Optimal Level Design in Video Games

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An important problem in video game design is how to sequence game elements within a level (or "chunk") of a game. Each element has two critical features: a *reward* (e.g., earning an item or being able to watch a cinematic) and a degree of *difficulty* (e.g., how much energy or focus needed to interact with the game element). The latter property is a distinctive feature in video games, which unlike services (like a trip to the spa) or passive entertainment (like sports or movies), often require concerted effort to consume. We study how to sequence game elements to maximize the overall experienced utility of the level, subject to the dynamics of adaptation to rewards and difficulty. Utility from rewards wain as players become accustomed to them while less effort needs to be expended to overcome challenges as players become accustomed to difficulty.

We find that the optimal design depends on the relationship between rewards and difficulty, leading to qualitatively different level designs. For example, when the proportion of reward-to-difficulty is high, the optimal design mimics that of more passive experiences like that studied in Das Gupta et al. (2016). By contrast, the optimal designs of games with low reward-to-difficulty ratios resemble work-out routines with "warm-ups" and "cool-downs" book-ending intense activity. Intermediate cases follow the classical "mini-boss, end-boss" design where difficulty has two peaks. Where the peaks appear depends on the degree to which players are reward-seeking or difficulty averse. This raises a salient distinction between games designed for entertainment (attracting players seeking rewards) versus "serious" games designed for educational and training purposes (attracting players who benefit from an incentive structure to face challenges). In the former case, for example, the optimal difficulty pattern follows an N-shape: it starts out easy, reaches an internal peak, then follows a U-shaped pattern thereafter. The design does not always follow a classical crescendo or U-shaped pattern uncovered in the previous literature because these designs can stress out players. Level designs with multiple peaks of difficulty are ubiquitous in video games. In summary, this paper provides practical guidance to game designers on how to match level design to the relationship between reward and difficulty inherent in their game's mechanics.

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1. Introduction

Video games are big business, representing the largest and fastest-growing segment of the entertainment industry.[1] However, not all games are successful. One example of a failed game — *E.T. The Extra Terrestrial* — is so notorious that it became the symbol of the video game crash in the early 1980s.[2] Post-mortem's of *E.T.*'s failure points to a number of causes, not the least of which its poor design. By *design*, we mean the various elements of the game experience: art, graphics, game mechanics, story, and level design. The focus of this paper is on the latter. Video games are different from movies in that they can be very long experiences, sometimes lasting dozens of hours. Accordingly, games are must be broken down into "sessions" that can be consumed in one sitting. The most common form of "sessioning" in games is by the notion of levels. A *level* is a discrete unit of gameplay with a beginning, middle, and an end that moves forward the game's story and often introduces new obstacles or game mechanics. Level design concerns finding the right balance of game elements and sequencing them in such a way that makes the level engaging and satisfying.

Some practicing game designers have proposed the use of optimization tools to assist them in designing levels. Paul Tozour, an experienced game designer, wrote about the challenge in an article for Gamasutra, a leading video game designer website.[3] As stated in this article, a major consideration when designing a level is balancing "reward" and "difficulty" in the arrangement of game elements. To make things concrete, consider the design of a side-scrolling action game like Capcom's classic *Mega Man 2*. Each level consists of platforming sections (i.e., sections that involve skilled jumping), standard enemy encounters, and one or more "boss" (i.e., difficult) enemy encounters. Standard and boss enemy encounters test the player's strategy and reflexes while platforming sections serve as tests of dexterity and hand-eye coordination. Different types of encounters also net different rewards. Defeating standard enemies may offer much-needed boosts to health or ammunition, while boss fights may earn the player new weapons or unlock new areas for exploration.

Level design can be seen as a problem of sequencing a set of given game elements (obstacles, enemy encounters, puzzles, etc.) to form a coherent experience. Two attributes are assigned to "defeating" or "passing" a game element: (i) a *reward* for completion and (ii) a degree of *difficulty*.

[1] https://www.reuters.com/sponsored/article/popularity-of-gaming

[2] https://www.npr.org/2017/05/31/530235165/total-failure-the-worlds-worst-video-game

[3] https://gamasutra.com/blogs/PaulTozour/20131201/206006/Decision_Modeling_and_Optimization_in_Game_Design_Part_9_Modular_Level_Design.php

Our analysis reveals that differences in the the reward-to-difficulty ratio leads to qualitatively different optimal level designs.

We see a number of canonical level designs across the history of video games. An intuitive design is one of increasing difficulty and reward as the level proceeds. As the player meets earlier tests they are more prepared to tackle later challenges. However, there is also logic for a U-shaped design where levels start difficult, become easier, then crescendo towards a difficult finish. This design was not uncommon in coin-operated video game arcades where having a rapid succession of failed attempts could drive up revenues. Social pressure and bragging rights among arcade patrons can drive players to "overcome" the initial challenge, only to be rewarded by a section of the game that is easier to handle, leading up to a "boss" of monumental difficulty.

Other designs are also typical. A classical design for console action games (like *Megaman 2* described above) is the "mini-boss-end-boss" structure, where levels start out easy, reach a peak of tension in the middle of the level with a "mini-boss" encounter, then easing off before another crescendo to an even more difficult "end-boss" encounter. Other level designs resemble more of a workout routine: starting out easy (warm-up) and ending easy (cool down) with an intermediate peak of difficulty.

Our research question is simple: under what conditions are these qualitatively different level designs optimal? The "conditions" refer to the nature of the game elements themselves, namely their rewards and difficulties. The notion of optimality is that of maximizing the player's experienced utility accrued up to the end of the level. This objective reflects the fact that player satisfaction is experienced dynamically over the course of the level and is assessed when the player decides whether or not to continue the game upon the completion of a level.

In order to answer this research question, we develop an optimization model for deciding the sequence of a given set of game elements to optimize the experienced utility of the player taking into account three psychological factors: accomplishment adaptation, stress adaptation, and memory decay. Accomplishment adaptation refers to the process by which utilities from rewards can wain as players become accustomed to them.[4] Stress adaptation refers to how disutility for expending effort diminishes as players become accustomed to certain challenges. This phenomenon is well-understood by game designers. Players can adapt to difficulty quickly as they become accustomed to challenges (Kalmpourtzis 2018, Schell 2019). Memory decay refers to the psychological fact that people tend to put more emphasis on recent experiences than older experiences.[5]

[4] Studies from the psychology literature that examine and measure the adaptation process are referenced in detail in Das Gupta et al. (2016), Li et al. (2020).

[5] Studies from the psychology literature that discuss memory decay are also discussed at length in Das Gupta et al. (2016), Li et al. (2020).

Other authors have studied related research questions leading to optimization problems with a similar structure. The most related papers to ours are Das Gupta et al. (2016), Li et al. (2020) that study the optimal design of experiential services considering both memory decay and adaptation to rewards. These papers find U-shaped and so-called IU-shaped structures for service quality against time. While our method of analysis draws much inspiration from these papers, our model and results are different. Most critically, difficulty and stress are essential characteristics of the video game experience that are not considered in these previous models. Video games are not passive and so it is not a surprise that models that assume a passive consumer (as in Das Gupta et al. (2016), Li et al. (2020)) do not suffice to capture the tradeoffs that interest us. Indeed, we find many level designs (like the classical "mini-boss-end-boss" design) that are not predicted by existing models. We should also mention the recent work by Roels (2020) that has a dynamic process of fatigue that bears some resemblance to our stress adaptation process. This paper tackles a very different research question than ours with a different type of resulting optimization model, but nonetheless provide inspiration for some of our analysis.

As for our findings, we analyze our proposed mathematical model of optimal level design to characterize when different qualitative designs are optimal. Our strongest analytical results are in the case when reward and difficulty are proportional; that is, easy game elements give small rewards while hard game elements give large rewards. This is common in the design of individual game elements, as this is consistent with the psychological theory of "flow" championed by psychologist Csikszentmihalyi (1990), whose ideas have significant influence among video game designers[6] and academic researchers of video game design (see, for instance, Cowley et al. (2008)).

In the theory of flow, the difficulty and reward for experiences should be balanced to help the participant achieve 'optimal' experience called *flow*. If an activity requires little effort, an outsized reward feels hollow and unearned. Meanwhile, a task that is very difficult but reaps little reward leads to frustration. In the "sweet spot" of flow, the participant feels sensations of timelessness, happiness, and acute focus. Indeed, video games are often cited as an example of an experience highly adept at achieving "flow" in players, something of concern to parents, policy-makers, and researchers (see, for instance, the review article Kuss and Griffiths (2012)).

A practical implication of Csikszentmihalyi's theory is that "flow" is best achieved when rewards and difficulty are proportional. It turns out that a key analytical driver of results in our model is precisely the reward-to-difficulty ratio. For example, when the proportion of reward-to-difficulty is high, the optimal design mimics that of more passive experiences like that studied in Das Gupta et al. (2016). This is intuitive since when difficulty is low, the gaming experience is not unlike a

[6] See for instance this article on Gamasutra on the concept of flow: `https://www.gamasutra.com/view/feature/166972/cognitive_flow_the_psychology_of_.php`

passive service experience. Classic games like *Dragon's Lair* — which is essentially an animated movie with very simple interactive elements separating scenes — is an example of a game with high reward-to-difficulty ratio.

Intermediate cases give rise to the possibility of optimal "mini-boss, end-boss"-like designs — what we call N-shaped designs since the difficulty, in this case, follows an 'N'-shaped pattern (see Figure 1 for an illustration). The intuition here is that a crescendo of sustained difficulty from the beginning of the level to the end builds up too much stress in the player, which can negatively impact their remembered utility given memory decay. Instead, the design starts with a crescendo of difficulty and rewards, so that the player adjusts to difficulty slowly and diminishes the amount of disutility accrued due to stress. Once accustomed to a certain level of difficulty at the peak of the crescendo (where the "mini-boss" is encountered), the remaining pattern is similar to a pure entertainment experience with a U-shaped design. The diminuendo subsequence at the middle of the level serves to reset the reference point for rewards and helps the player relax. The final crescendo sequence help to create a grand ending experience accentuated in the memory of the player.

As mentioned earlier, N-shaped designs are ubiquitous in video games. Our analysis shows that such designs are optimal under easy-to-accept assumptions of player behavior. Our model can also pinpoint when and how to place a peak.

We also show conditions under which an *inverted* N-shaped design is optimal. This is the case when players adapt more quickly to difficulty than to rewards. Such a setting can prevail in "serious" games designed for educational and training purposes. Here, players expect the game to be challenging and are in a mood to learn and adapt to difficulty, but unlike inverted U-shaped designs where rewards are very low, rewards for difficulty are significant enough so that, at the outset of the level, high rewards get the player "going" with positive reinforcement. We see these types of designs in educational games like the mathematics-based role-playing game *Prodigy*.

We also presents a number of extensions of our base model, including allowing for repeating of game elements and rewards that are not proportional to difficulties. In these scenarios, mathematical analysis is more challenging, but under certain restrictions, we show that the basic tradeoffs and design philosophies behind U-shaped, inverted U-shaped, N-shaped, and inverted N-shaped are robust.

In summary, this paper provides practical guidance to game designers on how to match level design to the relationship between reward and difficulty inherent in their game's mechanics. We make the following contributions.

- To our knowledge, this is the first paper to introduce a formal mathematical model for solving the sequencing problem inherent in video game level design.

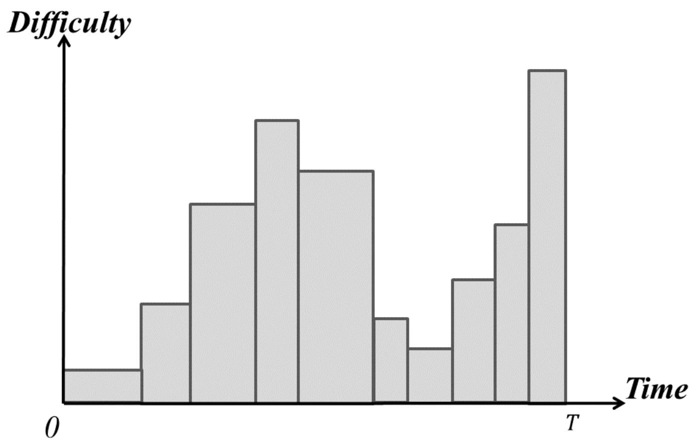


Figure 1 The N-shaped Game Design

- We provide mathematical justification for common level designs seen in practice, including N-shaped level designs, showing that they are optimal under certain conditions. Previous models for studying the design of experiential services (such as Das Gupta et al. (2016), Li et al. (2020)) are unable to justify the optimality of these types of designs.
- We incorporate behavioral elements into our models that are acknowledged as being significant by game designers in a way that is mathematically elegant and tractable. This includes behavioral elements not studied in the literature on the design of experiential services.
- We show that the essence of our findings is robust to several generalizations that add mathematical complexity at the cost of tractability, but also capture more general game design scenarios.

The paper is organized as follows. Section 2 summarizes related work on video games and the design of experiential services. Section 3 presents our main mathematical model of level design that is grounded in the behavioral theories of reward-seeking, difficulty aversion, and memory loss. Section 4 presents our main theoretical findings, including characterizations of when U-shaped, inverted U-shaped, N-shaped, and inverted N-shaped designs are optimal. Section 5 looks at an extension where game elements can be repeated when constructing levels, a natural generalization in the video game setting. Here we show that the basic optimal structures are robust to this generalization, allowing for similar insights. Section 6 concludes.

2. Related Work

This paper is related to two burgeoning streams of research in operations management, information systems, and marketing. The first is on business and design questions motivated by the video game context. Many of these papers are motivated by a similar central question — how game design relates to player engagement, retention, and monetization? — but none specifically look at the question of level design. The second stream of research concerns the optimal design of service operations that take into consideration some of the behavioral factors we deliberate on in our modeling exercise. These papers form the main methodological inspiration for our work.

Research primarily motivated by video games is a new and rapidly developing area in business research, crossing the disciplinary boundaries of operations management, information systems, and marketing. This includes research on the design of in-game advertising (Turner et al. 2011, Guo et al. 2019b, Sheng et al. 2020), the design of virtual currency systems (Guo et al. 2019a, Meng et al. 2021), and the selling of virtual items (Huang et al. 2020, Jiao et al. 2020, Runge et al. 2021, Chen et al. 2020, Vu et al. 2020).

We mention three papers that are arguably the most related to the current study. Huang et al. (2020) study how the concept of player "engagement" can be used to improve the design of games (specifically in protocols for matching players), leading to increased play and improved revenues. Sheng et al. (2020) also formalize the concept of engagement in a dynamic model for determining the optimal deployment of revenue-generating in-game advertising. (The concept of engagement in video games is also studied by Huang et al. (2019).) Ascarza et al. (2020) conduct a large-scale field experiment to draw empirical connections between game difficulty and player retention. All three studies examine the connection between game design and player motivation. Ascarza et al. (2020) relates these concepts to the notion of difficulty of a game.

In a high-level sense, our work also relates to player motivation and progression, but with a different lens. While the target practical audience of the previous papers might be those at game companies working on the business side of revenue generation, our focus here is to provide tactical insights to "front line" design staff in charge of structuring game content. We consider the issue of engagement in the design unit of a "level" and ask what we can do to maximize the utility of the player (a proxy for enjoyment) given a set of more granular design elements. In this sense, we build on the findings of previous research — the importance of engagement in games — and move towards tactical level-design questions.

Our work also builds on a growing literature concerned with behavioral aspects of offering experiential services based on the seminal work of Das Gupta et al. (2016). Connections between our work on this literature were described already at some length in the introduction, so we will not belabor the connection here. It is worth mentioning a related literature in theoretical information economics that was initiated by Ely et al. (2015), with a growing literature of applications and extensions (see, for instance, Nalbantis and Pawlowski (2019), Buraimo et al. (2020), Renault et al. (2017)). The major distinction between this line of research and the work following Das Gupta et al. (2016) is that the former focus on "forward-looking" behavioral concepts like "suspense" and "surprise", while the latter tends to focus on behaviors that are backward-looking. Forward-looking concepts have the added analytical complication of tracking beliefs, something we feel clouds the tradeoffs of interest in the current study.

3. Model

A game designer seeks to optimally sequence a collection of n given game elements into a level that maximizes the satisfaction of a representative player. Game elements refer to incremental units of a video game level, including enemy encounters, puzzles, or obstacles like a set of platforms to traverse. Each game element $i \in [n] \triangleq \{1, 2, \ldots, n\}$ has an associated reward r_i, a fixed duration τ_i, and a difficulty level d_i. The reward r_i can represent in-game "loot" that the player unlocks when passing game element i, or some more psychological notion of utility experienced by the player associated with the "fun" of the element or sense of accomplishment in completing it. The duration τ_i is the expected time it takes to pass game element i. The difficulty level d_i indicates how much mental and physical energy a player exhausts to pass game element i.

It is important to emphasize that the game elements and their data (rewards, difficulties, and duration) are all given. The decision that concerns us in this paper is how to sequence these game elements. The question of designing game elements is equally as interesting but beyond the scope of our study here. The assumption that game elements are given and then assembled into levels is consistent with game design practice. Consider, for example, the classic video game *Mario Brothers 3* by Nintendo that contains multiple "worlds" that consist of themed collection of levels with similar enemies and encounter styles. The enemies and encounter types are designed at the "world" level, whereas individual levels within the world sequence these enemies and encounter types. See Tozour (2013) for further discussion.

The game designer selects a permutation π of the set $[n]$ where $\boldsymbol{\pi} = (\pi[1], \ldots, \pi[n])$ where $\pi[i]$ is the ith game element in the sequence. For example, if there are three game elements indexed by the set $\{1, 2, 3\}$ then the sequence $\boldsymbol{\pi} = (2, 1, 3)$ designs the level with game element 2 first, followed by game element 1, and finally, game element 3. We assume that the elements are indexed in an increasing order of rewards; that is $r_1 \leq r_2 \leq \cdots \leq r_{n-1} \leq r_n$. We consider a level design problem with fixed duration $T = \sum_{i=1}^{n} \tau_i$. We denote by $t_{\pi[i]} = \sum_{j=1}^{i} \tau_{\pi[j]}$ the completion time of game element $\pi[i]$, and by $\bar{t}_{\pi[i]} = \sum_{j=i}^{n} \tau_{\pi[j]}$ the duration from the starting time of game element $\pi[i]$ until the end of the level. Observe that $T = \bar{t}_{\pi[i]} + t_{\pi[i]} - \tau_{\pi[i]}$. For simplicity of notation, we omit π in the subscripts and use $u_{[i]}$, $d_{[i]}$, $\tau_{[i]}$, t_i, and $\bar{t}_i$ to represent $r_{\pi[i]}$, $d_{\pi[i]}$, $\tau_{\pi[i]}$, $t_{\pi[i]}$ and $\bar{t}_{\pi[i]}$, respectively. Figure 2 provides a graphical representation of this notation.

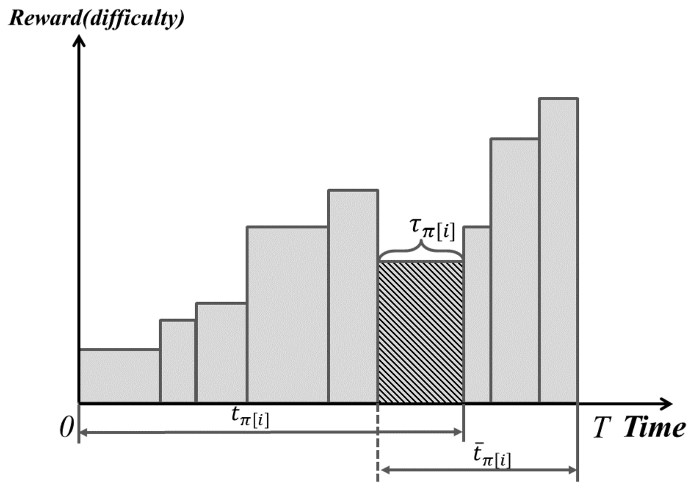


Figure 2 **Time Intervals for game element** $\pi[i]$**:** $\tau_{\pi[i]}, \bar{t}_{\pi[i]}, t_{\pi[i]}$.

3.1 Gameplay Satisfaction

We adopt a framework similar to Das Gupta et al. (2016) and Li et al. (2020) to quantify the player's retrospective perception of a level as the remembered utility accumulated from time 0 to time T. Our expression for this remembered utility draws on three psychological concepts, namely, (a) accomplishment adaptation, (b) stress adaptation, and (c) memory decay. These concepts reflect three typical behaviors in gameplay: reward-seeking, difficulty aversion, and memory loss.

The accomplishment process reflects the player's passion for seeking rewards. While people are attracted by the sensation of "winning", players also experience negative feelings during gameplay. If the game is difficult, players can come to feel anxious or frustrated, particularly when exposed to extended durations of difficulty (Chen 2007). Accordingly, we introduce a stress process that reflects the dynamics of a player's aversion to difficulty.

Another factor that affects gameplay satisfaction is memory capacity. As is well accepted by the literature (e.g., Kahneman et al. 1993, Fredrickson 2000), players cannot remember all the game elements they experience with perfect clarity. In our study, we follow the memory decay model of Das Gupta et al. (2016) and Li et al. (2020), which adapt the exponential memory decay model proposed by Ebbinghaus (1913). This model consistent with the game design literature on the limits of attention and memory capacity (e.g., Hiwiller 2015, Hodent 2017, 2020).

We introduce a memory decay process to reflect the player's memory loss due to a player's limited ability to remember what happened during the experience of a level. We must therefore examine a player's remembered utility of a level when assessing his appreciation of the design. Table 1 summarizes the relationship between the psychological process and the player behavior in the gameplay.

