Part IB — Methods Example Sheet 1

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QUESTION 1

Let Ω be the region

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}$$

We have Laplace's equation $\nabla^2 \phi = 0$ inside Ω, with the Dirichlet boundary conditions $\phi = 1$ on the z surface and $\phi = 0$ on all other surfaces:

Assume $\phi(x,y,z) = X(x)Y(y)\sinh[l(c-z)]$, so we have

$$\frac{X''}{X} + \frac{Y''}{Y} + l^2 = 0$$

Moving along the x direction for fixed $y, z, \frac{Y''}{Y}, l^2$ both constant, hence $\frac{X''}{X} = -\frac{Y''}{Y} - l^2$ is also equal to some constant, $-\lambda$.

Solving $X'' = -\lambda X$ with boundary conditions $X(0) = X(a) = 0$ restricts values of λ to

$$\lambda_p = \frac{p^2\pi^2}{a^2}, X_p = A\sin\left(\frac{p\pi x}{a}\right), p = 1,2,3,\cdots$$

Similarly, solving $Y'' = -\mu Y$, such that $Y(0) = Y(b) = 0$ implies that

$$\mu_q = \frac{q^2\pi^2}{b^2}, Y_q = B\sin\left(\frac{q\pi x}{b}\right), q = 1,2,3,\cdots$$

At this point we have a family of solutions

$$\psi_{p,q}(x,y,z) := \sin\left(\frac{p\pi}{a}x\right)\sin\left(\frac{q\pi}{b}y\right)\sinh\left[l(c-z)\right]$$

Laplaces equation linear with boundary conditions homogenous, the general solution is

$$\psi(x,y,z) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{p,q}\sin\left(\frac{p\pi}{a}x\right)\sin\left(\frac{q\pi}{b}y\right)\sinh\left[l(c-z)\right]$$

for some constants $A_{p,q}$. Setting $z = 0$ we require

$$1 = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{p,q}\sin\left(\frac{p\pi}{a}x\right)\sin\left(\frac{q\pi}{b}y\right)\sinh cl$$

Now using orthogonality relations

$$\int_0^a \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi x}{a}\right)\,\mathrm{d}x = \frac{a}{2}\delta_{p,q}$$

we deduce

$$A_{p,q} = \frac{4}{ab\sinh cl} \int_0^a \int_0^b \sin\left(\frac{q\pi x}{a}\right) \sin\left(\frac{p\pi y}{b}\right) \, \mathrm{d}x\mathrm{d}y$$

$$= \frac{4}{ab\sinh cl} \left[-\frac{a}{q\pi}\cos\left(\frac{q\pi x}{a}\right)\right]_0^a \left[-\frac{b}{p\pi}\cos\left(\frac{p\pi x}{b}\right)\right]_0^b$$

$$= \frac{4}{ab\sinh cl} \frac{ab}{\pi^2 pq} \left((-1)^q - 1\right)\left((-1)^p - 1\right)$$

$$= \begin{cases} \frac{16}{\pi^2 pq \sinh cl} & \text{if } q \text{ and } p \text{ are both odd} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the solution satisfying these boundary conditions is

$$\psi(x,y,z) = \frac{16}{\pi^2} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\sinh\left[l(c-z)\right]\sin\left((2p+1)\pi x/a\right)\sin\left((2q+1)\pi y/b\right)}{(2p+1)(2q+1)\sinh cl}$$

as required.
As $c \to \infty$, note that

$$\frac{\sinh(L(c-z))}{\sinh(Lc)} = \frac{\exp[L(c-z)] - \exp[-L(c-z)]}{\exp(Lc) - \exp(-Lc)}$$

$$\to \frac{\exp[L(c-z)]}{\exp(Lc)}$$

$$\to \exp(-Lz)$$

QUESTION 2

The potential satisfies

$$\nabla^2 \phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

with Dirichlet boundary conditions

$$\phi(r = 1, \theta) = \begin{cases} \pi/2 & \text{if } 0 \le \theta < \pi \\ -\pi/2 & \text{if } \pi \le \theta < 2\pi \end{cases}$$

