A general solution method for moral hazard problems

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Abstract

Principal-agent models are pervasive in theoretical and applied economics, but their analysis has largely been limited to the "first-order approach" where incentive compatibility is replaced by a first-order condition. However, there are numerous important examples of setting where the first-order approach fails. Researchers have proposed methods to solve principal agent problems in such settings, but these methods remain unsatisfactory for both theoretical and practical reasons. This paper presents a new and tractable approach to solving a wide class of principal agent problems that satisfy certain monotonicity assumptions (such as the monotone likelihood ratio property) but not assuming that the number of output scenarios is finite (as required by some other methods). This approach solves a max-min-max formulation over agent actions, alternate best responses by the agent, and contracts. The entire set of incentive compatibility constraints (of which there are infinitely many in general) are not required in this formulation. Instead a single "no-jump" constraint involving an optimal alternate best responses suffices. This allows a convenient characterization of optimal contracts that facilitates analysis and ensures the tractability of our approach.

Key Words: Principal-agent, Maxmin, Moral hazard, Solution method

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1 Introduction

Moral hazard principal-agent problems are well-studied, but unresolved technical difficulties persist. The essential difficulty is finding a tractable and general enough method to deal with the incentive compatibility (IC) constraints that capture the strategic behavior of the agent. We handle this difficulty using a novel methodology that provides a tractable method to solve a broad class of moral hazard problems.

Incentive compatibility is a challenging issues for at least two reasons. First, when the agent's action space is continuous there are, in principle, infinitely many IC constraints. Second, these constraints make the principal's decision into an optimization problem over a nonconvex set. Much attention has been given to finding structure in special cases that overcome these issues. The *first-order approach*, where the IC constraints are replaced by the first order condition of the agent's problem (Jewitt (1988), Rogerson (1985)), applies when the agent's objective function is concave in the agent's choice of action. When this property fails – as it does in many important important applications – more elaborate methods have been proposed. These methods must deal with the fact that first order conditions can admit non-optimal "critical points" that are not incentive compatible.

Grossman and Hart (1983) explore settings where there are finitely many output scenarios and reduce incentive compatibility to a finite number of linear constraints. However, their method does not apply when the agent's output takes on infinitely many values. An alternate approach that applies to such settings is due to Mirrlees (1999) and refined in Mirrlees (1986) and Araujo and Moreira (2001). This approach overcomes the deficiencies of the first-order approach by reintroducing a subset of the incentive compatibility constraints, in addition to the first-order condition, to eliminate alternate best responses. These reintroduced constraints – called *no-jump constraints* – isolate attention to contract-action pairs that are incentive compatible.

Mirrlees's approach suffers from its own drawbacks. There can numerous (possibly infinitely many) required no-jump constraints so little is saved over the original set of IC constraints. Even calculating which no-jump constraints are necessary is problematic. Their construction presumes *a priori* information about which contracts implement which actions.

Having such knowledge is difficult to justify in practice.

The procedure described in this paper systematically overcomes the deficiencies of Mirrlees's approach in a wide class of principal agent problems. Determining which no-jump constraints are needed can be recast as a minimization problem that identifies the hardest-to-satisfy no-jump constraint over the set of alternate best responses. This makes the original principal agent problem equivalent to an optimization problem that involves three sequential optimal decisions: maximizing over the contract, maximizing over the agent's action, and minimizing over alternate best responses to the chosen action. We then propose a tractable relaxation to this problem by changing the order of optimization to "max-min-max" where the former maximization is over agent actions and the latter maximization is over contracts. The analytical benefits of this new order are clear. The map that describes which optimal contracts support a given action against deviation to a specific alternate best response has desirable topological properties and can be used to determine the "minimizing" alternative best response without resort to enumeration, as is done in the Mirrlees approach. We call this "max-min-max" relaxation the "sandwich" problem because the inner minimization is "sandwiched" between two outer maximizations.

The main technical work of the paper is uncovering when the sandwich relaxation is tight. It turns out that this involves careful consideration of what utility can be guaranteed to the agent at an optimal solution to the principal-agent problem. In particular, if the individual rationality constraint is not binding, a family of sandwich relaxations indexed by lower bounds on agent utility that are larger than the reservation utility must be examined in order to find a relaxation that is tight. Constructing the appropriate bound and guaranteeing that the resulting relaxation is tight is the main focus of our development. Our guarantees of the tightness of the sandwich relaxation involve monotonicity conditions on the output distribution. These include the monotone likelihood ratio property (MLRP) and related properties.

The paper is organized as follows. Section 2 contains the model and discussion of existing approaches to solve the principal-agent problem when the first-order approach fails. Section 3 describes the sandwich relaxation and gives sufficient conditions to show the relaxation is tight given an appropriately chosen lower bound on agent utility. Section 4 describes the methodology to construct such a lower bound. Proofs of technical results are found in Appendix A.

2 Model and existing approaches

2.1 Principal-agent model

We study the classic moral hazard principal-agent problem with a single task and multi-dimensional output. The agent takes a private action $a \in \mathbb{A} = [\underline{a}, \overline{a}]$ that influences the distribution of the output $x \in \mathcal{X} \subseteq \mathbb{R}^K$ through the probability density function $f(x, a)$. The random output X is a continuous random variable and f is a continuous and twice differentiable in a.[1] The support of $f(\cdot, a)$ does not depend on a. The agent has a smooth Bernoulli utility function $u(w)$ and a cost $c(a)$ where w is the wage offered by the principal. The agent's separable utility for taking action a and receiving wage w is $u(w) - c(a)$. The Bernoulli utility u is assumed to be an increasing concave function and the cost function is assumed to be increasing. The principal chooses contract $w : \mathcal{X} \to [\underline{w}, \infty)$ in anticipation of the agent's action, where $\underline{w}$ is an exogenously given minimum wage. Let the value of output be given by the function $\pi : \mathcal{X} \to \mathbb{R}$. The principal has a smooth, strictly increasing and weakly concave utility function v over the net value $\pi - w$. The principal's expected utility is $V(w, a) = \int v(\pi(x) - w(x)) f(x, a) dx$ and the agent's expected utility is $U(w, a) = \int u(w(x)) f(x, a) dx - c(a)$.

The principal faces the optimization problem:

$$\max_{w \geq \underline{w}, a \in \mathbb{A}} V(w, a) \tag{P}$$

subject to the following conditions

$$U(w, a) - U(w, \hat{a}) \geq 0 \qquad \text{for all } \hat{a} \in \mathbb{A} \tag{IC}$$

$$U(w, a) \geq \underline{U} \tag{IR}$$

where (IC) are the incentive compatibility constraints that ensure the agent responds optimally and (IR) is the individual rationality constraint that guarantees participation of the

[1] In particular, $\mathcal{X}$ is not a finite set and so the procedure of Grossman and Hart (1983) does not apply.

agent reservation utility $\underline{U} > -\infty$. Throughout we will assume existence of optimal solutions to (P).[2]

Terminology and notation

Let $a^{BR}(w)$ denote the set of actions that satisfy the (IC) constraint for a given contract w. That is, $a^{BR}(w) \equiv \arg\max_{a'} U(w, a')$. Let $\mathcal{F}$ denote the set of feasible solutions to (P). That is, $\mathcal{F} := \left\{(w, a) : w \geq \underline{w}, a \in a^{BR}(w), U(w, a) \geq \underline{U}\right\}$. Given an action a, contract w is said to *implement* a if $(w, a) \in \mathcal{F}$. An action a is *implementable* if there exists a w that implements a. Let val$(*)$ denote the optimal value of the optimization problem $(*)$. In particular, val(P) denotes the optimal value of problem (P). The single constraint in (IC) of the form

$$U(w, a) - U(w, \hat{a}) \geq 0, \tag{NJ$(a, \hat{a})$}$$

is called the *no-jump* constraint at $\hat{a}$.

2.2 Existing approaches

We discuss the approaches to solve (P) that appear in the literature and their inherent deficiencies. The standard-bearer is the first-order approach, which replaces (IC) with first order conditions. Every implementable action a satisfies (IC) and so solves $\max_a U(w, a)$ for every contract w that implements a. In particular, a satisfies the first-order condition necessary conditions:

$$U_a(w, a) = 0 \text{ if } a \in (\underline{a}, \bar{a}),\ U_a(w, a) \leq 0 \text{ if } a = \underline{a}, \text{ and } U_a(w, a) \geq 0 \text{ if } a = \bar{a} \tag{FOC}$$

where the subscripts denote partial derivatives. Replacing (IC) with (FOC), problem (P) becomes

$$\max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq \underline{U} \text{ and (FOC)}\}, \tag{FOA}$$

which is called the first-order approach. When (FOA) and (P) have the same value (that is, val(P) = val(FOA)) the first-order approach is valid. As the following simple example illustrates, the first-order approach is very often invalid.

[2]See Kadan et al. (2014)and Ke (2014) for recent discussions of existence of solutions in moral hazard problems.

Example 1. Following Mirrlees (1999), we consider a special case of our model that facilitates a geometric understanding of the issues. We return to this example later to illustrate the alternate approaches to solve our principal-agent.

Suppose the principal chooses contract $z \in \mathbb{R}$ and the agent chooses an action $a \in [-2, 2]$. Assume the action of the agent is observable and so z can be expressed as a function of a. Assume the principal obtains utility $v(z, a) = za - 2a^2$ and the agent receives benefit $-za$, minus action cost $c(a) = (a^2 - 1)^2$, with total utility

$$u(z, a) = -za - (a^2 - 1)^2.$$

The principal's problem is

$$\max_{(z,a)} \{v(z, a) : u(z, a) \geq -2 \text{ and } a \in \arg\max_{a'} u(z, a')\}. \tag{1}$$

If we use the first-order approach, the solutions are $(z, a) = (\frac{3}{2}, \frac{1}{2})$ or $(-\frac{3}{2}, -\frac{1}{2})$ which are not incentive compatible. Thus, the first-order approach is invalid.

Since this problem is so simple we can solve it by inspection. As any difference of the signs between z and a will make the principal worse off, the solutions to problem (1) are $\{(0, 1), (0, -1)\}$. There are two maxima of the agent's utility at the optimal $z^* = 0$. This implies the no-jump constraint (NJ($a, \hat{a}$)) is binding at $a, \hat{a} \in \{1, -1\}$. Knowing this optimal solution will be useful for illustrative purposes when we return to this example in later sections. ◀

Previous studies have proposed various sufficient conditions for the first-order approach to be valid, including conditions for the agent's expected utility $U(w^a, a)$ to be globally concave in a, where w^a is a contract that implements a. Global concavity ensures that (FOC) is necessary and sufficient for (IC) (see, e.g., Conlon (2009), Jewitt (1988), Rogerson (1985), Sinclair-Desgagné (1994)). There are also recent non-global concavity approaches by Ke (2013), Kirkegaard (2013). These conditions impose strong conditions on the problem that are undesirable for many applications.

In response to these difficulties, Mirrlees (1999) (which originally appeared in 1975) proposed an approach (later refined in Mirrlees (1986) and Araujo and Moreira (2001)) to handle situations where the first-order approach fails.[3] Mirrlees recognized that difficulties arise when there are pairs (w, a) that satisfy (FOC) but w nonetheless fails to implement a. To combat this, Mirrlees suggested reintroducing constraints from (IC) that eliminate such pairs. The resulting problem (cf. Mirrlees (1986)) is:

$$\max_{(w,a)} \quad V(w,a) \tag{2}$$

$$\text{subject to} \quad U_a(w,a) = 0 \tag{3}$$

$$U(w,a) \geq \underline{U}, \tag{4}$$

$$U(w,a) - U(w,\hat{a}) \geq 0, \ \forall \hat{a} \text{ s.t. } U_a(w,\hat{a}) = 0. \tag{5}$$

The structure of this problem is understood as follows. Let (w', a') satisfy (3) and (4). This implies that w' is a candidate contract to implement a' and it remains only to establish a' is a best response to w'. This is achieved by the no-jump constraints (5). If our candidate contract w' violates the no-jump constraint (NJ$(a, \hat{a})$) then w' does not implement a', since an optimizing agent can improve his expected utility by "jumping" from a' to $\hat{a}$. The *best* choice of alternate action $\hat{a}^*$ given w' is included among the no-jump constraints, since such an $\hat{a}^*$ satisfies the first order condition $U_a(w', \hat{a}^*) = 0$. Hence if the candidate w' satisfies all no-jump constraints it must implement a'. The following example demonstrates how Mirrlees's approach can overcome the failure of the first-order approach.

