Part IB — Complex Methods Example Sheet 1

Supervised by Prof. Haynes (P.H.Haynes@damtp.cam.ac.uk)
Examples worked through by Christopher Turnbull

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QUESTION 1

(i) *[For each of the following we let $f(z) = u(x,y) + iv(x,y)$ and check the Cauchy-Riemann equations.]*

- $f(z) = \operatorname{Im} z$. This has $u = y$, $v = 0$. But

$$\frac{\partial u}{\partial y} = 1 \neq 0 = -\frac{\partial u}{\partial x}$$

  So $\operatorname{Im} z$ is nowhere differentiable, and hence nowhere analytic.

- $f(z) = |z|^2 = x^2 + y^2$. This has $u = x^2 + y^2$, $y = 0$. Have

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = 0$$

  Hence the Cauchy-Riemann equations are only satisfied at the origin. So f in only differentiable at $z = 0$, however it is not analytic since there is no neighbourhood of 0 throughout which f is differentiable.

- $f(z) = \operatorname{sech} z$. First note that if $f(z) = u + iv \neq 0$, then

$$\frac{1}{f(z)} = \frac{u}{u^2 + v^2} - \frac{iv}{u^2 + v^2}$$

  So if $f(z)$ is analytic, then $\frac{1}{f(z)}$ is analytic provided $f(z) \neq 0$.
  $g(z) := \cosh(z) = \frac{1}{2}(e^z + e^{-z})$ is entire since e^z is entire (from lectures). Checking when g is zero gives us $z = \frac{1}{2}\log(-1) = \frac{1}{2}[\log(1) + (2n+1)i\pi]$ for integer n.
  Hence $\operatorname{sech}(z)$ is differentiable at all points expect those at $(0, (n + \frac{1}{2})\pi)$ for integer n, and hence also analytic everywhere but these points.

(ii) Writing $z = r(\cos\theta + i\sin\theta)$, we obtain

$$u = r\cos 5\theta \quad v = r\sin 5\theta$$

Using the chain rule with $r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}$,
~~This is going to be messy~~ First note that

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$$

and

$$\frac{\partial r}{\partial x} = x(x^2 + y^2)^{-1/2}, \qquad \frac{\partial r}{\partial y} y(x^2 + y^2)^{-1/2}$$

The first Cauchy Riemann equation is

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\
&= \cos(5\theta)x(x^2+y^2)^{-1/2} - r\sin 5\theta \frac{-y}{x^2+y^2} \\
&= \cos(5\theta)r\cos\theta r^{-1} - r\sin 5\theta \frac{-r\sin\theta}{r^2} \\
&= \cos 4\theta
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} \\
&= \sin(5\theta)y(x^2+y^2)^{-1/2} + r\cos 5\theta \frac{x}{x^2+y^2} \\
&= \sin(5\theta)r\sin\theta r^{-1} + r\cos 5\theta \frac{r\cos\theta}{r^2} \\
&= \cos 4\theta
\end{aligned}$$

For second CR equation,

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} \\
&= \sin(5\theta)x(x^2+y^2)^{-1/2} + r\cos 5\theta \frac{-y}{x^2+y^2} \\
&= \sin(5\theta)r\cos\theta r^{-1} + r\cos 5\theta \frac{-r\sin\theta}{r^2} \\
&= \sin 4\theta
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \\
&= \cos(5\theta)y(x^2+y^2)^{-1/2} + -\sin 5\theta \frac{x}{x^2+y^2} \\
&= \cos(5\theta)r\sin\theta r^{-1} - r\sin 5\theta \frac{r\cos\theta}{r^2} \\
&= -\sin 4\theta
\end{aligned}$$

We conclude in fact that the Cauchy-Riemann equations are satisfied everywhere.

Now, looking closer at $\frac{\partial u}{\partial x}$, we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \cos 4\theta \\
&= \cos^2 2\theta - \sin^2 2\theta \\
&= (\cos^2\theta - \sin^2\theta)^2 - 4\sin^2\theta\cos^2\theta \\
&= (\cos^2\theta + \sin^2\theta)^2 - 8\sin^2\theta\cos^2\theta \\
&= 1 - \frac{8x^2y^2}{r^4} \\
&= 1 - \frac{8x^2y^2}{(x^2+y^2)^2}
\end{aligned}$$

Now we see that when $x = y = 0$, $\frac{\partial u}{\partial x}$ is not defined, which is enough to show that f is not differentiable at the origin.