Separating variables by writing $\phi(r, \theta) = R(r)\Theta(\theta)$,

$$\frac{1}{r}\frac{\partial}{\partial r}(rR'\Theta) + \frac{1}{r^2}R\Theta'' = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}(rR') + \frac{\Theta''}{\Theta} = 0$$

Keeping r fixed and varying θ, we see that $\frac{\Theta''}{\Theta} = -\lambda$ constant. Solving $\Theta'' = -\Theta X$, if ϕ single valued, must have $\Theta(\theta + 2\pi) = \Theta(\theta)$, thus $\lambda = n^2$ for some integer n, and

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

Next we solve

$$r\frac{\mathrm{d}}{\mathrm{d}r}(rR_n') - n^2 R_n = 0$$

For $n \ne 0$, assuming that $R_n \propto r^\beta$, we have

$$\beta^2 - n^2 = 0 \Rightarrow \beta = \pm n$$

Thus

$$R_n(r) = c_n r^n + d_n r^{-n}, \quad n = 1, 2, 3, \cdots$$

Therefore, the family of particular solutions is

$$\phi_n(r, \theta) = (a_n \cos n\theta + b_n \sin n\theta)(c_n r^n + d_n r^{-n}) \quad n = 1, 2, 3, \cdots$$

For $n = 0$, we solve $r\frac{\mathrm{d}}{\mathrm{d}r}(rR_0') = 0$, thus $rR_0' = $ constant, and we have $R_0 = d_0 \log r + c_0$

Hence by linearity, the general solution for Laplace's equation in polar coordinates is

$$\phi(r, \theta) = c_0 + d_0 \log r + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)(c_n r^n + d_n r^{-n})$$

Now, requiring regularity at the origin, $d_0 = 0, d_n = 0$, then absorb c_n as a general rescaling, we can write this solution as

$$\phi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)r^n$$

Using boundary conditions,

$$f(\theta) = \phi(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

The function is odd, hence

$$f(\theta) = \phi(1,\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

By orthogonality of sines,

$$
\begin{aligned}
b_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta \\
&= \frac{1}{\pi}\left(\int_0^{\pi} \frac{\pi}{2}\sin m\theta\,\mathrm{d}\theta + \int_{\pi}^{2\pi} -\frac{\pi}{2}\sin m\theta\,\mathrm{d}\theta\right) \\
&= \frac{1}{2m}\left([-\cos m\theta]_0^{\pi} + [\cos m\theta]_{\pi}^{2\pi}\right) \\
&= \frac{1}{m}(-1 - (-1)^m) \\
&= \begin{cases} \frac{2}{m} & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even} \end{cases}
\end{aligned}
$$

This gives $b_m = 2/m$, hence

$$\phi(r,\theta) = 2\sum_{n \text{ odd}} \frac{r^n \sin n\theta}{n}$$

Next, under the substitution $z = re^{i\theta}$, we have $z^n = r^n \cos n\theta + i\sin n\theta$

$$
\begin{aligned}
\phi(r,\theta) &= 2\sum_{n \text{ odd}} \frac{r^n \sin n\theta}{n} \\
&= 2\,\mathrm{Im}\left(\sum_{n \text{ odd}} \frac{z^n}{n}\right) \\
&= 2\,\mathrm{Im}\left(\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}\right) \\
&= 2\,\mathrm{Im}\left(\sum_{n=0}^{\infty} \int_0^z t^{2n}\,\mathrm{d}t\right)
\end{aligned}
$$

and, assuming we can swap the order of summation and integration, we have

$$= 2\operatorname{Im}\left(\int_0^z \sum_{n=0}^{\infty} t^{2n} \, \mathrm{d}t\right)$$

