Information-trigger contracts*

Rongzhu Ke† Christopher Thomas Ryan‡

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Abstract

We study the moral-hazard problem where the principal and agent are both risk neutral with lower and upper bounds on agent compensation. We show the optimality of *information-trigger contracts*, where the agent receives the highest possible payment when the likelihood ratio of an output signal between two actions is higher than a certain cut-off (trigger) and is otherwise paid the minimum. The likelihood ratio maps multi-dimensional outputs into an ordered information space where information-triggers provide a simple and economically intuitive "bonus" structure. Starting with the binary-action case, we reformulate the original problem to a covariance (between likelihood ratio and payment) maximization problem while controlling the expected payment, to establish the optimality of information-trigger contracts. For the case when there are more than two actions, we demonstrate equivalence with a relaxed problem where the order of optimization is swapped to where the principal responds optimally *given* two distinct actions, allowing us to leverage the binary-action result.

Moreover, under additional monotonicity assumptions, we adapt our machinery to show the optimality of quota-bonus contracts under weak conditions. When output satisfies the monotone likelihood ratio property, optimality follows by directly leveraging the information-trigger structure. This requires no additional assumptions, such as monotonicity of the contract space or the validity of the first-order approach, which are common in the existing literature. We also provide weak conditions for the optimality quota-bonus contracts when the monotonicity of contracts is assumed.

1 Introduction

The classical moral-hazard problem where both agent and principal are risk neutral, and the agent has limited liability, is among the simplest non-trivial incentive problems that is well-studied in the literature. Despite this prevalence, basic questions remain. The majority of work concerns the optimality of quota-bonus contracts, where the agent's wage is flat before jumping by some discrete bonus when an output quota is met. Much of the motivation for studying the risk-neutral setting has been to provide some explanation of the prevalence of such contracts in practice (Park, 1995; Kim, 1997; Oyer, 2000; Innes, 1990; Herweg et al., 2010; Dai and Jerath, 2016; Poblete and Spulber, 2012).

This work has been limited to a fairly restrictive subclass of risk-neutral problems. The quota-bonus structure arises in the setting of a single-dimensional signal, where it is straightforward to

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†Hong Kong Baptist University, rongzhuke@hkbu.edu.hk

‡University of Chicago, chris.ryan@chicagobooth.edu

compare outputs to a single quota value. The precise meaning of the quota-bonus structure is not obvious when the set of signals is not totally ordered. Moreover, analytical guarantees of the optimality of quota-bonus structure have typically included the validity of the first-order approach (FOA) or strong monotonicity conditions.

This paper concerns an optimality structure that can be identified for multidimensional signals and without additional assumptions beyond the basic set-up: a risk-neutral principal and risk-neutral agent with limited agent liability and a resource constraint that upper-bounds agent compensation. This optimal structure is based on the likelihood ratio of output signals between a high-cost agent action and a low-cost agent action. Given a choice of two such actions, if a signal's likelihood ratio is sufficiently high, the agent is maximally rewarded (given the upper bound on compensation). Otherwise, the agent is maximally punished (given the lower bound on compensation). We call this an *information-trigger* contract. The main result of this paper is that there always exists an optimal information-trigger contract in our basic set-up. We do not invoke the FOA nor make any monotonicity assumptions. We believe that the information-trigger structure cleanly captures the strength and impact of the risk-neutrality compensation-boundedness conditions, unclouded by additional technical assumptions.

Compared to quota-bonus contracts, information-trigger contracts also apply more generally. Whereas quota-bonus contracts concern single-dimensional signals, here multidimensional signals are mapped and ordered in the real line by evaluating likelihood ratios. This mapping uncovers an optimal "quota-bonus" structure in information space. The likelihood ratio can thus be seen as the "right way" to order multi-dimensional outputs, with the information-trigger providing an intuitive economic structure in that ordering.

Moreover, information-trigger contracts provide new practical and analytical insights into the design of contracts. We demonstrate this below with an application to CEO compensation with two output signals – stock price and profits. Bolton and Dewatripont (2005) study this problem in the risk-averse setting and derive an optimal contract when restricting to the space of linear contracts. In the risk-neutral setting with limited liability and a resource constraint, we construct a simple optimal contract that provides the CEO with a bonus if a weighted sum of stock price and profit exceed a cut-off value.

When returning to the study of classical quota-bonus contracts with single-dimensional outputs, the optimality of information-trigger contracts also sheds new light. Under additional monotonicity assumptions (typically weaker than in the existing literature), we apply our machinery to show the optimality of quota-bonus contracts. Our approach is simple and novel. When output satisfies the monotone likelihood ratio property (MLRP), optimality follows by directly leveraging the information-trigger structure – an information-trigger translates to an output-signal trigger (i.e. a quota-bonus structure). This requires no additional assumptions. We also provide weak conditions for the optimality quota-bonus contracts when the contract space is monotone. A quota-bonus contract is shown to be optimal in the binary-action case. If we further assume that the output distribution satisfies hazard rate dominance (a monotonicity condition in the hazard rates across actions), this binary-action result can be extended to the general-action case. Alternatively, assuming the convexity of the distribution function (CDFC) and a single-crossing property of hazard rates is also sufficient for the optimality of quota-bonus contracts in the general-action case. For a careful comparison of our assumptions with those in the literature we defer discussion to the closing section of the paper.

Finally, we also believe the elegance of the information-trigger structure paves the way for combining moral hazard with adverse selection without resort to the FOA. It could also prove useful in examining dynamic moral-hazard problems where simplicity of the static optimal contract is paramount. These are areas for potential future investigation that go beyond our scope.

The centrality of likelihood ratios that appears in the information-trigger structure has important antecedents in the literature. In their seminal monograph, Hart and Holmstrom (1987) uncover an insightful connection between likelihood ratios and contract design. They show, in a binary-action setting, that the structure of the optimal contract depends in a natural way on the likelihood ratio. The likelihood ratio appears in the formula for Bayesian updating the principal's beliefs about the agent's action. Agents are rewarded for outputs that revise upwards beliefs that the agent took the costlier action. In the risk-averse setting, this observation, in combination with first-order conditions, determine the shape of the optimal contract. When utility functions are sufficiently smooth, the optimal contract reacts in a smooth way to changes in the likelihood ratio.

These themes are echoed in our findings. In the risk-neutral setting, the contracting problem is linear in the contract and thus we expect an extremal solution. It is intuitive, then, that in the binary-action case there exists an optimal contract that is highly sensitive to the likelihood ratio of the signal. In fact, we first show that the information-trigger is optimal in this setting. This result is intuitive. When there are only two actions to consider, outputs can be labeled as either "good" (associated with the costlier action) or "bad" (associated with the cheaper action) and treated accordingly in as ruthless a manner possible by the principal. This is precisely an information-trigger structure. The optimality of this structure in the binary-action setting is a key building block for our more general result.

In fact, contract theory results that assume binary actions abound (see for instance Fudenberg and Tirole (1990); Chaigneau et al. (2016, 2014); Dai and Jerath (2013); Diamond (1998)). Such results are often suggestive of more general results. As Hart and Holmstrom write:

> Much of the general insights obtained form studying hidden action models can be conveyed in the simplest setting where the agent has only two actions to choose from. (Hart and Holmstrom, 1987)

However, things are not always as straightforward as we would like, even in the risk-neutral setting. It is precisely the possibility of many alternate best responses to a contract where the intuition from the binary-action case may fail to carry over. This motivates the widespread popularity of assuming the validity of the FOA, where the agent's incentive compatibility constraint can be replaced without loss with a first-order condition, when attempting to establish simple optimal contracts. The possibility for counter-intuition and the need for sophisticated analyses when the FOA fails is well-documented (Mirrlees, 1999), often revealing a stark contrast with the simplicity of the binary-action setting. Strong monotonicity assumptions or other technical conditions (such as in Innes (1990); Poblete and Spulber (2012); Wang and Hu (2016)) on the contract are typically needed to maintain simplicity when the FOA is not valid.

A main theoretical contribution of this paper is to show that information-trigger contracts can generalize our intuition beyond the binary-action setting. Our method of proof is to demonstrate the equivalence of the moral-hazard problem with a relaxed problem where the order of optimization is swapped to where the principal responds optimally *given* two distinct actions. Here we can leverage optimality of information-trigger contracts in the binary-action case. It remains to show that a contract of this structure is implementable to the original ("unswapped") problem. By linearity, the incentive constraint binds at optimality. This "pegs" the relationship between optimal trigger values and optimal agent responses in a way that is invariant under the order of optimization. This makes the original problem and its "swapped" relaxation equivalent, yielding the optimality of information-trigger contracts in the general setting.

Related literature

This paper touches on a number of additional themes in the theory of contracts.

First, there has long been interest in understanding situations where simple contract structures are optimal, given the prevalence of simple contracts in practice. Recently, Carroll (2015) showed the optimality of linear contracts under certain robustness properties, while Herweg et al. (2010) show the optimality of quota-bonus contracts under loss-aversion-based agent utility. Information-trigger contracts are another form of contract simplicity with a well-grounded economic justification, which, to our knowledge, has not been systematically studied in the literature.

Second, the centrality of the likelihood ratio and its relationship to statistical inference, draws a connection between this paper and recent work on informativeness in the risk-neutral limited liability setting (Chaigneau et al., 2016, 2014), which aims to refine our understanding of Holmström's informativeness principle (Holmstrom, 1979). Demougin and Fluet (1998) study the question of informativeness and sufficient statistics for output in the discrete-output setting. We note that Jung and Kim (2015) also explore the significance on the likelihood ratio in determining contract structure in the case where the principal (but not necessarily the agent) is risk neutral.

Third, our method of swapping the order of optimization draws inspiration from recent work on the general risk-averse problem (Ke and Ryan, 2017a,b). In the risk-averse case, justifying a change in the order of optimization is more involved, involving penalty function and perturbation arguments. By contrast, the argument in the risk-neutral case in made simpler by leveraging the underlying linearity of the problem. It should be noted that the results in Ke and Ryan (2017a,b) do not apply here since they require the strict risk aversion of the agent.

Fourth, this paper adds to the growing literature on analyzing moral-hazard problems without use of the FOA (Araujo and Moreira, 2001; Innes, 1990; Poblete and Spulber, 2012; Mirrlees, 1986; Wang and Hu, 2016; Ke and Ryan, 2017a). This is a particularly important issue in the risk-neutral setting, where assuming the FOA can yield counter-intuitive results. Oyer (2000) shows that his method to analyze the risk-neutral setting using the FOA rules out the existence of quota-bonus contracts for problems where the output distribution has additive noise with an increasing hazard rate (this limitation was recently highlighted by Dai and Jerath (2016)). Our method of analysis has no such limitation. We treat additive noise with increasing hazard rate as a clean special case.

We now summarize our results and provide a guide to the content of the paper. Section 2 introduces our model and sets notation. Section 3 contains our main result on the optimality of information-trigger contracts. Section 4 contains applications of our theory to the study of quota-bonus contracts under the MLRP and a CEO compensation problem with two output signals. Section 5 extends our analytical approach to the setting where contracts are restricted to be monotone. Lastly, we conclude with a discussion in Section 6. Here we provide a detailed comparison between our quota-bonus results and other related results in the literature.

2 Model

A principal hires an agent, who takes a costly action with a stochastic output. The action a taken by the agent cannot be observed by the principal, only its resulting output. The principal earns revenue $\pi(x)$ from output x and pays wage $w(x)$ to the agent as a function of the output x. The agent chooses an action in his best interest, given the offered wage contract. We assume both the agent and principal are risk-neutral.

Let $\mathcal{X}$ denote the set of possible outputs in $\mathbb{R}^n$ for some n. Actions are chosen from a compact set $\mathcal{A}$ with associated cost function $c(a)$ that is increasing in a. The resulting output of the agent's action is random and governed by a probability density function $f(x, a)$ that is continuously differentiable in x for all a. We also assume that the support of $f(x, a)$ is $\mathcal{X}$ for all a. Let $F(\cdot, a)$, $\mathbb{E}_a[\cdot]$, and $\mathbb{P}_a[\cdot]$ denote the cumulative distribution function, expectation operator and probability measure associated with the probability density function $f(\cdot, a)$. Where the action a is clear, we simply denote expectation by $\mathbb{E}$ and the probability measure by $\mathbb{P}$.

Given the principal's choice of contract w, the agent's choice of action a, and output realization x, the principal receives a utility of $\pi(x) - w(x)$ and the agent receives a utility of $w(x) - c(a)$. Both principal and agent maximize their expected utilities. The agent must receive an expected utility no less than the value $\underline{U}$ of his best outside alternative (forming his participation constraint) and cannot receive a wage less than $\underline{w}$ under any output realization (limited liability constraint). Without loss of generality we set $\underline{w} = 0$. We also assume an upper bound function m on agent compensation. A common example is to set $m(x) = \pi(x)$ so that agent compensation is financed from the value of the outputs.

A feasible contract w thus satisfies $0 \leq w(x) \leq m(x)$ for almost all $x \in \mathcal{X}$. We will typically shorten the statement of these bounds to simply $\underline{w} \leq w \leq m$. The assumption that the contract has both and lower and upper bound is not uncommon in the literature, see for instance (Holmstrom, 1979; Page, 1987; Innes, 1990). We also assume that an optimal contract exists.[1]

Notation 1 *We make the following conventions. When defining an integral over the whole set of outputs $\mathcal{X}$, we drop the associated limits on the integral. That is, $\int w(x)f(x, a)dx$ will be taken to mean $\int_{x\in\mathcal{X}} w(x)f(x, a)dx$. For integration that occurs over subsets of $\mathcal{X}$, this will always be specified in the limits of the integral. A similar convention holds for optimization problems and the* max *and* min *operators. For instance,* $\max\limits_{a}$ *will have the same meaning as* $\max\limits_{a\in\mathcal{A}}$.

Let $U(w, a) = \int w(x)f(x, a)dx - c(a)$ denote the expected utility of the agent and $V(w, a) = \int(\pi(x) - w(x))f(x, a)dx$ denote the expected utility of the principal under contract w and action a. Observe that $U(w, a) = \mathbb{E}_a[w(X)] - c(a)$ and $V(w, a) = \mathbb{E}_a[\pi(X)] - \mathbb{E}_a[w(X)]$ where X is the random output variable under action a, demonstrating the linearity of expected utility in the underlying contract.

The principal solves the following problem

$$\max_{w,a} V(w, a) \tag{1a}$$

$$\text{subject to } U(w, a) \geq \underline{U} \tag{1b}$$

$$U(w, a) - U(w, \hat{a}) \geq 0 \text{ for all } \hat{a} \in \mathcal{A} \tag{1c}$$

$$0 \leq w \leq m \tag{1d}$$

where the agent's optimizing behavior is captured by the incentive compatibility (IC) constraint (1c). Constraint (1b) is known as the individual rationality (IR) constraint. Of course, m must be such that a feasible contract exists.

Notation 2 *Throughout the paper we use val$(*)$ to denote the optimal value of an optimization problem $(*)$. For instance, val(1) denotes the optimal value of optimization problem* (1).

A word about the set of feasible contracts to (1). If w is a feasible contract then any adjustment of the values of w on a set of measure zero results in a new feasible contract. For this reason, we restrict attention to contracts that are right continuous; that is, where if w is a contract with a discontinuity at y, then $\lim_{x\to y^+} w(x) = w(y)$.[2]

[1] A sufficient condition for this beyond what has already been stated is that m is an integrable function (Ke, 2014).

