Part IB — Complex Methods Example Sheet 2

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QUESTION 1

(i)

$$\begin{aligned} z/\log(1+z) &= z\left[z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots\right]^{-1} \\ &= \left[1-\frac{z}{2}+\frac{z^2}{3}-\cdots\right]^{-1} \\ &= \left(1-\frac{z}{2}\right)^{-1}+O(z^2) \\ &= 1+\frac{z}{2}+O(z^2) \end{aligned}$$

Owing to the $\log(1+z)$ term, this series expansion converges if $|z|<1$.

(ii)

$$\begin{aligned} (\cos z)^{1/2}-1 &= \left[1-\frac{z^2}{2}+\frac{z^4}{4!}-\cdots\right]^{1/2}-1 \\ &= \left(1-\left(\frac{z^2}{2!}-\frac{z^4}{4!}\right)\right)^{1/2}-1+O(z^6) \\ &= -\frac{1}{4}z+\frac{5}{96}z^4+O(z^6) \end{aligned}$$

Converges for all $z\in\mathbb{C}$.

(iii) For $|e^z|<1 \iff |e^x e^{iy}|<1 \iff x<0$, we have, comparing coefficients of powers of z,

$$\begin{aligned} \log(1+e^z) &= e^z-\frac{e^{2z}}{2}+\frac{e^{3z}}{3}-\frac{e^{4z}}{4}+\cdots \\ &= \underbrace{\left(1-\frac{1}{2}+\frac{1}{3}-\cdots\right)}_{=\log 2}+(1-1+1-1+\cdots)\,z \\ &\quad+\frac{1}{2}(1-2+3-4+\cdots)\,z^2+\frac{1}{3!}\left(1^2-2^2+3^2-4^2+\cdots\right)z^3+O(z^4) \end{aligned}$$

And if $x>0$, have

$$\begin{aligned} \log(1+e^z) &= \log(e^z(1+e^{-z})) = z+e^{-z}-\frac{e^{-2z}}{2}+\frac{e^{-3z}}{3}-\frac{e^{-4z}}{4}+\cdots \\ &= \underbrace{\left(1-\frac{1}{2}+\frac{1}{3}-\cdots\right)}_{=\log 2}+(1-1+1-1+\cdots)\,z \\ &\quad+\frac{1}{2}(1-2+3-4+\cdots)\,z^2+\frac{1}{3!}\left(1^2-2^2+3^2-4^2+\cdots\right)z^3+O(z^4) \end{aligned}$$

(iv)

$$\begin{aligned} e^{e^z} &= 1 + e^z + \frac{1}{2!}e^{2z} + \frac{1}{3!}e^{3z} + \cdots \\ &= \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) + \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) z \\ &\quad + \frac{1}{2!}\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) z^2 + \cdots \end{aligned}$$

which seems to be valid for all z.

QUESTION 2

Using partial fractions,

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)$$

Have $0 < |a| < |b|$. In the region $|z| < |a|$, we have no singularities, ie our function is analytic here, and we can calculate the Taylor series about $z_0 = 0$. Note that (for $|z| < |a|$)

$$\frac{1}{z-a} = -\frac{1}{a}\left(1 - \frac{z}{a}\right)^{-1} = -\sum_{n=0}^{\infty} \frac{1}{a^{n+1}} z^n$$

Hence

$$\frac{1}{(z-a)(z-b)} = -\frac{1}{a-b}\sum_{n=0}^{\infty}\left(\frac{1}{a^{n+1}} - \frac{1}{b^{n+1}}\right) z^n$$

In the region $|a| < |z| < |b|$ we can determine a Laurent series for $\frac{1}{z-a}$ in this annulus, (but $\frac{1}{z-b}$ still has a Taylor series). Note that

$$\frac{1}{z-a} = \frac{1}{z}\left(1 - \frac{a}{z}\right)^{-1} = \sum_{m=0}^{\infty} \frac{a^m}{z^{m+1}} = \sum_{n=-\infty}^{-1} a^{-n-1} z^n.$$

Hence

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\sum_{n=-\infty}^{-1} a^{-n-1} z^n + \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} z^n\right)$$

Finally, in the region $|z| > |b|$, this is an annulus, that goes from $|b|$ to infinity. So it has a Laurent series, given by

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\sum_{n=-\infty}^{-1} (a^{-n-1} + b^{-n-1}) z^n\right)$$

QUESTION 3

We note that $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ has zeros at $e^{2iz} = 1$ ie. $z = n\pi$ for integer n.

