Part IB — Numerical Analysis Example Sheet 3

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QUESTION 1

(use col pivoting)

First, $\mathbf{u}_1^T$ is just the first row of A, ie $(10, 6, -2, 1)$, and $\mathbf{l}_1$ is the first column of A scaled so that $L_{1,1} = 1$, ie. $(1, 1, -\frac{1}{5}, \frac{1}{10})$. Calculating

$$\mathbf{l}_1\mathbf{u}_1^T = \begin{pmatrix} 10 & 6 & -2 & 1 \\ 10 & 6 & -2 & 1 \\ -2 & -\frac{6}{5} & \frac{2}{5} & -\frac{1}{5} \\ 1 & \frac{3}{5} & -\frac{1}{5} & \frac{1}{10} \end{pmatrix}$$

Now

$$\begin{aligned} \mathbf{A}_1 &:= A - \mathbf{l}_1\mathbf{u}_1^T \\ &= \begin{pmatrix} 0 & & & \\ & 4 & -3 & -1 \\ & \frac{16}{5} & -\frac{12}{5} & \frac{6}{5} \\ & \frac{12}{5} & -\frac{9}{5} & \frac{29}{10} \end{pmatrix} \end{aligned}$$

And so $\mathbf{u}_2^T = (0, 4, -3, -1)$, $\mathbf{l}_2 = (0, 1, \frac{4}{5}, \frac{3}{5})$.
Next,

$$\mathbf{l}_2\mathbf{u}_2^T = \begin{pmatrix} 0 & & & \\ & 4 & -3 & -1 \\ & \frac{16}{5} & -\frac{12}{5} & -\frac{4}{5} \\ & \frac{12}{5} & -\frac{9}{5} & -\frac{3}{5} \end{pmatrix}$$

Thus

$$\begin{aligned} \mathbf{A}_2 &:= A_1 - \mathbf{l}_2\mathbf{u}_2^T \\ &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 2 \\ & & & \frac{7}{2} \end{pmatrix} \end{aligned}$$

Here $(A_2)_{3,3} = 0$, so we have a problem. In fact, all the entries in the third column of A_2 are zero. So we pick $\mathbf{l}_3 = (0, 0, 1, 0)$, $\mathbf{u}_3^T = (0, 0, 0, 2)$. ~~(Is this how to proceed?)~~ Hence

$$\begin{aligned} \mathbf{A}_3 &:= A_2 - \mathbf{l}_3\mathbf{u}_3^T \\ &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \frac{7}{2} \end{pmatrix} \end{aligned}$$

Finally giving $\mathbf{l}_4 = (0, 0, 0, 1)$, $\mathbf{u}_4^T = (0, 0, 0, \frac{7}{2})$. Hence one factorization gives

$$L = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -\frac{1}{5} & \frac{4}{5} & 1 & \\ \frac{1}{10} & \frac{3}{5} & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 10 & 6 & -2 & 1 \\ 0 & 4 & -3 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \frac{7}{2} \end{pmatrix}$$

Solving $A\mathbf{x} = \mathbf{b}$ is the same as $L(U\mathbf{x}) = \mathbf{b}$, which we can decompose into $L\mathbf{y} = \mathbf{b}$, $U\mathbf{x} = \mathbf{y}$ (Both of which are triangular and easily solved).

First consider $L\mathbf{y} = \mathbf{b}$, with $\mathbf{b}^T = (-2, 0, 2, 1)$. This gives

$$1y_1 = -2 \Rightarrow y_1 = -2$$

$$1y_1 + 1y_2 = 0 \Rightarrow y_2 = 2$$

$$-\frac{1}{5}y_1 + \frac{4}{5}y_2 + y_3 = 2 \Rightarrow y_3 = 0$$

$$\frac{1}{10}y_1 + \frac{3}{5}y_2 + y_4 = 1 \Rightarrow y_4 = 0$$

Having found $\mathbf{y}$, we solve in reverse order for $U\mathbf{x} = \mathbf{y}$.