[2] For simplicity we assume that feasible contracts are almost everywhere continuous. This assumption streamlines

3 Information-trigger contracts

In this section we prove this existence of an optimal information-trigger contract. For the purposes of exposition, we first study the case of a simple uniform resource constraint $m(x) = \bar{w}$ for all x. This case avoids complications that distract from the novel aspects of our analysis.

The proof of the more general case appears in Section 3.3. The first two subsections tackle the simpler case. We break the proof across two subsections to highlight two distinct stages of the argument. In the first stage we establish the optimality of the information-trigger structure when there are only two actions. Several key insights used for the general case are best understood in this simplified setting. The second stage tackles the more general setting of multiple actions.

3.1 Binary-action case

Consider the special case where $\mathcal{A}$ consists of two distinct actions, denoted a and $\hat{a}$. The contracting problem to implement action a is

$$\max_{w} V(w,a) \tag{2a}$$

$$\text{subject to } U(w,a) \geq \underline{U} \tag{2b}$$

$$U(w,a) - U(w,\hat{a}) \geq 0 \tag{2c}$$

$$0 \leq w \leq \bar{w} \tag{2d}$$

where there is a single incentive compatibility constraint. Constraint (2d) reflects the simple upper bound $m(x) = \bar{w}$ for all x. Let w^* be an arbitrary optimal contract to (2) (at least initially, it need not be an information-trigger contract). The goal of this subsection is to describe sufficient conditions for the existence of an optimal information-trigger contract to (2).

We start with some preliminary lemmas.

Lemma 1 *If $\hat{a} > a$ then the IR constraint* (2b) *is binding at optimality and there exists a first-best contract to* (2)*.*

Hereafter we ignore this trivial case and assume that $a > \hat{a}$. The next result captures a straightforward observation.

Lemma 2 *If the IR constraint* (2b) *is not binding at optimality, then the IC constraint* (2c) *is binding at optimality.*

We derive a useful reformulation of (2) in terms of the following ratio function:

$$R(x) := 1 - \tfrac{f(x,\hat{a})}{f(x,a)}. \tag{3}$$

We comment on the interpretation of R. In the introduction of the paper, we referred to the likelihood ratio $\frac{f(x,a)}{f(x,\hat{a})}$. If this likelihood ratio increases or decreases, then the same is true of R. In other words, R is another measure of the extent to which a signal x is more associated with the costlier action a. Since $f(x,a)$ has support $\mathcal{X}$ for all a and f is continuously differentiable in x for all a, R itself is continuously differentiable.

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the proofs since we avoid dealing with positive measure sets that do not contain intervals (such as fat Cantor sets). Our results apply more generally at the cost of less transparent exposition in several proofs.

A key observation is that the incentive constraint (2c) can be expressed as

$$\int R(x)w(x)f(x,a)dx - [c(a) - c(\hat{a})] \geq 0. \tag{4}$$

We may interpret the first term on the left-hand side as the incremental benefit of taking action a over $\hat{a}$ and the second term is the incremental cost of taking action a over action $\hat{a}$. The larger the positive covariance between $R(x)$ and $w(x)$, the greater is the incremental benefit, enhancing the incentive for the agent to opt for action a. The significance of this observation is best understood in the following related problem to (2). Given an optimal contract w^* to (2), define the problem

$$\begin{aligned} \max_{w} \ & U(w,a) - U(w,\hat{a}) \\ \text{subject to } & U(w,a) \geq \underline{U} \\ & V(w,a) = V(w^*,a) \\ & 0 \leq w \leq \bar{w}. \end{aligned}$$

This problem can equivalently be rewritten as

$$\max_{w} \int R(x)w(x)f(x,a)dx \tag{5a}$$

$$\text{subject to } \mathbb{E}[w(X)] = \mathbb{E}[w^*(X)] \tag{5b}$$

$$0 \leq w \leq \bar{w}, \tag{5c}$$

dropping the constraint $U(w,a) \geq \underline{U}$ and $V(w,a) = V(w^*,a)$ since they are implied by (5b).

Lemma 3 *An optimal solution to* (5) *is an optimal solution to* (2). *Moreover, if there exists an optimal contract to* (2) *where the participation constraint* (2b) *is not binding, then problems* (2) *and* (5) *are equivalent.*

Observe that the objective function of (5) is the covariance between R and w, motivating interest in feasible perturbations of contracts that maximize the covariance between $R(x)$ and the perturbed contract. We call a function h a *mean-preserving perturbation* if $\mathbb{E}[h(X)] = 0$. Perturbed contracts of the form $w + h$ where h is mean-preserving automatically satisfy (5b). If, in addition, the covariance of h with R is greater than zero, then w cannot be optimal to (5). This yields a necessary condition for optimality to (5) that can be used to learn about the structure of optimal contracts. The key structure of interest is as follows.

Definition 1 *An information-trigger contract with bonus b and trigger t is a contract of the form:*

$$w^{b,t}(x) = \begin{cases} 0 & \text{if } R(x) < t \\ b & \text{if } R(x) \geq t. \end{cases} \tag{6}$$

Theorem 1 *The binary-action moral-hazard problem* (2) *has an optimal information-trigger contract $w^{\bar{w},t^*}$ with bonus $\bar{w}$ and trigger*

$$t^* = \max\left\{ t : \bar{w} \geq \frac{c(a) - c(\hat{a})}{\mathbb{P}_a[R(X) \geq t] - \mathbb{P}_{\hat{a}}[R(X) \geq t]} \text{ and } U(w^{\bar{w},t},a) \geq \underline{U} \right\}. \tag{7}$$

Moreover, if there exists an optimal contract to (2) *where the participation constraint* (2b) *is not binding then $w^{\bar{w},t^*}$ is the unique optimal contract to* (2).

Proof. Let w be an arbitrary feasible contract to (2). Suppose that w is not of the form $w^{\bar{w},t}$. We claim that w cannot be an optimal solution to the reformulation problem (5). Since w is not in information-trigger form, one of three cases hold.

Case 1: There exists a $t < 1$ and a positive measure subset $S_1 \subseteq \{x : R(x) < t\}$ where $w(x) > 0$ and a positive measure subset $S_2 \subseteq \{x : R(x) \geq t\}$ where $w(x) < \bar{w}$.

Define the following perturbation

$$h(x) = \begin{cases} -\alpha_1(x) & \text{if } x \in S_1 \\ \alpha_2(x) & \text{if } x \in S_2 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

where $0 < \alpha_1(x) \leq w(x)$ for all $x \in S_1$ and $0 \leq \alpha_2(x) \leq \bar{w} - w(x)$ for all $x \in S_2$. The functions α_1 and α_2 are specified so that $\mathbb{E}[h(X)] = 0$; that is,

$$-\int_{S_1} \alpha_1(x) f(x,a)dx + \int_{S_2} \alpha_2(x) f(x,a)dx = 0. \tag{9}$$

The perturbed contract $w + h$ is clearly feasible to (5). To show w is not optimal to (5) it suffices to show that $\int R(x)h(x)f(x,a)dx > 0$. This follows since

$$\begin{aligned} \int R(x)h(x)f(x,a)dx &= -\int_{S_1} \alpha_1(x)R(x)f(x,a)dx + \int_{S_2} \alpha_2(x)R(x)f(x,a)dx \\ &> -\int_{S_1} \alpha_1(x)tf(x,a)dx + \int_{S_2} \alpha_2(x)tf(x,a)dx \\ &= t\left(-\int_{S_1} \alpha_1(x)f(x,a)dx + \int_{S_2} \alpha_2(x)f(x,a)dx\right) \\ &= 0, \end{aligned}$$

where the inequality holds since $R(x) < t$ for $x \in S_1$ and $R(x) \geq t$ for $x \in S_2$ and the final equality holds using the mean-preserving equality (9).

Case 2: There exists a $t < 1$ and a positive measure subset $S_1 \subseteq \{x : R(x) < t\}$ where $w(x) > 0$ but for all x such that $R(x) \geq t$, $w(x) = \bar{w}$.

Let t be the minimal value where the conditions of Case 2 hold. That is, for any smaller t' we have $w(x) < \bar{w}$ for some x with $t' \leq R(x) < t$. Since w is almost everywhere continuous this implies there is a positive measure subset $S_2 \subseteq \{x : t' \leq R(x) < t\}$ where $w(x) < \bar{w}$. We can then define a perturbation exactly as in Case 1 (with this new definition of S_2) with the new trigger t' that is mean preserving and has strictly greater covariance than w.

Case 3: There exists a $t \leq 1$ and an positive measure set $S_2 \subseteq \{x : R(x) \geq t\}$ where $w(x) < \bar{w}$ but for all x such that $R(x) < t$, $w(x) = 0$.

Let t be the maximal value where the conditions of Case 3 hold. Analogous to Case 2, we can increase t slightly to t' to define positive measure subsets S_1 and S_2 with the properties of Case 1 with trigger t'. A mean-preserving perturbation can be constructed to increase the covariance, implying that w is not optimal.

Hence, there exists an optimal contract to (5) (and thus also (2) by Lemma 3) of the form $w^{\bar{w},t}$ for some trigger value t. Noting

$$V(w^{\bar{w},t}, a) = \mathbb{E}_a[\pi(X)] - \bar{w}\mathbb{P}_a[R(X) \geq t] \quad \text{and} \quad U(w^{\bar{w},t}, a) = \bar{w}\mathbb{P}_a[R(X) \geq t], \tag{10}$$

we plug in the contract form $w^{\bar{w},t}$ into (2) to yield:

$$\max_t \mathbb{E}_a[\pi(X)] - \bar{w}\mathbb{P}_a[R(X) \geq t] \tag{11a}$$

$$\text{subject to } \bar{w}\mathbb{P}_a[R(X) \geq t] - c(a) \geq \underline{U} \tag{11b}$$

$$\bar{w}\mathbb{P}_a[R(X) \geq t] - c(a) \geq \bar{w}\mathbb{P}_{\hat{a}}[R(X) \geq t] - c(\hat{a}). \tag{11c}$$

The objective (11a) is increasing in t and hence an optimal choice for t^* solves (7). Observe that in

$$w \geq \frac{c(a) - c(\hat{a})}{\mathbb{P}_a[R(X) \geq t] - \mathbb{P}_{\hat{a}}[R(X) \geq t]}.$$

we are not dividing by 0 since $c(a) - c(\hat{a}) > 0$ by Lemma 1 and the fact c is nondecreasing.[3]

Note that for t^* to be chosen maximally subject to the two constraints given, at least one of two constraints in (7) is tight.

Finally, we return to the question of the uniqueness of optimal solutions to the original problem (2). If the IR constraint (2b) is not binding, then by Lemma 3 then (2) and (5) are equivalent. We have just shown that (5) has a unique optimal solution equal to $w^{\bar{w},t^*}$, completing the proof. ■

The following result gives bounds on the possible values of the optimal trigger.

Proposition 1 *The optimal trigger t^* defined in* (7) *satisfies* $0 < t^* < 1$.

3.2 Leveraging the binary-action case

The binary-action problem yields an optimal information-trigger contract that distinguishes signals that are associated with high and low effort. In this section we show that such a contract is optimal for the general problem (1) with a compact set of actions $\mathcal{A}$.

The first task is to relate the general problem to the binary-action problem. We do this through swapping the order of optimization to yield a relaxed optimization problem. This relaxation admits optimal information-trigger contracts. Finally, we show that this relaxation is, in fact, tight. This allows us to conclude that the general problem also has an optimal information-trigger contract. Throughout the subsection we continue to assume that $m(x) = \bar{w}$ for all x.

3.2.1 A relaxation

We reformulate (1) to a "single level" optimization problem in w, a, and an additional auxiliary variable $\hat{a}$. The methodology has been used elsewhere in Ke and Ryan (2017a) and Ke and Ryan (2017b) where the agent is strictly risk averse. The current treatment is designed to handle the risk-neutral setting. It can be understood without reference to these earlier papers.

[3] In fact the denominator is positive since both the numerator and $\bar{w}$ are strictly positive. The economic intuition is that the higher reward should be given to the action that yields higher informativeness.

Observe that (1) is equivalent to[4]

$$
\begin{aligned}
\max_{w,a} \quad & V(w,a) \\
\text{subject to} \quad & U(w,a) \geq \underline{U} \\
& \min_{\hat{a}\in\mathcal{A}} \{U(w,a) - U(w,\hat{a})\} \geq 0 \qquad (12) \\
& 0 \leq w \leq \bar{w}
\end{aligned}
$$

We move the choice of $\hat{a}$ to the objective function. This requires handling the possibility that a choice of w does not implement the chosen a, in which case (12) is violated. Define the set

$$
\mathcal{W}(a,\hat{a}) \equiv \{(w,a) : U(w,a) \geq \underline{U} \text{ and } U(w,a) - U(w,\hat{a}) \geq 0\}, \tag{13}
$$

and the characteristic function

$$
V^I(w,a,\hat{a}) \equiv \begin{cases} V(w,a) & \text{if } (w,a) \in \mathcal{W}(a,\hat{a}) \\ -\infty & \text{otherwise.} \end{cases}
$$

Maximizing $V^I(w,a,\hat{a})$ over (w,a) results in a finite objective value if $(w,a) \in \mathcal{W}(a,\hat{a})$. Similarly, maximizing $\min_{\hat{a}\in\mathcal{A}} V^I(w,a,\hat{a})$ over (w,a) results in a finite objective value when (w,a) lies in $\mathcal{W}(a,\hat{a})$ for all $\hat{a} \in \mathcal{A}$. This implies (w,a) is feasible to (1) and demonstrates that

$$
\max_a \max_w \min_{\hat{a}} V^I(w,a,\hat{a})
$$

is equivalent to (1).

Now consider swapping the order of optimization in (3.2.1) so that $\hat{a}$ is chosen *before* w:

$$
\max_a \min_{\hat{a}} \max_w V^I(w,a,\hat{a})
$$

which is equivalent to

$$
\max_a \min_{\hat{a}} \max_w \{V(w,a) : (w,a) \in \mathcal{W}(a,\hat{a})\}, \tag{SAND}
$$

since an optimal choice of a precludes a subsequent optimal choice of $\hat{a}$ that sets $\mathcal{W}(a,\hat{a}) = \emptyset$, implying $V^I(w,a,\hat{a}) = V(w,a)$ when w is optimally chosen. The notation (SAND) connotes the fact that the minimization over $\hat{a}$ is "sandwiched" between two maximizations. We call it the *sandwich problem* associated with (1). The following is a basic consequence of the max-min inequality.

Lemma 4 (SAND) *is a relaxation of* (1). *That is,*

$$
\max_a \max_w \min_{\hat{a}} \{V(w,a) : (w,a) \in \mathcal{W}(a,\hat{a})\} \leq \max_a \min_{\hat{a}} \max_w \{V(w,a) : (w,a) \in \mathcal{W}(a,\hat{a})\}. \tag{14}
$$

The development of the binary-action case becomes relevant in analyzing (SAND). Indeed, for a given a and $\hat{a}$ the inner maximization problem over w

$$
\max_w \{V(w,a) : U(w,a) \geq \underline{U}, U(w,a) - U(w,\hat{a}) \geq 0\} \tag{SAND$|a,\hat{a}$}
$$

is precisely a binary-action problem of the form (2).

