Part IB — Methods Example Sheet 1

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QUESTION 1

$$\frac{f(x_+ + f(x_-))}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

For $f(x) = (x-1)^2$ on the interval $-1 \leq x \leq 1$, $f(x)$ is an even function, thus $b_n = 0$. We have $L = 1$, and

$$\begin{aligned}
\frac{1}{2}a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\ \mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} x^4 - 2x^2 + 1\ \mathrm{d}x \\
&= \int_{0}^{1} x^4 - 2x^2 + 1\ \mathrm{d}x \\
&= \frac{8}{15}
\end{aligned}$$

and

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \\
&= \int_{-1}^{1} x^4 \cos n\pi x\ \mathrm{d}x - 2\int_{-1}^{1} x^2 \cos n\pi x\ \mathrm{d}x + \int_{-1}^{1} \cos n\pi x\ \mathrm{d}x
\end{aligned}$$

Evaluating each integral separately, we have:

(i)

$$\int_{-1}^{1} \cos n\pi x\ \mathrm{d}x = \left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{-1} = 0$$

as $\sin n\pi x = 0 \ \forall\ n$

(ii) By parts,

$$\begin{aligned}
\int_{-1}^{1} x^2 \cos n\pi x\ \mathrm{d}x &= \left[\frac{x^2 \sin n\pi x}{n\pi} - \frac{1}{n\pi}\int 2x \sin n\pi x\ \mathrm{d}x\right]_{-1}^{1} \\
&= -\frac{2}{n\pi}\int_{-1}^{1} x \sin n\pi x\ \mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\int_{-1}^{1} x \sin n\pi x\ \mathrm{d}x &= \left[\frac{-x\cos n\pi x}{n\pi} + \frac{1}{n\pi}\int \cos n\pi x\ \mathrm{d}x\right]_{-1}^{1} \\
&= \frac{-2\cos n\pi x}{(n\pi)^2}
\end{aligned}$$

Thus the second integral contributes to give

$$-\frac{8cosn\pi x}{(n\pi)^2}$$

(iii)

$$\begin{aligned}\int_{-1}^{1} x^4 \cos n\pi x \, \mathrm{d}x &= \left[\frac{x^4 \sin n\pi x}{n\pi} - \frac{1}{n\pi}\int 4x^3 \sin n\pi x \, \mathrm{d}x\right]_{-1}^{1} \\ &= -\frac{4}{n\pi}\int_{-1}^{1} x^3 \sin n\pi x \, \mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{1} x^3 \sin n\pi x \, \mathrm{d}x &= \left[\frac{-x^3 \cos n\pi x}{n\pi} + \frac{1}{n\pi}\int 3x^2 \cos n\pi x\right]_{-1}^{1} \\ &= \frac{-2\cos n\pi}{n\pi} + \frac{3}{n\pi}\int_{-1}^{1} x^2 \cos n\pi x \, \mathrm{d}x\end{aligned}$$

Whence

$$\begin{aligned}\int_{-1}^{1} x^4 \cos n\pi x \, \mathrm{d}x &= \frac{8\cos n\pi}{n\pi} - \frac{12}{(n\pi)^2}\int_{-1}^{1} x^2 \cos n\pi x \, \mathrm{d}x \\ &= \frac{8\cos n\pi}{n\pi} - \frac{48\cos n\pi}{(n\pi)^4}\end{aligned}$$