Table 1 Psychological Process and Player Behavior

Psychological Process	Player Behavior	Outcome
Accomplishment process	Reward seeking	Utility
Stress process	Difficulty aversion	Disutility
Memory decay process	Memory loss	Remembered utility

To formalize the accomplishment and stress processes, we follow the adaptation model of Aflaki and Popescu (2013) in a similar pattern to Das Gupta et al. (2016), where experienced utility and disutility are functions of deviations from a reference point and this reference point evolves according to a differential equation akin to Newton's law of cooling. In our model, the accomplishment process is the source of utility and the stress process is the source of disutility. Each of these processes evolves according to its own adaptive process with given parameters. These two processes are described in the next two subsections.

As the player has both positive and negative feelings from gameplay, utility from rewards and disutility from difficulty jointly affect the game experience. Following our description of the accomplishment and stress processes, we combine them to determine a (net) utility process in Section 3.1.3. At time t during the player's experience of the level, we determine an instantaneous (net) utility at time t by subtracting the instantaneous disutility from the instantaneous utility at time t. In doing so, we adopt a linear-form model similar to Roels (2020), which is based on the athletic performance model by Banister et al. (1975) with two effects, one positive (fitness) and one negative (fatigue). While performance is not the subject of our study, we believe a linear-form model is justified here because it similarly weighs two psychological impacts on utility, one positive (reward) and one negative (difficulty). This matches with game design theory, which states that the game should balance its reward scheme and the degree of challenge (Schell 2019).

For the memory decay process, we consider a memory decay process with exponential memory decay as Das Gupta et al. (2016) and Li et al. (2020). The memory decay process determines the relative weight of each game element, and converts the instantaneous utility to remembered utility. Therefore, the sequence of game elements will affect the perception of the game experience.

Combining the three psychological effects, we present the framework of our study in Figure 3.

3.1.1 The Accomplishment Process The accomplishment process reflects the psychological phenomenon of reward-seeking and adaptation to rewards. On one hand, the player prefers to receive more rewards, on the other hand, the player will gradually adapt to the gain, and seek greater rewards (Plass et al. 2015).

To model the accomplishment process, we follow the adaptation model of Aflaki and Popescu (2013). For a given schedule $\boldsymbol{\pi}$, we denote by $f_{\boldsymbol{\pi}}(t)$ the player's reference reward at time t. For simplicity of notation, we omit π in the subscript and use $f(t)$. The instantaneous utility experienced

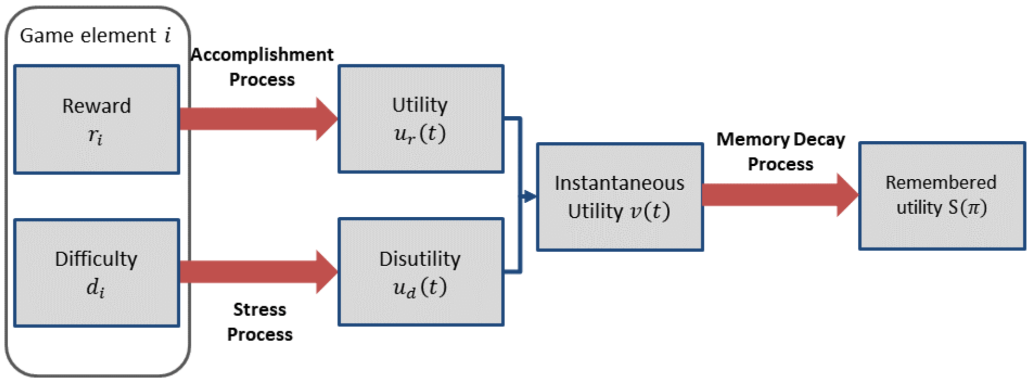


Figure 3 **The Combining Effects of Accomplishment, Stress, and Memory Decay**

at time $t \in [t_{i-1}, t_i]$ is a function of the difference between the current reward and the reference reward, which is given by:

$$u_r(t) = U_r\left(r_{[i]} - f(t)\right), \tag{1}$$

where $U_r(\cdot)$ is the player's utility function for rewards. We assume that utility function $U_r(\cdot)$ is linear, such that $U_r\left(r_{[i]} - f(t)\right) = u_{r,0} + a\left(r_{[i]} - f(t)\right)$, where $u_{r,0}$ is the initial utility from the experience and a is coefficient. We can normalize $u_{r,0}$ to 0 and a to 1 without loss as a simple rescaling of utilities.

We assume the rate of change of the reference reward is proportional to the instantaneous utility $u_r(t)$; i.e., the change rate of reference reward $f(t)$ at time $t \in [t_{i-1}, t_i]$ is:

$$\begin{aligned}\frac{df(t)}{dt} &= \alpha u_r(t),\\ &= \alpha\left(r_{[i]} - f(t)\right),\end{aligned}$$

where we refer to $\alpha > 0$ as the *degree of reward-seeking* of the player. Parameter α depicts the speed of adaptation to rewards. The larger is the risk-seeking degree α, the faster the reference reward accumulates. Players with very large α have insatiable appetites for rewards, even as they earn rewards they require even greater rewards to stay happy.

The reference reward at time $t \in [t_{i-1}, t_i]$ for a player with risk-seeking degree α is:

$$f(t) = r_{[i]} - \left(\left(r_{[1]} - f(0)\right) + \sum_{j=2}^{i}\left(r_{[j]} - r_{[j-1]}\right)e^{\alpha t_{j-1}}\right)e^{-\alpha t}. \tag{2}$$

With (1) and (2), the utility at $t \in [t_{i-1}, t_i]$ can be expressed as:

$$u_r(t) = \left(\left(r_{[1]} - f(0)\right) + \sum_{j=2}^{i}\left(r_{[j]} - r_{[j-1]}\right)e^{\alpha t_{j-1}}\right)e^{-\alpha t}. \tag{3}$$

3.1.2 The Stress Process The stress process reflects the psychological phenomenon of difficulty aversion and adaptation. A game is not an unbroken sequence of rewards. Effort must be exerted in order to earn rewards and this effort is proportional to the difficulty of the game element. We assume that players adapt to difficulty analogously to how they adapt to rewards. This fits the common understanding of game design, which suggests that players learn from playing and find that challenge diminishes when faced with equally difficult game elements (Kalmpourtzis 2018, Schell 2019).

As illustrated by Figure 3, the stress process governs disutility due to effort exerted in overcoming difficulty. For a given schedule $\boldsymbol{\pi}$, we denote by $g_{\boldsymbol{\pi}}(t)$ the player's reference difficulty at time t. For simplicity of notation, we omit π in the subscripts and use $g(t)$ instead. The disutility at time $t \in [t_{i-1}, t_i]$ is a function of the difference between the current difficulty and the reference difficulty, which is given by:

$$u_d(t) = U_d\left(d_{[i]} - g(t)\right), \tag{4}$$

where $U_d(\cdot)$ is the disutility function. As before, we assume that U_d is linear and let $U_d(d-g) = \delta(d-g)$, where δ is a given positive constant. We can scale δ to 1 without loss for simplicity of the analysis. For completeness, we verify this assertion in Lemma A.8 in Appendix A.[7]

Same as the accomplishment process, we assume the change rate of the reference difficulty is proportional to the disutility at time t; that is, the change rate of the reference difficulty $g(t)$ at time $t \in [t_{i-1}, t_i]$ is:

$$\begin{aligned}\frac{dg(t)}{dt} &= \beta u_d(t), \\ &= \beta\left(d_{[i]} - g(t)\right),\end{aligned}$$

where $\beta > 0$ (with $\beta \neq \alpha$) is the *degree of difficulty-aversion*. Parameter β depicts the speed of adaptation to difficulty. The larger is the difficulty-aversion degree β, the faster the reference difficulty accumulates.

The reference difficulty at time $t \in [t_{i-1}, t_i]$ is:

$$g(t) = d_{[i]} - \left(\left(d_{[1]} - g(0)\right) + \sum_{j=2}^{i}\left(d_{[j]} - d_{[j-1]}\right)e^{\beta t_{j-1}}\right)e^{-\beta t}. \tag{5}$$

With (4) and (5), the disutility at $t \in [t_{i-1}, t_i]$ can be expressed as:

$$u_d(t) = \left(\left(d_{[1]} - g(0)\right) + \sum_{j=2}^{i}\left(d_{[j]} - d_{[j-1]}\right)e^{\beta t_{j-1}}\right)e^{-\beta t}. \tag{6}$$

[7] Note that arguing for a simple normalization of utility without loss (recall that utilities are only defined up to affine scaling (Mas-Colell et al. 1995)) does not suffice here because we have already executed a normalization of the utilities for rewards in the previous subsection. This is why we introduce a secondary argument for why we may assume $\delta = 1$ without loss found in Lemma A.8.

3.1.3 The Memory Decay Process The memory decay process reflects the psychological phenomenon of memory loss and it converts the instantaneous utility into remembered utility. This works on the (net) utility derived from the instantaneous utility from rewards and disutility from difficulty. Instantaneous utility is affected by both the accomplishment and the stress processes.

As shown in Figure 3, we assume the instantaneous utility experienced at time $t \in [t_{i-1}, t_i]$ is a function of the utility from rewards and disutility from difficulty given by

$$v(t) \triangleq V(u_r(t), u_d(t)), \tag{7}$$

where $V(\cdot)$ is an aggregate utility function over utilities u_r and disutilities u_d.

We assume that the utility function $V(\cdot)$ is linear with

$$V(u_r(t), u_d(t)) = v_0 + \delta_r u_r(t) - \delta_d u_d(t),$$

where v_0 is the initial instantaneous utility and $\delta_r, \delta_d > 0$ are given coefficients of the utility.

Same as before, we normalize v_0 to 0 and scale δ_r and δ_d to 1 without loss for simplicity of the analysis. This is verified in Lemma A.8 in Appendix A.

As shown in Figure 3, the memory decay process affects the remembered utility originating from the instantaneous utility. We take the model that the player has an exponential memory decay process with rate $\gamma > 0$ and $\gamma \neq \alpha, \beta$. Then the player's cumulative remembered utility $S(\boldsymbol{\pi})$ is given by:

$$S(\boldsymbol{\pi}) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} v(t) e^{-\gamma(T-t)} dt, \tag{8}$$

where $v(t)$ is as defined in (7). Therefore, the latest game element will weigh more when players recall the game journey.

Combining (3), (6), (7), and (8) shows that the remembered utility of a level can be expressed as:

$$\begin{aligned} S(\boldsymbol{\pi}) = & \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\left(r_{[1]} - f(0)\right) e^{-\alpha t} + \sum_{j=2}^{i} \left(r_{[j]} - r_{[j-1]}\right) e^{-\alpha\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt \\ & - \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\left(d_{[1]} - g(0)\right) e^{-\beta t} + \sum_{j=2}^{i} \left(d_{[j]} - d_{[j-1]}\right) e^{-\beta\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt. \end{aligned} \tag{9}$$

We assume that the player does not have any previous gameplay experience, such that $f(0) = 0$ and $g(0) = 0$. Let $r_{[0]} = 0$ and $d_{[0]} = 0$. Continuing from (9) we have:

$$S(\boldsymbol{\pi}) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\sum_{j=1}^{i} \left(r_{[j]} - r_{[j-1]}\right) e^{-\alpha\left(t - t_{j-1}\right)} \right) e^{-\gamma(T-t)} dt$$

$$
\begin{aligned}
&-\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\sum_{j=1}^{i}\left(d_{[j]}-d_{[j-1]}\right)e^{-\beta\left(t-t_{j-1}\right)}\right)e^{-\gamma(T-t)}dt,\\
&=\sum_{i=1}^{n}\left(r_{[i]}-r_{[i-1]}\right)\frac{e^{-\alpha\bar{t}_i}-e^{-\gamma\bar{t}_i}}{\gamma-\alpha}-\sum_{i=1}^{n}\left(d_{[i]}-d_{[i-1]}\right)\frac{e^{-\beta\bar{t}_i}-e^{-\gamma\bar{t}_i}}{\gamma-\beta},\\
&=\sum_{i=1}^{n}r_{[i]}\left(\frac{e^{-\alpha\bar{t}_i}-e^{-\gamma\bar{t}_i}}{\gamma-\alpha}-\frac{e^{-\alpha\bar{t}_{i+1}}-e^{-\gamma\bar{t}_{i+1}}}{\gamma-\alpha}\right)-\sum_{i=1}^{n}d_{[i]}\left(\frac{e^{-\beta\bar{t}_i}-e^{-\gamma\bar{t}_i}}{\gamma-\beta}-\frac{e^{-\beta\bar{t}_{i+1}}-e^{-\gamma\bar{t}_{i+1}}}{\gamma-\beta}\right).
\end{aligned}
\tag{10}
$$

3.2 The Level Design Problem

In this section, we formulate the level design problem. To simplify the expression of game satisfaction, we first introduce the function $\Phi\left(t|\theta,\gamma\right)$, where θ can be either the risk-seeking degree α or the difficulty-aversion degree β. The function $\Phi\left(t|\theta,\gamma\right)$ is given by:

$$\Phi\left(t|\theta,\gamma\right)=\frac{e^{-\theta t}-e^{-\gamma t}}{\gamma-\theta}, \tag{11}$$

where $\theta\neq\gamma$, $\theta,\gamma>0$ $\Phi\left(t|\theta,\gamma\right)\geq 0$.

It is straightforward to see that $\Phi\left(t|\theta,\gamma\right)$ is continuous and twice differentiable in t. As shown in Lemma A.1 in Appendix A, $\Phi\left(t|\theta,\gamma\right)$ is a concave-convex function with one inflection point and one stationary point. Let $T_0\left(\theta,\gamma\right)$ be the inflection point and $T_0'\left(\theta,\gamma\right)$ be the stationary point, whose formulation is shown in (A.1) in Appendix A. These inflection and stationary points are important indicators of structure discussed later in Theorem 3 and Propositions 1 and 2.

By (10) and (11), we can formulate the level design problem as:

$$\max_{\boldsymbol{\pi}} S\left(\boldsymbol{\pi}\right)=\sum_{i=1}^{n}r_{[i]}\left(\Phi\left(\bar{t}_i|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right)-\sum_{i=1}^{n}d_{[i]}\left(\Phi\left(\bar{t}_i|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\right). \tag{12}$$

4. Optimal Structure of Game Design

We now examine the structural properties of optimal solutions to (12). This section contains two subsections. The first considers the special case where rewards and difficulties are proportional. As mentioned in the introduction, this is consistent with the concept of "flow" and is a common design principle in video games (Chen 2007). In the second subsection, we examine the case of the more general reward and difficulty patterns.

4.1 Sequencing Game Elements with Proportional Reward

In this section, we consider the case that the reward is proportional to the difficulty of the game element, with a uniform reward ratio $k>0$:

$$r_i=kd_i. \tag{13}$$

With (13), the player's remembered utility with proportional reward can be expressed by:

$$S(\boldsymbol{\pi}) = \sum_{i=1}^{n} kd_{[i]} \left(\Phi(\bar{t}_i|\alpha,\gamma) - \Phi(\bar{t}_{i+1}|\alpha,\gamma)\right) - \sum_{i=1}^{n} d_{[i]} \left(\Phi(\bar{t}_i|\beta,\gamma) - \Phi(\bar{t}_{i+1}|\beta,\gamma)\right),$$
$$= \sum_{i=1}^{n} d_{[i]} \left(\left(k\Phi(\bar{t}_i|\alpha,\gamma) - \Phi(\bar{t}_i|\beta,\gamma)\right) - \left(k\Phi(\bar{t}_{i+1}|\alpha,\gamma) - \Phi(\bar{t}_{i+1}|\beta,\gamma)\right)\right).$$

To simplify the expression, we introduce function $\Psi(t|\alpha,\beta,\gamma,k)$, which is given by:

$$\Psi(t|\alpha,\beta,\gamma,k) = k\Phi(t|\alpha,\gamma) - \Phi(t|\beta,\gamma).$$

Thus, we can rewrite the level design problem with proportional reward (LDPP) as

$$\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) = \sum_{i=1}^{n} d_{[i]} \left(\Psi(\bar{t}_i|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k)\right).$$

It is straightforward to see that $\Psi(t|\alpha,\beta,\gamma,k)$ is continuous and twice differentiable in t. As shown in Lemmas A.2 and A.3 in Appendix A, we prove that $\Psi(t|\alpha,\beta,\gamma,k)$ can have one or two of inflection point(s) and one or two stationary point(s).

Let $T_1(\alpha,\beta,\gamma,k)$ and $T_2(\alpha,\beta,\gamma,k)$ be the inflection points when there are two points, and $T_2(\alpha,\beta,\gamma,k)$ be the unique inflection point when there is only one inflection point. For simplicity, we will use T_1 and T_2 instead. They are important indicators in the optimal structure of the game elements discussed in Section 4.1.

To express the structural property of the optimal solution, we define the two thresholds

$$\underline{k} \triangleq \begin{cases} \frac{\beta+\gamma}{\alpha+\gamma}, & \text{if } \alpha > \beta, \\ \frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha < \beta \text{ and } \alpha > \gamma, \\ 0, & \text{if } \alpha < \beta \text{ and } \alpha < \gamma \end{cases} \tag{14}$$

$$\overline{k} \triangleq \begin{cases} \frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha > \beta \text{ and } \beta > \gamma, \\ +\infty, & \text{if } \alpha > \beta \text{ and } \beta < \gamma, \\ \frac{\beta+\gamma}{\alpha+\gamma}, & \text{if } \alpha < \beta. \end{cases} \tag{15}$$

First, we describe properties of the optimal sequence when the game duration is sufficiently long (i.e., $T > T_2$).

THEOREM 1. *When the game duration is sufficiently enough (i.e., $T > T_2$), in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP, the elements' rewards (difficulties) are in the following structure.*

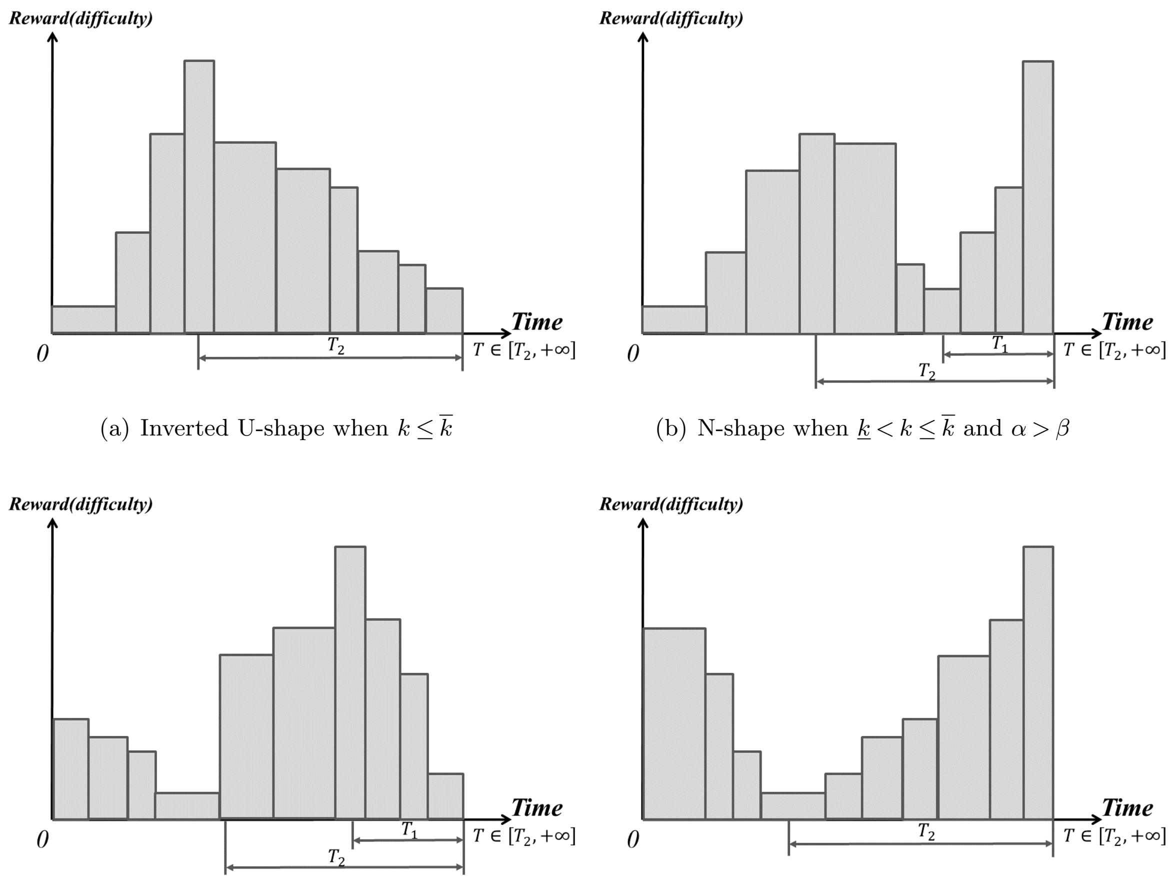
(i) When $k \le \underline{k}$, the optimal structure is an inverted U-shaped sequence.

(ii) When $\underline{k} < k \le \overline{k}$, there are two cases of the optimal structure.

(iia) If $\alpha > \beta$, the optimal structure is a N-shaped sequence.

(iib) If $\alpha < \beta$, the optimal structure is an inverted N-shaped sequence.

(iii) When $k > \overline{k}$, the optimal structure is a U-shaped sequence.