(iii) Using the Cauchy-Riemann equations, g is differentiable iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

QUESTION 2

[Each of the following analytical functions will be of the form $f(z) = u(x,y) + iv(x,y)$. Given u, we find v using the Cauchy-Riemann equations, and thus f.]

(i) $u = xy$, so the first Cauchy Riemann equation determines

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = y \implies v = \frac{1}{2}y^2 + g(x)$$

The other Cauchy Riemann equation gives

$$-x = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = g'(x)$$

So $g'(x) = -x$, giving us $g(x) = -\frac{1}{2}x^2 + \alpha$ for some constant α, wlog 0. The corresponding analytic function is therefore

$$\begin{aligned} f(z) &= xy + \frac{1}{2}i(y^2 - x^2) \\ &= \frac{1}{2}i\left(y^2 - 2ixy - x^2\right) \\ &= -\frac{1}{2}i\left(x^2 + 2ixy - y^2\right) \\ &= -\frac{1}{2}i\left(x + iy\right)^2 \\ &= -\frac{1}{2}iz^2 \end{aligned}$$

(ii) $u = \sin x \cosh y$, so the first Cauchy Riemann equation determines

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \cos x \cosh y \implies v = \cos x \sinh y + g(x)$$

The other Cauchy Riemann equation gives

$$\sin x \sinh y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \sin x \sinh y + g'(x)$$

So $g'(x) = 0$, giving us $v(x) = \cos x \sinh y + \alpha$ for some constant α (wlog set it to zero). The corresponding analytic function is therefore

$$\begin{aligned} f(z) &= \sin x \cosh y + i \cos x \sinh y \\ &= \frac{1}{2}e^y(\sin x + i\cos x) + \frac{1}{2}e^{-y}(\sin x - i\cos x) \\ &= \frac{1}{2}i[e^{y-ix} - e^{ix-y}] \\ &= i\sinh(z^*) \end{aligned}$$

(iii) $u = \log(x^2 + y^2)$, so Cauchy Riemann determine that
Recall $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan(x/a)$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} \implies v = 2\arctan(y/x) + g(x)$$

Next,

$$\frac{2y}{x^2+y^2} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{2y}{x^2+y^2} + g'(x)$$

Hence $g'(x) = 0$, set $g(x) = 0$ wlog, have that

$$\begin{aligned} f(z) &= \log(x^2+y^2) + i2\arctan(y/x) \\ &= \log(|z|^2) + 2i\mathrm{sgn}(x) arg(z) \end{aligned}$$

(iv) $u = e^{y^2-x^2}\cos 2xy$, so Cauchy Riemann determine that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -2(x\cos 2xy + y\sin 2xy)e^{y^2-x^2} \implies v = -e^{y^2-x^2}\sin 2xy + g(x)$$

Set $g(x) = 0$ wlog, have that

$$\begin{aligned} f(z) &= e^{y^2-x^2}\cos 2xy - ie^{y^2-x^2}\sin 2xy \\ &= \end{aligned}$$

(v) $u = \frac{y}{(x+1)^2+y^2}$, so Cauchy Riemann determine that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{-2y(x+1)}{[(x+1)^2+y^2]^2}$$

$$\begin{aligned} \implies v &= -(x+1)\int 2y[(x+1)^2+y^2]^{-2}\,\mathrm{d}y \\ &= \frac{-(x+1)}{(x+1)^2+y^2} + g(x) \end{aligned}$$

Setting $g(x) = 0$, the corresponding analytic function is therefore

$$\begin{aligned} f(z) &= \frac{y}{(x+1)^2+y^2} + -i\frac{(x+1)}{(x+1)^2+y^2} \\ &= y + i(x+1) \\ &= i(x - iy) + i \\ &= iz^* + i \end{aligned}$$

(vi) $u = \arctan\left(\frac{2xy}{x^2-y^2}\right)$, so Cauchy Riemann determine that

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} &= \frac{(x^2-y^2)2y - 2xy(2x)}{(x^2-y^2)^2+(2xy)^2} \\
&= \frac{2y[(x^2-y^2)-2x^2]}{(x^2+y^2)^2} \\
&= \frac{-2y(x^2+y^2)}{(x^2+y^2)^2} \\
&= \frac{-2y}{x^2+y^2}
\end{aligned}$$

$$\begin{aligned}
\implies v &= \int \frac{-2y}{x^2+y^2}\,\mathrm{d}y \\
&= -\log(x^2+y^2) + g(x)
\end{aligned}$$

