Part IB — Numerical Analysis Example Sheet 1

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QUESTION 1

We seek some polynomial interpolant $p \in \mathbb{P}_3[x]$. Using the Lagrange formula we have that

$$p(x) = \sum_{k=0}^{3} f(k) l_k$$

where

$$l_k = \prod_{i=0, i \neq k}^{3} \frac{x-i}{k-i}$$

that is,

$$p(x) = f(0)\frac{(x-1)(x-2)(x-3)}{-6} + f(1)\frac{x(x-2)(x-3)}{2} + f(2)\frac{x(x-1)(x-3)}{-2} + f(3)\frac{x(x-1)(x-2)}{6}$$

(i) The approximant $p(6)$

We have

$$\begin{aligned} p(6) &= f(0)\frac{5 \cdot 4 \cdot 3}{-6} + f(1)\frac{6 \cdot 4 \cdot 3}{2} + f(2)\frac{6 \cdot 5 \cdot 3}{-2} + f(3)\frac{6 \cdot 5 \cdot 4}{6} \\ &= -10f(0) + 36f(1) + -45f(2) + 20f(3) \end{aligned}$$

(ii) The approximant $p'(0)$

Taking the derivative of each term individually, we then plug in $x = 0$. We deduce that

$$p'(0) = -\frac{11}{6}f(0) + 3f(1) - \frac{3}{2}f(2) + \frac{1}{3}f(3)$$

(iii) The approximant $\int_0^3 p(x)\, \mathrm{d}x$

Expanding each term and integrating (I can't see a shorter way) we have that

$$p(x) = f(0)\frac{x^3 - 6x^2 + 11x - 6}{-6} + f(1)\frac{x^3 - 5x^2 + 6x}{2} + f(2)\frac{x^3 - 4x^2 + 3x}{-2} + f(3)\frac{x^3 - 3x^2 + 2x}{6}$$

(*I've cheated a bit with Mathematica here*)

$$\int_0^3 p(x)\, \mathrm{d}x = \frac{3}{8}f(0) + \frac{9}{8}f(1) + \frac{9}{8}f(2) + \frac{3}{8}f(3)$$

We can check this by supposing $f(x) = x$, so that $f(k) = k$ for each k. Indeed, $p(6) = 6, p'(0) = 1$, and $\int_0^3 p(x)\, \mathrm{d}x = 9/2$,

QUESTION 2

The formula is true when $x = 0, 1$ since both sides of the equation vanish. Let $x \in (0, 1)$ be any other point and define (for x fixed).

$$\phi(t) := [f(t) - p(t)] \prod_{i=0}^{3} (x - x_i) - [f(x) - p(x)] \prod_{i=0}^{3} (t - x_i), \quad t \in (0, 1)$$

where $x_0 = x_1 = 0, x_2 = x_3 = 1$. Note $\phi(0) = \phi(1) = 0$, and also $\phi(x) = 0$. Hence, ϕ has at least 3 zeroes. Applying Rolle's theorem, and using the condition that $f'(0) = f'(1) = 0$, we deduce that $\phi'(t)$ has at least 4 zeroes: one at $x_0 = 0$, one at $x_2 = 1$, and two more: one in the interval $(0, x)$, and the other in $(x, 1)$.

Then $\phi''(t)$ has at least 3 zeroes in $(0, 1)$, and... $\phi^{(4)}(x)$ has at least one zero in $(0, 1)$; call it ξ. Then

$$0 = \phi^{(4)}(\xi) = \left[f^{(n+1)}(\xi) - p^{(n+1)}(\xi) \right] \prod_{i=0}^{3} (x - x_i) - [f(x) - p(x)] \left. \frac{\mathrm{d}^4}{\mathrm{d}t^4} \right|_{t=\xi} \prod_{i=0}^{3} (t - x_i)$$