[4]To be entirely precise, the minimization in (12) should be an inf. However, the only time when the infimum may not be attained is when there is an infimizing sequence of $\hat{a}$'s converging to a. In this case, (12) can be replaced by a first-order condition. We ignore this straightforward case. For further details we refer to reader to Ke and Ryan (2017a).

3.2.2 Choosing the optimal trigger

Our method to establish sufficient conditions for optimal information-trigger contracts to exist for (1) is to leverage Theorem 1. Given a and $\hat{a}$, Theorem 1 implies that (SAND$|a, \hat{a}$) has an optimal contract of the form:

$$w_{a,\hat{a}}^{b,t}(x) := \begin{cases} 0 & \text{if } R(x|a,\hat{a}) < t \\ b & \text{if } R(x|a,\hat{a}) \geq t, \end{cases} \tag{15}$$

where $b = \bar{w}$ and

$$t(a,\hat{a}) := \max\left\{ t : \bar{w} \geq \frac{c(a) - c(\hat{a})}{\mathbb{P}_a[R(X) \geq t] - \mathbb{P}_{\hat{a}}[R(X) \geq t]} \text{ and } U(w_{a,\hat{a}}^{b,t}, a) \geq \underline{U} \right\}. \tag{16}$$

using the notation $R(x|a,\hat{a}) := 1 - \frac{f(x,\hat{a})}{f(x,a)}$ to distinguish that the ratio R depends on the choice of a and $\hat{a}$.

We now fix an optimal agent action a^* and derive results parametrically in a^*. We can conclude that there exists an optimal information-trigger contract to (SAND) of the form $w_{a^*,\hat{a}^*}^{\bar{w},t(a^*,\hat{a}^*)}$ (using the notation of (15)) where a^* and $\hat{a}^*$ solve the outer maximization and minimization, respectively, in (SAND). Then, plugging in $t(a^*,\hat{a})$ for t in (SAND$|a^*,\hat{a}$) yields a maximization problem only in $\hat{a}$. Let $\hat{a}^*$ denote an optimal solution to this problem. It has associated optimal trigger $t(a^*,\hat{a}^*)$ and optimal contract $w_{a^*,\hat{a}^*}^{\bar{w},t(a^*,\hat{a}^*)}$.

Notation 3 *To lighten notation, we let $t^* := t(a^*,\hat{a}^*)$ and $w^* := w_{a^*,\hat{a}^*}^{t(a^*,\hat{a}^*)}$. Moreover, we drop a^* from the ratio function $R(x|a^*,\hat{a})$ and write it instead as $R(x|\hat{a})$. Finally, we shorten $\mathbb{P}_{a^*}[\cdot]$ to $\mathbb{P}[\cdot]$.*

We can thus conclude that $(a^*, \hat{a}^*, w^*)$ is an optimal solution to (SAND). It only remains to show that (SAND) is, in fact, a *tight* relaxation. It suffices to show that (w^*, a^*) is a feasible solution to (1). In light of Lemma 4, this implies (SAND) is a tight relaxation and $(a^*, w^*, \hat{a}^*)$ is an optimal solution to (1).

3.2.3 Implementability

The contract w^* satisfies constraints (1b) and (1d) from its feasibility to (SAND). It remains to establish the incentive-compatibility constraint (1c). Noting that

$$V(w_{a^*,\hat{a}}^t, a^*) = \mathbb{E}[\pi(X)] - \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] \quad \text{and} \quad U(w_{a^*,\hat{a}}^t, a^*) = \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t], \tag{17}$$

we must establish

$$\bar{w}\mathbb{P}[R(X|\hat{a}^*) \geq t^*] - c(a^*) - (\bar{w}\mathbb{P}_{\hat{a}}[R(X|\hat{a}^*) \geq t^*] - c(\hat{a})) \geq 0$$

for all $\hat{a} \in \mathcal{A}$, by plugging w^* into (1c). We may ignore those $\hat{a}$ such that $c(\hat{a}) > c(a^*)$ since any optimal choice of $\hat{a}$ must satisfy $c(\hat{a}) \leq c(a^*)$ by Lemma 1 and the fact that c is increasing. Hence we must show

$$\bar{w} \geq \tfrac{c(a^*) - c(\hat{a})}{\mathbb{P}[R(X|\hat{a}^*) \geq t^*] - \mathbb{P}_{\hat{a}}[R(X|\hat{a}^*) \geq t^*]} \tag{18}$$

for all $\hat{a} \in \mathcal{A}$ such that $c(\hat{a}) < c(a^*)$. Note that this condition implies that implementability boils down to establishing that the bonus $\bar{w}$ is sufficiently large. The fraction on the right-hand side of (18) that provides lower bounds on this bonus plays a pivotal role in our development. Indeed, it bears a strong resemblance to the fraction in (16), except for the fact that in (18), some of the $\hat{a}$

are fixed at $\hat{a}^*$ and other $\hat{a}$ can take on different values, while in (16) all the $\hat{a}$ are the same. We will need to distinguish between these two types of alternate actions.

To progress towards showing (18), we use (17), to write (SAND) as

$$\begin{aligned}
&\min_{\hat{a}} \max_{t} \ \mathbb{E}[\pi(X)] - \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] \\
&\text{subject to } \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] - c(a^*) \geq \underline{U} \\
&\qquad\qquad\quad \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] - c(a^*) \geq \bar{w}\mathbb{P}_{\hat{a}}[R(X|\hat{a}) \geq t] - c(\hat{a}).
\end{aligned}$$

Since we have fixed a^*, $\mathbb{E}[\pi(X)]$ is a constant and can be removed from the principal's objective. Moreover, taking out the negative sign in the objective and swapping maxes for mins yields the equivalent problem to (SAND):

$$\max_{\hat{a}} \min_{t} \ \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] \tag{19a}$$

$$\text{subject to } \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] \geq c(a^*) + \underline{U} \tag{19b}$$

$$\bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] - \bar{w}\mathbb{P}_{\hat{a}}[R(X|\hat{a}) \geq t] \geq c(a^*) - c(\hat{a}). \tag{19c}$$

We note that $(\hat{a}^*, t^*)$ as defined above is an optimal solution to (19). There are two cases to consider, concerning whether constraint (19c) is slack or tight when evaluated at $(\hat{a}^*, t^*)$. For the case where (19c) is slack observe that t^* is an optimal solution to

$$\min_{t} \{\bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] : \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] \geq c(a) + \underline{U}\} \tag{20}$$

where the incentive constraint (19c) is dropped from (19). Then, observe that for any $\hat{a}^{\#}$ that is a best response to contract w^*, we have

$$\max_{\hat{a}} \min_{t} \left\{ \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] : \begin{array}{rl} \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] & \geq c(a^*) + \underline{U}, \\ \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] - \bar{w}\mathbb{P}_{\hat{a}}[R(X|\hat{a}) \geq t] & \geq c(a^*) - c(\hat{a}) \end{array} \right\} \tag{21}$$

$$\geq \min_{t} \left\{ \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] : \begin{array}{rl} \bar{w}\mathbb{P}[R(X|\hat{a}^{\#}) \geq t] & \geq c(a^*) + \underline{U}, \\ \bar{w}\mathbb{P}[R(X|\hat{a}^{\#}) \geq t] - \bar{w}\mathbb{P}_{\hat{a}}[R(X|\hat{a}^{\#}) \geq t] & \geq c(a^*) - c(\hat{a}^{\#}) \end{array} \right\} \tag{22}$$

$$\geq \min_{t} \{\bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] : \bar{w}\mathbb{P}[R(X|\hat{a}^{\#}) \geq t] \geq c(a^*) + \underline{U}\} \tag{23}$$

where the first inequality follows by evaluating $\hat{a}$ at $\hat{a}^{\#}$ and the second inequality follows by dropping a constraint. However, as we argued in (20) above, (21) and (23) are equal in value, and hence (22) must also share the same value. In particular t^* solves (22) and hence

$$\bar{w}\mathbb{P}[R(X|\hat{a}^{\#}) \geq t^*] - \bar{w}\mathbb{P}_{\hat{a}}[R(X|\hat{a}^{\#}) \geq t^*] \geq c(a^*) - c(\hat{a}^{\#}) \tag{24}$$

holds. This immediately implies (18) since $\hat{a}^{\#}$ is a best response to w^*.

For the case where (19c) is tight at $(\hat{a}^*, t^*)$, additional work is needed. The following notation clarifies our development.

Notation 4 *Define the intermediate bonus (value not necessarily equal to $\bar{w}$)*

$$b(\hat{a}_1, \hat{a}_2, t) := \tfrac{c(a^*) - c(\hat{a}_1)}{\mathbb{P}[R(X|\hat{a}_2) \geq t] - \mathbb{P}_{\hat{a}_1}[R(X|\hat{a}_2) \geq t]} \tag{25}$$

that is associated with information-trigger contract $w_{a^*, \hat{a}_2}^{b(\hat{a}_1, \hat{a}_2, t), t}$.

Since (19c) is assumed tight, $\bar{w} = b(\hat{a}^*, \hat{a}^*, t^*)$. In other words, at optimality in (SAND) the intermediate bonus is equal to $\bar{w}$. Our target condition (18) using this notation is

$$\bar{w} = b(\hat{a}^*, \hat{a}^*, t^*) \geq b(\hat{a}, \hat{a}^*, t^*) \text{ for all } \hat{a} \in \mathcal{A}, \tag{26}$$

which, as we have detailed, ensures that w^* is an optimal contract for the original problem (1).

We reformulate (SAND) in a way that explicitly incorporates a dependence on $\hat{a}^*$ to aid in establishing (26). This reformulation greatly facilitates swapping the order of optimization in (19) without loss of optimality.

Lemma 5 *The sandwich relaxation* (SAND) *is equivalent to the problem*

$$\max_{\hat{a}} \min_{t} \{ b(\hat{a}, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] : b(\hat{a}, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] \geq c(a^*) + \underline{U}, \bar{w} \geq b(\hat{a}, \hat{a}^*, t)\}, \tag{27}$$

where in b, $\hat{a}_2$ is fixed at $\hat{a}^*$. In particular $(\hat{a}^*, t^*)$ is an optimal solution to (27)*.*

In other words, to solve (SAND), it suffices to look at contracts of form $w_{a^*, \hat{a}_2}^{b(\hat{a}_1, \hat{a}_2, t), t}$ with $\hat{a}_2$ fixed at $\hat{a}^*$. Moreover, w^* is an optimal contract within this class.

In problem (27), t is chosen *after* $\hat{a}$ is determined. If, by contrast, t is chosen first, we optimize $b(\hat{a}, \hat{a}^*, t)$ over $\hat{a}$ (subject to constraints) since, in the objective function of (27), $\mathbb{P}[R(X|\hat{a}^*) \geq t]$ does not depend on $\hat{a}$. This optimization over $\hat{a}$ gets us closer to establishing (26). This inspires us to consider swapping max and min in (27):

$$\min_{t} \max_{\hat{a}} \; b(\hat{a}, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] \tag{28a}$$

$$\text{subject to } b(\hat{a}, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] \geq c(a) + \underline{U} \tag{28b}$$

$$\bar{w} \geq b(\hat{a}, \hat{a}^*, t)\}. \tag{28c}$$

In fact, this swap can be taken without loss of optimality.

Lemma 6 *Problems* (27) *and* (28) *are equivalent; that is, the order of optimization over $\hat{a}$ and t can be swapped without loss of optimality. In particular, $(t^*, \hat{a}^*)$ is an optimal solution to* (28)*.*

We now leverage the fact that $(t^*, \hat{a}^*)$ is an optimal solution to (28) to show that our target condition (26) holds. Suppose, by way of contradiction, that

$$\max_{\hat{a}} b(\hat{a}, \hat{a}^*, t^*) > b(\hat{a}^*, \hat{a}^*, t^*) = \bar{w}. \tag{29}$$

Observing that $b(\hat{a}, \hat{a}^*, t^*)$ is a continuous function of t, there exists an $\epsilon > 0$ sufficiently small such that

$$\max_{\hat{a}} b(\hat{a}, \hat{a}^*, t^* + \epsilon) \geq \bar{w}.$$

this is possible since $t^* < 1$ by Proposition 1. Therefore, in problem (28) when taking $t = t^* + \epsilon$, the maximization over $\hat{a}$ must bind constraint (28c); that is, there exists an $\hat{a}'$ such that $b(\hat{a}', \hat{a}^*, t^* + \epsilon) = \bar{w}$ (this follows by the continuity of b in its first argument). However, note since $\mathbb{P}[R(X|\hat{a}^*) \geq t^* + \epsilon] < \mathbb{P}[R(X|\hat{a}^*) \geq t^*]$ (by the piecewise continuity of $R(\cdot|\hat{a}^*)$) we have $\bar{w}\mathbb{P}[R(X|\hat{a}^*) \geq t^* + \epsilon] < \bar{w}\mathbb{P}[R(X|\hat{a}^*) \geq t^*]$ and thus the objective value of the solution $(t^* + \epsilon, \hat{a}')$ in (28) is strictly less than the objective value of solution $(t^*, \hat{a}^*)$. This contradicts the optimality of $(t^*, \hat{a}^*)$ established in Lemma 6. We have shown the main result of the paper in the special case where there is a uniform bound on agent compensation.

Theorem 2 *There exists an optimal information-trigger contract to moral-hazard problem* (1) *where $m(x) = \bar{w}$. In particular, contract $w^{\bar{w}, t^*}$ (as defined in* (6)*), where $t^* := (a^*, \hat{a}^*)$ is defined in* (16)*, is an optimal contract that implements agent action a^*.*

3.3 General resource constraint

We return now to the setting of a general resource constraint $m(x)$ with compensation bound $0 \le w(x) \le m(x)$ for $x \in \mathcal{X}$. Our main result is to show there exists an optimal information-trigger contract that takes the more general form:

$$w^{m,t}(x) = \begin{cases} 0 & \text{if } R(x|a,\hat{a}) < t \\ m(x) & \text{if } R(x|a,\hat{a}) \ge t. \end{cases} \tag{30}$$

for appropriately-chosen a, $\hat{a}$ and t.

Theorem 3 *There exists an optimal information-trigger contract of the form* (30) *for the risk-neutral moral-hazard problem* (1)*.*

This result suggests that the optimal contract appears as a truncation of the resource constraint $m(x)$. For signals with a weak association with the target action a (that is, with small R value) the agent receives his base compensation. For signals with a strong association with the target action, the agent receives a bonus $m(x)$ that exhausts the resource modeled by the function m.

The proof of Theorem 3 follows the template discussed in the previous two subsections. First, we establish the result in the case of binary actions. Second, we introduce a sandwich relaxation to leverage the binary-action result. The complication here is that additional work is needed to show that the sandwich relaxation is tight. When $m(x) = \bar{w}$, we are able to simplify the incentive-compatibility condition to be simply a bound satisfied by $\bar{w}$ in (18). In this case, we need to work more carefully since the resource constraint $m(x)$ is not uniform across outputs. To handle this, we introduce *four* equivalent problems to the sandwich relaxation in order to establish the implementability of its optimal solution. Complete details are found in the appendix.

