Part IB — Methods Example Sheet 1

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QUESTION 1

We have $\nabla^2\phi = 0$ on $0 < x < a$, $0 < y < b$, $0 < z < c$ with $\phi = 1$ on the z surface and $\phi = 0$ on all other surfaces:

Assume $\phi(x,y,z) = X(x)Y(y)Z(z)$, so we have

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Solving $X'' = -\lambda_p X$ such that $X(0) = X(a) = 0$ implies that

$$\lambda_p = \frac{p^2\pi^2}{a^2}, X_l = \sqrt{\frac{2}{a}} \sin\left(\frac{p\pi x}{a}\right), l = 1, 2, 3, \cdots$$

Similarly, solving $Y'' = -\mu_q Y$, such that $Y(0) = Y(b) = 0$ implies that

$$\mu_q = \frac{q^2\pi^2}{b^2}, Y_q = \sqrt{\frac{2}{b}} \sin\left(\frac{q\pi x}{b}\right), m = 1, 2, 3, \cdots$$

Now solving for Z using the eigenvalues:

$$Z'' = \left(\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2}\right) Z,$$

$$Z = \alpha \cosh\left[\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)^{1/2} \pi z\right] + \beta \sinh\left[\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)^{1/2} \pi z\right]$$

We can rewrite the z-dependent part (why??) as

$$Z = \alpha \cosh\left[l(c-z)\right] + \beta \sinh\left[l(c-z)\right]$$

Using the boundary conditions,
$Z(c) = 0 \Rightarrow \alpha = 0$, $Z(0) = 1 \Rightarrow \beta = \frac{1}{\sinh cl}$
Therefore, the general solution is

$$\psi(x,y,z) = \frac{2}{\sqrt{ab}} \sum_{p=0} \sum_{q=0} a_{pq} \sin\left(\frac{p\pi}{a}x\right) \sin\left(\frac{q\pi}{b}y\right) \sinh\left[l(c-z)\right] \frac{1}{\sinh cl}$$

QUESTION 2

The general solution for Laplace's equation in polar coordinates is

$$\phi(r,\theta) = c_0 + d_0 \log r + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)(c_n r^n + d_n r^{-n})$$

by requiring regularity at the origin, $d_0 = 0, d_n = 0$, then absorb c_n as a general rescaling, we can write this solution as

$$\psi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n$$

Using boundary conditions,

$$f(\theta) = \psi(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

The function is odd, hence

$$f(\theta) = \psi(1,\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

Then

$$\int_0^{2\pi} f(\theta) \sin m\theta \, \mathrm{d}\theta = \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin n\theta \sin m\theta \, \mathrm{d}\theta$$

$$\int_0^{\pi} \frac{\pi}{2} \sin m\theta \, \mathrm{d}\theta + \int_{\pi}^{2\pi} -\frac{\pi}{2} \sin m\theta \, \mathrm{d}\theta = b_m$$

This gives $b_m = \pi/m$, hence

$$\psi(r,\theta) = \sum_{n=1}^{\infty} \frac{\pi r^n \sin n\theta}{n}$$

QUESTION 3

$\psi(r,\theta)$ satisfies $\nabla^2 = 0$, in spherical polars this is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\psi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta\psi}\right) = 0$$

and $\psi(r,\theta)$ satisfies the boundary conditions

$$\psi(r,\theta) = \begin{cases} V & \text{if } 0 \le \theta < \frac{\pi}{2} \\ -V & \text{if } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

Assume

$$\psi(r,\theta) = R(r)\Theta(\theta)$$

we obtain

$$(\sin\theta\Theta')' + \lambda\sin\theta\Theta = 0$$
$$(r^2R')' - \lambda R = 0$$

Making the substitution $x = \cos\theta$ in the first equation yields Legendre's equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\Theta\right] = \lambda\Theta$$

substituting $\Theta = \sum_{n=0}^{\infty} a_n x^n$ yielding Legendre Polynomials as the answer:

$$\Theta_e = a_0\left[1 + \frac{(-\lambda)x^2}{2!}\cdots\right]$$
$$\Theta_o = a_1\left[x + \frac{(2-\lambda)x^3}{3!} + \cdots\right]$$