Since $p^{(4)} \equiv 0$, and $\left. \frac{\mathrm{d}^4}{\mathrm{d}t^4} \right|_{t=\xi} \prod_{i=0}^{3} (t - x_i) = 4!$, we obtain

$$\begin{aligned} f(x) - p(x) &= \frac{1}{4!} f^{(4)}(\xi) \prod_{i=0}^{3} (x - x_i) \\ &= \frac{1}{24} x^2 (1 - x)^2 f^{(4)}(\xi) \end{aligned}$$

QUESTION 3

Seeking a contradiction we suppose there exists some nonzero polynomial $p \in \mathbb{P}_4[x]$ st.

$$p(a) = p(b) = p'(a) = p'(b) = p'(c) = 0 \qquad (*)$$

Suppose that $q_1 \in \mathbb{P}_4[x]$ and $q_2 \in \mathbb{P}_4[x]$ both interpolate the data, then $q_1 - q_2$ vanishes at these points. Hence, we have

$$q_1 = q_2 + kp$$

for some $k \in \mathbb{R}$, so the solution of this interpolation problem is not unique.
To pick a value of c that satisfies $(*)$, try

$$p(x) = (x-a)(x-b) + (x-a)(x-b)(x-c)$$

Immediately we have $p(a) = p(b) = 0$. Now,

$$p'(x) = (x-a) + (x-b)(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$$

QUESTION 4

By definition, $f[x_0, x_1, \cdots, x_n, x]$ is the coefficient of x^{n+1} in $q \in \mathbb{P}_{n+1}[x]$ that interpolates the $n+2$ points $[x_0, x_1, \cdots, x_n, x]$ with f.

From the definition of the divided difference we have that

$$f[x_0, x_1, \cdots, x_n, x] = \frac{f[x_1, x_2, \cdots, x_{n+1}, x] - f[x_0, x_1, \cdots, x_n]}{x - x_0}$$

But also,

$$f[x_1, \cdots, x_n, x] = \frac{f[x_2, \cdots, x_{n+1}, x] - f[x_1, \cdots, x_n]}{x - x_1}$$

When $n = 0$ the identity reads

$$\begin{aligned} f(x) - p(x) &= f[x_0, x](x - x_0) \\ &= f(x) - f(x_0) \end{aligned}$$

When $n = 1$ the identity reads

$$\begin{aligned} f(x) - p(x) &= f[x_0, x_1, x](x - x_0)(x - x_1) \\ &= \frac{f[x_0, x_1] - f[x_1, x]}{x - x_0} \end{aligned}$$

Nah fuck that, third attempt

QUESTION 5

The Newton divided difference table for Question 5 is shown below.

x_i	f_i	$f[*,*]$	$f[*,*,*]$	$f[*,*,*,*]$
0	$f[0] = 0$			
		$f[0, 0.1] = 0.998$		
0.1	$f[0.1] = 0.0998$		$f[0, 0.1, 0.4]$	
		$f[0.1, 0.4] = 0.9687$	$= 0.0733$	$f[0, 0.1, 0.4, 0.7]$
0.4	$f[0.4] = 0.3894$		$f[0.1, 0.4, 0.7]$	$= -0.389$
		$f[0.4, 0.7] = 0.8493$	$= -0.1990$	
0.7	$f[0.7] = 0.6442$			

Using Newton's formula, the polynomial interpolating these points is given as

$$\begin{aligned}
p(x) &= f[0] + f[0, 0.1]x + f[0, 0.1, 0.4]x(x - 0.1) + f[0, 0.1, 0.4, 0.7]x(x - 0.1)(x - 0.4) \\
&= 0.998x + 0.0733(x^2 - 0.1x) - 0.389(x^3 - 0.5x^2 + 0.04x) \\
&= 0.9751x+
\end{aligned}$$

QUESTION 6

QUESTION 7

QUESTION 8

QUESTION 9

QUESTION 10

QUESTION 11

QUESTION 12