

Part IB — Methods Example Sheet 1

Supervised by ?

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Michaelmas 2017

QUESTION 1

$$\frac{f(x_+) + f(x_-)}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

For $f(x) = (x-1)^2$ on the interval $-1 \leq x \leq 1$, $f(x)$ is an even function, thus $b_n = 0$. We have $L = 1$, and

$$\begin{aligned} \frac{1}{2}a_0 &= \frac{1}{2L} \int_{-L}^L f(x) \, dx \\ &= \frac{1}{2} \int_{-1}^1 x^4 - 2x^2 + 1 \, dx \\ &= \int_0^1 x^4 - 2x^2 + 1 \, dx \\ &= \frac{8}{15} \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-1}^1 f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx \\ &= \int_{-1}^1 x^4 \cos n\pi x \, dx - 2 \int_{-1}^1 x^2 \cos n\pi x \, dx + \int_{-1}^1 \cos n\pi x \, dx \end{aligned}$$

Evaluating each integral separately, we have:

(i)

$$\int_{-1}^1 \cos n\pi x \, dx = \left[\frac{\sin n\pi x}{n\pi} \right]_{-1}^{-1} = 0$$

as $\sin n\pi x = 0 \, \forall \, n$

(ii) By parts,

$$\begin{aligned} \int_{-1}^1 x^2 \cos n\pi x \, dx &= \left[\frac{x^2 \sin n\pi x}{n\pi} - \frac{1}{n\pi} \int 2x \sin n\pi x \, dx \right]_{-1}^1 \\ &= -\frac{2}{n\pi} \int_{-1}^1 x \sin n\pi x \, dx \end{aligned}$$

and

$$\begin{aligned} \int_{-1}^1 x \sin n\pi x \, dx &= \left[\frac{-x \cos n\pi x}{n\pi} + \frac{1}{n\pi} \int \cos n\pi x \, dx \right]_{-1}^1 \\ &= \frac{-2 \cos n\pi x}{(n\pi)^2} \end{aligned}$$

Thus the second integral contributes to give

$$-\frac{8\cos n\pi x}{(n\pi)^2}$$

(iii)

$$\begin{aligned}\int_{-1}^1 x^4 \cos n\pi x \, dx &= \left[\frac{x^4 \sin n\pi x}{n\pi} - \frac{1}{n\pi} \int 4x^3 \sin n\pi x \, dx \right]_{-1}^1 \\ &= -\frac{4}{n\pi} \int_{-1}^1 x^3 \sin n\pi x \, dx\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^1 x^3 \sin n\pi x \, dx &= \left[\frac{-x^3 \cos n\pi x}{n\pi} + \frac{1}{n\pi} \int 3x^2 \cos n\pi x \, dx \right]_{-1}^1 \\ &= \frac{-2 \cos n\pi}{n\pi} + \frac{3}{n\pi} \int_{-1}^1 x^2 \cos n\pi x \, dx\end{aligned}$$

Whence

$$\begin{aligned}\int_{-1}^1 x^4 \cos n\pi x \, dx &= \frac{8 \cos n\pi}{n\pi} - \frac{12}{(n\pi)^2} \int_{-1}^1 x^2 \cos n\pi x \, dx \\ &= \frac{8 \cos n\pi}{n\pi} - \frac{48 \cos n\pi}{(n\pi)^4}\end{aligned}$$

using (ii).

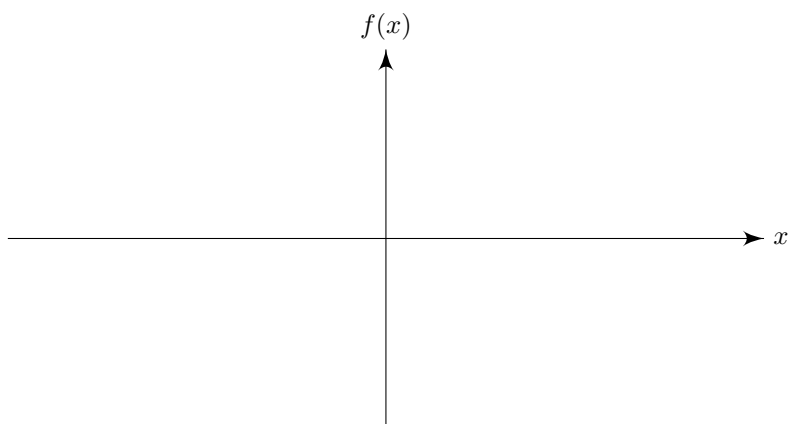
Finally,

$$\begin{aligned}a_n &= -\frac{48 \cos n\pi}{(n\pi)^4} \\ &= \frac{48(-1)^{n+1}}{(n\pi)^4}\end{aligned}$$

as $\cos n\pi x = (-1)^n$

Hence the Fourier Series is given by

$$\begin{aligned}f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \\ &= \frac{8}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos n\pi x\end{aligned}$$



$f(x)$ satisfies the Dirichlet conditions. The 1st derivative is the lowest derivative which is discontinuous (at the endpoints, as $f(x)$ even fn $\Rightarrow f'(x)$ odd), so Fourier coefficients are $\mathcal{O}(\frac{1}{n^2})$ as $n \rightarrow \infty$