Solution must remain bounded at $x = \pm 1$, so $\lambda = m(m+1)$ for some integer m. Call this $\Theta_n(\theta) = P_n(x) = P_n(\cos\theta)$, with $\lambda = n(n+1)$. Then the DE for R becomes

$$(r^2R_n')' - n(n+1)R_n = 0$$

Have

$$\psi_n(r,\theta) = (a_n r^n + b_n r^{-(n+1)})P_n(\cos\theta)$$

So GS is

$$\psi(r,\theta) = \sum_{n=0}^{\infty}(a_n r^n + b_n r^{-(n+1)})P_n(\cos\theta)$$

for solution to be regular at origin must have $b_n = 0$, so

$$\psi(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)$$

We can apply our boundary condition easiest by setting $r = 1$, then

$$\begin{aligned}
f(\theta) &= \sum_{n=0}^{\infty} a_n P_n \cos(\theta), \quad 0 \leq \theta \leq \pi \\
F(x) &= \sum_{n=0}^{\infty} a_n P_n(x), \qquad x = \cos\theta, -1 \leq x \leq 1 \\
a_n &= \frac{(2n+1)}{2} \int_{-1}^{1} F(x) P_n(x) \, \mathrm{d}x
\end{aligned}$$

For $0 \leq \theta < \pi/2$ we have $f(\theta) = V$, ie. for $0 \leq x < 1$ we have $F(x) = V$. Similarly $F(x) = -V$ if $-1 \leq x < 0$. So

$$\begin{aligned}
a_n &= \frac{(2n+1)}{2} V \int_{-1}^{1} P_n(x) \, \mathrm{d}x \qquad \text{for } 0 \leq x < 1 \\
a_n &= -\frac{(2n+1)}{2} V \int_{-1}^{1} P_n(x) \, \mathrm{d}x \qquad \text{for } -1 \leq x < 0
\end{aligned}$$

QUESTION 4

y_m is an eigenfunction, hence satisfies the Sturm-Liouville equation, so we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(py_m'\right) = -(\lambda_m - q)y_m$$

Then, integrating by parts,

$$=$$
$$=$$

QUESTION 5

(a)

(i)

$$q_n = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\underbrace{\left(x^{2n} - \binom{n}{2}x^{2n-2} + \cdots\right)}_{(*)}$$

Differentiating (*) n times produces a polynomial with highest power $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^{2n}) = x^n$. Hence $q_n(x)$ is of degree n.

(ii) By induction:

$$\begin{aligned} q_1(1) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\Big|_{x=1} \\ &= \frac{1}{2}2x\Big|_{x=1} \\ &= 1 \end{aligned}$$

True for $n = 1$. Now suppose $q_k(1) = 1$ for some $k > 0$.

$$\begin{aligned} q_{k+1}(x) &= \frac{1}{2^{k+1}(k+1)!}\frac{\mathrm{d}^n}{\mathrm{d}x^{k+1}}(x^2-1)^{k+1} \\ &= \frac{1}{2(k+1)}\left[\frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(2(k+1)x(x^2-1)^k\right)\right] \\ &= \frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x(x^2-1)^k\right) \end{aligned}$$

QUESTION 6

$y(x,t)$ satisfies the 1D wave equation

$$\frac{\partial^2}{\partial t^2}y(x,t) = c^2\frac{\partial^2}{\partial x^2}y(x,t) \qquad c^2 = \frac{T}{\mu}$$

Assume $y(x,t) = X(x)T(t)$ and separate variables:

$$\begin{aligned} X\ddot{T} &= c^2 X''T \\ \Rightarrow \frac{1}{c^2}\frac{\ddot{T}}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

for some $\lambda > 0$, so we have

$$\begin{aligned} X'' + \lambda X &= 0 \\ \ddot{T} + \lambda c^2 T &= 0 \end{aligned}$$

Solving the spatial equation first,

$$X = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$$

Applying initial conditions

- $X(0) = 0 \Rightarrow \alpha = 0$
- $X(l) = 0 \Rightarrow \beta\sin(\sqrt{\lambda}x) = 0 \Rightarrow \lambda = n^2\pi^2/l^2$, for integer n

