Part IB — Numerical Analysis Example Sheet 2

Supervised by Dr. Saxton

Examples worked through by Christopher Turnbull

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QUESTION 1

The differential equations, with initial condition $y(0) = 1$ have exact solutions given by

$$y = \frac{1}{1+t} \quad \text{and} \quad y = (1+t)^2, \quad 0 \le t \le 1$$

respectively.
Using the Euler method for the first ODE we have $f(t, y) = -\frac{y}{1+t}$.
Here, $y_0 = 1, t_m = mh$. For $n \ge 1$,

$$\begin{aligned} y_n &= y_{n-1} + hf(t_{n-1}, y_{n-1}) \\ &= y_{n-1}\left(1 - \frac{h}{1+(n-1)h}\right) \\ &= y_{n-1} \cdot \frac{1+(n-2)h}{1+(n-1)h} \end{aligned}$$

Have that $y_1 = 1 - h$, thus

$$\begin{aligned} y_n &= 1 \cdot (1-h)\left(\frac{1}{1+h}\right)\left(\frac{1+h}{1+2h}\right)\cdots\left(\frac{1+(n-3)h}{1+(n-2)h}\right)\left(\frac{1+(n-2)h}{1+(n-1)h}\right) \\ &= \frac{1-h}{1+(n-1)h} \end{aligned}$$

As $h \to 0$, $n \to \infty$ in such a way that $nh \to t$. So we deduce $y = 1/(1+t)$ as required.
Moreover, the error is

$$y_n - y(nh) = \frac{1-h}{1+(n-1)h} - \frac{1}{1+nh}$$

which is clearly $O(h)$.
For the second ODE we have $f(t, y) = \frac{2y}{1+t}$.
Calculating the first few terms we find that

$$\begin{aligned} y_1 &= y_0\left(1 + \frac{2h}{1+t_0}\right) \qquad t_0 = 0 \\ &= (1+2h) \end{aligned}$$

$$\begin{aligned} y_2 &= y_1\left(1 + \frac{2h}{1+t_1}\right) \qquad t_1 = h \\ &= (1+2h) \cdot \left(\frac{1+3h}{1+h}\right) \end{aligned}$$

$$
\begin{aligned}
y_3 &= y_2 \left(1 + \frac{2h}{1+t_1}\right) \qquad t_2 = 2h \\
&= (1+2h) \cdot \left(\frac{1+3h}{1+h}\right) \cdot \left(\frac{1+4h}{1+2h}\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
y_n &= (1+2h) \cdot \left(\frac{1+3h}{1+h}\right) \cdot \left(\frac{1+4h}{1+2h}\right) \cdots \left(\frac{1+(n+1)h}{1+(n-1)h}\right) \\
&= \frac{(1+nh)(1+(n+1)h)}{1+h}
\end{aligned}
$$

again $nh \to t$, so we have the result as required.
Here, the error is

$$
\begin{aligned}
y_n - y(nh) &= \frac{(1+nh)(1+(n+1)h)}{1+h} - (1+nh)^2 \\
&= \frac{(1+nh)^2 + h(1+nh) - (1+h)(1+nh)^2}{1+h}
\end{aligned}
$$

which is clearly $O(h)$.

QUESTION 2

The trapezoidal rule states that the numerical solution to the differential equation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}_n) \qquad (2.1)$$

is given by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})] \qquad (2.2)$$

Assuming that $\mathbf{f}$ satisfies the Lipschitz condition: there exists $\lambda \geq 0$ such that

$$||\mathbf{f}(t, \mathbf{v}) - \mathbf{f}(t, \mathbf{w})|| \leq \lambda ||\mathbf{v} - \mathbf{w}||, \quad t \in [0, t^*], \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^N$$

we will prove that the trapezoidal rule converges; ie.

$$\lim_{h \to 0} \max_{n=0,\cdots,\lfloor t^*/h \rfloor} ||\mathbf{y}_n(h) - \mathbf{y}(nh)|| = 0$$

where $\mathbf{y}(nh)$ is the evaluation at time $t = nh$ of the exact solution of (2.1).