4 Applications

We consider two illustrative applications of our main Theorem 3. First, we study the setting of a single-dimensional signal, where we derive the optimality of a quota-bonus contract under a weak monotonicity assumption on the output distribution. Second, we consider a two-signal CEO compensation problem inspired by a model in Bolton and Dewatripont (2005). There we use information-trigger contracts to derive a particularly elegant form of optimal contract.

4.1 Quota-bonus contracts

As we have shown in the previous section, information-trigger contracts possess an elegant theory and straightforward economic interpretation. However, in the single-dimensional output setting, we can further leverage our results.

Assumption 1 *The set of output signals $\mathcal{X}$ is an interval $[\underline{x}, \bar{x}]$ of the real line totally ordered in the usual fashion. We allow the possibility that $\underline{x} = -\infty$ and $\bar{x} = +\infty$.*

The single-dimensional output setting is most commonly studied in the risk-neutral literature. Written as a function of the output signal x itself, an information-trigger contract can jump back and forth between a value of 0 and $\bar{w}$. Contracts on single-dimensional signals seen in practice are rarely non-monotone, which is likely here without additional conditions on R.

We say an output distribution has the monotone likelihood ratio property (MLRP) if $\frac{f(x,\hat{a})}{f(x,a)}$ is nonincreasing for all actions $a \leq \hat{a}$ (as defined in Rogerson (1985); Holmstrom (1979)). The MLRP is a standard assumption in the moral hazard literature.

Under the MLRP assumption, R is a monotone nondecreasing function in x. Hence, there exists a q such that $R(x) < t$ for $x < q$ and $R(x) \geq t$ for $x \geq q$. In other words, we may rewrite the optimal information-trigger contract in Theorem 2 as

$$w^*(x) = \begin{cases} 0 & \text{if } x < q \\ m(x) & \text{if } x \geq q. \end{cases} \tag{31}$$

The following result is thus immediate.

Theorem 4 *If f satisfies the MLRP then an optimal quota-bonus contract exists with bonus $m(x)$ for outputs x larger than the quota.*

4.2 CEO compensation with two output signals

Consider the following model setup from Section 4.6.1 of (Bolton and Dewatripont, 2005). A CEO contracts with a single representative shareholder. The CEO takes hidden action $a \in \mathcal{A}$ that affects two output signals, profits x_1 and stock price x_2. Random profits X_1 and stock price X_2 follow a bivariate normal distribution with density function

$$f(x_1, x_2, a) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-a)^2}{\sigma_1^2} + \frac{(x_2-a)^2}{\sigma_2^2} - \frac{2\rho(x_1-a)(x_2-a)}{\sigma_1\sigma_2}\right]\right) \tag{32}$$

where the mean and standard deviation of X_i is a and σ_i respectively (for $i = 1, 2$) and ρ is correlation of X_1 and X_2.

Bolton and Dewatripont (2005) study this set-up with a risk-neutral CEO and restrict attention to linear contracts, which may not necessarily be optimal. We assume risk-neutrality, but include limited liability and resource constraints to partially capture risk aversion. A direct application of Theorem 3 reveals the existence of an optimal information-trigger contract of the form:

$$w^{m,t}(x) = \begin{cases} 0 & \text{if } R(x) < t \\ m(x) & \text{if } R(x) \geq t. \end{cases} \tag{33}$$

where $R(x) = 1 - \frac{f(x_1,x_2,\hat{a})}{f(x_1,x_2,a)}$ for appropriately chosen a, $\hat{a}$, and t where $a > \hat{a}$. Given the specific structure of $f(x_1, x_2, a)$ in (32) we can view the trigger contract in the space of the signals x_1 and x_2. Some algebra reveals that

$$1 - \frac{f(x_1, x_2, \hat{a})}{f(x_1, x_2, a)} \geq t$$

is equivalent to

$$\left(\frac{1}{\sigma_1^2} - \frac{\rho}{\sigma_1\sigma_2}\right)x + \left(\frac{1}{\sigma_2^2} - \frac{\rho}{\sigma_1\sigma_2}\right)y \geq \frac{-(1-\rho^2)\log(1-t)}{a - \hat{a}} + \frac{1}{2}(a + \hat{a})\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} - \frac{2\rho}{\sigma_1\sigma_2}\right)$$

In other words, an optimal contract for the principal is to provide the CEO with a bonus $m(x_1, x_2)$ if the weighted sum $\left(\frac{1}{\sigma_1^2} - \frac{\rho}{\sigma_1\sigma_2}\right)x + \left(\frac{1}{\sigma_2^2} - \frac{\rho}{\sigma_1\sigma_2}\right)y$ of profit and stock price exceed the cut-off value $\frac{-(1-\rho^2)\log(1-t)}{a-\hat{a}} + \frac{1}{2}(a + \hat{a})\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} - \frac{2\rho}{\sigma_1\sigma_2}\right)$.

5 Monotone contracts

A common assumption in the risk-neutral moral-hazard literature (see, for instance, Oyer (2000); Innes (1990)) is that contracts are restricted to be monotone (in a single-dimensional output). The usual justification for assuming monotonicity *a priori* is the notion that the agent can dispose of excess output without cost if it benefits him to do so. As already mentioned, one drawback of information-trigger contracts is an information-trigger contract need not be monotone in the single-dimensional setting. For this reason, a direct application of our results in Section 3 in the monotone contract case is not feasible.

Instead, we extend our analytical framework to work within the monotone setting. We proceed by mirroring the methodology we used to show the optimality of information-trigger contracts in this setting. We first study the binary-action case and leverage results there for more general-action sets using a "sandwich" procedure. In the binary-action case we continue to use reformulation (5) to derive our results, this time being sure to account for the monotonicity constraint (34).

Concretely, we add the constraint

$$w \text{ is monotone nondecreasing} \tag{34}$$

to our formulation of the moral-hazard problem (1). We will also make Assumption 1, restricting attention to single-dimensional signals. Moreover, we restrict development to the setting where $m(x) = \bar{w}$ for some uniform upper bound $\bar{w}$ on compensation. The methodology here can be adapted to the setting of a general resource constraint, at the cost of additional analytical complications. The results are designed to be analogous to Sections 3.1 and 3.2. We believe this best illustrates the complications of the monotone case and how we can adapt the main ideas of our methodology.

5.1 Binary action case

The main result in this section is to show the existence of a quota-bonus contract in the binary-action monotone case. We start with a preliminary lemma.

Lemma 7 *There exists a monotone nondecreasing and piecewise-constant contract w^* that is optimal to* (2) *with the additional monotonicity constraint* (34)*.*

The proof is analogous to the proof of Theorem 2, where we design perturbations that either maintain or increase covariance of the starting contract. The difference here is the need to ensure that the perturbations maintain monotonicity. Details are in the appendix.

Lemma 7 is only an intermediate result – monotonicity further allows us to show that there exists an optimal quota-bonus contract (that is, an optimal contract with only two constant pieces).

Theorem 5 *There exists an optimal quota-bonus contract to* (2) *with monotonicity constraint* (34)*.*

The proof of this result is quite technical. It requires additional notation and lemmas captured in the appendix. The overall idea of the proof is as follows. First, we show how to reduce consideration to a three-piece contract. The idea here is straightforward. Two consecutive constant pieces that takes values between 0 and $\bar{w}$ can collapsed into a single piece through a mean-preserving perturbation that increases the covariance between R and the perturbed contract by pulling the "lower" piece upwards and pushing the "higher" piece downwards. Repeating this argument iteratively, we come to an optimal contract with one piece at constant value 0, another at constant value

$\bar{w}$, and a third at some intermediate value. Next, we characterize the mathematical properties of "jump" points, leading to a mathematical absurdity when there is more than a single jump. Full details are in the appendix.

In the monotone case, our result implies the possibility that the bonus of an optimal contract need not be $\bar{w}$. A sufficient condition for a full bonus of $\bar{w}$ is follows. A distribution with probability density function f and cumulative density function F is *hazard rate dominant* (HRD) if $f(x,a)/(1-F(x,a))$ is increasing in a for all $x \in \mathcal{X}$. Hazard rate dominance is implied by the MLRP, but not vice versa.

Lemma 8 *If the output distribution f is HRD, then there exists an optimal quota-bonus contract with bonus $\bar{w}$ in the binary-action problem* (2) *with monotonicity constraint* (34).

The proof of the lemma leverages the fact that HRD implies that there exists regions closer and closer to $\bar{x}$ where R is increasing, allowing us to make covariance-improving perturbations until the contract has quota-bonus form.

5.2 Leveraging the binary-action case

We first of all treat the case discussed in Lemma 8; that is, we assume f is HRD and that an optimal quota-bonus contract with bonus $\bar{w}$ exists. To do so we return to the "sandwich" methodology used in Section 3.2. The development in Section 3.2.1 does not include the monotonicity condition. However, that analysis will follow through when adding constraint (34) to each formulation without any complication. Hence, we will reference results in that section in what follows, keeping this adjustment in mind, without further explanation.

By Lemma 8, for every a and $\hat{a}$, (SAND$|a,\hat{a}$) possesses an optimal quota-bonus contract with quota q and bonus $\bar{w}$, denoted

$$w^{b,q} := \begin{cases} 0 & \text{if } x < q \\ b & \text{if } x \geq q. \end{cases} \tag{35}$$

This is an abuse of notation when compared to the definition of information-trigger contracts in (6), but we believe there is no possibility of confusion since information-trigger contracts are not referred to in this section of the paper.

Noting that

$$V(w^{b,q},a) = \mathbb{E}_a[\pi(X)] - b(1-F(q,a)) \quad \text{and} \quad U(w^{b,q},a) = b(1-F(q,a)) - c(a), \tag{36}$$

problem (SAND$|a,\hat{a}$) becomes

$$\max_{q \in \mathcal{X}} \; -\bar{w}(1-F(q,a)) \tag{37a}$$

$$\text{subject to } \bar{w}(1-F(q,a)) - c(a) \geq \underline{U} \tag{37b}$$

$$\bar{w}(1-F(q,a)) - c(a) \geq \bar{w}(1-F(q,\hat{a})) - c(\hat{a}), \tag{37c}$$

since $\mathbb{E}_a[\pi(X)]$ is a constant in the principal's objective. Given this formulation, we can also determine the quota as a function a and $\hat{a}$. We restrict attention to those a and $\hat{a}$ where the incentive constraint is binding; that is,

$$\bar{w}(1-F(q,a)) - c(a) = \bar{w}(1-F(q,\hat{a})) - c(\hat{a}) \tag{38}$$

If this constraint does not bind then argument similar to that found between equations (20) and (24) applies and the result is straightforward. Since objective function of (37) is increasing in q and so the optimal quota $q(a, \hat{a})$ is the largest solution of q in (38) where $w^{\bar{w},q}$ remains feasible (that is, satisfies the rationality constraint (37b)) and is a *function* $q(a, \hat{a})$ of a and $\hat{a}$. Specifically,

$$q(a, \hat{a}) := \max_q \{q : (37\text{b}), (38)\}$$

Hence the optimal solution to (SAND) has the form $w^{\bar{w},q(a^*,\hat{a}^*)}$ where a^* and $\hat{a}^*$ solve

$$\max_a \min_{\hat{a}} \left\{ V(w^{\bar{w},q(a,\hat{a})}, a) : U(w^{\bar{w},q(a,\hat{a})}, a) \geq \bar{U}, U(w^{\bar{w},q(a,\hat{a})}, a) - U(w^{\bar{w},q(a,\hat{a})}, \hat{a}) \geq 0 \right\}. \tag{39}$$

We now show that this problem is equivalent to the original problem (1) under our assumptions. If we restrict attention to contracts of the form $w^{\bar{w},q}$ we clearly have

$$\max_a \max_w \min_{\hat{a}} \{V(w, a) : (w, a) \in \mathcal{W}(a, \hat{a})\} \geq \max_a \max_q \min_{\hat{a}} \{V(w^{\bar{w},q}, a) : (w^{\bar{w},q}, a) \in \mathcal{W}(a, \hat{a})\},$$

where $\mathcal{W}(a, \hat{a})$ is defined in (13).

Thus, it suffices to show (in light of this inequality and Lemma 4) that

$$\max_a \max_q \min_{\hat{a}} \{V(w^{\bar{w},q}, a) : (w^{\bar{w},q}, a) \in \mathcal{W}(a, \hat{a})\} \geq \max_a \min_{\hat{a}} \max_q \{V(w^{\bar{w},q}, a) : (w^{\bar{w},q}, a) \in \mathcal{W}(a, \hat{a})\}, \tag{40}$$

where the right-hand side is precisely problem (39). We have

$$\begin{aligned} &\max_{\hat{a}} \min_q \{\bar{w}(1 - F(q, a)) : \bar{w}(1 - F(q, a)) \geq c(a) + \underline{U}, \bar{w}(F(q, \hat{a})) - F(q, \hat{a})) = c(a) - c(\hat{a})\} \\ \geq &\min_q \max_{\hat{a}} \{\bar{w}(1 - F(q, a)) : \bar{w}(1 - F(q, a)) \geq c(a) + \underline{U}, \bar{w}(F(q, \hat{a})) - F(q, \hat{a})) = c(a) - c(\hat{a})\}, \end{aligned} \tag{41}$$

since (38) holds. Note that we took out a negative from the objective and so max is swapped with min (and vice versa).

Let q^* and $\hat{a}^*$ be an optimal solution the right-hand side problem in (41). By feasibility, $\hat{a}^*$ solves $\bar{w}(F(q, \hat{a})) - F(q, \hat{a})) = c(a) - c(\hat{a})$. Taking $\hat{a} = \hat{a}^*$ in the left-hand side problem of (41) yields

$$\begin{aligned} &\max_{\hat{a}} \min_q \{\bar{w}(1 - F(q, a)) : \bar{w}(1 - F(q, a)) \geq c(a) + \underline{U}, \bar{w}(F(q, \hat{a})) - F(q, \hat{a})) = c(a) - c(\hat{a})\} \\ &\geq \min_q \{\bar{w}(1 - F(q, a)) : \bar{w}(1 - F(q, a)) \geq c(a) + \underline{U}, \bar{w}(F(q, \hat{a})) - F(q, \hat{a})) = c(a) - c(\hat{a})\}. \end{aligned} \tag{42}$$

The resulting minimization over q takes the largest value for q that solves $\bar{w}(F(q, \hat{a})) - F(q, \hat{a})) = c(a) - c(\hat{a})$, since the objective function is decreasing in q and the first constraint will not bind under our choice of a. This yields precisely $q(a, \hat{a}^*)$ as an optimal solution to the right-hand side of (42). Thus, the right-hand side problem (42) is equivalent to the right-hand problem of (41), establishing the inequality in (41). We have thus shown the following:

Theorem 6 *If the output distribution f satisfies HRD then there exists an optimal quota-bonus contract with bonus $\bar{w}$ in the general-action problem* (1) *with monotonicity constraint* (34).

Finally, we consider the possibility of an optimal quota-bonus contract with bonus less than $\bar{w}$. The theory we have established so far only considers contracts with bonus $\bar{w}$ and so we need an alternate approach. We take a more standard approach and work with assumptions commonly associated with the FOA. A distribution function f has the *convex distribution function condition* (CDFC) if $F(x, a)$ is convex in a for all x. Additionally we say f has the *single-crossing hazard rate property* (SCHRP) if $\frac{f(x,a)}{1-F(x,a)}$ and $\frac{f(x,\hat{a})}{1-F(x,\hat{a})}$ have at most a single crossing along x for all $a > \hat{a}$.

