Part IA — Variational Principles Example Sheet 2

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QUESTION 1

$$\begin{aligned} F[x+\delta x]-F[x] &= \int_{t_1}^{t_2} f(x+\delta x, \dot{x}+\delta\dot{x}, \ddot{x}+\delta\ddot{x}, t)\ \mathrm{d}t - \int_{t_1}^{t_2} f(t,x,\dot{x},\ddot{x})\ \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left\{ \delta x \frac{\partial f}{\partial x} + (\delta\dot{x})\frac{\partial f}{\partial \dot{x}} + (\delta\ddot{x})\frac{\partial f}{\partial \ddot{x}} \right\}\ \mathrm{d}t + O(t^2) \end{aligned}$$

Discarding the (small)terms of $O(t^2)$, we call the first order variation $\delta F[x]$ and integrating by parts (twice), we have

$$\begin{aligned} \delta F[x] &= \int_{t_1}^{t_2} \left\{ \delta x \left[\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)\right] - (\delta\dot{x})\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \ddot{x}}\right)\right\}\ \mathrm{d}t + \left[\delta x \frac{\partial f}{\partial \dot{x}} + (\delta\dot{x})\frac{\partial f}{\partial \ddot{x}}\right]_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \left\{ \delta x \left[\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right)\right]\right\}\ \mathrm{d}t \\ &\quad + \left[\delta x \left\{\frac{\partial f}{\partial \dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \ddot{x}}\right)\right\} + (\delta\dot{x})\frac{\partial f}{\partial \ddot{x}}\right]_{t_1}^{t_2} \end{aligned}$$

We have fixed end boundary conditions, so $\delta x(t_1) = \delta x(t_2) = 0$ and also $\delta\dot{x}(t_1) = \delta\dot{x}(t_2) = 0$. Thus the boundary term is zero and we can write $\delta F[x]$ in the form

$$\delta F[x] = \int_{t_1}^{t_2} \left\{ \delta x(t) \frac{\delta F[x]}{\delta x(t)} \right\}\ \mathrm{d}t$$

where the *functional derivative* $\frac{\delta F[x]}{\delta x(t)}$ is defined as

$$\frac{\delta F[x]}{\delta x(t)} := \frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right)$$

The functional F is stationary when its functional derivative is zero (assuming that this derivative is defined on (t_1, t_2)) and the condition for this to be true is the following Euler-Lagrange equation:

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) = 0, \qquad t_1 < t < t_2$$

Given the functional

$$L[x] = \int_1^2 t^4[\ddot{x}(t)]^2\ \mathrm{d}t$$

In this case, $f = t^4[\ddot{x}(t)]^2$, so $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \dot{x}} = 0$ and the EL equation can be immediately twice integrated to give

$$\frac{\partial}{\partial \ddot{x}}\left[t^4[\ddot{x}(t)]^2\right] = At + B$$

for some constants A and B.
ie.

$$2t^4(\ddot{x})(t) = At + B$$

$$\Rightarrow \ddot{x}(t) = At^{-3} + Bt^{-4} \qquad t \neq 0$$

$$\Rightarrow \dot{x}(t) = A't^{-2} + B't^{-3} + C$$

Using b.c.s we have

$$-2 = A' + B' + C$$
$$-\frac{1}{4} = \frac{1}{4}A' + \frac{1}{8}B' + C \Rightarrow -2 = 2A' + B' + 8C$$

Subtracting immediately yields $A' = 7C$:

$$-2 = 8C + B'$$
$$-2 = 22C + B'$$

Therefore $C' = 0$, $B' = -2$, and $\dot{x}(n) = -2t^{-3}$.
Integrating, we get

$$x(t) = t^{-2} + D$$

b.c.s $\Rightarrow D = 0$, so $x(t) = \frac{1}{t^2}$.
This function is a global minimum as it is convex.
Note: $\delta^2 F > 0 \not\Longrightarrow$ global min. Need to use convexity. of $\int_1^2 t^4[\ddot{x}(t)]^2 \, \mathrm{d}t$. (Clearly convex as we're just doing linear things, then squaring.)