$$\frac{7}{2}x_4 = 0 \Rightarrow x_4 = 0$$

$$4x_2 - 3x_3 = 2$$

$$10x_1 - 6x_2 - 2x_3 = -2$$

$$\Rightarrow$$

QUESTION 2

Consider the LU factorization of the real $n \times n$ symmetric matrix A. We wish to prove that the elements of U satisfy the condition

$$|u_{ij}| < 2^{i-1}\alpha, \quad j = 1, \cdots, n$$

where $\alpha = \max(|A_{ij}|)$.
We proceed by induction:

- for $i = 1$, the condition is $|u_{1j}| \leq \alpha$. Now u_{1j} are elements of $\mathbf{u}_1^T$, which by the LU algorithm is just the first row of A, thus clearly all the elements satisfy $|A_{1j}| \leq \alpha$, so base done.
- for the inductive step we assume that $|u_{rj}| \leq 2^{r-1}\alpha$. The LU decomposition algorithm says that $\mathbf{u}_{r+1}^T$ is the $(r+1)$th row of A_r, where

$$A_r = A_{r-1} - \mathbf{l}_r \mathbf{u}_r^T$$

Want to show that $\left|u_{(r+1),j}\right| < 2^r\alpha$.

We see the condition for a 2x2 matrix to have this is such that, in $A_1 := A - \mathbf{l}_1\mathbf{u}_1^T$, we want the $2,2$ entry in $\mathbf{l}_1\mathbf{u}_1^T$ to be $-A_{2,2}$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
then $\mathbf{l}_1\mathbf{u}_1^T$ is formed by $\begin{pmatrix} 1 \\ \frac{c}{a} \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix}$
so if $\max(A_{ij}) = d$, we want $\frac{bc}{a} = -d$, ie. $ad + bc = 0$. So,

$$A = \begin{pmatrix} 3 & -6 \\ 2 & 4 \end{pmatrix}$$

will do. Similarly considering

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Our condition becomes

$$3(ah - bg)(af - cd) = (ae - bd)(-ia)$$

QUESTION 3

As $\mathbf{u}_1^T$ is just the first row of A, this is trivially true for $k = 1$.

For $k = 2$, we suppose the first 2 rows of A span some plane (and are therefore linearly independent).

In our LU decomposition algorithm, we simply have $A_1 = A - \mathbf{l}_1\mathbf{u}_1^T$, and the second row of $U, \mathbf{u}_2^T$, is the second row of $A_{1,1}$.

(general linear combination arguments)

(do general case).

QUESTION 4

$$A = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 5 & 1 \\ & & & & 1 & \lambda \end{pmatrix}$$

Define $D^{1/2}$ as the diagonal matrix whose (k,k) element is $D_{k,k}^{1/2}$, hence $D^{1/2}D^{1/2} = D$. Then, A being positive definite, we can write

$$A = (LD^{1/2})(D^{1/2}L^T) = (LD^{1/2})(LD^{1/2})^T$$

In other words, letting $\tilde{L} = LD^{1/2}$, we obtain the *Cholesky factorisation* $A = \tilde{L}\tilde{L}^T$.

Proceeding with our usual LU algorithm on A, we have $\mathbf{l}_1 = (1,1,0,0,0,0)$, $D_{1,1} = 1$, thus

$$\begin{aligned} A_1 &= A - D_{1,1}\mathbf{l}_1\mathbf{l}_1^T \\ &= \begin{pmatrix} 0 & & & & & \\ & 1 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 5 & 1 \\ & & & & 1 & \lambda \end{pmatrix} \end{aligned}$$

Next $\mathbf{l}_2 = (0,1,1,0,0,0)$, $D_{2,2} = 1$, thus

$$\begin{aligned} A_2 &= A_1 - D_{2,2}\mathbf{l}_2\mathbf{l}_2^T \\ &= \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 2 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 5 & 1 \\ & & & & 1 & \lambda \end{pmatrix} \end{aligned}$$

