Incentivized actions in freemium games

Mahesh Nagarajan[*]   Christopher Thomas Ryan[†]   Lifei Sheng[‡]

Yuan Cheng [§]   Chunyang Tong[¶]

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Abstract

We explore the phenomena of game companies offering to pay users in "virtual" benefits to take actions in-game that earn the game company revenue from third parties. Examples of such "incentivized actions" include paying users in "gold coins" to watch video advertising or speeding in-game progression in exchange for filling out a survey. These are common practices in mobile games that use a freemium business model, where users download and play for free and only a relatively small percentage of total users pay out-of-pocket when playing the game.

We develop a dynamic optimization model that looks at the costs and benefits of offering incentivized actions to users as they progress in their engagement with the game. We find sufficient conditions for the optimality of a threshold strategy of offering incentivized actions to low-engagement users and then removing incentivized actions to encourage real-money purchases once a player is sufficiently engaged. Our model also provides insights into what types of games can most benefit from offering incentivized actions. For instance, our analysis suggests that social games with strong network effects have more to gain from offering incentivized actions than solitary games.

1   Introduction

Games represent the fastest growing sector of the entertainment industry globally, which includes music, movies, and print publishing (McKinsey, 2013). Moreover, the online/mobile space is the fastest growing segment of games, which is itself dominated by games employing a "freemium" business model. Freemium games are free to download and play and earn revenue through advertising or selling game enhancements to dedicated players. When accessed on 23 April 2015, Apple Inc.'s App Store showed 190 out of the 200 top revenue-generating games (and all of the top 20) were free to download.[1] On Google Play, the other major mobile games platform, 297 out of the 300 top revenue-generating games were freemium.[2] Moreover, games are the dominant revenue generators

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[*]Sauder School of Business, University of British Columbia, E-mail: mahesh.nagarajan@sauder.ubc.ca

[†]Booth School of Business, University of Chicago, E-mail: chris.ryan@chicagobooth.edu

[‡]College of Business, University of Houston-Clear Lake, E-mail: fay.sheng@sauder.ubc.ca

[§]School of Economics and Management, Tsinghua University, E-mail: chengyn@sem.tsinghua.edu.cn

[¶]Shanghai University of Finance and Economics

in the global app market. Revenue from mobile games accounts for 79 percent of total app revenue on Apple's App Store and 92 percent of revenue on Google Play.[3]

The concept behind freemium is to attract large pools of players, many of whom might never make an in-app purchase. Players that do pay out of pocket are said to *monetize*. In general, successful games have a monetization rate of between 2 and 10 percent, with the average much closer to 2 percent.[4] When game publishers cannot earn directly from the pockets of consumers, they turn to other sources of revenue. This is largely through third parties who pay for delivering advertising content and having players download other apps, fill out surveys or apply for services, such as credit cards. This stream of revenue is less lucrative per conversion than in-app purchases. Typically, delivering a video earns pennies on the dollar compared to in-app purchases.

However, players can become irritated by advertising, especially when it interrupts the flow or breaks the fiction of a game. A recent innovation is to offer "incentives" for players to click on a banner ad, watch a video or fill out a survey. These are collectively called *incentivized actions*, or as it is commonly shortened, *incented actions*. To get a clearer sense of how incented actions work we examine a concrete example.

Candy Crush Saga is a puzzle game, published by King. King was recently acquired by Activision-Blizzard for 5.9 billion USD based on the enduring popularity of *Candy Crush Saga* and its portfolio of successful games.[5] In *Candy Crush Saga*, a player attempts to solve a progression of increasingly challenging puzzles. At the higher levels, players get stuck for extended periods on a single puzzle. Player progression is further hindered by a "lives" mechanic where each failed attempt at a puzzle consumes one of at most five total lives. Lives are regenerated either through waiting long periods of real-time or by purchasing additional lives with real money. Players may also buy items that enhance their chances of completing a puzzle.

Early versions of *Candy Crush Saga* had incented actions, including advertising. A player could take an incented actions to earn lives or items without using real money. However, in June of 2013, six months after *Candy Crush Saga* was launched on Apple iOS, King decided to drop all forms of in-game advertising in the game.[6] King's choice was surprising to many observers. What was the logic for removing a potential revenue stream? How did this move affect the monetization rate?

This raises other more tactical questions: when in the lifetime of a player is it best to offer incented actions? Should some players be offered incented actions and others not? Finally, there are related strategic questions: what types of games are best suited to offering incented actions?

In this paper, we present an analytical model to explore the use of incented actions and attempt to answer the above questions. In particular, we are interested in a game publisher's decision of when to offer incented actions to players, and when to remove this option. Our model emphasizes the connection of incented actions to two other useful concepts often discussed in the game industry – engagement and retention. Highly engaged players are more likely to make in-app purchases and less likely to quit. The longer a player is retained in the game, the more likely they are to become engaged and monetize. Analytically, player engagement levels are modeled as states in a Markov

chain and retention is the time a player stays in the system before being absorbed into a "quit" state. The more engaged a player, the more likely they are retained.

The concept of engagement is common to both games industry practitioners and academics who study the video game industry. Although all observers emphasize the importance of measuring player engagement, there is little consensus on how to measure it. Among practitioners (see, for instance, Lovell (2012)) a common measurement is the ratio of daily active users (DAU) over monthly active users (MAU). DAU measures the average number of users who play the game at least once per day, and MAU measures the average number of users who play the game at least once per month. This aggregate measure says relatively little about the behavior of an individual player. However, it is easy to calculate for publicly traded games companies that typically publish their DAU and MAU numbers and thus can be used as a benchmark. It also has the virtue of being universal across many game designs where the mechanics of the game (being a puzzle, sports or adventure game) may otherwise have little in common.

There is also substantial research on engagement in academic video game development journals. A recent study by Abbasi et al. (2017) describes a variety of quantitative and qualitative measurements of engagement including notions of immersion, flow, presence, etc. These are typically measured via questionnaires (such as the game-experience questionnaire (GEQ) proposed in Jennett et al. (2008)). This notion of engagement focuses on the psychological and cognitive state of the player and thus presents challenges to operationalize from the game designer's perspective.

Another view of engagement connects to notions of progress made by the player in the game. Players can invest significant time and resources to reach higher "levels" of progression. It can be difficult for players to walk away from playing a game they have heavily invested in. Thus, a simple measure of engagement is the extent to which a player has invested her energy in a game. In discussions with game developers, the primary focus of game design is to improve *retention*, with the empirically well-founded belief that the longer the relationship a player has with a game, the greater their willingness to spend.[7] The association of engagement with levels has the benefit of having both a psychological component and also be based on measurable game data. Progression suggests an investment of time and achievement of mastery that correlate well with psychological notions of engagement. On the other hand, progression can often be directly measured. For games like *Candy Crush Saga*, there are concrete and distinct *levels* that are reached by the player. In *Candy Crush Saga*, the current level is the number of puzzles completed by the player.

Not every game has a clear level-based design. For such games, other measures of engagement are typically employed. In Section 5 we analyze a proprietary data set for the game *Survival of Primitive* (or *Primitive*, for short) developed by Ebo games. This game has a less linear notion of progression than *Candy Crush*, where "survival" depends on a variety of resource collection activities with many possible paths towards success. In this setting, we abandon the "level"-based notion of engagement and adopt Ebo's measure of engagement based on the duration of daily play.

Our results. Our results come in two forms; first, analytical results based on a model with a level-based notion of engagement (inspire by *Candy Crush Saga* and second, a data-driven of the optimality of incented actions using our proprietary dataset for *Primitive*. Interestingly, both our analytical and numerical results share a common theme: an optimal deployment of incented actions (under a broad set of data specifications) is to offer incented actions to players with low engagement until they reach a threshold level of engagement, after which incented actions are removed. We also show how the optimal threshold level depends on changes in game parameters.

Our analytical results depend on a careful analysis of three main effects of incented actions. These effects are described in with greater precision below, but we mention them here at a conceptual level. First is the *revenue effect*. By offering incented actions, game publishers open up another channel of revenue. However, the net revenue of offering incented may nonetheless be *negative* if one accounts for the opportunity costs of players not making in-app purchases. This captures the possibility that a player would have made an in-app purchase if an incented action was not available. Second, the *retention effect* measures how effective an incented action is at keeping players from quitting. In other words, incented actions can delay a player's decision to quit the game. Third, the *progression effect* refers to the effectiveness of an incented action in deepening the engagement level of the player. It refers to an incented actions ability to increase the player's attachment to the game. These three effects are intuitively understood by game developers and the topic of much discussion and debate in the gaming industry.

Gaming companies grapple with the issue of understanding how these effects interact with each other in the context of specific games. As we shall see in concrete examples below, all three effects can act to either improve or erode the overall revenue available to the publisher. Each effect is connected and often move in similar directions as players progress. Part of our analysis is to describe situations where the effects move in different, sometimes counter-intuitive, directions.

We can analytically characterize each effect, allowing us to gain insights into how to optimally design a policy for offering incented actions. To understand the interactions between these effects and to capture the dynamics in a game, we use *Markov chains* to model player engagement and how they transition from one level of engagement to another. Then, using a Markov Decision Process (MDP) model we study the effect of specific decisions or policies of the game publisher. For example, we provide sufficient conditions for when a *threshold* policy is optimal. In a threshold policy incented actions are offered until a player reaches a target engagement level, after which incented actions are removed. The intuition of these policies is clear. By offering incented actions, the retention effect and progression effect keep the player in for longer by providing a non-monetizing option for progression. However, once a player is sufficiently engaged, the revenue effect becomes less beneficial and the retention effect less significant because highly engaged players are more likely to buy in-app purchases and keep playing the game. This suggests that it is optimal to remove incented actions and attempt to extract revenue directly from the player through monetization. Our sufficient conditions justify this logic, but we also explore settings where this basic intuition

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breaks down. For instance, it is possible that the retention effect remains a dominant concern even at higher engagement levels. Indeed, a highly engaged player may be quite likely to monetize, and so there is a strong desire on the part of the publisher to keep the player in the system for longer by offering incented actions to bolster retention.

The relative strengths of these three effects depend on the characteristics the game, including all the parameters in our MDP model. We examine this dependence by tracking how the threshold in an optimal threshold policy changes with the parameters. This analysis provides insights into the nature of optimal incented action policies.

For instance, we show analytically that the more able players are at attracting their friends into playing the game, the greater should be the threshold for offering incented actions. This suggests that social games that include player interaction as part of their design should offer incented actions more broadly, particularly when the retention effect is strongly positive since keeping players in the game for longer gives them more opportunities to invite friends. Indeed, a common incented action is to contact friends in your social network or to build a social network to earn in-game rewards. This managerial insight can assist game publishers in targeting what types of games in a portfolio of game projects can take the most advantage of delivering incented actions.

We also discuss different effects of the design of incented actions, in particular, their "strength" at attracting and engaging players. "Strength" here refers to how powerful the reward of the incented action is in the game. For instance, this could be the number of "coins" given to the player when an incented action is taken. If this reward is powerful, in comparison to in-app purchases, then it can help players progress, strengthening the progression effect. On the other hand, a stronger incented action may dissuade players further from monetizing, strengthening cannibalization. Through numerical examples, we illustrate a variety of possible effects that tradeoff the behavioral effects of players responding to the nature of the incented action reward and show that whether or not to offer incented actions to highly engaged players depends in a nonmonotonic way on the parameters of our model that indicate the strength of incented actions.

Finally, we analyze the data we gathered on *Survival of Primitive* from our industry partner. This game does not fully fit the main analytical setting explored in the previous settings because Primitive does not have a level-based design. Nonetheless, we use this data to calibrate a Markov Decision process to compute optimal policies. Interestingly, the optimality of threshold policies persists under almost all of our simulated values for parameters we could not clearly define using the provided data. Moreover, the sensitivity of the threshold to changes in various game parameters also follows the pattern predicted by our analytical results. Another way to interpret these findings is that the main intuition of our analytical findings is supported by a robustness check using data from a real game that satisfy a more general set of assumptions. Much of this was shared with our partner and the main insights from our analysis guided a their subsequent data collection and design policies.

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2 Related literature

As freemium business models have grown in prominence, so has interest in studying various aspects of freemium in the management literature. While papers in the marketing literature on freemium business models has been largely empirical (see for instance Gupta et al. (2009) and Lee et al. (2017)), our work connects most directly to a stream of analytical studies in the information systems literature that explores how "free" is used in the software industry. Two important papers for our context are Niculescu and Wu (2014) and Cheng et al. (2015) that together establish a taxonomy of different freemium strategies and examine in what situations a given strategy is most advantageous. *Seeding* is a strategy where some are given away entirely for free, to build a user base that attracts new users through word-of-mouth and network effects. Previous studies explored the seeding strategy by adapting the Bass model (Bass, 1969) to the software setting (see for instance Jiang and Sarkar (2009)). Another strategy is *time-limited freemium* where all users are given access to a complete product for a limited time, after which access is restricted (see Cheng and Liu (2012) for more details). The *feature-limited freemium* category best fits our setting, where a functional base product can always be accessed by users, with additional features available for purchase by users. In freemium mobile games, a base game is available freely for download with additional items and features for sale through accumulated virtual currency or real-money purchases.

Our work departs from this established literature in at least two dimensions. First, we focus on how to tactically implement a freemium strategy, in particular, when and how to offer incented actions to drive player retention and monetization. By contrast, the existing literature has largely focused on comparing different freemium strategies and their advantage over conventional software sales. This previous work is, of course, essential to understanding the business case for freemium. Our work contributes to a layer of tactical questions of interest to firms committed to a freemium strategy in search of additional insights into its deployment.

Second, games present a specific context that may be at odds with some common conceptualizations of a freemium software product. For a productivity-focused product, such as a PDF editor, a typical implementation of freemium is to put certain advanced features behind a pay-wall, such as the ability to make handwritten edits on files using a stylus. Once purchased, features are typically unlocked either in perpetuity or for a fixed duration by the paying player. By contrast, virtual items or currency that may enhance the in-game experience, speed progression, or provide some competitive advantage are typically purchased in games. These purchases are often *consumables*, meaning that they are depleted through use. This is true, for instance, of all purchases in *Candy Crush Saga*. Our model allows for a player to make repeated purchases and the degree of intensity of monetization to evolve over the course of play.

Other researchers have examined the specific context offered by games, as opposed to general software products, and have adapted specialized theory to this specific context. Guo et al. (2016) examine how the sale of virtual currencies in digital games can create a win-win scenario for players

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and publishers from a social welfare perspective. They make a strong case for the value created by games offering virtual currency systems. Our work adds a layer by examining how virtual currencies can be used to incentivize players to take actions that are profitable to the firm that does not involve a real-money exchange. A third-party, such as an advertiser, can create a mutually beneficial situation where the player earns additional virtual currency, the publisher earns revenue from the advertiser, and the advertiser promotes their product. Also, Guo et al. (2016) develop a static model where players decide on how to allocate a budget between play and purchasing virtual currency. We relate a player's willingness to take incented actions or monetize as their engagement with the game evolves, necessitating the use of a dynamic model. This allows us to explore how a freemium design can respond to the actions of players over time. This idea of progression in games has been explored empirically in Albuquerque and Nevskaya (2012). We adapt similar notions to derive analytical insights in our setting.

The dynamic nature of our model also shares similarities with threads of the vast customer relationship management (CRM) literature in marketing. In this literature, researchers are interested in how firms balance acquisition, retention, and monetization of players through the pricing and design of their product or service over time. For example, Libai et al. (2009) adapt Bass's model to the diffusion of services where player retention is an essential ingredient in the spread of the popularity of a platform. Fruchter and Sigué (2013) provide insight into how a service can be priced to maximize revenue over its lifespan. Both studies employ continuous-time and continuous-state models that are well-suited to examine the overall flow of player population. Our focus of analysis is at the player level and asks how to design the game (i.e., service) to balance retention and monetization through offering incented actions for a given acquired player. Indeed, game designs on mobile platforms can, in principle, be specialized down to a specific player. With the increasing availability of individual player level data, examination of how to tailor design with more granularity is worthy of exploration. By contrast, existing continuous models treat a single player's choice with measure zero significance.