Deduce that $g'(x) = 0$, set the constant to zero, so we have

$$\begin{aligned}
f(z) &= \arctan\left(\frac{2xy}{x^2-y^2}\right) - i\log(x^2+y^2) \\
&=
\end{aligned}$$

Now, if these $f = u + iv$ are analytic, (and therefore satisfy the Cauchy-Riemann equations) we can compute

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \\
&= -\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Therefore, when the CR equations are satisfied, the function u is harmonic. Hence, for the above questions, we have u harmonic on $\mathbb{R}^2$.

QUESTION 3

QUESTION 4

$$\phi(x, y) = e^x(x\cos y - y\sin y)$$

Calculating the partial derivatives,

$$\begin{aligned}
\partial_x\phi &= \phi + e^x \cos y \\
\partial_{xx}\phi &= \partial_x\phi + e^x \cos y \\
&= \phi + 2e^x \cos y
\end{aligned}$$

$$\begin{aligned}
\partial_y\phi &= e^x(-x\sin y - \sin y - y\cos y) \\
\partial_{yy}\phi &= e^x(-x\cos y - 2\cos y + y\sin y) \\
&= -\phi - 2e^x \cos y
\end{aligned}$$

Hence $\partial_{xx}\phi + \partial_{yy}\phi = 0$ and the function is indeed harmonic.
The harmonic conjugate $\psi(x, y)$ satisfies the Cauchy Riemann equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

The first of these gives

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = e^x(x\cos y - y\sin y) + e^x \cos y$$

Noting $\int y\sin y \mathrm{d}y = -y\cos y + \sin y$, we must have $\psi = e^x(x\sin y + y\cos y) + g(x)$. The other Cauchy Riemann equation gives

$$e^x(x\sin y + \sin y + y\cos y) = -\frac{\partial\phi}{\partial y} = \frac{\partial v}{\partial x} = e^x(x\sin y + \sin y + y\cos y) + g'(x)$$

So g must be a constant, say 0, so the harmonic conjugate of ϕ is

$$\psi(x, y) = e^x(x\sin y + y\cos y)$$

Can now show that $\nabla\phi \cdot \nabla\psi = 0$ (by the CR equations), ie. contours of harmonic conjugate function are perpendicular (in 2D).

Recall that a gradient of a function is perpendicular to its contours. Not sure how to finish.

QUESTION 5

The principle branch of $\log z$ is formed by introducing the branch cut from $-\infty$ on the real axis to the origin, which is used to fix values of θ lying in the range $(-\pi, \pi]$.

z^i has a branch point at the origin; consider a circle of radius r_0 centred at 0, starting from $-\pi$ and going round once anticlockwise, as we approach π there will be a jump from $r_0^i e^{\pi}$ to $r_0^i e^{-\pi}$. Hence using the same branch cut as before, this branch of z^i is single-valued and continuous on any curve C that does not cross the cut. This branch is in fact analytic everywhere, with $\frac{\mathrm{d}}{\mathrm{d}z} z^i = i z^{i-1}$.

We have

$$z^i = r^i e^{-\theta} = e^{-\theta + i \log r} \qquad \theta \in (-\pi, \pi]$$

Hence we can see that $i^i = e^{-\pi/2}$ and moreover, this branch of z^i maps onto the annulus with inner radius $e^{-\pi}$ and outer radius e^{π} infinitely often as r increases (not that the inner radius is not actually included in the mapping).

Using different branches will produce an annuli of different radii and for each annulus, $i^i = e^{-\theta}$. Eg. with $\theta \in (\pi, 3\pi]$, we have the annulus define from $(e^{\pi}, e^{3\pi}]$ with $i^i = e^{-5\pi/2}$.

