Pricing fast-moving products

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March 13, 2019

Abstract

Traditional models in dynamic pricing take an "isolated view" of customers by assuming each customer's action (to buy or not) is independent of others' actions and the firm can change price after each purchase. With the rise of "social" technologies that enable quick personal connections with potential customers, this traditional view of treating consumers as "isolated entities" no longer holds for some markets. For instance, in the virtual commerce (Weishang and influencer-driven) markets, consumers who subscribe to a social media channel can significantly accelerate new product adoptions and create a demand pattern that can be described as "hype". In these markets, future demand is naturally dependent on past demand and sales, and thus, demand changes rapidly from one period to next in response to past demand and sales. These cases are different from traditional dynamic pricing problems in two ways: (i) future demand is no longer independent from what customers did in the past, (ii) many people buy at the same time, thus, it is not possible to frequently adjust the price upon each sale. In this paper, we will call this "fast-moving" markets.

Motivated by a number of examples (most of fast-moving products are sold through virtual commerce markets), we propose a dynamic pricing model where future demand is dependent on current and past adoptions and inventory under limited price changes. In this setting, price not only affects the current revenue and sales, but also, future demand patterns and revenues as well. Thus, a good pricing policy should utilize this "demand hype" early after a product is introduced and account for the effect of the current price on both revenue and demand. An exact stochastic optimization formulation of this problem is not practically applicable due to the curse of dimensionality and lack of knowledge about full demand distribution. We propose a certainty equivalent (CE) policy with re-optimization. This policy has several advantages. First, it does not require the firm to have complete knowledge about the demand distribution. Instead, the only requirement is that the firm knows the average demand for a given price and state. Second, the policy requires only a limited number of price changes and so practically applicable in fast-moving markets. Finally, the CE policy with re-optimization is asymptotically optimal: as the market size m increases, the percentage loss as compared to the optimal policy decreases at the rate of $\mathcal{O}(1/\sqrt{m})$. Our numerical results are even more promising. Even with a relatively small market size and very few price changes, our policy performs well and often results in revenues that are only few percentages lower than the optimal. Lastly, we look at the joint optimization of pricing and initial inventory and leverage our CE analysis to provide asymptotically optimal recommendations on how much to restrict initial inventory to achieve the best profit.

1 Introduction

In recent years, companies large and small increasingly leverage dynamic pricing to increase their revenue and profit. Starting from the travel and hospitality industries with perishable inventory, it is now adopted in other industries such as retail, logistics, service, and so on.

The vast majority of existing literature in the field of dynamic pricing makes a few common modeling assumptions. First, demand is assumed to follow a stochastic process that depends only on the current price. Second, they assume that each customer is an isolated decision maker independent of other customers' actions. Third, price can be changed frequently enough so that the seller can offer a different price to each arriving customer (Gallego and Van Ryzin, 1997, 1994; Jasin and Kumar, 2012, 2013; Jasin, 2014). As a result, a customer's purchase affects the seller's current revenue but not the future demand. Indeed, there are many applications where an isolated perspective of customers is warranted. For example, individual airline passengers decide independently whether to buy a ticket or not through various channels. For some products (such as seasonal or fashion goods), the selling horizon may be too short for any significant knowledge to spread over a consumer population to have an impact on the demand (Elmaghraby and Keskinocak, 2003). In such settings, these assumptions capture essential features of the underlying market, thus, the established models that use these assumptions provide valuable insights into key drivers and trade-offs (e.g., the marginal value of inventory or the evolution of price over time) and the results of these models can be modified and deployed to construct pricing policies in practice.

In recent years, however, technology and social networks create new types of markets that did not exist before. One prominent example of this is "Weishang" — social media "influencers" who promote and sell products directly to their social media followers. Many different products are now sold by Weishangs (cosmetics, electronics, and even tourism products). In other countries, a growing number of YouTube and Instagram "influencers" with a large subscriber base become sellers of their own or endorsed products. This practice, broadly labeled as V-Commerce (virtual commerce) not only attracts interest in a product from pre-selected active customers but also provides convenient purchase options through the influencers' platforms. In fact, the sales of many beauty and cosmetic products in South Korea, Japan, and China are largely driven by "influencers" and "content creators." Some of them have millions of active followers worldwide. For instance, Michelle Phan (a US beauty influencer) and PonySyndrome (a beauty influencer in South Korea) each have more than 5 million YouTube and Instagram followers and have their own lines of products (their own brand or through collaboration with existing brands). In addition, the followers of a powerful creator form a community of their own and often interact and influence each other as well. Because of the size and tremendous market potential of these followers, there has been a proliferation of companies that specialize in managing and monetizing content creators, such as CJ E&M and TreasureHunter.

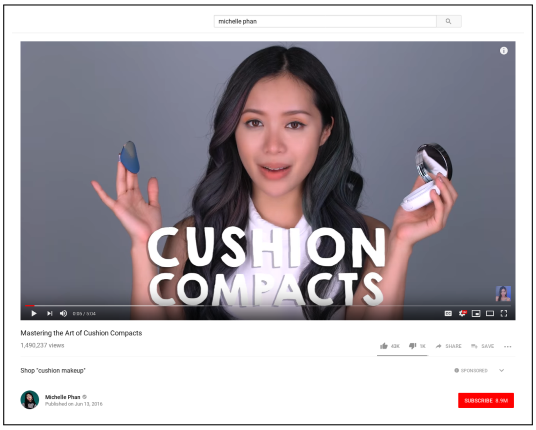

Figure 1: Michelle Phan's YouTube video recommending cushion compacts.

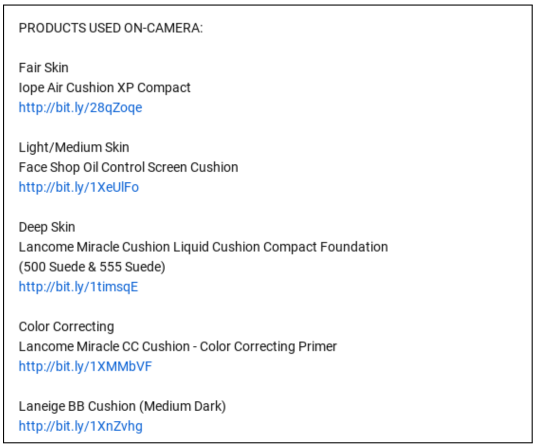

Figure 2: Names of products and purchase links shown under the YouTube video.

What is special about V-commerce markets is how fast they can move through creating demand hype. Information about a product can quickly be generated as followers receive synchronized notification about products directly from influencers and other consumers in the influencer's following. Consumers are no longer independent, and the future demand is dependent of the current cumulative sales. Note that in these examples, many customers buy around the same time triggered by social media contents or notifications. For instance, Weishangs make live product pitches to all of her/his followers at the same time. Thus, it is not possible to change the price to each arriving customer, which is another feature that is different from traditional dynamic pricing problems.

Moreover, sellers in the V-commerce setting often have limited inventory and the members of her following are often prompted to purchase "before it's too late" or potentially be "left out" of the rest of the group. This phenomenon is known as the *scarcity effect* (Yang and Zhang, 2014; Cachon et al., 2018), where the limited product availability increases the perceived value because of exclusivity, creating a sense of urgency among customers to "act fast".

We call the aforementioned settings *fast-moving markets* where demand moves quickly in response to past purchases and scarce inventories, so quickly that price changes cannot keep pace with movement in demand. A new modeling framework that embraces the salient features of fast-moving demand is required.

Note that some of the technical assumptions in the classical revenue management literature that made analysis feasible do not reflect the practice of V-commerce products. For instance, many existing models assume that prices can be adjusted in continuous time or at least that each arriving customer will see a different price (Gallego and Van Ryzin, 1994; Jasin, 2014). Such a pricing policy cannot be implemented in the V-commerce setting if many consumers participate in social media and are vying to purchase the product at essentially the same time, possibly triggered by notifications on their mobile devices. In such a case, the seller is not able to change the price frequently. This is a further aspect of the fast-moving nature of V-commerce markets.

In this paper, we propose a framework to model the pricing problem where demand is influenced by past sales and inventory availability. This allows us, in particular, to capture the demand patterns we observe in V-commerce markets. Our goal is to create a pricing policy that can take advantage of the fast-moving demand patterns described above. Our model assumes that demand can depend on past sales, initial availability, and current inventory (we refer to such demand as "path-dependent") and captures dynamic factors that influence current and future demand, such as *popularity, rarity, or scarcity*. We solve this in a periodic review setting, acknowledging the fact that the firm's ability to make frequent price changes is limited. Our framework can be used in other settings that have sales- and inventory-dependent demand (several of which are described in detail in Section 3.1). Furthermore, we can even recover classic results in dynamic pricing (such as Gallego and Van Ryzin 1994) as special cases.

When we introduce the drivers of fast-moving markets in a dynamic pricing model, the resultant stochastic optimization problem becomes too complicated for exact analysis, even for simple cases. Instead, we propose a policy that is based on the certainty equivalent formulation (CE) and re-optimization. Because our policy is based on certainty equivalence, the information that is required to solve the pricing problem is reduced considerably. For instance, we can solve the problem without knowing the full distribution of future demand. Instead, it suffices to know the average demand for a given price, previous sales, and current inventory. In other words, we replace the fully stochastic demand distribution with its conditional mean. Another distinctive feature of our policy is that it explicitly calls for re-optimization after each price change. We show that, when the demand is path dependent, re-optimization is critical to guarantee performance. Without re-optimization, prices can "go wild" and deviate from the optimal policy. To capture the variety of possible sales and inventory dependencies that drive fast-moving demand, we allow the conditional demand to be nonlinear. Our main result is that the CE policy with re-optimization performs well in the asymptotic sense: as the potential market size m for the product increases, the percentage revenue loss as compared to optimal policy decreases in rate $\mathcal{O}(1/\sqrt{m})$,[1] matching the asymptotic performance of policies established for the *i.i.d.* demand settings (Gallego and Van Ryzin, 1994).

Our numerical results are even more encouraging. We show that our policy performs very well even when the market size is small or the number of price changes is very limited. In our numerical experiments, we apply the policy to a market size as small as several hundred results in a revenue that is only one percent lower than the optimal revenue. This suggests our approach can be used in large and small markets, including moderately-sized social media followings typical in V-commerce.

To derive a policy that can be used in practice, our policy explicitly takes account of the number of price changes allowed during a selling season. Indeed, we show through large-scale numerical experiments that just a few price changes are needed to recover what a policy derived from a continuous time model can achieve. Our numerical results show that even a single price

[1]Throughout the paper we use the big $\mathcal{O}$ notation, where by definition $f(x) = \mathcal{O}(g(x))$ for positive real-valued functions f and g if there exists an $r \in \mathbb{R}$ such that $f(x) < rg(x)$ for x sufficiently large. Similarly, if $f(x) = \Omega(g(x))$, then $f(x) > rg(x)$. When $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$, it is represented by $f(x) = \Theta(g(x))$.

change captures, on average, 95% of the potential revenue from a continuous-time model. This result clearly shows that even a little bit of price flexibility can recover a good portion of the possible benefits of selling a product through social marketing. This may be reminiscent of the result of Gallego and Van Ryzin (1994) that a single static price was sufficient to recover the potential benefits of dynamic pricing at the asymptotic rate $\mathcal{O}(1/\sqrt{m})$ for stationary demand. But there is one notable difference. Because future demand depends on price and cumulative sales (and inventory) in our model, such performance can be achieved only when there is at least one price change. We show that a static pricing policy performs poorly in our demand settings.

We extend our analysis to optimizing the starting inventory level. This fits well within the context where short production runs or limited inventory is quite common. The certainty equivalence formulation allows us to solve this problem and establish an asymptotic performance guarantee for a joint pricing and inventory problem, which is the first in the literature.

2 Literature Review

Our work is related to the literature in dynamic pricing with limited inventory. Among them, Gallego and Van Ryzin (1994), a seminal paper in the field of dynamic pricing, characterizes the structure of the optimal policy under the assumption that future demands are independent and each arriving customer is making a decision independently. In their model, the paper assumes that a seller has the ability to make frequent price changes. Since then, many papers extend their model in a number of meaningful directions: multiple products (Gallego and Van Ryzin, 1997; Maglaras and Meissner, 2006), choice model (Talluri and Van Ryzin, 2004), and non-homogeneous demand (Zhao and Zheng, 2000). One element that is common in all of these papers is how arriving customers (and their choices) are modeled: Future arrivals are independent of the present and each arriving customer makes a purchase decision free of any influence. Feng and Gallego (2000) relaxed this assumption by allowing demand in a current period to depend on the previous period's demand through a continuous time Markov chain.

Another common feature in the vast majority of the existing literature is that the seller is allowed to make many price changes. Typically, the corresponding optimization problem is formulated as a continuous-time optimization problem or as a discrete-time Markov Decision Process in which there exists at most one customer between two periods. A number of papers, including Feng and Gallego (2000), Chatwin (2000) and Feng and Xiao (2000), relax this assumption and consider a pricing policy with a limited number of price changes. Bitran and Mondschein (1997) also recognizes that a discrete time framework is more appropriate in some contexts, and allow the possibility of lost sales between periods. They show that a few price changes suffice to recover much of the benefit of deploying a pricing policy in continuous time. With these set-ups, they study the impact of optimally choosing the initial inventory assuming that per-period demands are *i.i.d.* Poisson random variables. Our paper studies the same problem as these past works (a pricing problem with limited inventory) but differs in the following key features. We assume that the future demand is path (history) dependent, we allow batch

Table 1: An overview of closely related papers in the literature.

	Dependent demand	Periodic pricing	Partial demand information	Inventory decision
Gallego and Van Ryzin (1994)	No	No	Yes[(a)]	No
Bitran and Mondschein (1997)	No	Yes	No	Initial
Feng and Gallego (2000)	Yes	No[(b)]	No	No
Shen et al. (2014)	Yes	No	No	Replenish
Yang and Zhang (2014)	Yes	Yes	No	Replenish
This paper	Yes	Yes	Yes	Initial

[(a)] Original paper assumes Poisson demand distribution but can be extended to only require information about conditional expected demand.
[(b)] Considers only finitely many price levels.

arrivals between two periods, we consider a limited number of price changes, and we do not assume a specific demand distribution. Consequently, standard techniques used in the existing literature regarding optimal policy and its performance do not hold for our case.

Our paper is also related to literature that considers the demand phenomenon of product "hype" caused by previous sales (and inventory). The well-known Bass model (Bass, 1969) is a deterministic and continuous-time model that captures the effects of word-of-mouth and network effects in a diffusion model. The Bass model and its variants allow us to see the impact of price and inventory decisions on sales and revenue. A number of papers study the negative and positive effects of inventory availability– scarcity effect (negative) (Van Herpen et al., 2009; Balachander et al., 2009; Cui et al., 2016), and billboard effect (positive) (Wolfe, 1968; Achabal et al., 1990; Larson and DeMarais, 1990; Caro and Gallien, 2012). These problems are typically modeled as deterministic (or fluid) optimization problems or stochastic problems where the firm has full knowledge of demand. For instance, Mahajan et al. (1991) and Shen et al. (2014) use deterministic models to derive the optimal policy. On the other hand, by assuming that the firm knows the exact demand distribution, Yang and Zhang (2014) study a joint pricing and inventory problem when the scarcity effect exists. The exact stochastic dynamic program is computationally difficult to solve and it requires information about the demand distribution that many firms do not have. Instead, our focus blends deterministic optimization problems (which we call a certainty equivalent approximation) with reoptimization to solve the problem. Thus, it is not necessary to have full demand information. In addition, unlike other papers that use deterministic approximations, our primary focus is to derive an implementable policy (i.e., a policy that requires a few price changes and requires as little information as possible) with a provable performance guarantee. In fact, our policy provides the first provable guarantee of a pricing policy when demand is driven by past sales and inventory availability: two features that are distinctive in social commerce.

A summary of the difference between our work and representative revenue management literature is found in Table 1, where salient features of each respective model are highlighted.

The last stream of literature related to our work is on certainty equivalent approximations. It is known that the certainty equivalent pricing policy (analogous to our own) performs well under *i.i.d.* demands. Gallego and Van Ryzin (1994) show that in a single product decision problem, a fixed price policy (where price is fixed at the optimal certainty equivalent price) is asymptotically optimal when demand follows a Poisson process with a constant arrival rate. They show the revenue loss of the static pricing policy (certainty equivalent) is $\mathcal{O}(\sqrt{k})$ when the system size is linearly scaled by k, where k is the number of potential price changes and the market size. Gallego and Van Ryzin (1997) and Jasin and Kumar (2013) give the performance guarantee of a certainty equivalent control in the multi-product (network revenue management) setting. Because the demand processes are assumed to be multiple homogeneous Poisson processes, the deterministic approximation of their problem can be formulated as a linear or convex optimization problem. However, the deterministic approximation in the history-dependent demand setting is highly nonlinear and, thus, cannot utilize the existing tools and theory based on the linear or convex cases (e.g., strong duality). Many of those studies show that a dynamic pricing policy that does not re-optimize (i.e. can be computed *a priori* to any demand realizations) nonetheless performs well (Jasin and Kumar, 2012, 2013; Jasin, 2014). The attractive performance of these *a priori* constructed policies is also a function of the underlying independence of demand. When demand is dependent, re-optimization becomes necessary. Indeed, our results heavily rely on re-optimization to establish the connection between the stochastic problem and the certainty equivalent problem used in proving asymptotic bounds. Existing literature leaves open the question of whether the certainty equivalent policy still gives an $\mathcal{O}(\sqrt{k})$ bound when demand is dependent and reoptimization is used. We prove asymptotic optimality in the dependent setting under fairly general conditions.

3 Modeling framework

A monopolist sells a good over a finite time horizon T to a market of size m. At time 0, the seller decides an initial inventory $I_0 = \alpha m$ by choosing $\alpha \in [0, 1]$, and incurs a procurement cost c for each unit of inventory. Note that α represents the proportion of the market that is covered by inventory, hence we also refer to α as the *proportion of market coverage* (or simply, *market coverage*). At each discrete time period $t \in \mathcal{T} \triangleq \{1, 2, \dots, T\}$, the seller chooses a price $\pi_t \geq 0$. Demand is realized and then satisfied up to the remaining inventory. Goods that are not sold by the end of time T are salvaged at a (normalized) value of 0. The goal of the seller is to maximize the expected profit by jointly optimizing the starting inventory level and the finite-horizon pricing policy.

A distinctive feature of our model is that the demand in each period is a random variable whose distribution depends on several factors, which include the market size, past adoptions (cumulative sales), the inventory level (availability), and price. To be precise, let P_t denote the total cumulative sales up to time t. Define $\mathcal{F}_t = \sigma(P_0, P_1, \dots, P_t)$ to be the smallest σ-field where variables $P_0, P_1, \dots, P_t$ are measurable and let $\mathcal{F} = \{\mathcal{F}_0, \mathcal{F}_1, \dots\}$ to be a filtration. We assume that the (random) demand D_t in period t has conditional mean

$$\mathbb{E}[D_t|\mathcal{F}_{t-1}] \triangleq H(m, N_{t-1}, I_0, \pi_t), \tag{1}$$

where, if $P_t \triangleq \sum_{s=1}^{t} D_s$ is the cumulative sales at time t, then $N_{t-1} \triangleq I_0 - P_{t-1}$ is the remaining inventory at the start of time t. The functional form $H(m, N, I_0, \pi)$ captures dependence of demand on the current price π, the market size m, the remaining inventory N, and the cumulative sales $I_0 - N$. The dependence of demand on past sales can model network effects (e.g., increase in demand due to social influences from past adopters). Thus, unlike most dynamic pricing settings in the literature, our demand process is non-*i.i.d.* since current demand depends on demand in past periods. Inventory effects (e.g., billboard effect or bandwagon effect driven by scarcity) are captured in the model by the dependence of demand on the initial and the remaining inventories. Thus, our model captures the case where the effect on demand of $N = 20$ inventory units remaining can be different if the starting inventory I_0 was 30 or 30000.

Assumption 1. The conditional mean demand function H in (1) is multiplicatively separable in price with

$$H(m, N, I_0, \pi) = \lambda(m, N, I_0)x(\pi) \tag{2}$$

for some functions λ and x, where λ is homogeneous of degree 1.

Since λ is a homogeneous function by Assumption 1, then $\lambda(m, N, I_0) = m\lambda(1, n, \alpha)$ where $n \triangleq N/m$. We always analyze the function λ in the form $m\lambda(1, n, \alpha)$ and accordingly drop the first component of λ to write it simply as $m\lambda(n, \alpha)$ (slightly abusing notation). In other words, given the remaining inventory and the current sales, $\lambda(\cdot, \cdot)$ scales demand representing how the demand rate is affected by the past sales and inventory.

From (1) and Assumption 1, we have that the conditional mean of demand in period t is

$$\mathbb{E}[D_t|\mathcal{F}_{t-1}] = m\lambda(n_{t-1}, \alpha)x(\pi_t), \tag{3}$$

where $n_{t-1} = N_{t-1}/m$ can be interpreted as the proportion of the market that is covered by inventory remaining at the start of time t. Note that only the term $m\lambda(n_{t-1}, \alpha)$ depends on the remaining inventory and cumulative sales.[2] Hence, we refer to λ as the *sales and inventory sensitivity (SIS)* function since it captures the effect that past sales and available inventory have on demand. In Section 3.1.4, we provide an example of the SIS function λ that models demand hype in the social network context discussed in the introduction. Since in (3), price affects only the term $x(\pi_t)$, we refer to the function x as the *price sensitivity* of demand, capturing the effect of price on the demand. We will assume throughout the paper that the SIS function λ and price-sensitivity function x are known to the seller. Table 2 summarizes the notation of our framework, working from (3) as a primitive.