Finally, our modeling approach of using a discrete time Markov decision process model in search of threshold policies is a standard-bearer of analysis in the operations management literature. We have mentioned the advantages of this approach earlier. Threshold policies, which we work to establish, have the benefit of being easily implementable and thus draw favor in studies of tactical decision-making that is common in multiple areas including the economics and operations management literature. The intuition for their ease of use is somewhat easy to understand. The simplest type of threshold policies allows the system designer to simply keep track of nothing but the threshold (target) level and monitor the state of the system and take the appropriate action to reap the benefits of optimality. This is in contrast to situations where the optimal policy can be complex and has nontrivial state and parameter dependencies. Examples of effective use of this approach in dynamic settings include inventory and capacity management and control (Zipkin, 2000) and revenue management (Talluri and Van Ryzin, 2006).

3 Model

We take the perspective of a game publisher who is deciding how to optimally deploy incented actions. Incented actions can be offered (or not) at different times during a player's experience with the game. For example, a novice player may be able to watch video ads for rewards during the first few hours of gameplay, only later to have this option removed.

Our model has two agents: the game publisher and a single player. This assumes that the game publisher can offer a customized policy to each player, or at least customized policies to different classes of players. In other words, the "player" in our model can be seen as the representative of a class of players who behave similarly. The publisher may need to decide on several different policies for different classes of players for an overall optimal design.

We assume that the player behaves stochastically according to the options presented to her by the game publisher. The player model is a Markov chain with engagement level as the state variable. The hope is that this model will allow for many personal interpretations of what "engagement" specifically means. We do not model down to the specifics of a particular game and instead provide what we feel is a robust approach to engagement. The game publisher's decision problem is a Markov Decision Problem (MDP) where the stochasticity is a function of the underlying player model, and the publisher's decision is whether or not to offer incented actions. The player model is described in detail in the next subsection. The publisher's problem is detailed in Section 3.2.

3.1 Player model. The player can take three actions while playing the game. The first is to *monetize* (denoted M) by making an in-app purchase with real money. The second is to *quit* (denoted Q). Once a player takes the quit action, she never returns to playing the game. Third, the player can take an *incented action* (denoted I). The set of available actions is determined by whether the publisher offers an incented action or not. We let $A_1 = \{M, I, Q\}$ denote the set of available actions when an incented action is offered and $A_0 = \{M, Q\}$ otherwise.

The probability that the player takes a particular action depends on her engagement level (or simply *level*). These levels form the states of the Markov Chain. The set E of engagement levels is a discrete set (possibly countable), while -1 denotes a "quit" state where the player no longer plays the game. That is, the quit state is an absorbing state. The probability that the player takes a particular action also depends on what actions are available to her. We used the letter "p" to denote probabilities when an incented action is available and write $p_a(e)$ to denote the probability of taking action $a \in A_1$ at level $e \in E$. For example, $p_M(2)$ is the probability of monetizing at level 2 while $p_I(0)$ is the probability of taking an incented action at level 0. We use the letter "q" to denote action probabilities when the incented action is unavailable and write $q_a(e)$ for the probability of taking action $a \in A_0$ at level $e \in E$. By definition $p_M(e) + p_I(e) + p_Q(e) = 1$ and $q_M(e) + q_Q(e) = 1$ for all $e \in E$.

There is a relationship between $p_a(e)$ and $q_a(e)$. When an incentivized action is not available the probability $p_I(e)$ is allocated to the remaining two actions M and Q. For each $e \in E$ we assume

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that there exists a parameter $\alpha(e) \in [0,1]$ such that:

$$q_M(e) = p_M(e) + \alpha(e)p_I(e) \tag{1}$$

$$q_Q(e) = p_Q(e) + (1 - \alpha(e))p_I(e). \tag{2}$$

We call $\alpha(e)$ the *cannibalization parameter at level* e, since $\alpha(e)$ measures the impact of removing an incented action on the probability of monetizing and thus captures the degree to which incented actions cannibalize demand for in-app purchases. A large $\alpha(e)$ (close to 1) implies strong cannibalization whereas a small $\alpha(e)$ (close to 0) signifies weak cannibalization.

It remains to consider how a player transitions from one level to another. We must first describe the time epochs where actions and transitions take place. The decision epochs where actions are undertaken occur when the player is assessing whether or not they want to continue playing the game. The real elapsed time between decision epochs is not constant since it depends on the behavior of the player between sessions of play. Some players frequently play, others play only for a few minutes per day. A player might be highly engaged but have little time to play due to other life obligations. This reality suggests that the elapsed time between decision epochs should not be a critical factor in our model. We denote the level at decision epoch t by e_t and the action at decision epoch t by a_t.

Returning to the question of transitioning from level to level, in principle we would need to determine individually each transition probability $\mathbb{P}(e_{t+1} = e'|e_t = e$ and $a_t = a)$. For actions $a \in \{M, I\}$, we will assume that transition probabilities are stationary and set $\mathbb{P}(e_{t+1} = e'|e_t = e$ and $a_t = a) = \tau_a(e'|e)$ for all times t, where τ is a $[0,1]$-valued function such that $\sum_{e' \in E} \tau_a(e'|a) = 1$ for all $e \in E$ and $a \in \{M, I\}$. For the quit action, $\mathbb{P}(e_{t+1} = -1|e_t = e'$ and $a_t = Q) = 1$ for all times t and engagement levels e'. In other words, there are no "failed attempts" at quitting.

Taken together we get aggregate transition probabilities from state to state, depending on whether incented ads are available or not. If incented ads are available, the transition probability from engagement level e to engagement level e' is

$$\mathbb{P}_1(e'|e) := \begin{cases} p_M(e)\tau(e'|e, M) + p_I(e)\tau(e'|e, I) & \text{if } e, e' \in E \\ p_Q(e) & \text{if } e \in E, e = -1 \\ 1 & \text{if } e, e' = -1 \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and if incented ads are not available

$$\mathbb{P}_1(e'|e) := \begin{cases} q_M(e)\tau(e'|e, M) & \text{if } e, e' \in E \\ q_Q(e) & \text{if } e \in E, e = -1 \\ 1 & \text{if } e, e' = -1 \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

Assumption 1. *No matter how engaged, there is always a positive probability that a player will*

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322 *quit; i.e., $p_Q(e), q_Q(e) > 0$ for all $e \in E$.*

323 This acknowledges the fact that games are entertainment activities, and there are numerous
324 reasons for a player to quit due to factors in their daily lives, even when engrossed in the game.
325 This is also an important technical assumption since it implies the publisher's problem (see the
326 next section) is an absorbing Markov decision process.

327 **3.2 The publisher's problem.** We model the publisher's problem as an infinite horizon Markov
328 decision process under a total reward criterion (for details see Puterman (1994)). A Markov decision
329 process is specified by a set of states, controls in each state, transition probabilities under pairs of
330 states and controls, and rewards for each transition.

331 Specifically in our setting based on the description of the dynamics we have laid out thus far,
332 the set of states is $\{-1\} \cup E$ and the set of controls $U = \{0, 1\}$ is independent of the state, where
333 1 represents offering an incented action and 0 not offering an incented action. The transition
334 probabilities are given by (3) when $u = 1$ and (4) when $u = 0$. The reward depends on the action
335 of the player. When the player quits, the publisher earns no revenue, denoted by $\mu_Q = 0$. When
336 the player takes an incented action, the publisher earns μ_I, while a monetization action earns μ_M.

337 **Assumption 2.** *We assume $\mu_I < \mu_M$.*

338 This assumption is in concert with practice, as discussed in the introduction.

339 The expected reward in state e under control u is:

340
$$r(e, u) = \begin{cases} p_M(e)\mu_M + p_I(e)\mu_I & \text{if } e \in E \text{ and } u = 1 \\ q_M(e)\mu_M & \text{if } e \in E \text{ and } u = 0 \\ 0 & \text{if } e = -1. \end{cases}$$

341
342 Note that expected rewards do not depend on whether the player transitions to a higher level and
343 so the probabilities $\tau_a(e'|e)$ do not appear in $r(e, u)$.

344 A *policy* y for the publisher is a mapping from E to U. On occasion we will express a policy
345 by the vector form of its image. That is, the vector $y = (1, 0, 1)$ denotes offering incented actions
346 in engagement levels 1 and 3. Each policy y induces a stochastic process over rewards, allowing us
347 to write its value as:

348
$$W^y(e) := \mathbb{E}_e^y \left[\sum_{t=1}^{\infty} r(e_t, y(e_t)) \right] \tag{5}$$

349
350 where e is the player's initial level, and the expectation $\mathbb{E}_e^y[\cdot]$ derives from the induced stochastic
351 process. One may assume that all players start at level 0, but we also consider the possibility that
352 players can start at higher levels of the game for a couple of reasons. First, the time horizon of the
353 available may only capture the situation where some existing players have already started playing
354 the game. Second, we reason inductively where it is valuable to think of the process of restarting
355 at a higher level. For these reasons we allow the initial level of the player to be different than 0.

356 In many Markov decision processes, the sum in (5) does not converge, but under Assumption 1,

the expected total reward converges for every policy y. In fact, our problem has a special structure that we can exploit to derive a convenient analytical form for (5) as follows:

$$W^y(e) = \sum_{e' \in E} n^y_{e,e'} r(e', y(e')) \tag{6}$$

where $n^y_{e,e'}$ is the expected number of visits to engagement level e' starting in engagement level e. We derive closed-form expressions for $n_{e,e'}$ that facilitate analysis. For details see Appendix A.1.

The game publisher chooses a policy to solve the optimization problem: given a starting engagement level e solve:

$$\max_{y \in \{0,1\}^E} W^y(e). \tag{7}$$

This problem can be solved numerically using tools such as policy iteration (see, for instance, Puterman (1994)). These results are standard in the case of a finite number of engagement levels. The case of countably-many engagement levels also permits algorithms under additional conditions on the data (see, for instance, Hinderer and Waldmann (2005)). In our setting, rewards are bounded (equal to μ_I, μ_M or 0 for every e), which simplifies analysis. In this paper, we do not explore the countable engagement-level case.

The challenge, of course, in solving (3.2) is fitting the data to the model. This is taken up later in Section 5 for a specific game of interest. For now, we aim to learn more about the analytical structure of optimal solutions to (3.2). In general, the situation is hopeless. Although the decision of the publisher is a simple $\{0,1\}$-vector, the transition law (3)–(4) is quite general and evades the standard analysis needed to leverage existing structural results (such as monotonicity or submodularity). We consider a special case in the next section that will help us, nonetheless, get some structural insight into (7). In that setting we are able to show the optimality of threshold policies and conduct sensitivity analysis on the threshold level.

Game companies can make use of our results in a number of ways. By numerically solving (3.2) using an approach similiar to our Section 5, a detailed policy for offering incented actions can be devised. However, even for new games with little user data to estimate the parameters of this optimization problem, structural results from the analytical model can provide guidance. Our results suggest it is quite justified to restrict to threshold policies, which are easy to understand for game designers and easy to implement in practice. Also, sensitivity analysis yields insights into what general types of games may want to include incented actions or not (see this discussion in Section 4.3).

4 Analytical results for a special case

To facilitate analysis, we make the following simplifying assumption about state transitions (these assumptions are relaxed in the data-driven Section 5 below): (i) players progress at most one level at a time and never digress, (ii) the transition probability is independent of the current level and depends only on the action taken by the player, and (iii) there are finitely many engagement
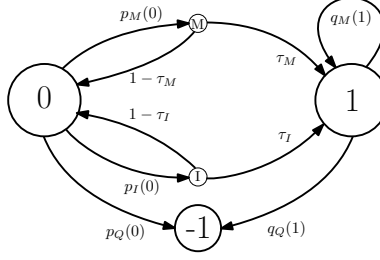
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Figure 1: A visual representations of the Markov chain model of player behavior with two levels and incented actions available at level 0.

levels $E = \{0, 1, \ldots, N\}$. These assumptions are consistent with the "game level" interpretation of engagement, where players can advance a single level at a time.

Assumption 3. *The engagement level transition probabilities satisfy the following conditions:*

$$\mathbb{P}(e'|e, a) = \begin{cases} \tau_a & \text{if } e' = e + 1 \text{ and } e < N \\ 1 - \tau_a & \text{if } e' = e < N \\ 1 & \text{if } e = e' = N \\ 0 & \text{otherwise} \end{cases}$$

for $a \in \{M, I\}$. For $a = Q$ the player transitions with probability one to a quit state denoted -1.

This structure simplifies the transition probabilities in (3) and (4). Figure 1 provides a visual representation of the Markov chain describing player behavior when there are two levels, with incented action only offered at level 0. Additionally we make some monotoncity assumptions on the problem data.

Assumption 4. *We make the following assumptions:*

(A4.1) $p_M(e)$ and $q_M(e)$ increase in e,

(A4.2) $p_Q(e)$ and $q_Q(e)$ decrease in e,

(A4.3) $p_I(e)$ decreases in e,

(A4.4) $\tau_M > \tau_I$, and

(A4.5) $\alpha(e)$ is increasing in e.

Assumption (A4.1) and Assumption (A4.2) ensure that players at higher levels are more likely to make in-app purchases and less likely to quit. The more invested a player is in a game, the more likely they are to spend and the less likely they are to quit. Assumption (A4.3) ensures that players are less likely to take an incented action as their level increases. One interpretation of this is that the rewards associated with an incented action are less valuable as a player progresses, decreasing the probability of taking such an action. Observe that Assumptions (A4.1)–(A4.3) put implicit assumptions on the cannibalization parameter $\alpha(e)$ via (1) and (2).

Assumption (A4.4) implies that a player is more likely to progress a level when monetizing than when taking an incented action. Again, the rewards for incented actions are typically less powerful

than what can be purchased for real money and so monetizing more likely leads to an increase in level. The example in games such as *Crossy Road* (by Hipster Whale), playable characters in the game can be directly bought with real money, but watching video ads can only contributes to random draws for characters.

Finally, (A4.5) implies that a greater share of the probability of taking an incented actions when offered is allocated to monetization when an incented ad is removed (see (1)). As a player moves higher up in levels, the monetization option becomes relatively more attractive than quitting once the incented action is removed. Indeed, quitting has the player walking away from a potentially significant investment of time and mastery captured by a high level in the game.

4.1 Understanding the effects of incented actions. In this section, we show how our analytical model under our additional assumptions helps sharpen our insight into the costs and benefits of offering incented actions in games. In particular, we give precise analytical definitions of the *revenue*, *retention* and *progression* effects discussed in the introduction.

Let $y_{\bar{e}}^1$ be a given policy with $y_{\bar{e}}^1(\bar{e}) = 0$ for some engagement level $\bar{e}$. Consider a local change to a new policy $y_{\bar{e}}^2$ where $y_{\bar{e}}^2(\bar{e}) = 1$ but $y_{\bar{e}}^2(e) = y_{\bar{e}}^1(e)$ for $e \neq \bar{e}$. We call $y_{\bar{e}}^1$ and $y_{\bar{e}}^2$ *paired policies with a local change at* $\bar{e}$. Analyzing this local change at the target engagement level $\bar{e}$ gives insight into the effect of starting to offer an incented action at a given engagement level. Moreover, this flavor of analysis suffices to determine an optimal threshold policy, as discussed in Section 4.2 below. For ease of notation, let $W^1(e) = W^{y_{\bar{e}}^1}(e)$ and $W^2(e) = W^{y_{\bar{e}}^2}(e)$.

Our goal is to understand the change in expected revenue moving from policy $y_{\bar{e}}^1$ to policy $y_{\bar{e}}^2$ where the player starts (or has reached) engagement level $\bar{e}$. Indeed, because the engagement does not decrease (before the player quits) if the player has reached engagement level $\bar{e}$ the result is the same as if the player just started at engagement level $\bar{e}$ by the Markovian property of the player model. Understanding when, and for what reasons, this change has a positive impact on revenue provides insights into the value of incented actions.