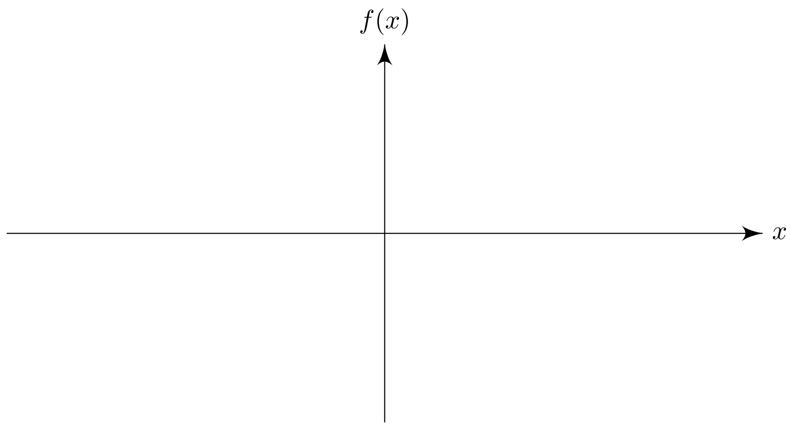
using (ii).

Finally,

$$\begin{aligned}a_n &= -\frac{48\cos n\pi}{(n\pi)^4} \\ &= \frac{48(-1)^{n+1}}{(n\pi)^4}\end{aligned}$$

as $\cos n\pi x = (-1)^n$
Hence the Fourier Series is given by

$$\begin{aligned}f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \\ &= \frac{8}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos n\pi x\end{aligned}$$



$f(x)$ satisfies the Dirichlet conditions. The 1$^{\text{st}}$ derivative is the lowest derivative which is discontinuous (at the endpoints, as $f(x)$ even fn $\Rightarrow$ $f'(x)$ odd), so Fourier coefficients are $\mathcal{O}(\frac{1}{n^2})$ as $n \to \infty$

QUESTION 2

Extending on range $(-\pi, \pi)$ so $L = \pi$ and

(a)

$$\frac{f(x_+ + f(x_-))}{2} = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,\mathrm{d}x \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int_0^{\pi} x^2 \sin nx\,\mathrm{d}x &= \left[\frac{-x^2\cos nx}{n} + \frac{1}{n}\int 2x\cos nx\,\mathrm{d}x\right]_0^{\pi} \\ &= \frac{-\pi^2\cos n\pi}{n} + \frac{2}{n}\int_0^{\pi} x\cos nx\,\mathrm{d}x \end{aligned}$$

and once again,

$$\begin{aligned} \int_0^{\pi} x\cos nx\,\mathrm{d}x &= \left[\frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx\,\mathrm{d}x\right]_0^{\pi} \\ &= -\frac{1}{n}\int_0^{\pi}\sin nx\,\mathrm{d}x \\ &= -\frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi} \\ &= \frac{1}{n^2}(\cos n\pi - 1) \end{aligned}$$

Back substituting in,

$$\begin{aligned} b_n &= \frac{2}{\pi}\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2}{n^3}(\cos n\pi - 1)\right) \\ &= \frac{2}{\pi n^3}\left(-2 + (2 - (\pi n)^2)\cos n\pi\right) \end{aligned}$$

Hence Fourier sine series given by:

$$\begin{aligned} f(x)_s &= \sum_{n=1}^{\infty} \frac{2}{\pi n^3}\left(-2 + (2-(\pi n)^2)(-1)^n\right)\sin nx \\ &= \sum_{n=1}^{\infty}\left\{\frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{n^3\pi}\right\}\sin nx \end{aligned}$$

(b) Similarly,

$$\frac{f(x_+ + f(x_-))}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$\begin{aligned}\frac{a_0}{2} &= \frac{1}{L}\int_0^L f(x)\ \mathrm{d}x \\ &= \frac{1}{\pi}\int_0^\pi x^2\ \mathrm{d}x \\ &= \frac{\pi^2}{3}\end{aligned}$$

and

$$\begin{aligned}a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\pi x^2 \cos nx\ \mathrm{d}x\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_0^\pi x^2 \cos nx\ \mathrm{d}x &= \left[\frac{x^2 \sin nx}{n} - \frac{1}{n}\int 2x \sin nx\ \mathrm{d}x\right]_0^\pi \\ &= \frac{-2}{n}\int_0^\pi x \sin nx\ \mathrm{d}x\end{aligned}$$