(a) Inverted U-shape when $k \leq \overline{k}$

(b) N-shape when $\underline{k} < k \leq \overline{k}$ and $\alpha > \beta$

(c) Inverted N-shape when $\underline{k} < k \leq \overline{k}$ and $\alpha < \beta$

(d) U-shape when $k > \underline{k}$

Figure 4 Optimal Structures of the LDPP

We present the optimal structures in Figure 4 and summarize the mathematical expressions of the optimal structure in Table A.1 in Appendix B.

Theorem 1 is arguably the central result of the paper, and so we deliberate on its meaning in the next few paragraphs.

In case (i), the rewards are so low in proportion to difficulty that utilities from the stress adaptation process dominate the accomplishment process. Here, we need to think about managing the stress of the player, so a huge jump in difficulty will cause a lot of disutility, so we have a warm up and cool down to avoid jumps. In other words, when the reward ratio is low (i.e., $k < \underline{k}$), the problem will become a workout design problem, whose optimal structure is an inverted U-shaped sequence regardless of the values of α and β. This is easy to understand. When you do a workout, the player gets tired very easily. We see this game design in genres that require intensive body movement like the arcade classics *Dance Dance Revolution* or *Whack-A-Mole*.

Conversely, when the reward ratio is high (i.e., $k \geq \overline{k}$), the problem essentially becomes the design of an entertainment or service experience as studied in Das Gupta et al. (2016). Accordingly, the optimal structure follows the U-shaped pattern identified in Das Gupta et al. (2016). This optimal structure is well-suited to games in which the plot is the most important issue (e.g., interactive fiction like the classic arcade game *Dragon's Lair*).

These two extreme cases are well-covered by previous literature, while the intermediate case (ii) yields fresh insights. In case (ii), rewards and difficulty are roughly even in weight (with k between $\underline{k}$ and $\bar{k}$) and so the degrees of reward-seeking α and difficulty aversion β start to play a pivotal role. Because this "second order affect" has bite, we no longer see the "extreme" cases of U-shaped and inverted U-shape. (We will see the even more extreme designs of pure crescendo and diminuendo arise in short duration levels in Theorem 2.) Indeed, Case (iia) yields an N-shaped design that starts with a preliminary crescendo of difficulty followed by a U-shaped finish. Case (iib) has an inverted N-shaped design that starts with a diminuendo of difficulty followed by an inverted U-shaped finish. In both of these cases, we see a more even distribution of hard and easy elements, common to many popular video games. Let's examine these two subcases in turn.

Case (iia) is distinguished by game designs with similar rewards and difficulties, but where the degree of reward-seeking outstrips the degree of difficulty aversion; that is, $\alpha > \beta$. It is our contention that $\alpha > \beta$ is a common case for players who play games largely as a form of entertainment. The player adjusts quickly to rewards and so demands increasing rewards to maintain a given level of utility. On the other hand, the players adapt slowly to difficulty and so suffer a lot of disutility if there is a sudden spike in challenge.

This is reflected in the optimal N-shaped design. The design starts with a crescendo of difficulty and rewards so that the player adjusts to difficulty slowly and diminishes the amount of disutility accrued. Once accustomed to a certain level of difficulty at the peak of the crescendo, the remaining pattern is similar to a pure entertainment experience with a U-shaped design. The diminuendo subsequence at the middle of the level serves to reset the reference point for rewards and helps the player relax. The final crescendo sequence helps to create a grand ending experience accentuated in the memory of the player who otherwise adapts quickly to rewards.

By contrast, Case (iib) is distinguished by game designs with similar rewards and difficulties, but now where the degree of difficulty aversion outstrips the degree of reward-seeking; that is, $\alpha < \beta$. We believe this scenario is common in "serious" games which are played not purely for entertainment, but for educational, training, and adherence purposes (see, for example, Plass et al. (2015) and Kalmpourtzis (2018) for discussions of study games, Sardi et al. (2017) for medical programs, and Seaborn and Fels (2015) for workplace incentive programs). The online game *Prodigy* is designed for school-age children to learn mathematics in a role-playing game (RPG) style environment. In a game like *Prodigy*, players are in a learning mode (no one mistakes Prodigy for a pure entertainment game) so they can adjust quickly to difficulty, whereas they are pleasantly surprised to be getting rewards while learning math and so adjust slowly in their expectations of rewards.

The inverted N-shaped design is intuitive under these conditions. The initial diminuendo subsequence at the beginning provides the player with a spike of initial rewards, which translates into a spike of utility because adaptation to rewards is slow. On the other hand, an initial spike of difficulty that slowly diminishes is expected in an educational game whose goal is to teach a difficult topic like mathematics. Players quickly adjust to these expectations as they figure out the types of questions or problems they are being presented with. As the player moves to the later part of the level, the inverted U-shaped subsequence is reminiscent of Case (i). Players have experienced already enough reward to undertake an ascending peak of rewards and difficulties, followed by a cool down. The decrescendo at the end takes advantage of a steady decline in disutility as the game elements become easier.

It is important to appreciate the differences between Case (iia) and Case (iib). In Case (iia), ending the level with a U-shaped subsequence will create high utilities, but we need a crescendo subsequence in the beginning to let the player adapt to difficulty first. In Case (iib), difficulty plays a more important role. This time, ending the level with an inverted U-shaped sequence will create high utilities, but we need a diminuendo subsequence in the beginning to give the player an initial sense of accomplishment at the outset. This design takes advantage of their fresh mind at the outset to get some of the difficult tasks under their belt, then reset their nerves for a final inverted U-shaped push.

From both the theoretical results and the practical use, we can tell that the game designer will have to understand the difficulty of the game to make a better design. If he is designing a low-difficulty game, then he can create an experience similar to pure entertainment. If he is designing a high-difficulty game, then he can forge an experience like a workout. When the designer is designing a game with medium difficulty, then the distribution of easy and hard elements should be more balanced and follows the characteristics of the players. For games leaning towards a more entertainment purpose, an N-shaped design with a mini-boss-end-boss structure is optimal. For games designed to educate or train, an inverted N-shaped design should be considered.

We complete the analysis initiated in Theorem 1 by investigating the case of short levels (i.e., when $T < T_2$).

THEOREM 2. *When the game duration is short (i.e., $T < T_2$), the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP exhibits the following structure:*

(i) When $k \leq \overline{k}$, the optimal structure degenerates to an diminuendo sequence if $T < T_2$.

(ii) When $\underline{k} < k \leq \overline{k}$, there are two cases of the optimal structure.

(iia) When $\alpha > \beta$, the optimal structure degenerates to a U-shaped sequence if $T_1 < T < T_2$, and a crescendo sequence if $T < T_1$.

(iib) When $\beta > \alpha$, the optimal structure degenerates to an inverted U-shaped sequence if $T_1 < T < T_2$, and a diminuendo sequence if $T < T_1$.

(iii) When $k > \underline{k}$, the optimal structure degenerates to a crescendo sequence if $T < T_2$.

The above proposition suggests that game duration is another key issue. If the game is designed with a compact duration, then you can only forge part of the optimal sequence to the players. This echos the findings in Das Gupta et al. (2016) and Li et al. (2020) that the optimal structure may degenerate when the duration is not long enough. Crescendo and diminuendo designs are also common in games. Mobile games in the "endless runner" genre (like the popular *Jetpack Joyride*) start out easy and quickly build towards greater and greater difficultly, reflecting a crescendo design. By contrast, many of the original arcade games, like *Donkey Kong*, start out punishingly difficult. This reflects the different types of players that the games were designed to attract. In the arcades of the late 1970s and early 1980s, video gaming had a public and competitive feel (captured, for example, in the 2007 documentary *King of Kong: A Fistful of Quarters*). Games the presented a stern challenge were favored by players as a way to "rank" the gaming abilities of those in the arcades.

Table 2 summarizes results in Theorems 1 and 2 on the optimal structure of levels based on our model.

Table 2 Optimal Structures of the LDPP

k	Reward Ratio	α, β	T	Duration	Optimal Structure
$k \le \underline{k}$	Low	$\alpha, \beta > 0$	$T > T_2$	Long	Inverted U-shape
			$0 < T \le T_2$	Short	Diminuendo
$\underline{k} < k \le \overline{k}$	Medium	$0 < \beta < \alpha$	$T > T_2$	Long	N-shape
			$T_1 < T \le T_2$	Medium	U-shape
			$0 < T \le T_1$	Short	Crescendo
		$0 < \alpha < \beta$	$T > T_2$	Long	Inverted N-shape
			$T_1 < T \le T_2$	Medium	Inverted U-shape
			$0 < T \le T_1$	Short	Diminuendo
$k > \overline{k}$	High	$\alpha, \beta > 0$	$T > T_2$	Long	U-shape
			$0 < T \le T_2$	Short	Crescendo

We can see that the value of reward ratio k, parameters α, β, and game duration T can jointly affect the optimal structure. When $T < T_2$, the optimal structure starts to degenerate. N-shaped and inverted N-shaped sequences can be optimal only when the reward ratio is in the medium level $\underline{k} < k \le \overline{k}$.

Finally, we consider the special case where reward equals difficulty (i.e., $k = 1$). In this case, we can interpret that the player accumulates a sense of accomplishment purely by the challenge of the game elements. The following corollary gives a very compact breakdown of how all six possible game designs (crescendo, diminuendo, inverted U-shape, U-shape, N-shape, inverted N-shape) are possible as the remaining parameters (besides k) change.

COROLLARY 1. *When the reward equals the difficulty of each game element, in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPP, the elements' rewards (difficulties) are in the following structure.*

(i) When $\alpha > \beta$, the optimal structure is a N-shaped sequence if $T > T_2$, a U-shaped sequence if $T_1 < T < T_2$, and a crescendo sequence if $T < T_1$.

(ii) When $\beta > \alpha$, the optimal structure is an inverted N-shaped sequence if $T > T_2$, an inverted U-shaped sequence if $T_1 < T < T_2$, and a diminuendo sequence if $T < T_1$.

One takeaway here is how pivotal a role is played by the parameters α and β. As discussed earlier, one can associate $\alpha > \beta$ with audiences that are looking for more entertainment experiences, while $\alpha < \beta$ is associated with more learning or training experiences. Corollary 1 highlights that these two orientations support fundamentally different optimal level designs. This is a nontrivial design insight for developers of educational games who might otherwise benchmark their level design against entertainment-focused games.

4.2 Game Design with General-reward Scheme

In this section, we consider the general case that there is no proportional reward-difficulty relationship as asserted in (13). In this case, we refer (12) as the level design problem with general reward (LDPG). When there is no proportional relationship, we were, for the most part, only able to analyze (12) numerically as an integer optimization problem. When only numerical methods were available, it proved too complex to prove the structure of the optimal sequence directly.

One special case we were able to analyze was when all the elements shared a common reward (or alternatively, a common difficulty). When the elements share a common difficulty $d_i = d$ for all $i \in [n]$, the level design problem with general reward and fixed difficulty (LDPGFD) can be expressed by:

$$\begin{aligned}\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} r_{[i]} \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - \sum_{i=1}^{n} d \left(\Phi\left(\bar{t}_i|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\right), \\ &= \sum_{i=1}^{n} r_{[i]} \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - d\Phi\left(T|\beta,\gamma\right). \end{aligned} \tag{16}$$

When the elements share a fixed reward $r_i = r$ for all $i \in [n]$ the level design problem with general reward and fixed reward (LDPGFR) can be expressed by:

$$\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) = \sum_{i=1}^{n} r \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - \sum_{i=1}^{n} d_{[i]} \left(\Phi\left(\bar{t}_i|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\right),$$

$$= r\Phi\left(T|\alpha,\gamma\right) + \sum_{i=1}^{n} d_{[i]}\left(\Phi\left(\bar{t}_{i+1}|\beta,\gamma\right) - \Phi\left(\bar{t}_{i}|\beta,\gamma\right)\right). \tag{17}$$

In these two settings, we were able to show the following structural result.

THEOREM 3. *Recall that $T_0(\alpha,\gamma)$ and $T_0(\beta,\gamma)$ are the unique inflection point of function $\Phi(\alpha,\gamma)$ and $\Phi(\beta,\gamma)$ respectively.*

(i) When the elements share a fixed difficulty, in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGFD, the elements' rewards are in a U-shaped sequence if $T > T_0(\alpha,\gamma)$, and a crescendo sequence if $T < T_0(\alpha,\gamma)$.

(ii) When the elements share a fixed reward, in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGFR, the elements' difficulties are in an inverted U-shaped sequence if $T > T_0(\beta,\gamma)$, and a diminuendo sequence if $T < T_0(\beta,\gamma)$.

While we cannot prove the properties of optimal structure for the general level design problem (12), we found some interesting results that are commonly observed in video game level designs seen in practice. We found a numerical instance with an optimal sequence as illustrated in Figure 5. The parameters defining the instance can be found in Table A.3 in Appendix C.

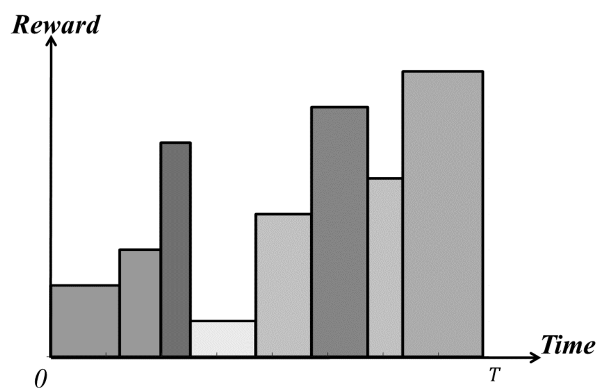


Figure 5 An Illustration of the Optimal Sequence of the LDPG.

In Figure 5, the heights of the shaded bars mark the rewards of the game elements and the gray-scale shadings represent the difficulty of the game elements. The taller is the bar, the greater the reward. The darker is the shading, the more difficult is the element.

The higher a bar is, the more reward can be obtained by completing the element. The darker a bar is, the more difficult it is to pass the game element.

In this example, the optimal schedule exhibits a "wave-like" structure. Difficulty increases for a certain period (e.g., the first three elements in Figure 5), then the game turns easy in a short time, followed by another crescendo subsequence of difficulty (e.g., the middle three elements in Figure 5). To match the changes in difficulty, the reward sequence also follows a wave-like structure. This design pattern matches recommendations by game designers like Hiwiller (2015) and Hodent (2017), and service designers like Lawrence (2014), which indicate that a structure with multiple peaks and drops is preferred by the players and customers.

We may develop an intuition for how the wave structure arises via the discussion that follows Theorem 1. Peaks in difficulty are followed by a "cooling" off period to slow the stress process. Rewards also work in patterns of crescendos and diminuendos so that players do not become "numb" to high rewards by adjusting their expectations. Players look for a challenging and rewarding experience, making intermittent crescendos of difficulty reward attractive, but an ever-increasing crescendo makes players increasingly stressed at the same time of becoming inured to the rewards. An example of a popular game with this "wave-like" pattern of difficulty and rewards is the *Plants vs Zombies* series of mobile games. In games in this series, the player makes defenses using plants to ward off waves of attacking zombies. The zombies come in waves of varying difficulties.

Table 3 summarizes the optimal structures and the conditions, and we summarize the mathematical expressions of the optimal structure in Table A.2 in Appendix B.

Table 3 Optimal Structures of the LDPG

Problem	Situation	T	Duration	Optimal Structure
LDPGFD	Fixed difficulty	$T > T_0$	Long	U-shape
		$0 < T \leq T_0$	Short	Crescendo
LDPGFR	Fixed reward	$T > T_0$	Long	Inverted U-shape
		$0 < T \leq T_0$	Short	Diminuendo
LDPG	General	$T > 0$	Any duration	Wave-like

5. Extension: Repeated Use of Game Elements

In this section, we study the level design problem with repeated use of game elements. One of the distinguishing features of games is that game elements are virtual, meaning that they can be reproduced costlessly multiple times within a level. This is in contrast with service design problems, like those studied in Das Gupta et al. (2016), where repeating a service element may be costly or not possible.

The possibility of repeating elements creates a new design problem that goes beyond sequencing, which has been the focus of previous papers in the literature. Here the decision space is extended to allow the level designer to choose the number of each game element to deploy (within a given time limit) as well as how to sequence these elements. We study the optimal structure of the final sequence of game elements (allowing for repeats) in both the proportional reward and general reward settings.

5.1 Sequencing Game Elements with Proportional-reward Scheme

In this section, we consider the level design problem, allowing for repeated elements when rewards are proportional to difficulties for all game elements, following the assumption in (13). The model is the same as that studied in Section 3, except that now each of the n game elements can be used multiple times in determining a level.

We include a few constraints on the level design problem for realism. We impose that every level design must include each of the n elements at least once. This reflects the fact that the designers of the game elements have designed them thematically to suit the level and expect them to be used at least once to contribute to the level's overall aesthetic and coherence. In addition, we impose a condition that the most difficult element can only be used once. This is the usual design aesthetic that every level should have at most one "boss", the hardest enemy or task within the level. This is important for narrative and climax.

Also, for tractability, we assume that the game elements have identical duration $\tau_i = \tau$ for all $i \in [n]$ and m periods, such that $m\tau = T$ and $\bar{t}_i = (m+1-i)\,\tau$. This makes it easily for different choices of game elements to add up to duration T.

To simplify the expressions, we define the weight w_i of game element i as follows:

$$w_i \triangleq \left(\Psi\left((m+1-i)\,\tau|\alpha,\beta,\gamma,k\right) - \Psi\left((m-i)\,\tau|\alpha,\beta,\gamma,k\right)\right) \ \forall i \in \{1,\ldots,m\}\,. \tag{18}$$

Then the remembered utility can be rewritten as:

$$\begin{aligned} S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} d_{[i]} \left(\Psi\left(\bar{t}_i|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1}|\alpha,\beta,\gamma,k\right)\right), \\ &= \sum_{i=1}^{m} w_i d_{[i]}. \end{aligned}$$

Using this notation, the level design problem with proportional rewards and repeated assignment (LDPPR) is:

$$\begin{aligned} \max_{\boldsymbol{\pi}} \ & S(\boldsymbol{\pi}) = \sum_{i=1}^{m} w_i d_{[i]}, \\ s.t. \ & \sum_{i=1}^{m} \mathbb{1}_{\pi[i]=j} \geq 1 \ \forall j \in [n-1], \\ & \sum_{i=1}^{m} \mathbb{1}_{\pi[i]=n} = 1. \end{aligned}$$

where now $\boldsymbol{\pi}$ is an integral vector in the set $\{1,2,\ldots,n\}^m$ where $\pi[i]=j$ if element j is the ith element encountered when playing the level.

We then prove the following Lemma on the elements that are repeatedly used.

Lemma 1. *In the optimal solution π^* of the LDPPR, only the lowest difficulty and second highest difficulty elements (i.e., elements 1 and $n-1$) are used repeated. The slots assigned with element 1 have negative weights (i.e., $w_i < 0$, if $\pi[i] = 1$ for all $i \in [m]$) , and the slots assigned with element $n-1$ have positive weights (i.e., $w_i > 0$, if $\pi[i] = n-1$ for all $i \in [m]$).*

Lemma 1 suggests that the game designer should use a mixture of both high-difficulty and low-difficulty elements. Li et al. (2020) showed a similar result that the highest-utility and lowest-utility activities should be selected to create an ideal experiential service. The reason that a mixture of high-difficulty and low-difficulty elements are selected follows similar reasoning. Low-difficulty elements can help players relax, which resets their reference points; high-difficulty elements provide the player a challenging experience, which leaves the player with a high-intensity remembered utility contradiction.

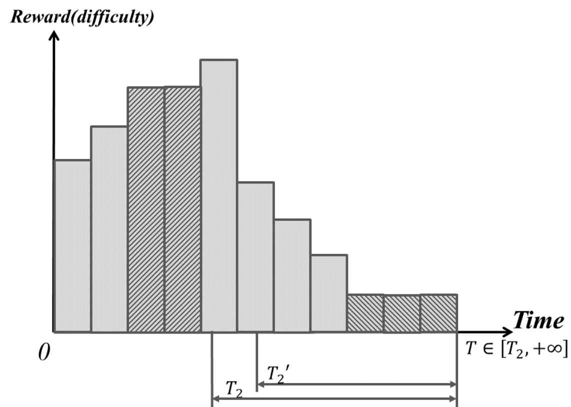
With this lemma in hand, we can prove the following proposition on the optimal structure.

Theorem 4. *The optimal schedule $\boldsymbol{\pi}^*$ of the LDPPR follows the same optimal structures as the LDPP, which are presented in Theorems 1 and 2.*

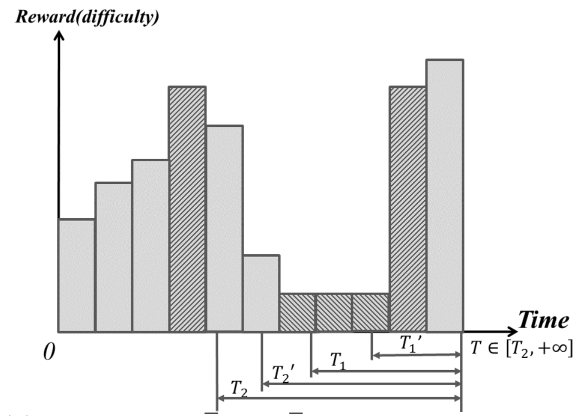
Theorem 4 shows that the optimal solution of LDPPR shares the same structural properties as optimal solutions to LDPP. In fact, we prove that LDPPR can be converted into an equivalent LDPP with m elements in Lemma A.5 in Appendix A. The game designer can enhance the game experience with the repeated use of game elements, but the designer should still follow structure properties mentioned in Theorems 1 and 2 for a given type of player, because allowed for the repeated use of game elements does not fundamentally alter the psychological processes of the player.