Theorem 7 *If CDFC and SCHRP hold and c is convex, then there exists an optimal quota-bonus contract to the moral-hazard problem* (1) *with monotonicity constraint* (34)*.*

We remark that SCHRP is a weaker condition than the HRD. Indeed, HRD implies that $\mathbb{E}[R(X)|X > q]$ is an increasing function of q, while the SCHRP allows for this conditional expectation to be inverse U-shaped. Of course, for SCHRP to suffice we need to additionally assume the CDFC.

The single-crossing and CDFC conditions imply that the mappings for the optimal quota q as a function of an alternate action $\hat{a}$, and the optimal $\hat{a}$ as a function of q, are both continuous maps. Then Brouwer's fixed-point theorem implies the equivalence of the max-min and min-max problems where we maximize over $\hat{a}$ and minimize over q. This shows the sandwich relaxation is tight and allows us to conclude that a quota-bonus contract is optimal to the moral-hazard problem (1) with monotonicity constraint (34). Complete details are found in the appendix.

We remark that the CDFC condition suffices for the FOA to be valid when restricted to monotone contracts, but it alone does suffice to show the optimality of a quota-bonus contract.

6 Discussion

This paper describes a novel type of contract simplicity in the form of information triggers. These contracts possess strong economic intuition and are optimal under weak assumptions. The information-trigger structure is a direct consequence of the risk-neutrality assumption unclouded by additional assumptions.

Paper	Contract assumption	Distribution assumption	Structure
(Kim, 1997)	limited liability only	CDFC and MLRP	quota-bonus
(Oyer, 2000)	w nondecreasing	FOA[5] valid hazard rate inverse U-shaped	quota-bonus
(Innes, 1990)	$x - w(x)$ nondecreasing $w(x) \leq x$	MLRP	debt[6]
(Poblete and Spulber, 2012)	w nondecreasing $x - w(x)$ nondecreasing	monotone critical ratio[7]	debt
(Wang and Hu, 2016)	w nondecreasing	various technical conditions[8]	single or two tier bonus
Theorem 4	resource constraint	MLRP	quota-bonus
Theorem 6	upper bound $\bar{w}$ w nondecreasing	HRD	quota-bonus
Theorem 7	w nondecreasing	CDFC single-crossing hazard rate	quota-bonus

Table 1: Other results on quota-bonus contracts. In all cases both the principal and agent are assumed to be risk neutral and the agent has limited liability.

[5]The FOA (see, for instance, (Rogerson, 1985)).

[6]A debt contract is equivalent to a quota-bonus contract in the setting where the agent offers the contract to the principal.

[7]The monotone critical ratio is the product of the hazard rate of the output and the marginal rate of technical substitution of the agent's effort for that output.

[8]See Propositions 1–3 in Wang and Hu (2016). Conditions include log-supermodularity of the distribution function, etc. The authors show that several common distributions (including normal, lognormal, exponential, Weibull, etc.)

Moreover, we show that the analytical framework we employ (the covariance reformation (5) of the binary-action problem and the sandwich relaxation (SAND)) is powerful and robust. The same essential methodology we used to show the existence of information-trigger contracts was also used to show the optimality of quota-bonus contracts in the monotone contract setting. One can view the "sandwich" approach as a tool to "squeeze" as much information as possible from the binary-action setting. It is a concrete tool to bolster the quote from Hart and Holmstrom (1987) cited in the introduction on the sufficiency of the binary-action setting for studying moral hazard.

Since we also explore the implications of our approach for the optimality of quota-bonus contracts, we make some comparisons between our results and those in the literature. Table 1 (which adapts and expands on Table 1 in Wang and Hu (2016)) captures various sufficient conditions discussed in the literature. Theorem 4 shows that assuming the MLRP alone (in addition to an upper bound on the contract) yields a quota-bonus contract. In comparison to Innes's result, this suggests that part of the underlying strength of the MLRP assumption might be leveraged to weaken his monotonicity assumption. Similarly for Kim's result, assuming the CDFC appears to be a strong assumption, given that the MLRP is also assumed.

It is difficult to provide a direct comparison of Oyer's result, since he provides no specific conditions to assure the validity of the FOA. For purposes of comparison with our Theorem 7 we suppose that Oyer's result assumes the CDFC. Moreover, Oyer assumes a specific form of model uncertainty as follows. The random output

$$x = h(a) + \eta \tag{43}$$

depends on a deterministic function h of the agent's action and a noise term η, where $h'(a) > 0$. The assumption detailed in Table 1 that the hazard rate is inverse U-shaped refers then to the distribution of η. This directly implies the single-crossing hazard rate property of Theorem 7, and hence we can view Theorem 7 as a strengthening of Oyer's result.

Moreover, in Oyer's Proposition 3, he argues that if the hazard rate of η is increasing, then no optimal quota-bonus contract exists. Dai and Jerath (2016) noted this and introduced inventory constraints on the problem to recover the quota-bonus structure in the increasing hazard rate setting (which is a very common assumption, particularly in the operations management literature that Dai and Jerath (2016) discuss).

We believe that this difficulty with increasing hazard rate distributions is an artifact of analyzing the model using the FOA. In fact, assuming η has an increasing hazard rate implies the output distribution satisfies HRD.[9] Hence, our Theorem 6 reasserts the existence of an optimal quota-bonus contracts, this time without using the FOA.

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satisfy their conditions.

[9]Letting k and K denote the probability density function and cumulative distribution function of η, respectively, we have

$$\begin{aligned} \frac{f(x,\hat{a})}{1-F(x,\hat{a})} &= \frac{k(x-h(\hat{a})}{1-K(x-h(\hat{a}))} \\ &\geq \frac{k(x-h(a))}{1-K(x-h(a))} \\ &= \frac{f(x,a)}{1-F(x,a)} \end{aligned}$$

for $a > \hat{a}$, where the inequality uses the increasing hazard rate property of η and the fact that h is an increasing function.

A Proofs

A.1 Proof of Lemma 1

Let w^* denote an arbitrary optimal solution to (2). Since the goal is to implement action a we may assume that $U(w^*, a) \geq U(w^*, \hat{a})$. In particular, this implies that $\mathbb{E}_a[w^*(X)] \leq \mathbb{E}_{\hat{a}}[w^*(X)]$. Also, since $\hat{a} > a$ and c is monotonically increasing, this implies $c(\hat{a}) > c(a)$. Together, this implies that the incentive constraint (2c) is satisfied strictly since we may write $U(w^*, a) = \mathbb{E}_a[w^*(X)] - c(a)$. Thus, only the IR constraint (2b) constrains the choice of w and thus we may assume that (2b) is satisfied with equality. Thus, there exists an optimal first-best contract.

A.2 Proof of Lemma 3

Let $\hat{w}$ be an optimal solution to (5). By (5b), $\hat{w}$ and w^* have the same objective value for both principal and agent. Moreover, $U(\hat{w}, a) - U(\hat{w}, \hat{a}) \geq U(w^*, a) - U(w^*, \hat{a}) \geq 0$ since $\hat{w}$ optimizes (5a) and w^* is feasible to (2) and so (2c) holds.

Conversely, suppose (2b) at an optimal solution w^* to (2) that is not optimal to (5). Then there exists an optimal solution $\hat{w}$ to (5) (and hence also (2)) such that

$$\begin{aligned}\int R(x)\hat{w}(x)f(x,a)dx &> \int R(x)w^*(x)f(x,a)dx \\ &\geq c(a) - c(\hat{a}).\end{aligned}$$

This implies, in (2c) is slack at $\hat{w}$ in (2), but (2b) is binding. Thus,

$$\begin{aligned}V(w^*, a) &= \mathbb{E}[\pi(X)] - \mathbb{E}[w^*(X)] \\ &< \mathbb{E}[\pi(X)] - (c(a) + \underline{U}) \\ &= \mathbb{E}[\pi(X)] - \mathbb{E}[\hat{w}(X)] \\ &= V(\hat{w}, a)\end{aligned}$$

since (2b) is slack at w^* and binds at $\hat{w}$. This is a contradiction of the optimality of w^*.

A.3 Proof of Proposition 1

First we show $t^* > 0$. Suppose otherwise that $t^* \leq 0$. Such a contract will not maximize covariance in (5). Indeed, since $w(x)$ and $f(x,a)$ are both nonnegative, covariance (seen in (5a)) is no worse by increasing t^* slightly. However, this will reduce the expected payout to the agent. Thus, such a choice for t^* cannot be optimal.

To show that $t^* < 1$ observe that $f(x,\hat{a})/f(x,a) > 0$ for all x since the support of $f(\cdot|a)$ is all of $\mathcal{X}$ for all a. Thus, we may assume that $t^* < 1$.

A.4 Proof of Lemma 5

We first show an intermediate equivalence. Consider the problem

$$\max_{\hat{a}} \min_{t} \{b(\hat{a},\hat{a},t)\mathbb{P}[R(X|\hat{a}) \geq t] : b(\hat{a},\hat{a},t)\mathbb{P}[R(X|\hat{a}) \geq t] \geq c(a) + \underline{U}, \bar{w} \geq b(\hat{a},\hat{a},t)\}. \tag{44}$$

We claim that (19) is equivalent to this problem. First, observe that (19) can be rewritten as

$$\max_{\hat{a}} \min_{t} \{\bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] : \bar{w}\mathbb{P}[R(X|\hat{a}) \geq t] \geq c(a) + \underline{U}, \bar{w} \geq b(\hat{a},\hat{a},t)\}. \tag{45}$$

For every feasible $(\hat{a}, t)$ in (44) we have $b(\hat{a}, \hat{a}, t) \leq \bar{w}$. This implies that $(\hat{a}, t)$ is also feasible to (45). In particular, $(\hat{a}^*, t^*)$ is feasible to both problems. Since the objective value $b(\hat{a}, \hat{a}, t)\mathbb{P}[R(X|\hat{a}) \geq t]$ in (44) for feasible solution $(\hat{a}, t)$ is less than or equal to that solutions value $\bar{w}\mathbb{P}[R(X|\hat{a}) \geq t]$ in (45), we can conclude that val(45) $\geq$ val(44).

It remains to show the converse that val(45) $\leq$ val(44). From (44) we know

$$\begin{aligned} \text{val}(44) &= \max_{\hat{a}} \min_{t} \{ b(\hat{a}, \hat{a}, t)\mathbb{P}[R(X|\hat{a}) \geq t] : b(\hat{a}, \hat{a}, t)\mathbb{P}[R(X|\hat{a}) \geq t] \geq c(a) + \underline{U}, \bar{w} \geq b(\hat{a}, \hat{a}, t) \} \\ &\geq \min_{t} \{ b(\hat{a}^*, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] : b(\hat{a}^*, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] \geq c(a) + \underline{U}, \bar{w} \geq b(\hat{a}^*, \hat{a}^*, t) \} \end{aligned} \tag{46}$$

by evaluating the outer maximization at $\hat{a}^*$. Let $t^\#$ be an optimal solution to the right-hand side problem. It suffices to argue that $b(\hat{a}^*, \hat{a}^*, t^\#)\mathbb{P}[R(X|\hat{a}^*) \geq t^\#] \geq \hat{w}\mathbb{P}[R(X|\hat{a}^*) \geq t^*] = \text{val}(45)$. By way of contradiction, suppose the converse is true. This implies that the following contract

$$w(x) = \begin{cases} 0 & \text{if } R(x|\hat{a}^*) < t^\# \\ b(\hat{a}^*, \hat{a}^*, t^\#) & \text{if } R(x|\hat{a}^*) \geq t^\# \end{cases} \tag{47}$$

pays out less than the agent $(b(\hat{a}^*, \hat{a}^*, t^\#)\mathbb{P}[R(X|\hat{a}^*) \geq t^\#] < \hat{w}\mathbb{P}[R(X|\hat{a}^*) \geq t^*])$ in the binary-action problem with actions a^* and $\hat{a}^*$ than the optimal contract w^* to that problem. This violates the optimality of w^*. Hence, (19) is equivalent to (44).

Now, we show (45) is equivalent to (27). Again, every feasible solution to (27) is also feasible to (45), and in particular (a^*, t^*) is feasible to both, and so it suffices to show that both problems have the some optimal objective value. The argument that val(27) $\leq$ val(45) follows analogous logic to part (i) and is thus omitted. For the other direction, we leverage the fact that the inequality in (46) is actually an equality by the the equivalence of (19) and (44). This equality is useful since:

$$\begin{aligned} \text{val}(27) &= \max_{\hat{a}} \min_{t} \{ b(\hat{a}, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] : b(\hat{a}, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] \geq c(a) + \underline{U}, \bar{w} \geq b(\hat{a}, \hat{a}^*, t) \} \\ &\geq \min_{t} \{ b(\hat{a}^*, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] : b(\hat{a}^*, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] \geq c(a) + \underline{U}, \bar{w} \geq b(\hat{a}^*, \hat{a}^*, t) \} \\ &= \text{val}(44) \\ &= \text{val}(45), \end{aligned}$$

where the inequality follows by plugging in $\hat{a}^*$, the first equality follows from the fact that the inequality in (46) is tight, and the second equality follows from the equivalence of (19) and (44).

We can therefore conclude that (SAND) is equivalent to (27) and $(\hat{a}^*, t^*)$ is an optimal solution to the latter.

A.5 Proof of Lemma 6

It is straightforward to see that $(t^*, \hat{a}^*)$ is a feasible solution to (28). It remains to show that val(27) = val(28). The "$\leq$" follows from the max-min inequality. Conversely,

$$\begin{aligned} \text{val}(28) &= \min_{t} \max_{\hat{a}^*} \{ b(\hat{a}, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] : b(\hat{a}, \hat{a}^*, t)\mathbb{P}[R(X|\hat{a}^*) \geq t] \geq c(a) + \underline{U}, \bar{w} \geq b(\hat{a}, \hat{a}^*, t) \} \\ &\leq \max_{\hat{a}} \{ b(\hat{a}, \hat{a}^*, t^*)\mathbb{P}[R(X|\hat{a}^*) \geq t^*] : b(\hat{a}, \hat{a}^*, t^*)\mathbb{P}[R(X|\hat{a}^*) \geq t^*] \geq c(a) + \underline{U}, \bar{w} \geq b(\hat{a}, \hat{a}^*, t^*) \} \\ &= b(\hat{a}^*, \hat{a}^*, t^*)\mathbb{P}[R(X|\hat{a}^*) \geq t^*] = \text{val}(27), \end{aligned}$$

where the inequality holds by plugging in t for t^* and the second equality holds since $(\hat{a}^*, t^*)$ is an optimal solution to (27) and so $\hat{a}^*$ is a maximizer when t^* is fixed.

A.6 Proof of Theorem 3

We first examine the binary-action case. For fixed a and $\hat{a}$, as in Section 3.1 we have the reformulation

$$\begin{aligned} \max_{w} \quad & \int R(x)w(x)f(x,a)dx \\ \text{subject to } & \mathbb{E}[w(X)] = \mathbb{E}[w^*(X)] \\ & 0 \le w(x) \le m(x) \text{ for almost all } x \in \mathcal{X}, \end{aligned}$$

which yields an optimal contract to the binary-action problem of the form

$$w^{m,t}_{a,\hat{a}}(x) = \begin{cases} 0 & \text{if } R(x|a,\hat{a}) < t \\ m(x) & \text{if } R(x|a,\hat{a}) \ge t. \end{cases}$$

Establishing the optimality of a contract of this form follows nearly identical reasoning to that of Theorem 1. One only major difference is in the specification of α_2 in the perturbation (8). We must define $0 \le \alpha_2(x) \le m(x) - w(x)$ in order to maintain feasibility of the perturbed contract. For the purposes of this proof we will not be as careful in defining t^* as in (7) nor claim uniqueness when the participation constraint is not binding, as in Theorem 1.