and once again,

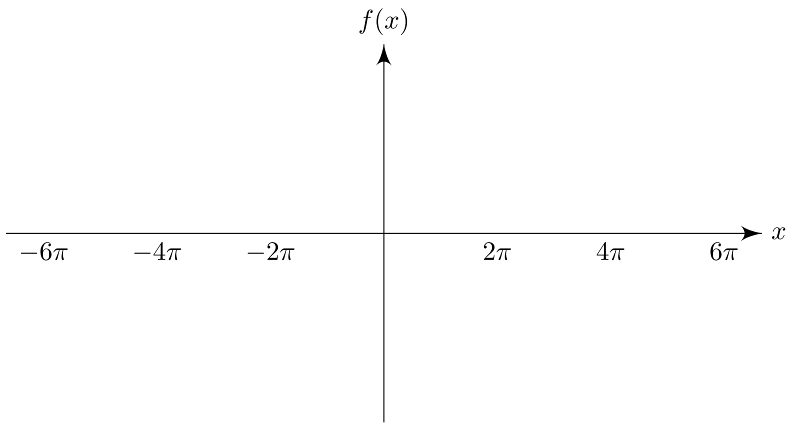
$$\begin{aligned}\int_0^\pi x \sin nx\ \mathrm{d}x &= \left[\frac{-x\cos nx}{n} + \frac{1}{n}\int \cos nx\ \mathrm{d}x\right]_0^\pi \\ &= \frac{-\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n\pi}\right]_0^\pi \\ &= \frac{-\pi\cos n\pi}{n}\end{aligned}$$

Thus

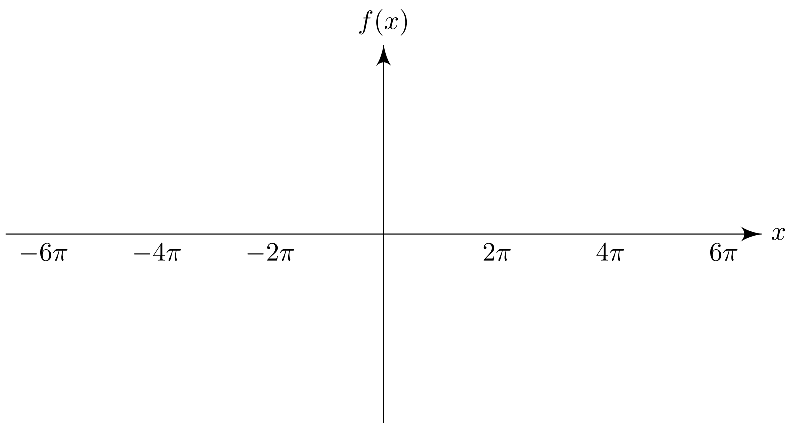
$$a_n = \frac{4}{n^2}\cos n\pi$$

and the Fourier cosine series is given by

$$f(x) = \frac{\pi^2}{3} + \sum_{n=0}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$



(i)



(ii)

Fourier series for $g(x) = 2x$ (odd function) in the range $(-\pi, \pi)$ given by

$$\frac{f(x_+ + f(x_-))}{2} = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_{-\pi}^{\pi} x \sin nx \, \mathrm{d}x \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int_{-\pi}^{\pi} x \sin nx \, \mathrm{d}x &= \left[\frac{-x \cos nx}{n} + \frac{1}{n} \int \cos nx \, \mathrm{d}x\right]_{-\pi}^{\pi} \\ &= \frac{-2\pi \cos n\pi}{n} + \frac{1}{n} \left[\frac{\sin nx}{n\pi}\right]_{-\pi}^{\pi} \\ &= \frac{2\pi(-1)^{n+1}}{n} \end{aligned}$$

