Part IB — Linear Algebra Sheet 1

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QUESTION 1

As all of the following basis are of order n, we need only check for linear independence (or spanning).

(a)
$$\alpha_1(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_2(\mathbf{e}_2 + \mathbf{e}_3) + \cdots + \alpha_{n-1}(\mathbf{e}_{n-1} + \mathbf{e}_n) + \alpha_n \mathbf{e}_n = \mathbf{0}$$

The first vector is the only one that contains $\mathbf{e}_1$, so $\alpha_1 = 0$. But then $\alpha_2 = 0, \cdots, \alpha_n = 0$ so this set is linearly independent, and thus a basis.

(b)

$$\alpha_1(\mathbf{e}_1 + \mathbf{e}_2) + \alpha_2(\mathbf{e}_2 + \mathbf{e}_3) + \cdots + \alpha_{n-1}(\mathbf{e}_{n-1} + \mathbf{e}_n) + \alpha_n(\mathbf{e}_n + \mathbf{e}_1) = \mathbf{0}$$

Then $\alpha_2 = -\alpha_1, \alpha_3 = \alpha_1, \cdots, \alpha_n = (-1)^{n+1}\alpha_1$. Thus for n even, it is possible to cancel out the $\mathbf{e}_1$ and have linear dependence, but not when n is odd. Thus

$$\begin{cases} \text{basis} & \text{if } n \text{ odd} \\ \text{not a basis} & \text{if } n \text{ even} \end{cases}$$

(c) Vectors in this basis are of the form $\mathbf{e}_i + (-1)^i \mathbf{e}_{n-i}$. If n is odd, say $n = 2k + 1$, setting

- $\alpha_{k+1} = 0$ (middle coefficient), only vector containing $\mathbf{e}_{k+1}$
- $\alpha_1 = -\alpha_n, \alpha_2 = -\alpha_{n-1}, \cdots$

is enough to show linear dependence.

If n is even, the first and last vector are $\mathbf{e}_1 - \mathbf{e}_n$ and $\mathbf{e}_1 + \mathbf{e}_n$, so these coefficients must both be set so zero. Likewise for $\mathbf{e}_2 - \mathbf{e}_{n-1}$ and $\mathbf{e}_2 + \mathbf{e}_{n-1}, \cdots$ etc, all the coefficients are zero, thus linear independence, thus this set is a basis when n is even.

QUESTION 2

(i)

Proposition. $T \cup U$ is a subspace of V only if either $T \leq U$ or $U \leq T$

Proof. – Choose $v_1 \in T \setminus U$, $v_2 \in U \setminus T$

- As $T \cup U$ is a subspace of V. $v_1, v_2 \in T \cup U \Rightarrow v_1 + v_2 \in T \cup U$
- $\Rightarrow v_1 + v_2 \in T$ or U
- If $v_1 + v_2 \in T$, then $v_2 \in T$. But we said $v_2 \in U \setminus T$. Contradiction.
- Hence $U \setminus T$ is empty and $U \leq T$.
- Similarly, $v_1 + v_2 \in U$ then $T \leq U$

□

(ii) (a) Choose

$$T = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \subset \mathbb{R}^2 \mid x \in \mathbb{R} \right\}, U = \left\{ \begin{pmatrix} x \\ 2x \end{pmatrix} \subset \mathbb{R}^2 \mid x \in \mathbb{R} \right\}$$

and

$$W = \left\{ \begin{pmatrix} x \\ 3x \end{pmatrix} \subset \mathbb{R}^2 \mid x \in \mathbb{R} \right\}$$

Then LHS $= T + (U \cap W) = T + \mathbf{0} = T$, and RHS $= (\mathbb{R}^2) \cap (\mathbb{R}^2) = \mathbb{R}^2$

(b) Choosing T, U and W as before, LHS $= (\mathbb{R}^2) \cap W = W$, and RHS $= \mathbf{0} + \mathbf{0} = \mathbf{0}$

(iii) The counter examples suggest which way the inclusions are:

Proposition. $T + (U \cap W) \subset (T + U) \cap (T + W)$

Proof. – Let $a + b \in T + (U \cap W)$

- $a \in T$, $b \in U \cap W$
- Then $b \in U$ and $b \in W$
- $a \in T$, $b \in U \Rightarrow a + b \in (T + U)$
- $a \in T, b \in W \Rightarrow a + b \in (T + W)$
- Thus $a + b \in (T + U) \cap (T + W)$

□

Proposition. $(T + U) \cap W \supset (T \cap W) + (U \cap W)$

Proof. – Similarly, let $a + b \in$ RHS

- so $a \in (T \cap W), b \in (U \cap W)$
- In particular, $a \in T$, $b \in U \Rightarrow a + b \in (T + U)$
- And $a \in W$, $b \in W \Rightarrow a + b \in W + W = W$
- Thus $a + b \in (T + U) \cap W$

□

QUESTION 3

Hint to show isomorphism: Guess an explicit inverse, compose both with right and left to get the identity

(a) Let $T : V \to W$ be defined by

$$T \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ -v_1 - v_2 - v_3 - v_4 \end{pmatrix}$$

It is straightforward to see that $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y})$ and $T(\alpha\mathbf{x}) = \alpha T(x)$, thus T is linear.