Figure 6 shows the possible structures of optimal schedules. The repeated elements are illustrated with a shaded bar. The tall shaded bars correspond to element $n-1$ and the short shaded bars correspond to element 1. Similar to what we saw in Figure 4, inverted U-shape, N-shape, inverted N-shape, and U-shape are optimal structures in different situations. Repeated elements are placed around the climax and low tide of the game. Element $n-1$ is used to extend the experience of a peak (e.g., Figure 6(c)), or act as a mini-boss in the middle (e.g., Figure 6(b)). Element 1 is used to extend the experience of low tide (e.g., Figure 6(b)). As discussed in the paragraph following Lemma 1, repeated use of element $n-1$ and 1 enhance the experience because they accentuate peaks and troughs in the game experience. Element 1 resets the reference point and element $n-1$ punctuates a challenging and rewarding section of a level.

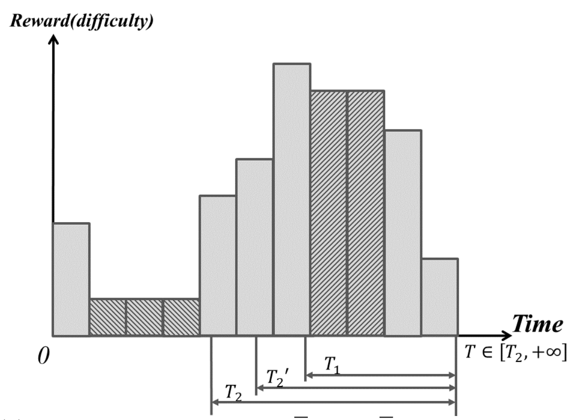
We can further prove structural results on the locations of repeated use of game elements (see Corollary A.2 in the Appendix). This result characterizes when all of element 1 and element $n-1$ are "bunched together" (as in Figure 6(d)) or sequenced apart from each other (as is the case for the two $n-1$ elements in Figure 6(b)). The conditions are rather technical, so we leave the statement and proof of this result in Appendix A.



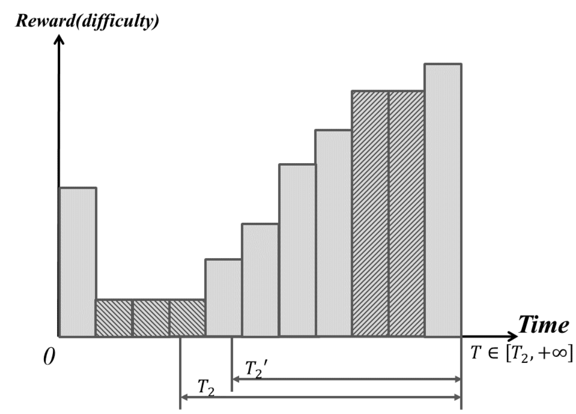
(a) Inverted U-shape when $k < \underline{k}$



(b) N-shape when $\overline{k} < k \leq \overline{k}$ and $\alpha > \beta$



(c) Inverted N-shape when $\overline{k} < k \leq \overline{k}$ and $\alpha < \beta$



(d) U-shape when $k > \underline{k}$

Figure 6 **Optimal Structures of the LDPPR**

5.2 Sequencing Game Elements with General-reward Scheme

In this subsection, we consider the level design problem, now allowing for the elements to be repeatedly used and with general rewards. As before, we make the game duration fixed to T for any feasible schedule, we assume that the game elements have identical duration $\tau_i = \tau$ for all $i \in [n]$ and we consider a game with m periods, such that $m\tau = T$ and $\bar{t}_i = (m+1-i)\tau$.

As this setting is quite general, we assume for tractability that there are only four types of elements. Element 1 (LL) has low reward and low difficulty (r_L and d_L), element 2 (LH) has low reward and high difficulty (r_L and d_H), element 3 (HL) has high reward and low difficulty (r_H and

d_L), and element 4 (HH) has high reward and high difficulty (r_H and d_H), where $0 < r_L < r_H$ and $0 < d_L < d_H$. These four types of game elements are qualitatively representative of the possibilities one may consider in the design of a level.

To simplify the expressions, we define the weight of the ith element as

$$w_i'(\theta) = \left(\Phi\left((m+1-i)\tau|\theta,\gamma\right) - \Phi\left((m-i)\tau|\theta,\gamma\right)\right) \ \forall i \in \{1,\dots,m\}. \tag{19}$$

The remembered utility can be rewritten as:

$$\begin{aligned}
S(\boldsymbol{\pi}) &= \sum_{i=1}^{n} \left(r_{[i]}\left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - d_{[i]}\left(\Phi\left(\bar{t}_i|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\right)\right), \\
&= \sum_{i=1}^{n} \left(r_{[i]}\left(\Phi\left(m+1-i|\alpha,\gamma\right) - \Phi\left(m-i|\alpha,\gamma\right)\right) - d_{[i]}\left(\Phi\left(m+1-i|\beta,\gamma\right) - \Phi\left(m-i|\beta,\gamma\right)\right)\right), \\
&= \sum_{i=1}^{m} \left(r_{[i]} w_i'(\alpha) - d_{[i]} w_i'(\beta)\right).
\end{aligned}$$

The level design problem with general reward and repeated assignment (LDPGR) can be expressed by:

$$\begin{aligned}
\max_{\boldsymbol{\pi}} \ & S(\boldsymbol{\pi}) = \sum_{i=1}^{m} \left(r_{[i]} w_i'(\alpha) - d_{[i]} w_i'(\beta)\right), \\
s.t. \ & \pi[i] \in \{1,2,3,4\} \, \forall i \in [m].
\end{aligned}$$

Note that in this formulation we have removed the constraints (imposed in the previous subsection) that each element must be used at least once and the last element at most once. We do this for simplicity of the analysis, these constraints could be added without much complication.

The following result reveals the optimal structure with general reward and repeated assignment, when the game duration is sufficiently long (i.e., $T > T_0'(\alpha,\gamma), T_0'(\beta,\gamma)$).

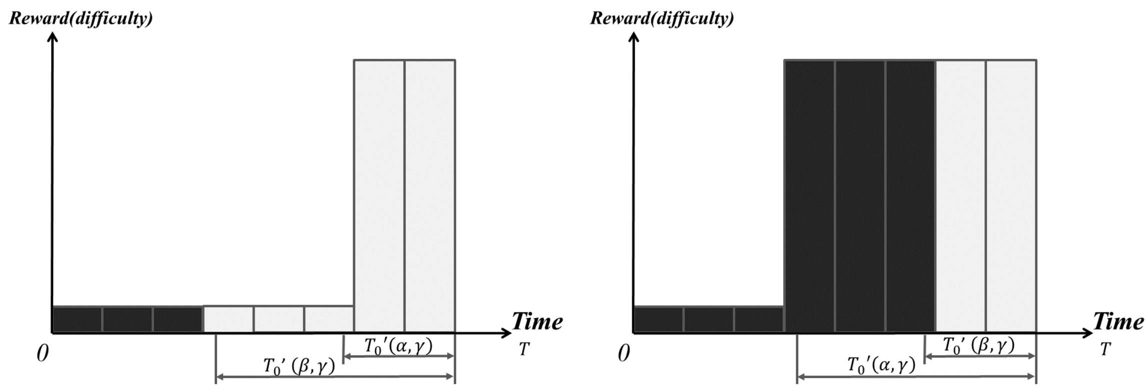
Proposition 1. *When the game duration is long enough (i.e., $T > T_0'(\alpha,\gamma), T_0'(\beta,\gamma)$), in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGR, the elements' rewards (difficulties) are in the following structure.*

(i) When $T_0'(\alpha,\gamma) < T_0'(\beta,\gamma)$ (i.e., $\alpha > \beta$), the optimal structure is a LH-LL-HL sequence.

(ii) When $T_0'(\alpha,\gamma) > T_0'(\beta,\gamma)$ (i.e., $\alpha < \beta$), the optimal structure is a LH-HH-HL sequence.

Figure 7 helps us visualize Proposition 1.

The heights of the bar mark the reward of a game element, and the grayscale marks the difficulty of a game element. A dark-shaded bar is of high difficulty d_H a light bar is of low difficulty d_L. We can tell from Figure 7 that levels should begin with low-reward and high-difficulty elements and end with high-reward and low-difficulty elements. This implies that memory decay plays an



(a) LH-LL-HL shape when $\alpha > \beta$

(b) LH-HH-HL shape when $\alpha < \beta$

Figure 7 Optimal Structures of the LDPGR

important role in the player's perception. Because the player tends to memorize the elements at the end, rewards at the end will maximize the player's remembered utility. On the other hand, difficult elements are placed at the beginning to utilize the fact that the player's mind is more clear at the moment. The results echo the analysis in Figure 5 that the service designer should value the endpoint of the game.

We further extend Proposition 1 by investigating the degenerate cases when the game duration is not long enough (i.e., when $T < T_2'$) in Proposition 2.

PROPOSITION 2. *When the game duration is not long enough (i.e., $T < \max\{T_0'(\alpha,\gamma), T_0'(\beta,\gamma)\}$), in the optimal schedule $\boldsymbol{\pi}^*$ of the LDPGR, the elements' rewards (difficulties) are in the following structure.*

(i) The optimal structure degenerates to an LL-HL sequence when $T_0'(\alpha,\gamma) < T < T_0'(\beta,\gamma)$, and an HL sequence when $T < T_0'(\alpha,\gamma) < T_0'(\beta,\gamma)$.

(ii) The optimal structure degenerates to an HH-HL sequence when $T_0'(\beta,\gamma) < T < T_0'(\alpha,\gamma)$, and an HL sequence when $T < T_0'(\beta,\gamma) < T_0'(\alpha,\gamma)$.

The following table summarizes the optimal structures and the conditions.

Table 4 Optimal Structures of the LDPGR

α, β	T	Duration	Optimal Structure
$0 < \beta < \alpha$	$T_0'(\alpha,\gamma) < T_0'(\beta,\gamma) < T$	Long	LH-LL-HL
	$T_0'(\alpha,\gamma) < T < T_0'(\beta,\gamma)$	Medium	LL-HL
	$T < T_0'(\alpha,\gamma) < T_0'(\beta,\gamma)$	Short	HL
$0 < \alpha < \beta$	$T_0'(\beta,\gamma) < T_0'(\alpha,\gamma) < T$	Long	LH-HH-HL
	$T_0'(\beta,\gamma) < T < T_0'(\alpha,\gamma)$	Medium	HH-HL
	$T < T_0'(\beta,\gamma) < T_0'(\alpha,\gamma)$	Short	HL

We can tell from Table 4, that it is optimal to place low-reward and high-difficulty elements at the beginning, and place high-reward and low-difficulty elements at the end. The duration of the game will also affect the optimal structure.

6. Conclusion

In this paper, we presented a mathematical model to analyze the problem of designing video game levels for players who are reward-seeking, difficulty averse, and suffer from memory decay. Our analysis shows that the relative strengths of these factors, and the properties of the game elements used to sequence a level, give rise to a variety of different level designs. Appendix B summarizes these findings in two convenient tables. We believe that future research into level design can further explore some of the complexities that we see in practice but are beyond the scope of the current model. First, in this model, we have assumed that players assess utility in a backwards-looking manner at the end of the level. This is consistent with the experiential services literature initiated by Das Gupta et al. (2016), but alternative "forward-looking" models like that found in Ely et al. (2015) offer other modeling opportunities to examine the optimal structure of video game levels. It would be interesting to see if these alternative theoretical foundations could provide additional insight into why certain level designs are prevalent in practice.

Also, our model assumes that players "stick around" until the end of the level before deciding whether or not to continue playing the game. We did this for tractability purposes, because otherwise we would need to track some "forwarding looking" information about what the player thinks will happen later when deciding if to quit a level mid-stream. We believe an extension that incorporates quitting behavior would be a major contribution, since retention of players is the core concern of game design, particularly in free-to-play games that are so prevalent in the video game industry today.

Some of the results we have may have promise for understanding the design of games in the "endless runner" genre, typified by the highly revenue-generating *Jetpack Joyride* on mobile platforms. In endless runners, levels are procedurally-generated (meaning generated randomly as they are encountered) and, in principle, have no end (hence the adjective "endless"). An infinite horizon dynamic model would be needed to study this problem, but we believe many of the insights we have developed here would be applicable in this setting, particularly the notion of how "peaks" and "valleys" of difficulty manage reward-seeking and difficulty-aversion behaviors of players.

Finally, there are applications of game design that extend beyond the classical entertainment setting that was largely the focus on this paper. The concept of gamification — using games to help people learn or comply with medical regimes, for example — is a growing area of application (see, for example, (Plass et al. 2015, Kalmpourtzis 2018, Sardi et al. 2017, Seaborn and Fels 2015)).

We expect this trend to continue into the future, as games become more widely accepted as a form of meaningful interaction in society.

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Online Appendix for "Optimal Level Design in Video Games"

Appendix A: Technical Proofs

To analyze the property of the optimal structure, we first define the formulation of $T_0'(\theta,\gamma)$ and $T_0(\theta,\gamma)$:

$$\begin{aligned} T_0'(\theta,\gamma) &= \frac{\ln\gamma - \ln\theta}{\gamma-\theta}, \\ T_0(\theta,\gamma) &= 2\frac{\ln\gamma - \ln\theta}{\gamma-\theta}. \end{aligned} \tag{A.1}$$

We prove the property of $\Phi(t|\theta,\gamma)$ in the following lemma.

Lemma A.1. *(i) $\Phi(t|\theta,\gamma)$ is an increasing function in t for $t \in [0, T_0'(\theta,\gamma)]$; $\Phi(t|\theta,\gamma)$ is a decreasing function otherwise.*

(ii) $\Phi(t|\theta,\gamma)$ is a concave-convex function with inflection point $T_0(\theta,\gamma)$.

Proof of Lemma A.1. By Lemma A3 in Das Gupta et al. (2016), we have $\frac{\partial\Phi(t|\theta,\gamma)}{\partial t} \geq 0$ when $t \in [0, T_0'(\theta,\gamma)]$, and $\frac{\partial\Phi(t|\theta,\gamma)}{\partial t} \leq 0$ when $t \in [T_0'(\theta,\gamma), T]$; we have $\frac{\partial^2\Phi(t|\theta,\gamma)}{\partial t^2} \leq 0$ when $t \in [0, T_0(\theta,\gamma)]$, and $\frac{\partial^2\Phi(t|\theta,\gamma)}{\partial t^2} \geq 0$ when $t \in [T_0(\theta,\gamma), T]$. The function is a concave-convex function with inflection point $T_0(\theta,\gamma)$ and stationary point $T_0'(\theta,\gamma)$. *Q.E.D.*

We then prove the property of function $\Psi(t|\alpha,\beta,\gamma,k)$ in the following two lemmas.

Lemma A.2. *Consider α, β, γ be three (different) positive parameters, $t \geq 0$, and k be arbitrary real numbers.*

(1) Suppose $\alpha > \beta$. Define $\underline{k} = \frac{\beta+\gamma}{\alpha+\gamma}$ and $\overline{k} = \begin{cases} \frac{\alpha-\gamma}{\beta-\gamma}, & \beta > \gamma, \\ +\infty, & \beta < \gamma. \end{cases}$

(1.1) When $k \leq \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave function.

(1.2) When $\underline{k} < k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex-concave function.

(1.3) When $k \geq \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex function.

(2) Suppose $\alpha < \beta$. Define $\underline{k} = \begin{cases} \frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha > \gamma \\ 0, & \text{if } \alpha < \gamma \end{cases}$ and $\overline{k} = \frac{\beta+\gamma}{\alpha+\gamma}$.

(2.1) When $k \leq \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave function.

(2.2) When $\underline{k} < k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave-convex function.

(2.3) When $k \geq \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex function.

Proof. By definition, $\Psi(t|\alpha,\beta,\gamma,k) = k\Phi(t|\alpha,\gamma) - \Phi(t|\beta,\gamma)$ where $\Phi(t|\theta,\gamma) = \frac{e^{-\theta t} - e^{-\gamma t}}{\gamma-\theta}$. Then

$$\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2} - \frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2} = k\frac{\alpha^2 e^{-\alpha t} - \gamma^2 e^{-\gamma t}}{\gamma-\alpha} - \frac{\beta^2 e^{-\beta t} - \gamma^2 e^{-\gamma t}}{\gamma-\beta}\ . \tag{A.2}$$

Following Lemma A.1, we know that $\Phi(t|\theta,\gamma)$ is a concave-convex function with inflection point $T_0(\theta,\gamma) = \frac{2(\ln s - \ln\gamma)}{s-\gamma}$. Specifically, $\frac{\partial^2\Phi(T_0(\theta,\gamma)|\theta,\gamma)}{\partial t^2} = 0$; $\frac{\partial^2\Phi(t|\theta,\gamma)}{\partial t^2} < 0$ if $t < T_0(\theta,\gamma)$; and $\frac{\partial^2\Phi(t|\theta,\gamma)}{\partial t^2} > 0$

if $t > T_0(\theta,\gamma)$. We further have $\frac{\partial T_0(\theta,\gamma)}{\partial\theta} = \frac{2(1-\frac{\gamma}{s}+\ln\frac{\gamma}{s})}{(s-\gamma)^2} < 0$, given that $1+\ln\xi < \xi$ for $0<\xi<1$ and $\xi>1$. Thus, $T_0(\theta,\gamma)$ decreases in θ.

We start with two special cases. First, suppose $k=0$. Then, $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = -\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}$. By Lemma A.1, we know that $\Psi(t|\alpha,\beta,\gamma,0)$ is a convex-concave function. Second, suppose $t = T_0(\beta,\gamma)$, resulting in $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2} = 0$. Then, $\frac{\partial^2\Psi(T_0(\beta,\gamma)|\alpha,\beta,\gamma,k)}{\partial t^2} = k\frac{\partial^2\Phi(T_0(\beta,\gamma)|\alpha,\gamma)}{\partial t^2}$, whose sign is determined by k and $\frac{\partial^2\Phi(T_0(\beta,\gamma)|\alpha,\gamma)}{\partial t^2}$. In particular, if $\alpha>\beta$, we have $T_0(\alpha,\gamma) < T_0(\beta,\gamma)$. Following Lemma A.1, we conclude that $\frac{\partial^2\Phi(T_0(\beta,\gamma)|\alpha,\gamma)}{\partial t^2} > 0$. If $\alpha<\beta$, we have $T_0(\alpha,\gamma) > T_0(\beta,\gamma)$ and we conclude that $\frac{\partial^2\Phi(T_0(\beta,\gamma)|\alpha,\gamma)}{\partial t^2} < 0$.

In the following, we focus on the case when $k \neq 0$ and $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2} \neq 0$. We rewrite (A.2) to be

$$\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{\frac{\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}}{\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}} - \frac{1}{k}\right\}. \tag{A.3}$$

Our goal is to determine the sign of $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}$ for any given t. We have already shown that the component $\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)$ is first negative and then positive. Below, we investigate the ratio $\frac{\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}}{\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}}$. For demonstration purposes, we denote

$$r_1(t,\alpha,\beta|\gamma) = \frac{\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}}{\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}} = \frac{\left(\frac{\alpha^2 e^{-\alpha t}-\gamma^2 e^{-\gamma t}}{\gamma-\alpha}\right)}{\left(\frac{\beta^2 e^{-\beta t}-\gamma^2 e^{-\gamma t}}{\gamma-\beta}\right)} = \left(\frac{\gamma-\beta}{\gamma-\alpha}\right)\left(\frac{\alpha^2 e^{-\alpha t}-\gamma^2 e^{-\gamma t}}{\beta^2 e^{-\beta t}-\gamma^2 e^{-\gamma t}}\right).$$

First of all, we examine its monotonicity. The first-order derivative is given by

$$\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} = -\frac{(\beta-\gamma)e^{-(\alpha-\beta)t}\left\{\alpha^2\beta^2(\alpha-\beta)+\gamma^2[\beta^2(\beta-\gamma)e^{(\alpha-\gamma)t}-\alpha^2(\alpha-\gamma)e^{(\beta-\gamma)t}]\right\}}{(\alpha-\gamma)(\beta^2-\gamma^2 e^{(\beta-\gamma)t})^2}.$$

Denote $r_2(t,\alpha,\beta|\gamma) = \alpha^2\beta^2(\alpha-\beta)+\gamma^2[\beta^2(\beta-\gamma)e^{(\alpha-\gamma)t}-\alpha^2(\alpha-\gamma)e^{(\beta-\gamma)t}]$. Clearly, $r_2(t,\alpha,\beta|\gamma)$ plays an important role in determining the sign of the derivative $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t}$. Moreover, we have

$$\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t} = e^{-\gamma t}(\beta^2 e^{\alpha t}-\alpha^2 e^{\beta t})\gamma^2(\gamma-\alpha)(\gamma-\beta) = e^{-\gamma t}e^{\beta t}\beta^2(e^{(\alpha-\beta)t}-\frac{\alpha^2}{\beta^2})\gamma^2(\gamma-\alpha)(\gamma-\beta). \tag{A.4}$$

It is straightforward to see from (A.4) that there exist 6 possible scenarios as below:

(a) If $\alpha>\beta>\gamma$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first negative and then positive.

(b) If $\alpha>\gamma>\beta$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first positive and then negative.

(c) If $\gamma>\alpha>\beta$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first negative and then positive.

(d) If $\beta>\alpha>\gamma$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first positive and then negative.

(e) If $\beta>\gamma>\alpha$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first negative and then positive.

(f) If $\gamma>\beta>\alpha$, $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t}$ is first positive and then negative.