QUESTION 2

Our aim is to maximise $A[x,y]$ subject to the constraint $P[x,y] = L$, where L is the fixed length and $P[x,y] = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\ \mathrm{d}\theta$

Using a Lagrange multiplier λ to impose this, we seek to maximize the functional

$$\begin{aligned}\phi_\lambda[x,y] &= A[x,y] - \lambda(P[y] - L)\\ &= \int_0^{2\pi} \underbrace{\frac{1}{2}(xy' - yx') - \lambda\sqrt{(x')^2 + (y')^2}}_{=f_\lambda(\mathbf{x},\mathbf{x}')}\ \mathrm{d}\theta + \lambda L\end{aligned}$$

where $\mathbf{x} = (x,y)$. The boundary conditions imply that $x(0) = x(2\pi), x'(0) = x'(2\pi)$ and similarly for y at the endpoints, so boundary terms vanish and functional is stationary for solutions of the E-L equation. $f_\lambda(\mathbf{x},\mathbf{x}')$ has no explicit θ dependence, so considering the y E-L equation we have that:

$$f - x'\frac{\partial f_\lambda}{\partial x'} - y'\frac{\partial f_\lambda}{\partial y'} = \text{ constant}$$

$$\begin{aligned}&\Rightarrow f - y'\left(\frac{1}{2}x - \frac{\lambda y'}{\sqrt{(x')^2 + (y')^2}}\right) - x'\left(-\frac{1}{2}y - \frac{\lambda x'}{\sqrt{(x')^2 + (y')^2}}\right) = \text{constant}\\ &\Rightarrow -\lambda\frac{((x')^2 + (y')^2)}{\sqrt{(x')^2 + (y')^2}} + \lambda\frac{(y')^2}{\sqrt{(x')^2 + (y')^2}} + \lambda\frac{(x')^2}{\sqrt{(x')^2 + (y')^2}} = \text{constant}\\ &\Rightarrow 0 = C\end{aligned}$$

Hmm? Try E-L directly...

QUESTION 3

Using Lagrange multiplier λ, wish to minimize

$$\begin{aligned}\Phi_\lambda[\psi] &= I[\psi] - \lambda\left[\int_{-\infty}^{\infty} \psi^2 \, \mathrm{d}x = 1\right] \\ &= \int_{-\infty}^{\infty} \underbrace{(\psi')^2 + (x^2-\lambda)\psi^2}_{f_\lambda(\psi,\psi';x)} \, \mathrm{d}x + \lambda\end{aligned}$$

Normalisation condition, can assume $\phi = 0$ at endpoints, so the functional is stationary for solutions of the E-L equation. Euler-Lagrange equations imply:

$$2(x^2-\lambda)\psi - \frac{\mathrm{d}}{\mathrm{d}x}\left[2\psi'\right] = 0$$

$$\Rightarrow \psi'' + (x^2-\lambda)\psi = 0$$

Note that

$$\begin{aligned}I[\psi] &= \int_{-\infty}^{\infty} (\psi' + x^2\psi^2)^2 - 2x\psi\psi' \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} (\psi' + x^2\psi^2)^2 \, \mathrm{d}x - \int_{-\infty}^{\infty} x\frac{\mathrm{d}}{\mathrm{d}x}[\psi^2] \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} (\psi' + x^2\psi^2)^2 \, \mathrm{d}x - \left[x\psi^2\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \psi^2 \, \mathrm{d}x \quad \text{by parts} \\ &= \int_{-\infty}^{\infty} (\psi' + x^2\psi^2)^2 \, \mathrm{d}x + 1\end{aligned}$$

As $(\psi' + x^2\psi^2)$ is real valued, its square gives a positive function, thus the integral is positive and $I[\psi] \geq 1$. Equality holds for

$$\begin{aligned}&(\psi' + x\psi) = 0 \\ \Rightarrow\ &\psi' + x\psi = 0 \\ \Rightarrow\ &\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi e^{x^2/2}\right) = 0 \\ \Rightarrow\ &\psi = Ce^{-x^2/2}\end{aligned}$$

for some constant C (which we recognise as the Gaussian Wave Function). The normalisation condition implies that $C = \left(\frac{1}{\pi}\right)^{1/4}$.

Note: We're minimising the energy of a harmonic oscillator. What does λ represent here?

QUESTION 4

Using Lagrange multiplier λ with constraint $|\mathbf{x}| = 1$ wish to minimize

$$\begin{aligned}\Phi_\lambda[\mathbf{x}] &= I[\mathbf{x}] - \lambda(|\mathbf{x}| - 1) \\ &= \int_{t_1}^{t_2} \underbrace{|\dot{\mathbf{x}}|^2 - \lambda|\mathbf{x}|}_{f_\lambda(\mathbf{x},\dot{\mathbf{x}};t)} \, \mathrm{d}t + \lambda\end{aligned}$$

The E-L equation for x_i component is $\frac{\partial f}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}_i}\right) = 0$, giving

$$-\frac{\lambda x_i}{|\mathbf{x}|} - \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{x}_i) = 0$$

$$2\ddot{x}_i + \lambda x_i = 0 \qquad \text{as } |\mathbf{x}| = 1$$

$$\Rightarrow \ddot{\mathbf{x}} + \frac{\lambda}{2}\mathbf{x} = 0$$

Also, there is no t dependence in f_λ, so E-L equations imply that

$$f - \dot{x}_i \frac{\partial f}{\partial \dot{x}_i} = \text{ constant}$$

$$f - 2\dot{x}_i^2 = \text{ constant}$$

Not too sure what to do with these.