Example 2 (Example 1 continued)**.** If we knew the two best responses *ex ante*, we could follow Mirrlees (1986) or Araujo and Moreira (2001) to solve (1) in the following manner:

$$\max_{(z,a)} v(z,a)$$

subject to the first-order condition

$$u_a(z,a) = -4a(a^2 - 1) - z = 0$$

and no–jump constraints

$$u(z,a) - u(z,\hat{a}) \geq 0$$

for $\hat{a} \in \{1, -1\}$. Figure 1 demonstrates the constraint sets and the optimal solutions.

[3] Note that the approach of Grossman and Hart (1983) does not apply since X is not a discrete set.

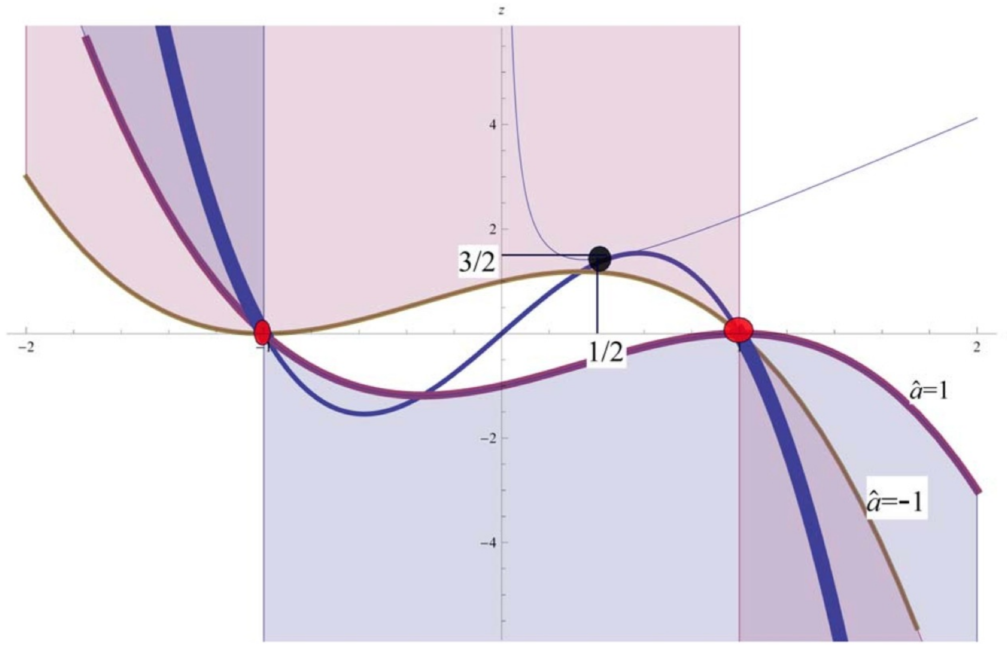

Figure 1: Figure for Example 2.

We plot the first-order condition curve (blue line), the best response set (bold part of blue line) and the regions for the two constraints (the shaded regions in the graph):

$$\begin{aligned} u(z,a) - u(z,1) &\geq 0 \\ u(z,a) - u(z,-1) &\geq 0. \end{aligned}$$

The region $\{(z,a) : u(z,a) - u(z,\hat{a}) \geq 0\}$ lies below the curve

$$z = -(\hat{a} + a)(\hat{a}^2 + a^2 - 2)$$

for $a > \hat{a}$ and above the curve for $a < \hat{a}$. These constraints preclude the optimal solution of the first-order approach: $(z,a) = (\frac{3}{2}, \frac{1}{2})$ and $(-\frac{3}{2}, -\frac{1}{2})$. The only contract-action pairs that satisfy the above constraints (including the (FOA) constraint) is the set that satsfies the original (IC) constraints. ◀

The main deficiency in Mirrlees's approach is in producing the no-jump constraints (5). This demands *a priori* knowledge of all alternate best responses to all candidate contracts w. A general approach to generating all necessary no-jump constraints remains open. Araujo and Moreira (2001) improve Mirrlees's approach by simplifying constraint (5) using second-order information. However, Araujo and Moreira (2001)'s characterization still suffers from of requiring *a priori* information about the set of best responses. Even when this information is known there remains the issue that infinitely many (indeed a whole continuum of) no-jump constraints may be needed.

Our proposed method – what we call the *sandwich procedure* – overcomes these difficulties. No *a priori* information is necessary. Moreover, the optimal contract can be characterized using a single, appropriately chosen, no-jump constraint. This single constraint is found by solving a tractable optimization problem in the alternate action $\hat{a}$. The sandwich procedure applies to a wide range of principal agent problems that exhibit some form of monotonicity in the output distribution. Details of the requirements for the procedure to apply are found in the statement of Lemma 2 below. The next two sections describe and justify the sandwich procedure.

3 The sandwich relaxation

We propose a relaxation of (P) and explore when this relaxation can be analyzed to solve our principal-agent problem. The first step in our development is to formally restate (P) using an inner minimization over $\hat{a}$. Observe that (P) is equivalent to

$$\begin{aligned} \max_{w \geq \underline{w}, a \in \mathbb{A}} \quad & V(w,a) \\ \text{subject to} \quad & \min_{\hat{a} \in \mathbb{A}} \{U(w,a) - U(w,\hat{a})\} \geq 0 \qquad (6) \\ & U(w,a) \geq \underline{U} \end{aligned}$$

since the (IC) constraint imposes that a optimizes agent utility given w since $U(w,a) \geq U(w,\hat{a})$ for all $\hat{a} \in \mathbb{A}$. To clarify the relationships between w, a, and $\hat{a}$, our goal is to pull the minimization operator out from the constraint (6) and behind the objective function. This requires handling the possibility that a choice of w does not implement the chosen a, in which case (6) is violated. We handle this issue as follows. Consider the set

$$\mathcal{W}(\hat{a}) \equiv \{(w,a) : U(a,w) \geq \underline{U} \text{ and } U(w,a) - U(w,\hat{a}) \geq 0\}. \qquad (7)$$

Define the characteristic function

$$V^I(w,a|\hat{a}) \equiv \begin{cases} V(w,a) & \text{if } (w,a) \in \mathcal{W}(\hat{a}) \\ -\infty & \text{otherwise} \end{cases}.$$

This is constructed so that if maximizing $V^I(w,a|\hat{a})$ over (w,a) results in a finite objective value then $(w,a) \in \mathcal{W}(\hat{a})$. Similarly, if maximizing $\min_{\hat{a}} V^I(w,a|\hat{a})$ over (w,a) results in a finite objective value then (w,a) lies in $\mathcal{W}(\hat{a})$ for all $\hat{a} \in \mathbb{A}$. This implies $(w,a) \in \mathcal{F}$ and demonstrates the equivalence of (P) and the problem:

$$\max_a \max_{w \geq \underline{w}} \min_{\hat{a} \in \mathbb{A}} \left\{ V^I(w,a|\hat{a}) : (w,a) \in \mathcal{W}(\hat{a}) \right\}. \tag{Max-Max-Min}$$

The basic idea of our relaxation is to explore what transpires when swapping the order of optimization in (Max-Max-Min) so that $\hat{a}$ is chosen *before* w. That is, we consider the problem

$$\max_a \min_{\hat{a}} \max_{w \geq \underline{w}} \left\{ V^I(w,a|\hat{a}) : U(w,a) \geq \underline{U}, U(w,a) - U(w,\hat{a}) \geq 0 \right\}$$

which is equivalent to

$$\max_a \min_{\hat{a}} \max_{w \geq \underline{w}} \{ V(w,a) : U(w,a) \geq \underline{U}, U(w,a) - U(w,\hat{a}) \geq 0 \} \tag{Max-Min-Max}$$

since an optimal choice of a precludes a subsequent optimal choice of $\hat{a}$ that sets $\mathcal{W}(\hat{a}) = \emptyset$. This implies $V^I(w,a|\hat{a}) = V(w,a)$ when w is optimally chosen.

The relationship between (Max-Max-Min) and (Max-Min-Max) is connected to the development of our companion paper Ke and Ryan (2015). In that paper, an implementable action a is fixed and the reservation utility $\underline{U}$ is assumed to be equal $U(w^a, a)$ where w^a is the optimal contract implementing a. Sufficient conditions (reproduced in detail below) are provided to assure

$$\max_{w \geq \underline{w}} \min_{\hat{a} \in \mathbb{A}} \left\{ V^I(w,a|\hat{a}) : (w,a) \in \mathcal{W}(\hat{a}) \right\} = \min_{\hat{a} \in \mathbb{A}} \max_{w \geq \underline{w}} \{ V(w,a) : (w,a) \in \mathcal{W}(\hat{a}) \} \tag{8}$$

for the given a. Note that we can write $V(w,a)$ instead of $V^I(w,a|\hat{a})$ on the right-hand side since a is assumed to implementable, implying there exists a w such that $\mathcal{W}(\hat{a})$ is nonempty for every choice of $\hat{a}$.

This result provides some hope for elucidating sufficient conditions that guarantee val(P) = val(Max-Min-Max). Unfortunately, there may not exist an optimal solution (w,a) to (P) where $U(w, a)$ equals the reservation utility $\underline{U}$. This renders a direct application of (8) useless. All hope is not lost. Careful consideration to adjusting the right-hand side of the (IR) constraint recovers the use of an adapted version of (8). This motivates recasting our development to allow for changes in the bound on agent utility. In particular, we are interested in analyzing the problem:

$$\max_{a} \min_{\hat{a}} \max_{w \geq \underline{w}} \left\{ V(w, a) : U(w, a) \geq b, U(w, a) - U(w, \hat{a}) \geq 0 \right\}. \tag{SAND|b}$$

where b is a real number no less than $\underline{U}$. We call (SAND|b) the sandwich problem given bound b, where "sandwich" refers to the fact that the minimization of $\hat{a}$ is sandwiched between two maximizations. Note that (Max-Min-Max) is the problem (SAND|$\underline{U}$). The next result shows that when b is appropriately chosen (SAND|b) is a relaxation of (P). For this reason we also call (SAND|b) the sandwich *relaxation* given b. The proof uses the following notation. Given the bound $b \geq \underline{U}$, define

$$\mathcal{W}(\hat{a}, b) \equiv \left\{ (w, a) : U(w, a) \geq b \text{ and } U(w, a) - U(w, \hat{a}) \geq 0 \right\}.$$

Note that $\mathcal{W}(\hat{a})$ defined in (7) is precisely $\mathcal{W}(\hat{a}, \underline{U})$.