QUESTION 6

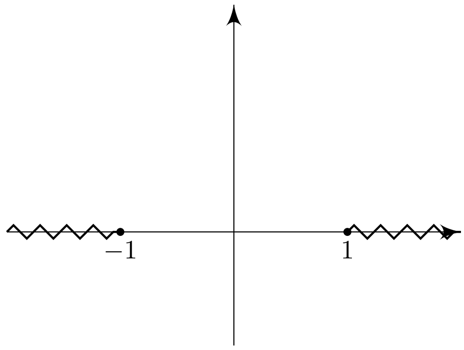
Introducing the branch cut from $-\infty$ on the real axis to the origin, once branch is given below as

$$z^{3/2} = r^{3/2} e^{3i\theta/2} \qquad \theta \in (-\pi, \pi]$$

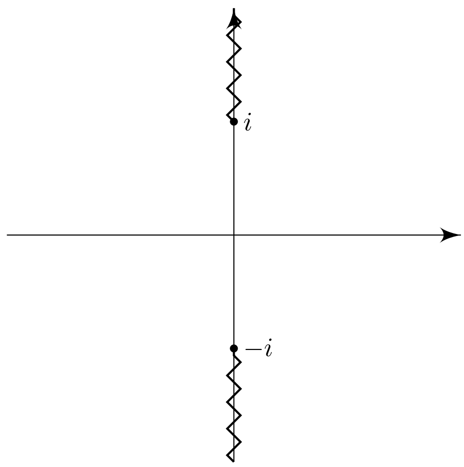
and we can define another two branches by choosing $\theta \in (\pi, 3\pi], \theta \in (3\pi, 5\pi]$.

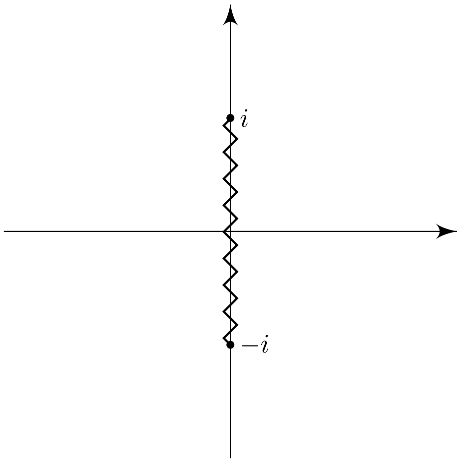
$f(z) = [z(z+1)]^{1/3}$ has two branch points, one at $z = 0$ and one at $z = 1$. So we now need two branch cuts. One possibility is shown below.

Note that we cannot make use of the branch cut $[-1, 1]$ as $z^{1/3}$ has a branch point at ∞.



Next, $g(z) = (z^2 + 1)^{1/2}$ has two different branch points, one at $z = i$ and one at $z = -i$. This time there is no branch point at infinity. Two different possibilities are shown below:





i
$-i$

QUESTION 7

Writing $z+1 = re^{i\theta}$ and $z-1 = r_1 e^{i\theta_1}$, we can write this as

$$\begin{aligned} f(z) &= (z-1)^{1/2}(z+1)^{1/2} \\ &= \sqrt{rr_1}e^{i(\theta+\theta_1)/2}. \end{aligned}$$

This has branch points at ± 1.

I'm not sure I'm understanding what the question is asking;

QUESTION 8

This mapping is a Möbius map, which sends circlines to circlines.

Looking at how the boundary of the disc is mapped, we see that $0, 2, 1 + i$ are mapped to $\infty, 1/2$ and $1/2 - i/2$, ie. this disc is mapped to the straight line intersecting the real axis at $1/2$.

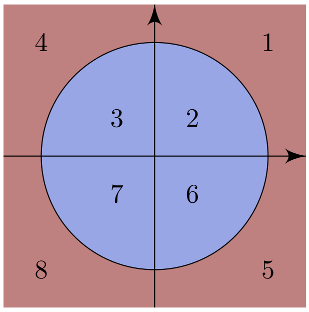
Inside the disc, the point 1 gets mapped to itself. Hence this disc is mapped to the RHP, right of (but not including) the line through $1/2$.

Next, using Möbius map ideas, we can consider the map $z \mapsto \frac{1}{1-z}$ acting on the unit disc $|z| < 1$. Now $-1, i, 1$ are mapped to $1/2, 1/2 + i/2, \infty$, ie. the straight line intersecting the real axis at $1/2$. The point 0 is mapped to 1. Hence under this map, the disc is transformed to all of $\mathbb{C}$ left of (but not including)this line.