By definition, $T_0(\alpha,\beta)$ is such that $\beta^2 e^{\alpha T_0(\alpha,\beta)} - \alpha^2 e^{\beta T_0(\alpha,\beta)} = 0$. From (A.4), we observe that $\frac{\partial r_2(t,\alpha,\beta|\gamma)}{\partial t} = 0$ at $t = T_0(\alpha,\beta)$. Given its monotonicity, we conclude that when $t = T_0(\alpha,\beta)$, $r_2(t,\alpha,\beta|\gamma)$ reaches the lowest point in cases (a), (c) and (e); while $r_2(t,\alpha,\beta|\gamma)$ reaches the highest point in cases (b), (d), and (f). In addition, the sign of $r_2(T_0(\alpha,\beta),\alpha,\beta|\gamma)$ follows from Lemma A.1 and the fact that $T_0(\alpha,\beta) < T_0(\beta,\gamma)$ if $\alpha > \gamma$ or $T_0(\alpha,\beta) > T_0(\beta,\gamma)$ if $\alpha < \gamma$. Specifically,

$$
\begin{aligned}
r_2(T_0(\alpha,\beta),\alpha,\beta|\gamma) &= \alpha^2\beta^2(\alpha-\beta) + \gamma^2[\beta^2(\beta-\gamma)e^{(\alpha-\gamma)T_0(\alpha,\beta)} - \alpha^2(\alpha-\gamma)e^{(\beta-\gamma)T_0(\alpha,\beta)}] \\
&= \alpha^2\beta^2(\alpha-\beta) + \gamma^2[\alpha^2(\beta-\gamma)e^{(\beta-\gamma)T_0(\alpha,\beta)} - \alpha^2(\alpha-\gamma)e^{(\beta-\gamma)T_0(\alpha,\beta)}] \\
&= \alpha^2\beta^2(\alpha-\beta) + \gamma^2\alpha^2(\beta-\alpha)e^{(\beta-\gamma)T_0(\alpha,\beta)} \\
&= \alpha^2(\alpha-\beta)[\frac{\beta^2}{\gamma^2} - e^{(\beta-\gamma)T_0(\alpha,\beta)}] \\
&= \begin{cases} >0, & \text{in case (a)} \\ <0, & \text{in case (b)} \\ >0, & \text{in case (c)} \\ <0, & \text{in case (d)} \\ >0, & \text{in case (e)} \\ <0. & \text{in case (f)} \end{cases}
\end{aligned}
$$

In summary, in cases (a), (c) and (e), we have shown that $r_2(t,\alpha,\beta|\gamma)$ first decreases in $t < T_0(\alpha,\beta)$ and then increases in $t \geq T_0(\alpha,\beta)$. In addition, the minimum value $r_2(T_0(\alpha,\beta),\alpha,\beta|\gamma)$ is positive. Therefore, we conclude that $r_2(t,\alpha,\beta|\gamma) > 0$ for $t \geq 0$ in cases (a), (c) and (e). On the other hand, in cases (b), (d), and (f), we have shown that $r_2(t,\alpha,\beta|\gamma)$ first increases in $t < T_0(\alpha,\beta)$ and then decreases in $t \geq T_0(\alpha,\beta)$. In addition, the maximum value $r_2(T_0(\alpha,\beta),\alpha,\beta|\gamma)$ is negative. Therefore, we conclude that $r_2(t,\alpha,\beta|\gamma) < 0$ for $t \geq 0$ in cases (b), (d), and (f).

Recall that

$$
\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} = -\frac{(\beta-\gamma)e^{-(\alpha-\beta)t}}{(\alpha-\gamma)(\beta^2-\gamma^2 e^{(\beta-\gamma)t})^2} r_2(t,\alpha,\beta|\gamma).
$$

We are able to make the following conclusion.

(a) If $\alpha > \beta > \gamma$, $r_2(t,\alpha,\beta|\gamma) > 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} < 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ decreases in t;

(b) If $\alpha > \gamma > \beta$, $r_2(t,\alpha,\beta|\gamma) < 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} < 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ decreases in t;

(c) If $\gamma > \alpha > \beta$, $r_2(t,\alpha,\beta|\gamma) > 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} < 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ decreases in t;

(d) If $\beta > \alpha > \gamma$, $r_2(t,\alpha,\beta|\gamma) < 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} > 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ increases in t;

(e) If $\beta > \gamma > \alpha$, $r_2(t,\alpha,\beta|\gamma) > 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} > 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ increases in t;

(f) If $\gamma > \beta > \alpha$, $r_2(t,\alpha,\beta|\gamma) < 0$ for $t \geq 0$, then $\frac{\partial r_1(t,\alpha,\beta|\gamma)}{\partial t} > 0$ for $t \geq 0$. That is, $r_1(t,\alpha,\beta|\gamma)$ increases in t;

In short, we have proven that $r_1(t,\alpha,\beta|\gamma)$ decreases in t if $\alpha > \beta$ or increases in t if $\alpha < \beta$.

Next, we explore the sign of $r_1(t,\alpha,\beta|\gamma)$. WLOG, we focus on the case that $\alpha > \beta$. It is straightforward to see that $r_1(0,\alpha,\beta|\gamma) = \frac{\alpha+\gamma}{\beta+\gamma} > 0$. Second, we have

$$r_1(+\infty,\alpha,\beta|\gamma) = \left(\frac{\beta-\gamma}{\alpha-\gamma}\right) \lim_{t\to+\infty} \frac{\alpha^2 e^{(\gamma-\alpha)t} - \gamma^2}{\beta^2 e^{(\gamma-\beta)t} - \gamma^2} = \begin{cases} \frac{\beta-\gamma}{\alpha-\gamma}, & \text{in case (a): } \alpha > \beta > \gamma \\ 0, & \text{in case (b): } \alpha > \gamma > \beta \\ 0, & \text{in case (c): } \gamma > \alpha > \beta \end{cases}$$

Furthermore, when $\alpha > \beta$, we have $T_0(\alpha,\gamma) < T_0(\beta,\gamma)$. Since $r_1(t,\alpha,\beta|\gamma) = \frac{\frac{\partial^2 \Phi(t|\alpha,\gamma)}{\partial t^2}}{\frac{\partial^2 \Phi(t|\beta,\gamma)}{\partial t^2}}$, by the definitions of $T_0(\alpha,\gamma)$ and $T_0(\beta,\gamma)$ as well as Lemma A.1, we obtain that $r_1(t,\alpha,\beta|\gamma) > 0$ when $t < T_0(\alpha,\gamma)$; $r_1(t,\alpha,\beta|\gamma) = 0$ at $t = T_0(\alpha,\gamma)$; $r_1(t,\alpha,\beta|\gamma) < 0$ when $T_0(\alpha,\gamma) < t < T_0(\beta,\gamma)$; and $r_1(t,\alpha,\beta|\gamma) > 0$ when $t > T_0(\beta,\gamma)$. Lastly, as t approaches to $T_0(\beta,\gamma)$ from the left, we have $\lim_{t\uparrow T_0(\beta,\gamma)} r_1(t,\alpha,\beta|\gamma) = -\infty$. As t approaches to $T_0(\beta,\gamma)$ from the right, we have $\lim_{t\downarrow T_0(\beta,\gamma)} r_1(t,\alpha,\beta|\gamma) = +\infty$.

The left column of Figure 8 illustrates the function $r_1(t,\alpha,\beta|\gamma)$ when $\alpha > \beta$. Following Equation (A.3), in order to determine the sign of $\frac{\partial^2 \Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}$, we need to compare $r_1(t,\alpha,\beta|\gamma)$ with any given k. For ease of understanding, we denote $w = \frac{1}{k}$ and compare $r_1(t,\alpha,\beta|\gamma)$ with w. As suggested from Figure 8, we identify two thresholds for w: upper threshold $\bar{w}$ and lower threshold $\underline{w}$ where

$$\bar{w} = \frac{\alpha+\gamma}{\beta+\gamma}$$

$$\underline{w} = \begin{cases} \frac{\beta-\gamma}{\alpha-\gamma}, & \text{in case (a): } \alpha > \beta > \gamma \\ 0, & \text{in case (b) and (c): } \alpha > \gamma > \beta \text{ or } \gamma > \alpha > \beta \end{cases}$$

- If $w > \bar{w}$, the equation $r_1(t,\alpha,\beta|\gamma) = w$ has only one solution at $t > T_0(\beta,\gamma)$. Let the solution be T_1. Then $r_1(t,\alpha,\beta|\gamma) < w$ when $t < T_0(\beta,\gamma)$; $r_1(t,\alpha,\beta|\gamma) > w$ when $T_0(\beta,\gamma) < t < T_1$; and $r_1(t,\alpha,\beta|\gamma) < w$ when $t > T_1$.
- If $w \leq \underline{w}$, the equation $r_1(t,\alpha,\beta|\gamma) = w$ has only one solution at $t < T_0(\beta,\gamma)$. Let the solution be T_2. Then $r_1(t,\alpha,\beta|\gamma) > w$ when $t < T_2$; $r_1(t,\alpha,\beta|\gamma) < w$ when $T_2 < t < T_0(\beta,\gamma)$; and $r_1(t,\alpha,\beta|\gamma) > w$ when $t > T_0(\beta,\gamma)$.
- If $\underline{w} < w \leq \bar{w}$, the equation $r_1(t,\alpha,\beta|\gamma) = w$ has two solutions. One is at $t < T_0(\beta,\gamma)$ and is denoted as T_3. The other is at $t < T_0(\beta,\gamma)$ and is denoted as T_4. Then $r_1(t,\alpha,\beta|\gamma) > w$ when $t < T_3$; $r_1(t,\alpha,\beta|\gamma) < w$ when $T_3 < t < T_0(\beta,\gamma)$; $r_1(t,\alpha,\beta|\gamma) > w$ when $T_0(\beta,\gamma) < t < T_4$; and $r_1(t,\alpha,\beta|\gamma) < w$ when $t > T_4$.

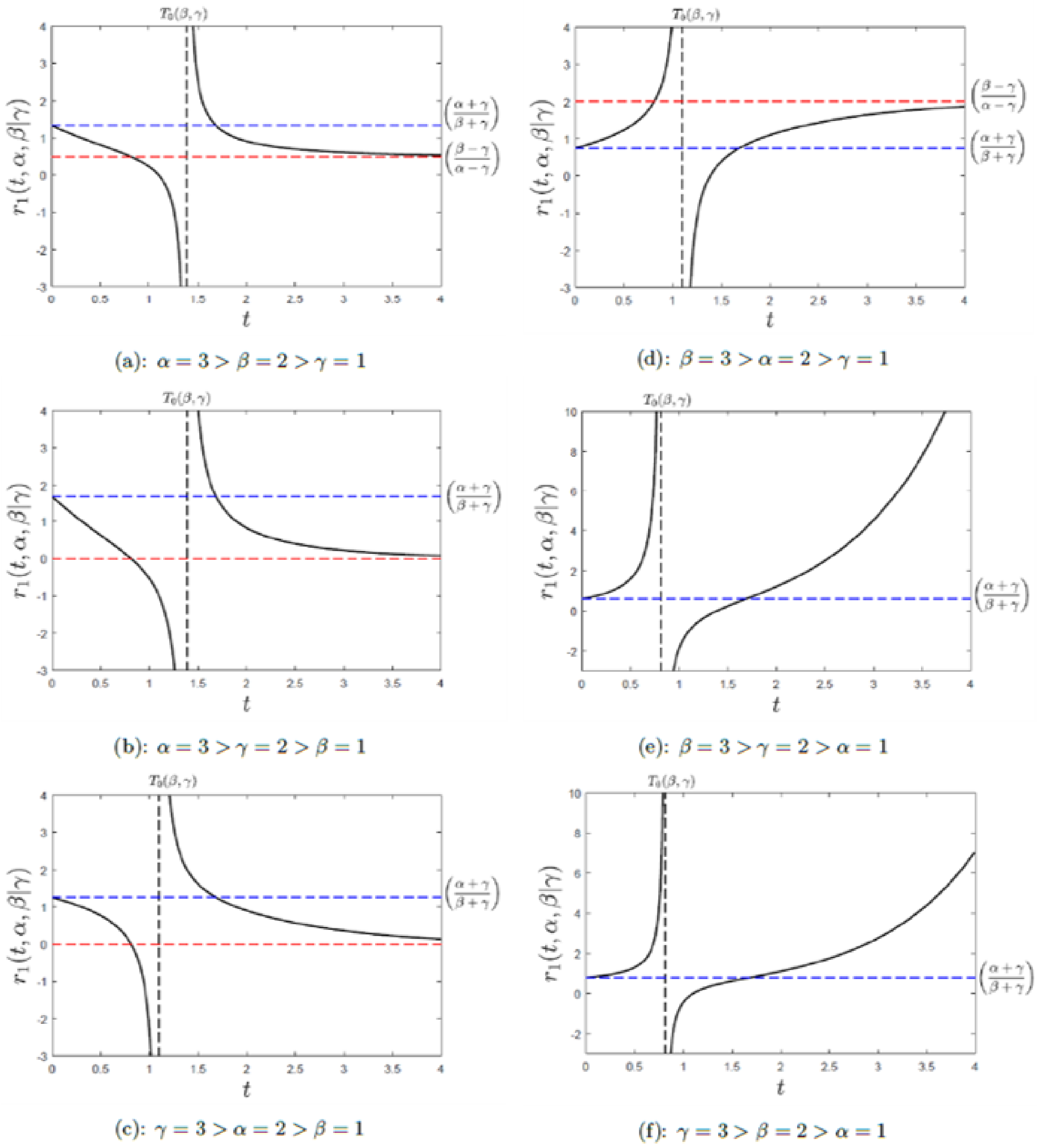


Figure 8 **Demonstration of the function $r_1(t, \alpha, \beta|\gamma)$. The subfigures on the left assume $\alpha > \beta$ and the subfigures on the right assume $\alpha < \beta$.**

Finally, recall that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ and $w=\frac{1}{k}$ by definition. In addition, $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}<0$ when $t<T_0(\beta,\gamma)$; $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}=0$ when $t=T_0(\beta,\gamma)$; and $\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}>0$ when $t>T_0(\beta,\gamma)$. From the above analysis, we make the following conclusion:

- If $k<0$, which implies that $w<\underline{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is positive when $t<T_2$; it is zero when $t=T_2$; and it is negative when $t>T_2$. Thus, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave.
- If $k=0$, as mentioned at the beginning of the proof, $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2}=k\frac{\partial^2\Phi(t|\alpha,\gamma)}{\partial t^2}$ is convex-concave.
- If $0<k<\frac{\beta+\gamma}{\alpha+\gamma}$, which implies that $w\geq\bar{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is positive when $t<T_1$; it is zero when $t=T_1$; and it is negative when $t>T_1$. Thus, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave.
- If $k=\frac{\beta+\gamma}{\alpha+\gamma}$, which implies that $w=\bar{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is zero when $t=0$; it is positive when $0<t<T_1$; it is zero again when $t=T_1$; and it is negative when $t>T_1$. Thus, $\Psi(t|\alpha,\beta,\gamma,k)$ is still convex-concave.

So far, we have shown that when $k\leq\frac{\beta+\gamma}{\alpha+\gamma}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave. Next, we consider the case that $k>\frac{\beta+\gamma}{\alpha+\gamma}$.

- When $\beta<\gamma$, we have $\underline{w}=0$. In this case, if $k>\frac{\beta+\gamma}{\alpha+\gamma}$, which implies that $\underline{w}<w<\bar{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is zero at $t=T_3$ and $t=t_4$; it is negative when $t<T_3$; it is positive when $T_3<t<T_4$; and it is negative when $t>T_4$. Therefore, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex-concave.

When $\beta>\gamma$, we have $\underline{w}=\frac{\beta-\gamma}{\alpha-\gamma}$. In addition, $\frac{\beta+\gamma}{\alpha+\gamma}<\frac{\alpha-\gamma}{\beta-\gamma}$. There exists a k satisfying $\frac{\beta+\gamma}{\alpha+\gamma}<k<\frac{\alpha-\gamma}{\beta-\gamma}$.

- If $\frac{\beta+\gamma}{\alpha+\gamma}<k<\frac{\alpha-\gamma}{\beta-\gamma}$, which implies that $\underline{w}<w<\bar{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is zero at $t=T_3$ and $t=t_4$; it is negative when $t<T_3$; it is positive when $T_3<t<T_4$; and it is negative when $t>T_4$. Therefore, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex-concave.
- If $k\geq\frac{\alpha-\gamma}{\beta-\gamma}$, which implies that $w\leq\underline{w}$, we conclude that $\frac{\partial^2\Psi(t|\alpha,\beta,\gamma,k)}{\partial t^2} = k\left(\frac{\partial^2\Phi(t|\beta,\gamma)}{\partial t^2}\right)\cdot\left\{r_1(t,\alpha,\beta|\gamma)-\frac{1}{k}\right\}$ is negative when $t<T_2$ and is positive when $t>T_2$. That is, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex function.

To wrap up, when $\alpha>\beta$, we define two thresholds for k to be

$$\underline{k}=\frac{\beta+\gamma}{\alpha+\gamma}\quad\text{and}\quad\overline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \beta>\gamma,\\ +\infty, & \beta<\gamma.\end{cases}$$

When $k\leq\underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave function. When $k\geq\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex function. When $\underline{k}<k<\overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex-concave function.

We can apply a similar proof for the case that $\alpha<\beta$. The right column of Figure 8 displays the function $r_1(t,\alpha,\beta|\gamma)$ when $\alpha<\beta$. Similarly as above, when $\alpha<\beta$, we define two thresholds for k to be

$$\underline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha>\gamma\\ 0, & \text{if } \alpha<\gamma\end{cases}\quad\text{and}\quad\overline{k}=\frac{\beta+\gamma}{\alpha+\gamma}$$

When $k \le \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave function. When $k \ge \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex function. When $\underline{k} < k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave-convex function. *Q.E.D.*

LEMMA A.3. *Consider α, β, γ be three (different) positive parameters, $t \ge 0$, and k be arbitrary real numbers.*

(1) Suppose $\alpha > \beta$. Recall $\overline{k} = \begin{cases} \frac{\alpha-\gamma}{\beta-\gamma}, & \beta > \gamma, \\ +\infty, & \beta < \gamma. \end{cases}$

(1.1) When $k \le 1$, $\Psi(t|\alpha,\beta,\gamma,k)$ decreases and then increases in t.

(1.2) When $1 < k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ increases, then decreases, and finally increases in t.

(1.3) When $k \ge \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ increases and then decreases in t.

(2) Suppose $\alpha < \beta$. Recall $\underline{k} = \begin{cases} \frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha > \gamma. \\ 0, & \text{if } \alpha < \gamma. \end{cases}$

(2.1) When $k \le \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ decreases and then increases in t.

(2.2) When $\underline{k} < k < 1$, $\Psi(t|\alpha,\beta,\gamma,k)$ decreases, then increases, and finally decreases in t.

(2.3) When $k \ge 1$, $\Psi(t|\alpha,\beta,\gamma,k)$ increases and then decreases in t.

Proof. We examine the first-order derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ that is given by

$$\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = k\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t} - \frac{\partial\Phi(t|\beta,\gamma)}{\partial t} = k\frac{\alpha e^{-\alpha t} - \gamma e^{-\gamma t}}{\alpha-\gamma} - \frac{\beta e^{-\beta t} - \gamma e^{-\gamma t}}{\beta-\gamma}. \tag{A.5}$$

Suppose $k = 0$. Then $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = -\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}$. Following Lemma A.1, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases and then increases in t. In the following, we focus on the case that $k \ne 0$. First, we have $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k-1$. Hence, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = \begin{cases} \ge 0, & k \ge 1 \\ < 0, & k < 1 \end{cases}$.

Next, we determine the sign of $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ when t is sufficiently large. When $\frac{\partial\Phi(t|\beta,\gamma)}{\partial t} \ne 0$, we can rewrite (A.5) to be

$$\begin{aligned} \frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} &= k\left(\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}\right)\cdot\left\{\frac{\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t}}{\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}} - \frac{1}{k}\right\} \\ &= k\left(\frac{\beta e^{-\beta t} - \gamma e^{-\gamma t}}{\beta-\gamma}\right)\left\{\frac{(\beta-\gamma)}{(\alpha-\gamma)}\cdot\frac{(\alpha e^{-\alpha t} - \gamma e^{-\gamma t})}{(\beta e^{-\beta t} - \gamma e^{-\gamma t})} - \frac{1}{k}\right\} \\ &= k\left(\frac{\beta e^{-\beta t} - \gamma e^{-\gamma t}}{\beta-\gamma}\right)\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k}\right\}. \end{aligned}$$

Lemma A.1 indicates that $\frac{\partial\Phi(t|\beta,\gamma)}{\partial t} = \frac{\beta e^{-\beta t} - \gamma e^{-\gamma t}}{\beta-\gamma}$ is first positive and then negative. Thus, $\left(\frac{\beta e^{-\beta t} - \gamma e^{-\gamma t}}{\beta-\gamma}\right)$ must be negative when t is sufficiently large. Then we consider the ratio $\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot$

$\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}$. We have the following result:

$$\lim_{t\to\infty}\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}=\begin{cases}\frac{(\beta-\gamma)}{(\alpha-\gamma)}, & \text{if } \alpha>\beta>\gamma\\ 0, & \text{if } \alpha>\gamma>\beta\\ 0, & \text{if } \gamma>\alpha>\beta\\ \frac{(\beta-\gamma)}{(\alpha-\gamma)}, & \text{if } \beta>\alpha>\gamma\\ +\infty, & \text{if } \beta>\gamma>\alpha\\ +\infty. & \text{if } \gamma>\beta>\alpha\end{cases}$$

We start with the case when $\alpha>\beta$. Recall that in this case, we defined $\underline{k}=\frac{\beta+\gamma}{\alpha+\gamma}$ and $\overline{k}=\begin{cases}\frac{\alpha-\gamma}{\beta-\gamma}, & \beta>\gamma,\\ +\infty, & \beta<\gamma.\end{cases}$. From above, we have

$$\lim_{t\to\infty}\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}=\begin{cases}\frac{(\beta-\gamma)}{(\alpha-\gamma)}-\frac{1}{k}, & \beta>\gamma\\ 0-\frac{1}{k}. & \beta<\gamma\end{cases}$$

As a result, when $k<0$, $\lim_{t\to\infty}\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}>0$, implying that $\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}$ is positive when t is sufficiently large. When $0<k<\overline{k}$, $\lim_{t\to\infty}\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}<0$, implying that $\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}$ is negative when t is sufficiently large. When $k\geq\overline{k}$, $\lim_{t\to\infty}\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}\geq 0$, implying that $\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}$ is positive when t is sufficiently large.

