \frac{f^{(n)}(a)}{n!}$$
  - Lagrange's Remainder Theorem becomes: $\exists c \in (a, x)$ where
    $$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$
- We still can't say in general whether a Taylor series will always converge to the function that generated it... and that's bc it's not true.

Let $g(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Taylor coefficients: $a_0 = 0$,

$$a_1 = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = \lim_{x \to 0} \frac{1/x}{e^{1/x^2}} = \frac{\infty}{\infty} \xrightarrow{\text{L'hospital}} \lim_{x \to 0} \frac{-1/x^2}{(-2/x^3 e^{-1/x^2})} = \lim_{x \to 0} \frac{x}{2e^{-1/x^2}} = 0$$

Exercise 6.6.6 shows $g^{(n)}(0) = 0 \ \forall n \in \mathbb{N}$. Therefore the Taylor series trivially converges uniformly on $\mathbb{R}$ to 0 function... But other than $x = 0$, $g(x)$ is never equal to 0. Thus the taylor series for g converges, but it never equals $g(x)$ except for $x = 0$.

☆ Not every infinitely differentiable function can be represented by its Taylor series.

6.6 Exercises

1) The Taylor series for arctan(x) is valid for $x \in (-1,1)$. Notice the series also converges when $x=1$. Assuming arctan(x) is continuous, explain why the value of the series at $x=1$ must be arctan(1). What identity do we get in this case?

$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ $\forall x \in (-1,1)$. At $x=1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by the AST. Because both arctan and its power series are continuous on $(-1,1]$,

$$\lim_{x \to 1^-} \arctan(x) = \lim_{x \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1).$$

This gives the identity $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$

2. Manipulate existing series to give Taylor series representations for the following. For what values of x do the series hold?

a) $x\cos(x^2)$

Start with $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Differentiate to get $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Substitute $x = x^2$ and multiply by x to get

$$x\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!}.$$ Holds for $x \in \mathbb{R}$.

b) $\frac{x}{(1+4x^2)^2}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1+4x^2} = \sum_{n=0}^{\infty} (-1)^n (4x^2)^n \Rightarrow \frac{-8x}{(1+4x^2)^2} = \sum_{n=1}^{\infty} 8x(-1)^n n (4x^2)^{n-1}$$

$$\Rightarrow \frac{x}{(1+4x^2)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n 4^{n-1} x^{2n-1}$$ for $|x| < 1/2$

c) $\log(1+x^2)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n \Rightarrow \ln(x) = \sum_{n=0}^{\infty} \frac{-(1-x)^{n+1}}{n+1}$$

$$\Rightarrow \ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2(n+1)}}{n+1}$$

3) Derive the formula for Taylor's coefficients in Theorem 6.6.2

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$. To isolate a_0, simply plug in $x=0$ & $f(0) = a_0$. To isolate a_1, differentiate once to get rid of a_0, then plug in $x=0$. $f'(0) = a_1$. If you differentiate n times, ~~term~~ a_0 through a_{n-1} will disappear, and plugging in $x=0$ will get rid of terms a_{n+1} on. This leaves $n!\,a_n$. So to get a_n just divide by $n!$. The whole process put together results in $a_n = \frac{f^{(n)}(0)}{n!}$

?

4) Explain how Lagrange's Remainder Theorem can be modified to prove $1 - 1/2 + 1/3 - 1/4 + \dots = \log(2)$

Don't know about using LRT, but from 2c we know that $\ln(x) = \sum_{n=0}^{\infty} \frac{-(1-x)^{n+1}}{n+1}$, and $x=2$ gives $\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

* Apparently sol'n manual says use $f(x) = \ln(1+x)$ with $x=1$ in LRT and show $E_n(1) \to 0$.