The construction of the sandwich relaxation and its optimal solution also follows the development of Sections 3.2.1 and 3.2.2 closely. Throughout, we generalize the maximization over w to be for $0 \le w(x) \le m(x)$. Also, the function $t(a, \hat{a})$ is defined more generally as:

$$t(a,\hat{a}) := \max \left\{ t : U(w^{m,t}_{a,\hat{a}}, a) - U(w^{m,t}_{a,\hat{a}}, \hat{a}) \ge 0 \text{ and } U(w^{m,t}_{a,\hat{a}}, a) \ge \underline{U} \right\}.$$

Again using the convention of Notation 3, we get an optimal solution $(a^*, \hat{a}^*, w^*)$ to (SAND) and the remaining task of the proof is to show that (w^*, a^*) is a feasible solution to (1).

Implementability follows by examining a sequence of four equivalent problems for determining the optimal solutions. First, solving (SAND) is equivalent to solving (cf. (19))

$$\max_{\hat{a}} \min_{t} \ \mathbb{E}[m(X)|R(x|\hat{a}) \ge t] \tag{48a}$$

$$\text{subject to } \mathbb{E}[m(X)|R(x|\hat{a}) \ge t] \ge c(a^*) + \underline{U} \tag{48b}$$

$$\mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}) \ge t] \ge c(a^*) - c(\hat{a}), \tag{48c}$$

which evaluates the sandwich relaxation when restricting to contracts at the information-trigger structure $w^{m,t}_{a^*,\hat{a}}$ and solving the cost-minimization version of the problem instead of the profit maximization problem for the principal.

Before continuing, we remark that if (48c) is not binding then we can establish implementability directly. An identical argument to that provided between equations (20) and (24) suffices to establish implementability. Thus, in the remainder of the proof we assume that (48c) is binding.

Next, we show that solving (48) is equivalent to solving

$$\max_{\hat{a}} \min_{t} \ \mathbb{E}[m(X)|R(x|\hat{a}) \ge t] \tag{49a}$$

$$\text{subject to } \mathbb{E}[m(X)|R(x|\hat{a}) \ge t] \ge c(a^*) + U^{a^*} \tag{49b}$$

$$\mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}) \ge t] \ge c(a^*) - c(\hat{a}), \tag{49c}$$

where $U^{a^*} = U(w^*, a^*)$ is the utility of the agent at optimal contract for the sandwich problem. The difference from the previous formulation is that the participation constraint (49b) has been adjusted to become stricter. This helps in guaranteeing the tightness of the participation constraint in subsequent formulations.

Next, we show that solving (50) is equivalent to solving (cf. (27))

$$\max_{\hat{a}} \min_{t} \ \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \tag{50a}$$

$$\text{subject to } \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \geq c(a^*) + U^{a^*} \tag{50b}$$

$$\mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}^*) \geq t] \geq c(a^*) - c(\hat{a}), \tag{50c}$$

The difference from the previous formulation is that in the ratio function R, $\hat{a}$ has been evaluated at $\hat{a}^*$ (this change is reflected in all of (50a)–(50c). This allows us to work towards calculations to evaluate the IC (1c) of the original problem. The reasoning is similar to that explored at the outset of Section 3.2.3. The IC constraint includes both the choice of $\hat{a}^*$ as well as alternate choices $\hat{a} \in \mathcal{A}$. This reformulation is how we introduce these two "types" of $\hat{a}$ into our analysis.

Next, we show that solving (50) is equivalent to solving

$$\max_{\hat{a}} \min_{t} \ \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \tag{51a}$$

$$\text{subject to } \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \geq c(a^*) + U^{a^*} \tag{51b}$$

$$\mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}^*) \geq t] \leq c(a^*) - c(\hat{a}), \tag{51c}$$

The difference from the previous formulation is that we have flipped the direction of the inequality in the incentive constraint (51c). This underscores the fact that at optimality this constraint is tight, which helps us in determining the relationship between the optimal choices of $\hat{a}$ and t. The flip in direction is related to the change in direction illustrated in the inequality (29) in the proof of the simpler case. It will also be used in a proof by contradiction in a similar vein below.

Finally, we show that solving (51) is equivalent to solving

$$\min_{t} \max_{\hat{a}} \ \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \tag{52a}$$

$$\text{subject to } \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \geq c(a^*) + U^{a^*} \tag{52b}$$

$$\mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}^*) \geq t] \leq c(a^*) - c(\hat{a}), \tag{52c}$$

The difference from the previous formulation is that we have changed the order of optimization. This change is essential in establishing implementability.

We give details of the equivalence of problems (48) through (52) below. For now we take as given that an optimal solution to (52) solves the sandwich relaxation (SAND) (and vice versa).

We now show that w^* implements a^*, where $w^* = w^{m,t^*}_{a^*,\hat{a}^*}$ and $(t^*, \hat{a}^*)$ is an optimal solution to (52). To this end, let's consider a perturbed problem, for small $\epsilon > 0$,

$$\min_{t} \max_{\hat{a}} \ \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \tag{53a}$$

$$\text{subject to } \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \geq c(a^*) + U^{a^*} - \epsilon \tag{53b}$$

$$\mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}^*) \geq t] \leq c(a^*) - c(\hat{a}), \tag{53c}$$

Clearly, the perturbed problem will have a larger optimal value than $c(a)+U^{a*}$ since $(t^*, \hat{a}^*)$ remains feasible to (53). Therefore, the participation constraint (53b) is not binding. Let t^ϵ be an optimal choice of t for (53).

We first claim that $w^{t^\epsilon,\hat{a}^*}_{a^*,\hat{a}^*}$ implements a^*. This will help us in attaining our eventual goal of showing that w^* implements a^* (when we eventually take $\epsilon \to 0$). We proceed by contradiction.

Suppose $w^{t^\epsilon,\hat{a}^*}_{a^*,\hat{a}^*}$ does not implement a^*. Then there exists an $\hat{a}'$ such that

$$\mathbb{E}[m(X)R(X|\hat{a}')|R(X|\hat{a}') \geq t^\epsilon] < c(a^*) - c(\hat{a}) \tag{54}$$

Then, by continuity, we can perturb t^ϵ by $\gamma > 0$, to get

$$\mathbb{E}[m(X)R(X|\hat{a}')|R(X|\hat{a}') \geq t^\epsilon + \gamma] \leq c(a^*) - c(\hat{a}) \tag{55}$$

and

$$c(a^*) + U^* - \epsilon < \mathbb{E}[m(X)|R(X|\hat{a}') \geq t^\epsilon + \gamma] \tag{56}$$

$$< \mathbb{E}[m(X)|R(X|\hat{a}') \geq t^\epsilon] \tag{57}$$

This is possible since the participation constraint (53b) is not binding.

Conditions (55) and (56) imply that $(t^\epsilon + \gamma, \hat{a}')$ is feasible to (53). Thus, plugging $t^\epsilon + \gamma$ into (53) yields

$$\begin{aligned}
&\max_{\hat{a}} \left\{ \mathbb{E}[m(X)|R(X|\hat{a}) \geq t^\epsilon + \gamma] : \begin{array}{rl} \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t^\epsilon + \gamma] & \geq c(a^*) + U^{a^*} - \epsilon \\ \mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}^*) \geq t^\epsilon + \gamma] & \leq c(a^*) - c(\hat{a}) \end{array} \right\} \\
&= \mathbb{E}[m(X)|R(X|\hat{a}') \geq t^\epsilon + \gamma] \\
&< \mathbb{E}[m(X)|R(X|\hat{a}') \geq t^\epsilon] \\
&\leq \max_{\hat{a}} \left\{ \mathbb{E}[m(X)|R(X|\hat{a}) \geq t^\epsilon] : \begin{array}{rl} \mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t^\epsilon] & \geq c(a^*) + U^{a^*} - \epsilon \\ \mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}^*) \geq t^\epsilon] & \leq c(a^*) - c(\hat{a}) \end{array} \right\}
\end{aligned}$$

where the equality holds since $(t^\epsilon + \gamma, \hat{a}')$ is feasible to (53) and the objective is constant once t is fixed, the first inequality holds by (57), and the second inequality holds by the definition of maximum and the fact that $(t^\epsilon, \hat{a}')$ is also feasible (using (54)). This contradicts the optimality of t^ϵ.

Therefore, no such $\hat{a}'$ exists and thus from (54) we have

$$\mathbb{E}[m(X)R(X|\hat{a}')|R(X|\hat{a}') \geq t^\epsilon] < c(a^*) - c(\hat{a})$$

for all $\hat{a}' \in \mathcal{A}$. This implies $w^{t^\epsilon,\hat{a}^*}_{a^*,\hat{a}^*}$ implements a^*. Equivalently, we obtain

$$\max_{a'} \int_{R(x|\hat{a}^*) \geq t^\epsilon} m(x) f(x, a') dx - c(a') = \mathbb{E}[m(X)|R(X|\hat{a}^*) \geq t^\epsilon] - c(a^*). \tag{58}$$

Thus, for any $\epsilon > 0$, by the continuity the right-hand side of (58) in t^ϵ, as $\epsilon \to 0$, the equality also holds. Moreover, by the upper hemicontinuity of the maximizer set $a^{BR}(w^{t^\epsilon,\hat{a}^*})$ in ϵ (by the Theorem of Maximum), we have that w^* implements a^*, as required.

It remains to show the equivalence of problems (48) through (52). We handle each in turn.

Equivalence of (48) *and* (49): If $U^{a^*} = \underline{U}$ we are done. Thus, we may assume that $U^{a^*} > \underline{U}$. In this case, (48c) is binding at optimality in (48).

Now consider problem (49). If (49b) is binding at optimality then val(49) $= c(a^*) + U^{a^*} =$ val(48), by the definition of U^{a^*}. Otherwise, (49b) is not binding at optimality and thus (49c) binds in (49). As we just mentioned above, we may also assume that (48c) is also binding at optimality in (48). Hence, problems (48) and (49) are equivalent.

Equivalence of (49) *and* (50): From the previous result we know val(49) $= c(a^*) + U^{a^*}$. Constraint (50b) says $\mathbb{E}[m(X)|R(x|\hat{a}^*) \geq t] \geq c(a^*) + U^{a^*} =$ val(49). This immediately implies that val(50) $\geq$ (49) by the definition of the objective in (50).

By the equivalence of (48) and (49) we know $(\hat{a}^*, t^*)$ is an optimal solution to (49). Plugging t^* in for t in (50) and optimizing for $\hat{a}$ yields problem:

$$\mathrm{val}(50) \le \max_{\hat{a}} \left\{ \mathbb{E}[m(X)|R(x|\hat{a}^*) \ge t^*] : \begin{array}{rl} \mathbb{E}[m(X)|R(x|\hat{a}^*) \ge t^*] & \ge c(a^*) + U^{a^*} \\ \mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}^*) \ge t^*] & \ge c(a^*) - c(\hat{a}) \end{array} \right\}. \quad (59)$$

Observe that the objective is constant in the choice of $\hat{a}$ and so any solution that is feasible to the constraints is an optimal solution. The solution $(\hat{a}^*, t^*)$ is feasible and thus the right-hand side of (59) has the same optimal value as problem (49). Thus, $\mathrm{val}(50) \le \mathrm{val}(49)$ and $(\hat{a}^*, t^*)$ is an optimal solution to (50).

Equivalence of (50) *and* (51): The fact that $\mathrm{val}(51) \ge \mathrm{val}(50)$ follows by identical reasoning as the previous equivalence. For the converse, this follows since when evaluating the minimization in t at t^* in (51), the incentive constraint (48c) binds at $(\hat{a}^*, t^*)$, as argued above.

Equivalence of (51) *and* (52): The fact that $\mathrm{val}(51) \le \mathrm{val}(52)$ follows by the max-min inequality. For the converse, observe that plugging t^* in for t in (51) and optimizing for $\hat{a}$ yields problem:

$$\mathrm{val}(52) \le \max_{\hat{a}} \left\{ \mathbb{E}[m(X)|R(x|\hat{a}^*) \ge t^*] : \begin{array}{rl} \mathbb{E}[m(X)|R(x|\hat{a}^*) \ge t^*] & \ge c(a^*) + U^{a^*} \\ \mathbb{E}[m(X)R(X|\hat{a})|R(X|\hat{a}^*) \ge t^*] & \le c(a^*) - c(\hat{a}) \end{array} \right\}. \quad (60)$$

As observed above, the objective is constant in the choice of $\hat{a}$ and so any solution that is feasible to the constraints is an optimal solution. The solution $(\hat{a}^*, t^*)$ is feasible and thus the right-hand side of (60) has the same optimal value as problem (51). Thus, $\mathrm{val}(52) \le \mathrm{val}(51)$ and $(\hat{a}^*, t^*)$ is an optimal solution to (52).

A.7 Proof of Lemma 7

Let w^* be an arbitrary optimal monotone nondecreasing contract. Divide the domain $\mathcal{X}$ into three disjoint regions. Let $I := \{x : R'(x) > 0\}$ denote region where R is increasing, $D := \{x : R'(x) < 0\}$ denote the region where R is decreasing and $F := \{x : R'(x) = 0\}$ denote the region where R is flat. Since R is continuously differentiable, I, D, and F are either empty or consist of a union of intervals.

We first claim that there always exists an optimal contract that is piecewise constant and monotone nondecreasing when restricted to the region F. If F is empty the result is trivial. Otherwise, let $[x_1, x_2]$ denote an arbitrary interval in the union of intervals defining F. The function $R(x)$ equals a constant C for all $x \in [x_1, x_2]$. Define the following contract:

$$\hat{w}(x) = \begin{cases} \frac{\int_{x_1}^{x_2} w^*(t) f(t,a) dt}{F(x_2,a) - F(x_1,a)}, & x \in [x_1, x_2] \\ w^*(x), & \text{otherwise.} \end{cases}$$

Clearly, $\mathbb{E}[\hat{w}(X)] = \mathbb{E}[w(X)]$ and so (5b) holds. The limited liability constraint (5c) also holds trivially and it is clear that $\hat{w}$ is nondecreasing over F since w^* is nondecreasing over the whole domain. This follows since

$$\begin{aligned}
\int R(x)\hat{w}(x)f(x,a)dx &= \int_{x_1}^{x_2} \hat{w}(x)R(x)f(x)dx + \int_{x\notin[x_1,x_2]} \hat{R}(x)w(x)f(x)dx \\
&= C\int_{x_1}^{x_2} \hat{w}(x)f(x,a)dx + \int_{x\notin[x_1,x_2]} R(x)\hat{w}(x)f(x)dx \\
&= C\int_{x_1}^{x_2} w^*(x)f(x,a)dx + \int_{x\notin[x_1,x_2]} R(x)w^*(x)f(x)dx \\
&= \int_{x_1}^{x_2} R(x)w^*(x)f(x,a)dx + \int_{x\notin[x_1,x_2]} w^*(x)R(x)f(x)dx \\
&= \int R(x)w^*(x)f(x,a)dx.
\end{aligned}$$

We conclude that $\hat{w}$ is optimal. Repeat the argument taking $\hat{w}$ for w^* and a new interval $[x_1,x_2]$ that defines F. Repeat until all intervals defining F have been considered. The result is an optimal contract that is piecewise constant over F.