Since $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}=k\left(\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}\right)\left\{\frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}\cdot\frac{(e^{-(\alpha-\gamma)t}-\frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t}-\frac{\gamma}{\beta})}-\frac{1}{k}\right\}$ and $\left(\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}\right)$ is negative when t is sufficiently large, we conclude that when t is sufficiently large, $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is negative if $k\geq\overline{k}$ while $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is positive if $k<\overline{k}$.

So far, we have examined the sign of $\frac{\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}$ and $\lim_{t\to\infty}\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$. Below, to determine the sign of $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ for any t, we combine the results with the convexity and concavity results in Lemma A.2. We identify four possible scenarios, as illustrated in Figure 9.

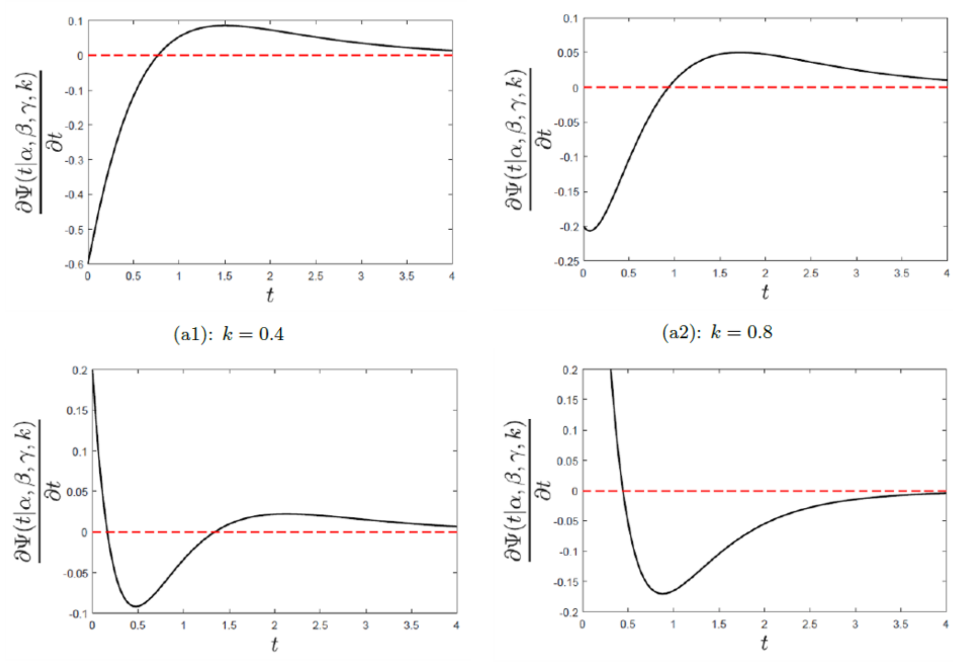
(a1) When $k\leq\underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}=k-1<0$ since $k\leq\underline{k}=\frac{\beta+\gamma}{\alpha+\gamma}<1$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}>0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first negative, increases in t, then becomes positive, finally decreases in t but stays positive. That is, when $k\leq\underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases and then increases in t.

(a2) When $\underline{k}<k<1$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex-concave since we have $\overline{k}>1$. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}=k-1<0$, and $\lim_{t\to\infty}\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}>0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first negative, decreases in t, then increases in t, becomes positive, and finally decreases in t but stays positive. That is, when $\underline{k}<k<1$, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases and then increases in t.

(a3) When $1 \leq k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex-concave. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k-1 \geq 0$, and $\lim_{t\to\infty} \frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} > 0$. Furthermore, when $\alpha > \beta$, we have $T_0(\alpha,\gamma) < T_0(\beta,\gamma)$. Following Lemma A.1, $\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t} > 0$ when $t < \frac{T_0(\alpha,\gamma)}{2}$ and $\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t} < 0$ when $t > \frac{T_0(\alpha,\gamma)}{2}$; while $\frac{\partial\Phi(t|\beta,\gamma)}{\partial t} > 0$ when $t < \frac{T_0(\beta,\gamma)}{2}$ and $\frac{\partial\Phi(t|\beta,\gamma)}{\partial t} < 0$ when $t > \frac{T_0(\beta,\gamma)}{2}$. Therefore, for $\frac{T_0(\alpha,\gamma)}{2} < t < \frac{T_0(\beta,\gamma)}{2}$ and $k \geq 1$, we obtain $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = k\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t} - \frac{\partial\Phi(t|\beta,\gamma)}{\partial t} < 0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis twice. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, decreases in t, becomes negative, then increases in t, becomes positive, finally decreases in t but stays positive. That is, when $1 \leq k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first increases, then decreases in t, and finally increases in t.

(a4) When $k \geq \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k-1 > 0$, and $\lim_{t\to\infty} \frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} < 0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, decreases in t, then becomes negative, finally increases in t but stays negative. That is, when $k \geq \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first increases and then decreases in t.

To sum up, when $\alpha > \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ decreases and then increases in t if $k \leq 1$; $\Psi(t|\alpha,\beta,\gamma,k)$ increases, then decreases, and finally increases in t if $1 < k < \overline{k}$; $\Psi(t|\alpha,\beta,\gamma,k)$ increases and then decreases in t if $k \geq \overline{k}$.



(a1): $k = 0.4$ (a2): $k = 0.8$

(a3): $k = 1.2$ (a4): $k = 2.4$

Figure 9 **Demonstration of the derivative** $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ **when** $\alpha = 3 > \beta = 2$**, and** $\gamma = 1$**. Then** $\underline{k} = \frac{\beta+\gamma}{\alpha+\gamma} = 3/4$ **and** $\overline{k} = \frac{\alpha-\gamma}{\beta-\gamma} = 2$**.**

Next, we consider the case when $\alpha < \beta$. Recall that in this case, we defined $\underline{k} = \begin{cases} \frac{\alpha-\gamma}{\beta-\gamma}, & \text{if } \alpha > \gamma \\ 0, & \text{if } \alpha < \gamma \end{cases}$ and $\overline{k} = \frac{\beta+\gamma}{\alpha+\gamma}$. Since we have

$$\lim_{t\to\infty} \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\} = \begin{cases} \frac{(\beta-\gamma)}{(\alpha-\gamma)}, & \alpha > \gamma \\ (+\infty) - \frac{1}{k}. & \alpha < \gamma \end{cases}$$

By similar analysis as above, we achieve that when $k < 0$, $\lim_{t\to\infty} \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\} > 0$. When $0 < k < \underline{k}$, $\lim_{t\to\infty} \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\} < 0$. When $k \geq \underline{k}$, $\lim_{t\to\infty} \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\} \geq 0$.

Therefore, we conclude that when t is sufficiently large, the first-order derivative $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = k\left(\frac{\partial\Phi(t|\beta,\gamma)}{\partial t}\right) \left\{ \frac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)} \cdot \frac{(e^{-(\alpha-\gamma)t} - \frac{\gamma}{\alpha})}{(e^{-(\beta-\gamma)t} - \frac{\gamma}{\beta})} - \frac{1}{k} \right\}$ is positive if $k \leq \underline{k}$ while $\frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is negative if $k > \underline{k}$.

Lastly, we combine the results about $\frac{\Psi(0|\alpha,\beta,\gamma,k)}{\partial t}$ and $\lim_{t\to\infty} \frac{\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ with Lemma A.2. Again, there exist four scenarios, as illustrated in Figure 10.

(b1) When $k \leq \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k - 1 < 0$ as $k \leq \underline{k} < 1$, and $\lim_{t\to\infty} \frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} > 0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, finally stay negative, then increases in t and becomes positive, finally decreases in t but stays positive. That is, when $k \leq \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases and then increases in t.

(b2) When $\underline{k} < k < 1$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave-convex since $\overline{k} >$. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k - 1 < 0$, and $\lim_{t\to\infty} \frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} < 0$. Furthermore, note that when $\alpha < \beta$, we have $T_0(\alpha,\gamma) > T_0(\beta,\gamma)$. Following Lemma A.1, for $\frac{T_0(\beta,\gamma)}{2} < t < \frac{T_0(\alpha,\gamma)}{2}$ and $0 \leq \underline{k} < k < 1$, we obtain $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} = k\frac{\partial\Phi(t|\alpha,\gamma)}{\partial t} - \frac{\partial\Phi(t|\beta,\gamma)}{\partial t} > 0$. Thus, the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis twice. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first negative, increases in t, becomes positive, then decreases in t, becomes negative, finally increases in t but stays negative. That is, when $\underline{k} < k < 1$, $\Psi(t|\alpha,\beta,\gamma,k)$ first decreases, then increases, and finally decreases in t.

(b3) When $1 \leq k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex-concave-convex. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k - 1 \geq 0$, and $\lim_{t\to\infty} \frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} < 0$. Thus, the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, increases in t, then decreases in t, becomes negative, finally increases in t but stays negative. That is, when $1 \leq k < \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first increases, and then decreases in t.

(b4) When $k \geq \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex. In addition, $\frac{\partial\Psi(0|\alpha,\beta,\gamma,k)}{\partial t} = k - 1 > 0$, and $\lim_{t\to\infty} \frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t} < 0$. These imply that the derivative $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ crosses the x-axis only once. We conclude that $\frac{\partial\Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ is first positive, decreases in t, then becomes negative, finally increases in t but stays negative. That is, when $k \geq \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ first increases, and then decreases in t.

To sum up, when $\alpha < \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ decreases and then increases in t if $k \le \underline{k}$; $\Psi(t|\alpha,\beta,\gamma,k)$ decreases, then increases, and finally decreases in t if $\underline{k} < k < 1$; $\Psi(t|\alpha,\beta,\gamma,k)$ increases and then decreases in t if $k \ge 1$. *Q.E.D.*

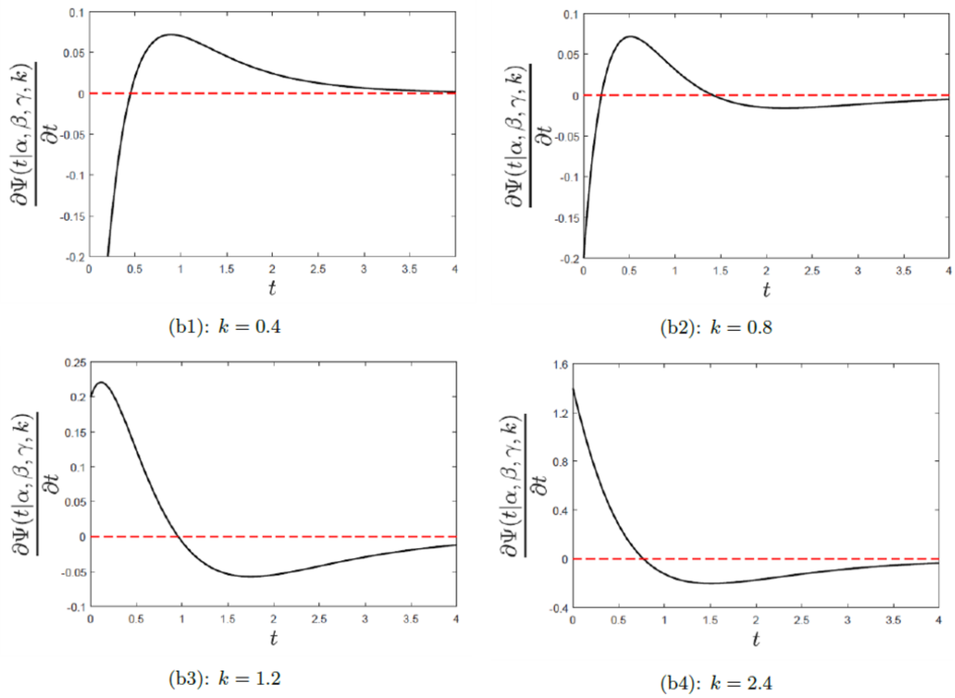


Figure 10 **Demonstration of the derivative** $\frac{\partial \Psi(t|\alpha,\beta,\gamma,k)}{\partial t}$ **when** $\alpha = 2 < \beta = 3$**, and** $\gamma = 1$**. Then** $\underline{k} = \frac{\alpha-\gamma}{\beta-\gamma} = 1/2$ **and** $\overline{k} = \frac{\beta+\gamma}{\alpha+\gamma} = 4/3$**.**

Let $T_1'(\alpha,\beta,\gamma,k)$ and $T_2'(\alpha,\beta,\gamma,k)$ be the stationary points when there are two points, and $T_2'(\alpha,\beta,\gamma,k)$ be the unique stationary point when there is only one stationary point. For simplicity, we will use T_1 and T_2 instead. Then we prove Corollary A.1.

Corollary A.1. *Consider α, β, γ be three (different) positive parameters, $t \ge 0$, and k be arbitrary real numbers.*

(1) When $k \le \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave function, which decreases and then increases in t. We have $T_2' < T_2$.

(2) When $k \le 1$,

(2.1) Suppose $\alpha > \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex-concave function, which decreases and then increases in t. We have $T_1 < T_2' < T_2$.

(2.2) Suppose $\alpha < \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave-convex function. $\Psi(t|\alpha,\beta,\gamma,k)$ decreases, then increases, and finally decreases in t. We have $T_1' < T_1 < T_2' < T_2$.

(3) When $1 < k < \overline{k}$,

(3.1) Suppose $\alpha > \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex-concave function. $\Psi(t|\alpha,\beta,\gamma,k)$ increases, then decreases, and finally increases in t. We have $T_1' < T_1 < T_2' < T_2$.

(3.2) Suppose $\alpha < \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a convex-concave-convex function. $\Psi(t|\alpha,\beta,\gamma,k)$ increases and then decreases in t. We have $T_1 < T_2' < T_2$.

(4) When $k \geq \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is a concave-convex function, which increases and then decreases in t. We have $T_2' < T_2$.

Proof. The proof follows Lemma A.2 and Lemma A.3. *Q.E.D.*

In the following lemma, we prove the relationship between structure and the property of function $\Psi(t|,\alpha,\beta,\gamma,k)$ in the proportional reward case.

LEMMA A.4. *In the proportional reward case, $\Psi(t|,\alpha,\beta,\gamma,k)$ and the optimal structure has the following relationship:*

(i) If $\Psi(t|,\alpha,\beta,\gamma,k)$ is convex $t \in [\bar{t}_a, \bar{t}_b]$, where $\bar{t}_a < \bar{t}_b$, in the optimal schedule, elements within $\left[(T-\bar{t}_b)^+, (T-\bar{t}_a)^+\right]$ are in a diminuendo subsequence.

(ii) If $\Psi(t|,\alpha,\beta,\gamma,k)$ is concave for $t \in [\bar{t}_a, \bar{t}_b]$, where $\bar{t}_a < \bar{t}_b$, in the optimal schedule, elements within $\left[(T-\bar{t}_b)^+, (T-\bar{t}_a)^+\right]$ are in a crescendo subsequence.

(iii) If $\Psi(t|,\alpha,\beta,\gamma,k)$ is convex for $t \in [\bar{t}_a, \bar{t}_b]$ and concave for $t \in [\bar{t}_b, T_c]$, where $\bar{t}_a < \bar{t}_b < T_c$, in the optimal schedule if there is a game element $\pi^*[i]$ start before but end after $(T-\bar{t}_b)^+$, then we have $r_{*[i]} \geq \min\left\{r_{*[i-1]}, r_{*[i+1]}\right\}$.

(iv) If $\Psi(t|,\alpha,\beta,\gamma,k)$ is concave in $t \in [\bar{t}_a, \bar{t}_b]$ and convex in $t \in [\bar{t}_b, T_c]$, where $\bar{t}_a < \bar{t}_b < T_c$, in the optimal schedule if there is a game element $\pi^*[i]$ start before but end after $(T-\bar{t}_b)^+$, then we have $r_{*[i]} \leq \max\left\{r_{*[i-1]}, r_{*[i+1]}\right\}$.

Proof of Lemma A.4. We use an interchange argument to prove this proposition. Let S^* be the satisfaction obtained from the optimal schedule $\boldsymbol{\pi}^*$, and let S_i^* be the satisfaction obtained by interchanging game element $\pi[i-1]^*$ and game element $\pi[i]^*$ in the optimal schedule. We have

$$S^* - S_i^* = (d_{[i-1]^*} - d_{[i]^*})\left(\left(\Psi(\bar{t}_i + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1} + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k)\right) - \left(\Psi(\bar{t}_i|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k)\right)\right). \tag{A.6}$$

(i) When $\Psi(t|\alpha,\beta,\gamma,k)$ is convex for $t \in [\bar{t}_a, \bar{t}_b]$, and both game element $\pi[i-1]^*$ and game element $\pi[i]^*$ start and finish within $\left[(T-\bar{t}_b)^+, (T-\bar{t}_a)^+\right]$.

As $\Psi'(t|\alpha,\beta,\gamma,k)$ is increasing for $t \in [\bar{t}_a, \bar{t}_b]$, we have $\Psi(\bar{t}_i|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k) \leq \Psi\left(\bar{t}_i + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1} + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k\right)$. By the optimality $S^* - S_i^* \geq 0$ and (A.6), we have $d_{[i-1]^*} \geq d_{[i]^*}$.

(ii) When $\Psi(t|\alpha,\beta,\gamma,k)$ is concave for $t \in [\bar{t}_a, \bar{t}_b]$, and both game element $\pi[i-1]^*$ and game element $\pi[i]^*$ start and finish within $\left[(T-\bar{t}_b)^+, (T-\bar{t}_a)^+\right]$.

As $\Psi'(t|\alpha,\beta,\gamma,k)$ is decreasing for $t \in [\bar{t}_a, \bar{t}_b]$, we have $\Psi(\bar{t}_i|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k) \geq \Psi\left(\bar{t}_i + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1} + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k\right)$. By the optimality $S^* - S_i^* \geq 0$ and (A.6), we have $d_{[i-1]^*} \leq d_{[i]^*}$.

(iii) When $\Psi(t|\alpha,\beta,\gamma,k)$ is convex for $t \in [\bar{t}_a, \bar{t}_b]$ and concave for $t \in [\bar{t}_b, T_c]$, a game element $\pi^*[i]$ starting before and finishing after $(T-\bar{t}_b)^+$ exists in the optimal schedule, where $1 < i < n$. Let S^*_{i+1} be the satisfaction obtained by interchanging game element $\pi[i]^*$ and game element $\pi[i+1]^*$; then, we have

$$S^* - S^*_{i+1} = (d_{[i]^*} - d_{[i+1]^*})\left(\left(\Psi(\bar{t}_{i+1} + \tau_{\pi[i]^*}|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+2} + \tau_{\pi[i]^*}|\alpha,\beta,\gamma,k)\right) - \left(\Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+2}|\alpha,\beta,\gamma,k)\right)\right). \tag{A.7}$$

Suppose otherwise that $d_{[i-1]^*} > d_{[i]^*}$ and $d_{[i]^*} < d_{[i+1]^*}$, by optimality $S^* - S^*_i \geq 0$ and $S^* - S^*_{i+1} \geq 0$, and by (A.6), (A.7) there must be $\Psi(\bar{t}_i|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k) \leq \Psi\left(\bar{t}_i + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1} + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k\right)$ and $\Psi(\bar{t}_{i+1}) - \Psi(\bar{t}_{i+2}|\alpha,\beta,\gamma,k) \geq \Psi\left(\bar{t}_{i+1} + \tau_{\pi[i]^*}|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+2} + \tau_{\pi[i]^*}|\alpha,\beta,\gamma,k\right)$. This contradicts the fact that $\Psi'(t|\alpha,\beta,\gamma,k)$ is increasing for $t \in [\bar{t}_a, \bar{t}_b]$ and $\Psi'(t|\alpha,\beta,\gamma,k)$ is decreasing for $t \in [\bar{t}_b, T_c]$. Therefore, $d_{*[i]} \geq \min\left\{d_{*[i-1]}, d_{*[i+1]}\right\}$.

(iv) When $\Psi(t|\alpha,\beta,\gamma,k)$ is concave for $t \in [\bar{t}_a, \bar{t}_b]$ and convex for $t \in [\bar{t}_b, T_c]$, a game element $\pi^*[i]$ starting before and finishing after $(T-\bar{t}_b)^+$ exists in the optimal schedule, where $1 < i < n$. Let S^*_{i+1} be the satisfaction obtained by interchanging game element $\pi[i]^*$ and game element $\pi[i+1]^*$; then we can also derive (A.7).