Assumption 2. Functions $x(\cdot)$ and $\lambda(\cdot, \cdot)$ have the following properties:

[2] Note that the cumulative sales at the start of time t is equal to $m(\alpha - n_{t-1})$.

Notation	Description
m	the total market size of the product
I_0	starting inventory
$\alpha = I_0/m$	market coverage
T	finite time horizon
D_t	stochastic demand in period t
$d_t = D_t/m$	proportion of market willing to buy in period t
N_t	remaining inventory at the end of period t
$n_t = N_t/m$	proportion of market covered by inventory remaining at the end of time t
π_t	price at time t
$\lambda(n_t, \alpha)$	sales and inventory sensitivity (SIS) function of demand
$x(\pi_t)$	price sensitivity function of demand
$\mathcal{T}$	total time interval $\{1, 2, \ldots, T\}$
c	unit procurement cost

Table 2: Notation for modeling framework.

(i) $x : [0, \infty) \mapsto [0, 1]$. Moreover, there exists a finite *choke price* π^c where $x(\pi^c) = 0$.

(ii) x is continuously differentiable and strictly decreasing (that is, $x'(\pi) < 0$). This implies x^{-1} exists and is a decreasing function.

(iii) $\pi + \frac{x(\pi)}{x'(\pi)}$ is increasing in π.[3]

(iv) $\rho(\pi) \triangleq \pi x(\pi)$ is continuously differentiable in π and $\rho''(\pi)$ exists for all $\pi < \pi^c$.

(v) $\lambda : [0, 1]^2 \mapsto [0, \overline{\lambda}]$ for some $\overline{\lambda} > 0$.

(vi) λ is jointly concave and continuously differentiable in both of its arguments.

(vii) $\pi_\ell(n) \triangleq x^{-1}(n/\lambda(n, \alpha))$ is differentiable in n for $n \in [0, 1]$.

The properties of the price sensitivity function x in Assumption 2(i)–(iv) are standard in the revenue management literature. The condition in Assumption 2(i) that $x(\pi) \in [0, 1]$ gives an interpretation of price sensitivity as proportionally "clawing back" on the "raw" demand $m\lambda(n, \alpha)$. The existence of the choke price in Assumption 2(i) implies that that if the price is too high, no one buys. Assumption 2(iii) is common in the inventory and revenue management literatures, as it facilitates establishing the concavity of value functions (for a discussion see Ziya et al. 2004; Lariviere 2006). Assumption 2(iv) implies that the effective revenue rate ρ is a strictly concave function and so has a unique maximizer $\bar{\pi}$ in $[0, \pi^c]$. That is, price $\bar{\pi}$ provides the optimal effective revenue rate. Assumption 2(vii) is an assumption on both λ and x. It states that the lowest price $\pi_\ell(n)$ that can be charged without stocking out a supply of n in expectation is differentiable in n. Later in Section 3.1, we illustrate how a variety of demand models studied in the literature satisfy our assumptions.

We now formally state the seller's decision problem, which is a stochastic dynamic optimization problem to decide the proportion of market coverage α and the pricing policy. The pricing

[3] Interpreting $x(\pi)$ as the cumulative distribution of the population's valuation of the product, the condition that $\pi + \frac{x(\pi)}{x'(\pi)}$ is increasing in π implies this distribution has an increasing generalized failure rate. See Lariviere (2006) for further discussion.

policy must determine a price π_t to charge at time t, which in light of (3), depends on the state $n_{t-1} = N_{t-1}/m$ where N_{t-1} is the remaining inventory at the start of time t (of course, this decision could also depend on m and α, but these are not included in the state since they are static). That is, a *pricing policy* $\boldsymbol{\pi} : [0,1] \times \mathcal{T} \mapsto \mathbb{R}_+$ (where $\mathbb{R}_+$ is the set of nonnegative real numbers) determines the price $\pi_t = \boldsymbol{\pi}(n_{t-1}, t)$ to charge at time $t \in \mathcal{T}$ for given state. The expected profit of a decision $(\alpha, \boldsymbol{\pi})$ is

$$Q^{\alpha,\boldsymbol{\pi}}(m,T) = \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}\left[\boldsymbol{\pi}(n_{t-1},t)(D_t - [D_t - N_{t-1}]^+) \mid \mathcal{F}_{t-1}\right]\right] - c\alpha m, \tag{4}$$

where $[q]^+ \triangleq \max\{0, q\}$, $\boldsymbol{\pi}$ is $\mathcal{F}_t$-adapted, $N_0 = \alpha m$, $N_t = [N_{t-1} - D_t]^+$ for all $t \in \mathcal{T}$, and $n_t = N_t/m$. The expression $D_t - [D_t - N_{t-1}]^+$ is the sales at time t. Observe that any unmet demand is lost, since if demand D_t exceeds the starting inventory N_{t-1} in period t, then only N_{t-1} units are sold in that period. Hence, the seller's decision problem is

$$\max_{\alpha\in[0,1]} \max_{\substack{\boldsymbol{\pi}:[0,1]\times\mathcal{T}\to\mathbb{R}_+ \\ \mathcal{F}_t \text{ adapted}}} Q^{\alpha,\boldsymbol{\pi}}(m,T). \tag{5}$$

In light of the properties of the price sensitivity function x, we will refine the formulation of our problem. Assumption 2(ii) allows us to introduce a new variable $y_t = x(\pi_t)$ called the *price-induced demand intensity* (or simply *intensity*) at time t. The price $\pi_t = x^{-1}(y_t)$ is uniquely determined by the intensity y_t, thus, the seller's problem of choosing a pricing policy $\boldsymbol{\pi}$ is equivalent to choosing a demand intensity policy $\mathbf{y} : [0,1] \times \mathcal{T} \mapsto [0,1]$. From the definition of a choke price (Assumption 2(i)), $x(\pi^c) = 0$ and so the choke price π^c is associated with an intensity of 0.

As the existing literature (Gallego and Van Ryzin, 1994) shows, intensity policies are easier to analyze. We thus recast the policy evaluation (4) in terms of $\mathbf{y}$. The value of choosing the proportional market coverage α and the intensity policy $\mathbf{y}$ is

$$Q^{\alpha,\mathbf{y}}(m,T) = \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}\left[x^{-1}\left(\mathbf{y}\left(n_{t-1},t\right)\right)\left(D_t - [D_t - N_{t-1}]^+\right) \mid \mathcal{F}_{t-1}\right]\right] - c\alpha m, \tag{6}$$

where now $\mathbf{y}$ is $\mathcal{F}_t$-adapted, $N_0 = \alpha$, $N_t = [N_{t-1} - D_t]^+$ for all $t \in \mathcal{T}$, and $n_t = N_t/m$.

Remark 1 (Notational conventions)**.** The superscripts (α, $\mathbf{y}$) of $Q^{\alpha,\mathbf{y}}(m,T)$ denote decisions by the seller. The arguments (m, T) are model primitives. Our key results are related to the asymptotic optimality of the CE policy in the market size, thus we do not suppress m. Also, one of our primary questions is how the number of price changes (T) affects the algorithm and resultant profits. For that reason, we do not suppress T.

Given the proportion of market coverage $\alpha \in [0,1]$, we assume that the seller will only consider policies that do not stock out *in expectation*; that is, intensity policies $\mathbf{y}$ that satisfy

$$\mathbb{E}\left[D_t \mid \mathcal{F}_{t-1}\right] \le N_{t-1} \qquad \text{for all } t \in \mathcal{T} \tag{7a}$$

$$\mathbb{E}\left[D_t \mid \mathcal{F}_{t-1}\right] = m\lambda(n_{t-1}, \alpha)\mathbf{y}(n_{t-1}, t) \qquad \text{for all } t \in \mathcal{T} \tag{7b}$$

$$N_t = [N_{t-1} - D_t]^+ \qquad \text{for all } t \in \mathcal{T} \tag{7c}$$

$$N_0 = \alpha. \tag{7d}$$

We call such policies *feasible* and we let $\mathbf{Y}^m \triangleq \{\mathbf{y} : [0,1] \times \mathcal{T} \to [0,1] \mid (7), \mathcal{F}_t\text{-adapted}\}$ denote the set of all feasible policies. The existence of a choke price by Assumption 2(i) ensures that $\mathbf{Y}^m$ is nonempty.

Considering only policies in $\mathbf{Y}^m$ facilitates our later analysis. In particular, it allows us to show that the optimal value of the stochastic problem can be bounded by its deterministic counterpart (Proposition 1). In order to derive a good implementable policy with a performance guarantee, a typical approach is to formulate the problem as a continuous time pricing problem and restricted to the set of policies that have no stockout *almost surely*. For example, Gallego and Van Ryzin (1994) and Jasin (2014) use this technical assumption to simplify the analysis of the optimal policy and to derive the asymptotic bounds for their algorithms. However, when many customers can arrive between price changes (as in our setting), the no stockout almost surely assumption is too conservative in the sense that it must set very high prices in order to avoid stockouts under any demand scenario, resulting in unsold inventory. Instead, we impose the less restrictive constraint set (7) of no stockout *in expectation* during each time period.

When $\mathbf{y} \in \mathbf{Y}^m$ is a feasible policy, its value can be evaluated as

$$Q^{\alpha,\mathbf{y}}(m,T) \triangleq \mathbb{E}\left[x^{-1}\left(\mathbf{y}\left(n_{t-1},t\right)\right)\left(D_t - [D_t - N_{t-1}]^+\right)\right] - c\alpha m. \tag{8}$$

Note that, because of the law of iterated expectations, we dropped the inside conditional expectation with respect to $\mathcal{F}_t$ since the outside expectation is on $\mathcal{F}_0$.

Hence, the seller's problem of jointly optimizing the initial inventory level and feasible intensity policy (and thus pricing policy) is equivalent to solving the following problem:

$$Q^*(m,T) \triangleq \max_{\alpha \in [0,1]} \max_{\mathbf{y} \in \mathbf{Y}^m} Q^{\alpha,\mathbf{y}}(m,T). \tag{$\mathbf{P}_m$}$$

We denote the optimal value of this optimization problem ($\mathbf{P}_m$) as $Q^*(m,T)$. Consistent with our discussion in Remark 1, parameters m and T remain as arguments of the function $Q^*(m,T)$.

Utilizing Assumption 1, much of our analysis will be done on a scaled version of ($\mathbf{P}_m$), where we divide by the market size m. Specifically, we denote $d_t \triangleq D_t/m$ which can be interpreted as the proportion of the market willing to buy at time t. Hence, a feasible policy $\mathbf{y}$ must satisfy

$$\mathbb{E}\left[d_t \mid \mathcal{F}_{t-1}\right] \leq n_{t-1} \qquad \text{for all } t \in \mathcal{T} \tag{9a}$$

$$\mathbb{E}\left[d_t \mid \mathcal{F}_{t-1}\right] = \lambda(n_{t-1}, \alpha)\mathbf{y}(n_{t-1}, t) \qquad \text{for all } t \in \mathcal{T} \tag{9b}$$

$$n_t = [n_{t-1} - d_t]^+ \qquad \text{for all } t \in \mathcal{T} \tag{9c}$$

$$n_0 = \alpha. \tag{9d}$$

The set of all feasible policies is $\mathbf{Y} = \{\mathbf{y} : [0,1] \times \mathcal{T} \to [0,1] \mid (9), \mathcal{F}_t\text{-adapted}\}$.

Because of the linearity of the constraints (7a)–(7b) in m, the sets $\mathbf{Y}$ and $\mathbf{Y}^m$ are equivalent. Therefore, solutions to ($\mathbf{P}_m$) can be easily derived from solutions to the following scaled problem:

$$Q^*(T) \triangleq \max_{\alpha \in [0,1]} \max_{\mathbf{y} \in \mathbf{Y}} Q^{\alpha,\mathbf{y}}(T) \tag{$\mathbf{P}$}$$

where

$$Q^{\alpha,\mathbf{y}}(T) \triangleq \mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}(n_{t-1},t)\right)\left(d_t - [d_t - n_{t-1}]^+\right)\right] - c\alpha. \tag{10}$$

Note that we simplify $Q^{\alpha,\mathbf{y}}(1,T)$ to $Q^{\alpha,\mathbf{y}}(T)$ and $Q^*(1,T)$ to $Q^*(T)$. This is again consistent with our notational convention since m is not a parameter that is allowed to change in ($\mathbf{P}$) and in (10) (it is permanently fixed at $m = 1$).

Before moving on, we draw out some properties of the following function

$$r(y) \triangleq x^{-1}(y)y. \tag{11}$$

Observe that if $y = x(\pi)$ then $\rho(\pi) = r(y)$, where ρ is defined in Assumption 2(iv). Hence, $r(y)$ also has the interpretation of being an effective *revenue intensity*. The following facts about y and r will prove critical in later analysis. Proofs are in Section A.1.

Lemma 1. Under Assumption 2, the following hold:

(i) $r(y)$ is continuously differentiable, strictly concave and r'' exists for all $y \in [0,1]$,

(ii) there exists a unique optimal solution $\bar{y}$ to the optimization problem $\max_{y \in [0,1]} r(y)$, and

(iii) $y_h(n) \triangleq n/\lambda(n,\alpha)$ is differentiable in n for $n \in [0,1]$. ($y_h(n)$ is the highest intensity not causing a stock out in expectation.)

3.1 Discussion on the demand model

The assumption of a known and fixed market size m is consistent with numerous demand models with network effects including the Bass model (Bass, 1969) and the generalized Bass model (Bass et al., 1994; Krishnan et al., 1999). This parameter plays an important role in defining the potential strength of sales-dependent demand effects, since the total population size has an important bearing on network effects. It is important to note that price (and other factors) do not affect the potential market m. In our model, the effect of price on demand is represented by a function $x(\pi)$.[4] Our expected demand is of a multiplicative form which separates the effect of the current period price from the effect of past sales and inventory (captured by the $\lambda(\cdot,\cdot)$ term). There are many papers that use multiplicative demand functions, for instance, Smith and Agrawal 2017; Bass et al. 1994; Krishnan et al. 1999. See the review paper Urban (2005) for additional discussion.

It should be noted that our model explicitly assumes the seller can change price a limited number of times. This is different from a discrete time representation of a continuous time Markov

[4]Our notation $x(\cdot)$ for modeling price effects is inspired by the notation in Bass et al. 1994.

chain (through uniformization). In these papers, it is typical to assume that each time period is very small that at most one customer can arrive between pricing periods (see, for instance, Sauré and Zeevi 2013; Chen et al. 2015), effectively allowing individualized pricing to arriving customers. By contrast, our framework allows multiple customers to arrive in each time period but requires that the mean and the variance of the number of arrivals can be manipulated by price (see Assumptions 1 and 3). The discrete-time setting allows us to consider demand models wherein multiple customers may arrive within a time period. This is consistent with demand resulting from social influence since customers decide together and see the same price.

We now show how the settings used in a number of papers can be captured with our framework and assumptions. We show that these examples satisfy the assumptions on the x and λ functions (Assumption 2).

3.1.1 Application: Dynamic pricing with sales-dependent demand

The generalized Bass model of Bass et al. (1994); Krishnan et al. (1999) describes demand that is influenced by customers who have previously bought the product. The expected demand under this model is

$$H(m, N, I_0, \pi) = m\lambda\left(\frac{N}{m}, \frac{I_0}{m}\right) x_t, \tag{12}$$

with

$$\lambda(n, \alpha) = (1 - \alpha + n)\left(p + q\left(\alpha - n\right)\right), \tag{13}$$

where x_t captures the effect of advertising or price on average demand. If $x_t = x(\pi_t)$ is a time-stationary function of price, then it is a price sensitivity function of the form we study in this paper, and (12) satisfies Assumption 1. Existing literature usually assumes the price sensitivity function x takes the form of an exponential (Shen et al., 2014) or linear (Raman and Chatterjee, 1995) function. In both these cases, x is consistent with Assumption 2. Note that λ in (13) also satisfies Assumption 2.

The factor $(1 - \alpha + n)$ in the SIS function (13) is the proportion of the market that has not yet purchased the product, while the second factor represents how the probability of purchasing is affected by innovation, p, and by imitation of customers that have already purchased the product, $q(\alpha - n)$. A demand pattern of a V-commerce product (where demand is highly affected by creators and their followers, as shown in Weishang) can be explained with the above model. When more people adopt the product early in the season, they tend to create a stronger word-of-mouth effect, which leads to a higher demand rate. As $\alpha - n$ approaches 1, the market tends to saturate so the demand rate drops. We call this *saturation effect*. Here, λ captures both of these saturation and word-of-mouth effects.

3.1.2 Application: Dynamic pricing with a "scarcity effect" on demand

As mentioned earlier, in certain markets, the limited product availability can affect the demand as inventory becomes more scarce. This scarcity effect is amplified with a market for fast-moving products where many consumers consider whether to purchase at the same time (e.g., during a Youtube live broadcast).

Yang and Zhang (2014) model the scarcity effect in an additive demand model. Note that the assumptions used in their paper satisfy all of Assumption 2, but their demand format is in additive form, thus violating Assumption 1. However, the multiplicative version of Yang and Zhang (2014) fits our framework and assumptions. To see this, the expected demand for a given state (written in our notation) is

$$H(m, N, I_0, \pi) = \lambda(m, N)x(\pi), \tag{14}$$

where $\lambda(m, N)$ is twice differentiable and concave decreasing in the remaining inventory N. The scarcity effect (the demand increases when the inventory becomes scarce) is captured since λ is decreasing in remaining inventory N.

3.1.3 Application: Dynamic pricing with a "display effect" on demand

Even with demand hype, inventory could have a positive effect on demand as more people notice and are aware of the product. In marketing literature, this is called a *display* or *billboard* effect (Wolfe, 1968; Achabal et al., 1990). In fact, a retailer like Walmart often requires a minimum number of products to be on display (defined as *psychic stock* by Larson and DeMarais 1990). In operations, one paper that captures the display effect is Smith and Agrawal (2017). Their model can be considered a special case of our framework.

Smith and Agrawal (2017) model inventory display effects through the following expected demand function (using our notation): [5]

$$H(t, N, \pi) = m(t)\lambda(N)x(\pi) \tag{15}$$

where N denotes remaining inventory and $m(t)$ is the time-variant market size. A canonical case that leads to several analytic result in Smith and Agrawal (2017) can be adapted to our framework with minor modification as follows:

$$\lambda(N) = (N/N_r)^{\beta} \tag{16a}$$

$$x(\pi) = e^{-\gamma\pi/c_e} - \epsilon_x \tag{16b}$$

where N_r and c_e are reference values, and $0 < \beta < 1$, $\gamma > 0$, and $\epsilon_x > 0$. Note that λ in (16a) is increasing in N, which captures the display effect. Moreover, λ is concave, thus reflects a diminishing marginal rate of return. The ϵ_x in (16b) is the only modification of Smith and Agrawal (2017) (which assumes $\epsilon_x = 0$) and is introduced so that there exists a finite choke price.

[5] Smith and Agrawal (2017) consider a multi-location inventory model where inventory is sold to customers in multiple locations and the seller must decide how to allocate a fixed inventory between locations. Our model is for a single location, so we adapt the single-location development (in Section 1) of Smith and Agrawal (2017). Focusing on Smith and Agrawal (2017) was largely an arbitrary choice, any number of display effect demand models could have been set into our framework (for example, Kopalle et al. (1999); Wang and Gerchak (2001)).

Since the choice of ϵ_x is arbitrary, it does not change the results and insights of their paper. It is straightforward to verify that these choices for λ and x satisfy Assumption 2 (note that one needs to normalize the argument of λ to be in $[0,1]$ in order the check these conditions).

3.1.4 Application: Dynamic pricing with sales- and inventory-dependent demand

Our model can capture a setting where several effects mentioned above (scarcity, display, network) co-exist. For example, the following demand function incorporates a social network effect and inventory effects:

$$\lambda(n,\alpha) = \left(w\lambda^{(1)}(n,\alpha) + (1-w)\lambda^{(2)}(n,\alpha)\right) \tag{17}$$

where $w \in [0,1]$, $\lambda^{(2)}(n,\alpha) = (1-\alpha+n)\,(p+q(\alpha-n))$ (cf. (13)) captures the sales-dependent demand, and $\lambda^{(1)}(n,\alpha) = \left(\frac{n-\alpha^2+1}{N_r}\right)^{\beta}$ (cf. (16a)) captures the inventory display effect and the initial inventory scarcity effect. Note that $\lambda^{(1)}$ is a modification of (16a) (by adding $1-\alpha^2$ to n) which ensures that $\lambda^{(1)}$ models the scarcity effect due to the initial inventory. Here, we only model the scarcity effect of initial inventory while the dynamic remaining inventory creates the display effect. Note that λ is jointly concave in (n,α), satisfying Assumption 2(vi).

4 Pricing policy with fixed starting inventory

In this section, we analyze problem (**P**) for a given initial inventory level (represented by a given market coverage α, in proportion to the market size m).