Proof. Let $\mathbf{e}_n = \mathbf{y}_n - \mathbf{y}(t_n)$, the error at step n, where $0 \leq n \leq t^*/h$, $t_n := nh$. Thus,

$$\mathbf{e}_{n+1} = \mathbf{y}_{n+1} - \mathbf{y}(t_{n+1}) = [\mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]] - [\mathbf{y}(t_n) + h\mathbf{y}'(t_n) + O(h^2)]$$

By the Taylor theorem, the $O(h^2)$ term can be bounded uniformly for all $[0, t^*]$ by ch^2, where $c > 0$. Thus, using (2.1) and the triangle inequality,

$$\begin{aligned} ||\mathbf{e}_{n+1}|| &\leq ||\mathbf{y}_n - \mathbf{y}(t_n)|| + h||\frac{1}{2}\{\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})\} - \mathbf{f}(t_n, \mathbf{y}(t_n))|| + ch^2 \\ &= \end{aligned}$$

Want this in terms of $||\mathbf{e}_n||$, but how? $\square$

QUESTION 3

The s-step Adams-Bashforth method is of order s and has the form

$$\mathbf{y}_{n+s} - \mathbf{y}_{n+s-1} = h\sum_{j=0}^{s-1} \sigma_j \mathbf{f}(t_{n+j}, \mathbf{y}_{n+j})$$

For $s = 3$ we have $\rho(w) = w^2(w-1)$. To maximize order, we let σ be the 2 degree polynomial ($\sigma_3 = 0$) arising from the truncation of the Taylor expanison of

$$\frac{\rho(w)}{\log w}$$

Letting $\xi = w - 1$ and expanding,

$$\begin{aligned}
\frac{w^2(w-1)}{\log w} &= \frac{(\xi+1)^2\xi}{\log(1+\xi)} = \frac{\xi + 2\xi^2 + \xi^3}{\xi - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3 - \cdots} \\
&= \frac{1 + 2\xi + \xi^2}{1 - \frac{1}{2}\xi + \frac{1}{3}\xi^2 - \cdots} \\
&= [1 + 2\xi + \xi^2][1 + (\frac{1}{2}\xi - \frac{1}{3}\xi^2) + (\frac{1}{2}\xi - \frac{1}{3}\xi^2)^2 + O(\xi^3)] \\
&= 1 + \frac{5}{2}\xi + \frac{5}{3}\xi^2 + O(\xi^3) \\
&= 1 + \frac{5}{2}(w-1) + \frac{5}{3}(w-1)^2 + O(|w-1|^3) \\
&= \frac{1}{6} - \frac{5}{3}w + \frac{5}{3}w^2 + O(|w-1|^3)
\end{aligned}$$

Therefore $\sigma_0 = \frac{1}{6}, \sigma_1 = -\frac{5}{3}, \sigma_2 = \frac{5}{3}, \sigma_3 = 0$

QUESTION 4

Applying the *explicit midpoint rule*

$$\mathbf{y}_{n+2} = \mathbf{y}_n + 2h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})$$

to the ODE $y' = -y$, we have

$$y_{n+2} = y_n - 2hy_{n+1}$$

Making the ansatz $y_n = k^n$ gives

$$k^2 + 2hk - 1 = 0$$

and hence

$$k = -h \pm \sqrt{h^2 - 1}$$

giving

$$y_n = A\left(-h - \sqrt{h^2-1}\right)^n + B\left(-h + \sqrt{h^2-1}\right)^n$$

Now $y_0 = 1 \Rightarrow A + B = 1$, and $y_1 = 1 - h \Rightarrow 1 = (B - A)\sqrt{h^2-1}$, thus

$$A = \frac{1}{2\sqrt{h^2-1}} + \frac{1}{2}$$

$$B = \frac{1}{2\sqrt{h^2-1}} - \frac{1}{2}$$

Now as $n \to \infty$, we wish to show that y_n diverges, ie. one of the terms blow up, and we want to show this happens for all $h > 0$. Can see that if $h > 1$, the $A\left(-h - \sqrt{h^2-1}\right)^n$ explodes as $-h - \sqrt{h^2-1} < -1$. If $0 < h < 1$, I don't know...

QUESTION 5

The multistep method

$$\sum_{j=0}^{3} \rho_j \mathbf{y}_{n+j} = h \sum_{j=0}^{2} \sigma_j \mathbf{f}(t_{n+j}, \mathbf{y}_{n+j}), \quad \rho_3 = 1$$

is of order 4 iff

$$\rho(e^z) - z\sigma(e^z) = O(z^5), \quad z \to 0$$

Expanding into Taylor series,

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + O(z^5)$$

$$e^{2z} = 1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{5}{8}z^4 + O(z^5)$$

$$e^{3z} = 1 + 3z + \frac{9}{2}z^2 + \frac{9}{2}z^3 + \frac{10}{3}z^4 + O(z^5)$$

$$\begin{aligned}\rho(e^z) - z\sigma(e^z) = {} & [1 + 3z + \frac{9}{2}z^2 + \frac{9}{2}z^3 + \frac{10}{3}z^4] + \rho_2[1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{5}{8}z^4] \\ & + \rho_1[1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4] + \rho_0 - z\sigma_2 \left[1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{5}{8}z^4\right] \\ & - z\sigma_1 \left[1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4\right] - z\sigma_0\end{aligned}$$

For this expression to be $O(z^5)$, looking at first order terms we deduce that $\rho_1 + \rho_2 + \rho_3 = -1$.

