Mich. 2016 **NUMBERS & SETS – EXAMPLES $1\frac{1}{2}$** G. Taylor

Some more questions, for those who like this sort of thing.

1. Prove that every positive integer has some multiple which, when written in base 10, consists only of the digits 0 and 1.

2. Find a positive integer a such that, for every integer n, the number $n^4 + a$ is not prime.

3. How many subsets of $\{1, 2, \ldots, n\}$ have even size?

   How many subsets of $\{1, 2, \ldots, n\}$ contain no consecutive numbers?

4. Find 100 consecutive natural numbers, each of which is composite.

5. Let x be the sum of the digits of 4444^{4444}, and let y be the sum of the digits of x. What is the sum of the digits of y?

6. Let S_n denote the sum of the first n primes. Prove that there is always a square number strictly between S_n and S_{n+1}.

7. Find a function from $\mathbb{N}$ to $\mathbb{N}$ that takes every value infinitely often.

8. Does there exist a subset $S \subset \mathbb{N}$ such that every distance occurs exactly once? (In other words, for every $n \in \mathbb{N}$ there is exactly one pair of points x, y in S with $|x - y| = n$.)

9. There are 99 boxes, each containing some positive number of red balls and some positive number of blue balls. Your task is to choose 50 of the boxes in such a way that you have claimed more than half of all the red balls, and also more than half of all the blue balls. Can you always do this, no matter how the balls are distributed?

10. In a *tournament* on n players, each pair play a game, with one or other player winning (there are no draws). Construct a tournament in which, for any two players, there is a player who beats both of them. Is it true that for any k there is a tournament in which, for any k players, there is a player who beats all of them?

11. Does there exist a cycle in $\mathbb{Z}^3$ (i.e., a path consisting of line segments joining neighbouring points which have integer coordinates, ending up where it started) such that none of the projections in the x, y and z directions contains a cycle?

Please send any comments to me at glt1000@cam.ac.uk

Some more questions, for those who like this sort of thing.

1. Let p be a prime such that $p+2$ and p^2+2 are also prime. Prove that p^3+2 is also prime.

2. Written in base 10, the number 2^{29} has nine digits, all distinct. Which digit is missing?

3. *If you did the first part of this on my introductory sheet, feel free to skip straight to the deductions at the end.*

   The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$.

   By induction on k, show that $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ for all $k > 1$.

   Deduce that $(F_m, F_n) = (F_{m-n}, F_n)$ and then that $(F_m, F_n) = F_{(m,n)}$.

4. An RSA encryption scheme (n, e) has modulus $n = 187$ and encoding exponent $e = 7$. Find a suitable decoding exponent d. Without using a calculator, check your answer by encoding the number 35 and then decoding the result.

5. What is the 5^{th}-last digit of $5^{5^{5^{5^5}}}$?

6. *Part (ii) needs a result from a question on N&S sheet 2.*

   (i) Prove that there are infinitely many primes of the form $6n-1$.

   (ii) Let p be a prime of the form $6n-1$. Prove that -3 is not a square modulo p.

   (iii) By considering numbers of the form $(2p_1 \ldots p_k)^2 + 3$, prove that there are infinitely many primes of the form $6n+1$.

7. *In this question, you may assume that for any prime p, the multiplicative group modulo p is cyclic. I.e., there exists some g in $\mathbb{Z}_p$ for which $g^k \equiv 1 \bmod p$ only if $p-1$ divides k.*

   An odd number n is called a *Carmichael number* if it is not prime, but every positive integer a satisfies $a^n \equiv a \bmod n$.

   (i) Show that a Carmichael number cannot be divisible by the square of a prime.

   (ii) Show that a product of two distinct odd primes cannot be a Carmichael number.

   (iii) Show that a product of three distinct odd primes p, q, r is a Carmichael number if and only if $p-1$ divides $qr-1$, $q-1$ divides $pr-1$ and $r-1$ divides $pq-1$.

   (iv) Deduce that 1729 is a Carmichael number.

8. For which positive integers n does $\phi(n)$ divide n?

+9. Let n and k be positive integers. Suppose that n is a k^{th} power modulo p for all primes p. Must n be a k^{th} power?

Please send any comments to me at glt1000@cam.ac.uk

Some more questions, for those who like this sort of thing.

1. Let (x_n) be a sequence of real numbers which have a decimal expansion that uses only the digits 0 and 1. If $x_n \to x$, must x have a decimal expansion that uses only the digits 0 and 1?

2. Suppose that $x \in \mathbb{R}$ and $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 = 0$, where $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$. Prove that either x is an integer or it is irrational.

3. Prove that $2^{1/3} + 2^{2/3}$ and $\sqrt{2} + \sqrt{3} + \sqrt{5}$ are irrational.

4. Define a sequence (x_n) by setting $x_1 = 1$ and $x_{n+1} = \frac{x_n}{1+\sqrt{x_n}}$ for all $n \geqslant 1$. Show that (x_n) converges, and determine its limit.