Whence

$$g(x) = \sum_{n=1}^{\infty} \frac{4\pi^2(-1)^{n+1}}{n^2} \sin nx$$

Fourier series for $h(x) = 2|x|$ (even function) in the range $(-\pi, \pi)$ given by

$$\frac{f(x_+ + f(x_-))}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$\begin{aligned}
\frac{1}{2}a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2|x|\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x \\
&= \pi
\end{aligned}$$

and

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,\mathrm{d}x
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
\int_{-\pi}^{\pi} |x|\cos nx\,\mathrm{d}x &= 2\int_{0}^{\pi} x\cos nx\,\mathrm{d}x \\
&= 2\left[\frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx\,\mathrm{d}x\right]_0^{\pi} \\
&= -\frac{2}{n}\int_0^{\pi}\sin nx\,\mathrm{d}x \\
&= -\frac{2}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi} \\
&= \frac{2}{n^2}(\cos n\pi - 1)
\end{aligned}$$

Whence

$$h(x) = \pi + \sum_{n=1}^{\infty} \frac{4[(-1)^n - 1]}{n^2\pi}\cos nx$$

Note that differentiating the Fourier sine series for x^2 gives

$$\frac{\mathrm{d}}{\mathrm{d}x}[f_s(x)] = \sum_{n=1}^{\infty}\left\{2\pi(-1)^{n+1} + \frac{4[(-1)^n - 1]}{n^2\pi}\right\}\cos nx$$

These don't quite match up: what is $\sum_{n=1}^{\infty} 2\pi(-1)^{n+1}\cos nx$ the Fourier series for?

Note that the cos coefficients $a_n = O(1)$ as $n \to \infty$, so this function is terrible. Using the direchlet conditions, $a_n = O(\frac{1}{n})$, so f is discontinuous.

This motivates us to check the Fourier series for the dirac delta function $\delta(x)$, with period $(-\pi, \pi)$

$$\delta(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

We find that

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-pi}^{\pi} \delta(x)\cos nx \, \mathrm{d}x \\ &= \frac{\cos 0}{\pi} \\ &= \frac{1}{\pi} \end{aligned}$$

and similarly

$$b_n = \frac{\sin 0}{\pi} = 0$$

ie.

$$\delta(x) \sim \sum_{n=1}^{\infty} \frac{1}{\pi}\cos nx + \frac{1}{2\pi}$$

This isn't quite what we wanted, but making a small adjustment:

$$\delta(x-\pi) : a_n = \frac{\cos n\pi}{\pi} = \frac{(-1)^n}{\pi} \quad b_n = 0$$

Finally, we conclude that

$$f_s'(x) = 2|x| - 2\pi^2\delta(x-\pi)$$

I guess the morale is, don't differentiate term by term if f is discontinuous...

QUESTION 3

$f(x) = e^x$ on $(-\pi, \pi)$ has Fourier series given by

$$\frac{f(x_+ + f(x_-))}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$\begin{aligned}\frac{1}{2}a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \,\mathrm{d}x \\ &= \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) \\ &= \frac{1}{\pi}\sinh \pi\end{aligned}$$

and

$$\begin{aligned}a_n &= \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} e^x \cos nx \,\mathrm{d}x}_{I_a} \\ I_a &= \left[e^x \cos nx + \int e^x n \sin nx \,\mathrm{d}x\right]_{-\pi}^{\pi} \\ &= (e^{\pi} - e^{-\pi})\cos n\pi + n\int_{-\pi}^{\pi} e^x \sin nx \,\mathrm{d}x \\ &= 2\sinh \pi\ (-1)^n + n\left[e^x \sin nx - \int e^x n \cos x \,\mathrm{d}x\right]_{-\pi}^{\pi} \\ &= 2\sinh \pi\ (-1)^n + -n^2\int_{-\pi}^{\pi} e^x \cos nx \,\mathrm{d}x \\ &= 2\sinh \pi\ (-1)^n + -n^2 I_a\end{aligned}$$

