Part IA — Markov Chains Example Sheet 1

Supervised by Prof Weber (rrw1@cam.ac.uk)
Examples worked through by Christopher Turnbull

Michaelmas 2017

QUESTION 1

$X = (X_n)$ is a Markov Chain, and so satisfies the Markov Property

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \cdots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

Given that $Z_k = X_{m+k}$, with $k \geq 0$, want to show that Z_k is a Markov Chain, ie.

$$\mathbb{P}(Z_{n+1} \mid Z_0, \cdots, Z_n) = \mathbb{P}(Z_{n+1} \mid Z_n) \qquad (*)$$

Substituting in our definition of Z_k,

$$\begin{aligned} \mathbb{P}(Z_{n+1} \mid Z_0, \cdots, Z_n) &= \mathbb{P}(X_{m+n+1} | X_m, \cdots, X_{m+n}) \\ &= \mathbb{P}(X_{m+n+1} | X_{m+n}) \qquad \text{as } X \text{ is Markov} \\ &= \mathbb{P}(Z_{n+1} \mid Z_n) \end{aligned}$$

Hence Z_k satisfies the Markov property, and is thus a Markov Chain.
$Z_0 = X_m = i$, so Z_k has starting state i.

QUESTION 2

Let $X_1, \cdots, X_n$ be independent random variables, so $\mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1})$. But then

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \cdots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1})$$

also by independence, so Markov property is satisfied.

Homogeneous if $\mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n)$ does not depend on the value of n, so must have X_n identically distributed.

QUESTION 3

As X_n is the *maximum* reading obtained after n throws, this depends only on the last maximum, so X_n is a Markov chain. In particular, the state transition matrix is

$$\begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 6/6 \end{pmatrix}$$

QUESTION 4

We have

$$S_{n+1} = \begin{cases} S_n + 1 & \text{with probability } p \\ S_n - 1 & \text{with probability } q \end{cases}$$

and $S_0 = 0$.

Let $X_n = |S_n|$

Clearly X_n satisfies the Markov property (where the walk is now only depends on where it was one step before), so X is a Markov Chain with initial state $X_0 = 0$. The transition probabilities are given by the transition matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ q & 0 & p & 0 & \cdots \\ 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Define $M_k := \max\{S_k \mid 0 \leq k \leq n\}$, and $Y_n = M_n - S_n$

QUESTION 5

X is a Markov chain, thus

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \cdots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

Have

$$\mathbb{P}(Y_{r+1} \mid Y_0, \cdots, Y_r) = \mathbb{P}(X_{n_{r+1}} \mid X_{n_0}, \cdots, X_{n_r})$$

If $n_r = 2r$ and X is a simple random walk on Z st.

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \\ X_n - 1 & \text{with probability } q \end{cases}$$

the allowed states are $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$. Of course, the probability of finding oneself in an odd state is always zero (we are always taking an even number of states).

It is difficult to visualise the state transition matrix as the state space is infinite, but a general row has the form

$$\begin{pmatrix} \cdots & 0 & q^2 & 0 & 2pq & 0 & p^2 & \cdots \end{pmatrix}$$

With probability $2pq$ of returning to the current state, p^2 moving two to the right and q^2 moving two to the left. (Thus rows summing to $(p+q)^2 = 1$)

QUESTION 6

I think the answer should be no. X and Y are both Markov chains, so the value of $X_{n+1} + Y_{n+1}$ only depends on X_n, and Y_n. But we only have the value of $X_n + Y_n$ (loss of information)

QUESTION 7

(i) Let the probability of the flea being on the original vertex after n hops be p_n. Then

$$p_n = \begin{cases} 0 & \text{if flea on first vertex after } n-1 \text{ hops} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Hence we set up the recurrence, with $p_0 = 1$

$$\begin{aligned} p_n &= 0 \cdot p_{n-1} + \frac{1}{2}(1 - p_{n-1}) \\ &= \frac{1}{2}(1 - p_{n-1}) \end{aligned}$$

Solving this difference equation gives general solution $p_n = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$.

(ii) Let p_n be defined as before. We have

$$p_n = \begin{cases} 0 & \text{if flea on first vertex after } n-1 \text{ hops} \\ \frac{2}{3} & \text{if flea on anticlockwise vertex ""} \\ \frac{1}{3} & \text{if flea on clockwise vertex ""} \end{cases}$$

This can be though of as a Markov chain on the state space of triangle vertices, with transition matrix

$$P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

We seek $(P^n)_{11}$

QUESTION 8

Let X_n be the score of the die on the n^{th} roll; this is a Markov Chain. Define Y_n as

$$Y_n = \begin{cases} 1 & \text{if } X_n = 6 \\ 0 & \text{if } X_n \neq 6 \end{cases}$$

This has transition matrix

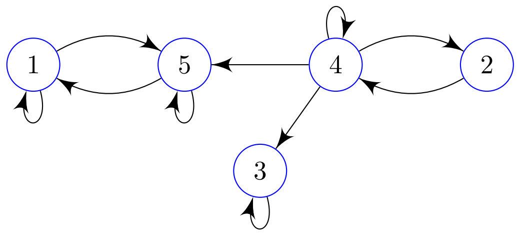
$$P = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ 1 & 0 \end{pmatrix}$$

We wish to find $P(Y_n = 1 \mid Y_0 = 0)$

QUESTION 9

QUESTION 10

Possible transitions of the chain are illustrated below:



The communicating classes are $C_1 = \{1, 5\}, C_2 = \{3\}$ and $C_3 = \{2, 4\}$. The classes C_1 and C_3 are not closed, but C_2 is closed.

QUESTION 11

QUESTION 12

QUESTION 13

QUESTION 14