For a given market coverage $\alpha \in [0,1]$, the value function (corresponding to (6) and representing the seller's expected revenue) can be written as:

$$V^{\alpha,\mathbf{y}}(m,T) \triangleq \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}\left[x^{-1}\left(\mathbf{y}\left(n_{t-1},t\right)\right)\left(D_t - [D_t - N_{t-1}]^+\right) \mid \mathcal{F}_{t-1}\right]\right], \tag{18}$$

where D_t is the demand at time t, $N_t = [N_{t-1} - D_t]^+$ is the remaining inventory at the end of time t (where $N_0 = \alpha m$), and $n_t = N_t/m$.

The resulting discrete-time dynamic pricing problem is

$$V^*(\alpha,m,T) \triangleq \max_{\mathbf{y}\in\mathbf{Y}} V^{\alpha,\mathbf{y}}(m,T). \tag{$\mathbf{P}'_m$}$$

When $m=1$, we write $V^{\alpha,\mathbf{y}}(T)$ instead of $V^{\alpha,\mathbf{y}}(1,T)$ and

$$V^*(\alpha,T) \triangleq \max_{\mathbf{y}\in\mathbf{Y}} V^{\alpha,\mathbf{y}}(T). \tag{$\mathbf{P}'$}$$

Note that we use V for revenue value functions and Q for profit value functions. Our notation in $V^*(\alpha,m,T)$ and $V^*(\alpha,T)$ is consistent with our notational conventions set in Remark 1. For now, the market coverage α is given and can be treated as a parameter for problems ($\mathbf{P}'_m$) and ($\mathbf{P}'$), and thus an argument of the functions.

Even when the proportion of market coverage α is given, solving (**P′**) is still challenging since the future demand depends on cumulative sales (or the amount of remaining inventory).[6] Even when the demand distribution is known, this path dependent nature of future demand makes the value function highly non-linear and makes a pricing problem difficult to solve (let alone the initial inventory problem). When the firm does not have full demand information, there is no easy way to solve (**P′**).

4.1 A certainty equivalent policy with re-optimization

In the absence of an exact optimal solution, one intuitive approach is to solve a deterministic counterpart of the stochastic problem on the basis of existing knowledge about the future demand (e.g., conditional mean of future demand). This is similar to the approach taken by Gallego and Van Ryzin (1994). Our *certainty equivalent* intensity policy is motivated by this.

For a given $\alpha \in [0, 1]$, we first introduce the following optimization program (**D′**), which is the deterministic approximation of the expected revenue-to-go with $T' \le T$ time periods left and with $u \in [0, \alpha]$ as the proportion of market that is covered by the *remaining* inventory.

$$V^{\mathsf{D}}(u, \alpha, T') \triangleq \max_{\substack{n \in \mathbb{R}^{T'+1} \\ y \in \mathbb{R}^{T'}}} \sum_{i=1}^{T'} x^{-1}(y_i)\lambda(n_{i-1}, \alpha) y_i \tag{D′a}$$

$$\text{s.t.} \quad \sum_{i=1}^{T'} \lambda(n_{i-1}, \alpha) y_i \le u \tag{D′b}$$

$$n_i = n_{i-1} - \lambda(n_{i-1}, \alpha) y_i \quad \text{for all } i = 1, \ldots, T' \tag{D′c}$$

$$n_0 = u \tag{D′d}$$

$$y_i \in [0, 1] \text{ for all } i = 1, \ldots, T'. \tag{D′e}$$

Observe that the optimal value $V^{\mathsf{D}}(u, \alpha, T')$ is indexed by both u and α since the SIS function depends on both the remaining and the initial inventory, so $V^{\mathsf{D}}(u, \alpha, T')$ is different from $V^{\mathsf{D}}(u, u, T')$. Note that constraint (**D′**b) means there is no stockout in expectation, and this is reflected in the objective function, which does not incorporate terms for stocking out.

(**D′**) is always feasible because of the existence of the choke price (Assumption 2(i)) which implies that $y = 0$ can always be chosen so (**D′**b) is satisfied. Observe that once a feasible vector y is chosen, the associated feasible vector n is immediately determined (and vice versa). Moreover, from our continuity assumptions on x and λ, the feasible region of (**D′**) is nonempty and compact and the objective function (**D′**a) is continuous, so at least one optimal solution exists (by Weierstrass's Theorem). We will prove later that optimal solutions to (**D′**) are unique (Theorem 2). We denote the unique optimal solution to (**D′**) by $(n^{\mathsf{D}}, y^{\mathsf{D}})$ where $n^{\mathsf{D}} = (n_0^{\mathsf{D}}, n_1^{\mathsf{D}}, \ldots, n_{T'}^{\mathsf{D}})$ and $y^{\mathsf{D}} = (y_1^{\mathsf{D}}, \ldots, y_{T'}^{\mathsf{D}})$.

[6]When α is given, knowing the remaining inventory level is equivalent to knowing cumulative sales.

Remark 2 (Notational conventions, continued)**.** We remark on the difference between $\mathbf{y}$ and y. The bold notation $\mathbf{y}$ is associated with intensity policy functions $\mathbf{y} : [0,1] \times \mathcal{T} \to [0,1]$ that form feasible solutions to the stochastic problems such as in (**P**) and (**P′**). By contrast, the non-bolded notation y is associated with finite-dimensional vectors of intensities, such as the partial feasible solution $y = (y_1, \ldots, y_{T'})$ to the deterministic problem (**D′**). We keep the underlying character 'y' consistent in both instances (bolded and non-bolded) because both have a common interpretation as intensities. The superscript $^{\mathsf{D}}$ will always be associated with optimality of the deterministic problem. By contrast, we reserve the superscript * for solutions and optimal values associated with the full stochastic version of the problem. ◁

We are now ready to describe the certainty equivalent (CE) intensity policy, which we denote as $\mathbf{y}^{\mathsf{CE}} : [0,1] \times \mathcal{T} \mapsto [0,1]$. Algorithm 1 is a description of the CE intensity policy. Suppose that N_{t-1} is the remaining inventory at the start of a period $t \in \mathcal{T}$. Then, the CE intensity policy will first solve (**D′**) with $u = N_{t-1}/m$ and $T' = T - t + 1$, then from the resulting vector of intensities $y^{\mathsf{D}} = (y_1^{\mathsf{D}}, \ldots, y_{T'}^{\mathsf{D}})$, it will set the intensity $\mathbf{y}^{\mathsf{CE}}(n_{t-1}, t) = y_1^{\mathsf{D}}$. It is important to note that at every period the certainty equivalent policy is re-optimizing the deterministic counterpart (**D′**) with u as the *updated* inventory information.

Algorithm 1: Intensity (price) sequence when applying policy $\mathbf{y}^{\mathsf{CE}}$.

procedure CERTAINTY EQUIVALENT PRICING(α, T)
    $n_0 \leftarrow \alpha$ ▷ initialize inventory
    for $t \leftarrow 1$ **to** T **do**
        $(n^{\mathsf{D}}, y^{\mathsf{D}}) \leftarrow$ optimal solution of (**D′**) with $u = n_{t-1}$ and $T' = T - t + 1$
        set intensity y_1^{D} by offering price $x^{-1}(y_1^{\mathsf{D}})$ ▷ set current intensity (price)
        observe demand D_t
        $u \leftarrow n_t = (n_{t-1} - D_t/m)^+$ ▷ update inventory
    end for
end procedure

Recall that our motivation in considering the certainty equivalent policy is that the stochastic version of the problem (**P′**) is difficult to solve. Hence, we next investigate the computational tractability of the CE intensity policy.

We first establish that the solution to the nonlinear program (**D′**) is unique. In order to do this, we need to show that its optimal value function V^{D} is strictly concave in u. This is established in the next result, whose proof (found in Section B.1) uses the concavity of the SIS function λ (Assumption 2(vi)) and of the revenue intensity function r (Lemma 1(i)).

Theorem 1 (Concavity of the value function)**.** The value function $V^{\mathsf{D}}(u, \alpha, T')$ defined in (**D′**) is strictly jointly concave in (u, α) for every fixed $T' \in \mathcal{T}$.

Next, we leverage the strict concavity of the value function of (**D′**) to show that there exists a unique optimal solution to (**D′**). This property implies that $\mathbf{y}^{\mathsf{CE}}$ is a proper intensity policy in that it maps each $(n_{t-1}, t) \in [0,1] \times \mathcal{T}$ to a unique intensity in $[0,1]$. The proof of this uniqueness result is not relegated to an appendix because it illustrates an efficient computational method for solving the nonlinear program (**D′**).

Theorem 2 (Uniqueness)**.** For any (u, α, T') with $0 \le u \le \alpha$ and $T' \in \mathcal{T}$, problem (**D′**) has a unique optimal solution.

Proof. Consider the case with $T' = 1$. By definition,

$$V^{\mathsf{D}}(u, \alpha, 1) = \max_{y \in [0,1]} \; x^{-1}(y)\lambda(u,\alpha)y \qquad \text{s.t. } \lambda(u,\alpha)y \le u. \tag{20}$$

By Lemma 1(i), the objective function is strictly concave. Since the set of feasible solutions is compact, this means a unique optimal solution exists to (20), and to (**D′**) when $T' = 1$.

Now consider the case with $T' \ge 2$. Note that we can reformulate $V^{\mathsf{D}}(u, \alpha, T')$ as

$$V^{\mathsf{D}}(u, \alpha, T') = \max_{y} \quad x^{-1}(y)\lambda(u,\alpha)y + V^{\mathsf{D}}\left(u - \lambda(u,\alpha)y, \alpha, T' - 1\right) \tag{21a}$$

$$\text{s.t.} \quad \lambda(u,\alpha)y \le u. \tag{21b}$$

Denote the objective function (21a) as

$$R^{y}(u, \alpha, T') = x^{-1}(y)\lambda(u,\alpha)y + V^{\mathsf{D}}\left(u - \lambda(u,\alpha)y, \alpha, T' - 1\right). \tag{22}$$

Since the feasible set of (21) is compact, to show that (21) has a unique solution, it suffices to show the following claim:

Claim 1. $R^{y}(u, \alpha, T')$ is strictly concave in y.

The first term of $R^{y}(u, \alpha, T')$ is strictly concave in y from Lemma 1(i). To see that the second term is also concave, its second order derivative with respect to y is

$$\lambda(u,\alpha)^2 \frac{\partial^2}{\partial n'^2} V^{\mathsf{D}}(n', \alpha, T' - 1)\Big|_{n' = u - \lambda(u,\alpha)y} \le 0, \tag{23}$$

where $|_{n'=n}$ means the term is evaluated at $n' = n$, and the inequality comes from Theorem 1. Then according to Claim 1, we know the objective of (21) is strictly concave in y. □

The proof of Theorem 2 suggests an efficient way to solve (**D′**) using backwards recursion on (21), which is the dynamic programming formulation of (**D′**). Hence, the CE intensity policy that uses solutions to (**D′**) with different values of (u, T') can be solved numerically.

Remark 3 (Setting the CE intensity policy)**.** If we know $V^{\mathsf{D}}(u, \alpha, T' - 1)$ for all $u \in [0, \alpha]$, then we can use (21) to find $V^{\mathsf{D}}(u, \alpha, T')$. Start with $T' = 0$ and set $V^{\mathsf{D}}(u, \alpha, 0) = 0$ for all $u \in [0, \alpha]$. Then, move backwards in time (i.e., increase T') and solve (21) for all u in a discretization of $[0, \alpha]$. Since the objective function of (21) is strictly concave by Claim 1, finding a stationary point is sufficient for solving (21). This can be accomplished efficiently with a root-finding method, such as Newton's method, that finds roots of the first-order conditions. $\triangleleft$

Uniqueness of the optimal solution to ($\mathbf{D}'$) has an interesting implication on the structure of $(n^{\mathsf{D}}, y^{\mathsf{D}})$. The next result shows that if the remaining inventory is nonzero, it is optimal for the deterministic model to have positive sales for *all* remaining periods. The proof is in Section B.2.

Theorem 3 (Positive intensity is optimal)**.** If $0 < u \leq \alpha$ and $T' \in \mathcal{T}$, then a solution (n, y) with $y > 0$ and n a strictly decreasing sequence is optimal to ($\mathbf{D}'$).

4.2 Asymptotic performance of the certainty equivalent pricing policy

We next turn to analyzing the performance of the CE intensity policy. In particular, we will show that when the market coverage $\alpha \in [0, 1]$ is fixed, the revenue loss of the CE intensity policy decreases in m with the rate $\mathcal{O}(1/\sqrt{m})$ (Theorem 5). The sketch of the proof is as follows: We first prove that the seller's optimal revenue $V^*(\alpha, T)$ is bounded above by its deterministic counterpart (Proposition 1). Then, we show that for a fixed α, the expected revenue of the CE intensity policy approaches this upper bound as m grows (Lemma 4). The combination of these two results imply that the CE intensity policy has a near optimal expected revenue in an asymptotic regime (i.e., as the market size m grows large) when α is fixed.

Before proceeding, we introduce simplified notation. We write $V^{\mathsf{D}}(\alpha, T)$ instead of $V^{\mathsf{D}}(\alpha, \alpha, T)$ for the deterministic approximation of the expected revenue with T periods to go and with full inventory α. We write $V^*(u, \alpha, T')$ to denote the optimal expected revenue under the stochastic model with T' periods to go and with remaining inventory u.

Proposition 1 (Upper bound)**.** Given $\alpha \in [0, 1]$ and $T \in \mathcal{T}$, the following hold:

(i) $V^*(u, \alpha, T') \leq V^{\mathsf{D}}(u, \alpha, T')$ for any $u \in [0, \alpha]$ and $T' \leq T$

(ii) $V^*(\alpha, T) \leq V^{\mathsf{D}}(\alpha, T)$

The proof of this result is in Section B.3. Note that Proposition 1(ii) is a special case of Proposition 1(i). Proposition 1(ii) is a statement about problem ($\mathbf{P}'$) but is equally true of ($\mathbf{P}'_m$); that is,

$$V^*(\alpha, m, T) \leq V^{\mathsf{D}}(\alpha, m, T),$$

where $V^{\mathsf{D}}(\alpha, m, T)$ is the optimal deterministic revenue if the market size is m. It is easy to check that $V^{\mathsf{D}}(\alpha, m, T) = mV^{\mathsf{D}}(\alpha, T)$.

Proposition 1(ii) states that the dynamic pricing problem ($\mathbf{P}'$) is bounded above by its deterministic equivalent. This result is in the same spirit as Theorem 2 in Gallego and Van Ryzin (1994) which studies dynamic pricing in a continuous time model under independent demand. However, it should be noted that the proof of Proposition 1(ii) significantly differs from how Gallego and Van Ryzin (1994) establish their result for two reasons. First, multiple customers can purchase in a given period, which necessitates a demand censoring term in the revenue function (18). This makes the optimal revenue of the stochastic problem harder to analyze than the expected revenue resulting from a no-stockout almost surely policy. Second, demand is history-dependent, hence the optimal price for the deterministic relaxation is time-varying since the price in a period indirectly influences future demand. Thus, we need to work on the dynamic evolution of sales and remaining inventory while Gallego and Van Ryzin (1994) dealt with *i.i.d.* demand, which means their deterministic problem can be analyzed in a myopic sense.

Our proof overcomes these challenges by using dynamic programming (DP) formulations of the stochastic and deterministic versions of the problem. Recall that (21) is the DP formulation of $V^{\mathsf{D}}(u,\alpha,T')$. The optimal stochastic revenue $V^*(u,\alpha,T')$ can also be formulated as a DP by focusing on policies in $\mathbf{Y}$, i.e., the price does not stock out the remaining inventory in expectation. We establish that $V^*(u,\alpha,T')$ is bounded by the Lagrangian relaxation to the DP version of $V^{\mathsf{D}}(u,\alpha,T')$. Specifically, $V^*(u,\alpha,T')$ is bounded by (25) for any $\mu \geq 0$. Hence, Proposition 1(i) follows strong duality holds. This is established in the next theorem. The proof is in Section B.4. We note that, typically, the stochastic version of the problem was formulated and analyzed under the assumption that stock-outs do not occur almost surely. On the other hand, in our work, this method is too restrictive in our periodic review setting where there can be multiple orders. We relax this assumption so that we consider a class of policies that do not result in stock out in expectation.

Theorem 4 (Strong duality)**.** For any $T' \leq T$ and any $u \in (0,\alpha]$:

$$V^{\mathsf{D}}(u,\alpha,T') = \inf_{\mu\geq 0} L^{\mathsf{D}}(u,\alpha,T',\mu), \tag{24}$$

where, for any $\mu \geq 0$, $L^{\mathsf{D}}(u,\alpha,T',\mu)$ is defined as:

$$L^{\mathsf{D}}(u,\alpha,T',\mu) \triangleq \max_{y\in[0,1]} \left\{x^{-1}(y)\lambda(u,\alpha)y + V^{\mathsf{D}}\left(u-\lambda(u,\alpha)y,\alpha,T'-1\right) + \mu\left(u-\lambda(u,\alpha)y\right)\right\}. \tag{25}$$

Our next goal is to show that for a fixed α, the expected revenue of the CE intensity policy (denoted by $V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m,T)$ using notation in (18)) closely approximates $V^{\mathsf{D}}(\alpha,m,T)$ under the asymptotic setting when m grows large. Combined with Proposition 1 which states that $V^*(\alpha,m,T) \leq V^{\mathsf{D}}(\alpha,m,T)$, this implies that the CE intensity policy is asymptotically optimal when α is fixed.

However, to prove this result is nontrivial since the expected demand rate of the CE intensity policy depends on the cumulative sales and the amount of remaining inventory. Hence, we first need to prove that at any time t, the remaining inventory under the CE intensity policy converges in expectation to n_t^{D}, where $(n^{\mathsf{D}},y^{\mathsf{D}})$ is the optimal solution to ($\mathbf{D'}$) (where $u=\alpha$ and $T'=T$) whose optimal value is $V^{\mathsf{D}}(\alpha,T)$. This is shown in the Lemma 2 below. The proof is in Section B.5 and the following assumption is required for the proof.

Assumption 3. There exists a constant $\sigma \geq 0$ [7] such that the variance of demand D_t for every period t is less than $\sigma\mathbb{E}(D_t \mid \mathcal{F}_{t-1})$.

[7] If $\sigma = 0$ then we have a deterministic problem, which is also handled by our analysis.

Assumption 3 restricts our analysis to demand distributions where the order of the variance does not exceed the order of the expectation. Clearly, the Poisson distribution that is commonly used in revenue management literature satisfies this assumption.

Lemma 2 (Convergence of SIS and demand)**.** Given $\alpha \in [0,1]$, let $N^{\mathsf{CE}} = (N_0^{\mathsf{CE}}, \dots, N_T^{\mathsf{CE}})$ be the stochastic sequence of remaining inventories under $\mathbf{y}^{\mathsf{CE}}$ and define $n_t^{\mathsf{CE}} \triangleq N_t^{\mathsf{CE}}/m$. Then,

$$\mathbb{E}\left|n_t^{\mathsf{CE}} - n_t^{\mathsf{D}}\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right), \quad \text{for all } t \in \mathcal{T} \tag{26}$$

$$\mathbb{E}\left|\lambda\left(n_t^{\mathsf{CE}}, \alpha\right) - \lambda\left(n_t^{\mathsf{D}}, \alpha\right)\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right), \quad \text{for all } t \in \mathcal{T} \tag{27}$$

where $n^{\mathsf{D}} = (n_1^{\mathsf{D}}, \dots, n_T^{\mathsf{D}})$ is the partial optimal solution of (**D′**) with $u = \alpha$ and $T' = T$.

Roughly speaking, Lemma 2 implies that, although our CE policy just implements a portion of the deterministic policy and re-optimizes in each period, a sample path of states under the CE policy (in particular, the remaining inventory) will be very similar to that under the policy that optimizes V^{D}. Convergence results in Lemma 2 are necessary to establish the convergence of expected revenue under CE to its optimal counterpart. In comparison, for classical studies in dynamic pricing where λ is a known constant, Lemma 2 holds trivially because the right-hand-side of (27) is always zero. However, with a history dependent demand, this is not necessarily true for all policies and needs to be proved.

The proof of Lemma 2 requires that $\mathbf{y}^{\mathsf{CE}}$ is a Lipschitz continuous function on $[0,1] \times \mathcal{T}$, which implies that the difference in the two policies is not too large when the inventory level changes. This is a desirable property in practice since it leads to a relatively stable pricing policy against inventory dynamics. This result is stated below and proved in Section B.6.