$$= 2\operatorname{Im}\int_0^z \frac{1}{1-t^2} \, \mathrm{d}t$$

$$= \operatorname{Im}\int_0^z \left(\frac{1}{1+t} + \frac{1}{1-t}\right) \, \mathrm{d}t$$

$$= \operatorname{Im}\left[\log(1+t) - \log(1-t)\right]_0^z$$

$$= \operatorname{Im}\left[\log(1+z) - \log(1-z)\right]$$

$$= \arg(1+z) - \arg(1-z)$$

which is some angle, see $w^2 = \frac{1+z}{1-z}$, V and M sheet 1

QUESTION 3

The potential satisfies

$$\nabla^2 \psi = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right)$$

and $\psi(r, \theta)$ satisfies the Dirichlet boundary conditions (hence solution existence and uniqueness)

$$\psi(r, \theta) = \begin{cases} V & \text{if } 0 \le \theta < \frac{\pi}{2} \\ -V & \text{if } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

Separating variables by writing $\phi(r, \theta) = R(r)\Theta(\theta)$, we have the two ODEs

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin \theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta} \right) + \lambda \sin \theta \, \Theta = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r} \right) - \lambda R = 0$$

with $\lambda \in \mathbb{R}$ separation constant.

Making the substitution $x = \cos \theta$ in the angular equation yields $\frac{\mathrm{d}}{\mathrm{d}\theta} = -\sin \theta \frac{\mathrm{d}}{\mathrm{d}x}$, and

$$-\sin \theta \frac{\mathrm{d}}{\mathrm{d}x} \left[\sin \theta \left(-\sin \theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta} \right) \right] + \lambda \sin \theta \, \Theta = 0$$

which becomes Legendre's equation:

$$-\frac{\mathrm{d}}{\mathrm{d}x} \left[(1 - x^2) \frac{\mathrm{d}}{\mathrm{d}x} \Theta \right] = \lambda \Theta$$

substituting $\Theta = \sum_{n=0}^{\infty} a_n x^n$ (only non-negative powers as we want solution to be regular at origin) yields

$$(1 - x^2) \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - 2 \sum_{n=1}^{\infty} a_n n x^n + \lambda \sum_{n=0}^{\infty} x^n = 0$$

which must hold for each power separately, thus we obtain the recursion relation:

$$0 = a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2a_n n + \lambda a_n$$

yielding the recursion relation

$$a_{n+2} = \left[\frac{n(n+1) - \lambda}{(n+1)(n+2)} \right] a_n$$

Thus we find two linearly independent (even and odd) solutions

$$\Theta_e = a_0 \left[1 + \frac{(-\lambda)x^2}{2!} \cdots \right]$$

$$\Theta_o = a_1 \left[x + \frac{(2 - \lambda)x^3}{3!} + \cdots \right]$$

Solution must remain bounded at $x = \pm 1$, so $\lambda = m(m+1)$ for some integer m. These $\Theta_n(\theta) = P_n(x) = P_n(\cos\theta)$, with $\lambda = n(n+1)$, are Legendre polynomials of order n, with the orthogonal property $\int_{-1}^{1} P_m(z)P_n(z)\,\mathrm{d}z = \frac{2}{2n+1}\delta_{mn}$. Then the DE for R becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R_n}{\mathrm{d}r}\right) - n(n+1)R_n = 0$$