Next we claim that there exists an optimal contract that is piecewise constant over I. Without loss we may assume that w^* is already piecewise constant over F by previous arguments. Consider the following perturbation:

$$h(x) = \begin{cases} w^*(x_1) - w^*(x) & \text{if } x \in [x_1,\hat{x}) \\ w^*(x_2) - w^*(x) & \text{if } x \in [\hat{x},x_2] \\ 0 & \text{otherwise,} \end{cases} \tag{61}$$

where $\hat{x} \in (x_1,x_2)$ is chosen so that $\mathbb{E}[h(X)] = 0$. Consider the perturbed contract $w^* + h$. Clearly it satisfies constraints (5b) and (5c) of the reformulated problem. By construction $w^* + h$ remains nondecreasing.

We now show that $w^* + h$ has greater covariance than w^*, contradicting the latter's optimality. It suffices to show that $\int h(x)R(x)f(x,a)dx > 0$. Observe that

$$\begin{aligned}
\int h(x)R(x)f(x,a)dx &= \int_{x_1}^{\hat{x}} (w^*(x_1) - w^*(x))R(x)f(x,a)dx + \int_{\hat{x}}^{x_2} (w^*(x_2) - w^*(x))(x)R(x)f(x,a)dx \\
&> \int_{x_1}^{\hat{x}} (w^*(x_1) - w^*(x))R(\hat{x})f(x,a)dx + \int_{\hat{x}}^{x_2} (w^*(x_2) - w^*(x))R(\hat{x})f(x,a)dx \\
&= R(\hat{x})\int_{x_1}^{x_2} h(x)f(x,a)dx = 0,
\end{aligned}$$

where the strict inequality holds since $R(x)$ is increasing in $[x_1,x_2]$ whereas $w^*(x_1) - w^*(x) < 0$ in $[x_1,\hat{x}]$ and $(w^*(x_2) - w^*(x)) > 0$ in $[\hat{x},x_2]$.

Next we claim that there exists an optimal contract that is piecewise constant over D. Without loss we may assume that w^* is already piecewise constant over F and D by previous arguments. An analogous perturbation to (61) can be constructed to complete the result. For brevity we suppress details.

A.8 Proof of Theorem 5

This argument requires the following additional notation and lemmas. Let w^* be a monotone nondecreasing and piecewise-constant contract that is optimal to (2) with the additional monotonicity constraint (34), guaranteed to exist by Lemma 7. Let $\mathcal{J}$ denote the set of $x \in \mathcal{X}$ where w^* "jumps" from one constant value to another.

Let $x_{\bar{w}}$ denote the smallest value of x such that $w(x) = \bar{w}$, if such a value exists, and set $x_{\bar{w}} = \infty$ otherwise. If $x_{\bar{w}}$ is finite then $\max \mathcal{J} = x_{\bar{w}}$ since, by monotonicity, w^* can no longer jump after $x_{\bar{w}}$. Let

$$\tilde{x} = \min\{\bar{x}, x_{\bar{w}}\}. \tag{62}$$

Finally, let $\mathcal{K}$ denote the subset of those $x \in \mathcal{X}$ that satisfy

$$R(x) = \mathbb{E}[R(X)|X \in (x, \tilde{x})] \tag{63}$$

where $\tilde{x}$ is defined in (62).

Lemma 9 *The set $\mathcal{J}$ of jump points (other than $x_{\bar{w}}$ if finite in value) is a subset of $\mathcal{K}$. Moreover, $x_{\bar{w}}$ satisfies $R(x_{\bar{w}}) \leq \mathbb{E}[R(X)|X \in (x_{\bar{w}}, \tilde{x})]$.*

Proof. Suppose by contradiction there exists a jumping point $x^c \in \mathcal{J}$ with $w(x^c) < \bar{w}$ such that $R(x^c) \neq \mathbb{E}[R(X)|X \in (x^c, \tilde{x})]$. In particular, x^c cannot be $x_{\bar{w}}$. There are two cases to consider.

Suppose $R(x^c) < \mathbb{E}[R(X)|X \in (x^c, \tilde{x})]$ (the case where $R(x^c) > \mathbb{E}[R(X)|X \in (x^c, \tilde{x})]$ can be argued analogously). For all $\epsilon > 0$ there exists positive α_1 and α_2 such that

$$h(x) = \begin{cases} -\alpha_1 & \text{if } x \in (x^c, x^c + \epsilon) \\ \alpha_2 & \text{if } x \in [x^c + \epsilon, \tilde{x}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{E}[h(X)] = 0$ and $w^* + h$ satisfies limited liability (2d) and remains nondecreasing. Note that $\mathbb{E}[h(X)] = 0$ implies

$$\alpha_1 \int_{x^c}^{x^c+\epsilon} f(x,a)dx = \alpha_2 \int_{x^c+\epsilon}^{\tilde{x}} f(x,a)dx. \tag{64}$$

As in previous proofs, we again argue that $w^* + h$ satisfies the IC constraint strictly for ϵ sufficiently small. It suffices to show $\int h(x)R(x)f(x,a)dx > 0$. Observe that

$$\begin{aligned} \int_{\underline{x}}^{\bar{x}} h(x)R(x)f(x,a)dx &= -\alpha_1 \int_{x_c}^{x_c+\epsilon} R(x)f(x,a)dx + \alpha_2 \int_{x_c+\epsilon}^{\tilde{x}} R(x)f(x,a)dx \\ &= \alpha_1 \left(-\int_{x_c}^{x_c+\epsilon} R(x)f(x,a)dx + \frac{\int_{x^c}^{x^c+\epsilon} f(x,a)dx}{\int_{x^c+\epsilon}^{\tilde{x}} f(x,a)dx} \int_{x_c+\epsilon}^{\tilde{x}} R(x)f(x,a)dx \right), \end{aligned}$$

where the second equality holds in light of (64). Since α_1 is strictly positive, $\int h(x)R(x)f(x,a)dx > 0$ holds if and only if

$$\frac{\int_{x^c+\epsilon}^{\tilde{x}} R(x)f(x,a)dx}{\int_{x^c+\epsilon}^{\tilde{x}} f(x,a)dx} > \frac{\int_{x^c}^{x^c+\epsilon} R(x)f(x,a)dx}{\int_{x^c+\epsilon}^{\tilde{x}} f(x,a)dx}. \tag{65}$$

Observe that as $\epsilon \to 0$, the left-hand side converges to $\mathbb{E}[R(X)|X \in (x^c, \tilde{x})]$ and the right-hand side converges to $R(x^c)$. Hence, under the condition of Case 1, (65) holds for ϵ sufficiently small, yielding our contradiction.

The above argument does not work to show that $R(x_{\bar{w}}) \leq \mathbb{E}[R(X)|X \in (x, \tilde{x})]$. However, assuming that $R(x_{\bar{w}}) > \mathbb{E}[R(X)|X \in (x_{\bar{w}}, \tilde{x})]$ and constructing, for every $\epsilon > 0$, a perturbation

$$h(x) = \begin{cases} \alpha_1 & \text{if } x \in (x_{\bar{w}} - \epsilon, x_{\bar{w}}) \\ -\alpha_2 & \text{if } x \in [x_{\bar{w}}, \tilde{x}) \\ 0 & \text{otherwise} \end{cases}$$

yields the same contradiction. Note that $w^* + h$ is a feasible perturbed contract. ■

Lemma 10 *There exists a $R^{\mathcal{J}}$ such that $R(x) = R^{\mathcal{J}}$ for every $x \in \mathcal{J}$.*

Proof. First, we argue that $R(x_i) = R(x_{i+1})$ for any two consecutive jumps $x_i, x_{i+1} \in \mathcal{J}$. The result then follows by applying to argument recursively.

Suppose $R(x_{i+1}) > R(x_i)$ (the case where $R(x_{i+1}) < R(x_i)$ can be argued analogously). For all $\epsilon > 0$ sufficiently small[10] there exists positive scalars α_1 and α_2 such that

$$h(x) = \begin{cases} -\alpha_1 & \text{if } x \in (x_i, x_i + \epsilon) \\ \alpha_2 & \text{if } x \in [x_{i+1} - \epsilon, x_{i+1}] \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{E}[h(X)] = 0$ and $w^* + h$ satisfies limited liability (2d) and remains nondecreasing. Following the same reasoning as Lemma 9 (surrounding (65)) that if we show

$$\frac{\int_{x_{i+1}-\epsilon}^{x_{i+1}} R(x) f(x,a) dx}{\int_{x_{i+1}-\epsilon}^{x_{i+1}} f(x,a) dx} > \frac{\int_{x_i}^{x_i+\epsilon} R(x) f(x,a) dx}{\int_{x_i}^{x_i+\epsilon} f(x,a) dx},$$

for sufficiently small ϵ then we derive a contradiction. This indeed holds by continuity, that the left-hand sides approaches $R(x_{i+1})$ and right-hand side approaches $R(x_i)$, and the supposition that $R(x_{i+1}) > R(x_i)$. ■

Based on the preceding two lemmas we can establish the following intermediate result.

Proposition 2 *There exists a piecewise constant contract with at most three pieces that is optimal to* (2) *with the additional monotonicity constraint* (34).

Proof. Let w^* be a monotone nondecreasing and piecewise-constant contract that is optimal to (2) with the additional monotonicity constraint (34), guaranteed to exist by Lemma 7. Consider any two consecutive constant pieces of w^* over respective intervals $[x_1, x_2)$ and $[x_2, x_3)$ with respective constant values $w_1 = w(x_1)$ and $w_2 = w(x_2)$, where $0 < w_1 < w_2 < \bar{w}$. We claim there exists a mean-preserving perturbation h such that $w^* + h$ remains optimal and takes a constant value on $[x_1, x_3)$. Iteratively repeating this argument then produces an optimal contract with at most three-pieces, one piece at value 0, one piece at $\bar{w}$, and one at a single intermediate value.

Since $x_1, x_2, x_3 \in \mathcal{J}$, by Lemma 9, we have $x_1, x_2, x_3 \in \mathcal{K}$. Moreover, $R(x_1) = R(x_2) = R(x_3)$ by Lemma 10. Thus,

$$\frac{\int_{x_1}^{\tilde{x}} R(x) f(x,a) dx}{\int_{x_1}^{\tilde{x}} f(x,a) dx} = \frac{\int_{x_2}^{\tilde{x}} R(x) f(x,a) dx}{\int_{x_2}^{\tilde{x}} f(x,a) dx} = \frac{\int_{x_3}^{\tilde{x}} R(x) f(x,a) dx}{\int_{x_3}^{\tilde{x}} f(x,a) dx} = R(x_1) = R(x_2) = R(x_3).$$

Therefore, we have (after some algebra)

$$\frac{\int_{x_1}^{x_2} R(x) f(x,a) dx}{\int_{x_1}^{x_2} f(x,a) dx} = \frac{\int_{x_2}^{x_3} R(x) f(x,a) dx}{\int_{x_2}^{x_3} f(x,a) dx}. \tag{66}$$

Then for any positive α_1 and α_2 such that

$$h(x) = \begin{cases} \alpha_1 & \text{if } x \in [x_1, x_2) \\ -\alpha_2 & \text{if } x \in [x_2, x_3] \\ 0 & \text{otherwise} \end{cases}$$

[10] More precisely, ϵ is chosen so that $x_i + \epsilon$ and $x_{i+1} - \epsilon$ both lie in the same "piece" of the domain as x_i.

satisfies $\mathbb{E}[h(X)] = 0$ and $w^* + h$ remains feasible, we have (by arguments similar to that found surrounding (65)) that $\int h(x)R(x)f(x,a)dx = 0$ if and only if

$$\frac{\int_{x_1}^{x_2} R(x)f(x,a)dx}{\int_{x_1}^{x_2} f(x,a)dx} = \frac{\int_{x_2}^{x_3} R(x)f(x,a)dx}{\int_{x_2}^{x_3} f(x,a)dx},$$

which is guaranteed by (66). Let $\hat{w} = (w(x_1) + w(x_2))/2$ and note that $\alpha_1 = \hat{w} - w(x_1)$ and $\alpha_2 = w(x_2) - \hat{w}$ is a legitimate choice of α_1 and α_2, and thus $w^* + h$ is optimal and constant with value $\hat{w}$ on $[x_1, x_3)$, as required. ■

To set notation, we parameterize the three-piece optimal contract described in Proposition 2. Outcome value q_1 denotes the first quota associated with the first bonus b_1. An even larger quota q_2 is associated with a total bonus of b_2. Our main result requires the following additional lemmas.

Lemma 11 *Let w^* a three-piece optimal solution to* (2) *with monotonicity constraint* (34) *parameterized by q_1, q_2, b_1, and b_2. (i) If $b_2 < \bar{w}$ then there exists an optimal quota-bonus contract. (ii) If $q_2 = \bar{x}$ then there exists an optimal quota-bonus contract.*

Proof. (i) If $b_2 < \bar{w}$ then the same reasoning as in the proof of Proposition 2 implies that a single piece can be constructed out of the two pieces at b_1 and b_2 to give an optimal quota-bonus contract. If (ii) holds then either $q_2 = \infty$ (if $\bar{x} = \infty$) in which case w^* already has only a single jump. If $q_2 = \bar{x} < \infty$ then redefine $w^*(q_2) = b_1$. Since this only changes a set of measure zero the resulting contract remains optimal and only has a single jump. ■

In light of Lemma 11 we may assume that $b_2 = \bar{w}$ and $q_2 < \bar{w}$ in the following analysis without loss of generality. With this condition we can specialize the reformulation (5) in the monotone case to the following:

$$\max_{q_1, b_1, q_2, \geq 0} \; b_1 \int_{x=q_1}^{q_2} R(x)f(x,a)dx + \bar{w} \int_{x=q_2}^{\bar{x}} R(x)f(x,a)dx \tag{67a}$$

$$\text{subject to } b_1 \int_{x=q_1}^{q_2} f(x,a)dx + \bar{w} \int_{x=q_2}^{\bar{x}} f(x,a)dx = \mathbb{E}[w^*(X)] \tag{67b}$$

$$b_1 \leq \bar{w}. \tag{67c}$$

Observe that constraint (67c) ensures monotonicity. This optimization problem in three variables (q_1, b_1 and q_2) will be leveraged to show that there exists a three-piece optimal contract with $q_1 = q_2$; that is, in fact, there exists an optimal quota-bonus contract.