Lemma 3 (Lipschitz continuous policy)**.** There exists C_y such that

$$\left|\mathbf{y}^{\mathsf{CE}}(u,t) - \mathbf{y}^{\mathsf{CE}}(u',t)\right| \le C_y \left|u - u'\right|, \quad \text{for all } t \in \mathcal{T}.$$

To prove Lemma 2, let N_t^{CE} be the random variable representing the remaining inventory at time t under policy $\mathbf{y}^{\mathsf{CE}}$. Because of the fact that the demand rate only depends on the state, remaining customers decisions depend on past purchases but are not affected by those who did not purchase until time t. One can approximate N_t^{CE} as m *i.i.d.* Bernoulli random variables (without loss of generality), each with a success probability $\mathbb{E}\left(n_t^{\mathsf{CE}}\right)$. This observations helps us find the gap between n_t^{CE} and its unconditional expectation $\mathbb{E}\left(n_t^{\mathsf{CE}}\right)$ without knowing the functional form of λ and its unconditional distribution. It is also crucial to the proof that $\mathbf{y}^{\mathsf{CE}}$ is a re-optimized policy so that the price it sets at each time does not stock out the remaining inventory in expectation. The re-optimized policy guarantees that the expected path of remaining inventory follows the same evolution as its deterministic path. If we do not re-optimize, the policy might result in stock-outs in expectation based on the current inventory level. This is not desirable as the linkage between the sequences $(n_0^{\mathsf{D}}, n_1^{\mathsf{D}}, \dots, n_T^{\mathsf{D}})$ and $(n_0^{\mathsf{CE}}, n_1^{\mathsf{CE}}, \dots, n_T^{\mathsf{CE}})$ critically relies on there being no censoring of demand.

An important implication of Lemma 2 is that the intensity policy $\mathbf{y}^{\mathsf{CE}}$ converges to the deterministic sequence $\mathbf{y}^{\mathsf{D}}$ since, with Lemma 3, we know that $\mathbf{y}^{\mathsf{CE}}$ is Lipschitz continuous. With this result, we now show that the *uncensored* expected revenue under $\mathbf{y}^{\mathsf{CE}}$ converges to $V^{\mathsf{D}}(\alpha, T)$. The uncensored revenue is computed assuming that all demands can be sold irrespective of the inventory level (and corresponds to the first sum in (28) below).

Lemma 4 (Convergence of uncensored revenue)**.** Given $\alpha \in [0, 1]$, let $N^{\mathsf{CE}} = (N_0^{\mathsf{CE}}, \ldots, N_T^{\mathsf{CE}})$ be the stochastic sequence of remaining inventories under $\mathbf{y}^{\mathsf{CE}}$ and define $n_t^{\mathsf{CE}} \triangleq N_t^{\mathsf{CE}}/m$. Then,

$$\left| \mathbb{E}\left(\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right) \lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right) \mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right) - V^{\mathsf{D}}(\alpha, T)\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right). \tag{28}$$

Proof. See Section B.7. □

We will show in the proof of Theorem 5 below that, because policy $\mathbf{y}^{\mathsf{CE}}$ is re-optimized in each period so that there is no stockout in expectation, its censored revenue $V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m, T)$ converges to the uncensored revenue as m grows large. Thus we will use Lemma 4 to establish the convergence of $V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m, T)$ to $V^{\mathsf{D}}(\alpha, m, T)$. Combined with Proposition 1 which states that $V^{\mathsf{D}}(\alpha, m, T)$ is an upper bound to the optimal expected revenue in ($\mathbf{P}'_m$), this proves that $\mathbf{y}^{\mathsf{CE}}$ is asymptotically optimal to ($\mathbf{P}'_m$).

Theorem 5 (CE intensity policy is asymptotically optimal)**.** Given $\alpha \in [0, 1]$,

$$1 - \frac{V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m, T)}{V^*(\alpha, m, T)} = \mathcal{O}\left(m^{-\frac{1}{2}}\right), \tag{29}$$

where $V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m, T)$ is the revenue of $\mathbf{y}^{\mathsf{CE}}$ and $V^*(\alpha, m, T)$ is the optimal revenue.

Proof. Since $V^*(\alpha, m, T) \le V^{\mathsf{D}}(\alpha, m, T)$ (Proposition 1), the LHS of (29) is bounded above by the LHS of (30). Hence, to prove the theorem, it is sufficient to show that

$$1 - \frac{V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m, T)}{V^{\mathsf{D}}(\alpha, m, T)} \le 1 - (1 - k)\left(1 - \frac{C}{2}\sqrt{\frac{\sigma}{m}}\right), \tag{30}$$

where $k = \Theta(m^{-\frac{1}{2}})$ and C is some constant that is independent of m.

Let $N^{\mathsf{CE}} = (N_0^{\mathsf{CE}}, \ldots, N_T^{\mathsf{CE}})$ be the stochastic sequence of remaining inventories under $\mathbf{y}^{\mathsf{CE}}$ and define $n_t^{\mathsf{CE}} \triangleq N_t^{\mathsf{CE}}/m$. Then from (18), we have

$$V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m, T) = \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}\left[x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)(d_t - [d_t - n_{t-1}^{\mathsf{CE}}]^+) \mid \mathcal{F}_{t-1}\right]\right]. \tag{31}$$

Using the fact that n_{t-1}^{CE} is not random when conditioning on the filtration $\mathcal{F}_{t-1}$, and using the

Scarf (1958) bound in (70), we get

$$
\begin{aligned}
&\mathbb{E}\left[\left[d_t - n^{\mathsf{CE}}_{t-1}\right]^+ \mid \mathcal{F}_{t-1}\right] \\
&\le \frac{\sqrt{\sigma m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right) + \left(N^{\mathsf{CE}}_{t-1} - m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right)^2}}{2} - \frac{\left(N^{\mathsf{CE}}_{t-1} - m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right)}{2} \\
&\le \frac{1}{2}\frac{\sigma m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)}{\sqrt{\sigma m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right) + (N^{\mathsf{CE}}_{t-1} - m\lambda(n^{\mathsf{CE}}_{t-1},\alpha)\mathbf{y}^{\mathsf{CE}}(n^{\mathsf{CE}}_{t-1},t))^2} + (N^{\mathsf{CE}}_{t-1} - m\lambda(n^{\mathsf{CE}}_{t-1},\alpha)\mathbf{y}^{\mathsf{CE}}(n^{\mathsf{CE}}_{t-1},t))} \\
&\le \frac{1}{2}\sqrt{\sigma m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)}. \qquad (32)
\end{aligned}
$$

The second inequality follows when we multiply the denominator and the numerator by

$$
\begin{aligned}
&\sqrt{\sigma m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right) + \left(N^{\mathsf{CE}}_{t-1} - m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right)^2} \\
&\qquad + \left(N^{\mathsf{CE}}_{t-1} - m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right).
\end{aligned}
$$

The last inequality comes from the fact that $N^{\mathsf{CE}}_{t-1} - m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right) \ge 0$ (since $\mathbf{y}^{\mathsf{CE}}$ guarantees that there are no stockouts in expectation).

Therefore, plugging (32) into the RHS of (31), we observe that

$$
\begin{aligned}
V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m,T) &\ge \mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right)\left(m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right) - \frac{1}{2}\sqrt{\sigma m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)}\right)\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right) m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right] \\
&\qquad - \frac{1}{2}\sqrt{\sigma m}\,\mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right)\sqrt{\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)}\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right) m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right] \\
&\quad \times \left(1 - \frac{1}{2}\sqrt{\frac{\sigma}{m}}\underbrace{\frac{\mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right)\sqrt{\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)}\right]}{\mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right)\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right]}}_{(**)}\right). \qquad (33)
\end{aligned}
$$

We get the first equality by multiplying x^{-1} term inside. The second equality comes from pulling out the first expectation term.

We first derive a lower bound for the first term in (33). Note that $\mathbf{y}^{\mathsf{CE}}$ does not scale with m since it it is constructed from solutions of (**D′**), which do not depend on m. From Lemma 4 and from $V^{\mathsf{D}}(\alpha,m,T) = mV^{\mathsf{D}}(\alpha,T)$, we know that the difference between the first term in (33) and $V^{\mathsf{D}}(\alpha,m,T)$ scales in $\mathcal{O}(\sqrt{m})$. This is slower than the speed of scaling $\Theta(m)$ of $V^{\mathsf{D}}(\alpha,m,T)$. Hence,

$$
\mathbb{E}\left(\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right) m\lambda\left(n^{\mathsf{CE}}_{t-1},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n^{\mathsf{CE}}_{t-1},t\right)\right) \ge V^{\mathsf{D}}(\alpha,m,T)(1-k), \qquad (34)
$$

where $k = \Theta(m^{-\frac{1}{2}})$.

Next, we want to derive an upper bound for the term $(**)$, which results in a lower bound for the second term in (34). Note that from Cauchy-Swartz inequality, the numerator of $(**)$ is bounded above by

$$
\begin{aligned}
&\mathbb{E}\left[\sqrt{\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)}\sqrt{\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)}\right] \\
&\leq \mathbb{E}\left[\sqrt{\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}} t\right)}\right]\sqrt{Tx^{-1}(0)} \\
&\leq \sqrt{\mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right]}\sqrt{Tx^{-1}(0)}, \qquad (35)
\end{aligned}
$$

where the first inequality comes from Assumption 2(ii), and the last inequality comes from Jensen's inequality and the fact that $\sqrt{z}$ is a concave function. Hence,

$$
(**) \leq \sqrt{\frac{Tx^{-1}(0)}{\mathbb{E}\left[\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right]}} \leq \sqrt{\frac{Tx^{-1}(0)}{V^{\mathsf{D}}(\alpha, T)(1-k)}}, \qquad (36)
$$

where the last inequality comes from (34).

Since $\Theta(m^{-\frac{1}{2}})$ decreases as m grows, we know there exists some constant $\Theta(1)$ unaffected by m such that $\Theta(m^{-\frac{1}{2}}) \leq \Theta(1)$. Therefore, we know

$$
\sqrt{\frac{1}{1-k}} = \sqrt{\frac{1}{1-\Theta(m^{-\frac{1}{2}})}} \leq \sqrt{\frac{1}{1-\Theta(1)}} = \Theta(1). \qquad (37)
$$

Hence, we have that

$$
(**) \leq \sqrt{\frac{Tx^{-1}(0)}{V^{\mathsf{D}}(\alpha, T)}}\Theta(1) \triangleq C. \qquad (38)
$$

Finally, we take (34) and (38) into (33), resulting in

$$
V^{\alpha, \mathbf{y}^{\mathsf{CE}}}(m, T) \geq V^{\mathsf{D}}(\alpha, m, T)(1-k)\left(1 - \frac{1}{2}\sqrt{\frac{\sigma}{m}}C\right). \qquad (39)
$$

This completes the proof. □

Theorem 5 states that applying the certainty equivalent policy $\mathbf{y}^{\mathsf{CE}}$ achieves asymptotically optimal revenue as m grows. This means that for products with a large addressable market, the certainty equivalent policy with re-optimization is close to optimal. In our simulations in Section 7, m greater than 1000 was "large", in the sense that the percentage revenue loss of $\mathbf{y}^{\mathsf{CE}}$ is no more than 0.4%.

We next discuss the similarities and differences of the asymptotic convergence of Theorem 5 to Theorem 3 of Gallego and Van Ryzin (1994). In the continuous time model of Gallego and Van Ryzin (1994), the certainty equivalent policy is asymptotically optimal as T grows, where the mean of cumulative demand scales in T and its standard deviation scales in $\sqrt{T}$. That is, in their setting, the market size also grows as T grows. Similarly, in our setting, the mean of cumulative demand scales in m and its standard deviation scales in $\sqrt{m}$ because of Assumption 3. In this sense, m and T play analogous roles in terms of demand asymptotics. Also, in both settings, the convergence results require the amount of initial inventory to scale. Despite these similarities, there are several key differences that are important to point out.

First, in our setting with history-dependent demand, Lemmas 2 and 4 are critical to establish the bound on asymptotic revenue loss. In contrast, in Gallego and Van Ryzin (1994) where the SIS function λ is a constant, Lemmas 2 and 4 are not needed since the per period revenue loss is independent of the demand path. However, when λ is affected by the historical demand, the difference in the demand path of the certainty equivalent policy compared to that of the optimal policy factors into the revenue loss. This means that if the expected path of remaining inventory $(n_0^{\mathsf{CE}}, n_1^{\mathsf{CE}}, \ldots, n_T^{\mathsf{CE}})$ does not converge to $(n_0^{\mathsf{D}}, n_1^{\mathsf{D}}, \ldots, n_T^{\mathsf{D}})$, it is not possible to establish the convergence of the expected revenue under $\mathbf{y}^{\mathsf{CE}}$ to $V^{\mathsf{D}}(\alpha, T)$. This is one of the notable difference of our analysis from traditional revenue management work assuming history-independent demand.

Second, in our setting, the number of price changes T does not need to grow with the market size m to guarantee asymptotic optimality of the certainty equivalent pricing policy. In contrast, because Gallego and Van Ryzin (1994) are dealing with a continuous-time pricing policy, the asymptotic convergence of the pricing policy with T requires *both* the number of price changes and the size of demand (and supply) to grow. The intuition behind their result is that as the problem size scales up, fluctuations in sales are "smoothed" out. Here, a growing number of customers arrive but sticking to few (i.e., a fixed number of) price change is asymptotically optimal.

Finally, Gallego and Van Ryzin (1994) considers a continuous time model and the optimal policy does not stock out *almost surely*. This is because, in their setting, the choke price can always be set as soon as the inventory stocks out. On the other hand, this is not possible with a discrete time setting, making the analysis more challenging. Nevertheless, we are able to prove asymptotic convergence of the certainty equivalent policy under the setting where there are no stockouts *in expectation*.

5 Joint optimization of starting inventory and pricing

In the previous section, we analyzed the pricing problem when the initial inventory is given (represented by the market coverage, α). We extend this to the case where the seller needs to place an order to build up its inventory. Recall that the expected profit under policy $(\alpha, \mathbf{y})$ is $Q^{\alpha,\mathbf{y}}(m, T)$ defined in (8) and the expected profit under the optimal policy is $Q^*(m, T)$ defined in $(\mathbf{P}_m)$. The only difference from the previous section is that now α is a decision.

We now define the certainty equivalent (CE) policy for jointly determining the starting inventory and prices. Algorithm 2 gives a description of the CE policy. Specifically, it sets the market coverage $\alpha^{\mathsf{CE}} \in [0,1]$ by solving the following problem:

$$Q^{\mathsf{D}}(T) \triangleq \max_{\alpha\in[0,1]} V^{\mathsf{D},\alpha}(T) - c\alpha, \tag{\textbf{D}}$$

where we write $V^{\mathsf{D},\alpha}(T)$ instead of $V^{\mathsf{D}}(\alpha, T)$ to emphasize that α is a decision variable. Note that $V^{\mathsf{D},\alpha}(T) - c\alpha$ is the deterministic counterpart of $Q^{\alpha,\mathbf{y}}(T)$, hence the CE policy is solving a deterministic version of (**P**). Given α^{CE}, it then sets $\mathbf{y}^{\mathsf{CE}} : [0,1] \times \mathcal{T} \mapsto [0,1]$ as the certainty equivalent intensity policy described in the previous section where $\alpha = \alpha^{\mathsf{CE}}$. Recall that in Theorem 1, we prove that the deterministic value function $V^{\mathsf{D}}(u,\alpha,T)$ is jointly concave in (u,α) for given T. This implies that solving for the certainty equivalent market coverage α^{CE} can be simply done by gradient methods like the Newton algorithm. We show that the regret under Algorithm 2 grows sub-linearly – this means the joint decision CE policy is asymptotically optimal.

Algorithm 2: Setting initial inventory and prices with the CE policy.

```
1: procedure CERTAINTY EQUIVALENT(T)
2:     α^CE ← optimal solution of (D)
3:     set I_0 = α^CE m                                         ▷ set initial inventory
4:     set prices with CERTAINTY EQUIVALENT PRICING(α^CE, T)    ▷ see Algorithm 1
5: end procedure
```

Theorem 6 below is the main result of this section. The proof is in Section C.1.

Theorem 6 (CE policy is asymptotically optimal)**.**

$$1 - \frac{Q^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T)}{Q^{*}(m,T)} = \mathcal{O}\left(m^{-\frac{1}{2}}\right), \tag{40}$$

where $Q^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T)$ is the profit of the certainty equivalent policy $(\alpha^{\mathsf{CE}}, \mathbf{y}^{\mathsf{CE}})$ and $Q^{*}(m,T)$ is the optimal profit.

This result shows that the CE policy guarantee a close-to-optimal profit in practice. This result is somewhat surprising since (**D**) is an approximation of the stochastic problem (**P**), α^{CE} is not necessarily equal to the optimal market coverage (which we denote by α^{*}). Indeed, when m is large, the initial inventory $\alpha^{\mathsf{CE}}m$ of the policy may be quite different from $\alpha^{*}m$. In particular, the fact that the CE policy may choose a different market coverage implies that the asymptotic optimality in Theorem 6 does not follow immediately from Theorem 5.

6 Comparison with static price

So far, we compared the performance of our CE policy to the optimal policy. We showed that our CE policy is implementable (while the optimal policy may not) and its performance is asymptotically optimal. In this section, we compare the CE policy to the a static pricing policy proposed by Gallego and Van Ryzin (1994). A static policy is easy to implement and it performs very well in a traditional dynamic pricing setting when demand in each period is *i.i.d.*. For instance, Gallego and Van Ryzin (1994) showed that a static pricing policy is asymptotically optimal in a setting that demand arrives according to a homogeneous Poisson process.

Similar to what we did for $\mathbf{y}^{\mathsf{CE}}$, we introduce the static policy $\mathbf{y}^{\mathsf{SP}}$ for a given market coverage $\alpha \in [0, 1]$. If α is sufficiently large, $\mathbf{y}^{\mathsf{SP}} = \bar{y}$, where $\bar{y} \in [0, 1]$ is the unique maximizer of the revenue function, i.e., $\bar{y} \triangleq \arg\max_{y\in[0,1]} x^{-1}(y)y$. In other words, if the inventory constraint is nonbinding, the static policy chooses the intensity that maximizes the current period revenue only, without considering the effects of inventory and sales on the demand. If the inventory constraint is binding, the static policy instead chooses the intensity so that the expected demand equals the initial inventory, i.e., the fixed point y^{so} of the equation:

$$\bar{y}^{\text{so}} = \frac{\alpha}{\sum_{t=1}^{T} \lambda\left(n_{t-1}^{\bar{y}^{\text{so}}}, \alpha\right)}, \tag{41}$$

where, for any $y \in [0, 1]$, $(n_0^y, n_1^y, \ldots, n_T^y)$ is defined as the deterministic sequence with $n_0^y = \alpha$ and $n_t^y = n_{t-1}^y - \lambda(n_{t-1}^y, y)y$ for all $t \in \mathcal{T}$. Note that $\bar{y}^{\text{so}}$ can be found by fixed point iteration. Mathematically, for a fixed α, the static policy $\mathbf{y}^{\mathsf{SP}}$ is defined for every $(n, t) \in (0, \alpha] \times \mathcal{T}$ as:

$$\mathbf{y}^{\mathsf{SP}}(n, t) = y^{\mathsf{SP}} \triangleq \begin{cases} \bar{y}, & \text{if } \alpha \geq \sum_{t=1}^{T} \lambda\left(n_{t-1}^{\bar{y}}, \alpha\right)\bar{y}, \\ \bar{y}^{\text{so}}, & \text{otherwise.} \end{cases} \tag{42}$$

To implement this, the seller will charge a price (y^{SP}) for T periods.

We are now ready to describe the static policy $(\alpha^{\mathsf{SP}}, \mathbf{y}^{\mathsf{SP}})$. As outlined in Algorithm 3, the static policy sets α^{SP} by solving

$$Q^{\mathsf{SP}}(T) \triangleq \max_{\alpha\in[0,1]} V^{\mathsf{SP}}(\alpha, T) - c\alpha, \tag{\textbf{S}}$$

where V^{SP} is the deterministic revenue under policy $\mathbf{y}^{\mathsf{SP}}$ given by

$$V^{\mathsf{SP}}(\alpha, T) \triangleq \sum_{t=1}^{T} x^{-1}\left(y^{\mathsf{SP}}\right)\lambda(n_{t-1}^{\mathsf{SP}}, \alpha)y^{\mathsf{SP}}, \tag{43}$$

where $n_0^{\mathsf{SP}} = \alpha$ and $n_t^{\mathsf{SP}} = n_{t-1}^{\mathsf{SP}} - \lambda(n_{t-1}^{\mathsf{SP}}, \alpha)y^{\mathsf{SP}}$ for all $t \leq T$. Then given α^{SP}, it sets $\mathbf{y}^{\mathsf{SP}}$ as the static intensity policy just described with $\alpha = \alpha^{\mathsf{SP}}$.

Algorithm 3: Setting the initial inventory and prices based on static policy.

procedure STATIC(T)
  $\alpha^{\mathsf{SP}} \leftarrow$ optimal solution of (**S**)
  set $I_0 = \alpha^{\mathsf{SP}} m$ ▷ set initial inventory
  set prices with STATIC PRICING(α^{SP}, T)
end procedure

procedure STATIC PRICING(α, T)
  $y^{\mathsf{SP}} \leftarrow \bar{y}$ or $\bar{y}^{\mathsf{so}}$ based on cases in (42) for α
  for $t \leftarrow 1$ **to** T **do**
    set intensity y^{SP} by offering price $x^{-1}(y^{\mathsf{SP}})$ ▷ set current intensity (price)
  end for
end procedure

We next state the main result of this section which describes the performance of the static policy under our demand model with sales- or inventory-dependent demand. Note that using notation introduced in (10) and (18), $Q^{\alpha,\mathbf{y}^{\mathsf{SP}}}(m,T)$ and $V^{\alpha,\mathbf{y}^{\mathsf{SP}}}(m,T)$ are the revenue and profit, respectively, under market coverage α and static intensity policy $\mathbf{y}^{\mathsf{SP}}$.