The change in total expected revenue from the policy change $y_{\bar{e}}^1$ to $y_{\bar{e}}^2$ at engagement level $\bar{e}$ is:

$$W^2(\bar{e}) - W^1(\bar{e}) = \underbrace{n_{\bar{e},\bar{e}}^2 r(\bar{e}, 1) - n_{\bar{e},\bar{e}}^1 r(\bar{e}, 0)}_{(C(\bar{e}))} + \underbrace{\sum_{e > \bar{e}} (n_{\bar{e},e}^2 - n_{\bar{e},e}^1) r(e, y(e))}_{(F(\bar{e}))} \tag{8}$$

Term $C(\bar{e})$ is the change of revenue accrued from visits to the current engagement level $\bar{e}$. We may think of $C(\bar{e})$ as denoting the *current* benefits of offering an incented action in state $\bar{e}$, where "current" means the current level of engagement. Term $F(\bar{e})$ captures the change due to visits to all other engagement levels. We may think of $F(\bar{e})$ as denoting the *future* benefits of visiting higher ("future") states of engagement. We can give explicit formulas for $C(\bar{e})$ and $F(\bar{e})$ for $e < N$ (after some work detailed in Appendix A.5) as follows:

$$C(\bar{e}) = \frac{p_M(\bar{e})\mu_M + p_I(\bar{e})\mu_I}{1 - p_M(\bar{e})(1 - \tau_M) - p_I(\bar{e})(1 - \tau_I)} - \frac{q_M(\bar{e})\mu_M}{1 - q_M(\bar{e})(1 - \tau_M)} \tag{9}$$

454   and

$$F(\bar{e}) = \{\frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} - \frac{q_M(\bar{e})\tau_M}{1 - q_M(\bar{e})(1-\tau_M)}\}\{\sum_{e' > \bar{e}} n_{\bar{e}+1,e'}^{y^1} r(e', y(e'))\}. \tag{10}$$

One interpretation of the formula $C(\bar{e})$ is that the two terms in (9) are conditional expected revenues associated with progressing to engagement level $\bar{e} + 1$ conditioned on the event that the player does not stay in engagement level e (by either quitting or advancing). Thus, $C(\bar{e})$ is the change in conditional expected revenue from offering incented actions. There is a similar interpretation of the expression $\frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} - \frac{q_M(\bar{e})\tau_M}{1 - q_M(\bar{e})(1-\tau_M)}$ in the definition of $F(\bar{e})$. Both terms are conditional probabilities of progressing from engagement level $\bar{e}$ to engagement level $\bar{e} + 1$ conditioned on the event that the player does not stay in engagement level $\bar{e}$ (by either quitting or advancing). Thus, $F(\bar{e})$ can be seen as the product of a term representing the increase in the conditional probability of progressing to engagement level $\bar{e}$ and the sum of revenues from expected visits from state $\bar{e} + 1$ to the higher engagement levels.

We provide some intuition behind what drives the benefits of offering incented actions, both current and future, not easily gleaned from these detailed formulas. In particular, we provide precise identification of three effects of incented actions that were discussed informally in the introduction. To this end, we introduce the notation:

$$\Delta_r(e|\bar{e}) := r(e, y_{\bar{e}}^2(e)) - r(e, y_{\bar{e}}^1(e)), \tag{11}$$

which expresses the change in the expected revenue per visit to engagement level e and

$$\Delta_n(e|\bar{e}) = n_{\bar{e},e}^2 - n_{\bar{e},e}^1, \tag{12}$$

which expresses the change in the number of expected visits to engagement level e (starting at engagement level $\bar{e}$) before quitting. Note that $\Delta_r(e|\bar{e}) = 0$ for $e \neq \bar{e}$ since we are only considering a local change in policy at engagement level $\bar{e}$. On the other hand,

$$\Delta_r(\bar{e}|\bar{e}) = -(q_M(\bar{e}) - p_M(\bar{e}))\mu_M + p_I(\bar{e})\mu_I. \tag{13}$$

The latter value is called the *revenue effect* as it expresses the change in the revenue per visit to the starting engagement level $\bar{e}$. The *retention effect* is the value $\Delta_n(\bar{e}|\bar{e})$ and expresses the change in the number of visits to the starting engagement level $\bar{e}$. Lastly, we refer to the value $\Delta_n(e|\bar{e})$ for $e > \bar{e}$ as the *progression effect at engagement level* e.

At first blush, it may seem possible for the progression effect to have different in sign at different engagement levels, but the following result shows that the progression effect is uniform in sign.

Proposition 1. *Under Assumptions 1 and 2, the progression effect is uniform in sign; that is, either* $\Delta_n(e|\bar{e}) \geq 0$ *for all* $e \neq \bar{e}$ *or* $\Delta_n(e|\bar{e}) \leq 0$ *for all* $e \neq \bar{e}$.

The intuition for the above result is simple. There is only a policy change at the starting engagement level $\bar{e}$. Thus, the probability of advancing from engagement level e to engagement level $e + 1$ is the same for policy $y_{\bar{e}}^1$ and $y_{\bar{e}}^2$ for $e > \bar{e}$. Hence, if $\Delta_n(\bar{e} + 1|\bar{e})$ is positive then $\Delta_n(e|\bar{e})$ is positive for $e > \bar{e} + 1$ since there will be more visits to engagement level $\bar{e} + 1$ and thus more visits to higher engagement levels since the transition probabilities at higher engagement levels are

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unchanged. Because of the consistency in sign, we may refer to the *progression effect* generally (without reference to a particular engagement level).

If both the revenue effect and retention effects are positive, $C(\bar{e})$ in (8) is positive, and there is a net increase in revenue due to visits to engagement level $\bar{e}$. Similarly, if both effects are negative, then $C(\bar{e})$ is negative. When one effect is positive, and other is negative, the sign of $C(\bar{e})$ is unclear. The sign of $F(\bar{e})$ is determined by the direction of the progression effect.

One practical motivation for incented actions is that relatively few players monetize in practice, and so opening up another channel of revenue the publisher can earn more from its players. If $q_M(\bar{e})$ and $p_M(\bar{e})$ are small (say in the order of 2%) then the first term in the revenue effect (13) is insignificant when compared to the second term $p_I(\bar{e})\mu_I$ and so it is most likely positive at low engagement levels. This motivation suggests that the retention and progression effects are also likely to be positive, particularly at early engagement levels when players are most likely to quit and least likely to invest money into playing a game.

However, our current assumptions do not fully capture the above logic. It is relatively straight-forward to construct specific scenarios that satisfy Assumptions 1–2 where the revenue and pro-gression effects are negative even at low engagement levels. Further refinements are needed (see Section 4.2 for further assumptions). This complexity is somewhat unexpected, given the parsimony on the model and structure already placed on the problem. Indeed, the assumptions do reveal a certain structure as demonstrated in the following result.

Proposition 2. *Under Assumptions 1 and 2, the retention effect is nonnegative; i.e.,* $\Delta_n(\bar{e}|\bar{e}) \geq 0$.

There are two separate reasons for why offering incented actions at engagement level $\bar{e}$ changes the number of visits to $\bar{e}$. This first comes from the fact that the quitting probability at engagement level $\bar{e}$ goes down from $q_Q(\bar{e})$ to $p_Q(\bar{e})$. The second is that the probability of progressing to a higher level engagement also changes from $q_M(\bar{e})\tau_M$ to $p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I$ when offering an ad. At a high level, the overall effect may seem unclear. However, observe that the probability of staying in engagement level e always improves when an incented action is offered:

$$p_M(\bar{e})(1 - \tau_M) + p_I(\bar{e})(1 - \tau_I) - q_M(\bar{e})(1 - \tau_M) = p_I(e)(-\alpha(e)(1 - \tau_M) + (1 - \tau_I)) > 0.$$

It is not always desirable for there to be more visits to engagement e if it is primarily at the expense of visits to more lucrative engagement levels. We must, therefore, consider the future benefits of the change in policy.

4.2 Optimal policies for the publisher. Recall the publisher's problem described in (7). This is a dynamic optimization problem where the publisher must decide on whether to deploy incented actions at *each* engagement level, with the knowledge that a change in policy at one engagement level can effect the behavior of the player at subsequent engagement levels. This "forward-looking" nature adds a great deal of complexity to the problem. A much simpler task would be to examine each engagement level in isolation, implying that the publisher need only consider term (i) of (8) at engagement level e to decide if $y(e) = 1$ or $y(e) = 0$ provides more revenue. A policy built in

this way is called *myopically optimal*. More precisely, policy y is myopically optimal if $y(e) = 1$ when $C(e) > 0$ and $y(e) = 0$ when $C(e) < 0$. The next result gives a sufficient condition for a myopically-optimal policy to be optimal.

Proposition 3. *Assumptions 1 and 2 and $\frac{\mu_I}{\mu_M} = \frac{\tau_I}{\tau_M}$ imply a myopically-optimal policy is optimal.*

This result is best understood by looking at the two terms in the change in revenue formula (8) discussed in the previous section. It is straightforward to see from (9) and (10) that when $\tau_I = \mu_I$ and $\tau_M = \mu_M$ that the sign of $C(\bar{e})$ and $F(\bar{e})$ are identical. That is, if the current benefit of offering the incented action has the same sign as the future benefit of offering an action then it suffices to consider the term first $C(\bar{e})$ only when determining an optimal policy. Given our interpretation of $C(\bar{e})$ and $F(\bar{e})$, the conditions of Proposition 3 imply that the conditional expected revenue from progressing one engagement level precisely equals the conditional probability of progressing one engagement level. This is a rather restrictive condition.

Since we know of only the above strict condition under which an optimal policy is myopic, in general we are in search of forward-looking optimal policies. Since the game publisher's problem is a Markov decision process, an optimal forward-looking policy y must satisfy the optimality equations for $e = 0, \ldots, N - 1$

$$W^y(e) = \begin{cases} r(e, 1) + \mathbb{P}_1(e|e)W(e) + \mathbb{P}_1(e+1|e)W(e+1) & \text{if } y(e) = 1 \\ r(e, 0) + \mathbb{P}_0(e|e)W(e) + \mathbb{P}_0(e+1|e)W(e+1) & \text{if } y(e) = 0 \end{cases}$$

and for $e = N$

$$W^y(N) = \begin{cases} r(N, 1) + \mathbb{P}_1(N|N)W^y(N) & \text{if } y(N) = 1 \\ r(N, 0) + \mathbb{P}_0(N|N)W^y(N) & \text{if } y(N) = 0, \end{cases}$$

where $\mathbb{P}_1$ and $\mathbb{P}_0$ are the transition probabilities when incented action are offered and not offered, respectively. The above structure shows that an optimal policy can be constructed by backwards induction (for details see Chapter 4 of Puterman (1994)): first determine an optimal choice of $y(N)$ and then successively find optimal choices for $y(N-1), \ldots, y(1)$ and finally $y(0)$. We use the notation $W(e)$ to denote the *optimal* revenue possible with a player starting at engagement level e, called the *optimal value function*. In addition we use the notation $W(e, y = 1)$ to denote the optimal expected total revenue possible when an incented action is offered at starting engagement level e. Similarly, we let $W(e, y = 0)$ denote the optimal expected revenue possible when an incented action is *not* offered at starting engagement level e. Then $W(e)$ must satisfy *Bellman's equation* for $e = 0, \ldots, N - 1$:

$$\begin{aligned} W(e) &= \max\left\{W(e, y = 1), W(e, y = 0)\right\} \\ &= \max\{r(e, 1) + \mathbb{P}_1(e|e)W(e) + \mathbb{P}_1(e+1|e)W(e+1), \\ &\qquad r(e, 0) + \mathbb{P}_0(e|e)W(e) + \mathbb{P}_0(e+1|e)W(e+1)\}. \end{aligned} \tag{14}$$

Lemma 1. *Under Assumptions 1–2, $W(e)$ is a nondecreasing function of e.*

The higher the engagement of a player, the more revenue can be extracted from them. This

Policy	$W^y(0)$	$W^y(1)$
$y = (0,0)$	0.723	1.13
$y = (1,0)$	0.691	1.13
$y = (1,1)$	0.701	1.15
$y^* = (0,1)$	0.727	1.15

Table 1: Total expected profit for Example 1.

567 result gives us a strong intuitive foundation on which to proceed.

568   The focus of our discussion is on optimal *forward threshold* policies that start by offering incented
569 action. Such a threshold policy y is determined by a single engagement level $\bar{e}$ where $y(e') = 1$
570 for $e' \leq \bar{e}$ and $y(e) = 0$ for $e' > \bar{e}$. According to (14) this happens when $W(\bar{e} + 1, y = 1) \leq$
571 $W(\bar{e} + 1, y = 0)$ implies $W(e', y = 1) \geq W(e', y = 0)$ for all $e' \leq \bar{e}$ and $W(e', y = 0) < W(e', y = 0)$
572 for all $e' > \bar{e} + 1$. In the general nomenclature of Markov decision processes, other policies would
573 be classified as threshold policies. This includes policies that start with not offering the incented
574 action until some point and thereafter offering the incented action.

575   Our interest in forward threshold policies comes from the following appealing logic, already
576 hinted at in the introduction. When players start out playing a game their engagement level is low
577 and they are likely to quit. Indeed, Lemma 1 says we get more value out of players at higher levels
578 of engagement. Hence, retaining players at early stages and progressing them to higher levels of
579 engagement is important for overall revenue. In Proposition 2, we see the retention effect of offering
580 incented actions is always positive, and intuitively, the revenue and progression effects are largest at
581 low levels of engagement because players are unlikely to monetize early on and the benefits derived
582 from increasing player engagement are likely to be at their greatest. This suggests it is optimal to
583 offer incented actions at low levels of engagement. However, once players are sufficiently engaged it
584 might make sense to removed incented actions to focus their attention on the monetization option.
585 If sufficiently engaged and $\alpha(e)$ is sufficiently large, most of the probability of taking the incented
586 action shifts to monetizing that drives greater revenue.

587   Despite this appealing logic, the following example shows that our current set of assumptions
588 are insufficient to guarantee the existence of an optimal forward threshold policy.

589 **Example 1.** *Consider the following two engagement level example. Assume $\mu_M = 1$, $\mu_I = 0.05$,*
590 *$\tau_M = 0.5$, $\tau_I = 0.4$. At level 0, $p_M(0) = 0.05$, $p_I(0) = 0.65$, $\alpha(0) = 0.5$ and thereby $q_M(0) = 0.375$.*
591 *At level 1, $p_M(1) = 0.2$, $p_I(1) = 0.6$, $\alpha(1) = 0.55$ and thereby $q_M(1) = 0.53$.*

592   *We solve the optimal policy by backward induction. At level 1, $W(1, y = 1) = \frac{p_M(1)\mu_M + p_I(1)\mu_I}{1 - p_M(1) - p_I(1)} =$*
593 *1.15 while $W(1, y = 0) = \frac{q_M(1)\mu_M}{1 - q_M(1)} \approx 1.13$. Therefore, $y^*(1) = 1$ and $W(1) = 1.15$. At level 0,*
594 *$W(0, y = 1) = 0.701$ and $W(0, y = 0) = 0.727$ hence $y^*(0) = 0$ and $W(0) = 0.727$.*

595   *Next, we show that $y^* = (0, 1)$ is the only optimal policy. In fact, we compute $W^y(0)$ and $W^y(1)$*
596 *under all possible policies in the following table. We observe that none of $(0, 0)$, $(1, 0)$ and $(1, 1)$*
597 *are optimal. This implies y^* is the only optimal policy. Since y^* is not a forward threshold policy,*

598   *this implies there is no optimal forward threshold policy.*   ◁

599   This example illustrates a break in the above logic. It is optimal to offer incented actions at the
600   higher engagement level because of the dramatic reduction in the quitting probability when offered,
601   reducing the quitting probability compared to a 0.47 quitting probability when not offering incented
602   actions. Although the expected revenue per period the player stays at the highest engagement level
603   is lower when incented actions are offered (0.23 as compared to 0.47), the player will stay longer and
604   thus earn additional revenue. However, at the lowest engagement level, the immediate reward of not
605   offering incented actions (0.462 versus 0.141) outweighs losses due to a lower chance of advancing
606   to the higher engagement level.

607   The goal for the remainder of this section is to devise additional assumptions that are relevant
608   to the settings of interest to our paper and that guarantee the existence of an optimal forward
609   threshold policy. The previous example shows how α plays a key role in determining whether a
610   threshold policy is optimal or not. When incentives actions are removed the probability $p_I(e)$ is
611   distributed to the monetization and quitting actions according to $\alpha(e)$. The associated increase
612   in the probability of monetizing from $p_M(e)$ to $q_M(e)$ makes removing incented actions attractive,
613   since the player is more likely to pay. However, the quitting probability increases from $p_Q(e)$ to
614   $q_Q(e)$, a downside of removing incented actions. Intuitively speaking, if $\alpha(e)$ grows sufficiently
615   quickly, the benefits will outweigh the costs of removing incented actions. From Assumption (A4.5)
616   we know that $\alpha(e)$ increases, but this alone is insufficient. Just how quickly we require $\alpha(e)$ to
617   grow to ensure a threshold policy requires careful analysis. This analysis results in lower bounds
618   on the growth of $\alpha(e)$ that culminates in Theorem 1 below.

