Part IB — Methods Example Sheet 4

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QUESTION 1

(i) Along characteristic curves,

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 1 \quad \frac{\mathrm{d}y}{\mathrm{d}s} = y$$

which has general solution $x = s + c$ and $y = Ae^s$ for some constants c, A. Cauchy data is $B(t) = \{(x = 0, y = t)\}$, intersect B at $s = 0$, thus characteristic curves are

$$x = s \quad y = te^s$$

$\frac{\partial u}{\partial s}\big|_t = 0$ so that $u = \text{cst}$ along these characteristics.
and the Cauchy data fixes $u(s,t) = t^3$ on the t^{th} curve. Inverting gives

$$s = x \quad t = ye^{-x}$$

and therefore the solution to our problem is

$$u(x, y) = y^3 e^{-3x}$$

throughout $\mathbb{R}^2$

(ii) Along characteristic curves

$$\frac{\mathrm{d}x}{\mathrm{d}s} = y \quad \frac{\mathrm{d}y}{\mathrm{d}s} = x$$

$$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}s^2} = x$$

$$x = A \sinh s + B \cosh s$$

Cauchy data is $B(t) = \{(x = 0, y = t)\}$, intersect at $s = 0$ gives $B = 0$, thus

$$x = t \sinh s \qquad y = t \cosh s$$

Then $\frac{\partial u}{\partial s}\Big|_t = 0 \Rightarrow u = \text{ cst.}$ along these characteristics, and Cauchy data fixes $u(s,t) = e^{-t^2}$, and have that

$$t^2 = (t \cosh s)^2 - (t \sinh s)^2 = y^2 - x^2$$

therefore solution is

$$u(x, y) = e^{x^2 - y^2}$$

which is uniquely defined only in the upper quadrant of the plane $\{(x, y) \subset R^2 : x \geq 0, y \geq 0\}$

(iii) PDE is $u_x + u_y = e^{x+2y} - u$.

Along characteristic curves

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 1 \quad \frac{\mathrm{d}y}{\mathrm{d}s} = 1$$

$$x = s + c \qquad y = s + d$$

Cauchy data is $B(t) = \{(x = t, y = 0)\}$, intersect at $s = 0$, thus

$$x = s + t \qquad y = s$$

Then $\left.\frac{\partial u}{\partial s}\right|_t = e^{2x+y} - u = e^{3s+t} - u$ along these characteristics, this is an ODE in s so multiplying by the integrating factor e^s gives

$$e^s \frac{\mathrm{d}u}{\mathrm{d}s} + e^s u = e^{4s+t}$$

$$\Rightarrow e^s u = \frac{1}{4} e^{4s+t} + \text{cst.}$$

Cauchy data $u(t, 0) = 0$ fixes cst $= -\frac{1}{4}e^t$, and have that

$$u(s,t) = \frac{1}{4} e^{t+3s} - \frac{1}{4} e^{t-s}$$

inverting the relations

$$s = y \qquad t = x - y$$

the solution is

$$\begin{aligned} u(x,y) &= \frac{1}{4} e^{x+2y} - \frac{1}{4} e^{x-2y} \\ &= \frac{1}{2} e^x \sinh 2y \end{aligned}$$

QUESTION 2

Separation of variables $\Rightarrow u(x,t) = X(x)T(t)$, thus

$$X''T + X\dot{T} = 0$$
$$\Rightarrow \frac{X''}{X} = -\frac{\dot{T}}{T}$$

LHS independent of x, RHS independent of t, so both sides constant. Setting $\lambda = \dot{T}/T$ we have

$$X'' + \lambda X = 0$$
$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

Boundary conditions $X(0) = X(\pi) = 0$ imply that $A = 0$, $\lambda = n^2$. Then solving $\dot{T} = \lambda T$ with condition $T(0) = U(x)$ gives

$$T(t) = U(x)e^{n^2 t}$$

Thus the unnormalised eigenfunctions of our problem are

$$u(x,t) = U(x)e^{n^2 t}\sin nx$$

For large n the solution then has oscillations with higher and higher wavenumber and larger and larger (indeed arbitrarily large) amplitude $U(x)e^{n^2 t}$, and so this problem is ill-posed.

QUESTION 3

(i) The principal part of the symbol of the differential operator is $\mathbf{k}^T\mathbf{A}\mathbf{k}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$$

$\det A$ is product of eigenvalues, therefore we have

$$\begin{array}{rl} \text{elliptic} & x > 0 \\ \text{parabolic} & x = 0 \\ \text{hyperbolic} & x < 0 \end{array}$$

In the hyperbolic region the negative eigenvector points in the y direction. Thus, if $f(x,y) = \text{cst.}$ is to be a characteristic surface, we need $\partial_y f = \pm\sqrt{\partial_x f A_{xx} \partial_x f/x} = \pm\sqrt{x}\partial_x f$. Letting $p = 2x^{1/2}$, this is $(\partial_y \pm \partial_p)f = 0$, so the characteristic surfaces are the two curves of constant $y \pm p$, that is

$$u = y + x^{1/2}$$
$$v = y - x^{1/2}$$

for u constant and v constant.