Proposition 2 (Profit loss of the static policy)**.** When $T \geq 2$, if the following conditions hold for a fixed $\alpha \in [0,1]$:

(i) $\frac{\partial}{\partial y} V^{\mathsf{D}}\left(\alpha - \lambda(\alpha,\alpha)y, \alpha, T-1\right)\Big|_{y=\bar{y}} \neq 0$, and

(ii) $\alpha \geq \sum_{t=1}^{T} \lambda(n_{t-1}^{\bar{y}}, \alpha)\bar{y}$,

then

$$V^*(\alpha, m, T) - V^{\alpha,\mathbf{y}^{\mathsf{SP}}}(m,T) = \Omega(m). \tag{44}$$

Moreover, if (i)–(ii) hold for $\alpha = \alpha^{\mathsf{SP}}$, then

$$Q^*(m,T) - Q^{\alpha^{\mathsf{SP}},\mathbf{y}^{\mathsf{SP}}}(m,T) = \Omega(m). \tag{45}$$

We relegate the proof to the end of this section. Proposition 2 shows that both profit loss and revenue loss of a static policy grows at least linearly in the market size m. One may think that the reason that the static policy performs poorly is that it may not start with the optimal market coverage, i.e., $\alpha^{\mathsf{SP}} \neq \alpha^*$. However, (45) shows that, regardless of the initial inventory level, the profit loss grows at least at a linear rate in the market size. This shows that the inability to adjust the price results in a much greater loss when demand is fast moving and history dependent.

In contrast, the CE policy with re-optimization allows the seller to adjust the price based on cumulative sales and remaining inventory. Thus, whether the future demand is driven by past sales or by inventory availability or by both, the seller can account for the current revenue as

$$\begin{tikzcd} \text{Lemma 5:} V^{\mathsf{D}}(\alpha, m, T) - V^{\mathsf{SP}}(\alpha, m, T) \arrow[d, <->, shift right=8ex, "\substack{\text{Theorem 5,}\\ \text{Proposition 1}}"'] \arrow[d, <->, shift left=8ex, "\text{Lemma 6}"] \\ \text{Proposition 2: } V^*(\alpha, m, T) - V^{\alpha, \mathbf{y}^{\mathsf{SP}}}(m, T) \end{tikzcd}$$

Figure 3: Illustration of proof logic of Proposition 2

well as the future revenue when setting the price. The difference in two pricing policies can be demonstrated when $T = 2$. Proposition 2 shows that a static pricing results in a loss growing at least at a linear rate in the market size. On the other hand, the loss of the CE policy is bounded by $\mathcal{O}(m^{\frac{1}{2}})$: see Theorems 5 and 6. This means, even a single opportunity to change the price (based on the sales and inventory) can substantially reduce the profit loss.

Since it is not possible to characterize the exact revenue difference between the optimal and the static policy, to prove Proposition 2, we utilize the bound established by V^{D}. To see this, an implication of our results in Section 4 is $0 \le V^{\mathsf{D}}(\alpha, m, T) - V^*(\alpha, m, T) \le \mathcal{O}(\sqrt{m})$ (Proposition 1, Theorem 5). In other words, $V^{\mathsf{D}}(\alpha, m, T)$ is a good approximation of the optimal revenue in an asymptotic regime. Hence, if we are able to show that

$$V^{\mathsf{D}}(\alpha, m, T) - V^{\alpha, \mathbf{y}^{\mathsf{SP}}}(m, T) = \Omega(m), \tag{46}$$

then this establishes (44) in Proposition 2. Note that this also proves (45) since the profit loss of the static policy $(\alpha^{\mathsf{SP}}, \mathbf{y}^{\mathsf{SP}})$ is bounded below by the revenue loss (44) of $\mathbf{y}^{\mathsf{SP}}$ with $\alpha = \alpha^{\mathsf{SP}}$.

As illustrated in Figure 3, we need two key results to prove (46), and, hence Proposition 2.

The first key result in establishing (46) is to show that $V^{\mathsf{D}}(\alpha, m, T) - V^{\mathsf{SP}}(\alpha, m, T) = \Theta(m)$, where $V^{\mathsf{SP}}(\alpha, m, T) \triangleq mV^{\mathsf{SP}}(\alpha, T)$. That is, the difference grows at a linear rate in m. This is formalized in the following lemma (whose proof is in Section D.1).

Lemma 5 (Revenue loss of static policy for deterministic problems)**.** When $T \ge 2$, for a fixed $\alpha \in [0, 1]$, if

(i) $\frac{\partial}{\partial y} V^{\mathsf{D}}\left(\alpha - \lambda(\alpha, \alpha) y, \alpha, T - 1\right)\Big|_{y=\bar{y}} \neq 0$, and

(ii) $\alpha \ge \sum_{t=1}^{T} \lambda(n_{t-1}^{\bar{y}}, \alpha)\bar{y}$,

then $V^{\mathsf{D}}(\alpha, m, T) - V^{\mathsf{SP}}(\alpha, m, T) = \Theta(m)$.

Condition (i) of Lemma 5 implies the myopic optimal intensity $\bar{y}$ is not the optimal deterministic price for $V^{\mathsf{D}}(\alpha, T)$. Condition (ii) means that we have a sufficient amount of initial inventory if we use to set the price at $x^{-1}(\bar{y})$.

The second key piece is the following lemma, which can be established from results in Section 4, is that the gap between $V^{\alpha, \mathbf{y}^{\mathsf{SP}}}(m, T)$ and $V^{\mathsf{SP}}(\alpha, m, T)$ is $\mathcal{O}(\sqrt{m})$. The proof is in Section D.4.

Lemma 6. For a fixed $\alpha \in [0, 1]$,

$$V^{\alpha,\mathbf{y}^{\mathsf{SP}}}(m, T) \le V^{\mathsf{SP}}(\alpha, m, T) + \mathcal{O}(\sqrt{m}), \tag{47}$$

where $V^{\mathsf{SP}}(\alpha, m, T) = mV^{\mathsf{SP}}(\alpha, T)$ and $V^{\mathsf{SP}}(\alpha, T)$ is defined in (43).

Now we are ready to prove the main result of this section.

Proof of Proposition 2. From the definition that $Q^*(m, T)$ is the optimal profit, we know $Q^*(m, T) \ge V^*(\alpha^{\mathsf{SP}}, m, T) - m\alpha^{\mathsf{SP}}c$. Then,

$$\begin{aligned} Q^*(m, T) - Q^{\alpha^{\mathsf{SP}},\mathbf{y}^{\mathsf{SP}}}(m, T) &\ge V^*(\alpha^{\mathsf{SP}}, m, T) - m\alpha^{\mathsf{SP}}c - \left(V^{\alpha^{\mathsf{SP}},\mathbf{y}^{\mathsf{SP}}}(m, T) - m\alpha^{\mathsf{SP}}c\right) \\ &= V^*(\alpha^{\mathsf{SP}}, m, T) - V^{\alpha^{\mathsf{SP}},\mathbf{y}^{\mathsf{SP}}}(m, T). \end{aligned} \tag{48}$$

Hence, to prove the proposition, it suffices to show (44) for any fixed $\alpha \in [0, 1]$.

We know that $V^*(\alpha, m, T)$ is bounded below by $V^{\alpha,\mathbf{y}^{\mathsf{CE}}}(m, T)$. Hence, by Theorem 5, we have that $V^*(\alpha, m, T) \ge V^{\mathsf{D}}(\alpha, m, T) - \mathcal{O}(\sqrt{m})$. This and Lemma 6 result in

$$V^*(\alpha, m, T) - V^{\alpha,\mathbf{y}^{\mathsf{SP}}}(m, T) \ge V^{\mathsf{D}}(\alpha, m, T) - \mathcal{O}(\sqrt{m}) - V^{\mathsf{SP}}(\alpha, m, T) - \mathcal{O}(\sqrt{m}). \tag{49}$$

Moreover, according to Lemma 5, we know the RHS of (49) equals to $\Theta(m) - \mathcal{O}(\sqrt{m})$, which is $\Omega(m)$. This concludes the proof. □

7 Numerical Study

In this section, we conduct several numerical experiments to demonstrate the performance of the CE policy with re-optimization. We first illustrate the analytic properties of the deterministic value function $V^{\mathsf{D}}(\alpha, T)$ and show why using V^{D} in pricing and initial inventory decisions makes the CE policy implementable. In Section 7.2, we show the CE converges fast numerically and can achieve close-to-optimal performance even in instances with small market size. In Sections 7.3 and 7.4, we experiment on the number of price changes and demonstrate the value of dynamic pricing.

7.1 The deterministic revenue $V^{\mathsf{D}}(\alpha, T)$ and the initial inventory problem

We illustrate the deterministic revenue function $V^{\mathsf{D}}(\alpha, T)$ with a concrete example adapted from the literature. We choose an intensity function $x(\pi) = e^{-\gamma\pi} - c_x$. We consider a case where the demand is influenced by both the past purchases and inventory availability which, as discussed in the introduction, are features of fast-moving products (e.g., V-commerce). Thus, we set the SIS function of this example λ to be a mixture of the SIS functions in Smith and Agrawal (2017) and Bass (1969) described in Sections 3.1.1 and 3.1.3, respectively. In particular,

$$\lambda(n, \alpha) = \left(w\lambda^{(1)}(n, \alpha) + (1 - w)\lambda^{(2)}(n, \alpha)\right)\Delta t \tag{50}$$

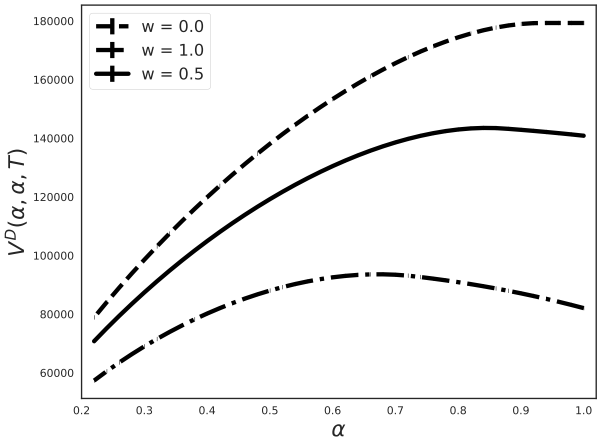

Figure 4: The deterministic revenue function (**D′**) plotted against the market coverage α, for different values of w in (50).

where $\lambda^{(1)}(n, \alpha) = \left((n - \alpha^2 + 1)/N_r\right)^\beta$ (cf. 16a) and $\lambda^{(2)}(n, \alpha) = (1 - (\alpha - n))\,(p + q(\alpha - n))$ (cf. 13). Note that we modified (16a) by adding $1 - \alpha^2$ to n so that the initial inventory affects demand and so that $\lambda(n, \alpha)$ is jointly concave in (n, α). Note that these modifications have no effect on the qualitative properties of the optimal prices in Smith and Agrawal (2017). Here Δt is the length of each time period and $\lambda(n, \alpha)$ in (50) is jointly concave in (n, α).

The parameters used in this example are

$$(p, q, N_r, \beta, \gamma, c_x, T, \Delta t) = (0.4, 0.6, 25, 0.6, 0.001, 0.01, 10, 2). \tag{51}$$

Figure 4 plots the optimal value function $V^{\mathsf{D}}(\alpha, T)$ as a function of the market coverage α with different weights w of the SIS function (50). The figure illustrates that $V^{\mathsf{D}}(\alpha, T)$ is concave in α, which agrees with Theorem 1. When $w = 0.0$, only network and saturation effects come into play and it is optimal to fully serve the market ($\alpha = 1$). When $w = 1.0$, only scarcity effects are felt and the optimal market coverage is $\alpha = 0.68$. When $w = 0.5$ (network effect, saturation effect, and scarcity effect are all present), the optimal choice of market coverage is $\alpha = 0.84$. This complex example with all three effects present shows that we should choose $\alpha < 1$ in the presence of scarcity effect of inventory. Based on Figure 4, we conjecture (possibly under additional assumptions) that the market coverage α which maximizes V^{D} is strictly less than 1 when demand is affected by inventory.

7.2 Revenue loss of the certainty equivalent policy

We next illustrate the performance of the CE policy with re-optimization on the demand pattern considered in Section 7.1. We set $w = 0.5$, thus both display effect and work-of-mouth effect are present. From the previous experiments, the CE policy sets an market coverage $\alpha^{\mathsf{CE}} = 0.84$. The dynamic pricing policy $\mathbf{y}^{\mathsf{CE}}$ is based on re-optimizing (**D′**) in each period with updated inventory levels.

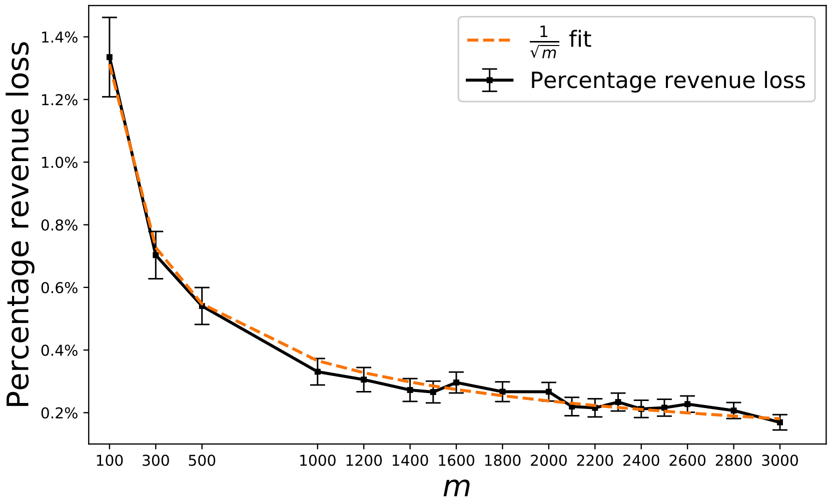

Figure 5: Upper bound of the percentage revenue loss of the certainty equivalent policy against the optimal value of the stochastic problem

We vary the population size m from 100 to 3000, with discretizations shown in the horizontal axis of Figure 5. For each m, we randomly generate 2×10^4 demand sample paths; we implement the dynamic pricing policy $\mathbf{y}^{\mathsf{CE}}$ and record the realized revenue on each path. The revenue averaged over the sample paths, which we denote by $\bar{V}^{\mathsf{CE}}(m,T)$, is an approximation for the expected revenue of the certainty equivalent policy, $V^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T)$. We also note the 95% confidence intervals of this sample average.

Since the optimal revenue $V^*(m,T)$ is impossible to compute for problems with large m, we compute $V^{\mathsf{D}}(m,T)$ (which is an upper bound of $V^*(m,T)$) for a comparison. Based on our sample approximation for $V^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T)$ for each m, we compute an upper bound for the revenue loss of the CE policy as $(V^{\mathsf{D}}(m,T) - \bar{V}^{\mathsf{CE}}(m,T))/V^{\mathsf{D}}(m,T)$, which are shown as the dots in Figure 5. The figure also shows the 95% confidence intervals of the revenue loss bound. From Theorem 5, we know that the upper bound on the revenue loss is $\mathcal{O}(m^{-\frac{1}{2}})$, which is tightly traced by the $m^{-\frac{1}{2}}$ fit, shown with a dashed line in Figure 5. We further observe that the revenue loss by implementing $\mathbf{y}^{\mathsf{CE}}$ is very small ($\sim 0.15\%$ when $m = 3000$). This implies that, for a product with market size even as small as 100–3000, the certainty equivalent policy $\mathbf{y}^{\mathsf{CE}}$ performs well. One may wonder how well the best static policy performs for the same problem. In all our examples, the static policy has a percentage revenue loss greater than or equal to 30% (we omit this from the figure to better highlight the difference between CE and the optimal policy).

7.3 Value of dynamic pricing

We next illustrate the value of dynamic pricing by comparing the dynamic policy $\mathbf{y}^{\mathsf{CE}}$ to the static policy $\mathbf{y}^{\mathsf{SP}}$. To isolate the value from dynamic pricing, we assume that both policies start with the same market coverage $\alpha = 0.84$. Using the same experimental setup as Section 7.2, for each m, we compute the realized revenue of the static policy $\mathbf{y}^{\mathsf{SP}}$ on the 2×10^4 demand

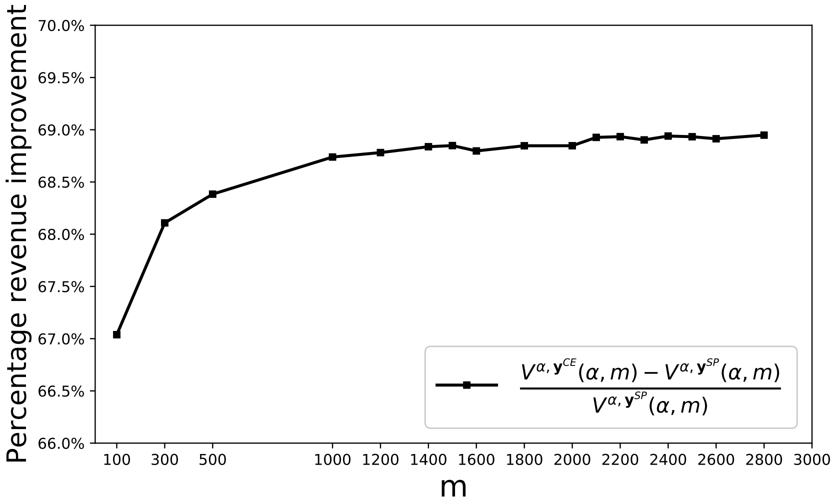

Figure 6: Value of dynamic pricing

sample paths. The revenue averaged over the samples, which we denote by $\bar{V}^{\mathsf{SP}}(m,T)$, is an approximation for the expected revenue of the static policy, $V^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T)$. Computing the percentage revenue improvement from $\bar{V}^{\mathsf{SP}}$ to $\bar{V}^{\mathsf{CE}}$ provides a proxy for the value of dynamic pricing, which we plot in Figure 6 as a function of m.

From this figure, we note that the percentage improvement over static pricing is consistently $\sim 69\%$. Further, even if changing prices only $T = 10$ times, there is already a significant revenue improvement compared with static pricing policy. Consistent with the findings in Yang and Zhang (2014) which only considers scarcity effect, our numerical experiment shows that the value of dynamic pricing is significant with both scarcity and network effect. This result is different from Gallego and Van Ryzin (1994) where they conclude that when demand intensity is stationary over time, a static pricing policy can extract almost all the revenue compared with the optimal case. The value of dynamic pricing is heightened by scarcity and network effects.

7.4 Revenue loss due to limited price changes

The certainty equivalent policy $(\alpha^{\mathsf{CE}}, \mathbf{y}^{\mathsf{CE}})$ is a discrete-time policy which assumes that the underlying demand is modeled as a discrete-time process. Hence, an interesting question to ask is: how much revenue can the discrete-time policy lose if the true demand is a continuous-time process? To answer this question, we run experiments on demand that is modeled as a continuous time Markov chain with the state variable $n = N/m$, where $N = \alpha m, \alpha m - 1, ..., 0$. The transition rate is $\lambda(n,\alpha)x(\pi)/\Delta t$, with $\lambda(n,\alpha)$ given in (50). That is, conditional on the remaining inventory being N, the probability of having one sale during a time period of length $o(t)$ is

$$\mathbb{P}\left(n_{t+o(t)} = \frac{N-1}{m} \middle| n_t = \frac{N}{m}\right) = \lambda\left(\frac{N}{m}, \alpha\right) o(t),$$

Table 3: The expected revenue of the discrete-time policy normalized with the expected revenue of a continuous-time policy

T: Number of price changes	1	2	4	5	10	17	22	35	45
95 % CI lower bound	70.3%	95.6%	97.9%	98.4%	99.4%	99.6%	99.8%	99.9%	100.0%
Expected normalized revenue	70.3%	95.6%	97.9%	98.5%	99.4%	99.7%	99.8%	100.0%	100.0%
95 % CI upper bound	70.3%	95.7%	98.0%	98.5%	99.4%	99.7%	99.9%	100.0%	100.0%

and there is $o(t)$ probability of having more than one sale during a time period of length $o(t)$.

To see the loss due to the discrete approximation, we experiment with different values for Δt, the length of time between price changes. We do this while keeping the total planning horizon length $\bar{T} = T\Delta t$ unchanged. In particular, the case when Δt approaches zero represents continuous price changes, which serves as a benchmark for the discrete-time model. For a given $(T, \Delta t)$ pair, we compute the certainty equivalent policy $(\alpha^{\mathsf{CE}}, \mathbf{y}^{\mathsf{CE}})$ and implement the discrete-time policy in 8×10^3 sample paths simulated from the continuous-time Markov chain process.

For the various values of T, Table 3 reports the average revenue (and 95% confidence intervals) of the certainty equivalent policy normalized against the average revenue with $T = 45$ price changes (i.e., the continuous-time policy benchmark). Notice that we can see diminishing marginal returns when increasing the number of price changes. Consistent with Section 7.2, we observe a sharp increase in revenue when the number of price changes increases from 1 to 10. However, we observe that 10 price changes is almost as good as continuous price changes. This gives us evidence that, sometimes a few price changes is good enough to capture the revenue from changing price continuously (which is very costly in practice).