619   Our first assumption on $\alpha(e)$ is a basic one:

620   **Assumption 5.** $\alpha(N) = 1$; *that is,* $q_Q(N) = p_Q(N)$ *and* $q_M(N) = p_M(N) + p_I(N)$.

621   It is straightforward to see that under this assumption it is never optimal to offer incented
622   action at the highest engagement level. This assumption also serves as an interpretation of what it
623   means to be in the highest engagement level, simply that players who are maximally engaged are no
624   more likely to quit when the incented action is removed. Under this assumption, and by Bellman's
625   equation (14), every optimal policy y^* has $y^*(N) = 0$. Note that this excludes the scenario in
626   Example 1 and also implies that "backward" threshold policies are not optimal (except possibly
627   the policy that $y(e) = 0$ for all $e \in E$ that is both a backward and forward threshold). Given this,
628   we restrict attention to forward threshold policies and drop the modifier "forward" in the rest of
629   our development.

630   The next step is to establish further sufficient conditions on the data that ensure that once the
631   revenue, retention, and progression effects are negative, they stay negative. As in Section 4.1, we
632   consider paired policies $y_{\bar{e}}^1$ and $y_{\bar{e}}^2$ with a local change at $\bar{e}$. Recall the notation $\Delta_r(e|\bar{e})$ and $\Delta_n(e|\bar{e})$
633   defined in (11) and (12), respectively. We are concerned with how $\Delta_r(e|\bar{e})$ and $\Delta_n(e|\bar{e})$ change with
634   the starting engagement level $\bar{e}$. It turns out that the revenue effect $\Delta_r(e|\bar{e})$ always behaves in a

635 way that is consistent with a threshold policy, without any additional assumptions.

Proposition 4. *Suppose Assumptions 1–2 hold. For every engagement level $\bar{e}$ let $y^1_{\bar{e}}$ and $y^2_{\bar{e}}$ be paired policies with a local change at $\bar{e}$. Then the revenue effect $\Delta_r(\bar{e}|\bar{e})$ is nonincreasing in $\bar{e}$ when $\Delta_r(\bar{e}|\bar{e}) \geq 0$. Moreover, if $\Delta_r(\bar{e}|\bar{e}) < 0$ for some $\bar{e}$ then $\Delta_r(e'|e') < 0$ for all $e' \geq \bar{e}$.*

This proposition says that the net revenue gain per visit to engagement level $\bar{e}$ is likely only to be positive (if it is ever positive) at lower engagement levels, confirming our basic intuition that incented actions can drive revenue from low engagement levels, but less so from highly engaged players. To show a similar result for the progression effect, we make the following assumption.

Assumption 6. $\alpha(e+1) - \alpha(e) > q_M(e+1) - q_M(e)$ *for all $e \in E$.*

This provides our first general lower bound on the growth of $\alpha(e)$. It says that $\alpha(e)$ must grow faster than the probability $q_M(e)$ of monetizing when the incented action is not offered.

Proposition 5. *Suppose Assumptions 1–6 hold. For every engagement level $\bar{e}$ let $y^1_{\bar{e}}$ and $y^2_{\bar{e}}$ be paired policies with a local change at $\bar{e}$ such that $y^1_{\bar{e}}$ and $y^2_{\bar{e}}$ are identical to some fixed policy y (fixed in the sense that y is not a function of $\bar{e}$) except at engagement level $\bar{e}$. Then*

(a) If $\Delta_n(e|\bar{e}) < 0$ for some $\bar{e}$ then $\Delta_n(e|e') < 0$ for all $e' \geq \bar{e}$.

(b) If $C(\bar{e}) < 0$ for some $\bar{e}$ then $C(e') < 0$ for all $e' \geq \bar{e}$, where C is as defined in (8).

This result implies that once the current and future benefits of offering an incented action are negative, they stay negative for higher engagement levels. Indeed, Proposition 5(a) ensures that the future benefits F in (8) stay negative once negative, while (b) ensures the current benefits C stay negative once negative. In other words, once the game publisher stops offering incented actions it is never optimal for them to return. Note that Proposition 4 does not immediately imply Proposition 5(b), Assumption 6 is needed to ensure the retention effect has similar properties, as guaranteed by Proposition 5(a) for $e' = \bar{e}$.

As mentioned above, the conditions established in Proposition 5 are necessary for the existence of an optimal threshold policy but does not imply that an threshold policies exists. This is because C and F in (8) may not switch sign from positive to negative at the same engagement level. An example of this is provided in Appendix A.10. We thus require one additional assumption:

Assumption 7. $1 - \alpha(e+1) \leq (1 - \alpha(e))\frac{p_M(e+1)\tau_M + p_I(e+1)\tau_I}{p_Q(e+1) + p_M(e+1)\tau_M + p_I(e+1)\tau_I}$ *for $e = 1, 2, \ldots, N-1$.*

Note that the fractional term in the assumption is the probability of advancing from engagement level $e+1$ to $e+2$ conditioned on leaving engagement level $e+1$ and is thus less than one. Hence, this is yet another lower bound on the rate of growth in $\alpha(e)$, complementing Assumptions 5 and 6. Which bound in Assumption 6 or Assumption 7 is tighter depends on the data specifications that arise from specific game settings.

Theorem 1. *Suppose Assumptions 1–7 hold. Then there exists an optimal threshold policy with threshold engagement level e^*. That is, there exists an optimal policy y^* with $y^*(e) = 1$ for any $e \leq e^*$ and $y^*(e) = 0$ for any $e > e^*$.*

The existence of an optimal threshold is the cornerstone analytical result of this paper. From

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our development above, it should be clear that obtaining a sensible threshold policy is far from a trivial task. We believe our assumptions are reasonable based on our understanding of the games, given the difficult standard of guaranteeing the existence of a threshold policy. Of course, such policies will be welcomed in practice, precisely because of their simplicity and (relatively) intuitive justification. We also remark that none of these assumptions are superfluous. In Appendix A.9 we show that if we drop Assumption 6 then a threshold policy may longer be optimal. Appendix A.10 shows that the same is true if Assumption 7 is dropped. As we see in some examples in the next section, our assumptions are sufficient but not necessary conditions for an optimal threshold policy to exist.

To simplify matters further, we also take the convention that when there is a tie in Bellman's equation (14) whether to offer an incented action or not, the publisher always chooses not to offer. This is consistent with the fact that there is a cost to offering incented actions. Although we do not model costs formally, we will use this reasoning to break ties. Under this tie-breaking rule there is, in fact, a *unique* optimal threshold policy guaranteed by Theorem 1. This unique threshold policy is our object of study in this section.

4.3 Game design and optimal use of incented actions. So far we have provided a detailed analytical description of the possible benefits of offering incented actions (in Section 4.1) and the optimality of certain classes of policies (in Section 4.2). There remains the question of what types of games most benefit from offering incented actions and how different types of games may qualitatively differ in their optimal policies. We focus on optimal threshold policies and concern ourselves with how changes in the parameters of the model affect the optimal threshold e^* of an optimal threshold policy y^* that is guaranteed to exist under Assumptions 1–7 by Theorem 1. Of course, these are only sufficient conditions, and so we do not restrict ourselves to that setting when conducting numerical experiments in this section.

We first consider how differences in the revenue parameters μ_I and μ_M affect e^*. Observe that only the revenue effect in (13) is impacted by changes in μ_I and μ_M, the retention and progression effects are unaffected. This suggests the following result:

Proposition 6. *The optimal threshold e^* is a nondecreasing function of the ratio $\frac{\mu_I}{\mu_M}$.*

Note that the revenue effect is nondecreasing in the ratio $\frac{\mu_I}{\mu_M}$. Since the other effects are unchanged, this implies that the benefit of offering incented actions at each engagement level is nondecreasing in $\frac{\mu_I}{\mu_M}$, thus establishing the monotonicity of e^* in μ_I/μ_M.

To interpret this result, we consider what types of games have a large or small ratio $\frac{\mu_I}{\mu_M}$. From the introduction in Section 1 we know that incented actions typically deliver far less revenue to the publisher than in-app purchases. This suggests that the ratio is small, favoring a lower threshold. However, this conclusion ignores how players in the game may influence each other. Although our model is a single player model, one way we can include the interactions among players is through the revenue terms μ_I and μ_M. In many cases, a core value of a player to the game publisher is the

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word-of-mouth a player spreads to their contacts. Indeed, this is the value of non-paying players that other researchers have mostly focused on (see, for instance, Lee et al. (2017), Jiang and Sarkar (2009), and Gupta et al. (2009)). In cases where this "social effect" is significant, it is plausible that the ratio of revenue terms is not so small. For instance, if δ is the revenue attributable to the word-of-mouth or network effects of a player, regardless of whether the player takes an incented actions or monetizes, then the ratio of interest is $\frac{\mu_I + \delta}{\mu_M + \delta}$. The larger is δ, the larger is this ratio, and according to Proposition 6, the larger is the optimal threshold.

This analysis suggests that games with a significant social component should offer incented actions more broadly in social games. For instance, if a game includes cooperative or competitive multi-player features, then spreading the player base is of particular value to the company. Thought of another way, in a social game it is important to have a large player base to create positive externalities for new players to join, and so having players quit is of greater concern in more social games. Hence, it is best to offer incented action until higher levels of engagement are reached. All of this intuition is confirmed by Proposition 6.

Besides the social nature of the game, other factors can greatly impact of the optimal threshold. Genre, intended audience, and structure of the game affect the other parameters of our model; particularly, τ_I, τ_M, and $\alpha(e)$. We first examine the progression probabilities τ_I and τ_M. As we did in the case of the revenue parameters, we focus on the ratio $\frac{\tau_I}{\tau_M}$. This ratio measures the relative probability of advancing through incented actions versus monetization. By item (A4.4), $\frac{\tau_I}{\tau_M} \leq 1$ but its precise value can depend on several factors. One is the relative importance of the reward granted to the player when taking an incented action.

Taking τ_M fixed, we note that increasing τ_I decreases the "current" benefit of offering incented actions, as seen by examining term $C(\bar{e})$ in (8). Indeed, the revenue effect is unchanged by τ_I, but the retention effect is weakened. The impact on future benefits is less obvious. Players are more likely to advance to a higher level of engagement with a larger τ_I. From Lemma 1 we also know higher engagement states are more valuable, and so we expect the future benefits of offering incented actions to be positive with a higher τ_I and even outweigh the loss in current benefits. This reasoning is confirmed by the next result.

Proposition 7. *The optimal threshold e^* is a nondecreasing function of the ratio $\frac{\tau_I}{\tau_M}$.*

One interpretation of this result is that the more effective an incented action is at increasing engagement of the player, the longer the incented action should be offered. This is indeed reasonable under the assumption that $p_I(e)$ and $p_M(e)$ are unaffected by changes in τ_I. However, if increasing τ_I necessarily increases $p_I(e)$ (for instance, if the reward of the incented action becomes more powerful and so drives the player to take the incented action with greater probability) the effect on the optimal threshold is less clear.

Example 2. *In this example we show that when the incented action is more effective it can lead to a decrease in the optimal threshold if $p_I(e)$ and $p_M(e)$. Consider the following two engagement*

level example. In the base case let $\mu_M = 1$, $\mu_I = 0.05$, $\tau_M = 0.8$, $\tau_I = 0.2$. At level 0, $p_M(0) = 0.3$,
$p_I(0) = 0.5$, $\alpha(0) = 0.7$ and thereby $q_M(0) = 0.65$. At level 1, $p_M(1) = 0.5$, $p_I(1) = 0.4$, $\alpha(1) = 1$
and thereby $q_M(1) = 0.9$. Through basic calculations similiar to those shown in previous examples,
one can show that the unique optimal policy is $y^* = (0, 1)$.

Now change the parameters as follows: increase τ_I to 0.25, which affects the decision-making
of the player so that $p_M(0) = 0.1$, $p_I(0) = 0.7$, $p_M(1) = 0.3$ and $p_I(1) = 0.6$. The incented action
became so attractive it reduces the probability of monetizing while increasing the probability of taking
the incented action. One can show that the unique optimal policy in this setting is $y^* = (0, 0)$. Hence
the optimal threshold has decreased. In conclusion, a change in the effectiveness of the incented
action in driving engagement can lead to an increase or decrease in the optimal threshold policy,
depending on how the player's behavioral response. ◁

This leads to an important investigation of how changes in the degree of cannibalization between incented actions and monetization. Recall that $\alpha(e)$ is the vector of parameters that indicate the degree of cannibalization at each engagement level. For the sake of analysis, we assume that $\alpha(e)$ is an affine function of e with

$$\alpha(e) = \alpha(0) + \alpha_{\text{step}} e$$

where $\alpha(0)$ and α_{step} are nonnegative real numbers. A very high $\alpha(0)$ indicates a design where the reward of the incented action and the in-app purchase have a similar degree of attractiveness to the player so that when the incented action is removed, the player is likely to monetize. This suggests that the cost-to-reward ratio of the incented action is similar to that of the in-app purchase. If one is willing, for instance, to endure the inconvenience of watching a video ad to get some virtual currency, they should be similarly willing to pay real money for a proportionate amount of virtual currency. A very low $\alpha(0)$ is associated with a very attractive cost-to-reward ratio for the incented action that makes monetization seem expensive in comparison.

The rate of change α_{step} represents the strength of an increase in cannibalization as the player advances in engagement. A fast rate of increase is associated with a design where the value of the reward of the incented action quickly diminishes. Despite the reward weakening, this option still attracts a lot of attention from players, especially if they have formed a habit of advancing via this type of reward. If, however, the videos are removed, the value proposition of monetizing seems attractive in comparison to the diminished value of the reward for watching a video. Seen in this light, the rate at which the value of the reward diminishes is controlled by the parameter α_{step}.

Analysis of how different values for $\alpha(0)$ and α_{step} impact the optimal threshold is not straightforward. This is illustrated in the following two examples. The first considers the sensitivity of the optimal threshold to α_{step}.

Example 3. *Consider the following example with nine engagement levels and the following data:*
$\mu_m = 1$, $\mu_I = 1, 0.05$ $\tau_M = 0.8$, $\tau_I = 0.4$, $p_M(e) = 0.0001 + 0.00005e$ and $p_I(e) = 0.7 - 0.00001e$
for $e = 0, 1, \ldots, 8$. We have not yet specified $\alpha(e)$. We examine two scenarios: (a) where $\alpha(0) = 0$

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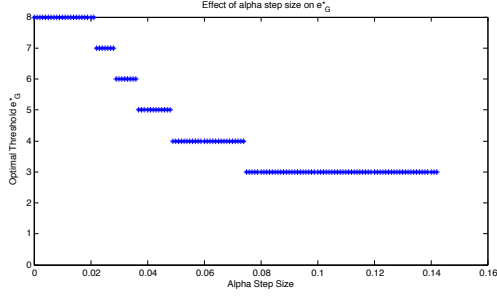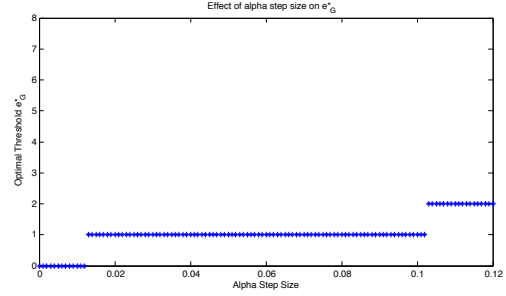
(a) Initial value $\alpha(0) = 0$. (b) Initial value $\alpha(0) = 0.16$.

Figure 2: Sensitivity of the optimal threshold to changes in α_{step}.