Lemma 1. If there exists an optimal solution (w^*, a^*) to (P) such that $b = U(w^*, a^*)$ then

$$\text{val(P)} \leq \text{val(SAND}|b) \leq \text{val(Max-Min-Max)} \leq \text{val(FOA)}. \tag{9}$$

Proof. We show the first inequality in (9). Define the characteristic function

$$V^I(w, a|\hat{a}, b) = \begin{cases} V(w, a) & \text{if } (w, a) \in \mathcal{W}(\hat{a}, b) \\ -\infty & \text{otherwise} \end{cases}.$$

The hypothesized optimal solution (w^*, a^*) that gives utility b to the agent lies in the set $\mathcal{W}(\hat{a}, b)$ since $a^* \in a^{BR}(w^*)$ and $U(w^*, a^*) = b \geq \underline{U}$. Hence

$$\max_{(w,a) \in \mathcal{F}} V(w, a) = \max_{(w,a)} \inf_{\hat{a}} V^I(w, a|\hat{a}, b).$$

By the minmax inequality,

$$\max_{(w,a)} \min_{\hat{a}} V^I(w, a|\hat{a}, b) \leq \max_{a} \min_{\hat{a}} \max_{w} V^I(w, a|\hat{a}, b).$$

Next we need to argue that the problem on the right-hand side of the above inequality is precisely (SAND|b). For any $(a, \hat{a})$, the choice of w to maximize $V^I(w, a|\hat{a}, b)$ will avoid $(w, a) \notin \mathcal{W}(\hat{a}, b)$. Therefore, any element of $\arg\max_w\{V(w, a) : (w, a) \in \mathcal{W}(\hat{a}, b)\}$ also maximizes $V^I(w, a|\hat{a}, b)$.

For the second inequality in (9), we note that for every $(a, \hat{a})$, the maximum value

$$\max_w\{V(w, a) : U(w, a) \geq b, and U(w, a) - U(w, \hat{a}) \geq 0\}$$

is non-increasing in b. Therefore, for $b \geq \underline{U}$,

$$\begin{aligned}
\text{val}(\text{SAND}|b) &= \min_{\hat{a}} \max_w\{V(w, a) : U(w, a) \geq b, U(w, a) - U(w, \hat{a}) \geq 0\} \\
&\leq \min_{\hat{a}} \max_w\{V(w, a) : U(w, a) \geq \underline{U}, U(w, a) - U(w, \hat{a}) \geq 0\} = \text{val}(\text{Max-Min-Max}).
\end{aligned}$$

The final inequality in (9) comes from Lemma 2 in Ke and Ryan (2015) and using similar reasoning as in the above. $\square$

The following example shows that the sandwich relaxation can in fact be tight when b is appropriately chosen.

Example 3 (Example 1 continued)**.** We solve the sandwich relaxation (SAND|b) for $b = 0$. Consider $a \geq 0$, since the case where $a \leq 0$ is symmetric. Note that the (IR) constraint requires

$$z \leq -\frac{(a^2-1)^2}{a},$$

and the no-jump constraint $u(z, a) - u(z, \hat{a}) \geq 0$ is equivalent to

$$z \begin{cases} \geq -(\hat{a}+a)(\hat{a}^2+a^2-2) \text{ for } \hat{a} > a \\ \leq -(\hat{a}+a)(\hat{a}^2+a^2-2) \text{ for } \hat{a} < a \\ \in (-\infty, \infty) \qquad\qquad \text{for } \hat{a} = a. \end{cases} \tag{10}$$

In this case $v(z, a)$ is increasing in z. The optimal $\hat{a}$ is chosen to minimize z. Hence, for any $a \in [0, 2]$ the optimal choice is $\hat{a} < a$ and the optimal z given $(a, \hat{a})$ is

$$-(\hat{a}+a)(\hat{a}^2+a^2-2),$$

whenever (IR) is satisfied. Therefore,

$$\begin{aligned} z^*(a) &= \min_{\hat{a}<a} -(\hat{a}+a)(\hat{a}^2+a^2-2) \\ &= \begin{cases} 4a-4a^3 & \text{for } 1 \le a \le 2 \\ -\frac{4}{27}(-9a+5a^3) - \frac{4}{27}\sqrt{2}\sqrt{27-27a^2+9a^4-a^6} & \text{for } 0 \le a \le 1. \end{cases} \end{aligned} \quad (11)$$

and

$$\hat{a}(a) = \arg\min_{\hat{a}<a} -(\hat{a}+a)(\hat{a}^2+a^2-2).$$

Note that for any $a \in [0,1)$, $z^*(a) < -\frac{(a^2-1)^2}{a}$ cannot satisfy the (IR) constraint. Therefore, the optimal choice of a is $a \ge 1$, where $z^*(a) = 4a - 4a^3$. The principal's utility $v(z^*(a), a)$ is decreasing in $a \in [1,2]$, then we obtain the solution $a^* = 1$. Note that this shows solving (SAND$|b=0$) gives the optimal value of the original problem (computed by inspection in Example 1):

$$\max_a \min_{\hat{a}} \max_z \{v(z,a) : u(z,a) \ge 0, u(z,a) - u(z,\hat{a}) \ge 0\} = -2 = \text{val(P)}. \qquad \blacktriangleleft$$

The key question now becomes whether there exists a b that makes the sandwich relaxation (SAND$|b$) a tight relaxation of (P) and whether there is a systematic way to find it (if it does). To this end, we isolate attention to a particular bound of interest. We say b is *tight at optimality* (or simply *tight* when the context is clear) if

$$b = \max\{U(w,a) : (w,a) \text{ is an optimal solution to (P)}\}. \quad (12)$$

Note that $\underline{U}$ itself may not be be tight at optimality. Refining this definition further, we say b is *tight at optimality for an implementable action* a' if $b = \max\{U(w,a') : (w,a') \text{ optimal for (P)}\}$.

Unfortunately, it is difficult to establish whether a given b is tight *a priori*. Constructing such a bound is the task of the sandwich procedure discussed in Section 4. However, if b is known to the tight, we can leverage the results of our companion paper Ke and Ryan (2015). To be self-contained, we restate one of the key results of that paper. This requires three properties of the output distribution given by density function f. More extended discussion of these properties can be found in Section 5 of Ke and Ryan (2015). The output distribution satisfies the *monotone likelihood ratio property* (MLRP) if for any a, $\frac{\partial \log f(\cdot,a)}{\partial a}$ is nondecreasing. A set $\mathbf{E}$ is an *increasing set* if for every $x \in \mathbf{E}$ and $y \ge x$ then $y \in \mathbf{E}$.

The output distribution satisfies *nondecreasing increasing-set probability* (NISP) if, for every increasing set $\mathbf{E}$, the probability $\Pr(\tilde{x} \in \mathbf{E} \,|a)$ is nondecreasing in a. The output distribution satisfies *conditional stochastic dominance* (CSD) if, for every x_{-i}, $\int_{\mathcal{X}_i} f(x_i, x_{-i}, a)dx_i$ is nondecreasing in a.

Lemma 2 (Theorem 4 of Ke and Ryan (2015))**.** Assume u is increasing and concave, v is increasing and weakly concave, and the output distribution f satisfies MLRP, CSD and NISP.[4] If given an implementable action a and bound $b \geq \underline{U}$ that is tight at optimality given $\bar{a}$ then

$$\max_{w \geq \underline{w}} \min_{\hat{a} \in \mathbb{A}} \left\{ V^I(w, a|\hat{a}, b) : (w, a) \in \mathcal{W}(\hat{a}, b) \right\} = \min_{\hat{a} \in \mathbb{A}} \max_{w \geq \underline{w}} \left\{ V(w, a) : (w, a) \in \mathcal{W}(\hat{a}, b) \right\}. \quad (13)$$

This lemma gives sufficient conditions for when it is legitimate to *swap* the order of optimization between w and $\hat{a}$ without loss. This powerful property gives us a sufficient condition for (SAND$|b$) to be a tight relaxation of (P).

Theorem 1. Suppose we are given a bound b that is tight at optimality and the assumptions of Lemma 2 hold. Then val(SAND$|b$) = val(P) and an optimal solution of (SAND$|b$) is an optimal solution to (P).

The proof of Theorem 1 is somewhat involved. The complete proof is in the appendix and relies on several intermediate results from the intervening sections. To build intuition for why the result holds we provide a proof of the case where $\mathcal{X} = \{x_0\}$ is a singleton and so contracts w are characterized by a single number $z = w(x_0)$. The proof of the general case relies on a calculus of variations arguments that build on the ideas of the single-dimensional case.

The single-dimensional assumption means $U(w, a) = u(z) - c(a)$ and $V(w, a) = v(\pi(x_0) - z)$. The fact that u (and thus U) is strictly increasing in z and v (and thus V) is strictly decreasing in z underscores that the principal and agent have a direct conflict of interest

[4] More relaxed but harder-to-state sufficient conditions are found in Proposition 2 of Ke and Ryan (2015). When output is single-dimensional the sufficient condition for f provided here can be simplified to requiring MLRP only (Theorem 2 of Ke and Ryan (2015)) or, when the principal is risk-neutral, requiring $\int \pi(x) f(x, a)dx$ is increasing in a, a weaker condition than MLRP (Theorem 3 of Ke and Ryan (2015)).

over the choice of z – the agent prefers higher z while the principal prefers lower z. This is an important idea in the proof. In particular, it implies that when a contract is proposed that tightly constrains the agent's utility (given by b in (SAND|b)), that contract cannot be adjusted to earn more utility for the principal and remain feasible for the agent. In what follows, we abuse notation slightly and replace the contract w with its image z by writing, for instance, $U(z,a)$ instead of $U(w,a)$.

Proof of Theorem 1 for a single-dimensional contract. By Lemma 1 we know val(P) $\leq$ val(SAND|b). It remains to show val(SAND|b) $\leq$ val(P). Let $(a^*, \hat{a}^*, z^*)$ be an optimal solution to (SAND|b). If we can show a^* is a best response to z^* then (z^*, a^*) is a feasible solution to (P) since (IC) holds and $U(z^*, a^*) \geq b \geq \underline{U}$. This then implies val(SAND|b) $\leq V(z^*, a^*) \leq$ val(P), as desired. Hence, it suffices to show that a^* is a best response to z^*.

By the optimality of $(a^*, \hat{a}^*, z^*)$ in (SAND|b) we know

$$V(z^*, a^*) = \min_{\hat{a}} \max_{z} \{V(z, a^*) : U(z, a) \geq b, U(z, a) - U(z, \hat{a}) \geq 0\} \tag{14}$$

Take a particular $\hat{a}'$ where $\hat{a}'$ is a best response to z^*. Then from the minimization over $\hat{a}$ in (14) this implies

$$V(z^*, a^*) \leq \max_{z} \{V(z, a^*) : U(z, a) \geq b, U(z, a) - U(z, \hat{a}') \geq 0\} \tag{15}$$

Since V is strictly descreasing in z, if (15) holding with equality implies $U(z^*, a^*) - U(z^*, \hat{a}') \geq 0$. Since $\hat{a}'$ is a best response to z^* this implies a^* is also a best response to z^*, as desired.

It suffices to show (15) is satisfied with equality; that is,

$$V(z^*, a^*) = \max_{z} \{V(z, a^*) : U(z, a) \geq b, U(z, a) - U(z, \hat{a}') \geq 0\} \tag{16}$$

Claim 1. $U(z^*, a^*) = b$.

This claim makes precise a standard intuition in principal-agent problems that the principal can drive utility of the agent down to some lower bound. That lower bound is precisely b when tight at optimality. A proof of the claim is provided below, but for now we assume it holds.

By way of contradiction, suppose (16) fails. Then there exists a

$$z' \in \operatorname{argmax} \{V(z, a^*) : U(z, a) \geq b, U(z, a) - U(z, \hat{a}') \geq 0\} \tag{17}$$

such that $V(z^*, a^*) < V(z', a^*)$. Since V is strictly decreasing in z this implies $z^* > z'$. However, since U is a strictly increasing function of z this means $U(z', a^*) < U(z^*, a^*) = b$, where the equality holds because of Claim 1. This contradicts the definition of z'. Hence, (16) holds and we are done.