QUESTION 5

Have the Lagrangian

$$L = \underbrace{\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\dot{\theta}^2}_{T} + \underbrace{mga\cos\theta}_{-V}$$

Note that $\frac{\partial L}{\partial \phi} = 0$ so we have the first integral

$$\text{const.} \; = \frac{\partial L}{\partial \dot{\phi}} = ma^2 \sin^2\theta$$

Note too that $\frac{\partial L}{\partial t} = 0$ (no t dependence) so we have another first integral

$$\begin{aligned}\text{const.} \; &= L - \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} \\ &= -T - V\end{aligned}$$

from which we deduce that

$$\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\dot{\theta}^2 - mga\cos\theta = E$$

for the total energy.

The Hamiltonian is defined as the Legendre transform of the Lagrangian with respect to velocity $\mathbf{v} = \dot{\mathbf{x}}$:

$$H(\mathbf{x}, \mathbf{p}; t) = [\mathbf{p}\cdot\mathbf{v} - L(\mathbf{x}, \mathbf{v})]_{\mathbf{v}=\mathbf{v}(\mathbf{p})}$$

where $\mathbf{v}(\mathbf{p})$ is the solution to $\frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$.

The momentum p_θ is given by

$$\begin{aligned}p_\theta &= \frac{\partial L}{\partial \dot{\theta}} \\ &= ma^2\dot{\theta}\end{aligned}$$

Similarly,

$$\begin{aligned}p_\phi &= \frac{\partial L}{\partial \dot{\phi}} \\ &= ma^2(\sin^2\theta)\dot{\phi}\end{aligned}$$

Thus $\mathbf{v} = (\dot{\theta}, \dot{\phi}) = \left(\frac{p_\theta}{ma^2}, \frac{p_\phi}{ma^2\sin^2\theta}\right)$ and

$$\begin{aligned}H &= \frac{p_\theta^2}{ma^2} + \frac{p_\phi^2}{ma^2\sin^2\theta} - \left(\frac{1}{2}ma^2\left(\frac{p_\theta}{ma^2}\right)^2 + \frac{1}{2}ma^2\left(\frac{p_\phi}{ma^2\sin^2\theta}\right)^2 + mga\cos\theta\right) \\ &= \frac{1}{2}\frac{p_\theta^2}{ma^2} + \frac{1}{2}\frac{p_\phi^2}{ma^2\sin^2\theta} - mga\cos\theta\end{aligned}$$

Hamilton's equations are given by

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}$$

ie.

$$\dot{\theta} = \frac{p_\theta}{ma^2}$$
$$\dot{\phi} = \frac{p_\phi}{ma^2 \sin^2 \theta}$$
$$\dot{p_\theta} = \frac{-p_\phi^2 \sin(2\theta)}{2ma^2 \sin^4 \theta} - mga \cos \theta$$
$$\dot{p_\phi} = 0$$

QUESTION 6

(i) Consider the variation directly, $u_t \to u_t + (\delta u)_t$ and $u_x \to u_x + (\delta u)_x$. Then

$$\begin{aligned}
\delta I[u] &= I[u+\delta u] - I[u] \\
&= \int \left[\frac{1}{2}\left(u_t + (\delta u)_t\right)^2 - F(u_x + (\delta u)_x)\right] \mathrm{d}x\ \mathrm{d}t - \int \left[\frac{1}{2}u_t^2 - F(u_x)\ \mathrm{d}x\ \mathrm{d}t\right] \\
&= \int u_t(\delta u)_t - (\delta u)_x \frac{\partial F(u_x)}{\partial u_x}\ \mathrm{d}x\ \mathrm{d}t + O(t^2) \\
&= \int u_{tt}(\delta u) - (\delta u)\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F(u_x)}{\partial u_x}\right] \mathrm{d}x\ \mathrm{d}t + \text{boundary terms} \qquad \text{(by parts)}
\end{aligned}$$

ignoring boundary terms

So the Euler-Lagrange equation is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F(u_x)}{\partial u_x}\right] = 0$$

ie $u_{tt} = u_{xx}\frac{\partial^2 F}{\partial u_x^2}$

(ii) Discarding second order terms throughout:

$$\begin{aligned}
\delta I[u] &= \int \left[(u_x + (\delta u)_x)^2 + (u_y + (\delta u)_y)^2 + e^{2(u+\delta u)}\right] \mathrm{d}x\ \mathrm{d}y - \int u_x^2 + u_y^2 + e^{2u}\ \mathrm{d}x\ \mathrm{d}y \\
&= \int 2u_x(\delta u)_x + 2u_y(\delta u)_y + \left(e^{2(u+\delta u)} - e^{2u}\right) \mathrm{d}x\ \mathrm{d}y + O(u^2) \\
&= \int 2u_x(\delta u)_x + 2u_y(\delta u)_y + e^{2u}\left(1 + 2\delta u\right) - e^{2u}\ \mathrm{d}x\ \mathrm{d}y \qquad \text{using } e^x \approx 1 + x \\
&= 2\int (\delta u)\{u_{xx} + u_{yy} + 2e^{2u}\}\ \mathrm{d}x\ \mathrm{d}y
\end{aligned}$$

So E-L equation is

$$\nabla^2 u = -2e^u$$

QUESTION 7

The E-L equations are of the form $\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$

We have

$$\frac{\partial L}{\partial x_i} = -q\nabla_i \phi + q\frac{\partial}{\partial x_i}(\mathbf{v}\cdot\mathbf{A})$$

and

$$\frac{\partial L}{\partial \dot{x}_i} = m\gamma v_i + qA_i$$

E-L equations imply

$$\begin{aligned} &-q\nabla_i\phi + q\frac{\partial}{\partial x_i}(\mathbf{v}\cdot\mathbf{A}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma v_i + qA_i\right) = 0 \\ \Rightarrow &\frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma v_i\right) = -q\nabla_i\phi + q\frac{\partial}{\partial x_i}(\mathbf{v}\cdot\mathbf{A}) - q\frac{\mathrm{d}}{\mathrm{d}t}(A_i) \end{aligned}$$

Using the chain rule:

$$\begin{aligned} \frac{\mathrm{d}A_i}{\mathrm{d}t} &= \frac{\partial A_i}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial A_i}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}t} \\ &= \frac{\partial A_i}{\partial t} + v_j\frac{\partial A_i}{\partial x_j} \end{aligned}$$

and

$$\frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A}) = v_j\frac{\partial A_j}{\partial x_i}$$

$$\begin{aligned} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma v_i\right) &= -q\nabla_i\phi + qv_j\frac{\partial A_j}{\partial x_i} - q\left[\frac{\partial A_i}{\partial t} + v_j\frac{\partial A_i}{\partial x_j}\right] \\ &= q\left(\underbrace{\nabla_i\phi - \frac{\partial A_i}{\partial t}}_{E_i} + \underbrace{\left[v_j\frac{\partial A_j}{\partial x_i} - v_j\frac{\partial A_i}{\partial x_j}\right]}_{(*)}\right) \end{aligned}$$

Lastly, want to show (*) is equal to $[\mathbf{v}\times\mathbf{B}]_i = [\mathbf{v}\times\nabla\times A]_i$

$$\begin{aligned} [\mathbf{v}\times\nabla\times A]_i &= \varepsilon_{ijk}v_j[nabla \times A]_k \\ &= \varepsilon_{kij}\varepsilon_{kpq}v_j\frac{\partial A_q}{\partial x_p} \\ &= \delta_{ip}\delta_{jq}v_j\frac{\partial A_q}{\partial x_p} - \delta_{iq}\delta_{jp}v_j\frac{\partial A_q}{\partial x_p} \\ &= \frac{\partial A_i}{\partial t} + v_j\frac{\partial A_i}{\partial x_j} \end{aligned}$$

as required.

QUESTION 8

$$S[\rho, \mathbf{v}, \phi] = \int \mathrm{d}t \int \mathrm{d}^3x \, \left\{ \frac{1}{2}\rho|\mathbf{v}|^2 - u(\rho) + \phi[\dot{\rho} + \nabla \cdot (\rho\mathbf{v})] \right\}$$

Note we need a Lagrange multiplier field $\phi(t, \mathbf{x})$ to impose the condition at an infinite number of points. This looks like an action for sure (KE + PE). We will have 2 E-L equations, one from varying ρ and the other from varying $\mathbf{v}$.

Taking variations directly,

$$\begin{aligned} S[\rho + \delta\rho, \mathbf{v}, \phi] &= S[\rho, \mathbf{v}, \phi] + \int \mathrm{d}t \int \mathrm{d}^3x \, \left\{ \frac{1}{2}\delta\rho|\mathbf{v}|^2 - \delta\rho\frac{\partial u}{\partial \rho} + \phi[\dot{\delta\rho} + \nabla \cdot (\delta\rho\mathbf{v})] \right\} \\ &= \end{aligned}$$

QUESTION 9

QUESTION 10

QUESTION 11