The annulus $0 < |z| < \pi$ contains no singularities, thus there exists a Laurent series for $\sin z$ in this annulus.

$$\begin{aligned}
\operatorname{cosec}^2 z &= \left[\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right]^{-1} \\
&= \left[z^2 - \frac{z^4}{3} + \frac{z^6}{60} - \cdots\right]^{-1} \\
&= z^{-2}\left[1 - \frac{1}{3}z^2 + \frac{1}{60}z^4 - \cdots\right]^{-1}
\end{aligned}$$

Not sure how to do the binomial expansion in a valid way.

Next part, if $0 < |z| < \pi$.

$$\begin{aligned}
g(z) &= f(z) - z^2 - \frac{1}{\pi^2}\left(1 + \frac{z}{\pi}\right)^{-2} - \frac{1}{\pi^2}\left(1 - \frac{z}{\pi}\right)^{-2} \\
&=
\end{aligned}$$

Can see z^{-2} term is removed by $f(z) - z^{-2}$, hence we $a_n = 0$ for all $n < 0$, and can remove the singularity at $z = 0$ by setting

$$G(z) = \begin{cases} g(z) & \text{if } z \neq 0 \\ \text{constant term in } g(z) & \text{if } z = 0 \end{cases}$$

Not sure why $z = |\pi|$ is fine?

QUESTION 4

$f(z)$ has a zero of order N at $z = z_0$ if $0 = f(z_0) = f'(z_0) = \cdots = f^{N-1}(z_0)$, but $f^{(N)}(z_0) \neq 0$.

If there is a $N > 0$ such that $a_n = 0$ for all $n < -N$ but $a_{-N} \neq 0$, then f has a pole of order N at z_0.

Not sure.

Write

$$f(z) = (z - z_0)^N G(z)$$

for some G with $G(z_0) \neq 0$. Then $\frac{1}{G(z)}$ has a Taylor series about z_0, and then the result follows.

QUESTION 5

(i) $\frac{1}{z^3(z-1)^2}$ has isolated singularities at $z = 0$ and $z = 1$.

(ii) $\tan z$ has isolated singularities at $z = (n + \frac{1}{2})\pi$.

(iii) $\sinh z$ has zeros where $\frac{1}{2}(e^z - e^{-z}) = 0$, i.e. $e^{2z} = 1$, i.e. $z = n\pi i$, where $n \in \mathbb{Z}$. Hence $z \coth z$ has isolated singularities here.

(iv) $\frac{e^z - e}{(1-z)^3}$ has a singularity at $z = 1$,

(v) $\exp(\tan z)$ has singularities $z = \frac{\pi}{2} + n\pi$.

(vi) $\sinh \frac{z}{z^2-1}$ has singularities at $z = \pm 1$

(vii) $\log(1 + e^z)$ has singularities $z = (1 + 2n)i\pi$.

(viii) $\tan(z^{-1})$ has singularities $1/(\frac{\pi}{2} + n\pi) = \frac{2}{\pi(2n+1)}$.

QUESTION 6

Firstly $\int_{-1}^{1} z \, \mathrm{d}z$ evaluated along γ_1, the straight line from -1 to $+1$ is simply $\left[\frac{z^2}{2}\right]_{-1}^{1} = 0$.

We integrate along the semicircular contour by making the substitution $z = e^{i\theta}$, $\mathrm{d}z = ie^{i\theta} \ \ d\theta$. Then

$$\begin{aligned}\int_{\gamma_2} z \, \mathrm{d}z &= \int_{\pi}^{0} e^{i\theta} \cdot ie^{i\theta} \, \mathrm{d}\theta \\ &= \int_{\pi}^{0} ie^{2i\theta} \, \mathrm{d}\theta \\ &= \left[\frac{1}{2}e^{2i\theta}\right]_{\pi}^{0} \\ &= 0\end{aligned}$$