Assuming that $R_n \propto r^\beta$,

$$\beta(\beta+1) = n(n+1)$$
$$\Rightarrow \beta = n \text{ or } -(n+1)$$

Thus our general solution takes the from

$$\psi(r,\theta) = \sum_{n=0}^{\infty}(a_n r^n + b_n r^{-(n+1)})P_n(\cos\theta)$$

for solution to be regular at origin must have $b_n = 0$, so

$$\psi(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)$$

We can apply our boundary condition easiest by setting $r = 1$, then

$$f(\theta) := \psi(r=1,\theta) = \sum_{n=0}^{\infty} a_n P_n \cos(\theta), \quad 0 \le \theta \le \pi$$

$$F(x) := \sum_{n=0}^{\infty} a_n P_n(x), \qquad x = \cos\theta, -1 \le x \le 1$$

$$\text{where } F(x) = \begin{cases} V & \text{if } 0 \le x < 1 \\ -V & \text{if } -1 \le x < 0 \end{cases}$$

$$\Rightarrow a_n = \frac{(2n+1)}{2}\int_{-1}^{1} F(x)P_n(x)\,\mathrm{d}x$$

$$= \frac{(2n+1)}{2}\int_{0}^{1} V P_n(x)\,\mathrm{d}x + \frac{(2n+1)}{2}\int_{-1}^{0} -V P_n(x)\,\mathrm{d}x$$

$$= \frac{V}{2}\int_{0}^{1} P'_{n+1}(x) - P'_{n-1}(x)\,\mathrm{d}x - \frac{V}{2}\int_{-1}^{0} P'_{n+1}(x) - P'_{n-1}(x)\,\mathrm{d}x$$

Note that for n even, the integrals cancel out and we have $a_n = 0$. Otherwise,

$$a_n = V\int_{0}^{1} P'_{n+1}(x) - P'_{n-1}(x)\,\mathrm{d}x \qquad \text{as } n \text{ odd}$$

$$= V[P_{n+1}(z) - P_{n-1}(z)]_{0}^{1}$$

$$= V(P_{n-1}(0) - P_{n+1}(0)) \quad \text{as } P_n(1) = 1 \ \forall\, n$$

Hence the potential inside the region is given by

$$\psi(r, \theta) = V \sum_{n=0}^{\infty} r^n (P_{n-1}(0) - P_{n+1}(0)) P_n(\cos \theta)$$

QUESTION 4

y_m is an eigenfunction, hence satisfies the Sturm-Liouville equation, so we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(py_m'\right) = -(\lambda_m - q)y_m$$

Starting with the left hand side

$$\int_a^b \left(py_m'y_n' + qy_my_n\right)\mathrm{d}x = \int_a^b py_m'y_n'\,\mathrm{d}x + \int_a^b qy_my_n\,\mathrm{d}x$$

and integrating by parts

$$\int_a^b py_m'y_n'\,\mathrm{d}x = \left[py_m'y_n\right]_a^b - \int_a^b -(\lambda_m - q)y_my_n\,\mathrm{d}x$$

Choosing suitable boundary conditions such that

$$\left[py_m'y_n\right]_a^b = 0$$

we have

$$\int_a^b \left(py_m'y_n' + qy_my_n\right)\mathrm{d}x = \int_a^b (\lambda_m - q)y_my_n\,\mathrm{d}x + \int_a^b qy_my_n\,\mathrm{d}x$$
$$= \int_a^b \lambda_m y_m y_n\,\mathrm{d}x$$
$$= \lambda_m \delta_{mn} \quad \text{by S-L orthogonality}$$

Last part? Get solution of Zimaras. Note that

$$y_n = P_n \implies \int y_n^2\,\mathrm{d}x = 1$$

QUESTION 5

(a)

  (i)

$$q_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} \underbrace{\left(x^{2n} - \binom{n}{2} x^{2n-2} + \cdots \right)}_{(*)}$$

    Differentiating $(*)$ n times produces a polynomial with highest power $\frac{d^n}{dx^n}(x^{2n}) = x^n$. Hence $q_n(x)$ is of degree n.