To show it is one-to-one, consider the map $T' : W \to V$ defined by

$$T' \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

Then $T \circ T' = T' \circ T = \mathrm{id}$.

(b) Note that $\{1, x, x^2, x^3, x^4, x^5\}$ is a spanning set for W. It is also linearly independent; suppose that

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 = \theta(x)$$

where $\theta(x)$ is the zero polynomial. If this holds for all values of x, then (since $\theta'(x) = \theta(x)$) we can differentiate both sides to obtain

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 = \theta(x)$$

Continuing differentiation in this fashion we arrive at

$$5!a_5 = \theta(x)$$

And we must have $a_5 = 0$. Going one diffentiation step back the previous equation insist $a_4 = 0$, and so we have $a_i = 0$ for all i, and thus $\{1, x, x^2, x^3, x^4, x^5\}$ is linearly independent in W.

Hence we have found a basis for W and conclude $\dim W = 6$. But $\dim V = 5$, and therefore there can be no such isomorphism.

(c) Define $T : W \to V$ as $T(f(x)) \mapsto f(2x + 1)$:

– Linear:

$$\begin{aligned} T(\lambda f_1(x) + \mu f_2(x) &= (\lambda f_1 + \mu f_2)(2x+1) \\ &= \lambda f_1(2x+1) + \mu f_2(2x+1) \\ &= \lambda T(f_1(x)) + \mu T(f_2(x)) \end{aligned}$$

– Bijective: Define $T' : W \to V$ as $T'(f(x)) = f(\frac{x-1}{2})$
  Show that $T \circ T' = T' \circ T = \mathrm{id}$

(d) Define $T : V \to W$ as $T(f(x)) \mapsto \int^x f(t)\, \mathrm{d}t$

(e) A natural basis for W is $\{A, B\}$ where solutions are of the form $A\cos t + B\sin t$. Hence define $T : V \to W$ as $T(v_1, v_2) = v_1 \cos t + v_2 \sin t$.

(f) Suppose $\varphi : R^4 \to C[0,1]$ is an isomorphism. Let e_1, e_2, e_3, e_4 be a basis for $\mathbb{R}^4$. Then

$$\{\varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)\}$$

is a basis for $C[0,1]$.

In particular, we have a spanning set of size 4. But, eg. $\{1, x, x^2, x^3, x^4, x^5\}$ is a linearly independent set of size 5. This is a contradiction (by Steinitz)

(g) Suppose $\phi : \mathcal{P} \to \mathbb{R}^{\mathbb{N}}$ is an isomorphism, with ϕ having the natural basis $\{1, x, x^2, \cdots, x^N\}$. Then

$$\underbrace{\{\phi(1), \phi(x), \cdots, \phi(x^N)\}}$$

is a countable basis for $\mathbb{R}^{\mathbb{N}}$. But, $\mathbb{R}^{\mathbb{N}}$ has no countable basis, no ϕ cannot be an isomorphism.

QUESTION 4

(i) Let α, β be linear maps from U to V. Then

$$\begin{aligned}(\alpha+\beta)(v_1+v_2) &= \alpha(v_1+v_2)+\beta(v_1+v_2)\\ &= \alpha(v_1)+\alpha(v_2)+\beta(v_1)+\beta(v_2)\\ &= (\alpha+\beta)(v_1)+(\alpha+\beta)(v_2)\end{aligned}$$

and

$$\begin{aligned}(\alpha+\beta)(\lambda v) &= \alpha(\lambda v)+\beta(\lambda v)\\ &= \lambda\alpha(v)+\lambda\beta(v)\\ &= \lambda(\alpha+\beta)(v)\end{aligned}$$

Thus $\alpha+\beta$ is also a linear map

(a) Let $\alpha, \beta : V \to V$ st. $\alpha = \mathrm{id}$, $\beta = -\alpha$.
Then $\mathrm{Im}(\alpha+\beta) = 0$, $\mathrm{Im}(\alpha) = V$, $\mathrm{Im}(\beta) = V$.