8 Conclusion

We consider a pricing problem for "fast-moving" products. In contrast to the demand models in traditional dynamic pricing model, we capture two distinct forces that critically influence future demand. The first force is that future demands are influenced by past sales (adoptions, or endorsements). In certain markets (Weishang, V-commerce), this effect often creates demand "hype" (rapid surge of demands driven by word-of-mouth or community endorsement). The second force is that future demand is influenced by inventory availability. This force is often manifested in forms of scarcity (in case of luxury or fad items) or billboard effect and can be found in fast-moving product markets. When these forces are strong, traditional dynamic pricing models that assume independent demand and frequent price changes can no longer capture the key features of such fast-moving markets. We propose a general framework to model the seller's pricing (and initial inventory) problem in such cases. We show that as the market size, m increases, the CE policy with re-optimization is asymptotically optimal with $\mathcal{O}(\sqrt{m})$ regret rate when compared with the optimal policy. We then extend our results to the case where the seller chooses initial inventory along with price in each period.

To highlight the difference of dynamic pricing for “fast-moving” products against traditional demand assumptions used in the dynamic pricing literature, we also evaluate the performance of the static pricing policy (which was proven to be optimal in classical settings) under our model with path dependent demand. We show that the revenue loss from static pricing can be huge and it grows at least at the rate of a linear function when demand is history-dependent.

An accompanying numerical study shows both performance and implementability of the CE policy. We show that the CE policy performs closely to the optimal policy even in cases where potential market size is not large. Furthermore, we show that significant revenue improvement can be achieved by just a few price changes.

There are several future directions for our work. One is to extend the framework to the multi-product case where those products share the same market. Another extension is to consider strategic customers. The customers can strategically wait until there is a discount. Sapra et al. (2010) touches on this with the wait-list effect, where here it may be that a customer registers some interest in the product (follows on Twitter) but is waiting for a sale. Another direction is to incorporate learning into our model. Here, we assume that the conditional expectation of the demand is known. It is possible to approximate the expectation using available data throughout the selling horizon.

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Appendix A Section 3 proofs

A.1 Proof of Lemma 1

Proof. We prove $r(y)$ is concave in y first. Using the product and inverse differentiation rules, and the fact that $\pi = x^{-1}(y)$, yields

$$\frac{d^2}{dy^2}[x^{-1}(y)y] = \frac{2 - \frac{x''(\pi)y}{x'(\pi)^2}}{x'(\pi)}.$$

By Assumption 2(ii) the denominator is negative. Assumption 2(iii) implies, after taking derivatives, that $2 - \frac{x''(\pi)x(\pi)}{x'(\pi)} > 0$. Since $y_i \in [0,1]$, this implies that the numerator is positive. Thus, $\frac{d^2}{dy_i^2}[x^{-1}(y)y] < 0$ and the concavity of $r(y)$ follows.

Besides the concavity of $r(y)$, the other properties are immediate from the relationships $y = x(\pi)$, $\rho(\pi) = r(y)$, the properties of x^{-1}, and Assumption 2(iv),(vii). □

Appendix B Section 4 proofs

B.1 Proof of Theorem 1

Proof. We prove strict concavity of $V^{\mathsf{D}}(u, \alpha, T')$ through a reformulation of (**D**$'$) using the transformation $d_t = \lambda(n_{t-1}, \alpha)y_t$ to yield:

$$\begin{aligned} V^{\mathsf{D}}(u, \alpha, T') = \max_{n,d} \; & \sum_{t=1}^{T'} x^{-1}\left(\frac{d_t}{\lambda(n_{t-1}, \alpha)}\right) \cdot d_t \\ \text{s.t.} \; & \sum_{t=1}^{T} d_t \le u \\ & n_t = n_{t-1} - d_t \quad \text{for all } t \ge 1 \\ & n_0 = u. \end{aligned} \tag{52}$$

For any $(u_1, \alpha_1) \ge 0$ and $(u_2, \alpha_2) \ge 0$, we denote the optimal solution of $V^{\mathsf{D}}(u_1, \alpha_1, T')$ and $V^{\mathsf{D}}(u_2, \alpha_2, T')$ by (n^1, d^1) and (n^2, d^2), respectively. We may assume without loss of generality that $(n^1, d^1) \neq (n^2, d^2)$. Given any $\theta \in (0,1)$, our goal is to construct a new solution from $(n^1, d^1), (n^2, d^2)$ that is feasible to (52) with $u = \bar{u} \triangleq \theta u_1 + (1-\theta)u_2$ and $\alpha = \bar{\alpha} \triangleq \theta\alpha_1 + (1-\theta)\alpha_2$, and whose objective value is strictly greater than $\theta V^{\mathsf{D}}(u_1, \alpha_1, T') + (1-\theta)V^{\mathsf{D}}(u_2, \alpha_2, T')$. Since $V^{\mathsf{D}}(\bar{u}, \bar{\alpha}, T')$ is no smaller than the objective value of any feasible solution, then it implies that $V^{\mathsf{D}}(\bar{u}, \bar{\alpha}, T') > \theta V^{\mathsf{D}}(u_1, \alpha_1, T') + (1-\theta)V^{\mathsf{D}}(u_2, \alpha_2, T')$, proving strict concavity of V^{D} in (u, α).

Set $\bar{n} \triangleq \theta n^1 + (1-\theta)n^2$ and $\bar{d} \triangleq \theta d^1 + (1-\theta)d^2$. It is easy to check that $(\bar{n}, \bar{d})$ is feasible to (52) with $u = \bar{u}$ and $\alpha = \bar{\alpha}$. It remains to show that this solution has a strictly better revenue than $\theta V^{\mathsf{D}}(u_1, \alpha_1, T') + (1-\theta) V^{\mathsf{D}}(u_2, \alpha_2, T')$. The revenue under $(\bar{n}, \bar{d})$ for period t is

$$g(\bar{d}_t, \bar{n}_t) \triangleq x^{-1}\left(\frac{\theta d_t^1 + (1-\theta) d_t^2}{\lambda(\theta n_t^1 + (1-\theta) n_t^2, \theta\alpha_1 + (1-\theta)\alpha_2)}\right) \cdot \left(\theta d_t^1 + (1-\theta) d_t^2\right).$$

Our goal is thus to show

$$\begin{aligned} \sum_{t=1}^{T'} g(\bar{d}_t, \bar{n}_t) > \theta \cdot \sum_{t=1}^{T'} x^{-1}\left(\frac{d_t^1}{\lambda(n_t^1, \alpha_1)}\right) d_t^1 + (1-\theta) \cdot \sum_{t=1}^{T'} x^{-1}\left(\frac{d_t^2}{\lambda(n_t^2, \alpha_2)}\right) d_t^2 \\ = \theta V^{\mathsf{D}}(\alpha_1, T') + (1-\theta) V^{\mathsf{D}}(\alpha_2, T'). \end{aligned} \tag{53}$$

In fact, we will show that there is a dominance of revenue in every period:

$$g(\bar{d}_t, \bar{n}_t) > \theta x^{-1}\left(\frac{d_t^1}{\lambda(n_t^1, \alpha_1)}\right) d_t^1 + (1-\theta) x^{-1}\left(\frac{d_t^2}{\lambda(n_t^2, \alpha_2)}\right) d_t^2. \tag{54}$$

To show (54), we note that $g(d, n) = \lambda(n, \alpha) \cdot r\left(\frac{d}{\lambda(n,\alpha)}\right)$, where r is the revenue intensity function defined in (11). To proceed, we require the following claim.

Claim 2. The function $(d, \lambda) \mapsto \lambda \cdot r\left(\frac{d}{\lambda}\right)$ is strictly concave in (d, λ).

The following argument proves the claim.

$$\begin{aligned} &(\theta\lambda_1 + (1-\theta)\lambda_2) \cdot r\left(\frac{\theta d_1 + (1-\theta) d_2}{\theta\lambda_1 + (1-\theta)\lambda_2}\right) \\ &= (\theta\lambda_1 + (1-\theta)\lambda_2) \cdot r\left(\frac{\theta\lambda_1}{\theta\lambda_1 + (1-\theta)\lambda_2}\frac{d_1}{\lambda_1} + \frac{(1-\theta)\lambda_2}{\theta\lambda_1 + (1-\theta)\lambda_2}\frac{d_2}{\lambda_2}\right) \\ &> (\theta\lambda_1 + (1-\theta)\lambda_2)\left[\frac{\theta\lambda_1}{\theta\lambda_1 + (1-\theta)\lambda_2} r\left(\frac{d_1}{\lambda_1}\right) + \frac{(1-\theta)\lambda_2}{\theta\lambda_1 + (1-\theta)\lambda_2} r\left(\frac{d_2}{\lambda_2}\right)\right] \\ &= \theta\lambda_1 r\left(\frac{d_1}{\lambda_1}\right) + (1-\theta)\lambda_2 r\left(\frac{d_2}{\lambda_2}\right), \end{aligned} \tag{55}$$

where the inequality follows from strict concavity of the function r by Lemma 1(i).

We can now show (54), because $\lambda(n, \alpha)$ is jointly concave in (n, α) by Assumption 2(vi), hence $\lambda(\bar{n}_t, \bar{\alpha}) \geq \theta\lambda(n_t^1, \alpha_1) + (1-\theta)\lambda(n_t^2, \alpha_2)$. Then because x^{-1} is a monotone decreasing function, we have

$$\begin{aligned} g(\bar{d}_t, \bar{n}_t) &\geq x^{-1}\left(\frac{\theta d_t^1 + (1-\theta) d_t^2}{\theta\lambda(n_t^1, \alpha_1) + (1-\theta)\lambda(n_t^2, \alpha_2)}\right) \cdot \left(\theta d_t^1 + (1-\theta) d_t^2\right) \\ &= \left(\theta\lambda(n_t^1, \alpha_1) + (1-\theta)\lambda(n_t^2, \alpha_2)\right) \cdot r\left(\frac{\theta d_t^1 + (1-\theta) d_t^2}{\theta\lambda(n_t^1, \alpha_1) + (1-\theta)\lambda(n_t^2, \alpha_2)}\right) \\ &> \theta\lambda(n_t^1, \alpha_1) r\left(\frac{d_t^1}{\lambda(n_t^1, \alpha_1)}\right) + (1-\theta)\lambda(n_t^2, \alpha_2) r\left(\frac{d_t^2}{\lambda(n_t^2, \alpha_2)}\right) \\ &= \theta x^{-1}\left(\frac{d_t^1}{\lambda(n_t^1, \alpha_1)}\right) d_t^1 + (1-\theta) x^{-1}\left(\frac{d_t^2}{\lambda(n_t^2, \alpha_2)}\right) d_t^2, \end{aligned}$$

where the first equality is from the definition of r in (11), and the last inequality is from Claim 2. This establishes (54), which in turn yields (53). This completes the proof. $\square$

B.2 Proof of Theorem 3

The proof requires the following lemma.

Lemma 7. Let (n, y) be a feasible solution to ($\mathbf{D'}$) where $y \neq 0$. If $y_i = 0$ for some index i, then there exists a feasible solution (n', y') where $(n', y') \neq (n, y)$ and whose objective value is the same as (n, y).

Proof of Lemma 7. We define the following procedure to move $y_i = 0$ to the last period T' to yield a solution (n', y') that gives the same objective value as (n, y).

```
procedure MOVE(i, n, y)                    ▷ Move y_i = 0 to end of sequence y
    (n'_t = n_t, y'_t = y_t) for all t ≤ i − 1
    (n'_t = n_{t+1}, y'_t = y_{t+1}) for all i ≤ t ≤ T' − 1
    (n'_{T'} = n_{T'}, y'_{T'} = 0)
    return (n', y')
end procedure
```

Since $y \neq 0$, the new policy generated from MOVE(i, n, y) for an appropriately chosen i results in $(n', y') \neq (n, y)$. (This is not true if the only nonzero entry of y is the first index; in which case, we modify the move procedure so that $y_i = 0$ is moved to the first period.) It is easy to check that (n', y') is a feasible solution to ($\mathbf{D'}$) since (n, y) is feasible.

Finally, we show that (n', y') has the same objective value as (n, y). Notice that n' is constructed by shifting every n_t with $t \geq i + 1$ to one index smaller. The ending period remaining inventory is $n'_{T'} = n_{T'}$. Hence,

$$\begin{aligned}
\sum_{t=1}^{T'} x^{-1}(y_t)\lambda(n_{t-1}, \alpha)y_t &= \sum_{t=1}^{i-1} x^{-1}(y_t)\lambda(n_{t-1}, \alpha)y_t + \sum_{t=i+1}^{T'} x^{-1}(y_t)\lambda(n_{t-1}, \alpha)y_t \\
&= \sum_{t=1}^{i-1} x^{-1}(y'_t)\lambda(n'_{t-1}, \alpha)y'_t + \sum_{t=i}^{T'-1} x^{-1}(y'_t)\lambda(n'_{t-1}, \alpha)y'_t.
\end{aligned}$$

Here, the first equality comes from $y_i = 0$. The second equality comes from how **??** (MOVE(i)) constructs y'. $\square$

Now we can proceed with the proof of the theorem.

Proof of Theorem 3. We first claim that for any $u \in (0, \alpha]$, the optimal partial solution y^{D} of ($\mathbf{D'}$) is such that $y^{\mathsf{D}} \neq 0$. This is because the objective value of $y = 0$ is 0. However, the objective value for y' where $y'_1 = u/\lambda(u, \alpha)$ and $y'_i = 0$ for $i \neq 1$ is $x^{-1}(u/\lambda(u, \alpha))\, u > 0$. Note that y' is feasible since y_1' is the intensity that depletes all remaining inventory u. Hence, $y = 0$ cannot be optimal, so $y^{\mathsf{D}} \neq 0$.

We prove that $y^{\mathsf{D}} > 0$ using contradiction. Assume there exists an i such that $y_i^{\mathsf{D}} = 0$. Then, according to Lemma 7, we can construct a different solution with the same objective value. This contradicts Theorem 2 that the optimal solution of ($\mathbf{D}'$) is unique. $\square$

B.3 Proof of Proposition 1

Proof. Let d^y denote the (random) demand proportion under the intensity y. In this proof, we define $V^{\mathbf{y}}(u, \alpha, T')$ as the expected revenue under policy $\mathbf{y}$ when the initial inventory coverage is α, the arbitrary remaining inventory proportion is $u \leq \alpha$, and the remaining number of time periods is T'. We will make use of induction on T' to prove inequality ((i)).

For the base case with $T' = 1$, we define the optimal expected revenue $V^*(u, \alpha, 1)$ for any given remaining inventory $u \leq \alpha$ as:

$$V^*(u, \alpha, 1) \triangleq \max_{y \in [0,1]} \mathbb{E}\left[x^{-1}(y)\left(d^y - [d^y - u]^+\right) \mid u, \alpha\right] \tag{56a}$$

$$\text{subject to } \mathbb{E}(d^y \mid u, \alpha) = \lambda(u, \alpha) y \leq u. \tag{56b}$$

Then, we can see $V^*(\alpha, \alpha, 1) = V^*(\alpha, 1)$ where $V^*(\alpha, 1)$ is defined in ($\mathbf{P}'$). For a given y, let us denote the objective value (56a) as $V^y(u, \alpha, 1)$.

Consider any $y \in [0, 1]$ that satisfies (56b). For any $\mu \geq 0$, we have that

$$\begin{aligned}
V^y(u, \alpha, T) &\leq \mathbb{E}\left[x^{-1}(y)\left(d^y - [d^y - u]^+\right) + \mu(u - \lambda(u, \alpha)y) \mid u, \alpha\right] && (57)\\
&\leq \max_{y_0 \in [0,1]} \mathbb{E}\left[x^{-1}(y_0)\left(d^{y_0} - [d^{y_0} - u]^+\right) + \mu(u - \lambda(u, \alpha)y_0) \mid u, \alpha\right] && (58)\\
&\leq \max_{y_0 \in [0,1]} \mathbb{E}\left[x^{-1}(y_0)d^{y_0} + \mu(u - \lambda(u, \alpha)y_0) \mid u, \alpha\right], && (59)\\
&= \max_{y_0 \in [0,1]} x^{-1}(y_0)\lambda(u, \alpha)y_0 + \mu(u - \lambda(u, \alpha)y_0). && (60)
\end{aligned}$$

Here (59) comes from the fact that $d \geq d - [d - u]^+$ and expectation preserves the inequality. From the definition of L^{D} in (25), the right-hand side of (60) is equal to $L^{\mathsf{D}}(u, \alpha, 1, \mu)$. Therefore, we have that

$$V^y(u, \alpha, 1) \leq L^{\mathsf{D}}(u, \alpha, 1, \mu). \tag{61}$$

Then taking infimum of the RHS of (61) over $\mu \geq 0$, by strong duality of Theorem 4, we have

$$V^y(u, \alpha, 1) \leq V^{\mathsf{D}}(u, \alpha, 1). \tag{62}$$

The last step to finish the base case of induction is to take the supremum of the left-hand side of (62) over all $y \in [0, 1]$ satisfying (56b), then set $u = \alpha$. This yields

$$V^*(\alpha, 1) \leq V^{\mathsf{D}}(\alpha, 1).$$

For the inductive step, we will work with the dynamic programming formulations of both V^* and V^{D}. We assume that for any given $u \le \alpha$, $V^*(u,\alpha,T') \le V^{\mathsf{D}}(u,\alpha,T')$ for all $T' \le T$ where $V^*(u,\alpha,T')$ is defined as follows

$$V^*(u,\alpha,T'+1) := \max_{y\in[0,1]} \mathbb{E}\left[x^{-1}(y)\left(d^y - [d^y - u]^+\right) + V^*([u-d^y]^+,\alpha,T') \mid u,\alpha\right] \tag{63a}$$

$$\text{s.t. } \mathbb{E}(d^y \mid u,\alpha) \le u \tag{63b}$$

$$\mathbb{E}(d^y \mid u,\alpha) = \lambda(u,\alpha)y. \tag{63c}$$

We will then prove $V^*(u,\alpha,T'+1) \le V^{\mathsf{D}}(u,\alpha,T'+1)$ to finish the inductive step.

Claim 3. The maximization problem (63) is feasible and $V^*(u,\alpha,T'+1)$ is bounded.

We know $y = 0$ is a feasible solution. Moreover, the objective function (63a) is bounded below by zero and bounded above by $x^{-1}(\bar{y})\lambda(u,\alpha)\bar{y} + \max_{\alpha\in[0,1]} V^{\mathsf{D}}(\alpha,T') < \infty$, where $\bar{y}$ is defined in Lemma 1(ii). This concludes the claim.

Now, consider any $y \in [0,1]$ feasible to (63b) and (63c). We denote its objective value (63a) as $V^y(u,\alpha,T'+1)$. Then for any $\mu \ge 0$ and γ, we have that

$$\begin{aligned} V^y(u,\alpha,T'+1) &\le \mathbb{E}\left[x^{-1}(y)\left(d^y - [d^y-u]^+\right) + V^*([u-d^y]^+,\alpha,T') \mid u,\alpha\right] + \mu(u - \mathbb{E}(d^y \mid u,\alpha)) \\ &\quad + \gamma\left(\mathbb{E}(d^y \mid u,\alpha) - \lambda(u,\alpha)y\right) && (64a) \\ &\le \max_{y_0\in[0,1]} \mathbb{E}\Big[x^{-1}(y_0)\left(d^{y_0} - [d^{y_0}-u]^+\right) + V^*([u-d^{y_0}]^+,\alpha,T') \\ &\qquad + \mu(u-d^{y_0}) + \gamma\left(d^{y_0} - \lambda(u,\alpha)y_0\right) \mid u,\alpha\Big] && (64b) \\ &\le \max_{y_0\in[0,1]} \mathbb{E}\Big[x^{-1}(y_0)\left(d^{y_0} - [d^{y_0}-u]^+\right) + V^{\mathsf{D}}([u-d^{y_0}]^+,\alpha,T') \\ &\qquad + \mu(u-d^{y_0}) + \gamma\left(d^{y_0} - \lambda(u,\alpha)y_0\right) \mid u,\alpha\Big]. && (64c) \end{aligned}$$

Here, (64c) comes from the assumption of induction. Since (64) is true of any feasible y, then taking the supremum of $V^y(u,\alpha,T'+1)$ over $y \in [0,1]$ satisfying (63b) and (63c), we have that $V^*(u,\alpha,T'+1)$ is bounded above by (64c).

Note that (64c), and hence $V^*(u,\alpha,T'+1)$, is bounded above by

$$\begin{aligned} &\max_{\substack{y_0\in[0,1],\\ d\in\Re}} \left\{x^{-1}(y_0)\left(d - [d-u]^+\right) + V^{\mathsf{D}}([u-d]^+,\alpha,T') + \mu(u-d) + \gamma\left(d - \lambda(u,\alpha)y_0\right)\right\} && (65a) \\ &\le \inf_{\mu\ge 0,\gamma} \max_{\substack{y_0\in[0,1],\\ d\in\Re}} \left\{x^{-1}(y_0)\left(d - [d-u]^+\right) + V^{\mathsf{D}}([u-d]^+,\alpha,T') + \mu(u-d) + \gamma\left(d - \lambda(u,\alpha)y_0\right)\right\}. && (65b) \end{aligned}$$

Note that (65a) is an upper bound because d, being a decision variable that can take any value, results in a larger value than (64c). Since it is an upper bound for any values of $\mu \ge 0$ and λ, we take the infimum over all possible values resulting in the upper bound (65b).