  (ii) By induction:

$$q_1(1) = \frac{1}{2} \frac{d}{dx}(x^2 - 1)\Big|_{x=1}$$
$$= \frac{1}{2} 2x \Big|_{x=1}$$
$$= 1$$

    True for $n = 1$. Now suppose $q_k(1) = 1$ for some $k > 0$.

$$q_{k+1}(x) = \frac{1}{2^{k+1}(k+1)!} \frac{d^n}{dx^{k+1}} (x^2 - 1)^{k+1}$$
$$= \frac{1}{2(k+1)} \left[\frac{1}{2^k k!} \frac{d^k}{dx^k} \left(2(k+1)x(x^2-1)^k \right) \right]$$
$$= \frac{1}{2^k k!} \frac{d^k}{dx^k} \left(x(x^2-1)^k \right)$$

  (b)

  (i) P_n are polynomial solutions to Legendre's equation with $\lambda_n = n(n+1)$, and q_n are polynomial solutions to Legendre's equation with $\lambda_n = n(n+1)$, we have $P_n \propto q_n$. But $P(1) = q(1) = 1$, so $P_n = q_n$ (are there non poly solutions to worry about? See Skinners notes)

  (ii) Hence we see that

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n$$

    Using this result, by parts,

$$\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \left(\frac{d}{dx} \right)^n (x^2-1)^n \left(\frac{d}{dx} \right)^n (x^2-1)^n \, dx$$
$$= \frac{1}{2^{2n}(n!)^2} \left(\left[\left(\frac{d}{dx} \right)^{n-1} (x^2-1)^n \left(\frac{d}{dx} \right)^n (x^2-1)^n \right]_{-1}^{1} \right)$$
$$- \frac{1}{2^{2n}(n!)^2} \left(\int_{-1}^{1} \left(\frac{d}{dx} \right)^{n-1} (x^2-1)^n \left(\frac{d}{dx} \right)^{n+1} (x^2-1)^n \, dx \right)$$

But we know that the boundary term vanishes since

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m (x^2 - 1)\Big|_{x=\pm 1} = 0 \text{ for } m < n$$

Thus, we integrate by parts iteratively, until

$$\int_{-1}^{1} [P_n(x)]^2 \, \mathrm{d}x = \cdots$$

$$= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2 - 1)^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2n} (x^2 - 1)^n \, \mathrm{d}x$$

$$= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2 - 1)^n \, \mathrm{d}x$$

Finally,

$$I_n = \int_{-1}^{1} (x^2 - 1)^n \, \mathrm{d}x \qquad \text{(by parts with 1)}$$

$$= \left[x(x^2 - 1)^n\right]_{-1}^{1} - \int_{-1}^{1} 2x^2 n (x^2 - 1)^{n-1} \, \mathrm{d}x$$

$$= -2n \int_{-1}^{1} (x^2 - 1)(x^2 - 1)^{n-1} + (x^2 - 1)^{n-1} \, \mathrm{d}x$$

$$= -2n(I_n + I_{n-1})$$

$$\Rightarrow I_n = \frac{2}{(2n+1)!!}$$

Arrive at (?) result.

QUESTION 6

$y(x, t)$ satisfies the 1D wave equation

$$\frac{\partial^2}{\partial t^2} y(x, t) = c^2 \frac{\partial^2}{\partial x^2} y(x, t) \qquad c^2 = \frac{T}{\mu}$$

Assume $y(x, t) = X(x)T(t)$ and separate variables:

$$X\ddot{T} = c^2 X'' T$$

$$\Rightarrow \frac{1}{c^2} \frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda$$

for some $\lambda > 0$, so we have

$$X'' + \lambda X = 0$$
$$\ddot{T} + \lambda c^2 T = 0$$

Solving the spatial equation first,

$$X = \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x)$$

Applying initial conditions

– $X(0) = 0 \Rightarrow \alpha = 0$

– $X(l) = 0 \Rightarrow \beta \sin(\sqrt{\lambda}x) = 0 \Rightarrow \lambda = n^2\pi^2/l^2$, for integer n