Suppose otherwise that $d_{[i-1]^*} < d_{[i]^*}$ and $d_{[i]^*} > d_{[i+1]^*}$, by optimality $S^* - S^*_i \geq 0$ and $S^* - S^*_{i+1} \geq 0$, and by (A.6), (A.7) there must be $\Psi(\bar{t}_i|\alpha,\beta,\gamma,k) - \Psi(\bar{t}_{i+1}|\alpha,\beta,\gamma,k) \geq \Psi\left(\bar{t}_i + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+1} + \tau_{\pi[i-1]^*}|\alpha,\beta,\gamma,k\right)$ and $\Psi(\bar{t}_{i+1}) - \Psi(\bar{t}_{i+2}) \leq \Psi\left(\bar{t}_{i+1} + \tau_{\pi[i]^*}|\alpha,\beta,\gamma,k\right) - \Psi\left(\bar{t}_{i+2} + \tau_{\pi[i]^*}|\alpha,\beta,\gamma,k\right)$. This contradicts the fact that $\Psi'(t|\alpha,\beta,\gamma,k)$ is decreasing for $t \in [\bar{t}_a, \bar{t}_b]$ and $\Psi'(t|\alpha,\beta,\gamma,k)$ is increasing for $t \in [\bar{t}_b, T_c]$. Therefore, $d_{*[i]} \leq \max\left\{d_{*[i-1]}, d_{*[i+1]}\right\}$. *Q.E.D.*

Proof of Theorem 1. We consider four situations to prove the proposition. By Lemma A.2, function $\Psi(t|\alpha,\beta,\gamma,k)$ is concave-convex with inflection point T_2 when $k \leq \underline{k}$; convex-concave with inflection point T_2 when $k > \underline{k}$; concave-convex-concave with inflection points T_1 and T_2 when $\alpha > \beta$ and $\underline{k} < k \leq \overline{k}$; convex-concave-convex with inflection points T_1 and T_2 when $\alpha < \beta$ and $\underline{k} < k \leq \overline{k}$.

(i) When $k \leq \overline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex for $t \in [0, T_2]$ and concave for $t \in [T_2, +\infty]$. By Lemma A.4, elements in $\left[0, (T-T_2)^+\right]$ are in a crescendo subsequence , and elements in $\left[(T-T_2)^+, T\right]$ are in a diminuendo subsequence . If there is a game element i starting before and ending after T_2, then we have $d_{*[i]} \geq \min\left\{d_{*[i-1]}, d_{*[i+1]}\right\}$. Therefore, the optimal structure is an inverted U-shaped sequence.

(ii) When $\underline{k} < k \leq \overline{k}$ and $\alpha > \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave for $t \in [0, T_1]$, convex for $t \in [T_1, T_2]$, and concave for $t \in [T_2, +\infty]$. By Lemma A.4, elements in $\left[0, (T-T_1)^+\right]$ are in a crescendo subsequence , elements in $\left[(T-T_1)^+, (T-T_2)^+\right]$ are in a diminuendo subsequence , and elements in

$\left[(T-T_2)^+, T\right]$ are in a crescendo subsequence . If there is a game element i starting before and ending after T_1, then we have $d_{*[i]} \geq \min\left\{d_{*[i-1]}, d_{*[i+1]}\right\}$. If there is a game element j starting before and ending after T_2, then we have $d_{*[j]} \leq \max\left\{d_{*[j-1]}, d_{*[j+1]}\right\}$. Therefore, the optimal structure is a N-shaped sequence.

(iii) When $\underline{k} < k \leq \overline{k}$ and $\alpha < \beta$, $\Psi(t|\alpha,\beta,\gamma,k)$ is convex for $t \in [0, T_1]$, concave for $t \in [T_1, T_2]$, and convex for $t \in [T_2, +\infty]$. By Lemma A.4, elements in $\left[0, (T-T_1)^+\right]$ are in a diminuendo subsequence , elements in $\left[(T-T_1)^+, (T-T_2)^+\right]$ are in a crescendo subsequence , and elements in $\left[(T-T_2)^+, T\right]$ are in a diminuendo subsequence . If there is a game element i starting before and ending after T_1, then we have $d_{*[i]} \leq \max\left\{d_{*[i-1]}, d_{*[i+1]}\right\}$. If there is a game element j starting before and ending after T_2, then we have $d_{*[j]} \geq \min\left\{d_{*[j-1]}, d_{*[j+1]}\right\}$. Therefore, the optimal structure is an inverted N-shaped sequence.

(iv) When $k > \underline{k}$, $\Psi(t|\alpha,\beta,\gamma,k)$ is concave for $t \in [0, T_2]$ and convex for $t \in [T_2, +\infty]$. By Lemma A.4, elements in $\left[0, (T-T_2)^+\right]$ are in a diminuendo subsequence, and elements in $\left[(T-T_2)^+, T\right]$ are in a crescendo subsequence . If there is a game element i starting before and ending after T_2, then we have $d_{*[i]} \leq \max\left\{d_{*[i-1]}, d_{*[i+1]}\right\}$. Therefore, the optimal structure is a U-shaped sequence. *Q.E.D.*

Proof of Theorem 2. The proof follows Theorem 1 with the situation when $T < T_2$. *Q.E.D.*

Proof of Corollary 1. By the definition of k in (14), when $\alpha > \beta$ and $\beta > \gamma$, $\underline{k} = \frac{\beta-\gamma}{\alpha-\gamma}$ and $\overline{k} = \frac{\alpha+\gamma}{\beta+\gamma}$, we have $\underline{k} < 1 < \overline{k}$; when $\alpha > \beta$ and $\beta < \gamma$, $\underline{k} = 0$ and $\overline{k} = \frac{\alpha+\gamma}{\beta+\gamma}$, we have $\underline{k} < 1 < \overline{k}$; when $\alpha < \beta$ and $\alpha > \gamma$, $\underline{k} = \frac{\alpha+\gamma}{\beta+\gamma}$ and $\overline{k} = \frac{\beta-\gamma}{\alpha-\gamma}$, we have $\underline{k} < 1 < \overline{k}$; when $\alpha < \beta$ and $\alpha > \gamma$, $\underline{k} = \frac{\alpha+\gamma}{\beta+\gamma}$ and $\overline{k} = +\infty$, we have $\underline{k} < 1 < \overline{k}$.

To sum up, $\underline{k} < 1 < \overline{k}$ for $\alpha, \beta, \gamma > 0$ and $\alpha \neq \beta \neq \gamma$. By Theorem 1, we have the optimal sequence is N-shaped when $\alpha > \beta$ and inverted N-shaped when $\beta > \alpha$. *Q.E.D.*

Proof of Theorem 3. We use an interchange argument to prove this proposition. Let S^* be the satisfaction obtained from the optimal schedule $\boldsymbol{\pi}^*$, and let S_i^* be the satisfaction obtained by interchanging game element $\pi[i-1]^*$ and game element $\pi[i]^*$ in the optimal schedule. We prove the proposition by the situations when the difficulties are fixed and the reward are fixed respectively.

(i) When the difficulties are fixed (i.e., $d_i = d$)

By (16), as $d\Phi(T|\beta,\gamma)$ is a constant with given parameters d, T, β, γ, we have

$$S^* - S_i^* = \left(d_{[i-1]^*} - d_{[i]^*}\right)\left(\left(\Phi\left(\bar{t}_i + \tau_{\pi[i-1]^*}|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1} + \tau_{\pi[i-1]^*}|\alpha,\gamma\right)\right) - \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right)\right). \tag{A.8}$$

We consider three cases to prove the proposition.

Case (ia): Both game element $\pi\left[i-1\right]^*$ and game element $\pi\left[i\right]^*$ start and finish within $\left[0,\left(T-T_0\left(\alpha,\gamma\right)\right)^+\right]$. As $\Phi'\left(t|\alpha,\gamma\right)$ is increasing for $t\in\left[\min\left\{T_0\left(\alpha,\gamma\right),T\right\},T\right]$, we have $\Phi\left(\bar{t}_i|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\le\Phi\left(\bar{t}_i+\tau_{\pi[i-1]^*}|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}+\tau_{\pi[i-1]^*}|\alpha,\gamma\right)$. By the optimality $S^*-S_i^*\ge 0$ and (A.6), we have $r_{[i-1]^*}\ge r_{[i]^*}$.

Case (ib): A game element $\pi^*\left[i\right]$ starting before and finishing after $\left(T-T_0\left(\alpha,\gamma\right)\right)^+$ exists in the optimal schedule, where $1<i<n$. Let S_{i+1}^* be the satisfaction obtained by interchanging game element $\pi\left[i\right]^*$ and game element $\pi\left[i+1\right]^*$; then, we have

$$S^*-S_{i+1}^*=\left(r_{[i]^*}-r_{[i+1]^*}\right)\left(\left(\Phi\left(\bar{t}_{i+1}+\tau_{\pi[i]^*}|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+2}+\tau_{\pi[i]^*}|\alpha,\gamma\right)\right)-\left(\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+2}|\alpha,\gamma\right)\right)\right). \tag{A.9}$$

Suppose otherwise that $r_{[i-1]^*}<r_{[i]^*}$ and $r_{[i]^*}>r_{[i+1]^*}$, by optimality $S^*-S_i^*\ge 0$ and $S^*-S_{i+1}^*\ge 0$, and by (A.8), (A.9) there must be $\Phi\left(\bar{t}_i|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\ge\Phi\left(\bar{t}_i+\tau_{\pi[i-1]^*}|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}+\tau_{\pi[i-1]^*}|\alpha,\gamma\right)$ and $\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+2}|\alpha,\gamma\right)\le\Phi\left(\bar{t}_{i+1}+\tau_{\pi[i]^*}|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+2}+\tau_{\pi[i]^*}|\alpha,\gamma\right)$. This contradicts the fact that $\Phi'\left(t|\alpha,\gamma\right)$ is increasing for $t\in\left[\min\left\{T_0\left(\alpha,\gamma\right),T\right\},T\right]$ and $\Psi'\left(t|\alpha,\gamma\right)$ is decreasing for $t\in\left[0,\min\left\{T_0\left(\alpha,\gamma\right),T\right\}\right]$. Therefore, $r_{*[i]}\le\max\left\{r_{*[i-1]},r_{*[i+1]}\right\}$.

Case (ic): Both game element $\pi\left[i-1\right]^*$ and game element $\pi\left[i\right]^*$ start and finish within $\left[\left(T-T_0\left(\alpha,\gamma\right)\right)^+,T\right]$. As $\Phi'\left(t|\alpha,\gamma\right)$ is decreasing for $t\in\left[0,\min\left\{T_0\left(\alpha,\gamma\right),T\right\}\right]$, we have $\Phi\left(\bar{t}_i|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\ge\Phi\left(\bar{t}_i+\tau_{\pi[i-1]^*}|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}+\tau_{\pi[i-1]^*}|\alpha,\gamma\right)$. By the optimality $S^*-S_i^*\ge 0$ and (A.6), we have $r_{[i-1]^*}\le r_{[i]^*}$.

(ii) When the rewards are fixed (i.e., $r_i=r$)

By (17), as $u\Phi\left(T|\alpha,\gamma\right)$ is a constant with given parameters u,T,α,γ, we have

$$S^*-S_i^*=\left(d_{[i-1]^*}-d_{[i]^*}\right)\left(\left(\Phi\left(\bar{t}_i|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\right)-\left(\Phi\left(\bar{t}_i+\tau_{\pi[i-1]^*}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+1}+\tau_{\pi[i-1]^*}|\beta,\gamma\right)\right)\right). \tag{A.10}$$

We consider three cases to prove the proposition.

Case (iia): Both game element $\pi\left[i-1\right]^*$ and game element $\pi\left[i\right]^*$ start and finish within $\left[0,\left(T-T_0\left(\beta,\gamma\right)\right)^+\right]$. As $\Phi'\left(t|\beta,\gamma\right)$ is increasing for $t\in\left[\min\left\{T_0\left(\beta,\gamma\right),T\right\},T\right]$, we have $\Phi\left(\bar{t}_i|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)\le\Phi\left(\bar{t}_i+\tau_{\pi[i-1]^*}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+1}+\tau_{\pi[i-1]^*}|\beta,\gamma\right)$. By the optimality $S^*-S_i^*\ge 0$ and (A.6), we have $d_{[i-1]^*}\le d_{[i]^*}$.

Case (iib): A game element $\pi^*\left[i\right]$ starting before and finishing after $\left(T-T_0\left(\beta,\gamma\right)\right)^+$ exists in the optimal schedule, where $1<i<n$. Let S_{i+1}^* be the satisfaction obtained by interchanging game element $\pi\left[i\right]^*$ and game element $\pi\left[i+1\right]^*$; then, we have

$$S^*-S_{i+1}^*=\left(d_{[i]^*}-d_{[i+1]^*}\right)\left(\left(\Phi\left(\bar{t}_{i+1}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+2}|\beta,\gamma\right)\right)-\left(\Phi\left(\bar{t}_{i+1}+\tau_{\pi[i]^*}|\beta,\gamma\right)-\Phi\left(\bar{t}_{i+2}+\tau_{\pi[i]^*}|\beta,\gamma\right)\right)\right). \tag{A.11}$$

Suppose otherwise that $d_{[i-1]^*} > d_{[i]^*}$ and $d_{[i]^*} < d_{[i+1]^*}$, by optimality $S^* - S_i^* \geq 0$ and $S^* - S_{i+1}^* \geq 0$, and by (A.10), (A.11) there must be $\Phi\left(\bar{t}_i|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta,\gamma\right) \geq \Phi\left(\bar{t}_i + \tau_{\pi[i-1]^*}|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1} + \tau_{\pi[i-1]^*}|\beta,\gamma\right)$ and $\Phi\left(\bar{t}_{i+1}|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+2}|\beta,\gamma\right) \leq \Phi\left(\bar{t}_{i+1} + \tau_{\pi[i]^*}|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+2} + \tau_{\pi[i]^*}|\beta,\gamma\right)$. This contradicts the fact that $\Phi'\left(t|\beta,\gamma\right)$ is increasing for $t \in \left[\min\left\{T_0\left(\beta,\gamma\right),T\right\},T\right]$ and $\Psi'\left(t|\beta,\gamma\right)$ is decreasing for $t \in \left[0, \min\left\{T_0\left(\beta,\gamma\right),T\right\}\right]$. Therefore, $r_{*[i]} \leq \max\left\{r_{*[i-1]}, r_{*[i+1]}\right\}$.

Case (ic): Both game element $\pi\left[i-1\right]^*$ and game element $\pi\left[i\right]^*$ start and finish within $\left[\left(T - T_0\left(\beta,\gamma\right)\right)^+, T\right]$. As $\Phi'\left(t|\beta,\gamma\right)$ is decreasing for $t \in \left[0, \min\left\{T_0\left(\beta,\gamma\right),T\right\}\right]$, we have $\Phi\left(\bar{t}_i|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta,\gamma\right) \geq \Phi\left(\bar{t}_i + \tau_{\pi[i-1]^*}|\beta,\gamma\right) - \Phi\left(\bar{t}_{i+1} + \tau_{\pi[i-1]^*}|\beta,\gamma\right)$. By the optimality $S^* - S_i^* \geq 0$ and (A.6), we have $r_{[i-1]^*} \geq r_{[i]^*}$.

In summary, the following applies in the optimal schedule of this situation: (i) Elements are in a U-shaped sequence of rewards when the difficulties are fixed. (ii) Elements are in an inverted U-shaped sequence of difficulties when the rewards are fixed. *Q.E.D.*

We then prove the optimal structural properties with the proportional reward scheme, when we allow repeated use of game elements.

Proof of Lemma 1. It is straightforward to see that the LDPPR is an integer optimization problem with linear objective function.

(i) Suppose $\boldsymbol{\pi}^*$ is the optimal solution of the LDPPR, with repeated use of elements $j \in \{2, \ldots, n-2\}$. For all the slots $i \in [m]$ in $\boldsymbol{\pi}^*$ assigned with repeated elements, we replace element $\pi\left[i\right]$ with element $n-1$ if $w_i > 0$, and with element 1 if $w_i < 0$, which formulates a schedule $\boldsymbol{\pi}_1$. There must be $S(\boldsymbol{\pi}_1) - S(\boldsymbol{\pi}^*) = \sum_{i=1}^n \left(d_{\pi_1[i]} - d_{\pi^*[i]}\right) w_i = \sum_{i=1}^n \left(d_{n-1} - d_{\pi^*[i]}\right) w_i \mathbb{1}_{w_i>0} + \sum_{i=1}^n \left(d_1 - d_{\pi^*[i]}\right) w_i \mathbb{1}_{w_i<0} > 0$, which contradicts with the assumption that $\boldsymbol{\pi}^*$ is the optimal solution. Thus, element 1 and $n-1$, but not other elements can be repeated used in the optimal solution.

(ii) Suppose $\boldsymbol{\pi}^*$ is the optimal solution of the LDPPR, in which some slots with positive weight is assigned with element 1 and some slots with negative weight is assigned with element $n-1$. For all the slots in $\boldsymbol{\pi}^*$ assigned with element 1, we replace it with element $n-1$ if $w_i > 0$; For all the slots in $\boldsymbol{\pi}^*$ assigned with element $n-1$, we replace it with element $n-1$ if $w_i < 0$. This formulates a schedule $\boldsymbol{\pi}_2$. There must be $S(\boldsymbol{\pi}_2) - S(\boldsymbol{\pi}^*) = \sum_{i=1}^n \left(d_{\pi_2[i]} - d_{\pi^*[i]}\right) w_i = \sum_{i=1}^n \left(d_{n-1} - d_1\right) w_i \mathbb{1}_{w_i>0} + \sum_{i=1}^n \left(d_1 - d_{n-1}\right) w_i \mathbb{1}_{w_i<0} > 0$, which contradicts with the assumption that $\boldsymbol{\pi}^*$ is the optimal solution. Thus, if element 1 is repeated used, it is assigned with slots have negative weights; if element $n-1$ is repeated used, it is assigned with slots have positive weights. *Q.E.D.*

LEMMA A.5. *Consider an LDPPR with a given difficulty sequence $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$. Suppose $\boldsymbol{\pi}^*$ is the optimal schedule of the LDPPR, and the optimal difficulty sequence is $\boldsymbol{d}_{\pi^*} = \left(d_{\pi^*[1]}, d_{\pi^*[2]}, \ldots, d_{\pi^*[m]}\right)$. Then $\boldsymbol{\pi}^*$ must also be the optimal solution of the LDPP with difficulty sequence $\boldsymbol{d}_{\pi^*}$.*

Proof of Lemma A.5.

Suppose there is another solution $\boldsymbol{\pi}$ of the LDPP with difficulty sequence $\boldsymbol{d}_{\pi^*}$, which offers remembered utility $S(\boldsymbol{\pi}) > S(\boldsymbol{\pi}^*)$. It is straightforward to see $\boldsymbol{\pi}$ is also a feasible solution of the LDPPR with difficulty sequence $\boldsymbol{d}$. As $S(\boldsymbol{\pi}) > S(\boldsymbol{\pi}^*)$, $\boldsymbol{\pi}^*$ can not be the optimal solution of the LDPPR, which contradicts with the assumption. Therefore, $\boldsymbol{\pi}^*$ must also be the optimal solution of the LDPP with difficulty sequence $\boldsymbol{d}_{\pi^*}$. *Q.E.D.*

Proof of Theorem 4.

Let $\boldsymbol{\pi}^*$ be the optimal solution of the LDPPR with difficulty sequence $\boldsymbol{d}$. By Lemma A.5, $\boldsymbol{\pi}^*$ is also the optimal solution of the LDPP with difficulty sequence $\boldsymbol{d}_{\pi^*}$. As the optimal solution of an LDPP shares the structural properties presented in Theorems 1 and 2, $\boldsymbol{\pi}^*$ must share the optimal structures presented in Theorems 1 and 2. *Q.E.D.*

To prove Corollary A.2, we first analyze the properties of the weight $\boldsymbol{w}$ in the following lemma.

LEMMA A.6. *For the LDPPR the weights $\boldsymbol{w}$ has the following properties:*

(i) If $\Psi(t|,\alpha,\beta,\gamma,k)$ is increasing for $t \in [\bar{t}_i, \bar{t}_{i+1}]$, we have $w_i > 0$.

(ii) If $\Psi(t|,\alpha,\beta,\gamma,k)$ is decreasing for $t \in [\bar{t}_i, \bar{t}_{i+1}]$, we have $w_i < 0$.

Proof of Lemma A.6.

It is straightforward to prove this lemma by the definition of the weight in (18). *Q.E.D.*

The following corollary shows some properties on the repeated assignment, which suggests repeated assignment to the climax and low tide of the game.

COROLLARY A.2. *In the optimal schedule $\boldsymbol{\pi}^*$ of the LDPPR, the repeated used elements have the following properties.*

(1) When $k \le \underline{k}$, if element 1 is repeated used, it is scheduled at interval $[(T-T_2')^+, T]$; if element $n-1$ is repeated used, it is scheduled at interval $[0, (T-T_2')^+]$.

(2) When $\underline{k} \le k \le 1$,

(2.1) Suppose $\alpha > \beta$, if element 1 is repeated used, it is scheduled at interval $[(T-T_2')^+, T]$, if element $n-1$ is repeated used, it is scheduled at interval $[0, (T-T_2')^+]$.

(2.2) Suppose $\alpha < \beta$, if element 1 is repeated used, it is scheduled at interval $[0, (T-T_2')^+]$ or $[(T-T_1')^+, T]$; if element $n-1$ is repeated used, it is scheduled at interval $[(T-T_2')^+, (T-T_1')^+]$.