It only remains to verify Claim 1. We proceed by assuming $U(z^*, a^*) > b$ and uncovering a contradiction of the definition of b being tight. The details of this argument rely on the following auxiliary lemma:

Lemma 3. Let $(a^*, \hat{a}^*, z^*)$ be an optimal solution to the single-dimensional vertion of (SAND$|b$) with $U(z^*, a^*) > b$. Then for sufficiently small $\varepsilon > 0$ the perturbed problem (SAND$|b + \varepsilon$) also has an optimal solution $(a^*_\varepsilon, \hat{a}^*_\varepsilon, z^*_\varepsilon)$ with $U(z^*_\varepsilon, a^*_\varepsilon) > b + \varepsilon$ and the same optimal value; that is, $V(z^*_\varepsilon, a^*_\varepsilon) = V(z^*, a^*) = \text{val}(\text{SAND}|b)$.

The proof of this lemma (found in the appendix) relies on strong duality and the fact that if a constraint is slack, the dual multiplier on that constraint is 0 by complementary slackness. A small perturbation of the right-hand side of a slack constraint does not impact the optimal value. This argument is standard (see for instance, Bertsekas (1999)) in the absence of the inner minimization problem $\min_{\hat{a}}$ in (SAND$|b$). With the inner minimization the proof becomes nontrivial. For now we take the claim as given.

Returning to our proof of Theorem 1 for the single-dimensional case, we have assumed (by way of contradiction) that $U(z^*, a^*) > b$. Lemma 3 assures us that there exists an $\bar{\varepsilon} > 0$ where (SAND$|b + \bar{\varepsilon}$) has an optimal solution $(a^*_{\bar{\varepsilon}}, \hat{a}^*_{\bar{\varepsilon}}, z^*_{\bar{\varepsilon}})$ with $U(z^*_{\bar{\varepsilon}}, a^*_{\bar{\varepsilon}}) = b + \bar{\varepsilon}$ and $V(z^*_{\bar{\varepsilon}}, a^*_{\bar{\varepsilon}}) = V(z^*, a^*)$. Use an identical argument to that found in the paragraph surrounding equation (17) (except now taking condition $U(z^*_{\bar{\varepsilon}}, a^*_{\bar{\varepsilon}}) = b + \bar{\varepsilon}$ in place of the condition $U(z^*, a^*) = b$ to conclude $a^*_{\bar{\varepsilon}}$ is a best response to $z^*_{\bar{\varepsilon}}$ and hence $(z^*_{\bar{\varepsilon}}, a^*_{\bar{\varepsilon}})$ is an optimal solution to (P). This violates the defintion of b being tight, since now there is an optimal solution to (P) with an agent utility strictly larger than b, namely $b + \bar{\varepsilon}$. This completes the proof. □

We provide here some intuition behind Theorem 1 in the single-dimensional setting. For a given target action a^* we can think of the contracting problem as a sequential game, where the the principal chooses z and the agent chooses $\hat{a}$. The original (IC) constraint is equivalent to the situation that the principal chooses z first, and the agent chooses $\hat{a}$

afterwards. So the optimal choice of z should take all possible $\hat{a}$ into consideration. The agent has a second-mover advantage. Now consider a change in the order of decisions and let the agent chooses $\hat{a}$ first, with the principal choosing z in response. In this case the principal has a second-mover advantage, since the principal need not consider all possible $\hat{a}$. This provides intuition behind the bound in Lemma 1. However, if the agent utility bound b is tight given a^*, the principal cannot gain an advantage by moving second. No choice of contract by the principal can drive the agent's utility down any further. Since the principal and agent have a direct conflict of interest over the direction of z, this means the principal cannot improve her utility. In other words, the order of decisions does not matter when b is tight and so the sandwich problem provides a tight relaxation. This argument relies on the dimensionality of w. In the more multidimensional case, we parameterize the payment function through a unidimensional z using calculus of variations. As long as a conflict of interest exists, we obtain a similar intuition and a similar result.

4 The sandwich procedure

Given Theorem 1, our search turns towards finding a bound b that is tight at optimality. Once such a b is determined, the goal is to find a systematic way to solve (SAND$|b$). We approach both tasks concurrently using what we call the *sandwich procedure*. The basic logic of the procedure is to use backwards induction to characterize an optimal contract w^*, an optimal agent action a^*, and an alternate best response $\hat{a}^*$ as functions of the bound b. These characterization are then leveraged to compute a tight b by solving a carefully designed optimization problem. The resulting optimization problem has the real number b as a decision variable. We reduce much of the complexity of solving the infinite-dimensional optimization problem (P) into solving a single-dimensional optimization problem. Below is a high-level description of the sandwich procedure.

THE SANDWICH PROCEDURE

(Step 1) CHARACTERIZE CONTRACT: Characterize an optimal solution to the innermost maximization:

$$\max_{w \geq \underline{w}} \{V(w,a) : U(w,a) \geq b, U(w,a) - U(w,\hat{a}) \geq 0\} \qquad (\text{SAND}|a,\hat{a},b)$$

as a function of $a \in \mathbb{A}$, $\hat{a} \in \mathbb{A}$ and $b \geq \underline{U}$. Denote the resulting optimal contract by $w(a,\hat{a},b)$.

(Step 2) Characterize actions: Determine optimal solutions to the outer maximization and minimization

$$\max_a \min_{\hat{a}} V(w(a,\hat{a},b),a) \tag{18}$$

as functions of b. Denote a resulting optimal solutions by $(a(b)$ and $\hat{a}(b)$ and let $w(b) := w(a(b),\hat{a}(b),b)$.

(Step 3) Compute a tight bound: Solve the one-dimensional optimization problem:

$$b^* := \min \left\{ \text{argmin}_{b \geq \underline{U}} \left\{ V(w(b),a(b)) - \max_{a \in a^{BR}(w(b))} V(w(b),a) \right\} \right\}. \tag{19}$$

Let $a^* := a(b^*)$, $\hat{a}^* := \hat{a}(b^*)$, and $w^* := w(b^*)$.

The work of this section is to provide justification for the validity and each step in the sandwich procedure and the prove the following result.

Theorem 2. Let b^*, a^*, $\hat{a}^*$, and w^* be as defined in Step 3 of the sandwich procedure and suppose the assumptions of Lemma 2 hold. Then b^* is tight at optimality, (w^*,a^*) is an optimal solution to (P), and $\text{val}(\text{SAND}|b^*) = \text{val}(\text{P})$.

Before tackling the general setting, we discuss some special cases. If b is known to be tight at optimality through some independent means, Step 3 can be avoided and Steps 1 and 2 determine an optimal solution to (P). The next result provides a sufficient condition for the reservation utility $\underline{U}$ to be tight at optimality.

Proposition 1. The reservation utility $\underline{U}$ is tight at optimality when the limited liability constraint $w \geq \underline{w}$ is not tight in (P).

Proof. It suffices to prove the (IR) constraint is binding in (P). This is because if there exists an optimal solution that gives the agent his reservation utility then no optimal solution to (P) can give the agent any higher utility. Thus, when the (IR) constraint is binding in (P), $\underline{U}$ is tight at optimality. The proof that (IR) is binding is essentially the same as Proposition 2 in Grossman and Hart (1983). Suppose to the contrary that (w^*, a^*) is an optimal contract in which (IR) is not binding. Consider the contract $\tilde{w}$ solving $u(\tilde{w}) = u(w^*) - \varepsilon$ for any constant $\varepsilon > 0$, whenever $w^* \neq \underline{w}$. We choose ε such that (IR) is binding. Note that when $a^* \in a^{BR}(w^*)$,

$$U(\tilde{w}, a^*) = \int u(w^*) f(x, a^*) dx - c(a^*) - \varepsilon \geq \int u(w^*) f(x, a) dx - c(a) - \varepsilon.$$

therefore, for any a,

$$U(\tilde{w}, a^*) = \int u(w^*) f(x, a^*) dx - c(a^*) - \varepsilon \geq \int u(\tilde{w}) f(x, a) dx - c(a) + \varepsilon - \varepsilon = U(\tilde{w}, a)$$

implying that $\tilde{w}$ implements the same action as w^*. As $\varepsilon > 0$, the principal is strictly better off with $\tilde{w}$. This contradicts the definition of w^*. $\square$

We use our motivating example to illustrate how Step 3 can be used to determine a b^* that is is tight at optimality.

Example 4 (Example 1 continued)**.** For given $b \geq -2$, consider the sandwich relaxation:

$$\max_a \min_{\hat{a}} \max_z \{v(z, a) : u(z, a) \geq b, u(z, a) - u(z, \hat{a}) \geq 0\}.$$

The constraint set $u(z, a) \geq b$ is equivalent to

$$z \begin{cases} \geq -\frac{b+(a^2-1)^2}{a} & \text{for } a < 0 \\ \leq -\frac{b+(a^2-1)^2}{a} & \text{for } a > 0. \end{cases}$$

Given that the principal's utility is increasing in z when $a > 0$ and decreasing in z when $a < 0$, the optimal choice z of the innermost problem only optimizes over the boundary of constraint set $u(z, a) \geq b$, i.e., $z = -\frac{b+(a^2-1)^2}{a}$.

Consider the choice of $\hat{a}$. The constraint $u(z, a) - u(z, \hat{a}) \geq 0$ is equivalent to the region defined in (10). We only discuss when $a \geq 0$, because the situation when $a \leq 0$ is symmetric. When $a > 0$, $v(z, a)$ is increasing in z. The optimal $\hat{a}$ is to minimize z. So it is impossible to

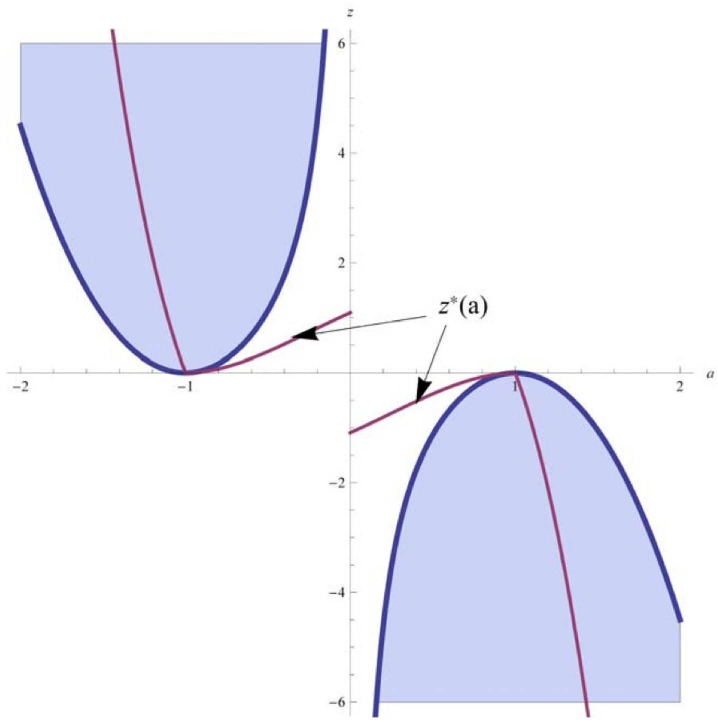

Figure 2: Figure for Example 4.

choose $\hat{a} \geq a$. Given $\hat{a} < a$, the maximum of $\{z : u(z,a) - u(z,\hat{a}) \geq 0\}$ is $-(\hat{a}+a)(\hat{a}^2+a^2-2)$, therefore, the optimal $\hat{a}$ is

$$\hat{a}(a,b) = \begin{cases} \arg\max_{\hat{a}} -(\hat{a}+a)(\hat{a}^2+a^2-2), \text{ if } \max_{\hat{a}} -(\hat{a}+a)(\hat{a}^2+a^2-2) > -\frac{b+(a^2-1)^2}{a}, \\ \arg\min_{\hat{a}<a} -(\hat{a}+a)(\hat{a}^2+a^2-2), \text{ if } \max_{\hat{a}} -(\hat{a}+a)(\hat{a}^2+a^2-2) \leq -\frac{b+(a^2-1)^2}{a}. \end{cases}$$

Therefore, it follows that (see Figure 2) $z^*(a,b) = z^*(a)$ if $z^*(a) \leq -\frac{b+(a^2-1)^2}{a}$, where $z^*(a)$ is defined in (11). Otherwise, $z^*(a)$ is not feasible and the constraint set

$$\{z : u(z,a) \geq b, u(z,a) - u(z,\hat{a}) \geq 0\}$$

is empty for some $\hat{a}(a)$, which should be avoided.