The normal modes are the associated eigenfunctions given by

$$X_n(x) = \beta_n \sin\left(\frac{n\pi x}{l}\right)$$

The associated $T_n(t)$ is given by

$$\ddot{T}_n + \frac{n^2\pi^2 c^2}{l^2} T_n = 0$$

$$\Rightarrow T_n(t) = \gamma_n \cos\left(\frac{n\pi ct}{l}\right) + \delta_n \sin\left(\frac{n\pi ct}{l}\right)$$

Hence the general solution is

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right)$$

Using BCs, $y(x, 0) = 0 \Rightarrow A_n = 0 \ \forall \ n$.
Next, have $y_t(x, 0) = \frac{4V}{l^2} x(l - x)$, so

$$\frac{4V}{l^2} x(l - x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow B_n = \frac{4V}{l^2}\left(\frac{l}{n\pi c}\right)\frac{2}{l}\int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x$$

$$= \frac{8V}{n\pi cl^2}\int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x$$

$$= \cdots$$

$$= \frac{16lV^2}{c\pi^4 n^4}\left(1-(-1)^n\right)$$

(Think should be V, and not V^2, do check later.)
Hence

$$y(x,t) = \frac{32lV^2}{c\pi^4}\sum_{n\text{ odd}}\frac{1}{n^4}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

Next, kinetic energy is given by

$$K = \int_0^L \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2\,\mathrm{d}x$$

Have

$$y_t(x,t) = \frac{32V^2}{\pi^3}\sum_{n\text{ odd}}\frac{1}{n^3}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

Thus

$$K = \frac{512V^4}{\pi^6}\mu\int_0^l\sum_{n,m\text{ odd}}\frac{1}{n^3m^3}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)\,\mathrm{d}x$$

$$= \frac{512V^4}{\pi^6}\mu\sum_{n,m\text{ odd}}\frac{1}{n^3m^3}\cos\left(\frac{n\pi ct}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)\underbrace{\int_0^l\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\,\mathrm{d}x}_{=\frac{l}{2}\delta_{mn}}$$

$$= \frac{256V^4l}{\pi^6}\mu\sum_{n\text{ odd}}\frac{1}{n^6}\cos^2\left(\frac{n\pi ct}{l}\right)$$

Similarly, PE given by

$$V = \int_0^l\frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2\,\mathrm{d}x$$

Have

$$y_x(x,t) = \frac{32V^2}{c\pi^3}\sum_{n\text{ odd}}\frac{1}{n^3}\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

Thus

$$V = \frac{512V^4}{c^2\pi^6}T\int_0^l \sum_{n,m \text{ odd}} \frac{1}{n^3m^3} \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{m\pi ct}{l}\right)\ \mathrm{d}x$$

$$= \frac{512V^4}{c^2\pi^6}T \sum_{n,m \text{ odd}} \frac{1}{n^3m^3}\sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{m\pi ct}{l}\right)\underbrace{\int_0^l \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)\ \mathrm{d}x}_{\frac{l}{2}\delta_{mn}}$$

$$= \frac{256V^4 l}{c^2\pi^6}T \sum_{n \text{ odd}} \frac{1}{n^6}\sin^2\left(\frac{n\pi ct}{l}\right)$$

Compare this with the initial energy

$$K = \int_0^l \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\Big|_{t=0}\right)^2\ \mathrm{d}x$$

$$= \int_0^l \frac{1}{2}\mu \left(\frac{4V}{l^2}x(l-x)\right)^2\ \mathrm{d}x$$

$$= \frac{8V^2\mu}{l^4}\int_0^l x^2(l-x)^2\ \mathrm{d}x$$

$$= \frac{8V^2\mu}{l^4}\left(\frac{l^5}{30}\right)$$

$$= \frac{4V^2\mu l}{15}$$

Setting $t = 0$ in our previous result gives

$$K = \frac{256V^4 l}{\pi^6}\mu \sum_{n \text{ odd}} \frac{1}{n^6}$$

Thus by comparison

$$\frac{256}{\pi^6}\sum_{n \text{ odd}}\frac{1}{n^6} = \frac{4}{15}$$

$$\Rightarrow \sum_{n \text{ odd}}\frac{1}{n^6} = \frac{\pi^6}{960}$$

QUESTION 7

(i)   – Assume all displacements are sufficiently small $(y \ll l)$

– Assume all displacements are vertical

– Consider two points x and $x + \delta x$. The angle of the string to the horizontal at x is θ_1, and the angle at $x + \delta x$ is θ_2.