So we have $\rho(w) = w^3 + \rho_2 w^2 - 9w + \rho_0$, $\rho_0 + \rho_2 = 8$ for this to satisfy the root condition we must have all zeros residing in $|w| \leq 1$, and all zeros of unit modulus simple.

QUESTION 6

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QUESTION 7

Consider the ODE $y' = y$ with $y(0) = 1$ whose solution is $y(t) = e^t$. For this ODE we can write the local error explicitly: indeed we have

$$\begin{aligned}
k_1 &= f(t_n, y(t_n)) = e^{t_n} \\
k_2 &= y(t_n) + \frac{1}{3}hk_1 = e^{t_n}(1 + \frac{1}{3}h) \\
k_3 &= y(t_n) - \frac{1}{3}hk_1 + hk_2 = e^{t_n}\left(1 + \frac{2}{3}h + \frac{1}{3}h^2\right) \\
k_4 &= y(t_n) + hk_1 - hk_2 + hk_3 = e^{t_n}\left(1 + h + \frac{1}{3}h^2 + \frac{1}{3}h^3\right)
\end{aligned}$$

Then the local error is

$$\begin{aligned}
y(t_{n+1}) - (y(t_n) + \frac{1}{8}hk_1 + \frac{3}{8}hk_2 + \frac{3}{8}hk_3 + \frac{1}{8}hk_4) &= e^{t_{n+1}} - e^{t_n} - e^{t_n} \\
&=
\end{aligned}$$

Let us now show that the method has order at least 3. To do this we restrict our attention to scalar, autonomous equations of the form $y' = f(y)$.

QUESTION 8

Consider the linear scalar system

$$\begin{cases} y' = \lambda y \\ y(0) = 1 \end{cases}$$

where $\lambda < 0$. The solution is $y(t) = e^{\lambda t}$ which decays to 0 as $t \to \infty$.

(i) For the explicit Euler method we get $y_{n+1} = y_n + h\lambda y_n$ whose solution is $y_n = (1+h\lambda)^n$, so $y_n \to 0$ iff $|1+h\lambda| < 1$, therefore $\mathcal{D} = \{z \in \mathbb{C} \ ; \ |1+z| < 1\}$, and $\mathcal{D} \cap \mathbb{R} =$

(ii) Considering now the trapezoidal rule we get $y_{n+1} = [(1+\frac{1}{2}h\lambda)/(1-\frac{1}{2}h\lambda)]y_n$, and thus by induction, $y_n = [(1+\frac{1}{2}h\lambda)/(1-\frac{1}{2}h\lambda)]^n y_0$. Therefore

$$z \in \mathcal{D} \iff \left| \frac{1+\frac{1}{2}z}{1-\frac{1}{2}z} \right| < 1 \iff \text{Re } z < 0$$

and we deduce that $\mathcal{D} = \mathbb{C}^-$. Hence the method is A-stable.

(iii)

(iv)

(v) Applying the RK method to $y' = \lambda y$ we have

$$\begin{aligned} hk_1 &= h\lambda y_n \\ hk_2 &= h\lambda(y_n + hk_1) \end{aligned}$$

therefore

$$y_{n+1} = y_n + \frac{1}{2}hk_1 + \frac{1}{2}hk_2 = (1 + h\lambda + \frac{1}{2}h^2\lambda^2)y_n$$

Let

$$r(z) = 1 + z + \frac{1}{2}z^2$$

Then $y_{n+1} = r(h\lambda)y_n$, therefore, by induction, $y_n = [r(h\lambda)]^n y_0$ and we deduce that

$$\mathcal{D} = \{z \in \mathbb{C} \ ; \ |r(z)| < 1\}$$

r is analytic in $\mathcal{V} = \{z \in \mathbb{C} \ ; \ \text{Re } z <\leq 0\}$. Therefore it attains its maximum on $\partial\mathcal{V} = i\mathbb{R}$.

QUESTION 9

Consider the two-step BDF method: $\mathbf{y}_{n+2} - \frac{4}{3}\mathbf{y}_{n+1} + \frac{1}{3}\mathbf{y}_n = \frac{2}{3}hf(t_{n+2}, \mathbf{y}_{n+2})$. Applied to $y' = \lambda y$ we get

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}h\lambda y_{n+2}$$

$$(3 - 2h\lambda)y_{n+2} - 4y_{n+1} + y_n = 0$$

We try $y_n = k^n$ and obtain

$$(3 - 2h\lambda)k^2 - 4k + 1 = 0$$

So

$$\begin{aligned} k &= \frac{4 \pm \sqrt{16 - 4(3 - 2h\lambda)}}{(6 - 4h\lambda)} \\ &= \frac{2 \pm \sqrt{1 + 2h\lambda}}{3 - 2h\lambda} \end{aligned}$$

Hence

$$y_n = A\left(\frac{2 + \sqrt{1 + 2h\lambda}}{3 - 2h\lambda}\right)^n + B\left(\frac{2 - \sqrt{1 + 2h\lambda}}{3 - 2h\lambda}\right)^n$$

Unsure what to deduce.