Next, consider

$$I_3 = \oint_{\gamma_3} \bar{z} \, \mathrm{d}z, \quad I_4 = \oint_{\gamma_4} \bar{z} \, \mathrm{d}z$$

where γ_3 is the unit circle $|z| = 1$, and γ_4 is the translated unit circle $|z-1| = 1$. For I_3 we again make the substitution $z = e^{i\theta}$, $\mathrm{d}z = ie^{i\theta} \ \ d\theta$, so

$$\begin{aligned}I_3 &= \int_{0}^{2\pi} e^{-i\theta} ie^{i\theta} \, \mathrm{d}\theta \\ &= 2\pi\end{aligned}$$

For I_4 we make the substitution $z = 1 + e^{i\theta}$, $\mathrm{d}z = ie^{i\theta} \ \ d\theta$, so

$$\begin{aligned}I_4 &= \int_{0}^{2\pi} (1 + e^{-i\theta}) ie^{i\theta} \, \mathrm{d}\theta \\ &= \int_{0}^{2\pi} i(1 + e^{i\theta}) \, \mathrm{d}\theta \\ &= 2\pi i\end{aligned}$$

QUESTION 7

At a *simple* pole, the residue is given by

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0)f(z).$$

Hence

(i)

$$\begin{aligned}\operatorname*{res}_{z=z_0} f(z)/(z-z_0) &= \lim_{z\to z_0} f(z)\\ &= f(z_0)\end{aligned}$$

(ii)

$$\begin{aligned}\operatorname*{res}_{z=z_0} f(z)/g(z) &= \lim_{z\to z_0} (z-z_0)f(z)/g(z)\\ &= f(z_0)\end{aligned}$$

(iii)

Proposition. At a pole of order N, the residue is given by

$$\lim_{z\to z_0} \frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}(z-z_0)^N f(z).$$

Proof. We can simply expand the right hand side to obtain

$$\begin{aligned}
&\lim_{z\to z_0} \frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}(z-z_0)^N\left(\frac{a_{-N}}{(z-z_0)^N}+\cdots+\frac{a_{-2}}{z-z_0}^2+\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+\cdots\right)\\
&\lim_{z\to z_0} \frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\left(a_{-N}+a_{-N+1}(z-z_0)+\cdots+a_{-1}(z-z_0)^{N-1}+a_0(z-z_0)^N+\cdots\right)\\
&\lim_{z\to z_0} \frac{1}{(N-1)!}\left((N-1)!a_{-1}+N!a_0(z-z_0)+\cdots\right)\\
&=\lim_{z\to z_0}\left(a_{-1}+Na_0(z-z_0)+\cdots\right)\\
&=a_{-1}
\end{aligned}$$

as required. □

We now compute the residues of the poles given in question 5.

(i) We can use the fact that f has a pole of order 3 at $z=0$. So we can use the formula to obtain

$$\operatorname*{res}_{z=0} f(z) = \lim_{z\to 0}\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(z^3 f(z)) = \lim_{z\to 0}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{1}{(z-1)^2} = \frac{1}{2}.$$

QUESTION 8

QUESTION 9

We shall evaluate

$$I = \int_\gamma \frac{z^n \, \mathrm{d}z}{(z-a)(z-a^{-1})},$$

where γ is the unit circle. Making the substitution $z = e^{i\theta}$ and traversing anticlockwise from $z = 1$ gives

$$\begin{aligned} I &= \int_0^{2\pi} \frac{e^{in\theta}}{(e^{i\theta}-a)(e^{i\theta}-a^{-1})} ie^{i\theta} \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{ie^{in\theta}}{e^{i\theta}-a-a^{-1}+e^{-i\theta}} \, \mathrm{d}\theta \\ &= -ia \int_0^{2\pi} \frac{\cos n\theta + i \sin n\theta}{1-2a\cos\theta + a^2} \, \mathrm{d}\theta \end{aligned}$$