8) Prove a weaker form of Lagrange's Remainder Theorem

a) Prove this lemma: If g, h are differentiable on $[0, x]$ with $g(0) = h(0)$ and $g'(t) \leq h'(t)$ $\forall t \in [0, x]$, then $g(t) \leq h(t)$ $\forall t \in [0, x]$

Proof:
Let $t \in [0, x]$. By the Generalized MVT, $\exists c \in (0, t)$ s.t.
$\frac{g'(c)}{h'(c)} = \frac{g(t) - g(0)}{h(t) - g(0)} \leq 1$ because $g'(c) \leq h'(c)$. Thus we get
$g(t) - g(0) \leq h(t) - g(0) \Rightarrow g(t) \leq h(t)$. If $h'(c) = 0$ on $(0, t)$, then we get $h(t) = g(0)$, which still satisfies $g(t) \leq h(t)$.

b) Let f, S_N, E_N be as Theorem 6.6.3, and take $0 < x < R$. If $|f^{(N+1)}(t)| \leq M$ $\forall t \in [0, x]$ show $|E_N(x)| \leq \frac{Mx^{N+1}}{(N+1)!}$

Proof:
Let $g(t) = |E_N^{(N)}(t)|$ and $h(t) = Mt$. $g(0) = h(0) = 0$, and $g'(t) = |f^{(N+1)}(t)| \leq M = h'(t)$ $\forall t \in [0, x]$. By the lemma in (a), it follows that $g(t) \leq h(t)$. Now let $g(t) = |E_N^{(N-1)}(t)|$ and $h(t) = Mt^2/2$ (antiderivatives). Again $g(0) = h(0) = 0$, and in the first step we showed $g'(t) \leq h'(t)$. Thus again $g(t) \leq h(t)$. Repeat the process inductively: set $g(t)$ and $h(t)$ to the antiderivatives of the current g and h, and apply the lemma to show $g(t) \leq h(t)$. Eventually we arrive at $|E_N(t)| \leq Mt^{N+1}/(N+1)!$ $\forall t \in [0, x]$, and so $|E_N(x)| \leq \frac{Mx^{N+1}}{(N+1)!}$.
This process works because $|E_N^{(n)}(0)| = 0$ $\forall n \in 0, 1, \ldots$ and $Mt^n = 0$ when $t = 0$ for $n \in 1, 2, 3, \ldots$ $\square$

9) (Cauchy's Remainder Theorem). Let f be differentiable $N+1$ times on $(-R,R)$. For each $a \in (-R,R)$, let $S_N(x,a)$ be the partial sum of the Taylor series for f centered on a. In other words:
$S_N(x,a) = \sum_{n=0}^{N} c_n (x-a)^n$ where $c_n = \frac{f^{(n)}(a)}{n!}$.

Let $E_N(x,a) = f(x) - S_N(x,a)$. Now fix $x \neq 0$ in $(-R,R)$ and consider $E_N(x,a)$ as a function of a.

a) Find $E_N(x,x)$

$$E_N(x,x) = f(x) - \sum_{n=0}^{\infty} c_n (x-x)^n = f(x) - c_0 = f(x) - f(x) = 0$$

b) Explain why $E_N(x)$ is differentiable wrt a and show $E_N'(x) = \frac{-f^{(N+1)}(a)}{N!}(x-a)^N$

Both $f(x)$ and $S_N(x,a)$ are differentiable with respect to a, so $E_N(x) = f(x) - S_N(x,a)$ must be as well.

$$E_N'(x) = \frac{d}{da}\left(f(x) - \sum_{n=0}^{N} c_n (x-a)^n\right) = -\frac{d}{da}\left(\sum_{n=0}^{N} c_n (x-a)^n\right) = -\frac{d}{da}\left(\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n\right)$$

$$= -\left[\sum_{n=0}^{N} \frac{f^{(n+1)}(a)}{n!}(x-a)^n - \sum_{n=1}^{N} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}\right] = \frac{-f^{(N+1)}(a)}{N!}(x-a)^N$$

c) Show $E_N(x) = E_N(x,0) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$ for some c between 0 and x. This is Cauchy's form of the remainder for a Taylor series centered at the origin.

Apply the MVT to $E_N(x,a)$ on $(0,x)$. Then $\exists c \in (0,x)$ s.t.

$$E_N'(c) = \frac{E_N(x,x) - E_N(x,0)}{x} \Rightarrow -x E_N'(c) = E_N(x,0).$$

$$-x E_N'(c) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x = E_N(x,0)$$

□