(3) When $1 < k < \overline{k}$,

(3.1) Suppose $\alpha > \beta$, if element 1 is repeated used, it is scheduled at interval $[(T-T_2')^+, (T-T_1')^+]$; if element $n-1$ is repeated used, it is scheduled at interval $[0, (T-T_2')^+]$ or interval $[(T-T_1')^+, T]$.

(3.2) Suppose $\alpha < \beta$, if element 1 is repeated used, it is scheduled at interval $[0,(T-T_2')^+]$; if element $n-1$ is repeated used, it is scheduled at interval $[(T-T_2')^+, T]$.

(4) When $k \geq \overline{k}$, if element 1 is repeated used, it is scheduled at interval $[0,(T-T_2')^+]$; if element $n-1$ is repeated used, it is scheduled at interval $[(T-T_2')^+, T]$.

Proof of Corollary A.2. By Corollary A.1 and Lemma A.6, we have the following properties on the value of the weight.

(1) When $k \leq \underline{k}$, elements in $[(T-T_2')^+, T]$ have negative weights, and elements in $[0,(T-T_2')^+]$ have positive weights.

(2) When $k \leq 1$,

(2.1) Suppose $\alpha > \beta$, elements in $[(T-T_2')^+, T]$ have negative weights; elements in $[0,(T-T_2')^+]$ have positive weights.

(2.2) Suppose $\alpha < \beta$, elements in $[0,(T-T_2')^+]$ or $[(T-T_1')^+, T]$ have negative weights; elements in $[(T-T_2')^+, (T-T_1')^+]$ have positive weights.

(3) When $1 < k < \overline{k}$,

(3.1) Suppose $\alpha > \beta$, elements in $[(T-T_2')^+, (T-T_1')^+]$ have negative weights; elements in $[0,(T-T_2')^+]$ or $[(T-T_1')^+, T]$ have positive weights.

(3.2) Suppose $\alpha < \beta$, elements in $[0,(T-T_2')^+]$ have negative weights; elements in $[(T-T_2')^+, T]$ have positive weights.

(4) When $k \geq \overline{k}$, elements in $[0,(T-T_2')^+]$ have negative weights; elements in $[(T-T_2')^+, T]$ have positive weights.

By Lemma A.6, element 1 can be assigned to slots with negative weights, and element $n-1$ can be assigned to slots with positive weights. We thus prove the Corollary A.2. *Q.E.D.*

We then prove the optimal structural property of the LDPGR.

LEMMA A.7. *In the optimal solution π^* of the LDPGR, for the ith element, if both $w_i'(\alpha)$ and $w_i'(\beta)$ are positive, then element HL is assigned to slot i; if both $w_i'(\alpha)$ is positive and $w_i'(\beta)$ is negative, then element HH is assigned to slot i; if both $w_i'(\alpha)$ is negative and $w_i'(\beta)$ is positive, then element LL is assigned to slot i; if both $w_i'(\alpha)$ and $w_i'(\beta)$ are negative, then element LH is assigned to slot i.*

Proof of Lemma A.7. Let $\boldsymbol{\pi}_1$ be the schedule in which element HL is assigned to slots with positive weights $\boldsymbol{w}'(\alpha)$ and $\boldsymbol{w}'(\beta)$; element HH is assigned to slots with positive weight $\boldsymbol{w}'(\alpha)$ and negative weight $\boldsymbol{w}'(\beta)$; element LL is assigned to slots with negative weight $\boldsymbol{w}'(\alpha)$ and positive weight $\boldsymbol{w}'(\beta)$; element LH is assigned to slots with negative weights $\boldsymbol{w}'(\alpha)$ and $\boldsymbol{w}'(\beta)$. For any element-slot assignment allowing repeated usage $\boldsymbol{\pi}_2 \neq \boldsymbol{\pi}_1$, there must be $S(\boldsymbol{\pi}_1) - S(\boldsymbol{\pi}_2) =$

$\sum_{i=1}^{n}\left(\left(r_{\pi_1[i]}-r_{\pi_2[i]}\right)w_i'(\alpha)-\left(d_{\pi_1[i]}-d_{\pi_2[i]}\right)w_i'(\beta)\right)>0$. Because there must be $r_{\pi_1[i]}-r_{\pi_2[i]}=r_H-r_{\pi_2[i]}\geq 0$ when $w_i'(\alpha)>0$; $r_{\pi_1[i]}-r_{\pi_2[i]}=r_L-r_{\pi_2[i]}\leq 0$ when $w_i'(\alpha)<0$; $d_{\pi_1[i]}-d_{\pi_2[i]}=d_L-r_{\pi_2[i]}\leq 0$ when $w_i'(\beta)>0$; $d_{\pi_1[i]}-d_{\pi_2[i]}=d_H-r_{\pi_2[i]}\geq 0$ when $w_i'(\beta)<0$. *Q.E.D.*

Next, we study the property with the relationship between function $\Phi(\cdot)$ and the optimal subsequence.

COROLLARY A.3. *In the optimal solution π^* of the LDPGR, if both $\Phi(\alpha)$ and $\Phi(\beta)$ are decreasing in for $t\in[\bar{t}_{i+1},\bar{t}_i]$, then the element LH is assigned to slot i; if $\Phi(\alpha)$ is decreasing and $\Phi(\beta)$ is increasing in for $t\in[\bar{t}_{i+1},\bar{t}_i]$, then the element LL is assigned to slot i; if $\Phi(\alpha)$ is increasing and $\Phi(\beta)$ is decreasing in for $t\in[\bar{t}_{i+1},\bar{t}_i]$, then the element HH is assigned to slot i; if both $\Phi(\alpha)$ and $\Phi(\beta)$ are increasing in for $t\in[\bar{t}_{i+1},\bar{t}_i]$, then the element HL is assigned to slot i.*

Proof of Corollary A.3. By Lemma A.7 and the definition of the weight in (19), we have the following properties:

(i) In $t\in[\bar{t}_{i+1},\bar{t}_i]$, if both $\Phi(t|\alpha,\gamma)$ and $\Phi(t|\beta,\gamma)$ are decreasing, then $w_i'(\alpha)<0$ and $w_i'(\alpha)<0$. Thus, element LH should be assigned to slot i.

(ii) In $t\in[\bar{t}_{i+1},\bar{t}_i]$, if $\Phi(t|\alpha,\gamma)$ is decreasing and $\Phi(t|\beta,\gamma)$ is increasing, then $w_i'(\alpha)<0$ and $w_i'(\beta)>0$. Thus, element LL should be assigned to slot i.

(iii) In $t\in[\bar{t}_{i+1},\bar{t}_i]$, if $\Phi(t|\alpha,\gamma)$ is increasing and $\Phi(t|\beta,\gamma)$ is decreasing, then $w_i'(\alpha)>0$ and $w_i'(\beta)<0$. Thus, element HH should be assigned to slot i.

(iv) In $t\in[\bar{t}_{i+1},\bar{t}_i]$, if both $\Phi(t|\alpha,\gamma)$ and $\Phi(t|\beta,\gamma)$ are increasing, then $w_i'(\alpha)>0$ and $w_i'(\alpha)>0$. Thus, element HL should be assigned to slot i. *Q.E.D.*

Then we prove the optimal structure of the level design problem with general rewards and repeated assignment.

Proof of Proposition 1. (i) When $\alpha>\beta$ (or when $T_0'(\alpha,\gamma)<T_0'(\beta,\gamma)$ by (A.1)), by Lemma A.1, there exist slots i and j $(i<j)$ where $\bar{t}_{i+2}<T_0'(\beta)<\bar{t}_{i+1}$ and $\bar{t}_{j+2}<T_0'(\alpha)<\bar{t}_{j+1}$, such that weight $\boldsymbol{w}'(\alpha)$ and weight $\boldsymbol{w}'(\beta)$ of slots $\{1,\ldots,i\}$ are negative, $\boldsymbol{w}'(\alpha)$, and weight $\boldsymbol{w}'(\beta)$ of slot $i+1$ can be either positive or negative; weight $\boldsymbol{w}'(\alpha)$ of slots $\{i+2,\ldots,j\}$ are negative, weight $\boldsymbol{w}'(\beta)$ of slots $\{i+2,\ldots,j\}$ are positive, $\boldsymbol{w}'(\alpha)$, and weight $\boldsymbol{w}'(\beta)$ of slot $j+1$ can be either positive or negative; weight $\boldsymbol{w}'(\alpha)$ and $\boldsymbol{w}'(\beta)$ of slots $\{i+2,\ldots,j\}$ are positive. By Corollary A.3, in the optimal schedule, slots $\{1,\ldots,i\}$ are assigned with element LH; slots $\{i+2,\ldots,j\}$ are assigned with element LL; slots $\{j+2,\ldots,n\}$ are assigned with element HL; slot $i+1$ is assigned with either LH or LL;slot $j+1$ is assigned with either LL or HL.

(ii) When $\alpha<\beta$ (or when $T_0'(\alpha,\gamma)>T_0'(\beta,\gamma)$ by (A.1)), by Lemma A.1, there exist slots i and j $(i<j)$ where $\bar{t}_{i+2}<T_0'(\alpha)<\bar{t}_{i+1}$ and $\bar{t}_{j+2}<T_0'(\beta)<\bar{t}_{j+1}$, such that weight $\boldsymbol{w}'(\alpha)$ and weight

$\boldsymbol{w}'(\beta)$ of slots $\{1,\dots,i\}$ are negative, $\boldsymbol{w}'(\alpha)$, and weight $\boldsymbol{w}'(\beta)$ of slot $i+1$ can be either positive or negative; weight $\boldsymbol{w}'(\alpha)$ of slots $\{i+2,\dots,j\}$ are positive, weight $\boldsymbol{w}'(\beta)$ of slots $\{i+2,\dots,j\}$ are negative, $\boldsymbol{w}'(\alpha)$, and weight $\boldsymbol{w}'(\beta)$ of slot $j+1$ can be either positive or negative; weight $\boldsymbol{w}'(\alpha)$ and $\boldsymbol{w}'(\beta)$ of slots $\{i+2,\dots,j\}$ are positive. By Corollary A.3, in the optimal schedule, slots $\{1,\dots,i\}$ are assigned with element LH; slots $\{i+2,\dots,j\}$ are assigned with element HH; slots $\{j+2,\dots,n\}$ are assigned with element HL; slot $i+1$ is assigned with either LH or HH;slot $j+1$ is assigned with either HH or HL.

Therefore, (i) When $\alpha>\beta$, the optimal structure is a LH-LL-HL sequence. (ii) When $\alpha<\beta$, the optimal structure is a LH-HH-HL sequence. *Q.E.D.*

Proof of Proposition 2. The proof follows Proposition 1 with the situation when $T<T_0'$. *Q.E.D.*

We then prove the following lemma regarding coefficient δ, δ_r and δ_d and the optimal structure

LEMMA A.8. *When $\delta\neq 1$, $\delta_r\neq 1$ and $\delta_d\neq 1$ we have:*

(i) Theorems 1, 2 and 4 and Lemma 1 still hold with reward ratio $k'=\frac{\delta_r}{\delta_d}k$ and difficulty-aversion degree $\beta'=\delta\beta$, where k is the reward ratio defined in (13).

(ii) Theorem 3, Propositions 1 and 2 still hold for any positive constants δ_r and δ_d, and the degree of difficulty-aversion $\beta'=\delta\beta$.

Proof of Lemma A.8. When $\delta\neq 1$, $\delta_r\neq 1$ and $\delta_d\neq 1$, we have

$$S(\boldsymbol{\pi})=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} v(t)\,e^{-\gamma(T-t)}dt, S(\boldsymbol{\pi}), \tag{A.12}$$

$$=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\delta_r u_r(t)-\delta_d u_d(t)\right)e^{-\gamma(T-t)}dt, \tag{A.13}$$

$$=\delta_r\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\left(r_{[1]}-f(0)\right)e^{-\alpha\bar{t}}+\sum_{j=2}^{i}\left(r_{[j]}-r_{[j-1]}\right)e^{-\alpha\left(\bar{t}-\bar{t}_{j-1}\right)}\right)e^{-\gamma(T-t)}dt \tag{A.14}$$

$$-\delta_d\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(\left(d_{[1]}-g(0)\right)e^{-\beta'\bar{t}}+\sum_{j=2}^{i}\left(d_{[j]}-d_{[j-1]}\right)e^{-\beta'\left(\bar{t}-\bar{t}_{j-1}\right)}\right)e^{-\gamma(T-t)}dt, \tag{A.15}$$

$$=\delta_r\sum_{i=1}^{n}r_{[i]}\left(\Phi\left(\bar{t}_i|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right)-\delta_d\sum_{i=1}^{n}d_{[i]}\left(\Phi\left(\bar{t}_i|\beta',\gamma\right)-\Phi\left(\bar{t}_{i+1}|\beta',\gamma\right)\right) \tag{A.16}$$

(i) When we consider a proportional reward as (13), let $k'=\frac{\delta_r}{\delta_d}k$ the player's remembered utility with proportional reward can be expressed by:

$$S(\boldsymbol{\pi})=\delta_d\left(\sum_{i=1}^{n}k'd_{[i]}\left(\Phi\left(\bar{t}_i|\alpha,\gamma\right)-\Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right)-\sum_{i=1}^{n}d_{[i]}\left(\Phi\left(\bar{t}_i|\beta',\gamma\right)-\Phi\left(\bar{t}_{i+1}|\beta',\gamma\right)\right)\right), \tag{A.17}$$

$$=\delta_d\sum_{i=1}^{n}d_{[i]}\left(\Psi\left(\bar{t}_i|\alpha,\beta',\gamma,k'\right)-\Psi\left(\bar{t}_{i+1}|\alpha,\beta',\gamma,k'\right)\right). \tag{A.18}$$

By the proofs of Theorems 1, 2 and 4 the optimal structure are the same as mentioned in Theorems 1, 2 and 4 with reward ratio k' and difficulty-aversion degree β'. By the proof of Lemma 1 the selection of the repeated used elements are the same as mentioned in Lemma 1 depending on the reward ratio k' and difficulty-aversion degree β'

(ii) When we consider a general reward scheme, and the elements share a fixed reward $r_i = r$ for all $i \in [n]$ the LDPGFD can be expressed by:

$$\begin{aligned}\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) &= \delta_r \sum_{i=1}^{n} r_{[i]} \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - \delta_d \sum_{i=1}^{n} d\left(\Phi\left(\bar{t}_i|\beta',\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta',\gamma\right)\right),\\ &= \delta_r \sum_{i=1}^{n} r_{[i]} \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - \delta_d d \Phi\left(T|\beta',\gamma\right). \end{aligned} \tag{A.19}$$

When the elements share a fixed reward $r_i = r$ for all $i \in [n]$ the level design problem with general reward and fixed reward (LDPGFR) can be expressed by

$$\begin{aligned}\max_{\boldsymbol{\pi}} S(\boldsymbol{\pi}) &= \delta_r \sum_{i=1}^{n} r \left(\Phi\left(\bar{t}_i|\alpha,\gamma\right) - \Phi\left(\bar{t}_{i+1}|\alpha,\gamma\right)\right) - \delta_d \sum_{i=1}^{n} d_{[i]}\left(\Phi\left(\bar{t}_i|\beta',\gamma\right) - \Phi\left(\bar{t}_{i+1}|\beta',\gamma\right)\right),\\ &= \delta_r r \Phi\left(T|\alpha,\gamma\right) + \sum_{i=1}^{n} d_{[i]} \left(\Phi\left(\bar{t}_{i+1}|\beta',\gamma\right) - \delta_d \Phi\left(\bar{t}_i|\beta',\gamma\right)\right). \end{aligned} \tag{A.20}$$

Given that $\delta_d d$ and $\delta_r r$ are positive constants, by the proof of Theorem 3, the optimal structure is exactly the same as mentioned in Theorem 3 with difficulty-aversion degree β'. By the same reason, the properties presented in Propositions 1 and 2 still hold with difficulty-aversion degree β' by their proofs. *Q.E.D.*

Appendix B: Mathematical Expressions of the Optimal Structures

Table A.1 summarizes the expressions of optimal structure of the level design problem with proportional reward. As the reward is proportional to the difficulty, difficulty sequence and reward sequence reveal the same structural properties. LDPP and LDPPR share the same structural properties. Table A.2 summarizes the optimal structure of the level design problem with general reward.

Table A.1 Mathematical Expressions of the Optimal Structures with Proportional Reward Scheme

Structure	Mathematical Expression
Crescendo sequence	$d_{[1]} \leq \cdots \leq d_{[n]}$.
Diminuendo sequence	$d_{[1]} \geq \cdots \geq d_{[n]}$.
U-shaped sequence	Some game element $\pi[i]$, $i \in \{1, \ldots, n\}$, exists such that $d_{[1]} \geq \cdots \geq d_{[i-1]} \geq d_{[i]} \leq d_{[i+1]} \cdots \leq d_{[n]}$.
Inverted U-shaped sequence	Some game element $\pi[i]$, $i \in \{1, \ldots, n\}$, exists such that $d_{[1]} \leq \cdots \leq d_{[i-1]} \leq d_{[i]} \geq d_{[i+1]} \cdots \geq d_{[n]}$.
N-shaped sequence	Some elements $\pi[i]$ and $\pi[j]$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $d_{[1]} \leq \cdots \leq d_{[i-1]} \leq d_{[i]} \geq d_{[i+1]} \geq \cdots \geq d_{[j-1]} \geq d_{[j]} \leq d_{[j+1]} \cdots \leq d_{[n]}$.
Inverted N-shaped sequence	Some elements $\pi[i]$ and $\pi[j]$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $d_{[1]} \geq \cdots \geq d_{[i-1]} \geq d_{[i]} \leq d_{[i+1]} \leq \cdots \leq d_{[j-1]} \leq d_{[j]} \geq d_{[j+1]} \cdots \geq d_{[n]}$.
Inverted N-shaped sequence	Some elements $\pi[i]$ and $\pi[j]$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $d_{[1]} \geq \cdots \geq d_{[i-1]} \geq d_{[i]} \leq d_{[i+1]} \leq \cdots \leq d_{[j-1]} \leq d_{[j]} \geq d_{[j+1]} \cdots \geq d_{[n]}$.

Table A.2 Mathematical Expressions of the Optimal Structures with General Reward Scheme

Problem	Structure	Mathematical Expression
LDPGFD	U-shaped reward Sequence	Some game element $\pi[i]$, $i \in \{1, \ldots, n\}$, exists such that $r_{[1]} \geq \cdots \geq r_{[i-]} \geq r_{[i]} \leq r_{[i+1]} \leq \cdots \leq r_{[n]}$. In addition, $d_{[1]} = \cdots = d_{[n]} = d$
LDPGFR	Inverted U-shaped difficulty sequence	Some game element $\pi[i]$, $i \in \{1, \ldots, n\}$, exists such that $d_{[1]} \leq \cdots \leq d_{[i-]} \leq d_{[i]} \geq d_{[i+1]} \geq \cdots \geq d_{[n]}$. In addition, $r_{[1]} = \cdots = r_{[n]} = r$
LDPGR	HL sequence	$r_{[1]} = \cdots = r_{[n]} = r_H$ and $d_{[1]} = \cdots = d_{[n]} = d_L$
	LL-HL sequence	Some game element $\pi[i]$, $i \in \{1, \ldots, n\}$, exists such that $r_{[1]} = \cdots = r_{[i]} = r_L$ and $r_{[i+1]} = \cdots = r_{[n]} = r_H$. In addition, $d_{[1]} = \cdots = d_{[n]} = d_L$
	HH-HL sequence	$r_{[1]} = \cdots = r_{[n]} = r_H$. In addition, Some game element $\pi[i]$, $i \in \{1, \ldots, n\}$, exists such that $d_{[1]} = \cdots = d_{[i]} = d_H$ and $d_{[i+1]} = \cdots = d_{[n]} = d_L$.
	LH-LL-HL sequence	Some elements $\pi[i]$ and $\pi[j]$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $r_{[1]} = \cdots = r_{[j]} = r_L$ and $r_{[j+1]} = \cdots = r_{[n]} = r_H$; and $d_{[1]} = \cdots = d_{[i]} = d_H$ and $d_{[i+1]} = \cdots = d_{[n]} = d_L$
	LH-HH-HL sequence	Some elements $\pi[i]$ and $\pi[j]$, $i \leq j$, $i, j \in \{1, \ldots, n\}$, exist such that $r_{[1]} = \cdots = r_{[i]} = r_L$ and $r_{[i+1]} = \cdots = r_{[n]} = r_H$; and $d_{[1]} = \cdots = d_{[j]} = d_H$ and $d_{[j+1]} = \cdots = d_{[n]} = d_L$

Appendix C: Parameters of the General Reward Scheme Example

Table A.3 summarize the parameters and optimal solution of the example problem we present in Figure 5.

Table A.3 Parameters and Optimal Schedule of the General Reward Scheme Example

Parameter	Value
Number of elements	$n = 8$
Vector of rewards	$\boldsymbol{r} = (1, 2, 3, 4, 5, 6, 7, 8)^{\mathrm{T}}$
Vector of difficulties	$\boldsymbol{d} = (1, 5, 5, 3, 3, 7, 6, 4)^{\mathrm{T}}$
Vector of durations	$\boldsymbol{t} = (5.90, 6.22, 3.71, 5.04, 3.16, 2.67, 5.07, 7.23)^{\mathrm{T}}$
Planning time	$T = 39.01$
Degree of reward seeking	$\alpha = 0.02$
Degree of difficulty aversion	$\beta = 0.01$
Memory-decay rate	$\gamma = 0.05$
Optimal sequence	$\boldsymbol{\pi}^* = (2, 3, 6, 1, 4, 7, 5, 8)$