784 *and we vary the value of α_{step} (see Figure 2a) and (b) where $\alpha(0) = 0.16$ and we vary the value*
785 *of α_{step} (see Figure 2b). The vertical axis of these figures is the optimal threshold of the unique*
786 *optimal threshold policy for that scenario. What is striking is that the threshold e^* is nonincreasing*
787 *in α_{step} when $\alpha(0) = 0$ but nondecreasing in α_{step} when $\alpha(0) = 0.16$.*

788 *One explanation in the difference in patterns between Figures 2a and 2b concerns whether it is*
789 *optimal to include incented actions initially or not. In Figure 2a the initial degree of cannibalization*
790 *is zero, making it costless to offer incented actions initially. When α_{step} is very small cannibalization*
791 *is never an issue, and incented actions are offered throughout. However, as α_{step} increases, the*
792 *degree of cannibalization eventually makes it optimal to stop offering incented actions to encourage*
793 *monetization. This explains the nonincreasing pattern in Figure 2a.*

794 *By contrast, in Figure 2b the initial degree of cannibalization is already quite high, making*
795 *it optimal to start by offering for low values of α_{step}. However, when α_{step} is sufficiently large,*
796 *there are benefits to encouraging the player to advance. Recall, $\alpha(e)$ affects both the probability of*
797 *monetization and the probability of quitting. In the case where α_{step} is sufficiently high, there are*
798 *greater benefits to the player progressing, making quitting early more costly. Hence it can be optimal*
799 *to offer incented actions initially to discourage quitting and encourage progression. This explains*
800 *the nondecreasing pattern in Figure 2b.* ◁

801 As the following example illustrates, adjusting for changes in $\alpha(0)$ reveals a different type of
802 complexity.

803 **Example 4.** *Consider the following two engagement level example. Assume $\mu_M = 1$, $\mu_I = 0.0001$,*
804 *$\tau_M = 0.01$, $\tau_I = 0.009$. At level 0, $p_M(0) = 0.05$, $p_I(0) = 0.68$. At level 1, $p_M(1) = 0.3$,*
805 *$p_I(1) = 0.65$. We set α step size be 0.6, i.e. $\alpha(1) = \alpha(0) + 0.6$. Figure 3 captures how changes*
806 *in $\alpha(0)$ leads to different optimal thresholds (details on how the figure is derived is suppressed for*
807 *brevity, but follows the same pattern as previous examples).*

808 *The striking feature of the figure is that the optimal threshold decreases, and the increases, as*
809 *$\alpha(0)$ becomes larger. This "U"-shaped pattern reveals competing effects associated with changes*
810 *in $\alpha(0)$. As $\alpha(0)$ increases, the benefit of increasing retention (at the cost of harming retention)*
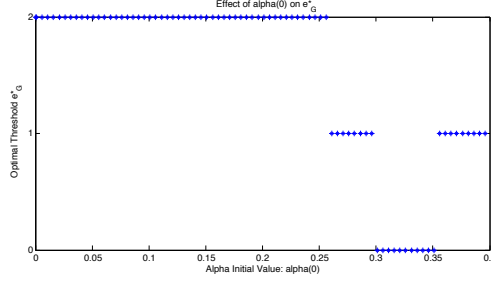
Figure 3: Sensitivity of the optimal threshold to changes in $\alpha(0)$.

weakens. This contributes to downward pressure downward pressure on the optimal threshold. On the other hand, increasing $\alpha(0)$ also increases $\alpha(1)$. This increases the attractiveness of reaching a higher engagement level and dropping the incented action. Indeed, $W^y(1)$ is increasing in $\alpha(1)$ when $y(0) = 1$. This puts upward pressure on the optimal threshold. This latter "future" benefit is weak for lower levels of $\alpha(0)$, where it may be optimal to offer an incented action in the last period. This provides justification for the "U"-shaped pattern. ◁

The scenarios in the above two examples provide a clear illustration of the complexity of our model. At different engagement levels, and with different prospects for the value of future benefits, the optimal strategy can be influenced in nonintuitive ways. This is particularly true for changes in $\alpha(e)$ as it impacts all three effects – revenue, retention, and progression. In some sense, cannibalization is the core issue in offering incented actions. This is evident in our examples and a careful examination of the definitions in Section 4.1 – the parameter $\alpha(e)$ is ubiquitous.

5   Data-driven results

The previous section considered a special case of our problem (7) under (admittedly) restrictive assumptions. The benefit was that it allowed us to establish the existence of optimal threshold policies and to develop analytical sensitivity results. The cost is a potential lack of realism. The goal is this section is to look at a realistic setting and ascertain, to what extent, our structural findings of the previous section carry over. This provides a real-world "robustness check" on our analytical results.

As mentioned in the introduction, the data comes from a mobile game entitled *Survival of Primitive* (or *Primitive*, for short) developed by Tianjin Ebo Digital Technology Co., Ltd. (or Ebo, for short) published under the Loco Games Studio imprint in July 2016. As described in the introduction, and with more detail below, Primitive does not perfectly fit the analytical model of the previous sections. Unlike Candy Crush, Primitive does not have "levels" that clearly demarcate progress. Ebo uses a duration-of-play based measure of engagement that fails to evolve as specified in previous sections. This gives rise to a different model of player behavior that we solve numerically using game data as input.

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5.1 Data. Our data set consists of the daily game behavior of 5000 randomly selected players starting from the release date through 31 March 2017. For each player we have daily entries for play time duration, the number of incented actions taken, and the number of in-app purchases made. The data used to build the model has a total of 1529,577 player-day observations.

We use this data to define engagement levels and estimate the parameters of the publisher's problem in (7). We consulted with Ebo on how they defined and tracked engagement. Based on these conversations we tied engagement level to how "active" the player is or how "frequently" the player plays the game, as captured by daily gameplay durations. We computed the (moving) average of play durations trailing some number of days. Engagement levels then consist of discrete buckets of play duration averages. For our main data analysis, we chose duration buckets of 0 to 30 minutes, 30 minutes to 60 minutes, 60 to 90 minutes and 90 to 120 minutes. That is, if the average time the player spent is less than 30 minutes, her engagement level is 0; between 30 minutes and 1 hour, her engagement level is 1, etc. No player had a moving average duration greater than two hours, so there were four engagement levels overall.

Another challenge is that, on a given day, there are players who neither monetize, take an incented action, or quit. According to our model, this day should not be considered a decision epoch for that player. We identified the quitting decision as follows: if the last log-in of a player was greater than seven days before the end of the data collection period (31 March 2017), then we assume that player took the quit action on the seventh data after her last log in. When we removed patient-day observation that did not correspond to decision epochs in our model we were left with 123,117 observations.

We then estimate p_M, p_I as follows. Given an engagement level e, let $TN(e)$ denote the total number of observations where the player was in engagement level e, let $MN(e)$ denote the number of these observations where a player monetized, and let $IN(e)$ denote the number of these observations where a player took an incentivized action. Then our estimate of $p_M(e)$ is the fraction $MN(e)/TN(e)$ and our estimate of $p_I(e)$ is the fraction $IN(e)/TN(e)$. The quit probability estimate is $p_Q(e) = 1 - p_M(e) - p_I(e)$. Table 2 captures the values we estimated from data.

Moving average play time (min)	Engagement Level (e)	p_I	p_M	p_Q	μ_M	μ_I
$[0, 5]$	1	0.7711	0.2176	0.0113	1	0.05
$(5, 30]$	2	0.7552	0.2143	0.0305		
$(30, 60]$	3	0.7576	0.2112	0.0312		
$(60, 90]$	4	0.7621	0.2064	0.0315		
$(90, 120]$	5	0.7534	0.2141	0.0325		

Table 2: Probabilities estimated using data from *Survival of Primitive*.

As in Section 3, we assume that given an action a, the probability of changing engagement does not depend on the current engagement level (other than restricting that engagement remain in $\{0, 1, \ldots, N-1, N\}$). However, we relax the assumption that engagement was restricted to go

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868 up by one level. This made sense in the context of level-based engagement, not so for *Primitive*
869 where engagement was calculated using play durations. To compute these conditional transition
870 probabilities, we first calculate the total number $MN = \sum_e MN(e)$ of player-days where a mon-
871 etization action occurs and the total number $IN = \sum_e IN(e)$ of player days where an incented
872 action is taken. Then we calculate the total number MN_{+i} of player-days where engagement in-
873 creased by i (for $i = 0, 1, \ldots, N$) engagement levels following a monetization action and the total
874 number of MN_{-i} of player-days where monetization was followed a decrease in engagement by
875 i (for $i = 1, \ldots, N$) engagement levels. The values IN_{+i} and IN_{-i} are defined analogously for
876 incented actions. The resulting conditional transition probabilities, which are calculated by taking
877 the appropriate ratio MN_{+i}/MN are captured in the Table 2.

Action	Changes in engagement level								
	-4	-3	-2	-1	0	1	2	3	4
$a = M$	0	0.0028	0.0123	0.1610	0.6303	0.1620	0.0203	0.0106	0.0006
$a = I$	0	0.0032	0.0145	0.1973	0.5496	0.1984	0.0231	0.0131	0.0008

Table 3: Conditional Transition Probabilities. Rows correspond to the two action M and I. Columns correspond to changes in engagement level: -4 means engagement level goes down by 4 from the current engagement level; 0 means engagement level remains the same; 4 means engagement level goes up by 4 levels, etc.

878 Ebo provided information on the value of a monetization action μ_M but did not provide infor-
879 mation about μ_I. Typically, μ_I depends on a variety of factors, including the nature of the action
880 (for instance the length of a video ad), and the nature of contacts with third-party providers. We
881 normalize $\mu_M = 1$ and set $\mu_I = 0.05$ (as reflected in Table 2). This choice of μ_I is reasonable, the
882 value of an incented action is typically cents on the dollar when compared to monetization. Other
883 choices for μ_I were considered, with the same qualitative findings as below.

884 Ideally, we would also like to provide an estimate of the cannibalization parameter α. However,
885 since incented actions were always available to players during the data collection period. Thus,
886 unlike estimates of μ_I which are available to Ebo but not part of our data set, α is not even
887 currently available to Ebo. In fact, one of the proposals we made to Ebo is to experiment with
888 removing incented actions for select players for a period of time to learn about α. Accordingly,
889 below we will often parameterize our results regarding the parameter α. Once Ebo gets an estimate
890 of α, then all the parameters of (7) can be estimated and solved numerically to optimality.

891 **5.2 Optimal policies.** In this section, we numerically examine the structure of optimal policies
892 to the publisher's problem (7) using the parameter values estimated in Table 2. We examine how
893 this optimal structure numerically depends on α, the degree of cannibalization in the game. One
894 feature that stands out is the prevalence of optimal *threshold* contracts, where incented actions are
895 offered up to a specified engagement level and thereafter removed. In a final subsection, we show
896 how the threshold level depends on the parameters of the model.

897 Given a value for α and the data in Table 2 we use policy iteration to determine optimal policies
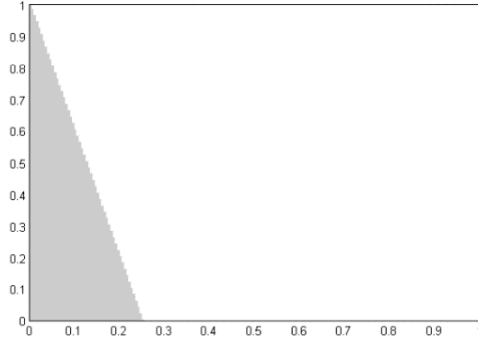
Figure 4: Check threshold policy when changing $\alpha(0)$ and α_{step}.

for (7). Our implementation of policy iteration is entirely standard (see for instance, Puterman (1994) for details). To give the reader a sense of how the optimal policies look, we take a particular instance of α (in fact it was chosen randomly among non-decreasing vectors), reflected in the second column on Table 4. The third column captures the associated optimal decision y^* to (7) given this specification of α and the data in Table 2. The fourth column provides the optimal value $W^*(e) = W^y(e)$ of the policy, as a function of the starting engagement level e. The optimal policy

e	$\alpha(e)$	$y^*(e)$	$W^*(e)$
1	0.0975	1	13.224
2	0.2785	1	12.656
3	0.5469	1	12.815
4	0.9575	0	13.792
5	0.9649	0	14.4

Table 4: Optimal policy and optimal value function for a specified α.

presented in Table 4 is a *threshold* policy. Incented actions are offered at engagement levels at or below 3. For higher engagement levels incented actions are removed. This type of policy is easy to implement from Ebo's perspective. Simply track the individual play of a user and have the game deliver ads only when average usage (as measured by a five-day moving average) is below a certain threshold. The technology to personalize the delivery of ads to individual players exists and is not uncommonly employed.

5.2.1 Prevalence of threshold policies. The optimal threshold structure in Table 4 is not atypical. In fact, *most* instances we generated had an optimal threshold structure. We present some of these instances here. For visualization purposes, we focused on *linear* functions for α; that is, $\alpha(e) = \alpha(1) + \alpha_{step}(e-1)$ where we systematically generated values for $\alpha(1)$ and α_{step} in $[0, 1]$ independently. For each generated α we computed the optimal policy. Figure 4 captures the result: gray means "the optimal policy is a threshold policy"; **black** means "the optimal policy is not a threshold"; while a white color means "not applicable" because $q_M(e) = p_M(e) + \alpha(e)p_I(e)$ becomes greater than 1 for some e which violates our assumptions. We note that no there is no "black" in
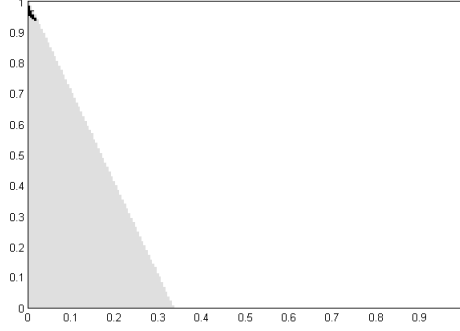
Figure 5: Threshold and non-threshold policies under engagement level breakdown $[15, 98, 108, 120]$.

Figure 4; that is, all instances generated in this way are threshold policies. This finding was even robust across different criterion for determining the engagement level (the first column of Table 2) and choosing a three- or seven-day moving average instead of a five-day moving average. It was only by choosing a "nonlinear" α (captured in Table 5) that we could find instances where policy iteration did not produce a threshold policy, this also after 8000 randomly drawn instances that all resulted in threshold policies. However, if we specify the engagement levels differently, there

e	$\alpha(e)$	$y^*(e)$	$W^*(e)$
1	0.0687	1	12.4
2	0.497	1	11.8
3	0.930	0	11.8
4	0.931	1	12.3
5	0.970	0	13.6

Table 5: An example of a non-threshold policy.

can be entire regions where there exist non-threshold optimal policies. By randomly generating engagement criteria we generated the scenario in Figure 5. The first engagement level was for 0 to 15 minutes of play per day, the second from 15 to 98 minutes, the third from 98 to 108 minutes, and the fourth from 108 to 120 minutes.

5.2.2 Data-driven sensitivity analysis. Next, we examine sensitivity results in this setting in the spirit of those conduction analytically in Section 4.3. Our main findings here are in concert with the analytical results in Section 4.3.

We explore how changes in the revenue parameters μ_I and μ_M affect the optimal threshold. In fact, what matters in the optimization problem is the ratio μ_I/μ_M since one of the parameters can always be normalized. Figure 6 illustrates a typical result for two different choices of α. Observe that the threshold increases in the ratio μ_I/μ_M. This result is intuitive. As incented actions yield more revenue (relative to monetization) it becomes more attractive to offer them more extensively; that is, for more engagement levels. Conversely, if monetization yields more revenue (relative to incented actions) then incented actions should be used less extensively. This provides clear guidance for Ebo. If incented actions are sufficiently lucrative, then they should always be offered. This is

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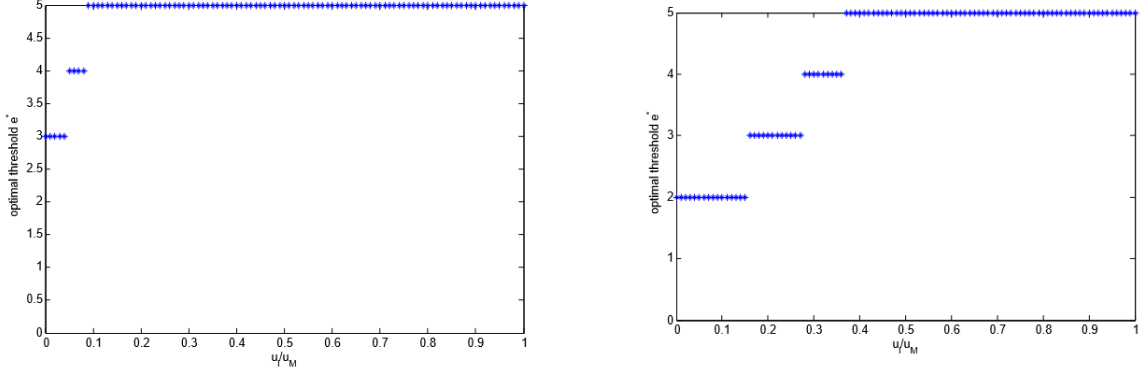
(a) $\alpha = [0.127, 0.632, 0.815, 0.906, 0.913]$.