Hence

$$a_n = \frac{1}{\pi}I_a \qquad I_a = \frac{2}{1+n^2}\sinh \pi(-1)^n$$

Also,

$$b_n = \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} e^x \sin nx \,\mathrm{d}x}_{I_b}$$

$$\begin{aligned}I_b &= \left[e^x \sin nx - \int e^x n \cos nx \,\mathrm{d}x\right]_{-\pi}^{\pi} \\ &= -nI_a\end{aligned}$$

$$b_n = -\frac{n}{\pi}I_a$$

Combining these results, the Fourier series for e^x is given by

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\sinh\pi + \sum_{n=1}^{\infty}\left[\left(\frac{1}{\pi}\cos nx - \frac{n}{\pi}\sin nx\right) I_a\right] \\
&= \frac{1}{\pi}\sinh\pi + \frac{2}{\pi}\sinh\pi\sum_{n=1}^{\infty}\left[(\cos nx - n\sin nx)\frac{(-1)^n}{1+n^2}\right]
\end{aligned}$$

Setting $x = \pi$ yields

$$e^{\pi} = \frac{1}{\pi}\sinh\pi + \frac{2}{\pi}\sinh\pi\sum_{n=1}^{\infty}\frac{1}{1+n^2}$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi e^{\pi} - \sinh\pi}{2\sinh\pi}$$

Setting $x = -\pi$ similarly yields

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi e^{-\pi} - \sinh\pi}{2\sinh\pi}$$

Adding and dividing by two,

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{1+n^2} &= \frac{\pi(e^{\pi} + e^{-\pi}) - 2\sinh\pi}{4\sinh\pi} \\
&= \frac{2\pi\cosh\pi - 2\sinh\pi}{4\sinh\pi} \\
&= \frac{1}{2}(\pi\coth\pi - 1)
\end{aligned}$$

QUESTION 4

(i) Reposing the Fourier Series of $f(t)$ using complex variables,

$$\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \left(e^{\frac{in\pi t}{L}} + e^{\frac{-in\pi t}{L}} \right) + \frac{b_n}{2i} \left(e^{\frac{in\pi t}{L}} - e^{\frac{-in\pi t}{L}} \right) \right] \\
&= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi t}{L}}, \\
c_n &= \frac{a_n - ib_n}{2} \; n > 0; \\
c_{-n} &= \frac{a_n + ib_n}{2} \; n > 0; \\
c_0 &= \frac{a_0}{2}
\end{aligned}$$

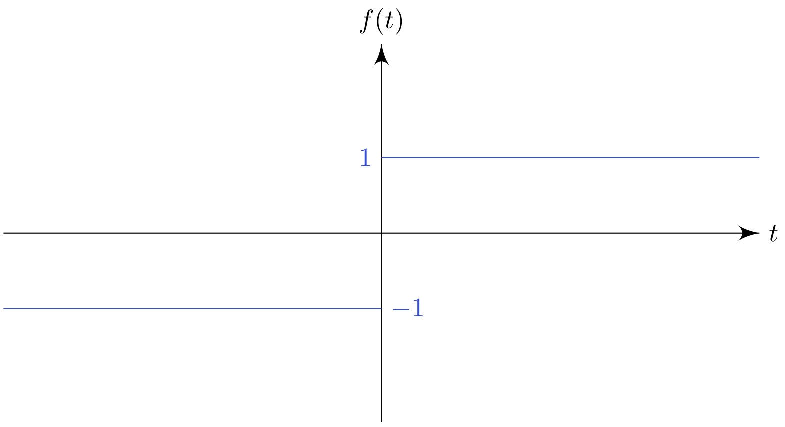
Using the orthogonality of complex exponentials and the properties of complex Fourier coefficients, we deduce that

$$\begin{aligned}
\int_{-L}^{L} [f(t)]^2 \; \mathrm{d}t &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \int_{-T}^{T} \exp\left[\frac{i\pi t(n+m)}{L} \right] \; \mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m 2T \delta_{n[-m]} \\
&= 2T \sum_{n=-\infty}^{\infty} c_n c_{-n} \\
&= 2T \sum_{n=-\infty}^{\infty} c_n c_n^* \\
&= 2T \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}$$

This can be then re-expressed in terms of the a_n and b_n as

$$\int_{-L}^{L} [f(t)]^2 \; \mathrm{d}t = L \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

as required.