$$\mathrm{Im}(\alpha+\beta) \neq \mathrm{Im}\,\alpha + \mathrm{Im}\,\beta$$

(b) Using the same maps, $\ker(\alpha+\beta) = V$, $\ker\alpha = \mathbf{0}$ and $\ker\beta = \mathbf{0}$, hence

$$\ker(\alpha+\beta) \neq \ker\alpha \cap \ker\beta$$

Proposition.

$$\mathrm{Im}(\alpha+\beta) \subset \mathrm{Im}\,\alpha + \mathrm{Im}\,\beta$$

Proof. Suppose $v \in$ LHS, that is

$$\begin{aligned}v &\in \{v \in V \mid v = (\alpha+\beta)(u),\ \text{some } u \in U\}\\ &= \{v \in V \mid v = \alpha(u)+\beta(u),\ \text{some } u \in U\}\\ &\subset \{v \in V \mid v = \alpha(u),\ \text{some } u \in U\} + \{v \in V \mid v = \beta(u),\ \text{some } u \in U\}\\ &= \mathrm{Im}\,\alpha + \mathrm{Im}\,\beta\end{aligned}$$

Hence $v \in$ RHS

□

Proposition.

$$\ker(\alpha+\beta) \supset \ker\alpha \cap \ker\beta$$

Proof. Suppose $u \in$ RHS □

Proof. Let $u \in$ RHS, ie

$$\begin{aligned} u &\in \{u \in U \mid \alpha(u) = \mathbf{0}\} \cap \{u \in U \mid \beta(u) = \mathbf{0}\} \\ &= \{u \in U \mid \alpha(u) = \beta(u) = \mathbf{0}\} \\ &\subset \{u \in U \mid \alpha(u) + \beta(u) = \mathbf{0}\} \\ &= \ker(\alpha + \beta) \end{aligned}$$

□

(ii) (Might be helpful to think of α geometrically as a projection). We want to prove that if $\alpha^2 = \alpha$, then

- $\operatorname{Im}\alpha \cap \ker\alpha = \{\mathbf{0}\}$
- $\operatorname{Im}\alpha + \ker\alpha = V$

Proof. – Given $v \in \operatorname{Im}\alpha \cap \ker\alpha$, there exists some w st. $v = \alpha(w)$. So

$$\begin{aligned} v &= \alpha(w) \\ &= \alpha^2(w) \\ &= \alpha(\alpha(w)) \\ &= \alpha(v) \qquad \in \ker\alpha \\ &= \mathbf{0} \end{aligned}$$

– Given $v \in V$, then

$$v = \underbrace{\alpha(v)}_{\in \operatorname{Im}\alpha} + \underbrace{(v - \alpha(v))}_{\in \ker\alpha}$$

since

$$\begin{aligned} \alpha(v - \alpha(v)) &= \alpha(v) - \alpha^2(v) \\ &= \alpha(v) - \alpha(v) \\ &= \mathbf{0} \end{aligned}$$

So $V = \ker \oplus \operatorname{im}\alpha$

□

QUESTION 5

$U \cap W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 = 0, x_2 = x_3 = x_4, x_1 + x_5 = 0\}$ by combining the conditions on U and W. Vectors in U, W and $U \cap W$ respectively have the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 - x_3 \\ -\frac{1}{2}(x_1 + x_2) \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_2 \\ x_2 \\ -x_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2x \\ x \\ x \\ x \\ 2x \end{pmatrix}$$

Thus a natural basis for $U \cap W$ is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Basis for U, W:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\} \qquad \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Now add the vector to each of these basis and perform Gaussian elimination. Or, note that we can switch it for the first vector in U and the second vector in W, as the first component is non-zero. Thus the required basis for U, W are:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\} \qquad \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Now the basis for $U + W$ is just basis for $U \cup$ basis for V, provided the basis for $U \cap W$ is a subset of both.

So a basis for $U + W$ is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

QUESTION 6

Let $\alpha : V \to V$ linear, and let $v_1 = \alpha(u_1)$, $v_2 = \alpha(v_2)$
From the first isomorphism theorem we have $\text{Im}(\alpha) \leq V$, $\ker(\alpha) \leq V$

Proposition. $\ker(\alpha) \leq V$

Proof.
- $\mathbf{0} \in \ker \alpha$
- Let $v_1, v_2 \in \ker \alpha$. Then

$$\begin{aligned}\alpha(\lambda v_1 + \mu v_2) &= \lambda\alpha(v_1) + \mu\alpha(v_2) \\ &= \mathbf{0} + \mathbf{0} = \mathbf{0} \qquad \text{as } v_1, v_2 \in \ker \alpha\end{aligned}$$