5. Let (F_n) denote the sequence of Fibonacci numbers, and let $x_n = F_{n+1}/F_n$. Show that $x_n - x_{n+1} = (-1)^n/F_nF_{n+1}$ for all n. Deduce that (x_n) converges to a limit τ as $n \to \infty$, and that τ is irrational and algebraic.

6. Let (a_n) be a sequence of integers with $a_1 \in \mathbb{Z}$ and $0 \leqslant a_n < n$ for $n > 1$. Show that the series $\sum_{n=1}^{\infty} a_n/n!$ converges. Conversely, show that any real number can be represented as a series of this form. Show that a number is rational if and only if it has two such representations.

7. A real number $r = 0 \cdot d_1d_2d_3\ldots$ is called *repetitive* if its decimal expansion contains arbitrarily long blocks that are the same; that is, for every k there exist distinct m and n such that $d_m = d_n$, $d_{m+1} = d_{n+1}$, $\ldots$, $d_{m+k} = d_{n+k}$. Prove that the square of a repetitive number is repetitive.

8. Let (x_n) be a sequence of positive reals such that $\sum_{n=1}^{\infty} x_n$ converges. Must $\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ converge?

9. Let (x_n) be a sequence of reals such that $\sum_{n=1}^{\infty} x_n$ converges. Must $\sum_{n=1}^{\infty} \frac{x_n}{n}$ converge?

10. Let (x_n) be a real sequence with $x_n \to 0$. Prove that we may choose a sequence (ε_n), with each $\varepsilon_n = \pm 1$, such that $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent. If $(y_n)_{n=1}^{\infty}$ is another real sequence tending to 0, can we choose the ε_n so that $\sum_{n=1}^{\infty} \varepsilon_n y_n$ is convergent as well?

+11. Let (x_n) be a real sequence such that $\sum_{n=1}^{\infty} |x_n|$ is convergent. Show that if $\sum_{n=1}^{\infty} x_{kn} = 0$ for every $k \in \mathbb{N}$ then $x_n = 0$ for all n. What if we drop the restriction that $\sum_{n=1}^{\infty} |x_n|$ is convergent?

Please send any comments to me at glt1000@cam.ac.uk

Some more questions, for those who like this sort of thing.

1. Let p be prime. Prove that if $0 < k < p$ then $\binom{p}{k} \equiv 0 \bmod p$ in two different ways: using the factorial formula (but argue correctly), and using the definition in terms of subsets.

2. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$. How many functions $A \to B$ are there? How many injections $A \to B$ are there? How many surjections $B \to A$ are there?

3. (i) Give eight relations on the set $\mathbb{Z}$, one for each subset of {reflexive, symmetric, transitive}. That is: one obeying all of R, S, T; one obeying R and S but not T; etc.
   (ii) For each of the eight subsets of {reflexive, symmetric, transitive}, what is the smallest set on which we can define such a relation?

4. The relation S *contains* the relation R if aSb whenever aRb. Let R be the relation on $\mathbb{Z}$ given by 'aRb if $b = a + 3$'. How many equivalence relations on $\mathbb{Z}$ contain R?

5. Let $f : X \to Y$ be a function. For $A \subseteq X$ let $f(A) = \{f(x) : x \in A\}$. For $B \subseteq Y$ let $f^{-1}(B) = \{x \in X : f(x) \in B\}$. Give proofs or counter-examples for the following claims:
   (a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$; (b) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$;
   (c) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$; (d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;
   (e) $f^{-1}(f(A)) = A$; (f) $f(f^{-1}(B)) = B$.

   Show that for each case which is not true, we can replace '$=$' with one of '$\subseteq$' and '$\supseteq$'.

6. For two sets A, B, we write A^B for the set of functions from B to A. Show that for (not necessarily finite) sets A, B, C, there is a bijection between $(A^B)^C$ and $A^{B \times C}$.

7. *The Schröder-Bernstein theorem states the following: if A, B are sets and there are injections $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.*
   (i) Use the Schröder-Bernstein theorem to show that there is a bijection between the open interval $(0, 1)$ and the closed interval $[0, 1]$. Now give an explicit bijection.
   (ii) Use the Schröder-Bernstein theorem to show that there is a bijection between the set $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ and the set $\mathbb{R}^{\mathbb{R}}$ of all functions $\mathbb{R} \to \mathbb{R}$.
   (iii) If you didn't prove the Schröder-Bernstein theorem in lectures, then try proving it!

8. Let A be the set of all bijections from $\mathbb{N}$ to $\mathbb{N}$, and let $B = \{f \in A : f(n) = n$ for all but finitely many $n \in \mathbb{N}\}$. Show that A is uncountable, but that B is countable.

9. Define a *cross* in the plane to be two open straight line segments intersecting at exactly one point. Is there an uncountable set of pairwise disjoint crosses in the plane?

+10. Each of an infinite sequence of dons has a favourite colour, namely red or blue, and each of finitely many first year mathematicians decides to buy a don a hat for Christmas. Each student may ask about the colour preferences of any (proper) subset of the dons, in any order, but to keep their presents a surprise, the students don't share this information with each other, and each must buy a hat for a don whose preference they haven't asked about. How many students can successfully buy their chosen don's favourite colour hat?

Please send any comments to me at glt1000@cam.ac.uk