Next, we will prove that (65b) equals to $V^{\mathsf{D}}(u,\alpha,T'+1)$. According to Claim 3, we know (65b) is bounded below. Hence, the optimal d in the inner problem of (65b) satisfies $d \le u$ and $d = \lambda(u,\alpha)y_0$; otherwise, (65b) will have a value of $-\infty$, which results in a contradiction.

Therefore, we can simplify (65b), resulting in

$$
\begin{aligned}
V^*(u,\alpha,T'+1) &\le \inf_{\mu\ge 0}\max_{y_0\in[0,1]}\left\{x^{-1}(y_0)\lambda(u,\alpha)y_0 + V^{\mathsf{D}}(u-\lambda(u,\alpha)y_0,\alpha,T') + \mu(u-\lambda(u,\alpha)y_0)\right\}\\
&= \inf_{\mu\ge 0} L^{\mathsf{D}}(u,\alpha,T'+1,\mu) \qquad \text{(from the definition of } L^{\mathsf{D}} \text{ in (25))}\\
&= V^{\mathsf{D}}(u,\alpha,T'+1). \qquad \text{(according to Theorem 4)}
\end{aligned}
\tag{66}
$$

This finishes our inductive step. Finally, by setting $u=\alpha$ in (66), we know

$$
V^*(\alpha,T'+1) \le V^{\mathsf{D}}(\alpha,T'+1).
$$

□

B.4 Proof of Theorem 4

Proof. To prove the result, we will use Slater's condition for convex programming duality (see page 226 in Boyd and Vandenberghe (2004)). Recall, to invoke the Slater condition we need to show that (21) is a convex optimization problem with a feasible point that satisfies its constraints strictly. Observe that all the constraints in (21) are affine in y. The objective function is concave in y, as established in Claim 1. Hence, (21) is a convex optimization problem

The next step is to demonstrate that there exists a feasible solution to (21) that satisfies the inequality constraint (21b) strictly. Notice that any $y\in(0,\min\{1,u/\lambda(u,\alpha)\})$ is strictly feasible to (21) because since $u>0$ and with Assumption 2(v), $u/\lambda(u,\alpha)>0$. Hence, Slater's condition implies (24) holds.

□

B.5 Proof of Lemma 2

Proof. We prove the lemma by induction. The base case is $t=0$, where all policies start with $n_0^{\mathsf{CE}} = n_0^{\mathsf{D}} = \alpha$, and hence $\lambda\left(n_0^{\mathsf{CE}},\alpha\right) = \lambda(n_0^{\mathsf{D}},\alpha) = \lambda(\alpha,\alpha)$. Therefore, (26) and (27) hold for $t=0$. For the induction proof, assume that (26) and (27) hold for $t-1$. We will next prove both properties hold for t.

To prove (26) for t, notice that by adding and subtracting $\mathbb{E}\left(n_t^{\mathsf{CE}}\right)$,

$$
\begin{aligned}
\mathbb{E}\left|n_t^{\mathsf{CE}} - n_t^{\mathsf{D}}\right| &= \mathbb{E}\left|n_t^{\mathsf{CE}} - \mathbb{E}\left(n_t^{\mathsf{CE}}\right) + \mathbb{E}\left(n_t^{\mathsf{CE}}\right) - n_t^{\mathsf{D}}\right|\\
&\le \mathbb{E}\left|n_t^{\mathsf{CE}} - \mathbb{E}\left(n_t^{\mathsf{CE}}\right)\right| + \left|\mathbb{E}\left(n_t^{\mathsf{CE}}\right) - n_t^{\mathsf{D}}\right|.
\end{aligned}
\tag{67}
$$

We will show that both terms in (67) are $\mathcal{O}(m^{-\frac{1}{2}})$.

We first show this for the first term in (67). Note that $\mathbb{P}\left(n_t^{\mathsf{CE}} = k/m\right)$ is the probability that the remaining inventory at time t is k. Put differently, it is also the probability that k customers in a population size of m have the opportunity to purchase (since inventory is available) but have not done so prior to time t. Further, we can interpret k/m as the probability that a randomly selected customer among a population of size m had the opportunity to purchase but had not done so prior to time t, conditioning on the fact that there are k such customers. Since

$$\mathbb{E}\left(n_t^{\mathsf{CE}}\right) = \sum_{k=0}^{m} \frac{k}{m} \cdot \mathbb{P}\left(n_t^{\mathsf{CE}} = \frac{k}{m}\right), \tag{68}$$

we can therefore interperet $\mathbb{E}(n_t^{\mathsf{CE}})$ as the *unconditional* probability that a randomly selected customer from a population of size m had the opportunity to purchase but has not done so prior to time t. Hence, the sum of m independent identical Bernoulli random variables $\xi_1, \xi_2, \ldots, \xi_m$, each with a success probability $\mathbb{E}(n_t^{\mathsf{CE}})$, is the random number of customers in the market who have the opportunity to buy (since inventory is available) but haven't done so prior to time t. Note that this is simply equivalent to N_t^{CE}, which is the random number of remaining inventory at time t. Since $\xi_i \le 1$, we can use the Hoeffding inequality to get

$$\mathbb{P}\left(\left|\frac{1}{m}\sum_{i=1}^{m} \xi_i - \mathbb{E}\left(\xi_i\right)\right| \ge \epsilon\right) = \mathbb{P}\left(\left|n_t^{\mathsf{CE}} - \mathbb{E}\left(n_t^{\mathsf{CE}}\right)\right| \ge \epsilon\right) \le 2e^{-2m\epsilon^2} \quad \text{for any } \epsilon \ge 0, \tag{69}$$

where the equality follows from our above argument and because $n_t^{\mathsf{CE}} = N_t^{\mathsf{CE}}/m$. By integrating (69) over $\epsilon \ge 0$, we have

$$\mathbb{E}\left|n_t^{\mathsf{CE}} - \mathbb{E}\left(n_t^{\mathsf{CE}}\right)\right| \le \int_0^\infty 2e^{-2m\epsilon^2} \mathrm{d}\epsilon = \sqrt{\frac{\pi}{2m}},$$

which gives us that the first term on the RHS of (67) is $\mathcal{O}(m^{-\frac{1}{2}})$.

For the second term in (67), we want to bound the difference between $\mathbb{E}\left(n_t^{\mathsf{CE}}\right)$ and n_t^{D}. From the definition of n_t^{CE}, we know

$$\mathbb{E}\left(n_t^{\mathsf{CE}} \mid \mathcal{F}_{t-1}\right) = \mathbb{E}\left(\left[n_{t-1}^{\mathsf{CE}} - d_t\right]^+ \mid \mathcal{F}_{t-1}\right),$$

with $\mathbb{E}(d_t \mid \mathcal{F}_{t-1}) = \lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right) \mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)$. Note that n_{t-1}^{CE} is not random when conditioning on the filtration $\mathcal{F}_{t-1}$. A well-known result by Scarf (1958) is that for any random variable x with mean μ and standard deviation σ,

$$\mathbb{E}([a-x]^+) \le \frac{1}{2}\left(\sqrt{\sigma^2 + (\mu - a)^2} - (\mu - a)\right). \tag{70}$$

Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left(\left[n_t^{\mathsf{CE}} - d_t\right]^+ \mid \mathcal{F}_{t-1}\right) \\
&\le \frac{1}{2}\left(\sqrt{\frac{\sigma\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)}{m} + \left(\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - n_{t-1}^{\mathsf{CE}}\right)^2} - \left(\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - n_{t-1}^{\mathsf{CE}}\right)\right) \\
&\le \frac{1}{2}\left(\sqrt{\frac{\sigma\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)}{m}} + \left|\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - n_{t-1}^{\mathsf{CE}}\right| - \left(\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - n_{t-1}^{\mathsf{CE}}\right)\right) \\
&\le n_{t-1}^{\mathsf{CE}} - \lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) + \frac{1}{2}\sqrt{\frac{\sigma\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)}{m}}.
\end{aligned}
$$

Note that the first inequality comes from our assumption on the variance of D_t (Assumption 3). The last inequality comes from the fact that at time t given inventory n_{t-1}^{CE}, the next price chosen by policy $\mathbf{y}^{\mathsf{CE}}$ obeys the no-stockout in expectation constraint: $\mathbb{E}(n_{t-1}^{\mathsf{CE}} - d_t \mid \mathcal{F}_{t-1}) \ge 0$, or equivalently, $n_{t-1}^{\mathsf{CE}} - \lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) \ge 0$. Therefore, we have

$$
\mathbb{E}\left(n_t^{\mathsf{CE}} \mid \mathcal{F}_{t-1}\right) = \mathbb{E}\left(\left[n_{t-1}^{\mathsf{CE}} - d_t\right]^+ \mid \mathcal{F}_{t-1}\right) \le n_{t-1}^{\mathsf{CE}} - \lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) + k \tag{71}
$$

where $k = \Theta(m^{-\frac{1}{2}})$.

Also from the definition of n_t^{D}, we know that

$$
n_t^{\mathsf{D}} = n_{t-1}^{\mathsf{D}} - \lambda(n_{t-1}^{\mathsf{D}},\alpha)y_t^{\mathsf{D}}. \tag{72}
$$

Therefore, plugging (71) and (72) into the second term of (67) yields

$$
\begin{aligned}
&\left|\mathbb{E}\left(n_t^{\mathsf{CE}}\right) - n_t^{\mathsf{D}}\right| \le \mathbb{E}\left|\left(n_{t-1}^{\mathsf{CE}} - \lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right) - \left(n_{t-1}^{\mathsf{D}} - \lambda(n_{t-1}^{\mathsf{D}},\alpha)y_t^{\mathsf{D}}\right)\right| + k \\
&\le \mathbb{E}\left|n_{t-1}^{\mathsf{CE}} - n_{t-1}^{\mathsf{D}}\right| + \mathbb{E}\left|\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - \lambda(n_{t-1}^{\mathsf{D}},\alpha)y_t^{\mathsf{D}}\right| + k \qquad (73) \\
&\le \mathbb{E}\left|n_{t-1}^{\mathsf{CE}} - n_{t-1}^{\mathsf{D}}\right| + \underbrace{\mathbb{E}\left|\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - \lambda(n_{t-1}^{\mathsf{D}},\alpha)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right|}_{(*)} \\
&\qquad + \underbrace{\mathbb{E}\left|\lambda(n_{t-1}^{\mathsf{D}},\alpha)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - \lambda(n_{t-1}^{\mathsf{D}},\alpha)y_t^{\mathsf{D}}\right|}_{(**)} + k, \qquad (74)
\end{aligned}
$$

where (73) comes from triangle inequality and monotonicity of expectation, (74) is derived by subtracting and adding $\lambda(n_{t-1}^{\mathsf{D}},\alpha)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)$ and using the triangle inequality.

To analyze the bound for $(*)$, we know λ is Lipschitz continuous. This is because λ is continuously differentiable in its two variables (Assumption 2(vi)), so there exists a C_λ such that $|\lambda(n,\alpha) - \lambda(n',\alpha)| \le C_\lambda |n - n'|$ for all n, n', and fixed α. Also for $(*)$, we know $\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) \le 1$. Therefore, we have $(*) \le (***)$ where $(***)$ is defined in (75). To analyze the bound for $(**)$, we know from Lemma 3 that $\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)$ is Lipschitz continuous and the Lipschitz constant is C_y. Also, λ is upper bounded by $\bar{\lambda}$ according to Assumption 2(v). Therefore, we have $(**) \le (****)$

where $(****)$ is defined in (75) below. Then we conclude

$$\left|\mathbb{E}\left(n_t^{\mathsf{CE}}\right) - n_t^{\mathsf{D}}\right| \leq \mathbb{E}\left|n_{t-1}^{\mathsf{CE}} - n_{t-1}^{\mathsf{D}}\right| + \underbrace{1 \cdot C_\lambda \mathbb{E}\left|n_{t-1}^{\mathsf{CE}} - n_{t-1}^{\mathsf{D}}\right|}_{(***)} + \underbrace{\bar{\lambda} C_y \mathbb{E}\left|n_{t-1}^{\mathsf{CE}} - n_{t-1}^{\mathsf{D}}\right|}_{(****)} + k, \tag{75}$$

$$= \mathcal{O}\left(m^{-\frac{1}{2}}\right), \tag{76}$$

where (76) comes from the inductive hypothesis (26) on $t-1$.

Therefore, we can conclude that the RHS two terms of (67) are both bounded by $\mathcal{O}\left(m^{-\frac{1}{2}}\right)$, thus giving us (26) for all t. For a given t, (27) follows by the Lipschitz continuity of λ and (26):

$$\mathbb{E}\left|\lambda\left(n_t^{\mathsf{CE}}, \alpha\right) - \lambda(n_t^{\mathsf{D}}, \alpha)\right| \leq C_\lambda \mathbb{E}\left|n_t^{\mathsf{CE}} - n_t^{\mathsf{D}}\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right).$$

This concludes the proof. □

B.6 Proof of Lemma 3

Proof. We prove the lemma by showing that $\mathbf{y}^{\mathsf{CE}}(u,t)$ has a bounded derivative with respect to u for $u \in [0,1]$ because

$$\left|\mathbf{y}^{\mathsf{CE}}(u,t) - \mathbf{y}^{\mathsf{CE}}(u',t)\right| = \left|\int_{u'}^{u} \frac{\partial \mathbf{y}^{\mathsf{CE}}(n,t)}{\partial n} \mathrm{d}n\right| \leq \max_{v\in[u',u]} \left|\frac{\partial \mathbf{y}^{\mathsf{CE}}(n,t)}{\partial n}\right| |u-u'|.$$

Because the analysis for $t = T$ (i.e., the last period) is different from the analysis for $t < T$, we analyze the two cases separately.

When $t = T$, we define the following partitions of the set $[0,1]$:

$$S_1 = \left\{u \in [0,1] : \frac{u}{\lambda(u,\alpha)} < \bar{y}\right\} \text{ and } S_2 = \left\{u \in [0,1] : \frac{u}{\lambda(u,\alpha)} \geq \bar{y}\right\}.$$

Note that when $t = T$, we have

$$\mathbf{y}^{\mathsf{CE}}(u,t) = \begin{cases} \frac{u}{\lambda(u,\alpha)} & \text{if } u \in S_1 \\ \bar{y} & \text{if } u \in S_2 \end{cases}$$

where $\bar{y}$ is defined in Lemma 1(ii). Note that when $u \in S_1$, $\mathbf{y}^{\mathsf{CE}}(u,t)$ has bounded derivative w.r.t. u because of Lemma 1(iii). For $u \in S_2$, the function is constant so the derivative is 0.

Now consider $t < T$. We will prove that the derivative of $\mathbf{y}^{\mathsf{CE}}(u,t)$ w.r.t. u is bounded for $u \in [0,1]$. Note that, by definition, $\mathbf{y}^{\mathsf{CE}}(u,t) = y_0^{\mathsf{D}}(u,\alpha,T-t+1)$ where

$$y_0^{\mathsf{D}}(u,\alpha,T-t+1) = \arg\max_{y \leq \frac{u}{\lambda(u,\alpha)}} R^y(u,\alpha,T-t+1)$$

where $R^y(u,\alpha,T') = x^{-1}(y)\lambda(u,\alpha)y + V^{\mathsf{D}}(u-\lambda(u,\alpha)y,\alpha,T')$ was defined in (22). By Claim 1, $R^y(u,\alpha,T-t+1)$ is strictly concave in y for a given $(u,\alpha,T-t+1)$. Let $\bar{y}_{t,u}$ to be the value that satisfies

$$\frac{\partial}{\partial y} R^y(u,\alpha,T-t+1)\Big|_{y=\bar{y}_{t,u}} = \lambda(u,\alpha)\frac{\partial}{\partial y}\left(x^{-1}(y)y\right)\Big|_{y=\bar{y}_{t,u}} - \lambda(u,\alpha)V_1^{\mathsf{D}}(u-\lambda(u,\alpha)\bar{y}_{t,u},\alpha,T-t) = 0,$$

so

$$\frac{\partial}{\partial y}\left(x^{-1}(y)y\right)\Big|_{y=\bar{y}_{t,u}} = V_1^{\mathsf{D}}(u-\lambda(u,\alpha)\bar{y}_{t,u},\alpha,T-t). \tag{77}$$

Then, by defining

$$S_1' = \left\{u \in [0,1] : \frac{u}{\lambda(u,\alpha)} < \bar{y}_{t,u}\right\} \text{ and } S_2' = \left\{u \in [0,1] : \frac{u}{\lambda(u,\alpha)} \geq \bar{y}_{t,u}\right\},$$

we know

$$\mathbf{y}^{\mathsf{CE}}(u,t) = \begin{cases} \frac{u}{\lambda(u,\alpha)} & \text{if } u \in S_1' \\ \bar{y}_{t,u} & \text{if } u \in S_2'. \end{cases}$$

From Lemma 1(iii), we have that the derivative of $\mathbf{y}^{\mathsf{CE}}(u,t)$ w.r.t. u is bounded when $u \in S_1'$. When $u \in S_2'$, we will next find the derivative of $\mathbf{y}^{\mathsf{CE}}(u,t) = \bar{y}_{t,u}$ w.r.t. u. We will do this by differentiating (77) with respect to u through chain rule. Specifically, we have

$$\frac{\partial \bar{y}_{t,u}}{\partial u}\left(x^{-1}(y)y\right)''\Big|_{y=\bar{y}_{t,u}} = \left(1-\lambda_1(u,\alpha)\bar{y}_{t,u} - \lambda(u,\alpha)\frac{\partial \bar{y}_{t,u}}{\partial u}\right)V_{11}^{\mathsf{D}}\left(u-\lambda(u,\alpha)\bar{y}_{t,u},\alpha,T-t\right). \tag{78}$$

Rearranging terms in (78) yields the following relationship:

$$\left|\frac{\partial \bar{y}_{t,u}}{\partial u}\right| = \left|\frac{\left(1-\lambda_1(u,\alpha)\bar{y}_{t,u}\right)V_{11}^{\mathsf{D}}\left(u-\lambda(u,\alpha)\bar{y}_{t,u},T-t\right)}{(x^{-1}(y)y)''\Big|_{y=\bar{y}_{t,u}} + \lambda(u,\alpha)V_{11}^{\mathsf{D}}\left(u-\lambda(u,\alpha)\bar{y}_{t,u},T-t\right)}\right|. \tag{79}$$

The term on the RHS of (79) is bounded (i.e., the denominator is nonzero) because $r''(y) < 0$ is defined for $y \in [0,1]$ according to Lemma 1(i), $V_{11}^{\mathsf{D}}(u',T-t) < 0$ is defined for $u' \in [0,1]$ (Theorem 1), and $\lambda(u,\alpha)$ is continuous differentiable for $(u,\alpha) \in [0,1]\times[0,1]$. This concludes our proof. □

B.7 Proof of Lemma 4

Proof. Note that, by definition, $V^{\mathsf{D}}(\alpha,T) = \sum_{t=1}^{T} x^{-1}(y_t^{\mathsf{D}})\lambda(n_{t-1}^{\mathsf{D}},\alpha)\mathbf{y}_t^{\mathsf{D}}$. Hence, we can write the LHS of (28) as

$$\left|\mathbb{E}\left[\sum_{t=1}^{T}\left(x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - x^{-1}(y_t^{\mathsf{D}})\lambda(n_{t-1}^{\mathsf{D}},\alpha)y_t^{\mathsf{D}}\right)\right]\right| \tag{80}$$

$$\leq \mathbb{E}\left|\sum_{t=1}^{T}\left(x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - x^{-1}(y_t^{\mathsf{D}})\lambda(n_{t-1}^{\mathsf{D}},\alpha)y_t^{\mathsf{D}}\right)\right| \tag{81}$$

$$\leq \sum_{t=1}^{T}\mathbb{E}\left|x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - x^{-1}(y_t^{\mathsf{D}})\lambda(n_{t-1}^{\mathsf{D}},\alpha)y_t^{\mathsf{D}}\right|. \tag{82}$$

Here, the first inequality comes from $|\mathbb{E}X| \le \mathbb{E}|X|$ as a result of Jensen's inequality. The second inequality comes from triangle inequality and linearity of expectation. To prove the proposition, since T is a finite number, it is sufficient to show each term inside the summation of (82) is $\mathcal{O}\left(m^{-\frac{1}{2}}\right)$.

Note that

$$\begin{aligned}
&\mathbb{E}\left|x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right) - x^{-1}(y_t^{\mathsf{D}})\lambda(n_{t-1}^{\mathsf{D}}, \alpha)y_t^{\mathsf{D}}\right| \\
&\quad = \mathbb{E}\left|r\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right) - r(y_t^{\mathsf{D}})\lambda(n_{t-1}^{\mathsf{D}}, \alpha)\right|, \qquad (83)
\end{aligned}$$

where r is the per-period revenue rate defined in (11). Our goal is to show that (83) is $\mathcal{O}(m^{-\frac{1}{2}})$.