Taking away $1/2$ we are covering the RHP. Now, squaring will give $\mathbb{C}\setminus(-\infty, 0]$ as we are not including the imaginary axis. Hence lastly, taking away $1/4$, we get the desired region.

QUESTION 9

$f(z) = \frac{z-1}{z+1}$ permutes the 8 divisions on the complex plane as follows:



The map sends $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1$ and $5 \mapsto 6 \mapsto 7 \mapsto 8 \mapsto 5$.

We can derive as follows. f is a Möbius map, and must map circlines to circlines. We can see that

$$\begin{aligned} \infty &\mapsto 1 \\ 0 &\mapsto -1 \\ 1 &\mapsto 0 \\ -1 &\mapsto \infty \\ i &\mapsto i \\ -i &\mapsto -i \end{aligned}$$

Consider $f(z) = \frac{z-1}{z+1}$ acting on the unit disk $U = \{z : |z| < 1\}$. The boundary of U is a circle. The three points $-1, i$ and $+1$ lie on this circle, and are mapped to ∞, i and 0 respectively.

Since Möbius maps take circlines to circlines, the image of ∂U is the imaginary axis. Since $f(0) = -1$, we see that the image of U is the left-hand half plane.

Similarly we can conclude the image of $\mathbb{C} \setminus U$ is the RHP, LHP is mapped to $\mathbb{C} \setminus U$, RHP is mapped to U, and UHP and DHP are mapped to themselves (repsectively). That is;

$$\begin{aligned} \{2,3,6,7\} &\mapsto \{3,4,6,7\} \\ \{1,4,5,8\} &\mapsto \{1,2,5,6\} \\ \{3,4,7,8\} &\mapsto \{1,4,5,8\} \\ \{1,2,5,6\} &\mapsto \{2,3,6,7\} \\ \{1,2,3,4\} &\mapsto \{1,2,3,4\} \\ \{5,6,7,8\} &\mapsto \{5,6,7,8\} \end{aligned}$$

Now consider the alternative Möbius map $g(z) = \frac{z-i}{z+i}$. We can see that

$$\begin{aligned} \infty &\mapsto 1 \\ 0 &\mapsto -1 \\ 1 &\mapsto i \\ -1 &\mapsto -i \\ i &\mapsto 0 \\ -i &\mapsto \infty \end{aligned}$$

Now $-1, i$ and $+1$ on ∂U are mapped to $-i$, 0 and i respectively, so the image of ∂U is the imaginary axis. Since $f(0) = -1$, we see that the image of U is the left-hand half plane (as before).

Similarly we can conclude the image of $\mathbb{C} \setminus U$ is the RHP, but now LHP is mapped to DHP, RHP is mapped to UHP, UHP is mapped to $\mathbb{C} \setminus U$ and DHP is mapped to U. In other words

$$\begin{aligned} \{2,3,6,7\} &\mapsto \{3,4,6,7\} \\ \{1,4,5,8\} &\mapsto \{1,2,5,6\} \\ \{3,4,7,8\} &\mapsto \{5,6,7,8\} \\ \{1,2,5,6\} &\mapsto \{1,2,3,4\} \\ \{1,2,3,4\} &\mapsto \{1,4,5,8\} \\ \{5,6,7,8\} &\mapsto \{2,3,6,7\} \end{aligned}$$

Thus, this map sends $1 \mapsto 1, 2 \mapsto 4 \mapsto 5 \mapsto 2$ and... slight slip somewhere?

QUESTION 10

Sketches are given below:

(i) $f_1 = \frac{1}{\alpha} \log z$, where the branch cut is made outside of the angular sector (doesn't matter too much where.)

(ii) $g_1(z) = e^z$ maps our half strip onto the upper half of the unit disc. We now apply $g_2(z) = \frac{z-1}{z+1}, g_3(z) = iz^2, g_2(z)$ in succession. Thus the desired conformal map is $g_2 \circ g_3 \circ g_2 \circ g_1$, illustrated in the sketches below.

(iii)

QUESTION 11

Write $g(z) = e^z = e^x e^{iy}$, so g maps to points with radius e^x and angle given by y. In our strip, as $x \to \pm\infty$, we can get arbitrarily small or large radius, and as $0 < \operatorname{Im} z < \pi$, we can only reach the negative real axis. So this maps our strip to the UHP.

The conformal map is $f(z) = \log(\sin z)$, being careful with our choice of branch cut?

QUESTION 12