Part IB — Numerical Analysis Example Sheet 2

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QUESTION 1

The differential equations, with initial condition $y(0) = 1$ have exact solutions given by

$$y = \frac{1}{1+t} \quad \text{and} \quad y = (1+t)^2, \quad 0 \le t \le 1$$

respectively.
The Euler method: we approximate the exact solution of the ODE:

$$y' = f(t, y)$$

by

$$y_{n+1} = y_n + hf(t, y_n), \quad n = 0, 1, \cdots$$

where y_n approximates $y(nh)$.
For the first ODE we have $f(t, y) = -\frac{y}{1+t}$.
Here, $y_0 = 1$,

$$\begin{aligned} y_1 &= y_0 + h\left(-\frac{y_0}{1+t}\right) \\ &= y_0\left(1 - \frac{h}{1+t}\right) \\ &= 1 - \frac{h}{1+t} \end{aligned}$$

$$y_2 = \left(1 - \frac{h}{1+t}\right)^2, \cdots, y_n = \left(1 - \frac{h}{1+t}\right)^n$$

For the second ODE we have $f(t, y) = \frac{2y}{1+t}$.
Here, $y_0 = 1$,

$$\begin{aligned} y_1 &= y_0 + h\left(\frac{2y_0}{1+t}\right) \\ &= y_0\left(1 + \frac{2h}{1+t}\right) \end{aligned}$$

Similarly, I'm just getting

$$y_n = \left(1 + \frac{2h}{1+t}\right)^n$$

QUESTION 2

The trapezoidal rule states that the numerical solution to the differential equation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}_n) \qquad (2.1)$$

is given by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})] \qquad (2.2)$$

Assuming that $\mathbf{f}$ satisfies the Lipschitz condition: there exists $\lambda \geq 0$ such that

$$||\mathbf{f}(t, \mathbf{v}) - \mathbf{f}(t, \mathbf{w})|| \leq \lambda||\mathbf{v} - \mathbf{w}||, \quad t \in [0, t^*], \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^N$$

we will prove that the trapezoidal rule converges; ie.

$$\lim_{h \to 0} \max_{n=0,\cdots,\lfloor t^*/h \rfloor} ||\mathbf{y}_n(h) - \mathbf{y}(nh)|| = 0$$

where $\mathbf{y}(nh)$ is the evaluation at time $t = nh$ of the exact solution of

Proof. Let $\mathbf{e}_n = \mathbf{y}_n - \mathbf{y}(t_n)$, the error at step n, where $0 \leq n \leq t^*/h$, $t_n := nh$. Thus,

$$\mathbf{e}_{n+1} = \mathbf{y}_{n+1} - \mathbf{y}(t_{n+1}) = [\mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]] - [\mathbf{y}(t_n) + h\mathbf{y}'(t_n) + O(h^2)]$$

By the Taylor theorem, the $O(h^2)$ term can be bounded uniformly for all $[0, t^*]$ by ch^2, where $c > 0$. □

QUESTION 3

QUESTION 4

QUESTION 5

QUESTION 6

QUESTION 7

QUESTION 8

QUESTION 9

QUESTION 10

QUESTION 11

QUESTION 12