– Resolving horizontally shows that $T(x) \cos \theta_1 = T(x + \delta x) \cos \theta_2$, since $\theta \ll 1$, the tension is approximately constant.

– Resolving vertically

$$T \sin \theta_2 - T \sin \theta_1 - \mu g \delta x - 2k\mu \delta x \frac{\partial}{\partial t} y = \mu \delta x \frac{\partial^2}{\partial t^2} y \qquad (*)$$

– Assume angles are small

$$\sin \theta_2 \approx \tan \theta_2 = \left. \frac{\partial y}{\partial x} \right|_{x+\delta x} \approx \left. \frac{\partial y}{\partial x} \right|_x + \delta x \left. \frac{\partial^2 y}{\partial x^2} \right|_x$$

$$\sin \theta_1 \approx \tan \theta_1 = \left. \frac{\partial y}{\partial x} \right|_x$$

– (*) becomes

$$T \delta x \frac{\partial^2 y}{\partial x^2} - \mu g \delta x - 2k\mu \delta x \frac{\partial}{\partial t} y = \mu \delta x \frac{\partial^2}{\partial t^2} y$$

$$\Rightarrow \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial t^2} y + 2k \frac{\partial}{\partial t} y + g$$

– Further assume the weight is insignificant $(g \to 0)$

– Hence arrive at the equation of motion

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial t^2} y + 2k \frac{\partial}{\partial t} y$$

where $c^2 = \frac{T}{\mu}$

Assume $y(x, t) = X(x)T(t)$ and separating variables gives

$$c^2 \frac{X''}{X} = \frac{\ddot{T}}{T} + 2k \frac{\dot{T}}{T}$$

$$\Rightarrow \frac{X''}{X} = \frac{\ddot{T} + 2k\dot{T}}{T} = -\lambda$$

for some $\lambda > 0$, so we have

$$X'' + \lambda X = 0$$
$$\ddot{T} + 2k\dot{T} + \lambda c^2 T = 0$$

Solving the spatial equation first,

$$X = \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x)$$

Applying initial conditions

– $X(0) = 0 \Rightarrow \alpha = 0$
– $X(l) = 0 \Rightarrow \beta \sin(\sqrt{\lambda}x) = 0 \Rightarrow \lambda = n^2\pi^2/l^2$, for integer n

These λ are eigenvalues, with associated eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

The associated $T_n(t)$ is given by

$$\ddot{T}_n + 2k\dot{T}_n + \frac{n^2\pi^2 c^2}{L^2}T_n = 0 \qquad k = \frac{\pi c}{l}$$
$$\Rightarrow T_n(t) = e^{-kt}\left(\gamma_n \cos\left(\sqrt{n^2-1}kt\right) + \delta_n \sin\left(\sqrt{n^2-1}kt\right)\right) \qquad (n \geq 2)$$

Being careful with the $n = 1$ case, must have

$$T_1(t) = e^{-kt}\left(\gamma_1 + \delta_1 t\right)$$

Hence the general solution is

$$y(x,t) = e^{-kt}\sin\left(\frac{\pi x}{l}\right)(A_1 + B_1 t)$$
$$+ \sum_{n=2}^{\infty} e^{-kt}\sin\left(\frac{n\pi x}{l}\right)\left(A_n \cos\left(\sqrt{n^2-1}kt\right) + B_n \sin\left(\sqrt{n^2-1}kt\right)\right)$$