Back to the choice of a. Given b, and based on the above analysis, a should satisfy

$$z^*(a) \leq -\frac{b+(a^2-1)^2}{a}$$

or equivalently,

$$b \leq -z^*(a)a - (a^2-1)^2.$$

Therefore, (SAND$|b$) is the problem

$$\max_{a\in[0,2]}\{v(z^*(a),a) : -z^*(a)a-(a^2-1)^2 \geq b\},$$

which is just a standard constraint optimization problem. Its optimal solution $a^*(b)$ is a solution to the equation $b = -z^*(a)a - (a^2-1)^2$. The corresponding value of z is $z^*(a^*(b))$. For any $b < 0$, $a^*(b) < 1$ and $z^*(a^*(b)) < 0$.[5] Hence we have

$$\max_{a\in a^{BR}(z^*(a^*(b))} v(z^*(a^*(b)),a) < \text{val}(\text{SAND}|b).$$

The smallest b to close the gap is $b^* = 0$. Thus, b^* is tight at optimality by Theorem 1. This tightness of $b^* = 0$ can also be verified by inspection. ◀

4.1 Analysis of Step 1

In this section we show how to characterize contracts $w(a,\hat{a},b)$ that solve (SAND$|a,\hat{a},b$). In addition, we show that $w(a,\hat{a},b)$ is a continuous function in $(a,\hat{a};b)$ and $V(w(a,\hat{a},b),a)$ is continuous and almost-everywhere differentiable function of $(a,\hat{a})$ given b. These topological properties establish the existence of solutions in Steps 2 and Step 3.

Construct the Lagrangian function to (SAND$|a,\hat{a},b$) with $b \geq \underline{U}$:

$$\mathcal{L}(w,\lambda,\delta|a,\hat{a},b) = V(w,a) + \lambda[U(w,a)-b] + \delta[U(w,a)-U(w,\hat{a})], \tag{20}$$

where $\lambda \geq 0$ and $\delta \geq 0$ are the Lagrangian multipliers with respect to constraints $U(w,a) \geq b$ and $U(w,a) - U(w,\hat{a}) \geq 0$, respectively. Note that $\lambda \geq 0$ and $\delta \geq 0$, and we assume this throughout the paper without further indication.

When the support of $f(\cdot,a)$ does not depend on a (as we assume), we can further write the Lagrangian as

$$\mathcal{L}(w,\lambda,\delta|a,\hat{a},b) = \int L(w,\lambda,\delta|a,\hat{a},b)f(x,a)dx,$$

where

$$L(w,\lambda,\delta|a,\hat{a},b) = v(\pi-w) + \lambda(u(w)-c(a)-b) + \delta[u(w)(1-\tfrac{f(x,\hat{a})}{f(x,a)}) - c(a) + c(\hat{a})]$$

[5] For example, in the Example 3, $b = -0.5$, $a^*(b) \approx 0.383$, $z^*(a^*(b)) \approx -0.541$, and $a^{BR}(z^*(a^*(b)) \approx 1.062$, the principal value $\max_{a\in a^{BR}(z^*(a^*(b))} v(z^*(a^*(b)),a) \approx -2.830$, which is much less than val(SAND$|b$) $= -0.501$.

The maximization of $\mathcal{L}(w, \lambda, \delta | a, \hat{a}, b)$ over w can be done pointwise through $L(w, \lambda, \delta | a, \hat{a}, b)$. The concavities of v and u imply that $L(w, \lambda, \delta | a, \hat{a}, b)$ is single-peaked in w given every other arguments. Therefore, there exists a unique w such that the first-order condition

$$\frac{\partial}{\partial w} L(w, \lambda, \delta | a, \hat{a}, b) = 0$$

is satisfied for $w > \underline{w}$, or $w = \underline{w}$ if $\frac{\partial}{\partial w} L(w, \lambda, \delta | a, \hat{a}, b) < 0$. This unique w can be solved by the equation (for almost every x)

$$\frac{v'(\pi - w)}{u'(w)} = \lambda + \delta \left(1 - \frac{f(x, \hat{a})}{f(x, a)}\right) \tag{21}$$

whenever $\lambda + \delta(1 - \frac{f(x,\hat{a})}{f(x,a)}) \geq \frac{v'(\pi - \underline{w})}{u'(\underline{w})}$. We denote this solution by $w_{\lambda,\delta}(a, \hat{a}, b)$ and refer to it as the *generalized Mirrlees-Holmstrom (GMH) contract* given $(\lambda, \delta, a, \hat{a}, b)$. Such contracts are generalized versions of Mirrlees-Holmstrom contracts in the special case of a binary action.

There are five parameters $(\lambda, \delta, a, \hat{a}, b)$ in a GMH contract. However, Lemma 4 below shows each GMH contract is really a function of three parameters: a, $\hat{a}$ and b. This is because there exist unique Lagrangian multipliers λ and δ that make $w_{\lambda,\delta}(a, \hat{a}, b)$ a solution to (SAND$|a, \hat{a}, b$) given a, $\hat{a}$ and b.

Lemma 4. If u is concave and v is weakly concave, then for every $(a, \hat{a}, b)$ there exists a unique Lagrangian multipliers λ^* and δ^* and associated GMH contract w^* such that (i) (21) holds given $(a, \hat{a}, b)$ and (ii) the complementary slackness conditions hold:

$$\lambda^* \geq 0,\, U(w^*, a) - b \geq 0 \quad \text{and} \quad \lambda^*[U(w^*, a) - b] = 0, \tag{ii-a}$$

$$\delta^* \geq 0,\, U(w^*, a) - U(w^*, \hat{a}) \geq 0 \quad \text{and} \quad \delta^*[U(w^*, a) - U(w^*, \hat{a})] = 0. \tag{ii-b}$$

Moreover, the following properties hold: (iii) $(\lambda^*, \delta^*) = (\lambda(a, \hat{a}, b), \delta(a, \hat{a}, b))$ depends on $(a, \hat{a}, b)$ in a continuous and almost everywhere differentiable manner and (iv)

$$w^* = w_{\lambda(a,\hat{a},b),\delta(a,\hat{a},b)}(a, \hat{a}, b)$$

is continuous and almost everywhere differentiable in $(a, \hat{a}, b)$.

The proof of the lemma is in the appendix. For brevity we let $w(a, \hat{a}, b) \equiv w_{\lambda(a,\hat{a},b),\delta(a,\hat{a},b)}(a, \hat{a}, b)$ where $w_{\lambda(a,\hat{a},b),\delta(a,\hat{a},b)}(a, \hat{a}, b)$ is the GMH contract defined in Lemma 4(iv). The principal's optimal utility given $(a, \hat{a}, b)$ is hence

$$V(w(a, \hat{a}, b), a) = \min_{(\lambda,\delta)} \max_{w} \mathcal{L}(w, \lambda, \delta | a, \hat{a}, b) = \mathcal{L}^*(a, \hat{a}|b)$$

where

$$\mathcal{L}^*(a, \hat{a}|b) \equiv \mathcal{L}(w(a, \hat{a}, b), \lambda(a, \hat{a}, b), \delta(a, \hat{a}, b)|a, \hat{a}b). \tag{22}$$

This uses the fact that complementary slackness in Lemma 4(ii) implies strong duality between (SAND$|a, \hat{a}, b$) and its Lagrangian dual. The theorem of maximum then implies that $\mathcal{L}^*(a, \hat{a}|b)$ is continuous and almost everywhere differentiable in $(a, \hat{a}|b)$ because both the maximizer of the Lagrangian over w and the Lagrangian multipliers are unique given $(a, \hat{a}|b)$.

4.2 Analysis of Step 2

This subsection provides necessary optimality conditions for a and $\hat{a}$ to optimize (SAND$|b$) given the characterized optimal contract $w(a, \hat{a}, b)$ and its associated dual multipliers $\lambda(a, \hat{a}, b)$ and $\delta(a, \hat{a}, b)$ as defined in Section 4.1. In particular, we solve (18) in Step 2 by solving:

$$\max_{a} \min_{\hat{a}} \mathcal{L}^*(a, \hat{a}|b) \tag{23}$$

using the definition of $\mathcal{L}^*$ in (22).

Proposition 2. Assume u is increasing and concave and v is increasing and weakly concave. Suppose that a^* and $\hat{a}^*$ solve (23) for a given $b \geq \underline{U}$. The following hold: (i) for an interior solution $\hat{a}^* \in (\underline{a}, \bar{a})$, $\frac{\partial}{\partial \hat{a}} \mathcal{L}^*(a^*, \hat{a}^*|b) = -\delta^*(a^*, \hat{a}^*, b) U_a(w(a^*, \hat{a}^*, b), \hat{a}^*) = 0$, and $U_a(w(a^*, \hat{a}^*, b), \hat{a}^*) \geq 0$ (≤ 0) for $\hat{a}^* = \bar{a}$ $(\hat{a}^* = \underline{a})$; (ii) for an interior solution $a^* \in (\underline{a}, \bar{a})$, the right derivative $\frac{\partial}{\partial a^+} (\min_{\hat{a}} \mathcal{L}^*(a^*, \hat{a}|b)) \leq 0$, and left derivative $\frac{\partial}{\partial a^-} (\min_{\hat{a}} \mathcal{L}^*(a^*, \hat{a}|b)) \geq 0$; and $\frac{\partial}{\partial \hat{a}} \mathcal{L}^*(a^*, \hat{a}^*|b) \leq 0$ (≥ 0) for $a^* = \underline{a}$ $(a^* = \bar{a})$.

Proof. For part (i), since

$$\min_{\hat{a}} \min_{\lambda,\delta} \max_{w} \mathcal{L}(w, \lambda, \delta | a, \hat{a}, b) = \min_{\lambda,\delta} \min_{\hat{a}} \max_{w} \mathcal{L}(w, \lambda, \delta | a, \hat{a}, b)$$

the desired result follows from the envelope theorem. For part (ii), note that $\min_{\hat{a}} \mathcal{L}^*(a, \hat{a}|b)$ is continuous and directionally differentiable in a (see e.g., Corollary 4.4 of Dempe (2002)). Since a^* is a maximum, then $\frac{\partial}{\partial a^+} (\min_{\hat{a}} \mathcal{L}^*(a^*, \hat{a}|b)) \leq 0$ and $\frac{\partial}{\partial a^-} (\min_{\hat{a}} \mathcal{L}^*(a^*, \hat{a}|b)) \geq 0$. $\square$

4.3 Analysis of Step 3

Steps 1 and 2 of the sandwich procedure show us how to characterize optimal solutions to the sandwich relaxation as a function of the bound b. However, these optimal solutions are only guaranteed to be optimal solutions to the original problem (P) when b is known to be tight at optimality. The goal of Step 3 is to find such a b. Given Steps 1 and 2, this boils down to the following single-dimensional optimization problem in b.

Lemma 5. Assume the conditions of Lemma 2 hold. Then, there exists a real number b^* that satisfies (19) and, furthermore, U^* is tight at optimality.