Next $\mathbf{l}_3 = (0,0,1,\frac{1}{2},0,0)$, $D_{3,3} = 2$, thus

$$\begin{aligned} A_3 &= A_2 - D_{3,3}\mathbf{l}_3\mathbf{l}_3^T \\ &= \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & \frac{7}{2} & 1 & \\ & & & 1 & 5 & 1 \\ & & & & 1 & \lambda \end{pmatrix} \end{aligned}$$

Next $\mathbf{l}_4 = (0,0,0,1,\frac{2}{7},0)$, $D_{4,4} = \frac{7}{2}$, thus

$$\begin{aligned} A_4 &= A_3 - D_{4,4}\mathbf{l}_4\mathbf{l}_4^T \\ &= \begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& \frac{33}{7} & 1 \\ &&&& 1 & \lambda \end{pmatrix} \end{aligned}$$

Next $\mathbf{l}_5 = (0,0,0,0,1,\frac{7}{33})$, $D_{5,5} = \frac{33}{7}$, thus

$$\begin{aligned} A_5 &= A_4 - D_{5,5}\mathbf{l}_5\mathbf{l}_5^T \\ &= \begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & \\ &&&&& \lambda - \frac{7}{33} \end{pmatrix} \end{aligned}$$

Finally giving $\mathbf{l}_6 = (0,0,0,0,0,1)$, $D_{6,6} = \frac{1}{\lambda - 33/7}$.
Hence A has the Cholesky factorisation $A = \tilde{L}\tilde{L}^T$, $\tilde{L} = LD^{1/2}$, where

$$L = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ & 1 & 1 &&& \\ && \frac{1}{2} & 1 && \\ &&& \frac{2}{7} & 1 & \\ &&&& \frac{7}{33} & 1 \end{pmatrix}, \quad D^{1/2} = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && \sqrt{2} &&& \\ &&& \sqrt{\frac{7}{2}} && \\ &&&& \sqrt{\frac{33}{7}} & \\ &&&&& \frac{1}{\sqrt{\lambda - 33/7}} \end{pmatrix}$$

Thus if

QUESTION 5

Draw up some different banded matrices for different n and r, think about gram-schmidt for each of them, generalize in terms of r and n. (how many subtractions would you need to do to get each column to be upper triangular).

QUESTION 6

Using the Gram-Schmidt algorithm we can factorize $A = QR$

$$\mathbf{a}_k = \sum_{j=1}^{k} r_{jk}\mathbf{q}_k, k = 1, 2, 3 \qquad (*)$$

Setting $k = 1$ in (*) tells us that we must have $R_{1,1} = ||\mathbf{a}_1|| = 7$, and

$$\mathbf{q}_1 = \mathbf{a}_1/R_{1,1} = \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix}$$

Next we form the vector $\mathbf{b} = \mathbf{a}_2 - \langle \mathbf{q}_1, \mathbf{a}_2 \rangle \mathbf{q}_1$, which is orthogonal to $\mathbf{q}_1$. Have

$$\begin{aligned} \mathbf{b} &= \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix} - 8 \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix} \\ &= \begin{pmatrix} -6/7 \\ 18/7 \\ -9/7 \end{pmatrix} \end{aligned}$$

with $R_{1,2} = \langle \mathbf{q}_1, \mathbf{a}_2 \rangle = 8$, $R_{2,2} = ||\mathbf{b}|| = \frac{21}{7}$, and

$$\mathbf{q}_2 = \mathbf{b}/R_{2,2} = \begin{pmatrix} -2/7 \\ 6/7 \\ -3/7 \end{pmatrix}$$