QUESTION 10

Given that $|y_n - y(t_n)| \leq 10^{-6}$, with Euler's method, setting $h = 2 \times 10^{-4}$, we have

$$
\begin{aligned}
|y_{n+1} - y(t_{n+1})| &= |y_n + h_n f(t_n, y_n) - t_n^{-1}| \\
&= |y_n + 2 \times 10^{-4}(-10^4(y_n - t_n^{-1}) - t_n^{-2}) - t_{n+1}^{-1}| \\
&= |(1 + 2 \times 10^{-8})y_n - (2 \times 10^{-8})t_n^{-1} - 2 \times 10^{-4})t_n^{-2} - t_{n+1}^{-1}|
\end{aligned}
$$

No idea what I'm doing here.

QUESTION 11

First consider the predictor; substituting the true solution

$$\mathbf{y}(t_{n+3}) - \{-\frac{1}{2}\mathbf{y}(t_n) + 3\mathbf{y}(t_{n+1}) - \frac{3}{2}\mathbf{y}(t_{n+2}) + 3h\mathbf{y}'(t_{n+2})\} \quad (*)$$

Performing Taylor expansions:

$$\mathbf{y}(t_{n+3}) = \mathbf{y}(t_n) + 3h\mathbf{y}'(t_n) + \frac{9}{2}h^2\mathbf{y}''(t_n) + \frac{9}{2}h^3\mathbf{y}'''(t_n) + \frac{27}{8}h^4\mathbf{y}''''(t_n) + O(h^5)$$

$$\mathbf{y}(t_{n+1}) = \mathbf{y}(t_n) + h\mathbf{y}'(t_n) + \frac{1}{2}h^2\mathbf{y}''(t_n) + \frac{1}{6}h^3\mathbf{y}'''(t_n) + \frac{1}{24}h^4\mathbf{y}''''(t_n) + O(h^5)$$

$$\mathbf{y}(t_{n+2}) = \mathbf{y}(t_n) + 2h\mathbf{y}'(t_n) + 2h^2\mathbf{y}''(t_n) + \frac{4}{3}h^3\mathbf{y}'''(t_n) + \frac{2}{3}h^4\mathbf{y}''''(t_n) + O(h^5)$$

$$h\mathbf{y}'(t_{n+2}) = h\mathbf{y}'(t_n) + 2h^2\mathbf{y}''(t_n) + 2h^3\mathbf{y}'''(t_n) + \frac{4}{3}h^4\mathbf{y}''''(t_n) + O(h^5)$$

Substituting these into $(*)$ it is clear that the predictor method is third order; moreover we deduce that

$$\mathbf{y}(t_{n+3}) - \{-\frac{1}{2}\mathbf{y}(t_n) + 3\mathbf{y}(t_{n+1}) - \frac{3}{2}\mathbf{y}(t_{n+2}) + 3h\mathbf{y}'(t_{n+2})\} = \frac{1}{4}h^4\mathbf{y}''''(t_n) + O(h^5)$$

and thus

$$\mathbf{y}^P_{n+3} \approx \mathbf{y}(t_{n+3}) - \frac{1}{4}h^4\mathbf{y}''''(t_n)$$

Similarly for the corrector; substituting the true solution

$$\mathbf{y}(t_{n+3}) - \frac{1}{11}\{2\mathbf{y}(t_n) - 9\mathbf{y}(t_{n+1}) + 18\mathbf{y}(t_{n+2}) + 6h\mathbf{y}'(t_{n+3})\} \quad (**)$$

Noting that

$$h\mathbf{y}'(t_{n+3}) = h\mathbf{y}'(t_n) + 3h^2\mathbf{y}''(t_n) + \frac{9}{2}h^3\mathbf{y}'''(t_n) + 9h^4\mathbf{y}''''(t_n) + O(h^5)$$

We again see this method is third order, and that

$$\mathbf{y}(t_{n+3}) - \frac{1}{11}\{2\mathbf{y}(t_n) - 9\mathbf{y}(t_{n+1}) + 18\mathbf{y}(t_{n+2}) + 6h\mathbf{y}'(t_{n+3})\} =$$

QUESTION 12

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