Part IB — Linear Algebra

Based on lectures by Dr. Keating
Notes taken by Christopher Turnbull

Michaelmas 2017

0 Introduction

1 Vector Spaces

Definition. An $\mathbb{F}$-Vector space (a vector space on $\mathbb{F}$) is an abelian group $(V, +)$ equipped with a function[1] $F \times V \to V$, $(\lambda, v) \mapsto \lambda V$

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$$

$$(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$$

$$\lambda(\mu v) = \lambda\mu v$$

$$1v = v$$

$$v + \mathbf{0} = v$$

for all $\lambda_i, \mu \in F$, $v_i \in V$

Note that we will not be underlining our vectors, as this is cumbersome here. We will however be using $\mathbf{0}$ to denote the zero vector.

Example. For all $n \in \mathbb{N}$, $\mathbb{F}^n$ = space of column vectors of length n, entries in $\mathbb{F}$. We understand the definition as entry-wise addition, entry-wise scalar multiplication

Example. $M_{m,m}(\mathbb{F})$, the set of $m \times m$ matrices with entries in $\mathbb{F}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

again all operations defined entry-wise

Example. For any set X, $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$ Addition and scalar multiplication defined pointwise $= f_1(x) + f_2(x)$.

Exercise. Show that the above examples satisfy the axioms

Proposition. $0v = \mathbf{0}$ for all $v \in V$.

Proof. ($(0+0)v = 0v \iff 0v + 0v = 0v \iff 0v = \mathbf{0}$) □

Exercise. Show[2] that $(-1)v = -v$

Definition. Let V be an $\mathbb{F}$-vector space. A subset U of V is a subspace ($U \leq V$) if:

(i) $\mathbf{0} \in U$

(ii) $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$ "U is closed under addition..."

(iii) $u \in U$, any $\lambda \in \mathbb{F} \Rightarrow \lambda u \in U$ "...and scalar multiplication"

Exercise. If U is a subspace of V, then U is also an $\mathbb{F}$-vector space.

[1] scalar multiplication
[2] Hint: Use the previous proposition

Example. Let $V = \mathbb{R}^{\mathbb{R}}$, then $f : R \to R$. The set of all continuous functions $C(\mathbb{R})$ are a subspace. An even smaller subspace is the set of all polynomials.

Exercise. Define $U \subseteq R^3$ as:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad a_1 + a_2 + a_3 = t$$

for some constant t. Check that this is a subspace of $\mathbb{R}^3$ if and only if $t = 0$.

Proposition. Let V be an F-vector space, $U, W \leq V$. Then $U \cap W \leq V$.

Proof. (i) $0 \in U$, $0 \in W \Rightarrow 0 \in U \cap W$

(ii) Suppose $u, v \in U \cap W$, $\lambda, \mu \in F$. U is a subspace $\Rightarrow \lambda u + \mu v \in U$. Similarly $\lambda u + \mu v \in U \in W$, so it is in the intersection.

□

Example. $V = \mathbb{R}^3$, $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0 \right\}$, $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = 0 \right\}$ then $U \cap W = U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0, y = 0 \right\}$ (intersect along the z-axis)

Note: union of family of subspaces is almost never a subspace itself.

Definition. Let V be an F-vector space, $U, W \leq V$. The *sum* of U and W is the set:

$$U + W = \{u + w \mid u \in U, w \in W\}$$

Proposition. $U + W \leq V$

Proof. $\mathbf{0} \in U, W \Rightarrow \mathbf{0} + \mathbf{0} = \mathbf{0} \in U + W$
$u_1, u_2 \in U$, $w_1, w_2 \in W$,

$$(u_1 + w_1) + (u_2 + w_2) = \underbrace{(u_1 + u_2)}_{\in U} + \underbrace{(w_1 + w_2)}_{\in W}$$

Similarly for scalar multiplication (ex.) □

Note: $U + W$ is the smallest subspace containing both U and W. (This is becaues all elements of the form $u + w$ ae forced to be in such a subspace by the "closed under addition" axiom)

Definition. V is an $\mathbb{F}$-vector space, $U \leq V$. The quotient space[3] V/U is the abelian group V/U equipped with scalar multiplication;

$$F \times V/U \to V/U$$

$$(\lambda, v + U) \mapsto \lambda v + U$$

[3]think of this as the collection of cosets of U in V

Proposition. This is well-defined, and V/U is an F-vector space.

Proof. Well-defined: Suppose $v_1 + U = v_2 + U \in V/U$. $\Rightarrow (v_2 - v_1) \in U \Rightarrow (\lambda v_2 - \lambda v_1) \in U \Rightarrow \lambda v_2 + U = \lambda v_1 + U \in V/U$

To show that it is an $\mathbb{F}$-vector space, we must show that the axioms hold. These follow from the axioms of V. $\lambda(\mu(v+U)) = \lambda(\mu v + U) = \lambda(uv) + U = (\lambda u)v + U = \lambda u(v \in U)$ (scalar multiplication on V/U).

Ex. Other axioms follow similarly from using vecton space axioms

$\square$

Definition. V is an $\mathbb{F}$-vector space, $S \subset V$. The *span* of S is denoted by

$$< S >= \left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{F} \right\}$$

ie. the set of all finite linear combinations, all but finitely many of the λ_s are zero.

Remark: $< S >$ is the smallest subspace of V which contains[4] all of the elements of S

Convention: $< \emptyset >= \{\mathbf{0}\}$.