(b) $\alpha = [0.805, 0.879, 0.947, 0.958, 0.965]$.

Figure 6: The effect of μ_I/μ_M on the optimal threshold.

the current policy of Ebo. However, Ebo's policy to always offer incented actions is uninformed by knowledge of the specific value of α. This provides another justification for Ebo to at least attempt to learn about α if only to test how suboptimal (if at all) is their policy of always offering incented actions. Notice also that this conclusion supports the conclusion of the analytical result Proposition 6.

We conducted similar sensitivity analysis studies as in Section 4.3 and found that our data-driven results confirmed the basic intuition discussed there. This illustrates a type of robustness of our conclusions for managing incented action in games across different defining notions of engagement.

6 Conclusion

In this paper, we investigated the use of incented actions in mobile games, a popular strategy for extracting additional revenue from players in freemium games where the vast majority of players are unlikely to monetize. We discussed the reasons for offering incented actions, and built an analytical model to assess the associated tradeoffs. This understanding leads us to define sufficient conditions for the optimality of threshold policies, which we later analyzed to provide managerial insights into what types of game designs are best suited to offering incented actions. Our approach of using an MDP has some direct benefit to practitioners. With player data and relevant game parameters that companies have access to in the age of big data, validating our model and using it to derive insights on the impact of certain policies is plausible.

Our analytical approach was to devise a parsimonious stylized model that abstracts a fair deal from reality and yet nonetheless maintained the salient features needed to assess the impact and effects of offering incented actions. For instance, we assume the publisher has complete knowledge about the player's transition probabilities and awareness of the engagement state. In the setting where transition probabilities are unknown, some statistical learning algorithm and classification of players into types would be required. Moreover, in the situation where engagement is difficult to define or measure, a partially observed Markov decision process (POMDP) model would be

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required, where only certain signals of the player's underlying engagement can be observed. There is also the question of micro-founding the player model that we explore, asking what random utility model could give rise to the transition probabilities that we take as given in our model. All these questions are outside of our current scope but could nonetheless add realism to our approach. Of course, the challenge of establishing the existence of threshold policies in these extensions is likely to be prohibitive. Indeed, discovering analytical properties of optimal policies of any form in a POMDP is challenging (Krishnamurthy and Djonin, 2007). It is likely that these extensions would produce studies that are more algorithmic and numerical, whereas in the current study we were interested in deriving analytical insights.

Finally, the current study ignores an important actor in the case of games hosted on mobile platforms – the platform holder. In the case of the iOS App Store, Apple has made several interventions that either limited or more closely monitored the practice of incented actions.[8] In fact, the platform holder and game publisher have misaligned incentives when it comes to incented actions. Typically, the revenue derived from incented actions is not processed through the platform, whereas in-app purchases are. We feel that investigation of the incentive misalignment problem between platform and publisher, possibly as a dynamic contracting problem, is a promising area of future research. The model developed here is a building block for such a study.

Acknowledgments

Notes

[1] http://appshopper.com/bestsellers/games/gros/?device=iphone
[2] https://play.google.com/store/apps/collection/topgrossing?hl=en
[3] https://www.engadget.com/2013/12/18/fun-app-store-facts-from-distimos-2013-year-in-review/
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A Electronic companion: Technical proofs

A.1 Derivation of expected total value in (6)**.** Given a policy y, the induced stochastic process underlying our problem is an *absorbing Markov chain* (for a discussion on absorbing Markov chains see Chapter III.4 of Taylor and Karlin (2014)). An absorbing Markov chain is one where every state can reach (with nonzero probability) an absorbing state. In our setting the absorbing state is the quit state -1 and Assumption 1 assures that the quit state is reachable from every engagement level. Absorbing Markov chains under the total reward criterion have been studied in the literature (see, for instance, Hinderer and Waldmann (2005) and the references therein).

The absorbing Markov chain structure allows for clean formulas for the total expected reward. Policy y induces a Markov chain transition matrix

$$P^y := \begin{bmatrix} S^y & s^y \\ 0 & 1 \end{bmatrix}$$

where S^y is an $n+1$ by $n+1$ matrix with entries corresponding to the transition probabilities between engagement levels, given the policy y. The vector s^y has entries corresponding to the quitting probabilities of the engagement levels, and the bottom right corner "1" indicates the absorbing nature of the quitting state -1.

Associated with policy y and the transition matrix P^y is a *fundamental matrix* $M^y := \sum_{k=0}^{\infty} S^k = (I_{n+1} - S)^{-1}$ where I_{n+1} is the $n+1$ by $n+1$ identity matrix. The fundamental matrix is a key ingredient for analyzing absorbing Markov chains. Its entries have the following useful interpretation: the (e, e')th entry $n_{e,e'}^y$ of M^y is the expected number of visits to engagement level e' starting in engagement level e before being absorbed in the quit state. Using the entries of the fundamental matrix we can write a closed-form formula for the total expected revenue of policy y:

$$W^y(e) = \sum_{e' \in E} n_{e,e'}^y r(e', y(e')).$$

An advantage of this expression over (5) is that the former is a finite sum over the number of engagement levels and does not explicitly involve the time index t.

A.2 Proof of Proposition 1**.** The following is an important lemma to understand the nature of the fundamental matrix in our setting:

Lemma 2. *The matrix Q is upper bidiagonal and its component is denoted by $k_{i,j}$,*

$$S = \begin{bmatrix} k_{1,1} & k_{1,2} & 0 & \ldots & 0 & 0 \\ 0 & k_{2,2} & k_{2,3} & 0 & \ldots & 0 \\ & & \ldots & \ldots & & \\ 0 & 0 & \ldots & 0 & k_{N-1,N-1} & k_{N-1,N} \\ 0 & 0 & 0 & 0 & \ldots & k_{N,N} \end{bmatrix}$$

the corresponding fundamental matrix $(I - S)^{-1}$ is upper triangular and its (i,j)-th entry is $\frac{1}{(1-k_{i,i})}$ if $i = j$ and $\frac{\prod_{v=i}^{j-1} k_{v,v+1}}{\prod_{v=i}^{j}(1-k_{v,v})}$ if $i < j$, i.e.

$$1072 \qquad (I-S)^{-1} = \begin{bmatrix} \frac{1}{(1-k_{1,1})} & \frac{k_{1,2}}{(1-k_{1,1})(1-k_{2,2})} & \cdots & \frac{\prod_{j=1}^{N-2} k_{j,j+1}}{\prod_{j=1}^{N-1}(1-k_{j,j})} & \frac{\prod_{j=1}^{N-1} k_{j,j+1}}{\prod_{j=1}^{N}(1-k_{j,j})} \\ 0 & \frac{1}{(1-k_{2,2})} & \frac{k_{2,3}}{(1-k_{2,2})(1-k_{3,3})} & \cdots & \frac{\prod_{j=2}^{N-1} k_{j,j+1}}{\prod_{j=2}^{N}(1-k_{j,j})} \\ & & \cdots & & \\ 0 & \cdots & 0 & \frac{1}{(1-k_{N-1,N-1})} & \frac{k_{N-1,N}}{(1-k_{N-1,N-1})(1-k_{N,N})} \\ 0 & 0 & 0 & \cdots & \frac{1}{(1-k_{N,N})} \end{bmatrix}$$

1073

1074 *Proof.* We prove the lemma by showing $(I-S) \times (I-S)^{-1} = I$ where $(I-S)^{-1}$ is proposed above.

1075 Denote R as the production of $(I-S)$ and $(I-S)^{-1}$. The (i,j)-th entry of R results from the

1076 multiplication of the i-th row of $(I-S)$ and the j-th column of $(I-S)^{-1}$.

1077 The i-th row of $(I-S)$ is $(0, \ldots, 0, \underbrace{1-k_{i,i}}_{i}, \underbrace{-k_{i,i+1}}_{i+1}, 0, \ldots, 0)$. For the j-th column of $(I-S)^{-1}$,

1078 we consider three possible cases:

1079 1) If $j < i$, the j-th column of $(I-S)^{-1}$ is $\left(\frac{\prod_{v=1}^{j-1} k_{v,v+1}}{\prod_{v=1}^{j}(1-k_{v,v})}, \ldots, \underbrace{0}_{i}, \underbrace{0}_{i+1}, 0, \ldots, 0 \right)^T$. Clearly the

1080 (i,j)-th entry of R is 0.

1081 2) If $j = i$, the j-th column of $(I-S)^{-1}$ is $\left(\frac{\prod_{v=1}^{j-1} k_{v,v+1}}{\prod_{v=1}^{j}(1-k_{v,v})}, \ldots, \underbrace{\frac{1}{(1-k_{i,i})}}_{i}, \underbrace{0}_{i+1}, 0, \ldots, 0 \right)^T$. So the

1082 (i,i)-th entry of R is 1.

1083 3) If $j > i$, the j-th column of $(I-S)^{-1}$ is $\left(\frac{\prod_{v=1}^{j-1} k_{v,v+1}}{\prod_{v=1}^{j}(1-k_{v,v})}, \ldots, \underbrace{\frac{\prod_{v=i}^{j-1} k_{v,v+1}}{\prod_{v=i}^{j}(1-k_{v,v})}}_{i}, \underbrace{\frac{\prod_{v=i+1}^{j-1} k_{v,v+1}}{\prod_{v=i+1}^{j}(1-k_{v,v})}}_{i+1}, 0, \ldots, 0 \right)^T$.

1084 By simply algebra, we obtain the (i,j)-th entry of R is 0. Thus, the (i,j)-th entry of R is 1 if $i = j$

1085 and is 0 otherwise and so R is an identity matrix. $\qquad \square$

1086 In our model, $k_{i,j}$ indicates the transition probability from state i to state j. Suppose the

1087 policy is y^1, we have $k_{\bar{e},\bar{e}} = q_M(\bar{e})(1-\tau_M)$ and $k_{\bar{e},\bar{e}+1} = q_M(\bar{e})\tau_M$. Suppose the policy is y^2, we have

1088 $k_{\bar{e},\bar{e}} = p_M(\bar{e})(1-\tau_M) + p_I(\bar{e})(1-\tau_I)$ and $k_{\bar{e},\bar{e}+1} = p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I$. By definition, the expected

1089 number of visits $n_{\bar{e},e}^y$ is the $(\bar{e},e)$-th entry of the fundamental matrix M^y. According to Lemma 2, we

1090 have $n_{\bar{e},e}^1 = \frac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)} \times \frac{\prod_{j=\bar{e}+1}^{e-1} k_{j,j+1}}{\prod_{j=\bar{e}+1}^{e}(1-k_{j,j})}$ and $n_{\bar{e},e}^2 = \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_M}{1-p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} \times \frac{\prod_{j=\bar{e}+1}^{e-1} k_{j,j+1}}{\prod_{j=\bar{e}+1}^{e}(1-k_{j,j})}$. (We

1091 assume $\prod j = \bar{e} + 1 \bar{e} k_{j,j+1} = 1$). Because we only make local change of the policy at engagement

1092 level $\bar{e}$, $n_{\bar{e},e}^1$ and $n_{\bar{e},e}^2$ share the same term $\frac{\prod_{j=\bar{e}+1}^{e-1} k_{j,j+1}}{\prod_{j=\bar{e}+1}^{e}(1-k_{j,j})}$ where $k_{j,j}$ and $k_{j,j+1}$ depend on the policy

$y^1(j)$ for $j > \bar{e}$. In fact,

$$k_{j,j} = \begin{cases} q_M(j)(1-\tau_M) & \text{if } y^*(j) = 0 \text{ and } j < N \\ p_M(j)(1-\tau_M) + p_I(j)(1-\tau_I) & \text{if } y^*(j) = 1 \text{ and } j < N \\ q_M(j) & \text{if } y^*(j) = 0 \text{ and } j = N \\ p_M(j) + p_I(j) & \text{if } y^*(j) = 1 \text{ and } j = N \end{cases}$$

$$k_{j,j+1} = \begin{cases} q_M(j)\tau_M & \text{if } y^*(j) = 0 \text{ and } j < N \\ p_M(j)\tau_M + p_I(j)\tau_I & \text{if } y^*(j) = 0 \text{ and } j < N \end{cases}$$

Moreover, we find out that $n_{\bar{e}+1,e}^{y^1(e+1)} = \frac{\prod_{j=\bar{e}+1}^{e-1} k_{j,j+1}}{\prod_{j=\bar{e}+1}^{e}(1-k_{j,j})}$. Hence, we rewrite $n_{\bar{e},e}^1 = \frac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}$

and $n_{\bar{e},e}^2 = \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_M}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}$ for all $e > \bar{e}$. Finally, the progression effect is equivalent to the following:

$$\Delta_n(e|\bar{e}) = n_{\bar{e},e}^2 - n_{\bar{e},e}^1$$

$$= \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_M}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)} - \frac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}$$

$$= [\frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_M}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} - \frac{q_M(\bar{e})\tau_M}{1-q_M(\bar{e})(1-\tau_M)}] n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}$$

$$= \frac{p_I(\bar{e})\{\tau_I-\alpha(\bar{e})\tau_M+q_M(\bar{e})[(1-\tau_I)\tau_M-(1-\tau_M)\tau_I]\}}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]} n_{\bar{e}+1,e}^{y^1(\bar{e}+1)} \tag{15}$$

Since the denominator of (15) is positive and $n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}$ is positive, the sign of $\Delta_n(e|\bar{e})$ is completely determined by the term $\tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I]$ for all $e > \bar{e}$. It is only affected by $\bar{e}$ but not e. It means that the progression effect is uniform in sign with respect to e.

A.3 Proof of Proposition 2. By definition, $n_{\bar{e},\bar{e}}^y$ is the $(\bar{e},\bar{e})$-th entry of the fundamental matrix N^y. According to Lemma 2 (in Appendix A.2) if the policy is y^1, $k_{\bar{e},\bar{e}}^1 = q_M(\bar{e})(1-\tau_M)$ and thereby $n_{\bar{e},\bar{e}}^1 = \frac{1}{1-q_M(\bar{e})(1-\tau_M)}$. If the policy is y^2, $k_{\bar{e},\bar{e}}^2 = p_M(\bar{e})(1-\tau_M) + p_I(\bar{e})(1-\tau_I)$ and consequently $n_{\bar{e},\bar{e}}^2 = \frac{1}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)}$.

Therefore, the retention effect is equal to

$$\Delta_n(\bar{e}|\bar{e}) = n_{\bar{e},\bar{e}}^2 - n_{\bar{e},\bar{e}}^1$$

$$= \frac{1}{1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)} - \frac{1}{1-q_M(\bar{e})(1-\tau_M)}$$

$$= \frac{[1-q_M(\bar{e})(1-\tau_M)]-[1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]}$$

$$= \frac{p_I(\bar{e})[(1-\tau_I)-\alpha(\bar{e})(1-\tau_M)]}{[1-q_M(\bar{e})(1-\tau_M)][1-p_M(\bar{e})(1-\tau_M)-p_I(\bar{e})(1-\tau_I)]}$$

where the last equality comes from the fact $q_M(\bar{e}) = p_M(\bar{e}) + \alpha(\bar{e})p_I(\bar{e})$. The sign of $\Delta_n(\bar{e}|\bar{e})$ completely depends on $(1 - \tau_I) - \alpha(\bar{e})(1 - \tau_M)$. Under Assumptions 1 to 3, we have $(1 - \tau_I) \geq (1 - \tau_M) \geq \alpha(\bar{e})(1 - \tau_M)$. Hence the retention effect is nonnegative, i.e $\Delta_n(\bar{e}|\bar{e}) \geq 0$ for all $\bar{e}$.

A.4 Proof of Lemma 1. In order to prove the Theorem, we first introduce the following lemma.