Using the boundary condition $y(x,0) = A\sin(\pi x/l)$, we have

$$A\sin(\pi x/l) = A_1 \sin\left(\frac{\pi x}{l}\right) + \sum_{n=2}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

Ah. So we conclude that $A_1 = A$, and $A_n = 0$ for $n \geq 2$, thus

$$y(x,t) = (A+B_1 t)e^{-kt}\sin\left(\frac{\pi x}{l}\right) + \sum_{n=2}^{\infty} e^{-kt}\sin\left(\frac{n\pi x}{l}\right)B_n \sin\left(\sqrt{n^2-1}kt\right)$$

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Next, use the boundary condition $y_t(x, 0) = 0$. We have that

$$y_t(x, t) = (-kA + B_1 - kB_1 t)e^{-kt}\sin\left(\frac{\pi x}{l}\right)$$
$$+ \sum_{n=2}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left[-kte^{-kt}B_n\sin\left(\sqrt{n^2-1}kt\right) + (\sqrt{n^2-1}k)e^{-kt}B_n\cos\left(\sqrt{n^2-1}kt\right)\right]$$

Thus

$$0 = y_t(x, 0) = (-kA + B_1)\sin\left(\frac{\pi x}{l}\right) + \sum_{n=2}^{\infty} B_n\sin\left(\frac{n\pi x}{l}\right)(\sqrt{n^2-1}k)$$

Thus we conclude that $B_1 = kA$ and $B_n = 0$ for $n \geq 2$.

Thus the general solution is given by

$$y(x, t) = A(1 + kt)e^{-kt}\sin\left(\frac{\pi x}{l}\right)$$

(ii)

QUESTION 8

(i)   – Assume all displacements are sufficiently small ($y \ll l$)

– Assume all displacements are vertical

– With mass M at $x = 0$, consider the tension of the string acting on the two points $-\varepsilon$ and ε either side, with the angle the string makes with the horizontal denoted by $\theta_{-\varepsilon}$ and $\theta_{+\varepsilon}$

– Resolving horizontally shows that the tension is constant.

– Using Newton's Second Law vertically, we have that

$$M \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\bigg|_{x=0} = T \sin \theta_\varepsilon - T \sin \theta_{-\varepsilon}$$

– Assume angles are small

$$\sin \theta_\varepsilon \approx \tan \theta_\varepsilon = \frac{\partial y}{\partial x}\bigg|_{x=\varepsilon}$$

– Hence

$$M \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\bigg|_{x=0} = T \frac{\partial y}{\partial x}\bigg|_{x=\varepsilon} - T \frac{\partial y}{\partial x}\bigg|_{x=-\varepsilon}$$
$$= T \left[\frac{\partial y}{\partial x}\right]_{x=0_-}^{x=0_+} \qquad \text{as } \varepsilon \to 0$$

(ii) The incident wave $W_I = \exp(i\omega(t - x/c))$ will produce a transmitted wave $W_T = T \exp(i\omega(t - x/c))$ and a reflected wave $W_R = R \exp(i\omega(t + x/c))$. Know that

$$y(x,t) = \begin{cases} W_I + W_R & \text{if } x < 0 \\ W_T & \text{if } x > 0 \end{cases}$$

with boundary conditions continuity at zero ($[y(0,t)]_{x=0_-}^{x=0_+} = 0$) and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\bigg|_{x=0} = \frac{T}{M} \left[\frac{\partial y}{\partial x}\right]_{x=0_-}^{x=0_+}$$

QUESTION 9

$y(x, t)$ define on $0 \le x \le l$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with $y(0, t) = y(l, t) = 0$ (fixed at endpoints). We can find the solution $y(x, t)$ for $t < 0$, given $y(x, 0) = 0$, and

$$\left[\frac{\partial y}{\partial t} \right]_{t=0_-}^{t=0_+} = \lambda \delta \left(x - \frac{l}{2} \right)$$

QUESTION 10