Example. $V = \mathbb{R}^3$,

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \right\}$$

5

$$< S >= \{ \begin{pmatrix} a \\ b \\ 2b \end{pmatrix} \} \mid a, b \in \mathbb{R}$$

Example. For X a set,

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$< \delta_x \mid x \in X >= \{ f \in \mathbb{R}^X \mid f \text{ has finite support} \}$$

$$=< x \in X \mid f(x) \neq 0 >$$

Definition. S *spans* V if $< S >= V$

Definition. V is *finite dimensiona*l over $\mathbb{F}$ if it is spanned by a set that is finite.

Definition. The vectors $v_1, \cdots, v_n$ are *linearly indepedent* over $\mathbb{F}$ if

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \Rightarrow \lambda_i \text{ for all } i$$

some coefficients $\lambda_i \in \mathbb{F}$. $S \subset V$ is linearly independent if every finite subset of it is.

[4]This is essentially a tautology

Example. The fist example, u, v, w are not linearly independent.

Example. The set $\{\delta_X \mid x \in X\}$ is linearly indepndent.

Definition. If *not* linearly indepnedent, say a set is linearly dependent.

Definition. S is a *basis* of V if it is linearly indepnedent and spans V

Example. $\mathbb{F}^n$ standard basis: $e_1, e_2, \cdots, e_n$.

Example. $V = \mathbb{C}$ over $\mathbb{C}$ has natural basis $\{1\}$, over $\mathbb{R}$ has natural basis $\{1, i\}$

Example. $V = \mathcal{P}(\mathbb{R})$ space of all polynomials, has natural basis

$$\{1, x, x^2, x^3, \cdots\}$$

Exercise. Check this carefully

Lemma. V is an $\mathbb{F}$-vector space. The vectors $v_1, \cdots, v_n$ form a basis of V iff each vector $v \in V$ has a unique expression

$$v = \sum_{i=1}^{n} \lambda_i v_i, \text{ with } \lambda_i \in \mathbb{F}$$

Proof. ($\Rightarrow$) Fix $v \in V$. The v_i span, so

$$\exists \lambda_i \in \mathbb{F} \text{ s.t. } v = \sum \lambda_i v_i$$

Suppose also $v = \sum \mu_i v_i$ for some $\mu_i \in \mathbb{F}$. $\sum (\mu_i - \lambda_i) v_i = \mathbf{0}$.

The v_i are linearly indepnedent so $\mu_i - \lambda_i = 0$ for all i, $\lambda_i = \mu_i$

($\Leftarrow$) The v_i span V, since any $v \in V$ is a linear combination of them. IF $\sum_{i=1}^{n} \lambda_i v_i = \mathbf{0}$. Note that $\mathbf{0} = \sum_{i=1}^{n} 0 v_i$. By uniqueness (applied to $\mathbf{0}$), $\lambda_i = 0$ for all i. ☐

Lemma. If $v_1, \cdots, v_n$ span V (over $\mathbb{F}$), then some subset of $v_1, \cdots, v_n$ is a basis for V (over $\mathbb{F}$).

Proof. If $v_1, \cdots, v_n$ ilnearly indepnedent, done. Otherwise for some l, there exist $\alpha_1, \cdots, \alpha_{l-1} \in \mathbb{F}$ such that

$$v_l = \alpha_1 v_1 + \cdots + \alpha_{l-1} v_{l-1}$$

(If $\sum \lambda_i v_i = \mathbf{0}$, not all $\lambda_i = 0$. Take l maximaml with $\lambda_i \neq 0$, just $\alpha_i = -\lambda_i / \lambda_l$).

Now $v_2, \cdots, v_{l-1}, v_{l+1}, \cdots, v_n$ still span V. Continue interatively until get linear independence.

☐

Theorem. (Steinitz exchange lemma) Let V be a finite dimensional vector space over $\mathbb{F}$. Take $v_1, \cdots, v_m$ to be linearly independent $w_1, \cdots, w_n$ to span V.

Then $m \leq n$, and reordering the spanning set if needed,

$$v_1, \cdots, v_m, w_{m+1}, \cdots, w_n$$

span V.

Proof. (Induction) Suppose that we've replaced $l(\geq 0)$ of the w_i. Reordering the w_i if needed, $v_1, \cdots, v_l, w_{l+1}, \cdots, w_n$ span V.

If $l = m$, done.

If $l < m$, then

$$v_{l+1} = \sum_{i=1}^{l} \alpha_i v_i + \sum_{i>l} \beta_i w_i$$

$\alpha_i, \beta_i \in \mathbb{F}$. As the v_i are lin. indep, $\beta_i \neq 0$ for some i. (After reordering, $\beta_{l+1} \neq 0$).

$$w_{l+1} = \frac{1}{\beta_{l+1}} \left(v_{l+1} - \sum_{i \leq l} \alpha_i v_i - \sum_{i > l+1} \beta_i w_i \right)$$

This $v_1, \cdots, v_{l+1}, w_{l+2}, \cdots, w_n$ also spans V. After m steps, w_i will have replaced m of the w_i by v_i. Thus $m \leq n$. □

Theorem. If V is a finite dimensional vector space over $\mathbb{F}$, then any two bases for V have the same number of elements. This is what we call the *dimension* of V, denoted $\dim_{\mathbb{F}} V$.

Proof. If $v_{\{1}, \cdots, v_n$ is a basis and $w_1, \cdots, w_m$ is another basis, the $\{v_i\}$ span and $\{w_i\}$ is linearly indepnedent' so by Steinitz $m \leq n$. Likewise, $n \leq m$. □

Example. $\dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{R}} \mathbb{C} = 2$

1.1