Now evaluating I by the residue theorem, the poles of the integrand are at $z_0 = a$ and $z_1 = a^{-1}$, with z_1 lying inside the contour with residue

$$\frac{a^{-n+1}}{1-a^2}$$

Hence we get

$$I = 2\pi i \frac{a^{-n+1}}{1-a^2}$$

QUESTION 10

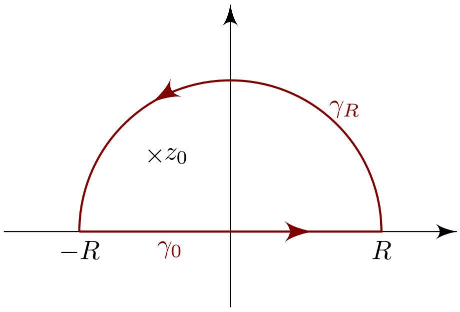
We shall evaluate

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x+x^2},$$

Consider

$$\oint_\gamma \frac{\mathrm{d}z}{1+z+z^2},$$

where γ is the contour "closing in the upper-half plane", shown: from $-R$ to R along the real axis (γ_0), then returning to $-R$ via a semicircle of radius R in the upper half plane (γ_R).



Now we have

$$\frac{1}{1+z+z^2} = \frac{1}{(z-z_0)(z-\bar{z_0})}.$$

where $z_0 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. So the only singularity enclosed by γ is a simple pole at $z = z_0$, where the residue is

$$\lim_{z \to z_0} \frac{1}{z - \bar{z_0}} = -\frac{1}{\sqrt{3}i}.$$

Hence

$$\int_{\gamma_0} \frac{\mathrm{d}z}{1+z+z^2} + \int_{\gamma_R} \frac{\mathrm{d}z}{1+z+z^2} = \int_\gamma \frac{\mathrm{d}z}{1+z+z^2} = 2\pi i \cdot -\frac{1}{\sqrt{3}i} = -\frac{2\sqrt{3}}{3}\pi.$$

Let's now look at the terms individually. We know

$$\int_{\gamma_0} \frac{\mathrm{d}z}{1+z^2} = \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^2} \to I$$

as $R \to \infty$. Also,

$$\int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2} \to 0$$

as $R \to \infty$ (see below). So we obtain in the limit

$$I + 0 = -\frac{2\sqrt{3}}{3}\pi.$$

So

$$I = -\frac{2\sqrt{3}}{3}\pi.$$

Finally, we need to show that the integral about γ_R vanishes as $R \to \infty$. We can also do this informally, by writing

$$\left| \int_{\gamma_R} \frac{\mathrm{d}z}{1+z+z^2} \right| \leq \pi R \sup_{z \in \gamma_R} \left| \frac{1}{1+z+z^2} \right| = \pi R \cdot O(R^{-2}) = O(R^{-1}) \to 0.$$

QUESTION 11

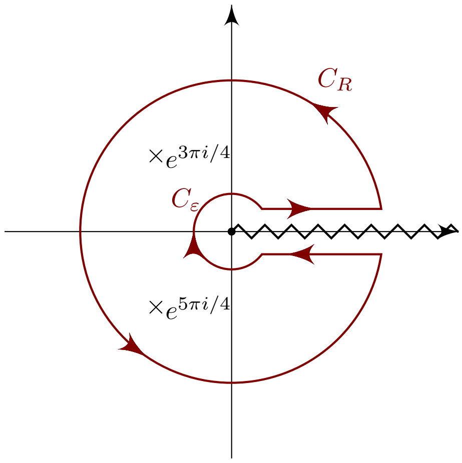
we want to integrate

$$I = \int_0^\infty \frac{x^{a-1}}{1+x}\,\mathrm{d}x,$$

with $0 < a < 1$ so that the integral converges. We need a branch cut for z^{a-1}. We take our branch cut to be along the positive real axis, and define

$$z^\alpha = r^\alpha e^{i\alpha\theta},$$

where $z = re^{i\theta}$ and $0 \le \theta < 2\pi$. We use the following keyhole contour:



This consists of a large circle C_R of radius R, a small circle C_ε of radius ε, and the two lines just above and below the branch cut. We will simultaneously take the limit $\varepsilon \to 0$ and $R \to \infty$.