QUESTION 2

(a)

$$\frac{f(x_+) + f(x_-)}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \sin nx \, dx \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int_0^{\pi} x^2 \sin nx \, dx &= \left[\frac{-x^2 \cos nx}{n} + \frac{1}{n} \int 2x \cos nx \, dx \right]_0^{\pi} \\ &= \frac{-\pi^2 \cos n\pi}{n} + \frac{2}{n} \int_0^{\pi} x \cos nx \, dx \end{aligned}$$

and once again,

$$\begin{aligned} \int_0^{\pi} x \cos nx \, dx &= \left[\frac{x \sin nx}{n} - \frac{1}{n} \int \sin nx \, dx \right]_0^{\pi} \\ &= -\frac{1}{n} \int_0^{\pi} \sin nx \, dx \\ &= -\frac{1}{n} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \\ &= \frac{1}{n^2} (\cos n\pi - 1) \end{aligned}$$

Back substituting in,

$$\begin{aligned} b_n &= \frac{2}{\pi} \left(\frac{-\pi^2 \cos n\pi}{n} + \frac{2}{n^3} (\cos n\pi - 1) \right) \\ &= \frac{2}{\pi n^3} (-2 + (2 - (\pi n)^2) \cos n\pi) \end{aligned}$$

Hence Fourier sine series given by:

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^3} (-2 + (2 - (\pi n)^2)(-1)^n) \sin nx$$

(b) Similarly,

$$\frac{f(x_+) + f(x_-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$\begin{aligned} \frac{a_0}{2} &= \frac{1}{L} \int_0^L f(x) \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x^2 \, dx \\ &= \frac{\pi^2}{3} \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int_0^{\pi} x^2 \cos nx \, dx &= \left[\frac{x^2 \sin nx}{n} - \frac{1}{n} \int 2x \sin nx \, dx \right]_0^{\pi} \\ &= \frac{-2}{n} \int_0^{\pi} x \sin nx \, dx \end{aligned}$$

and once again,

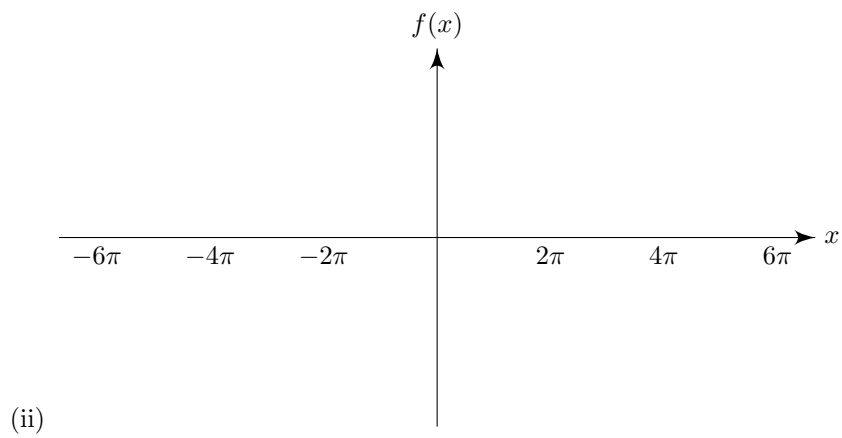
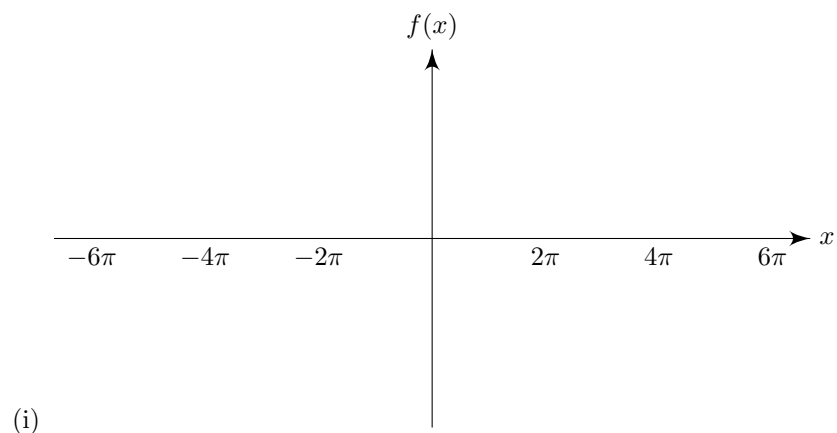
$$\begin{aligned} \int_0^{\pi} x \sin nx \, dx &= \left[\frac{-x \cos nx}{n} + \frac{1}{n} \int \cos nx \, dx \right]_0^{\pi} \\ &= \frac{-\pi \cos n\pi}{n} + \frac{1}{n} \left[\frac{\sin nx}{n\pi} \right]_0^{\pi} \\ &= \frac{-\pi \cos n\pi}{n} \end{aligned}$$

Thus

$$a_n = \frac{4}{n^2} \cos n\pi$$

and the Fourier cosine series is given by

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$



QUESTION 3

QUESTION 4

QUESTION 5