We first prove the Lipschitz continuity of the function $r(y)$. From Lemma 1(i), $r(y)$ is concave in y and is continuously differentiable for $y \in [0, 1]$. Therefore, there exists C_r such that

$$\left|r(y) - r(y')\right| \le C_r|y - y'|. \qquad (84)$$

Additionally, $r(y) \le \bar{f} = r(\bar{y})$ where $\bar{y}$ is defined in Lemma 1(ii). Then, subtracting and adding $r\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda(n_{t-1}^{\mathsf{D}}, \alpha)$ inside the absolute value in (83), and by triangle inequality, we get (83) is upper bounded by

$$\begin{aligned}
&\mathbb{E}\left|r\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right) - r\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda(n_{t-1}^{\mathsf{D}}, \alpha)\right| \\
&\quad + \mathbb{E}\left|r\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda(n_{t-1}^{\mathsf{D}}, \alpha) - r(y_t^{\mathsf{D}})\lambda(n_{t-1}^{\mathsf{D}}, \alpha)\right| \\
&\le \bar{f}\;\mathbb{E}\left|\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right) - \lambda(n_{t-1}^{\mathsf{D}}, \alpha)\right| + \bar{\lambda}C_r\;\mathbb{E}\left|\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right) - y_t^{\mathsf{D}}\right|, \qquad (85)
\end{aligned}$$

where the second term of (85) comes from (84) and Assumption 2(v). Hence, it suffices to show the two terms in (85) are bounded by $\mathcal{O}(m^{-\frac{1}{2}})$. This is true because, from (27) of Lemma 2, we know for any t,

$$\mathbb{E}\left|\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right) - \lambda(n_{t-1}^{\mathsf{D}}, \alpha)\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right).$$

Moreover, by definition, $\mathbf{y}^{\mathsf{CE}}$ results from re-optimizing the deterministic equivalent at each time period, hence we have that $y^{\mathsf{CE}}(n_{t-1}^{\mathsf{D}}, t) = y_t^{\mathsf{D}}$. Therefore, by (26) of Lemma 2 and by Lemma 3,

$$\mathbb{E}\left|\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right) - y_t^{\mathsf{D}}\right| = \mathbb{E}\left|\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right) - y_t^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{D}}, t\right)\right| \le C_y\mathbb{E}\left|n_{t-1}^{\mathsf{CE}} - n_{t-1}^{\mathsf{D}}\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right).$$

This concludes the proof. □

Appendix C Section 5 proofs

C.1 Proof of Theorem 6

Proof. Because $Q^{\alpha^{\mathsf{CE}}, \mathbf{y}^{\mathsf{CE}}}(m, T) = V^{\alpha^{\mathsf{CE}}, \mathbf{y}^{\mathsf{CE}}}(m, T) - c\alpha^{\mathsf{CE}}m$, we first analyze the bound for $V^{\alpha^{\mathsf{CE}}, \mathbf{y}^{\mathsf{CE}}}(m, T)$ and then get $Q^{\alpha^{\mathsf{CE}}, \mathbf{y}^{\mathsf{CE}}}(m, T)$ by subtracting $c\alpha^{\mathsf{CE}}m$.

Let $(N_0^{\mathsf{CE}}, N_1^{\mathsf{CE}}, \dots, N_T^{\mathsf{CE}})$ be the sequence of stochastic remaining inventories under $\mathbf{y}^{\mathsf{CE}}$ starting with market coverage α^{CE}. Define $n_t^{\mathsf{CE}} \triangleq N_t^{\mathsf{CE}}/m$. From (33) and (38), we know

$$V^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T) \geq \mathbb{E}\left(\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right) m\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha^{\mathsf{CE}}\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right)\left(1-\frac{1}{2}\sqrt{\frac{\sigma}{m}}C\right). \tag{86}$$

Note that Lemma 4 implies that

$$\mathbb{E}\left(\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right) m\lambda\left(n_{t-1}^{\mathsf{CE}},\alpha^{\mathsf{CE}}\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right)\right) \geq m\left(V^{\mathsf{D},\alpha^{\mathsf{CE}}}(T)-k\right), \tag{87}$$

where $k=\mathcal{O}\left(m^{-\frac{1}{2}}\right)$ and $k \geq 0$. Therefore, subtracting both sides of (86) by $c\alpha^{\mathsf{CE}}m$, and using (87), we have

$$\underbrace{V^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T)-c\alpha^{\mathsf{CE}}m}_{Q^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T)} \geq m\left(V^{\mathsf{D},\alpha^{\mathsf{CE}}}(T)-k\right)\left(1-\frac{1}{2}\sqrt{\frac{\sigma}{m}}C\right)-c\alpha^{\mathsf{CE}}m. \tag{88}$$

Now we analyze the RHS of (88) to connect it to $Q^*(m,T)$. Define $k_1=\frac{1}{2}\sqrt{\frac{\sigma}{m}}C$ where C is defined in (38) with $\alpha=\alpha^{\mathsf{CE}}$. Factoring out $m\,(1-k_1)$ in the RHS of (88) results in

$$\begin{aligned}
& m(1-k_1)\left(V^{\mathsf{D},\alpha^{\mathsf{CE}}}(T)-k-\frac{c\alpha^{\mathsf{CE}}}{1-k_1}\right) \\
&= m(1-k_1)\left(\underbrace{V^{\mathsf{D},\alpha^{\mathsf{CE}}}(T)-c\alpha^{\mathsf{CE}}}_{Q^{\mathsf{D},\alpha^{\mathsf{CE}}}(T)}-k+c\alpha^{\mathsf{CE}}-\frac{c\alpha^{\mathsf{CE}}}{1-k_1}\right) && \text{subtracting and adding } c\alpha^{\mathsf{CE}} \\
&\geq m(1-k_1)\left(\underbrace{V^{\mathsf{D},\alpha^*}(T)-c\alpha^*}_{Q^{\mathsf{D},\alpha^*}(T)}-k-c\alpha^{\mathsf{CE}}\frac{k_1}{1-k_1}\right) && \text{definition of } \alpha^{\mathsf{CE}} \text{ so } Q^{\mathsf{D},\alpha^{\mathsf{CE}}}(T)\geq Q^{\mathsf{D},\alpha^*}(T) \\
&= (1-k_1)\left(V^{\mathsf{D},\alpha^*}(m,T)-c\alpha^*m-mk-c\alpha^{\mathsf{CE}}m\frac{k_1}{1-k_1}\right) && \text{multiplying } m \text{ inside} \\
&\geq (1-k_1)\left(V^*(\alpha^*,m,T)-c\alpha^*m-mk-c\alpha^{\mathsf{CE}}m\frac{k_1}{1-k_1}\right) && \text{from Proposition 1 so } V^{\mathsf{D},\alpha^*}(m,T)\geq V^*(\alpha^*,m,T) \\
&= (1-k_1)\left(\underbrace{V^*(\alpha^*,m,T)-c\alpha^*m}_{Q^*(m,T)}-(m+c\alpha^{\mathsf{CE}}m)k_2\right)
\end{aligned} \tag{89}$$

with $k_2=\Theta\left(m^{-\frac{1}{2}}\right)$ because

$$\frac{k_1}{1-k_1}=\Theta\left(\frac{1}{\sqrt{m}-1}\right).$$

Dividing (88) and the RHS of the last equality of (89) by $Q^*(m,T)=V^*(\alpha^*,m,T)-c\alpha^*m$ yields

$$\frac{Q^{\alpha^{\mathsf{CE}},\mathbf{y}^{\mathsf{CE}}}(m,T)}{Q^*(m,T)} \geq (1-k_1)\left(1-k_2\frac{m+c\alpha^{\mathsf{CE}}m}{V^*(\alpha^*,m,T)-c\alpha^*m}\right).$$

Hence, to prove (40), it suffices to show

$$\frac{m + c\alpha^{\mathsf{CE}} m}{V^*(\alpha^*, m, T) - c\alpha^* m} = \mathcal{O}(1).$$

This is true because

$$\frac{m + c\alpha^{\mathsf{CE}} m}{V^*(\alpha^*, m, T) - c\alpha^* m} = \frac{m\left(1 + c\alpha^{\mathsf{CE}}\right)}{m\left(V^*(\alpha^*, T) - c\alpha^*\right)} = \frac{1 + c\alpha^{\mathsf{CE}}}{V^*(\alpha^*, T) - c\alpha^*},$$

which is constant in m. This concludes the proof. □

Appendix D Section 6 proofs

D.1 Proof of Lemma 5

Proof. By definition, we have

$$V^{\mathsf{D}}(\alpha, T) = \max_{y \le \frac{\alpha}{\lambda(\alpha,\alpha)}} \left\{ x^{-1}(y)\lambda(\alpha, \alpha)y + V^{\mathsf{D}}(\alpha - \lambda(\alpha, \alpha)y, \alpha, T - 1) \right\}. \tag{90}$$

Due to condition (ii) of the lemma and from (42), we have that $y^{\mathsf{SP}} = \bar{y}$ and

$$V^{\mathsf{SP}}(\alpha, T) = x^{-1}(\bar{y})\lambda(\alpha, \alpha)\bar{y} + V^{\mathsf{SP}}(\alpha - \lambda(\alpha, \alpha)\bar{y}, \alpha, T - 1),$$

where we define $V^{\mathsf{SP}}(u, \alpha, T')$ to be (43) with $n_0^{\mathsf{SP}} = u$ (instead of α).

We next define

$$\begin{aligned} R(y) &\triangleq x^{-1}(y)\lambda(\alpha, \alpha)y + V^{\mathsf{D}}(\alpha - \lambda(\alpha, \alpha)y, \alpha, T - 1), \\ R_{\mathsf{SP}}(y) &\triangleq x^{-1}(y)\lambda(\alpha, \alpha)y + V^{\mathsf{SP}}(\alpha - \lambda(\alpha, \alpha)y, \alpha, T - 1), \end{aligned}$$

Note that $R(y)$ is the objective in (90) which, if we recall from Section 4.1, achieves its maximum at y_1^{D}. We observe that

$$V^{\mathsf{D}}(\alpha, T) - V^{\mathsf{SP}}(\alpha, T) = \underbrace{R(y_1^{\mathsf{D}}) - R(\bar{y})}_{(a)} + \underbrace{R(\bar{y}) - R_{\mathsf{SP}}(\bar{y})}_{(b)}. \tag{91}$$

In (91), $(b) \ge 0$ because

$$(b) = V^{\mathsf{D}}\left(\alpha - \lambda(\alpha, \alpha)\bar{y}, \alpha, T - 1\right) - V^{\mathsf{SP}}\left(\alpha - \lambda(\alpha, \alpha)\bar{y}, \alpha, T - 1\right) \ge 0,$$

since V^{SP} is the certainty equivalent revenue from $\bar{y}$ and $\bar{y}$ is feasible to the deterministic problem (**D′**). Therefore, the RHS of (91) is lower bounded by (a).

Because $R(y)$ is strictly concave in y (Claim 1) and since $y_1^D > 0$ (Theorem 3), then we know

$$\frac{\partial R}{\partial y}\Big|_{y=y_1^{\mathsf{D}}} = \underbrace{\frac{\partial}{\partial y}x^{-1}(y)\lambda(\alpha,\alpha)y\Big|_{y=y_1^{\mathsf{D}}}}_{(c)} + \underbrace{\frac{\partial}{\partial y}V^{\mathsf{D}}(\alpha-\lambda(\alpha,\alpha)y,\alpha,T-1)\Big|_{y=y_1^{\mathsf{D}}}}_{(d)} = 0, \tag{92}$$

Condition (i) of Lemma 5 states that $(d) \neq 0$ which, combined with (92), implies that $(c) \neq 0$. Since $\bar{y}$ is the unique value that can make $\frac{\partial}{\partial y}x^{-1}(y)\lambda(\alpha,\alpha)y$ equal to zero (Lemma 1(ii)), we conclude $y_1^{\mathsf{D}} \neq \bar{y}$. Therefore, by the mean value theorem, there exists a $y' \in \left(\min\{\bar{y}, y_1^{\mathsf{D}}\}, \max\{\bar{y}, y_1^{\mathsf{D}}\}\right)$ such that

$$(a) = R(y_1^{\mathsf{D}}) - R(\bar{y}) = \frac{\partial R(y)}{\partial y}\Big|_{y=y'}(y_1^{\mathsf{D}} - \bar{y}). \tag{93}$$

Note that $(a) \geq 0$ because y_1^{D} is the maximizer of $R(y)$. Note that derivative term in (93) is nonzero because $y' \neq y_1^{\mathsf{D}}$ and y_1^{D} is the unique maximizer of $R(y)$ (Lemma 1(ii)). Further, since $y_1^{\mathsf{D}} \neq \bar{y}$, we have that $(a) > 0$. Hence, $V^{\mathsf{D}}(\alpha,T) - V^{\mathsf{SP}}(\alpha,T) > 0$. This implies that $V^{\mathsf{D}}(\alpha,m,T) - V^{\mathsf{SP}}(\alpha,m,T) = m\left(V^{\mathsf{D}}(\alpha,T) - V^{\mathsf{SP}}(\alpha,T)\right) = \Theta(m)$. This concludes our proof. □

D.2 Corollary 1 and proof

Corollary 1. Given $\alpha \in [0,1]$, let $(N_0^{\mathsf{SP}}, N_1^{\mathsf{SP}}, \ldots, N_T^{\mathsf{SP}})$ denote the sequence of stochastic remaining inventory under policy $\mathbf{y}^{\mathsf{SP}}$ with $N_0^{\mathsf{SP}} = \alpha m$. Define $n_t^{\mathsf{SP}} \triangleq N_t^{\mathsf{SP}}/m$. Let $n^{\mathsf{D,SP}} = (n_0^{\mathsf{D,SP}}, \ldots, n_T^{\mathsf{D,SP}})$ be the deterministic optimal solution of (**D′**) when fixing $y = (y^{\mathsf{SP}}, \ldots, y^{\mathsf{SP}})$. Then the following hold:

$$\mathbb{E}\left|n_t^{\mathsf{SP}} - n_t^{\mathsf{D,SP}}\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right)$$

and

$$\mathbb{E}\left|\lambda\left(n_t^{\mathsf{SP}},\alpha\right) - \lambda\left(n_t^{\mathsf{D,SP}},\alpha\right)\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right).$$

Proof. The only difference between Corollary 1 and Lemma 2 is the gap between the stochastic intensity sequence and the deterministic intensity sequence. In Lemma 2 (using the notation in the proof of Lemma 2), we apply $\mathbf{y}^{\mathsf{CE}}$ to the stochastic problem and accordingly get n^{CE}; and we apply y^{D} to the deterministic problem and accordingly have n^{D}. However, in Corollary 1, we apply $(y^{\mathsf{SP}}, \ldots, y^{\mathsf{SP}})$ to the stochastic problem and accordingly get n^{SP}; and we apply the same $(y^{\mathsf{SP}}, \ldots, y^{\mathsf{SP}})$ to the deterministic problem and accordingly have $n^{\mathsf{D,SP}}$. As a result, the key difference between the proofs of Lemma 2 and Corollary 1 is the logic to have the same $(**)$ in (74) upper bounded by $(****)$ in (75). Note that the definition of $\mathbf{y}^{\mathsf{SP}}$ in (42) also guarantees that inventory constraint is satisfied in expectation, so the logic in the proof stays the same as Lemma 2.

In Lemma 2, (using the notation in the proof of Lemma 2) we have the gap between $\mathbf{y}^{\mathsf{CE}}$ and y^{D} is

$$\mathbb{E}\left|\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}},t\right) - y_t^{\mathsf{D}}\right| \leq \bar{\lambda}C_y\mathbb{E}\left|n_{t-1}^{\mathsf{CE}} - n_{t-1}^{\mathsf{D}}\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right). \tag{94}$$

Note that (94) is the key to have $(**) \leq (****)$ in the proof of Lemma 2. To get (94), the crucial part is the Lipschitz continuity of policy y^{CE} proved in Lemma 3. Therefore, in Corollary 1, if we also have the gap between y sequences applied to the stochastic and deterministic problems is $\mathcal{O}(m^{-\frac{1}{2}})$, then we are done. In fact, for Corollary 1, we apply the same sequence $(y^{\mathsf{SP}}, \ldots, y^{\mathsf{SP}})$ to both stochastic and deterministic problems, so clearly

$$\mathbb{E}\left|\mathbf{y}^{\mathsf{SP}}\left(n_{t-1}^{\mathsf{SP}}, t\right) - y^{\mathsf{SP}}\right| = 0,$$

thus is $\mathcal{O}(m^{-\frac{1}{2}})$. Therefore, we get the same bound as (75) in the proof of Lemma 2. Then, Corollary 1 holds by applying the same logic as the proof of Lemma 2.

□

D.3 Corollary 2 and proof

Corollary 2. Given $\alpha \in [0,1]$, let $(N_0^{\mathsf{SP}}, N_1^{\mathsf{SP}}, \ldots, N_T^{\mathsf{SP}})$ denote the sequence of stochastic remaining inventory under policy $\mathbf{y}^{\mathsf{SP}}$ with $N_0^{\mathsf{SP}} = \alpha m$. Define $n_t^{\mathsf{SP}} \triangleq N_t^{\mathsf{SP}}/m$. Then,

$$\left|\mathbb{E}\left(\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{SP}}(n_{t-1}^{\mathsf{SP}}, t)\right)\lambda\left(n_{t-1}^{\mathsf{SP}}, \alpha\right)\mathbf{y}^{\mathsf{SP}}(n_{t-1}^{\mathsf{SP}}, t)\right) - V^{\mathsf{SP}}(\alpha, T)\right| = \mathcal{O}\left(m^{-\frac{1}{2}}\right).$$

Proof. Similar to the proof of Corollary 1 (see Section D.2), the only difference between Corollary 2 and Lemma 4 is the gap between the stochastic intensity sequence and the deterministic intensity sequence. In Lemma 4 (using the notation in the proof of Lemma 4), we apply $\mathbf{y}^{\mathsf{CE}}$ to the stochastic problem and accordingly get n^{CE} and the expected revenue

$$\mathbb{E}\left(\sum_{t=1}^{T} x^{-1}\left(\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right)\lambda\left(n_{t-1}^{\mathsf{CE}}, \alpha\right)\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right)\right);$$

and we apply y^{D} to the deterministic problem ($\mathbf{D}'$) and accordingly have n^{D} and the deterministic revenue $V^{\mathsf{D}}(\alpha, T)$. However, in Corollary 1, we apply $(y^{\mathsf{SP}}, \ldots, y^{\mathsf{SP}})$ to the stochastic problem and accordingly get n^{SP} and the expected revenue

$$\mathbb{E}\left(\sum_{t=1}^{T} x^{-1}(y^{\mathsf{SP}})\lambda\left(n_{t-1}^{\mathsf{SP}}, \alpha\right)y^{\mathsf{SP}}\right);$$

and we apply the same $(y^{\mathsf{SP}}, \ldots, y^{\mathsf{SP}})$ to the deterministic problem ($\mathbf{D}'$) and accordingly have $n^{\mathsf{D},\mathsf{SP}}$ and the deterministic revenue $V^{\mathsf{SP}}(\alpha, \alpha, T)$.

The proof of Corollary 2 follows exactly the same logic of the proof of Lemma 4. Whenever we use Lemma 2 in the proof of Lemma 4, we replace these with Corollary 1. Whenever we use Lemma 3 to bound $\mathbb{E}\left|\mathbf{y}^{\mathsf{CE}}\left(n_{t-1}^{\mathsf{CE}}, t\right) - y_t^{\mathsf{D}}\right|$, we do not need them because we have zero gap between two sequences of y, that is $\mathbb{E}\left|\mathbf{y}^{\mathsf{SP}}\left(n_{t-1}^{\mathsf{SP}}, t\right) - y^{\mathsf{SP}}\right| = 0$. □

D.4 Proof of Lemma 6

Proof. Given $\alpha \in [0,1]$, let $(N_0^{\mathsf{SP}}, N_1^{\mathsf{SP}}, \ldots, N_T^{\mathsf{SP}})$ denote the sequence of stochastic remaining inventory under policy $\mathbf{y}^{\mathsf{SP}}$ with $N_0^{\mathsf{SP}} = \alpha m$. Define $n_t^{\mathsf{SP}} \triangleq N_t^{\mathsf{SP}}/m$.

First we notice that

$$V^{\alpha,\mathbf{y}^{\mathsf{SP}}}(m,T) \leq m\mathbb{E}\left(\sum_{t=1}^{T} x^{-1}(\mathbf{y}^{\mathsf{SP}})\lambda\left(n_{t-1}^{\mathsf{SP}}, \alpha\right)\mathbf{y}^{\mathsf{SP}}\right) \tag{95}$$

because the RHS is the expected revenue under $\mathbf{y}^{\mathsf{SP}}$ without the inventory constraint.

According to Corollary 2 (see Section D.3), we know

$$mV^{\mathsf{SP}}(\alpha,T) - \mathcal{O}(\sqrt{m}) \leq m\mathbb{E}\left(\sum_{t=1}^{T} x^{-1}(\mathbf{y}^{\mathsf{SP}})\lambda\left(n_{t-1}^{\mathsf{SP}}, \alpha\right)\mathbf{y}^{\mathsf{SP}}\right) \leq mV^{\mathsf{SP}}(\alpha,T) + \mathcal{O}(\sqrt{m}). \tag{96}$$

Plugging (96) into RHS of (95), we get

$$V^{\alpha,\mathbf{y}^{\mathsf{SP}}}(m,T) \leq mV^{\mathsf{SP}}(\alpha,T) + \mathcal{O}(\sqrt{m}) = V^{\mathsf{SP}}(\alpha,m,T) + \mathcal{O}(\sqrt{m}). \tag{97}$$

□