Lemma 3. *For any $e = 1, \ldots, N$, $W(e, y = 0) \geq \frac{q_M(e)\mu_M}{1-q_M(e)}$ and $W(e, y = 1) \geq \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)-p_I(e)}$*

Proof of Lemma 3: The proof is by induction. Clearly, at the highest engagement level

$e = N$, we have $W(N, y = 0) = \frac{q_M(N)\mu_M}{1-q_M(N)}$ and $W(N, y = 1) = \frac{p_M(N)\mu_M+p_I(N)\mu_I}{1-p_M(N)-p_I(N)}$. Now suppose it holds for level $j \geq e + 1$, we would like to show the result still holds for level e.

$$W(e, y = 0) - \frac{q_M(e)\mu_M}{1-q_M(e)}$$

$$= \frac{q_M(e)\mu_M}{1-q_M(e)(1-\tau_M)} + \frac{q_M(e)\tau_M}{1-q_M(e)(1-\tau_M)}W(e+1) - \frac{q_M(e)\mu_M}{1-q_M(e)}$$

$$= \frac{[q_M(e)\mu_M+q_M(e)\tau_M W(e+1)][1-q_M(e)]-q_M(e)\mu_M[1-q_M(e)(1-\tau_M)]}{[1-q_M(e)(1-\tau_M)][1-q_M(e)]}$$

$$= \frac{q_M(e)\tau_M[W(e+1)(1-q_M(e))-q_M(e)\mu_M]}{[1-q_M(e)(1-\tau_M)][1-q_M(e)]}$$

By the inductive assumption, $W(e + 1) \geq W(e + 1, y = 0) \geq \frac{q_M(e+1)\mu_M}{1-q_M(e+1)} \geq \frac{q_M(e)\mu_M}{1-q_M(e)}$, we finally obtain $W(e, y = 0) - \frac{q_M(e)\mu_M}{1-q_M(e)} \geq 0$. Similarly,

$$W(e, y = 1) - \frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)-p_I(e)}$$

$$= \frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} + \frac{p_M(e)\tau_M+p_I\tau_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)}W(e+1) - \frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)-p_I(e)}$$

$$= \frac{(p_M(e)\tau_M+p_I\tau_I)[W(e+1)(1-p_M(e)-p_I(e))-p_M(e)\mu_M-p_I(e)\mu_I]}{[1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)][1-p_M(e)-p_I(e)]}$$

By the inductive assumption, we have $W(e + 1) \geq W(e + 1, y = 1) \geq \frac{p_M(e+1)\mu_M+p_I(e+1)\mu_I}{1-p_M(e+1)-p_I(e+1)} \geq \frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)-p_I(e)}$. Therefore, $W(e, y = 1) - \frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)-p_I(e)} \geq 0$. This completes the proof for Lemma 3.

We return to the proof of Lemma 1, to show the optimal value function $W(e)$ is increasing in e. It suffices to show $W(e, y = 1) \leq W(e + 1)$ and $W(e, y = 0) \leq W(e + 1)$ for any $e < N$, in that we will have $W(e) = \max\{W(e, y = 1), W(e, y = 0)\} \leq W(e + 1)$. First, we compare $W(e, y = 1)$ and $W(e + 1)$.

$$W(e, y = 0) - W(e + 1) = \frac{q_M(e)\mu_M}{1-q_M(e)(1-\tau_M)} + \frac{q_M(e)\tau_M}{1-q_M(e)(1-\tau_M)}W(e+1) - W(e+1)$$

$$= \frac{q_M(e)\mu_M}{1-q_M(e)(1-\tau_M)} - \frac{1-q_M(e)}{1-q_M(e)(1-\tau_M)}W(e+1)$$

$$= \frac{1-q_M(e)}{1-q_M(e)(1-\tau_M)}[\frac{q_M(e)\mu_M}{1-q_M(e)} - W(e+1)] \leq 0$$

where the last equality holds because $\frac{q_M(e)\mu_M}{1-q_M(e)} \leq \frac{q_M(e+1)\mu_M}{1-q_M(e+1)} \leq W(e + 1, y = 0) \leq W(e + 1)$. Similarly, we compare $W(e, y = 1)$ and $W(e + 1)$.

$$W(e, y = 1) - W(e + 1) = \frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} + \frac{p_M(e)\tau_M+p_I(e)\tau_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)}W(e+1) - W(e+1)$$

$$= \frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} - \frac{1-p_M(e)-p_I(e)}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)}W(e+1)$$

$$= \frac{1-p_M(e)-p_I(e)}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)}[\frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)-p_I(e)} - W(e+1)] \leq 0$$

where the last equality holds since $\frac{p_M(e)\mu_M+p_I(e)\mu_I}{1-p_M(e)-p_I(e)} \leq \frac{p_M(e+1)\mu_M+p_I(e+1)\mu_I}{1-p_M(e+1)-p_I(e+1)} \leq W(e + 1, y = 1) \leq W(e + 1)$.

Finally, since $W(e) = max\{W(e, y = 1), W(y = 0, e)\}$ and we have shown that both $W(e, y = 1)$ and $W(e, y = 0)$ are no greater than $W(e + 1)$, we conclude that $W(e) \leq W(e + 1)$.

A.5 Proof of Proposition 3. We denote $W^2(\bar{e}) - W^1(\bar{e}) = C(\bar{e}) + F(\bar{e})$, where $C(\bar{e})$ represents the "current" benefits of offering incented actions and $F(\bar{e})$ represents the "future" benefits. In

order to prove the optimality of the myopic policy, we first take a close look at $C(\bar{e})$ and $F(\bar{e})$. By definition,

$$C(\bar{e}) = \frac{p_M(\bar{e})\mu_M + p_I(\bar{e})\mu_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} - \frac{q_M(\bar{e})\mu_M}{1 - q_M(\bar{e})(1-\tau_M)}$$

$$= \frac{\{[p_M(\bar{e})\mu_M + p_I(\bar{e})\mu_I][1 - q_M(\bar{e})(1-\tau_M)] - q_M(\bar{e})\mu_M[1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]\}}{[1 - q_M(\bar{e})(1-\tau_M)][1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]}$$

$$= \frac{p_I(\bar{e})}{[1 - q_M(\bar{e})(1-\tau_M)][1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]}\{\mu_I - \alpha(\bar{e})\mu_M + q_M(\bar{e})[(1-\tau_I)\mu_M - (1-\tau_M)\mu_I]\}$$

$$F(\bar{e}) = \left\{\frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} - \frac{q_M(\bar{e})\tau_M}{1 - q_M(\bar{e})(1-\tau_M)}\right\}\left\{\sum_{e'>\bar{e}} n_{\bar{e}+1,e'}^{y^1} r(e', y(e'))\right\}$$

$$= \frac{\{[p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I][1 - q_M(\bar{e})(1-\tau_M)] - q_M(\bar{e})\tau_M[1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]\}}{[1 - q_M(\bar{e})(1-\tau_M)][1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]}\left\{\sum_{e'>\bar{e}} n_{\bar{e}+1,e'}^{y^1} r(e', y(e'))\right\}$$

$$= \frac{p_I(\bar{e})\{\sum_{e'>\bar{e}} n_{\bar{e}+1,e'}^{y^1(\bar{e}+1)} r(e', y(e'))\}}{[1 - q_M(\bar{e})(1-\tau_M)][1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]}\{\tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I]\}$$

We further define

$$\delta_1(\bar{e}) = \mu_I - \alpha(\bar{e})\mu_M + q_M(\bar{e})[(1-\tau_I)\mu_M - (1-\tau_M)\mu_I]$$

$$\delta_2(\bar{e}) = \tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I]$$

Clearly, the sign of $C(\bar{e})$ is determined by $\delta_1(\bar{e})$ and the sign of $F(\bar{e})$ is determined by $\delta_2(\bar{e})$. Now suppose $\tau_I/\tau_M = \mu_I/\mu_M$, it leads to $\delta_1(\bar{e})/\mu_M = \delta_2(\bar{e})/\tau_M$. Therefore, $\delta_1(\bar{e})$ and $\delta_2(\bar{e})$ must have the same sign. It implies that whenever $\delta_1(\bar{e})$ is positive, we must have $\delta_2(\bar{e})$ positive and thereby $W^2(\bar{e}) - W^1(\bar{e})$ positive. Similarly, whenever $\delta_1(\bar{e})$ is negative, we must have $\delta_2(\bar{e})$ negative and thereby $W^2(\bar{e}) - W^1(\bar{e})$ negative.

Notice that the previous analysis does not rely on the policy for higher engagement level $e > \bar{e}$. Even if we fix $y^1(e)$ to be the optimal action which is solved by backward induction for $e > \bar{e}$, we still have that the "current" benefit $C(\bar{e})$ and the "future" benefit $F(\bar{e})$ share the same sign. Therefore, the optimal action at engagement level $\bar{e}$ can be determined by whether $C(\bar{e})$ is positive or negative. Equivalently speaking, the myopically-optimal policy will be the optimal policy.

A.6 Proof of Proposition 4. By definition, the revenue effect is

$$\Delta_r(\bar{e}) = p_M(\bar{e})\mu_M + p_I(\bar{e})\mu_I - q_M(\bar{e})\mu_M = p_I(\bar{e})[\mu_I - \alpha(\bar{e})\mu_M]$$

where the last equality comes from the fact $q_M(\bar{e}) = p_M(\bar{e}) + \alpha(\bar{e})p_I(\bar{e})$.

First of all, as $p_I(\bar{e})$ is non-negative, the sign of $\Delta_r(\bar{e})$ is determined by $\mu_I - \alpha(\bar{e})\mu_M$ which is decreasing in $\bar{e}$. Thus if $\Delta_r(\bar{e}) < 0$ for some $\bar{e}$, we will also have $\Delta_r(e') < 0$ for all $e' \geq \bar{e}$.

Moreover, both $p_I(\bar{e})$ and $\mu_I - \alpha(\bar{e})\mu_M$ are nonincreasing in $\bar{e}$, so we conclude that $\Delta_r(\bar{e})$ is nonincreasing in $\bar{e}$ whenever $\mu_I - \alpha(\bar{e})\mu_M > 0$, i.e. whenever $\Delta_r(\bar{e}) > 0$.

A.7 Proof of Proposition 5. (a) Recall the expression for $\Delta_n(e|\bar{e})$.

$$\Delta_n(e|\bar{e}) = \frac{p_I(\bar{e})n_{\bar{e}+1,e}^{y^1(\bar{e}+1)}}{[1 - q_M(\bar{e})(1-\tau_M)][1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]}\{\tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I]\}$$

The sign of $\Delta_n(e|\bar{e})$ is completely decided by $\delta_2(\bar{e}) = \tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[(1-\tau_I)\tau_M - (1-\tau_M)\tau_I] = \tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[\tau_M - \tau_I]$.

1192   Assumption 6 indicates that $-\alpha(\bar{e})\tau_M + q_M(\bar{e})\tau_M$ will decrease in $\bar{e}$. Besides, $-q_M(\bar{e})\tau_I$ will also

1193   decrease in $\bar{e}$. Therefore, $\delta_2(\bar{e})$ will be a decreasing function of $\bar{e}$. It implies $\Delta_n(e|\bar{e})$ satisfies the

1194   following property: if $\Delta_n(e|\bar{e}) > 0$ for some $\bar{e}$, then $\Delta_n(e|e') > 0$ for all $e' \le \bar{e}$; and if $\Delta_n(e|\bar{e}) < 0$

1195   for some $\bar{e}$, then $\Delta_n(e|e') < 0$ for all $e' \ge \bar{e}$

1196   (b) From the previous analysis, we have already seen that the sign of $C(\bar{e})$ only depends on the

1197   term $\delta_1(\bar{e}) = \mu_I - \alpha(\bar{e})\mu_M + q_M(\bar{e})[(1-\tau_I)\mu_M - (1-\tau_M)\mu_I]$ and the sign of $F(\bar{e})$ only depends on

1198   the term $\delta_2(\bar{e}) = \tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[\tau_M - \tau_I]$. From (a), we have already shown that $\delta_2(\bar{e})$ will

1199   decrease in $\bar{e}$. Therefore, if $F(\bar{e}) > 0$ for some $\bar{e}$, then $F(\bar{e}) > 0$ for all $e' \le \bar{e}$; if $F(\bar{e}) < 0$ for some

1200 $\bar{e}$, then $F(\bar{e}) < 0$ for all $e' \ge \bar{e}$.

1201   Similarly, Assumption 6 ensures that $-\alpha(\bar{e})\mu_M + q_M(\bar{e})(1-\tau_I)\mu_M$ will also decrease in $\bar{e}$, in

1202   that $\alpha(\bar{e}+1)\mu_M - q_M(\bar{e}+1)(1-\tau_I)\mu_M - \alpha(\bar{e})\mu_M + q_M(\bar{e})(1-\tau_I)\mu_M = [\alpha(\bar{e}+1) - \alpha(\bar{e})]\mu_M -$

1203 $[q_M(\bar{e}+1) - q_M(\bar{e})]\mu_M(1-\tau_I) \ge \mu_M[(\alpha(\bar{e}+1) - \alpha(\bar{e})) - (q_M(\bar{e}+1) - q_M(\bar{e}))] \ge 0$. In addition,

1204 $-q_M(\bar{e})(1-\tau_M)\mu_I$ will decrease in $\bar{e}$ as well. As a result, $\delta_1(\bar{e})$ will also be an decreasing function

1205   of $\bar{e}$. Hence $C(\bar{e})$ satisfies the following property: if $C(\bar{e}) > 0$ for some $\bar{e}$, then $C(\bar{e}) > 0$ for all

1206 $e' \le \bar{e}$; if $C(\bar{e}) < 0$ for some $\bar{e}$, then $C(\bar{e}) < 0$ for all $e' \ge \bar{e}$.

1207   **A.8 Proof of Theorem 1.** In order to prove the optimal policy is a threshold policy, it suffices

1208   to show if there exists some $\bar{e}$ such that $W(\bar{e}, y = 1) - W(\bar{e}, y = 0) > 0$, then we must have

1209 $W(e, y = 1) - W(e, y = 0) > 0$ for all $e < \bar{e}$.

1210   First of all, Assumption 5 guarantees that it is optimal not to offer incented action at the highest

1211   engagement level. Because $W(N, y = 1) = \frac{p_M(N)\mu_M + p_I(N)\mu_I}{p_Q(N)} < \frac{p_M(N)\mu_M + p_I(N)\mu_M}{p_Q(N)} = \frac{q_M(N)\mu_M}{q_Q(N)} =$

1212 $W(N, y = 0)$. Suppose that $W(\bar{e}, y = 1) - W(\bar{e}, y = 0) > 0$ for some $\bar{e} < N$ where

1213
1214 $$W(\bar{e}, y = 1) - W(\bar{e}, y = 0) = \frac{p_I(\bar{e})}{[1 - q_M(\bar{e})(1-\tau_M)][1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)]}\{\delta_1(\bar{e}) + \delta_2(\bar{e}) \sum_{e > \bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e, y(e))\}$$

1215   so it is equivalent to assume $\delta_1(\bar{e}) + \delta_2(\bar{e}) \sum_{e > \bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e, y(e)) > 0$. It implies that at least

1216   one of $\delta_1(\bar{e})$ and $\delta_2(\bar{e})$ has to be positive.

1217   We would like to show $W(\bar{e}-1, y = 1) - W(\bar{e}-1, y = 0) > 0$. It suffices to show $\delta_1(\bar{e}-1) +$

1218 $\delta_2(\bar{e}-1) \sum_{e > \bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e, y(e)) > 0$. Now we consider three possible scenarios:

1219   (1.1) Suppose $\delta_1(\bar{e}) > 0$ and $\delta_2(\bar{e}) > 0$. We have already proven that $\delta_1(\bar{e})$ and $\delta_2(\bar{e})$ are

1220   decreasing functions in $\bar{e}$ under Assumption 6. Therefore, $\delta_1(\bar{e}-1) > \delta_1(\bar{e}) > 0$ and $\delta_2(\bar{e}-1) >$

1221 $\delta_2(\bar{e}) > 0$. Clearly, we have $W(\bar{e}-1, y = 1) - W(\bar{e}-1, y = 0) > 0$ in this case.