Proof. Let $U^* = \max\{U(w,a) : (w,a) \text{ is optimal to (P)}\}$. Clearly U^* exists and is, by definition, tight at optimality. The heart of the proof is to show that

$$U^* = \min\left\{\text{argmin}_{b \geq \underline{U}}\left(\mathcal{L}^*(a(b), \hat{a}(b)|b) - \max_{a \in a^{BR}(w(b))} V(w(b), a)\right)\right\} \tag{24}$$

where $a(b)$, $\hat{a}(b)$ and $w(b)$ are as defined in Step 2 of the sandwich procedure. That is, $(a(b), \hat{a}(b))$ satisfy $\mathcal{L}^*(a(b), \hat{a}(b)|b) = \max_a \min_{\hat{a}} \mathcal{L}^*(a, \hat{a}|b)$ and $w(b)$ denotes the GMH contract $w(a, \hat{a}, b) = w_{\lambda(a(b),\hat{a}(b),b),\delta(a(b),\hat{a}(b),b)}(a, \hat{a}, b)$.

Note that (19) is equivalent to

$$\min\left\{\text{argmin}_{b \geq \underline{U}}\left(\mathcal{L}^*(a(b), \hat{a}(b)|b) - \max_{a \in a^{BR}(w(b))} V(w(b), a)\right)\right\} \tag{25}$$

using the Lagrangian tools set up in Section 4.1. Hence (24) and (25) will imply $b^* = U^*$, which establishes the result, provided (24) can be shown. This is our task. By Lemma 1 the difference $\mathcal{L}^*(a(b), \hat{a}(b)|b) - \max_{a \in a^{BR}(w(b))} V(w(b), a) \geq 0$. Also, since U^* is tight at optimality we have $\mathcal{L}^*(a(b), \hat{a}(b)|b) - \max_{a \in a^{BR}(w(b))} V(w(b), a) = 0$ and so

$$U^* \in \text{argmin}_{b \geq \underline{U}}\left(\mathcal{L}^*(a(b), \hat{a}(b)|b) - \max_{a \in a^{BR}(w(b))} V(w(b), a)\right).$$

It remains to show that U^* is the minimum of the argmin set. Note that for any $b \in [\underline{U}, U^*)$

we have

$$\begin{aligned}\mathcal{L}^*(a(b),\hat{a}(b)|b) &> \mathcal{L}^*(a(U^*),\hat{a}(U^*)|U^*)\\ &\geq \max_{(w,a)}\{V(w,a) : U(w,a) \geq U^*, a \in a^{BR}(w)\}\\ &= \text{val}(\text{P})\\ &\geq \max_{a\in a^{BR}(w(b))} V(w(b),a)\end{aligned}$$

where the first strict inequality is due to the fact that $\mathcal{L}^*(a(b),\hat{a}(b),b)$ is decreasing in b, the second equality is from Lemma 1, the equality comes from the definition of U^*, and the final inequality follows since

$$(w(b), a^{BR}(w(b)) \subset \mathcal{F}.$$

This says every $b \in [\underline{U}, U^*)$ is not in argmin $\left(\mathcal{L}^*(a(b),\hat{a}(b)|b) - \max_{a\in a^{BR}(w(b))} V(w(b),a)\right)$ and so U^* is its minimum element. □

We are now ready to proof of our main result that the sandwich procedure produces an optimal solution to (P) when the conditions of Lemma 2 hold. The proof is a straightforward application of our analysis of Steps 1 to 3.

Proof of Theorem 2. By Lemma 5 there exists a b^* that satisfies (19) and is tight at optimality. Hence, by Theorem 1, val(SAND$|b^*$) = val(P) and every optimal solution $(w(b^*), a(b^*))$ to (SAND$|b^*$) is optimal to (P). The GMH contract $w(a(b^*),\hat{a}(b^*),b^*)$ resulting from Lemma 4 is precisely one such optimal contract where $a(b^*)$ and $\hat{a}(b^*)$ satisfy the optimality conditions of Proposition 2. □

4.4 An example

We consider an example from Araujo and Moreira (2001) to show how the sandwich procedure produces an optimal solution. In the process we demonstrate the advantage of our approach to finding an optimal solution in comparison to the method proposed by Araujo and Moreira (2001). They use an algorithm to analyze a nonlinear optimization problem with 20 non-linear constraints using Kuhn-Tucker conditions, whereas we use a straightforward calculation.

Example 5. The principal has expected utility $V(w,a) = \sum_{i=1}^{2} p_i(a)(x_i - w_i)$, where $p_1(a) = 1 - a^3$, $p_2(a) = a^3$ for $a \in [0, 0.9]$ for outcomes $x_1 = 1$, $x_2 = 5$. The minimum wage is $\underline{w} = 0$. The agent's expected utility is $U(w,a) = \sum_{i=1}^{2} p_i(a)\sqrt{w_i} - a^2$ with reservation utility $\underline{U} = 0$. It is easy to show that the first-order approach is invalid and the assumptions in Lemma 2 hold, so we apply the sandwich procedure to find an optimal solution.

Solving the first-order condition (21), the GMH contract is

$$w_{\lambda,\delta}(a,\hat{a},b)_i = \frac{1}{4}\left(\lambda + \delta\left[1 - \frac{p_i(\hat{a})}{p_i(a)}\right]\right)^2 \text{ for } i = 1,2.$$

Proposition 1 applies to this setting since $w_{\lambda,\delta}(a,\hat{a},b) \geq 0 = \underline{w}$ for all contracts so the minimum wage constraint is not binding. Hence, reservation utility $\underline{U} = 0$ is tight at optimality. By Lemma 4, we have

$$\lambda(a,\hat{a},0) = 2a^2 \text{ and } \delta(a,\hat{a},0) = \frac{2a^3(a+\hat{a})(1-a^3)}{(a-\hat{a})(a^2+a\hat{a}+\hat{a}^2)^2},$$

and

$$\begin{aligned}
\mathcal{L}^*(a,\hat{a}|0) &= \sum_{i=1}^{2} p_i(a)(x_i - w(a,\hat{a},0)_i) \\
&= (1-a^3) + 5a^3 - \frac{a^3(a\hat{a}^2(2a^2+2a\hat{a}+\hat{a}^2) + (a+\hat{a})^2)}{(a^2+a\hat{a}+\hat{a}^2)^2}.
\end{aligned}$$

As $\frac{\partial}{\partial \hat{a}}\mathcal{L}^*(a,\hat{a}|0) > 0$, we have $\hat{a}^* = 0$ by minimizing $\mathcal{L}^*(a,\hat{a}|0)$ over $\hat{a}$. Then,

$$\mathcal{L}^*(a,0|0) = (1-a^3) + 5a^3 - a$$

has a maximum at $a^* = 0.9$. We can check whether $a^* = 0.9$ is implementable as follows.

The agent's expected utility under the contract $w(a,\hat{a},0)$ is

$$U(w(a,\hat{a},0),\tilde{a}) = \frac{(a-\tilde{a})(\hat{a}-\tilde{a})(a\hat{a}+\tilde{a}(a+\hat{a}))}{(a^2+a\hat{a}+\hat{a}^2)}.$$

When $\tilde{a} > \frac{2(a^2+a\hat{a}+\hat{a}^2)}{3(a+\hat{a})}$, $U(w(a,\hat{a},0),\tilde{a})$ increases in $\tilde{a}$, and when $\tilde{a} < \frac{2(a^2+a\hat{a}+\hat{a}^2)}{3(a+\hat{a})}$, $U(w(a,\hat{a},0),\tilde{a})$ decreases in $\tilde{a}$. Therefore, the best response must be in the corner(s). When $\hat{a}^* = 0$ and $a^* = 0.9$, the best response is $a^* = 0.9$ which is implementable. ◀

5 Conclusion

We provide a general method to solve moral hazard problems when output is a continuous random variable with a distribution that satisfies certain monotonicity properties (see the conditions in Lemma 2). This involves solving a tractable relaxation of the original problem using a bound on agent utility derived from our proposed procedure. Optimal contracts have a simple (GMH) structure that looks at a single no-jump constraint from the set of IC constraints instead of infinitely many such constraints, as faced by other methods that tackle moral hazard problems when the first-order approach fails.

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A Appendix: Technical proofs of selected results

A.1 Proof of Lemma 3

We require the following preliminary claim:

Claim 2. Consider a maximization problem $\max_x\{f(x) : g(x) \geq 0\}$ where $f : \mathbb{X} \to \mathbb{R}$, and $g : \mathbb{X} \to \mathbb{R}^k$, for some compact subset $\mathbb{X} \subset \mathbb{R}^n$. Assume that both f and g are continuous and differentiable. If the Lagrangian $L(x, \lambda) = f(x) + \lambda \cdot g(x)$ is strictly concave in x, then $\max_x\{f(x) : g(x) \geq 0\} = \inf_{\lambda \geq 0} \max_x L(x, \lambda)$, where we assume the maximum of $L(x, \lambda)$ over x exists for any given λ.

Proof. If $L(x, \lambda)$ is strictly concave in x and $\max_x L(x, \lambda) < \infty$, then there exists a unique maximum $x^*(\lambda) = \arg\max_x L(x, \lambda)$. Then the dual $\max_x L(x, \lambda) = L(x^*(\lambda), \lambda)$ is continuous, convex and differentiable in λ. If the minimum of the dual $\max_x L(x, \lambda)$ exists, then it must satisfy the stationary condition $\frac{\partial}{\partial \lambda_i}(\max_x L(x, \lambda)) = 0$ for $\lambda_i > 0$ and otherwise $\lambda = 0$. This implies at the solution λ^* $\nabla_\lambda (\max_x L(x, \lambda^*)) = g(x^*(\lambda^*)) = 0$, which means no duality gap. If the minimum of the dual $\max_x L(x, \lambda)$ does not exist for at least in dimension i (suppose λ_{-i} is bounded) it must imply that $\frac{\partial}{\partial \lambda_i}(\max_x L(x, \lambda)) \leq 0$ (so that $\lambda_i \to \infty$). Then we have $g_i(x^*(\lambda_i, \lambda_{-i})) \leq 0$, and $\lim_{\lambda_i \to \infty} g_i(x^*(\lambda_i, \lambda_{-i})) = 0$, where the last equality holds since otherwise $\max_x L(x, \lambda^*) \to -\infty$.

Therefore, we have

$$\begin{aligned}
\inf_{\lambda \geq 0} \max_x L(x, \lambda) &= \lim_{\lambda_i \to \infty} [f(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))) + \lambda_i g_i(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))) + \lambda^*_{-i}(\lambda_i) \cdot g_{-i}(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i)))] \\
&= \lim_{\lambda_i \to \infty} [f(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))) + \lambda_i g_i(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))) \\
&\leq \lim_{\lambda_i \to \infty} f(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))).
\end{aligned}$$

where the second equality is from the complementary slackness: $\lambda^*_{-i}(\lambda_i) \cdot g_{-i}(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))) = 0$.