(ii)

The unit amplitude square wave has Fourier series (odd function)

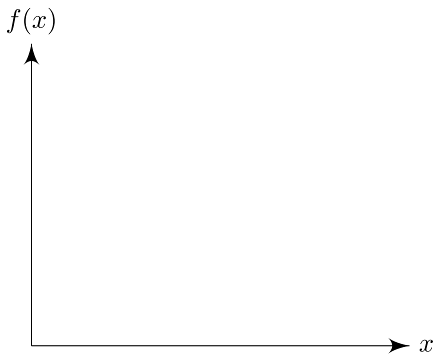
$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T}\right)$$

Frequencies less than $\frac{9}{2}\pi T^{-1}$ correspond to terms in the Fourier series with $\frac{n\pi}{T} < \frac{9}{2}\pi T^{-1}$, ie. $n = 1, 2, 3, 4$.

Also,

$$\begin{aligned} b_n &= \frac{1}{T}\int_{-T}^{T} \\ &= \end{aligned}$$

QUESTION 5



(i)

$f(x)$ on $(0, 2\pi)$ has Fourier series given by

$$\frac{f(x_+ + f(x_-))}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$\begin{aligned}
\frac{1}{2}a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x)\ \mathrm{d}x \\
&= \frac{1}{2\pi} \int_\pi^{2\pi} 1\ \mathrm{d}x \\
&= \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\ \mathrm{d}x \\
&= \frac{1}{\pi} \int_\pi^{2\pi} \cos nx\ \mathrm{d}x \\
&= \frac{1}{\pi} \left[\frac{1}{n} \sin nx \right]_\pi^{2\pi} \\
&= 0
\end{aligned}$$

and

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \,\mathrm{d}x \\
&= \frac{1}{\pi}\int_\pi^{2\pi} \sin nx \,\mathrm{d}x \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_\pi^{2\pi} \\
&= -\frac{1}{\pi n}\left[\cos nx\right]_\pi^{2\pi} \\
&= 0 \text{ if } n \text{ even or } -\frac{2}{n\pi} \text{ if } n \text{ odd}
\end{aligned}
$$

Hence

$$
f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}
$$

(ii) Taking the hint, differentiating term by term gives

$$
\frac{\mathrm{d}}{\mathrm{d}x}[S_n(x)] = \frac{2}{\pi}\sum_{n=1}^{N}\cos(2n-1)x
$$

Now

$$
\begin{aligned}
\sum_{n=1}^{N}\cos(2n-1)x &= \mathrm{Re}\left[\sum_{n=1}^{N} e^{(2n-1)i}\right] \\
&= \mathrm{Re}\left[\frac{e^{(2N+1)i} - e^{i}}{e^{2i}-1}\right]
\end{aligned}
$$

QUESTION 6

Assuming the solution takes the form $y \propto e^{\sigma x}$,, we have

$$y(x) = A\cos(\mu x) + B\sin(\mu x)$$

where A and B are constants, and $\mu^2 = \lambda$. Applying the boundary conditions, $y(0) = 0$ implies that $A = 0$. The other boundary condition implies

$$B\sin\mu + B\mu\cos\mu = 0$$

$$\mu = -\tan\mu$$

This eigenvalue equation has an infinite number of solutions, μ_n (and hence there an infinite number of positive eigenvalues $\lambda_n = \mu_n^2$).