We have four integrals to work out. The first is

$$\int_{\gamma_R} \frac{z^{a-1}}{1+z}\,\mathrm{d}z = O(R^{a-2}) \cdot 2\pi R = O(R^{a-1}) \to 0$$

as $R \to \infty$. To obtain the contribution from γ_ε, we substitute $z = \varepsilon e^{i\theta}$, and obtain

$$\int_{2\pi}^0 \frac{\varepsilon^{a-1}e^{i(a-1)\theta}}{1+\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\,\mathrm{d}\theta = O(\varepsilon^{a-1}) \to 0.$$

Finally, we look at the integrals above and below the branch cut. The contribution from just above the branch cut is

$$\int_\varepsilon^R \frac{x^{a-1}}{1+x}\,\mathrm{d}x \to I.$$

Similarly, the integral below is

$$\int_R^\varepsilon \frac{x^{a-1}e^{2a\pi i}}{1+x}\,\mathrm{d}x \to -e^{2a\pi i}I.$$

So we get

$$\oint_\gamma \frac{z^(a-1)}{1+z}\,\mathrm{d}z \to (1-e^{2a\pi i})I.$$

All that remains is to compute the residues. The only pole is the simple pole at $z = -1$, with residue

$$(-1)^{a-1}$$

Hence we know

$$(1 - e^{2a\pi i})I = 2\pi i \left((-1)^{a-1}\right).$$

In other words, we get

$$e^{a\pi i}\frac{1}{2i}(e^{-a\pi i} - e^{a\pi i})I = \pi(-1)^{a-1}e^{-\pi a i}$$

Thus we have

$$I = \frac{\pi}{\sin \pi a}$$

QUESTION 12

QUESTION 13

Not too sure about these trigonometric integrals, but will attempt before supervision.

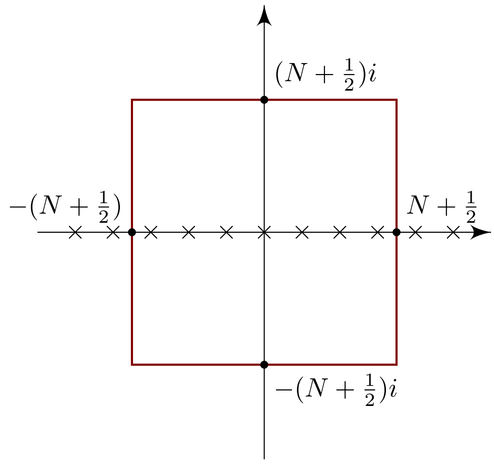
QUESTION 14

QUESTION 15

Consider the integral

$$\int_\gamma \frac{\cot z}{z^2+\pi^2 a^2}\,\mathrm{d}z,$$

where γ is the square contour shown with corners at $(N+\frac{1}{2})(\pm 1 \pm i)$, where N is a large integer, avoiding the singularities



There are simple poles at $z = n\pi,\ n \in \mathbb{Z}\setminus\{0\}$, with residues $\frac{1}{(n^2+a^2)\pi}$, and a two poles at $z = \pm i\pi a$ with residue $-\frac{1}{\pi a}\coth \pi a$ each. It turns out the integrals along the sides all vanish as $N \to \infty$ (see later). So we know

$$2\pi i\left(2\sum_{n=1}^{N}\frac{1}{(n^2+a^2)\pi} - \frac{2}{\pi a}\coth \pi a\right) \to 0$$

as $N \to \infty$. In other words,

$$\sum_{n=1}^{N}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth \pi a.$$

[Not sure of the details of calculating these residues...] Hence all that remains is to show that the integrals along the sides vanish. On the right-hand side, we can write $z = N + \frac{1}{2} + iy$. Then

$$|\cot z| = \left|\cot\left(\left(N+\frac{1}{2}\right)+iy\right)\right| = |-\tan iy| = |\tanh y| \le 1.$$

So $\cot \pi z$ is bounded on the vertical side. Since we are integrating $\frac{\cot \pi z}{z^2+\pi^2 a^2}$, the integral vanishes as $N \to \infty$.

Along the top, we get $z = x + \left(N+\frac{1}{2}\right)i$. This gives

$$|\cot z| = \frac{\sqrt{\cosh^2\left(N+\frac{1}{2}\right)-\sin^2 x}}{\sqrt{\sinh^2\left(N+\frac{1}{2}\right)+\sin^2 x}} \le \coth\left(N+\frac{1}{2}\right) \le \coth\frac{1}{2}.$$

So again $\cot \pi$ is bounded on the top side. So again, the integral vanishes as $N \to \infty$.

Similarly the left and bottom boundary both vanish too, hence the required result.