(ii) The PDE is hyperbolic in the $y < 0$ region. Here

$$\mathbf{A} = \text{diag}(1, y)$$

and $\mathbf{m}$ points in the y-direction, with $\mathbf{Am} = y\mathbf{m}$. Thus, if $f(x,y) = \text{cst.}$ is to be a characteristic surface, we need $\partial_y f = \pm\sqrt{\partial_x f A_{xx} \partial_x f/y} = \pm\sqrt{y}\partial_x f$. Letting $\xi = x, \nu = 2y^{1/2}$, this is $(\partial_\xi \pm \partial_p)f = 0$, so the characteristic surfaces are the two curves of constant $y \pm p$, that is

QUESTION 4

Green's second identity is

$$\int_{\Omega} \phi \nabla^2 \psi - \psi \nabla^2 \phi \, \mathrm{d}V = \int_{\delta\Omega} \phi(\mathbf{n} \cdot \nabla \psi) - \psi(\mathbf{n} \cdot \nabla \phi) \mathrm{d}S$$

where $\Omega \subset \mathbb{R}^n$ is a compact set, and $\phi, \psi : \Omega \to \mathbb{R}$ are a pair of functions on Ω regular throughout Ω.

QUESTION 5

$u(\mathbf{x})$ is harmonic and therefore satisfies $\nabla^2 \mathbf{u} = 0$. Consider a Dirichlet Green's function for the Laplace operator on D; we have

$$\nabla^2 G(\mathbf{r}; \mathbf{r}_0)$$

It can be shown that

$$G(\mathbf{x}; \mathbf{x}_0) = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0|$$

Using Green's second identity with $\mathbf{u}$ and G we have

$$\int_\Omega u\nabla^2 G - G\nabla^2 u \, \mathrm{d}V = \int_{\delta\Omega} G(\mathbf{n} \cdot \nabla u) - u(\mathbf{n} \cdot \nabla G) \mathrm{d}S$$

For some reason, $\mathbf{n} \cdot \nabla u = \frac{\partial u}{\partial n}$. Also, since G only depends on the outward normal, we have $\mathbf{n} \cdot \nabla G = \frac{\partial G}{\partial n}$.

Thus the equation becomes

$$\int_\Omega u\nabla^2 G - G\nabla^2 u \, \mathrm{d}V = \int_{\delta\Omega} G(\mathbf{n} \cdot \nabla u) - u(\mathbf{n} \cdot \nabla G) \mathrm{d}S$$

QUESTION 6

QUESTION 7

QUESTION 8

QUESTION 9

QUESTION 10

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \ : \ y \geq 0\}$ and suppose $\psi : \Omega \to \mathbb{R}$ solves Laplace's equation $\nabla^2 \psi = 0$ inside Ω, subject to

$$\psi(x, 0) = f(x) \quad \text{and} \quad \lim_{|\mathbf{x}| \to \infty} \psi = 0$$

(i) We must construct a Green's function that vanishes on $\delta\Omega$. As well as vanishing on the x-axis, we also require G vanishes as $|\mathbf{x}| \to \infty$. We'll set $\mathbf{x} = (x, y)$ and $\mathbf{y} := \mathbf{x}_0^+ = (x_0, y_0)$ in terms of Cartesian coordinates, with $y_0 > 0$. We know that the free-space Green's function

$$G_2(\mathbf{x}, \mathbf{x}_0^+) = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x_0}^+| + c_2$$

satisfies all conditions except that

$$G_2(\mathbf{x}, \mathbf{x}_0^+)|_{y=0} = \frac{1}{2\pi} \log |[(x - x_0)^2 + y_0^2]^{1/2}| + c_2 \neq 0$$

We need to cancel the nonzero boundary value of G_2 by adding on some function.

Let $\mathbf{x_0}^-$ be the point $(x_0, -y_0)$. The location $\mathbf{x_0}^- \notin \Omega$, so the Green's function $G_2(\mathbf{x}, \mathbf{x}_0^-)$ is regular everywhere within Ω, and so obeys Laplace's equation everywhere in the upper half-space. Also,

$$G_2(\mathbf{x}, \mathbf{x}_0^- |_{y=0} = \frac{1}{2\pi} \log |[(x - x_0)^2 + y_0^2]^{1/2}| + c_2'$$

QUESTION 11

QUESTION 12