As $n \to \infty$, $\mu_n \to \infty$, so μ is close to an odd multiple of $\frac{\pi}{2}$, ie. $\mu_n \approx (2n+1)\pi/2$, and hence $\lambda_n \approx (2n+1)^2\pi^2/4$

QUESTION 7

(i) $p(x) = \exp\left(\int^x \frac{-2u}{1-u^2}\,\mathrm{d}u\right) = (1-x^2)$, thus integrating factor is $-\frac{1}{1-x^2}\left((1-x^2)\right) = -1$. We can then rewrite the equation as

$$-(1-x^2)y'' + 2xy' - n(n+1)y = 0$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right) - n(n+1)y = 0$$

(ii)

$$\begin{aligned}
p(x) &= \exp\left(\int^x \frac{(1+a+b)u-c}{u(u-1)}\,\mathrm{d}u\right) \\
&= \exp\left(\int^x \frac{c}{u} + \frac{1+a+b-c}{u-1}\,\mathrm{d}u\right) \\
&= \exp\left(c\log x + (1+a+b-c)\log(x-1)\right) \\
&= x^c + (x-1)^{1+a+b-c}
\end{aligned}$$

Thus the required integrating factor is

$$-\frac{x^c + (x-1)^{1+a+b-c}}{x(x-1)}$$

The equation becomes

$$-(x^c+(x-1)^{1+a+b-c})y'' + -[(1+a+b)x-c]\frac{x^c+(x-1)^{1+a+b-c}}{x(x-1)}y' - \frac{x^c+(x-1)^{1+a+b-c}}{x(x-1)}aby = 0$$

which, in Sturm-Liouville form, is

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^c+(x-1)^{1+a+b-c})\frac{\mathrm{d}y}{\mathrm{d}x}\right] - \frac{x^c+(x-1)^{1+a+b-c}}{x(x-1)}aby = 0$$

(iii) Self-adjoint form, integrating factor $-e^{4x}$,

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{4x}\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 4e^{4x}y = \lambda e^{4x}y$$

weight function is hence e^{4x}.

Easier to consider original equation; assuming the solution takes the form $y \propto e^{\sigma x}$, σ satisfies the auxillary equation

$$\sigma^2 + 4\sigma + 4 + \lambda = 0 \Rightarrow \sigma = -2 \pm i\sqrt{\lambda},$$

$$y(x) = Ae^{-2x}\cos(\mu x) + Be^{-2x}\sin(\mu x)$$

where A and B are constants, and $\mu^2 = \lambda$. Applying the boundary conditions, $y(0) = 0$ implies that $A = 0$. The other boundary condition implies

$$\begin{aligned} &Be^{-2}\sin\mu = 0 \\ &\Rightarrow \mu = n\pi \end{aligned}$$

Thus infinite positive eigenvalues $\lambda_n = n^2\pi^2$

The associated eigenvectors are thus proportional to $e^{-2x}\sin(n\pi x)$.

Eigenvectors associated with distinct eigenvalues are indeed orthogonal on the interval, if the weight function e^{4x} is correctly included in the inner product integral I_{mn}, $(m \neq n)$ defined as

$$I_{mn} = \int_0^1 e^{4x} Y_n(x) Y_m(x)\,\mathrm{d}x$$

where Y_n are Y_m are normalized eigenfunction with distinct eigenvalues $\lambda_n = n^2\pi^2$ and $\lambda_m = m^2\pi^2$

QUESTION 8

(i) Using L to denote the operator,

$$L := \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

the Sturm-Liouville form is

$$Lu = -\lambda x u, \quad 0 < x < 1$$

hence with weight function x.

We seek a linear substitution to turn this into Bessel's equation of order zero. We cannot make a substitution for u as the linearity of the operator makes this redundant; after trying a few things, we see that $x = \frac{z}{\sqrt{\lambda}}$ is the way forward. Turns it into

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)y = -zy$$

which is Bessel's equation, and we are told the general solution is of the form

$$y(z) = AJ_0(z) + B[R(z) + J_0(z)\log(z)]$$

where A, B are constants and $R(z)$ is a 'regular function'.

