Part IB — Complex Methods Example Sheet 2

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QUESTION 1

Using partial fractions,

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)$$

Have $0 < |a| < |b|$. In the region $|z| < |a|$, we have no singularities, ie our function is analytic here, and we can calculate the Taylor series about $z_0 = 0$. Note that (for $|z| < |a|$)

$$\frac{1}{z-a} = -\frac{1}{a}\left(1 - \frac{z}{a}\right)^{-1} = -\sum_{n=0}^{\infty} \frac{1}{a^{n+1}} z^n$$

Hence

$$\frac{1}{(z-a)(z-b)} = \frac{1}{b-a}\sum_{n=0}^{\infty}\left(\frac{1}{a^{n+1}} - \frac{1}{b^{n+1}}\right) z^n$$

In the region $|a| < |z| < |b|$ we can determine a Laurent series for $\frac{1}{z-a}$ in this annulus, (but $\frac{1}{z-b}$ still has a Taylor series). Note that

$$\frac{1}{z-a} = \frac{1}{z}\left(1 - \frac{a}{z}\right)^{-1} = \sum_{m=0}^{\infty} \frac{a^m}{z^{m+1}} = \sum_{n=-\infty}^{-1} a^{-n-1} z^n.$$

Hence

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\sum_{n=-\infty}^{-1} a^{-n-1} z^n + \sum_{n=0}^{\infty} \frac{1}{a^{n+1}} z^n\right)$$

Finally, in the region $|z| > |b|$, we can determine a Laurent Series in the annulus $|b| < |z| < |R|$ for some $R \in \mathbb{C}$ st. $|R| > |b|$.

QUESTION 2

QUESTION 3

QUESTION 4

Firstly $\int_{-1}^{1} z \, \mathrm{d}z$ evaluated along γ_1, the straight line from -1 to $+1$ is simply $\left[\frac{z^2}{2}\right]_{-1}^{1} = 0$.

We integrate along the semicircular contour by making the substitution $z = e^{i\theta}$, $\mathrm{d}z = ie^{i\theta} \; d\theta$. Then

$$\begin{aligned}
\int_{\gamma_2} z \, \mathrm{d}z &= \int_{\pi}^{0} e^{i\theta} \cdot ie^{i\theta} \, \mathrm{d}\theta \\
&= \int_{\pi}^{0} ie^{2i\theta} \, \mathrm{d}\theta \\
&= \left[\frac{1}{2} e^{2i\theta}\right]_{\pi}^{0} \\
&= 0
\end{aligned}$$

Next, consider

$$I_3 = \oint_{\gamma_3} \bar{z} \, \mathrm{d}z, \quad I_4 = \oint_{\gamma_4} \bar{z} \, \mathrm{d}z$$

where γ_3 is the unit circle $|z| = 1$, and γ_4 is the translated unit circle $|z-1| = 1$. For I_3 we again make the substitution $z = e^{i\theta}$, $\mathrm{d}z = ie^{i\theta} \; d\theta$, so

$$\begin{aligned}
I_3 &= \int_{0}^{2\pi} e^{-i\theta} ie^{i\theta} \, \mathrm{d}\theta \\
&= 2\pi
\end{aligned}$$

For I_4 we make the substitution $z = 1 + e^{i\theta}$, $\mathrm{d}z = ie^{i\theta} \; d\theta$, so

$$\begin{aligned}
I_4 &= \int_{0}^{2\pi} (1 + e^{-i\theta}) ie^{i\theta} \, \mathrm{d}\theta \\
&= \int_{0}^{2\pi} i(1 + e^{i\theta}) \, \mathrm{d}\theta \\
&= 2\pi i
\end{aligned}$$

QUESTION 5

QUESTION 6

QUESTION 7

QUESTION 8

QUESTION 9

QUESTION 10

QUESTION 11

QUESTION 12