Lemma 12 *There exists an optimal three-piece nondecreasing contract w^* to* (2) *with monotonicity constraint* (34) *such that $R(x)$ is increasing in a neighborhood of q_2, except possibly at q_2 itself where it remains possible that $R'(q_2) = 0$.*

Proof. We prove by contradiction. First, suppose $R'(q_2) < 0$ where the contract w^* jumps from value w_1 to w_2 with $w_1 < w_2$. Let $\hat{w} = (w_1 + w_2)/2$. Then a sufficiently small $\epsilon > 0$ can be chosen so that $R(q_2 - \epsilon) > R(q_2) > R(q_2 + \epsilon)$ and

$$h(x) = \begin{cases} \hat{w} - w_1 & \text{if } x \in (q_2 - \epsilon, q_2) \\ \hat{w} - w_2 & \text{if } x \in [q_2, q_2 + \epsilon) \\ 0 & \text{otherwise} \end{cases}$$

is such that $\mathbb{E}[h(X)] = 0$ and $0 \leq w^* + h \leq \bar{w}$. Then,

$$\begin{aligned}\int h(x)R(x)f(x,a)dx &= \int_{q_2-\epsilon}^{q_2} h(x)R(x)f(x,a)dx + \int_{q_2}^{q_2+\epsilon} h(x)R(x)f(x,a)dx \\ &> R(q_2)\int_{q_2-\epsilon}^{q_2} h(x)f(x,a)dx + R(q_2)\int_{q_2}^{q_2+\epsilon} h(x)f(x,a)dx \\ &= R(q_2)\int h(x)f(x,a)dx = 0.\end{aligned}$$

This provides a contradiction of the optimality of w^*.

Next, suppose $R'(x) = 0$ for all $x \in (q_2, q_2 + t)$ where $t > 0$ (a similar argument holds for if $R'(x) = 0$ on an interval to the left of $\hat{x}$). Define the mean-preserving perturbation

$$h(x) = \begin{cases} \epsilon_1 > 0 & \text{if } x \in (q_1, q_2) \\ \epsilon_2 < 0 & \text{if } x \in [q_2, q_2 + t) \\ 0 & \text{otherwise,} \end{cases}$$

in which case $\epsilon_2 = \frac{\int_{q_1}^{q_2} f(x,a)dx}{\int_{q_2}^{q_2+t} f(x,a)dx}\epsilon_1$. Moreover, the covariance of the perturbation is

$$\begin{aligned}&\int h(x)R(x)f(x,a)dx \\ &= \epsilon_1 \int_{x^c}^{x^K} R(x)f(x,a)dx - \epsilon_1 \frac{\int_{x^c}^{x^K} f(x,a)dx}{\int_{x^K}^{x^K+t} f(x,a)dx}\int_{x^K}^{x^K+t} R(x)f(x,a)dx \\ &= \epsilon_1 R(x^K)\int_{x^c}^{x^K} f(x,a)dx - \epsilon_1 R(x^K)\frac{\int_{x^c}^{x^K} f(x,a)dx}{\int_{x^K}^{x^K+t} f(x,a)dx}\int_{x^K}^{x^K+t} f(x,a)dx = 0.\end{aligned}$$

This implies that the contract $w^* + h$ is also an optimal contract since it is an optimal solution to (67). Now, we take t sufficiently large so that $R'(q_2 + t) > 0$. If no such t exists then in fact we can find an optimal quota-bonus contract by taking $q_2 = \bar{x}$, a case we eliminated via Lemma 11. Hence we may assume such a t exists and thus clearly $w^* + h$ has the property expressed in the theorem statement. ■

Proof of Theorem 5. By Proposition 2, there exists an optimal three-piece contract w^* with parameters q_1, b_1 and q_2 that solve (67) (by Lemma 11 we have $b_2 = \bar{w}$). Using the equality (67b) to eliminate b_1 in (67a) and (67c) yields an equivalent problem in the decision variables q_1 and q_2:

$$\max_{q_1,q_2,\geq 0} \frac{\int_{q_1}^{q_2} R(x)f(x,a)dx}{\int_{q_1}^{q_2} f(x,a)dx}(\mathbb{E}[w^*(X)] - \bar{w}\int_{q_2}^{\bar{x}} f(x,a)dx) + \bar{w}\int_{q_2}^{\bar{x}} R(x)f(x,a)dx \tag{68a}$$

$$\text{subject to} \quad \bar{w}\int_{q_1}^{q_2} f(x,a)dx \geq \mathbb{E}[w^*(X)]. \tag{68b}$$

If (68b) is a tight constraint then $b_1 = \bar{w}$ and we are done. Hence we may suppose that (68b) is slack and hence $b_1 < \bar{w}$. We further suppose, by way of contradiction, that $q_1^* < q_2^*$. The contradiction comes in the form of showing that q_2^* in this case is ill-defined.

Since (68b) is slack, every optimal solution (q_1^*, q_2^*) to (68) is an optimal solution to the unconstrained problem (68a).[11] Moreover, observe that $q_1^* \in \arg\max_x \mathbb{E}[R(X)|X \in (x, q_2^*)]$ when (68b)

[11] This argument uses the fact that a Lagrangian multiplier exists for constraint (68b). However, the problem is finite dimensional in the two decision variables q_1 and q_2 and has a single constraint. Therefore, a Lagrange multiplier is certain to exist.

is not binding. Thus we may re-expressive the objective function (68a) as

$$\left(\max_x \mathbb{E}[R(X)|X \in (x, q_2)]\right) \left(\mathbb{E}[w^*(X)] - \bar{w} \int_{q_2}^{\bar{x}} f(x,a)dx\right) + \bar{w} \int_{q_2}^{\bar{x}} R(x) f(x,a)dx, \tag{69}$$

which we have assumed is maximized with respect to q_2 at q_2^*. We argue that this is a contradiction by showing, perversely, that (69) is strictly increasing at q_2^*. To do this, we use a perturbation argument and examine the right derivative of this objective (from the Theorem of Maximum, (69) is almost everywhere differentiable in q_2). Taking the right derivative of (69) with respect to q_2 yields, using the product rule,

$$\begin{aligned} \tfrac{\partial^+}{\partial q_2}(\max_x \mathbb{E}[R(X)|X \in (x, q_2)])(\mathbb{E}[w^*(X)] - \bar{w} \int_{q_2}^{\bar{x}} f(x,a)dx) \\ + (\max_x \mathbb{E}[R(X)|X \in (x, q_2)] - R(q_2))\bar{w} f(q_2, a). \end{aligned} \tag{70}$$

We argue that this expression is strictly positive at q_2^*.

We have already assumed that $\mathbb{E}[w^*(X)] - \bar{w} \int_{q_2^*}^{\bar{x}} f(x,a)dx > 0$. To analyze the second term we make the following observation. Since $w^*(q_2^*) = \bar{w}$ we have $\tilde{x} = x_{\underline{w}} = q_2$ in (62) and so Lemma 9 implies

$$R(q_1^*) = \mathbb{E}[R(X)|X \in (q_1^*, q_2^*)]. \tag{71}$$

Also, by Lemma 10, $R(q_1^*) = R(q_2^*)$ and so from (71) we can conclude $\mathbb{E}[R(X)|X \in (q_1^*, q_2^*)] = R(q_2^*)$. Hence

$$\max_x \mathbb{E}[R(X)|X \in (x, q_2^*)] \geq \mathbb{E}[R(X)|X \in (q_1^*, q_2^*)] = R(q_2^*),$$

and so the second term in (70) is nonnegative.

It remains to show that $\max_x \mathbb{E}[R(X)|X \in (x, q_2)]$ is strictly increasing at q_2^*. Let $t > 0$ be such that $q_2^* + t$ is in the neighborhood guaranteed in Lemma 12 where $R(x)$ is increasing. Now, we have

$$\begin{aligned} \max_x \mathbb{E}[R(X)|X \in (x, q_2^* + t)] &\geq \mathbb{E}[R(X)|X \in (q_1^*, q_2^* + t)] \\ &= \frac{\int_{q_1^*}^{q_2^*} R(x)f(x,a)dx + \int_{q_2^*}^{q_2^*+t} R(x)f(x,a)dx}{\int_{q_1^*}^{q_2^*} f(x,a)dx + \int_{q_2^*}^{q_2^*+t} f(x,a)dx} \\ &> \frac{\int_{q_1^*}^{q_2^*} R(x)f(x,a)dx}{\int_{q_1^*}^{q_2^*} f(x,a)dx} \\ &= \max_x \mathbb{E}[R(X)|X \in (x, q_2^*)], \end{aligned}$$

where the first equality uses the definition of maximum when evaluating x at q_1^*, the second equality is by definition, the inequality holds since

$$\frac{\int_{q_2^*}^{q_2^*+t} R(x)f(x,a)dx}{\int_{q_2^*}^{q_2^*+t} f(x,a)dx} > R(q_2^*) = \mathbb{E}[R(X)|X \in (q_1^*, q_2^*)] = \frac{\int_{q_1^*}^{q_2^*} R(x)f(x,a)dx}{\int_{q_1^*}^{q_2^*} f(x,a)dx},$$

using (71), and the final equality holds $q_1^* \in \arg\max_x \mathbb{E}[R(X)|X \in (x, q_2^*)]$. Thus, $\max_x \mathbb{E}[R(X)|X \in (x, q_2)]$ is strictly increasing at q_2^*, completing the proof. ■

A.9 Proof of Lemma 8

We use the following lemma.

Lemma 13 *If $R(\bar{x}) = \sup_x R(x)$ then there exists an optimal quota-bonus contract with bonus $\bar{w}$ for the binary-action problem* (2) *with monotonicity constraint* (34)*.*

Proof. Let w^* be an arbitrary optimal quota-bonus contract with quota q and bonus b and suppose $b < \bar{w}$. Since $R(\bar{x}) = \sup_x R(x)$ and R is an almost everywhere differentiable function, for all possible choices of quota q there exists a region $(x_1, x_2]$ with $q < x_1 < x_2 \leq \bar{x}$ in which $R(x)$ is strictly increasing.[12] A straightforward adjustment of the proof of Lemma 12 also shows that $R(x)$ must be increasing in a neighborhood of q and thus we can conclude that $R(x_2) > R(q)$. Now, consider the following perturbation

$$h(x) = \begin{cases} -\alpha_1 & \text{if } x \in (q, x_1) \\ \alpha_2 & \text{if } x \in [x_1, \bar{x}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_1 \in (0, b)$ and $\alpha_2 \in (0, \bar{w} - b)$ are chosen so that $\mathbb{E}[h(X)] = 0$. However, the covariance is increased since

$$-\alpha_1 \frac{\int_q^{x_1} R(x) f(x,a) dx}{\int_q^{x_1} f(x,a) dx} + \alpha_1 \frac{\int_{x_1}^{\bar{x}} R(x) f(x,a) dx}{\int_{x_1}^{\bar{x}} f(x,a) dx} > 0$$

as $\frac{\int_{x_1}^{\bar{x}} R(x) f(x,a) dx}{\int_{x_1}^{\bar{x}} f(x,a) dx} > R(q) > \frac{\int_q^{x_1} R(x) f(x,a) dx}{\int_q^{x_1} f(x,a) dx}$ since $R(q) = \mathbb{E}[R(X) | X \in (q, \bar{x})]$ by Lemma 9 and $R(x_1) > R(q)$. ■

With the lemma in hand the rest of the proof of Theorem 6 is straightforward. Indeed, HRD implies that $R(x) \leq \mathbb{E}[R(X) | X \in (x, \bar{x})]$ for all x and consequently $R(\bar{x}) = \sup_x R(x)$. Hence, the conditions of Lemma 13 hold, implying that there exists an optimal contract with bonus equal to $\bar{w}$ to the binary-action problem (2) for any choice of actions a and $\hat{a}$ (with $\hat{a} < a$).

A.10 Proof of Theorem 7

Following earlier arguments Section 5.2, it suffices to show

$$\max_{\hat{a}} \min_q \frac{c(a) - c(\hat{a})}{F(q, \hat{a}) - F(q, a)} (1 - F(q, a)) = \min_q \max_{\hat{a}} \frac{c(a) - c(\hat{a})}{F(q, \hat{a}) - F(q, a)} (1 - F(q, a)). \tag{72}$$

where we have removed the individual rationality constraint, since as previously argued it is always slack at the optimal solution. We now provide a sufficient condition based on a fixed-point argument. If there is a unique minimizer

$$q(\hat{a}) \in \arg\min_q \frac{c(a) - c(\hat{a})}{F(q, \hat{a}) - F(q, a)} (1 - F(q, a))$$

and a unique maximizer

$$\hat{a}(q) \in \arg\max_{\hat{a}} \frac{c(a) - c(\hat{a})}{F(q, \hat{a}) - F(q, a)},$$

[12] There is one small technical detail with the possibility that $\bar{x}$ is not an isolated maximizer. If $\bar{x}$ is finite the argument can easily be adjusted. If $\bar{x}$ is infinite, there cannot exist an interval of maximizers ending in $\bar{x}$ since this would violate the condition that $\int R(x) f(x,a) dx = 0$.

then the mappings $q(\hat{a})$ and $\hat{a}(q)$ are continuous by the Theorem of Maximum (and hence so is the composition $\hat{a} \circ q$. Then by Brouwer's fixed point theorem there is a fixed point $\hat{a}^* \in \hat{a}(q(\hat{a}^*))$ such that (72) holds at $\hat{a}^*$ and $q^* := q(\hat{a}^*)$.

We now show that the sufficient condition holds. We first show that $\frac{1-F(q,a)}{F(q,\hat{a})-F(q,a)}$has a unique minimizer (note that $c(a) - c(\hat{a})$ is constant with respect to q). By taking derivative w.r.t. q we obtain:

$$
\begin{aligned}
&\tfrac{(1-F(q,a))}{F(q,\hat{a})-F(q,a)}\big(-\tfrac{f(q,a)}{1-F(q,a)} - \tfrac{f(q,\hat{a})-f(q,a)}{F(q,\hat{a})-F(q,a)}\big)\\
&= \tfrac{f(q,a)}{F(q,\hat{a})-F(q,a)}\big(-1 - \tfrac{(1-F(q,a))}{F(q,\hat{a})-F(q,a)}\tfrac{f(q,\hat{a})-f(q,a)}{f(q,a)}\big)\\
&= \tfrac{f(q,a)}{F(q,\hat{a})-F(q,a)}\big(-1 - \tfrac{(1-F(q,a))}{F(q,\hat{a})-F(q,a)}\big(-1 + \tfrac{f(q,\hat{a})}{f(q,a)}\big)\big)\\
&= \tfrac{(1-F(q,\hat{a}))(1-F(q,a))}{[F(q,\hat{a})-F(q,a)]^2}\big(\tfrac{f(q,a)}{(1-F(q,a))} - \tfrac{f(q,\hat{a})}{(1-F(q,\hat{a}))}\big)
\end{aligned}
$$

Hence, the sign of the derivative depends on the sign of $\frac{f(q,a)}{(1-F(q,a))} - \frac{f(q,\hat{a})}{(1-F(q,\hat{a}))}$. Thus, by the single-crossing hazard rate assumption this function crosses the x-axis from above only once and so there is a unique maximizer.

Next, we justify that $\hat{a}(q) \in \arg\max_{\hat{a}} \frac{c(a)-c(\hat{a})}{F(q,\hat{a})-F(q,a)}$ is also unique. This is by the CDFC and convexity of c assumptions, which implies by standard reasoning $\frac{c(a)-c(\hat{a})}{F(q,\hat{a})-F(q,a)}$ is only maximized when $\hat{a} \to a$, that is $\frac{c(a)-c(\hat{a})}{F(q,\hat{a})-F(q,a)} \to -\frac{c'(a)}{F_a(q,a)}$. Therefore, we can apply the fixed-point theorem to show the equivalence.

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