The normal modes are the associated eigenfunctions given by

$$X_n(x) = \beta_n \sin\left(\frac{n\pi x}{L}\right)$$

The associated $T_n(t)$ is given by

$$\begin{aligned} \ddot{T}_n + \frac{n^2\pi^2c^2}{L^2}T_n = 0 \\ \Rightarrow T_n(t) = \gamma_n\cos\left(\frac{n\pi ct}{L}\right) + \delta_n\sin\left(\frac{n\pi ct}{L}\right) \end{aligned}$$

Hence the specific solution is

$$y_n = X(x)T(t) = \sin\left(\frac{n\pi x}{L}\right)\left(A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right)$$

QUESTION 7

(i)
- Assume all displacements are sufficiently small ($y << l$)
- Assume all displacements are vertical
- Consider two points x and $x + \delta x$. The angle of the string to the horizontal at x is θ_1, and the angle at $x + \delta x$ is θ_2.
- Resolving vertically

$$T \sin\theta_2 - T \sin\theta_1 - \mu g \delta x - 2k\mu\delta x \frac{\partial}{\partial t} y = \mu \delta x \frac{\partial^2}{\partial t^2} y \qquad (*)$$

- Assume angles are small

$$\sin\theta_2 \approx tan\theta_2 = \frac{\partial y}{\partial x}\Big|_{x+\delta x} \approx \frac{\partial y}{\partial x}\Big|_x + \delta x \frac{\partial^2 y}{\partial x^2}\Big|_x$$

$$\sin\theta_1 \approx \tan\theta_1 = \frac{\partial y}{\partial x}\Big|_x$$

- (*) becomes

$$T\delta x \frac{\partial^2 y}{\partial x^2} - \mu g \delta x - 2k\mu\delta x \frac{\partial}{\partial t} y = \mu\delta x \frac{\partial^2}{\partial t^2} y$$
$$\Rightarrow \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial t^2} y + 2k\frac{\partial}{\partial t} y + g$$

- Further assume the weight is insignificant ($g \to 0$)
- Hence arrive at the equation of motion

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial t^2} y + 2k \frac{\partial}{\partial t} y$$

where $c^2 = \frac{T}{\mu}$

Assume $y(x, t) = X(x)T(t)$ and separating variables gives

$$c^2 \frac{X''}{X} = \frac{\ddot{T}}{T} + 2k\frac{\dot{T}}{T}$$
$$\Rightarrow \frac{X''}{X} = \frac{\ddot{T} + 2k\dot{T}}{T} = -\lambda$$

for some $\lambda > 0$, so we have

$$X'' + \lambda X = 0$$
$$\ddot{T} + 2k\dot{T} + \lambda c^2 T = 0$$

Solving the spatial equation first,

$$X = \alpha \cos(\sqrt{\lambda} x) + \beta \sin(\sqrt{\lambda} x)$$

Applying initial conditions

- $X(0) = 0 \Rightarrow \alpha = 0$
- $X(l) = 0 \Rightarrow \beta \sin(\sqrt{\lambda} x) = 0 \Rightarrow \lambda = n^2\pi^2/l^2$, for integer n

These λ are eigenvalues, with associated eigenfunctions

$$X_n(x) = \beta_n \sin\left(\frac{n\pi x}{L}\right)$$

The associated $T_n(t)$ is given by

$$\ddot{T}_n + 2k\dot{T}_n + \frac{n^2\pi^2c^2}{L^2}T_n = 0 \qquad k = \frac{\pi c}{l}$$
$$\Rightarrow T_n(t) = e^{-kt}\left(\gamma_n \cos\left(\frac{\sqrt{n^2-1}\pi c}{L}t\right) + \delta_n \sin\left(\frac{\sqrt{n^2-1}\pi c}{L}t\right)\right)$$

Hence the specific solution is

$$y_n = e^{-kt}\sin\left(\frac{\sqrt{n^2-1}\pi c}{L}t\right)\left(A_n \cos\left(\frac{\sqrt{n^2-1}\pi c}{L}t\right) + B_n \sin\left(\frac{\sqrt{n^2-1}\pi c}{L}t\right)\right)$$

QUESTION 8

QUESTION 9

QUESTION 10