Now we form the vector $\mathbf{c} = \mathbf{a}_3 - \langle \mathbf{q}_1, \mathbf{a}_3 \rangle \mathbf{q}_1 - \langle \mathbf{q}_2, \mathbf{a}_3 \rangle \mathbf{q}_2$, which is orthogonal to both $\mathbf{q}_1$ and $\mathbf{q}_2$. Have

$$R_{1,3} = \langle \mathbf{q}_1, \mathbf{a}_3 \rangle = \frac{11}{7}, R_{2,3} = \langle \mathbf{q}_2, \mathbf{a}_3 \rangle = \frac{1}{7}$$

Thus

$$\begin{aligned} \mathbf{c} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{11}{7} \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix} - \frac{1}{7} \begin{pmatrix} -2/7 \\ 6/7 \\ -3/7 \end{pmatrix} \\ &= \begin{pmatrix} -6/7 \\ 18/7 \\ -9/7 \end{pmatrix} \end{aligned}$$

with $R_{3,3} = ||\mathbf{c}|| = \frac{21}{7}$

QUESTION 7

First pick $\Omega^{[1,2]}$ so that $(\Omega^{[1,2]}A)_{2,1} = 0$. The choice of θ is given by

$$\cos\theta = \frac{A_{1,1}}{\sqrt{A_{1,1}^2 + A_{2,1}^2}}, \quad \sin\theta = \frac{A_{1,1}}{\sqrt{A_{1,1}^2 + A_{2,1}^2}}$$

Hence $\cos\theta = 6/\sqrt{45}$, $\sin\theta = 3/\sqrt{45}$, and

$$\begin{aligned}
\Omega^{[1,2]}A &= \begin{pmatrix} 6/\sqrt{45} & 3/\sqrt{45} & 0 \\ -3/\sqrt{45} & 6/\sqrt{45} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3\sqrt{5} & \frac{18}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 2 & 1 & 1 \end{pmatrix}
\end{aligned}$$

Next, pick $\Omega^{[1,3]}$ so that $(\Omega^{[1,3]}\Omega^{[1,2]}A)_{3,1} = 0$. Say $B = \Omega^{[1,2]}A$, so the choice of θ is given by

$$\cos\theta = \frac{B_{1,1}}{\sqrt{B_{1,1}^2 + B_{3,1}^2}}, \quad \sin\theta = \frac{B_{3,1}}{\sqrt{B_{1,1}^2 + B_{3,1}^2}}$$

Hence $\cos\theta = \frac{3}{7}\sqrt{5}$, $\sin\theta = \frac{2}{7}\sqrt{5}$, and

$$\begin{aligned}
\Omega^{[1,3]}\Omega^{[1,2]}A &= \begin{pmatrix} \frac{3}{7}\sqrt{5} & 0 & \frac{2}{7} \\ 0 & 1 & 0 \\ -\frac{2}{7} & 0 & \frac{3}{7}\sqrt{5} \end{pmatrix} \begin{pmatrix} 3\sqrt{5} & \frac{18}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 2 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 7 & 8 & \frac{11}{7} \\ 0 & \frac{6}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & -\frac{36}{7\sqrt{5}} + \frac{3\sqrt{5}}{7} & -\frac{6}{7\sqrt{5}} + \frac{3\sqrt{5}}{7} \end{pmatrix}
\end{aligned}$$

Finally, pick $\Omega^{[2,3]}$ so that $(\Omega^{[2,3]}\Omega^{[1,3]}\Omega^{[1,2]}A)_{3,2} = 0$. Say $C = \Omega^{[1,3]}\Omega^{[1,2]}A$, so the choice of θ is given by

$$\cos\theta = \frac{C_{2,2}}{\sqrt{C_{2,2}^2 + C_{3,2}^2}}, \quad \sin\theta = \frac{C_{3,2}}{\sqrt{C_{2,2}^2 + C_{3,2}^2}}$$

Hence $\cos\theta = \frac{3}{7}\sqrt{5}$, $\sin\theta = \frac{2}{7}\sqrt{5}$, and

QUESTION 8

QUESTION 9

QUESTION 10