Hence $\lambda v_1 + \mu v_2 \in \ker \alpha$

□

$$\begin{aligned}\text{Im}(\alpha^{k+1}) &= \{v \in V \mid \alpha^{k+1}(u) \in V, \text{some } u \in V\} \\ &= \{v \in V \mid \alpha^{k}(\alpha(u)) \in V, \text{some } u \in V\} \\ &\subseteq \{v \in V \mid \alpha^{k}(v) \in V, \text{some } v \in V\} \qquad \text{as } \text{Im}(\alpha) \leq V \\ &= \text{Im}(\alpha^k)\end{aligned}$$

Hence

$$V \geq \text{Im}(\alpha) \geq \text{Im}(\alpha^2) \geq \cdots$$

Next, $\alpha(\mathbf{0}) = \mathbf{0}$, so trivially $\{0\} \leq \ker(\alpha)$, and

$$\begin{aligned}\ker(\alpha^{k+1}) &= \{v \in V \mid \alpha^{k+1}(v) = 0\} \\ &= \{v \in V \mid \alpha^{k}(\alpha(v)) = 0\} \\ &\subseteq \{v \in V \mid \alpha^{k}(v) = 0\} \qquad \text{as } \ker(\alpha) \leq V \\ &= \ker(\alpha^k)\end{aligned}$$

It now follows that

$$\{\mathbf{0}\} \leq \ker \alpha \leq \ker \alpha^2 \leq \cdots$$

Next, taking dim of the first inequality gives

$$\dim V \geq r_1 \geq r_2 \geq \cdots$$

Thus $r_k \geq r_{k+1}$. Similarly for $n_k = n(\alpha^k)$, we have $n_k \leq n_{k+1}$.
Let $\widetilde{\alpha}_k : \text{Im}\,\alpha_k \to V$ be defined by $v \mapsto \alpha(v)$. Note that $\text{Im}(\widetilde{\alpha}_k) = \text{Im}(\alpha^{k+1})$
Applying R-N to $\widetilde{\alpha}_k$,

$$\dim(\text{Im}(\alpha^k)) = r(\widetilde{\alpha}_k) + n(\widetilde{\alpha}_k)$$

So

$$n_{k+1} = r_k - r_{k+1}$$

QUESTION 7

With respect to the standard basis, α is represented by the matrix A, where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Change of basis matrix P and it's inverse are given by

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

So the matrix $\tilde{A}$ representing the linear map with respect to the new basis is given by

$$\begin{aligned} \tilde{A} &= P^{-1}AP \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \end{aligned}$$

EASIER: Given the basis

$$\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$$

for the domain, and the same one for the range, α maps

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So the first column of A is $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, etc.

QUESTION 8

(i) $\Rightarrow$ (iii) B spans, any $v \in V$ is $u_1 + \cdots + u_n$, for some $u_i \in U_i$, write u_i in terms of B_i, Then $u_1 + \cdots + u_n$ is a lin comb. of elements of B.

B indep?

$$\sum_{v \in B} \lambda_v v = \mathbf{0} = \mathbf{0}_{U_1} + \cdots + \mathbf{0}_{U_n}$$

$$\underbrace{\sum_{v \in B_1} \lambda_v v}_{\in U_1} + \cdots + \sum_{v \in B_n} \lambda_v v$$

By uniqueness of expressions,

$$\sum_{v \in B_1} \lambda_v v = \mathbf{0}_{U_1} \cdots \sum_{v \in B_n} \lambda_v v = \mathbf{0}_{U_n}$$

As $B_1, \cdots, B_n$ are basis, all of the λ_v are zero.

(iii) $\Rightarrow$ (ii).
Given $v \in U_j \cap \sum_{i \neq j} U_i$
Since $v \in U_j$, can write

$$v = \sum_{b_i \in B_i} \lambda_i b_i$$

Since $v \in \sum_{i \neq j} U_i$, can write

$$v = \sum_{b_i \in \cup_{k \neq j} B_k} \mu_i b_i$$

No b_i's in common because of the pairwise disjointness of the B_i.
But $\cup_k B_k$ is a basis, so by uniqueness of expression,
$\lambda_i = \mu_i = 0$ for all i.
So $v = \mathbf{0}$.
(ii) $\Rightarrow$ (i)
Suppose that

$$\sum u_i = \sum u_i'$$

Then for each j,

$$u_j - u_j' = \sum_{i \neq j} (u_i' - u_i) \in U_j \cap \sum_{i \neq j} U_i = \mathbf{0}$$

So $u_j = u_j'$, for all j. Thus uniqueness of expression