Since $\lim_{\lambda_i \to \infty} g_i(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))) = 0$, $\lim_{\lambda_i \to \infty} x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))$ satisfies all constraints. Therefore, we have $\lim_{\lambda_i \to \infty} f(x^*(\lambda_i, \lambda^*_{-i}(\lambda_i))) \leq \inf_{\lambda \geq 0} \max_x L(x, \lambda)$ by weak duality. Therefore, strong duality holds. The result should be true if the Lagrangian multiplier do not exist for more than one dimension. □

Proof of Lemma 3. Let $(a^*, \hat{a}^*, z^*)$ be an optimal solution (SAND$|b$); that is,

$$U(z^*, a^*) = \max_a \min_{\hat{a}} \max_z \{V(z, a) : U(z, a) \geq b,\, U(z, a) - U(z, \hat{a}) \geq 0\}.$$

Given a^* consider the Lagrangian dual of the final maximization problem in z; that is,

$$L(z, \lambda, \delta | a^*, \hat{a}, b) = V(z, a^*) + \lambda[U(z, a^*) - b] + \delta[U(z, a^*) - U(z, \hat{a})].$$

Since $U(z, a)$ is concave in z and $V(z, a)$ is weakly concave in z (implying $L(z, \lambda, \delta | a^*, \hat{a}, b)$ is concave in z), Claim 1 implies:

$$\min_{\hat{a}} \max_z \{V(z, a^*) : U(z, a^*) \geq b,\, U(z, a^*) - U(z, \hat{a}) \geq 0\} = \min_{\hat{a}} \inf_{\lambda \geq 0, \delta \geq 0} \max_{z \in Z} L(z, \lambda, \delta | a^*, \hat{a}, d)$$

for all $d \in [b, b + \varepsilon)$. We now consider three cases.

Case 1. The set $\cap_{\hat{a} \in \mathbb{A}} \{z : U(z, a^*) \geq b + \varepsilon, U(z, a^*) - U(z, \hat{a}) \geq 0\}$ is empty, for any arbitrarily small $\varepsilon > 0$. We want to rule out this case. Note that in this case, the Lagrangian multiplier

$$\lambda(a^*, \hat{a}^*_\varepsilon) \in \arg \inf_{\lambda \geq 0} \inf_{\delta \geq 0} \max_{z \in Z} L(z, \lambda, \delta | a^*, \hat{a}, b + \varepsilon)$$

is unbounded, where $\hat{a}^*_\varepsilon \in \arg\min_{\hat{a}} \inf_{\lambda \geq 0, \delta \geq 0} \max_{z \in Z} L(z, \lambda, \delta | a^*, \hat{a}, b + \varepsilon)$. Also, $U(z^*_\varepsilon, a^*) < b + \varepsilon$ for any z^*_ε such that

$$L(z^*_\varepsilon, \lambda(a^*_\varepsilon, \hat{a}^*_\varepsilon), \delta(a^*_\varepsilon, \hat{a}^*_\varepsilon) | a^*_\varepsilon, \hat{a}^*_\varepsilon, b + \varepsilon)) = \inf_{\lambda \geq 0} \inf_{\delta \geq 0} \max_{z \in Z} L(z, \lambda, \delta | a^*, \hat{a}, b + \varepsilon).$$

Therefore, we choose a sequence $\varepsilon_n = \frac{\varepsilon}{n}$, and we have

$$U(z^*_{\varepsilon_n}, a^*) - b - \varepsilon_n < 0,$$

where $z^*_{\varepsilon_n}$ is a sequence such that

$$V(z^*_{\varepsilon_n}, a^*) = \min_{\hat{a}} \inf_{\lambda \geq 0, \delta \geq 0} \max_{z \in Z} L(z, \lambda, \delta | a^*, \hat{a}, b + \varepsilon_n).$$

Note that $(z^*_\varepsilon, a^*_\varepsilon, \hat{a}^*_\varepsilon)$ is upper hemicontinuous in ε, as a solution to the optimization problem. Then as $n \to \infty$, the limit $(z^*_0, a^*_0, \hat{a}^*_0; \lambda(a^*_0, \hat{a}^*_0), \delta(a^*_0, \hat{a}^*_0))$ is a solution to the problem without perturbation ($\varepsilon = 0$). Without loss of generality, we choose

$$(z^*, a^*, \hat{a}^*; \lambda(a^*, \hat{a}^*), \delta(a^*, \hat{a}^*)) = (z^*_0, a^*_0, \hat{a}^*_0; \lambda(a^*_0, \hat{a}^*_0), \delta(a^*_0, \hat{a}^*_0)).$$

Then, passing to the limit (taking a subsequence if necessary), $z^*_{\varepsilon_n} \to z^*$, we have

$$\lim_{n \to \infty} \left[U(z^*_{\varepsilon_n}, a^*) - b - \varepsilon_n \right] = U(z^*, a^*) - b \leq 0$$

which contradicts of the supposition $U(z^*, a^*) > b$. Therefore, the set

$$\cap_{\hat{a} \in A} \{ z : U(z, a^*) \geq b + \varepsilon, U(z, a^*) - U(z, \hat{a}) \geq 0 \}$$

is non-empty for a sufficiently small ε.

Case 2. The set $\cap_{\hat{a} \in \mathbb{A}} \{ z : U(z, a^*) \geq b + \varepsilon, U(z, a^*) - U(z, \hat{a}) \geq 0 \}$ is nonempty and $\lambda(a^*, \hat{a}^*_\varepsilon) > 0$, for any $\varepsilon > 0$.

We also want to rule out this case. Note that $\lambda(a^*, \hat{a}^*_\varepsilon) > 0$ implies the constraint $U(z^*_\varepsilon, a^*) \geq b + \varepsilon$ is binding given strong duality. We choose a sequence $\varepsilon_n = \frac{\varepsilon}{n}$. Passing to the limit (taking a subsequence if necessary), $z^*_{\varepsilon_n} \to z^*$, we have

$$0 = \lim_{n \to \infty} \left[U(z^*_{\varepsilon_n}, a^*) - b - \varepsilon_n \right] = U(z^*, a^*) - b$$

which contradicts with the supposition $U(z^*, a^*) > b$.

Case 3. The set $\cap_{\hat{a} \in A} \{ z : U(z, a^*) \geq U^* + \varepsilon, U(z, a^*) - U(z, \hat{a}) \geq 0 \}$ is nonempty and $\lambda(a^*, \hat{a}^*_\varepsilon) = 0$, for some arbitrarily small $\varepsilon > 0$.

Given $\lambda(a^*, \hat{a}^*_\varepsilon) = 0$, then we have

$$\begin{aligned}
V(z^*_\varepsilon, a^*) &= \max_z V(z, a^*) + \lambda(a^*, \hat{a}^*_\varepsilon)(U(z, a^*) - b - \varepsilon) + \delta(a^*, \hat{a}^*_\varepsilon)(U(z, a^*) - U(z, \hat{a}^*_\varepsilon)) \\
&= \max_z V(z, a^*) + \lambda(a^*, \hat{a}^*_\varepsilon)(U(z, a^*) - b) + \delta(a^*, \hat{a}^*_\varepsilon)(U(z, a^*) - u(z, \hat{a}^*_\varepsilon)) \\
&\geq \min_{\hat{a}} \inf_{\lambda \geq 0, \delta \geq 0} \max_z V(z, a^*) + \lambda(a^*, \hat{a}^*_\varepsilon)(U(z, a^*) - b) + \delta(a^*, \hat{a}^*_\varepsilon)(U(z, a^*) - U(z, \hat{a}^*_\varepsilon)) \\
&= V(z^*, a^*).
\end{aligned}$$

We already know $V(z^*, a^*) \geq V(z^*_\varepsilon, a^*)$ by $\varepsilon > 0$. Therefore, we have shown $V(z^*_\varepsilon, a^*) = V(z^*, a^*)$, as required. □

A.2 Proof of Lemma 4

Proof. For the existence of λ,δ that satisfy (i) and (ii), we refer to our companion paper which establishes the existence of optimal Lagrangian solutions. As for uniqueness (part (iii)), note that $\mathcal{L}(w, \lambda, \delta \,|a, \hat{a}, b\,)$ has the unique maximum $w_{\lambda,\delta}(a, \hat{a}, b)$ and $w_{\lambda,\delta}(a, \hat{a}, b) = w_{\lambda',\delta'}(a, \hat{a}, b)$ implies $(\lambda, \delta) = (\lambda', \delta')$ both contracts are continuous functions of the variable x.

Therefore, $\max_{w \geq \underline{w}} \mathcal{L}(w, \lambda, \delta \,|a, \hat{a}, b\,)$ is strictly convex in (λ, δ) and so uniqueness follows. □

A.3 Proof of Theorem 1

We want to show a solution to the problem

$$a^{\#} \in \arg\max_{a} \left(\inf_{\hat{a}} \max_{w \geq \underline{w}} \{V(w, a) : U(w, a) \geq b,\, U(w, a) - U(w, \hat{a}) \geq 0\} \right)$$

is implementable, where $b = U(w^*, a^*)$ is tight at optimality. If $a^{\#}$ is implementable then there exists an optimal solution to (SAND$|b$) is feasible to (P) and then Lemma 1, this implies the result.

Let $(a^{\#}, \hat{a}^{\#}, w^{\#})$ be a solution to (SAND$|b$). Our goal is to show this $a^{\#}$ is implementable. We use a variational approach. Let a family of test function h given w be

$$\mathcal{H}_w \equiv \{h : h(x) = 0 \text{ for } x \in \{x : w(x) = \underline{w}\} \text{ and } 0 \leq h(x) \leq \min\{w(x) - \underline{w}, \bar{h}\}\}, \tag{26}$$

where we further restrict $|h(x)| \leq \bar{h}$ for some number $\bar{h} > 0$. The variational approach allows to leverage insights similar to those used to prove the one-dimensional case (see the proof following the statement of Theorem 1). In particular, the z defined below is analogous to the z used in the proof of Theorem 1 in the one-dimensional case. The intuition from the single-dimensional case guides our proof.

Define:

$$\min_{\hat{a}} \max_{h \in \mathcal{H}_{w^{\#}}} \max_{z \in [\underline{z}, \overline{z}]} \{V(w^{\#} + zh, a^{\#}) : (w^{\#} + zh, a^{\#}) \in \mathcal{W}(\hat{a}, b)\}, \tag{27}$$

where $\underline{z} = -\max_x\{(w(x) - \underline{w})/h(x)\}$ and $\overline{z}$ is a sufficiently large constant. These bounds are understood throughout and are thus not denoted. The existence of maximum over $h \in \mathcal{H}_{w^{\#}}$ follows since $\mathcal{H}_{w^{\#}}$ is weakly sequential compact. As we will show later (Claim 3), the above problem is equivalent to (SAND$|b$). Let us suppose that the equivalence is true. Now consider $\hat{a}' \in a^{BR}(w^{\#})$, we have

$$V(w^{\#}, a^{\#}) = \min_{\hat{a}} \max_{h \in \mathcal{H}_{w^{\#}}} \max_{z} \{V(w^{\#} + zh, a^{\#}) : (w^{\#} + zh, a^{\#}) \in \mathcal{W}(\hat{a}, b)\}$$
$$\leq \max_{h \in \mathcal{H}_{w^{\#}}} \max_{z} \{V(w^{\#} + zh, a^{\#}) : U(w^{\#} + zh, a^{\#}) \geq b, U(w^{\#} + zh, a^{\#}) - U(w^{\#} + zh, \hat{a}') \geq 0\}. \tag{28}$$

Denote a solution

$$(h'z') \in \arg \max_{h \in \mathcal{H}_{w^{\#}}} \max_{z} \{V(w^{\#}+zh, a^{\#}) : U(w^{\#}+zh, a^{\#}) \geq b, U(w^{\#}+zh, a^{\#}) - U(w^{\#}+zh, \hat{a}') \geq 0\}.$$

Denote $b^{\#} = U(w^{\#} + (zh)', a^{\#}) \geq b$. If inequality (28) is strict, then we have

$$V(w^{\#}, a^{\#}) < V(w^{\#} + (zh)', a^{\#}),$$

which implies $(zh)' < 0$ with positive measure or $z' < 0$ since $V(\cdot, a^{\#})$ is decreasing. However, it contradicts $U(w^{\#} + (zh)', a^{\#}) \geq b^{\#}$ given that $U(\cdot, a^{\#})$ is increasing.