Now we have $u(x)$ bounded as $x \to 0$, same must be true for $y(z)$ as $z \to 0$. From the series definition of J_0, we know that $J_0(0) = 1$. Hence as $\log(z) \to -\infty$ as $z \to 0$, we have $B = 0$, concluding that $y(z) = AJ_0(z)$, ie.

$$u(x) = AJ_0(\sqrt{\lambda}x)$$

and $u(1) = 0 \Rightarrow AJ_0(\sqrt{\lambda}) = 0$, so $\sqrt{\lambda} = j_0$ for $n = 1, 2, \cdots$.

Thus the operator L has eigenfunctions $u_n(x) = J_0(j_n x)$ with eigenvalues $\lambda_n = j_n^2$.

(ii) Acting the operator L on its eigenfunction $J_0(\alpha x)$ given

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)J_0(\alpha x) = -\alpha^2 x J_0(\alpha x)$$

Multiplying by $J_0(\beta x)$ and integrating gives

$$\int_0^1 J_0(\beta x)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)J_0(\alpha x)\,\mathrm{d}x = -\alpha^2\int_0^1 xJ_0(\alpha x)J_0(\beta x)\,\mathrm{d}x$$

$\cdots$

Next part: setting $\alpha = j_n$, $\beta = j_m$, we note that $J_0(j_n) = J_0(j_m) = 0$, thus the identity follows.

Next: note that our first result is only valid for when $\beta \neq \alpha$. So we set $\alpha = j_n$, $\beta = j_n + \varepsilon$, and the result should pop out.

(iii) Summarising what we have so far.

$$L := \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) \qquad \text{with weight } w = x$$

$$\lambda_n = j_n^2, \qquad u_n(x) = J_0(j_n x)$$

$$\text{orth. relation: } \int_0^1 xJ_0(j_n x)J_0(j_m x)\,\mathrm{d}x = \frac{1}{2}\left[J_0'(j_n)\right]^2 \delta_{mn}$$

To solve

$$Lu + \tilde{\lambda}xu = xf(x) \qquad (*)$$

Seek eigenfunction expansions

$$u = \sum_{n=1}^{\infty} a_n J_0(j_n x) \qquad f(x) = \sum_{n=1}^{\infty} b_n J_0(j_n x)$$

Substitute into $(*)$

$$\sum_{n=1}^{\infty} a_n \underbrace{L[J_0(j_n x)]}_{-j_n^2 x J_0(j_n x)} + \tilde{\lambda}x\sum_{n=1}^{\infty} a_n J_0(j_n x) = x\sum_{n=1}^{\infty} b_n J_0(j_n x)$$

Comparing coefficients (note how the x makes this easy)

$$\begin{aligned} &-a_n j_n^2 + \tilde{\lambda}a_n = b_n \\ &\Rightarrow a_n = \frac{b_n}{\tilde{\lambda} - j_n^2} \end{aligned}$$

Noting that $\tilde{\lambda}$ is not an eigenvalue, ie. $\tilde{\lambda} \neq \lambda_n = j_n^2$.

To find the eigenfunction expansion of u it remains to find b_n st.

$$f(x) = \sum_{n=1}^{\infty} b_n J_0(j_n x)$$

Multiply by $xJ_0(j_n x)$ and integrate, thus

$$\begin{aligned}\int_0^1 xf(x)J_0(j_m x)\,\mathrm{d}x &= \sum_{n=1}^{\infty} b_n \int_0^1 xJ_0(j_n x)J_0(j_m x)\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} b_n \frac{1}{2}[J_0'(j_n)]^2\delta_{mn} \\ &= b_m \frac{1}{2}[J_0'(j_m)]^2\end{aligned}$$

ie.

$$b_n = \frac{2\int_0^1 tf(t)J_0(j_m t)\,\mathrm{d}t}{[J_0 1(j_n)]^2}$$

Hence

$$u(x) = 2\sum_{n=1}^{\infty} \frac{\int_0^1 tf(t)J_0(j_m t)\,\mathrm{d}t}{[J_0 1(j_n)]^2(\tilde{\lambda} - j_n^2)} J_0(j_n x)$$

QUESTION 9