1222   (1.2) Suppose $\delta_1(\bar{e}) < 0$ but $\delta_2(\bar{e}) > 0$ (which may happen only if $\mu_I/\mu_M < \tau_I/\tau_M$). We first

1223   prove under Assumption 7, $\delta_2(\bar{e}-1) \sum_{e > \bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e, y(e)) > \delta_2(\bar{e}) \sum_{e > \bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e, y(e))$. Notice

1224   that

1225 $$\delta_2(\bar{e}-1) \sum_{e > \bar{e}-1} n_{\bar{e},e}^{y^*(\bar{e})} r(e, y(e)) = \delta_2(\bar{e}-1)\{n_{\bar{e},\bar{e}}^{y^*(\bar{e})} r(\bar{e}, y^*(\bar{e})) + n_{\bar{e},\bar{e}+1}^{y^*(\bar{e})} r(\bar{e}+1, y^*(\bar{e}+1)) + \cdots + n_{\bar{e},N}^{y^*(\bar{e})} r(N, y^*(N))\}$$

1226
1227 $$\delta_2(\bar{e}) \sum_{e > \bar{e}} n_{\bar{e}+1,e}^{y^*(\bar{e}+1)} r(e, y(e)) = \delta_2(\bar{e})\{n_{\bar{e}+1,\bar{e}+1}^{y^*(\bar{e}+1)} r(\bar{e}+1, y^*(\bar{e}+1)) + \cdots + n_{\bar{e}+1,N}^{y^*(\bar{e}+1)} r(N, y^*(N))\}$$

EC.6

For $e > \bar{e}$, we have such relationship $n^1_{\bar{e},e} = \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} n^{y^*(\bar{e}+1)}_{\bar{e}+1,e}$ since $y^*(\bar{e}) = 1$. As a result,

$$\delta_2(\bar{e}-1) \sum_{e > \bar{e}-1} n^{y^*(\bar{e})}_{\bar{e},e} r(e, y(e)) = \delta_2(\bar{e}-1) n^{y^*(\bar{e})}_{\bar{e},\bar{e}} r(\bar{e}, y^*(\bar{e})) + \delta_2(\bar{e}-1) \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} \sum_{e > \bar{e}} n^{y^*(\bar{e}+1)}_{\bar{e}+1,e} r(e, y(e))$$

Next we are going to show $\delta_2(\bar{e}-1) \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} > \delta_2(\bar{e})$ under Assumption 7. Since $\delta_2(\bar{e}-1) > \delta_2(\bar{e}) > 0$, we can compare $\frac{\delta_2(\bar{e})}{\delta_2(\bar{e}-1)}$ with $\frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)}$. For the ratio $\frac{\delta_2(\bar{e})}{\delta_2(\bar{e}-1)}$, we have

$$\frac{\delta_2(\bar{e})}{\delta_2(\bar{e}-1)} = \frac{\tau_I - \alpha(\bar{e})\tau_M + q_M(\bar{e})[\tau_M - \tau_I]}{\tau_I - \alpha(\bar{e}-1)\tau_M + q_M(\bar{e}-1)[\tau_M - \tau_I]} = \frac{\tau_I/\tau_M - \alpha(\bar{e}) + q_M(\bar{e})[1 - \tau_I/\tau_M]}{\tau_I/\tau_M - \alpha(\bar{e}-1) + q_M(\bar{e}-1)[1 - \tau_I\tau_M]}$$

$$= \frac{1 - q_M(\bar{e})}{1 - q_M(\bar{e}-1)} + \frac{[\alpha(\bar{e}-1) - q_M(\bar{e}-1)]\frac{1 - q_M(\bar{e})}{1 - q_M(\bar{e}-1)} - [\alpha(\bar{e}) - q_M(\bar{e})]}{\tau_I/\tau_M - \alpha(\bar{e}-1) + q_M(\bar{e}-1)[1 - \tau_I/\tau_M]} \leq \frac{1 - \alpha(\bar{e})}{1 - \alpha(\bar{e}-1)} \qquad (16)$$

Because $[\alpha(\bar{e}-1) - q_M(\bar{e}-1)]\frac{1 - q_M(\bar{e})}{1 - q_M(\bar{e}-1)} - [\alpha(\bar{e}) - q_M(\bar{e})] < 0$ and $1 - q_M(\bar{e}-1) > 0$, the ratio $\frac{\delta_2(\bar{e})}{\delta_2(\bar{e}-1)}$ will increase in τ_I/τ_M and reach maximum when $\tau_I/\tau_M = 1$. We achieve (16).

Finally, Assumption 7 claims that $\frac{1 - \alpha(\bar{e})}{1 - \alpha(\bar{e}-1)} \leq \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)}$. Hence, we end up with $\delta_2(\bar{e}-1) \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} > \delta_2(\bar{e})$. We further have

$$\delta_2(\bar{e}-1) \sum_{e > \bar{e}-1} n^{y^*(\bar{e})}_{\bar{e},e} r(e, y(e)) = \delta_2(\bar{e}-1) n^2_{\bar{e},\bar{e}} r(\bar{e}, y^*(\bar{e}))$$

$$+ \delta_2(\bar{e}-1) \frac{p_M(\bar{e})\tau_M + p_I(\bar{e})\tau_I}{1 - p_M(\bar{e})(1-\tau_M) - p_I(\bar{e})(1-\tau_I)} \sum_{e > \bar{e}} n^{y^*(\bar{e}+1)}_{\bar{e}+1,e} r(e, y(e))$$

$$> \delta_2(\bar{e}) \sum_{e > \bar{e}} n^{y^*(\bar{e}+1)}_{\bar{e}+1,e} r(e, y(e))$$

Finally we conclude

$$\delta_1(\bar{e}-1) + \delta_2(\bar{e}-1) \sum_{e > \bar{e}-1} n^{y^*(\bar{e})}_{\bar{e},e} r(e, y(e)) > \delta_1(\bar{e}) + \delta_2(\bar{e}) \sum_{e > \bar{e}} n^{y^*(\bar{e}+1)}_{\bar{e}+1,e} r(e, y(e)) > 0$$

equivalently, we have $W(\bar{e}-1, y=1) - W(\bar{e}-1, y=0) > 0$ in this case.

(1.3) Suppose $\delta_1(\bar{e}) > 0$ but $\delta_2(\bar{e}) < 0$ (which may happen only if $\mu_I/\mu_M > \tau_I/\tau_M$). If $\delta_2(\bar{e}-1) \geq 0$, we easily get $\delta_1(\bar{e}-1) + \delta_2(\bar{e}-1)W(\bar{e}) > \delta_1(\bar{e}) + \delta_2(\bar{e}-1)W(\bar{e}) > 0$. Else if $0 > \delta_2(\bar{e}-1) \geq \delta_2(\bar{e})$, Lemma 1 indicates that $0 \leq W(\bar{e}) \leq W(\bar{e}+1)$, therefore we have $\delta_2(\bar{e}-1)W(\bar{e}) > \delta_2(\bar{e})W(\bar{e}) > \delta_2(\bar{e})W(\bar{e}+1)$. Finally, $\delta_1(\bar{e}-1) + \delta_2(\bar{e}-1) \sum_{e > \bar{e}-1} n^{y^*(\bar{e})}_{\bar{e},e} r(e, y(e)) = \delta_1(\bar{e}-1) + \delta_2(\bar{e}-1)W(\bar{e}) > \delta_1(\bar{e}) + \delta_2(\bar{e})W(\bar{e}+1) = \delta_1(\bar{e}) + \delta_2(\bar{e}) \sum_{e > \bar{e}} n^{y^*(\bar{e}+1)}_{\bar{e}+1,e} r(e, y(e)) > 0$. Therefore, $W(\bar{e}-1, y=1) - W(\bar{e}-1, y=0) > 0$.

In conclusion, we have shown that once $W(\bar{e}, y=1) - W(\bar{e}, y=0) > 0$ for some $\bar{e}$, we must also have $W(\bar{e}-1, y=1) - W(\bar{e}-1, y=0) > 0$. As a result, the optimal policy should be a forward threshold policy.

A.9 Example of a non-threshold policy where Assumption 6 is violated. Consider the following two engagement level example. Assume $\mu_M = 1$, $\mu_I = 0.27$, $\tau_M = 0.99$, $\tau_I = 0.25$. At level 0, $p_M(0) = 0.23$, $p_I(0) = 0.54$, $\alpha(0) = 0.75$ and thereby $q_M(0) = 0.635$. At level 1, $p_M(1) = 0.34$, $p_I(1) = 0.52$, $\alpha(1) = 0.81$ and thereby $q_M(1) = 0.7612$. At level 2, $p_M(2) = 0.42$, $p_I(2) = 0.45$,

1263 $\alpha(2) = 1$ and thereby $q_M(1) = 0.87$.

1264   The optimal policy is $y^* = (0, 1, 0)$. We use backward induction. At the highest level 2, we

1265 have $y^*(2) = 0$ and $W(2) = 6.692$. At level 1, $W(1, y = 1) = 5.9395$ and $W(1, y = 0) = 5.849$,

1266 therefore $y^*(1) = 1$ and $W(1) = W(1, y = 1) = 5.9395$. Finally, at level 0, $W(0, y = 1) = 4.2684$

1267 and $W(0, y = 0) = 4.3969$ and so $y^*(0) = 0$ and $W(0) = W(0, y = 0) = 4.3969$. The optimal policy

1268 is not a threshold policy.

1269   In fact, Assumption 6 is violated because $\alpha(1) - \alpha(0) = 0.06$ while $q_M(1) - q_M(0) = 0.1262$.

1270 Assumption 7 is satisfied since $1 - \alpha(1) = 0.19$ and $(1 - \alpha(0))\frac{p_M(1)\tau_M + p_I(1)\tau_I}{(1 - p_M(1)(1 - \tau_M) - p_I(1)(1 - \tau_I))} = 0.1923$.

1271 **A.10 Example of non-threshold policy where Assumption 7 is violated.** Consider the

1272 following two engagement level example. Assume $\mu_M = 1$, $\mu_I = 0.2$, $\tau_M = 0.91$, $\tau_I = 0.47$. At level

1273 0, $p_M(0) = 0.03$, $p_I(0) = 0.51$, $\alpha(0) = 0.59$ and thereby $q_M(0) = 0.3309$. At level 1, $p_M(1) = 0.05$,

1274 $p_I(1) = 0.5$, $\alpha(1) = 0.62$ and thereby $q_M(1) = 0.36$. At level 2, $p_M(2) = 0.34$, $p_I(2) = 0.45$,

1275 $\alpha(2) = 1$ and thereby $q_M(1) = 0.79$.

1276   The optimal policy is $y^* = (0, 1, 0)$. We use backward induction to show this. At the highest

1277 level 2, we have $y^*(2) = 0$ and $W(2) = 3.7619$. At level 1, $W(1, y = 1) = 1.6498$ and $W(1, y =$

1278 $0) = 1.6459$, therefore $y^*(1) = 1$ and $W(1) = W(1, y = 1) = 1.6498$. Moreover, $C(1) = -0.1668$

1279 and $F(1) = 0.1708$. Finally, we look at level 0. Note that $W(0, y = 1) = 0.7876$ and $W(0, y =$

1280 $0) = 0.8532$, as we can see $y^*(0) = 0$ and $W(0) = W(0, y = 0) = 0.8532$. Besides, $C(0) = -0.1595$

1281 and $F(0) = 0.0939$. The optimal policy is not a threshold policy. In fact, Assumption 6 is satisfied

1282 since $\alpha(1) - \alpha(0) = 0.03$ while $q_M(1) - q_M(0) = 0.0291$.

1283 **A.11 Proof of Proposition 6.** We will restrict ourselves only to threshold policies. According

1284 to the backward induction, in order to prove the optimal threshold is non-decreasing in μ_I, we only

1285 need to show $W^2(e) - W^1(e)$ is non-decreasing in μ_I given that $y^1(e') = 0$ for $e' > e$. Because

1286 the optimal threshold e^* is solved by $W^2(e^*) - W^1(e^*) > 0$ where $y^1(e') = 0$ for $e' > e^*$ and

1287 $W^2(e^* + 1) - W^1(e^* + 1) \leq 0$ where $y^1(e') = 0$ for $e' > e^* + 1$.

1288   We have already characterized the explicit expression for $W^2(e) - W^1(e)$ which is

1289
$$W^2(e) - W^1(e) = \begin{cases} \frac{p_I(e)}{[1 - q_M(e)(1 - \tau_M)][1 - p_M(e)(1 - \tau_M) - p_I(e)(1 - \tau_I)]}\{\delta_1(e) + \delta_2(e)\sum_{e' > e} n_{e+1,e'}^{y^1(e+1)} r(e', y(e'))\} & \text{if } e < N \\ \frac{p_I(N)}{p_Q(N) q_Q(N)}\{\mu_I - \alpha(N)\mu_M + q_M(N)(\mu_M - \mu_I)\} & \text{if } e = N \end{cases}$$

1290   where $\delta_1(e) = [\mu_I - \alpha(e)\mu_M + q_M(e)(1 - \tau_I)\mu_M - q_M(e)(1 - \tau_M)\mu_I]$

1291 $\delta_2(e) = [\tau_I - \alpha(e)\tau_M + q_M(e)(\tau_M - \tau_I)]$

1293 Clearly, $W^2(N) - W^1(N)$ will increase in μ_I. For any $e < N$, $\delta_1(e)$ will increase in μ_I while

1294 $\delta_2(e)$ will keep constant. In addition, both $n_{e+1,e'}^{y^1(e+1)}$ and $r(e', y(e'))$ will remain the same since we

1295 fix $y^1(e') = 0$ unchanged for all $e' > e$. Hence, $W^2(e) - W^1(e)$ will increase in μ_I. Let $\hat{e}^*$ be

1296 the largest engagement level such that $W^2(e) - W^1(e) > 0$. By definition, $\hat{e}^*$ is actually the new

1297 optimal threshold under a larger μ_I. Since originally $W^2(e^*) - W^1(e^*) > 0$ and the difference

1298 $W^2(e) - W^1(e)$ is increasing in μ_I, we conclude that $\hat{e}^* \geq e^*$. The optimal threshold must be

non-decreasing in μ_I.

A.12 Proof of Proposition 7. Similar to Proposition 6, we will still restrict ourselves to threshold policies. It suffices to show $W^2(e) - W^1(e)$ is non-decreasing in τ_I given that $y^1(e') = 0$ for $e' > e$.

Obviously, $W^2(N) - W^1(N)$ does not depend on τ_I. For $e < N$, since $y^1(e') = 0$ for $e' > e$, we have $r(e', y(e'))$ and $n_{e+1,e'}^{y^1(e+1)}$ unrelated with τ_I. Therefore, $W^1(e) = \frac{q_M(e)\mu_M}{1-q_M(e)(1-\tau_M)} + \frac{q_M(e)\tau_M}{1-q_M(e)(1-\tau_M)} \sum_{e'>e} n_{e+1,e'}^{y^1(e+1)} r(e', y(e'))$ will not be affected by τ_I. Now we would like to show $W^2(e) = \frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} + \frac{p_M(e)\tau_M + p_I(e)\tau_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} \sum_{e'>e} n_{e+1,e'}^{y^1(e+1)} r(e', y(e'))$ will increase in τ_I. In fact, if τ_I increases by $\epsilon > 0$, we have

$$[\frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I-\epsilon)} + \frac{p_M(e)\tau_M + p_I(e)(\tau_I+\epsilon)}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I-\epsilon)} \sum_{e'>e} n_{e+1,e'}^{y^1(e+1)} r(e', y(e'))]$$

$$- [\frac{p_M(e)\mu_M + p_I(e)\mu_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} + \frac{p_M(e)\tau_M + p_I(e)\tau_I}{1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)} \sum_{e'>e} n_{e+1,e'}^{y^1(e+1)} r(e', y(e'))]$$

$$= \frac{\epsilon p_I(e)q_Q(e)}{[1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I-\epsilon)][1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)]} \{\sum_{e'>e} n_{e+1,e'}^{y^1(e+1)} r(e', y(e')) - \frac{p_M(e)\mu_M + p_I(e)\mu_I}{p_Q(e)}\}$$

$$= \frac{\epsilon p_I(e)q_Q(e)}{[1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I-\epsilon)][1-p_M(e)(1-\tau_M)-p_I(e)(1-\tau_I)]} \{W^{y^1}(e+1) - \frac{p_M(e)\mu_M + p_I(e)\mu_I}{p_Q(e)}\} > 0 \quad (17)$$

where $W^{y^1}(e+1)$ is the revenue at engagement level $e+1$ if Publisher follows the policy $y^1(e') = 0$ for all $e' > e$. The inequality (17) holds because of Lemma 3. As a result, $W^2(e)$ will increase in τ_I and consequently $W^2(e) - W^1(e)$ will increase in τ_I for all $e < N$. This implies that the optimal threshold must be non-decreasing in τ_I.