Therefore, under $\hat{a}' \in a^{BR}(w^{\#})$,

$$0 \in \arg \max_{h \in \mathcal{H}_{w^{\#}}} \max_{z} \{V(w^{\#} + zh, a^{\#}) : U(w^{\#} + zh, a^{\#}) \geq b, U(w^{\#} + zh, a^{\#}) - U(w^{\#} + zh, \hat{a}') \geq 0\}$$

is a solution. It follows by $U(w^{\#} + 0, a^{\#}) - U(w^{\#} + 0, \hat{a}') \geq 0$, $a^{\#}$ is also a best response to $w^{\#}$, which show the desired result.

It only remains to show the following claim, which was used in the above proof.

Claim 3. $\mathrm{val}(27) = \min_{\hat{a}} \max_{w \geq \underline{w}} \{V(w, a^{\#}) : (w, a^{\#}) \in \mathcal{W}(\hat{a}, b)\} = \mathrm{val}(\mathrm{SAND}|b)$.

Proof of Claim 3. By definition of optimality $\mathrm{val}(27) \leq \min_{\hat{a}} \max_{w \geq \underline{w}} \{V(w, a^{\#}) : (w, a^{\#}) \in \mathcal{W}(\hat{a}, b)\}$. We only need to show the other direction. Note that given any $h \in \mathcal{H}_{w^{\#}}$ (the selection of h may depend on $\hat{a}$), the Lagrangian

$$L(z, \lambda, \delta \,\Big|\, a^{\#}, \hat{a}, h) = V(w^{\#} + zh, a^{\#}) + \lambda[U(w^{\#} + zh, a^{\#}) - b] + \delta[U(w^{\#} + zh, a^{\#}) - U(w^{\#} + zh, \hat{a})]$$

is concave in z, since both $V(w^{\#} + zh, a^{\#})$ and $U(w^{\#} + zh, a^{\#})$ are concave in z. It follows by Claim 2 that

$$\max_{z} \{V(w^{\#}+zh, a^{\#}) : U(w^{\#}+zh, a^{\#}) \geq b, U(w^{\#}+zh, a^{\#}) - U(w^{\#}+zh, \hat{a}) \geq 0 = \inf_{\lambda, \delta} \max_{z} L(z, \lambda, \delta \,\Big|\, a^{\#}, \hat{a}, h)$$

Therefore, by an identical argument in the proof of Lemma 3, there exists a constant $b^* \geq b$ such that for a solution $(zh)^*$ to (27) the constraint $U(w^{\#} + (zh)^*, a^{\#}) \geq b^*$ is binding.

Let

$$\hat{a}^* \in \arg\min_{\hat{a}} \max_{h\in\mathcal{H}_{w^\#}} \max_{z}\{V(w^\# + zh, a^\#) : (w^\# + zh, a^\#) \in \mathcal{W}(\hat{a}, b^*)\}$$

and note that $\hat{a}^*$ does not depend on h and z since we have maximized over these arguments. We now solve the maximization problem

$$\max_{h\in\mathcal{H}_{w^\#}} \max_{z}\{V(w^\# + zh, a^\#) : (w^\# + zh, a^\#) \in \mathcal{W}(\hat{a}^*, b^*)\}, \tag{29}$$

in search of optimality conditions that characterize $w^\#$. Problem (29) is equivalent to a maximization problem that chooses function zh to maximize the functional $V(w^\# + zh, a^\#)$ subject to two constraints $U(w^\# + zh, a^\#) \geq b^*$ and $U(w^\# + zh, a^\#) - U(w^\# + zh, \hat{a}^*) \geq 0$. Note that the maximum over $h \in \mathcal{H}_{w^\#}$ in (29) is attained. This is because $\mathcal{H}_{w^\#}$ is weakly sequential compact by construction and for every $h \in \mathcal{H}_{w^\#}$, there is an optimal $z \in [\underline{z}, \overline{z}]$ solving (29), so the solution to maximization problem $\max_{h\in\mathcal{H}_{w^\#}} \max_z\{V(w^\# + zh, a^\#) : (w^\# + zh, a^\#) \in \mathcal{W}(\hat{a}^*, b^*)\}$ exists, since the constraint set is nonempty (we can find some h that is positively correlated to $(1 - \frac{f(x,\hat{a}^*)}{f(x,a^\#)})$ to satisfy the no-jump constraint).

We want to derive a necessary condition for a solution $(zh)^*$ that will ensure

$$\max_{h\in\mathcal{H}_{w^\#}} \max_{z}\{V(w^\# + zh, a^\#) : (w^\# + zh, a^\#) \in \mathcal{W}(\hat{a}^*, b^*)\} = \max_{w\geq\underline{w}}\{V(w, a^\#) : (w, a^\#) \in \mathcal{W}(\hat{a}^*, b^*)\}. \tag{30}$$

Let $(zh)^*$ be a solution to (29). We use the method of calculus of variations again. For any $\tilde{h} \in \tilde{\mathcal{H}}$, where

$$\tilde{\mathcal{H}} =\{\tilde{h}(x) : \tilde{h}(x) = 0 \text{ for } x \in \{x : w^\# + (zh)^* = \underline{w}\} \text{ and } 0 \leq \tilde{h}(x) \leq \min\{w^\# + (zh)^* - \underline{w}, \bar{h}\}\},$$

we have

$$V(w^\# + (zh)^*, a^\#) = \max_{\xi\geq\underline{\xi}}\{V(w^\# + (zh)^* + \xi\tilde{h}, a^\#) : (w^\# + (zh)^* + \xi\tilde{h}, a^\#) \in \mathcal{W}(\hat{a}^*, b^*)\}, \tag{31}$$

where $\underline{\xi} = -\sup_x\{(w^\#(x) + (zh(x))^* - \underline{w})/h(x)\}$. We have the solution

$$\xi^* = 0 \in \arg\max_{\xi\geq\underline{\xi}}\{V(w^\# + (zh)^* + \xi\tilde{h}, a^\#) : (w^\# + (zh)^* + \xi\tilde{h}, a^\#) \in \mathcal{W}(\hat{a}^*, b^*)\}.$$

We then form a exact penalty function for problem (31):

$$\tilde{v}^k(\xi) \equiv \tilde{v}(\xi, a^\#) - \frac{k}{2}(\min\{0, \tilde{u}(\xi, a^\#) - b^*\})^2 - \frac{k}{2}(\min\{0, \tilde{u}(\xi, a^\#) - \tilde{u}(\xi, \hat{a}^*)\})^2 - \frac{1}{2}\xi^2$$

where denote $\tilde{v}(\xi, a^{\#}) \equiv V(w^{\#} + (zh)^{*} + \xi\tilde{h}, a^{\#})$, $\tilde{u}(\xi, a^{\#}) \equiv U(w^{\#} + (zh)^{*} + \xi\tilde{h}, a^{\#})$. Denote $\xi_k \in \arg\max_{1 \geq \xi \geq \underline{\xi}} \tilde{v}^k(\xi)$ as a solution given k. Following a standard approach (see Berteskas, 1999), we can show that

$$\lim_{k\to\infty} \max_{1\geq\xi\geq\underline{\xi}} \tilde{v}^k(\xi) = \tilde{v}(0, a^{\#}),$$

which implies that there exists a convergent subsequence $\xi_k \to 0$. Then we can write the first order condition as:

$$\begin{aligned} 0 \;=\; & \int -v'(\pi - w^{\#} - (zh)^{*})\tilde{h}(x)f(x, a^{\#})dx \qquad (32) \\ & -k\left(\min\{0, \xi_k \int u'(w^{\#} + (zh)^{*})\tilde{h}(x)f(x, a^{\#})dx + o(\xi_k)\}\right)(\int u'(w^{\#} + (zh)^{*})\tilde{h}(x)f(x, a^{\#})dx + O(\xi_k)) \\ & -k\left(\min\{0, \xi_k \int u'(w^{\#} + (zh)^{*})(1 - \frac{f(x, \hat{a}^{*})}{f(x, a^{\#})})\tilde{h}(x)f(x, a^{\#})dx + o(\xi_k)\}\right) \times \\ & \left[\int u'(w^{\#} + (zh)^{*})(1 - \frac{f(x, \hat{a}^{*})}{f(x, a^{\#})})\tilde{h}(x)f(x, a^{\#})dx + O(z_k)\right] - \frac{1}{2}\xi_k. \end{aligned}$$

We further restrict $\tilde{h}$ to be such that

$$\int u'(w^{\#} + (zh)^{*})\tilde{h}(x)f(x, a^{\#})dx > 0$$

and

$$\int u'(w^{\#} + (zh)^{*})(1 - \frac{f(x, \hat{a}^{*})}{f(x, a^{\#})})\tilde{h}(x)f(x, a^{\#})dx > 0.$$

It follows that the sequence $-k\min\{0, \xi_k\}$ should be bounded. Suppose not. It follows $kz_k \to -\infty$. We divide both sides of (32) by $-\lim_{k\to\infty} k\min\{0, \xi_k\}$, and taking the advantage of the restriction on $\tilde{h}$, the major components of the first order condition (32) becomes

$$0 = (\int u'(w^{\#} + (zh)^{*})\tilde{h}(x)f(x, a^{\#})dx)^2 + \left(\int u'(w^{\#} + (zh)^{*})(1 - \frac{f(x, \hat{a}^{*})}{f(x, a^{\#})})\tilde{h}(x)f(x, a^{\#})dx\right)^2,$$

which is not possible. Now denote $\theta_{\tilde{h}} \equiv \lim_{n\to\infty} -k_n \min\{0, \xi_{k_n}\}$ by the limit of a convergent subsequence of $-k\min\{0, \xi_k\}$. The major terms of (32) can be written as

$$\begin{aligned} 0 &= \int -v'(\pi - w^{\#} - (zh)^{*})\tilde{h}(x)f(x, a^{\#})dx + \lambda_{\tilde{h}} \int u'(w^{\#} + (zh)^{*})\tilde{h}(x)f(x, a^{\#})dx \\ &\qquad + \delta_{\tilde{h}} \int u'(w^{\#} + (zh)^{*})(1 - \frac{f(x, \hat{a}^{*})}{f(x, a^{\#})})\tilde{h}(x)f(x, a^{\#})dx \\ &= \int \left(-v'(\pi - w^{\#} - (zh)^{*}) + u'(w^{\#} + (zh)^{*})(\lambda_{\tilde{h}} + \delta_{\tilde{h}}(1 - \frac{f(x, \hat{a}^{*})}{f(x, a^{\#})})\right)\tilde{h}(x)f(x, a^{\#})dx \qquad (33) \end{aligned}$$

where $\lambda_{\tilde{h}} = \theta_{\tilde{h}} \int u'(w^{\#}+(zh)^{*})\tilde{h}(x)f(x, a^{\#})dx$ and $\delta_{\tilde{h}} = \theta_{\tilde{h}} \int u'(w^{\#}+(zh)^{*})(1-\frac{f(x,\hat{a}^{*})}{f(x,a^{\#})})\tilde{h}(x)f(x, a^{\#})dx$. Following the same argument as Lemma 2 in Ke and Ryan (2015) (and using its assumptions) we know (33) implies the integrand equals zero and this means $\lambda_{\tilde{h}}$ and $\delta_{\tilde{h}}$ are constants. It follows that $w^{\#} + (zh)^*$ is the GMH contract $w(a^{\#}, \hat{a}^*, b)$. Hence, val(29) $\geq \min_{\hat{a}} \max_{w \geq \underline{w}} \{V(w, a^{\#}) : (w, a^{\#}) \in \mathcal{W}(\hat{a}, b)\} =$ val(SAND$|b$) since, as we have just shown, the optimal solution to (29) is a GMH contract for a fixed $\hat{a}^*$ while (SAND$|b$) optimized over GMH contracts by minimizing over $\hat{a}$. $\square$