Part IB — Linear Algebra Sheet 2

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QUESTION 1

The three types of elementary matrices are:

(i)
$$\begin{pmatrix}
1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & 1 & & & & & & & & \\
& & & 0 & & & & 1 & & & \\
& & & & 1 & & & & & & \\
& & & & & \ddots & & & & & \\
& & & & & & 1 & & & & \\
& & & 1 & & & & 0 & & & \\
& & & & & & & & 1 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 1
\end{pmatrix}$$

The zeros appear in row i, row j. This swaps column i and column j, and is self-inverse.

(ii)
$$\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & & \lambda & & & \\
& & & & 1 & & \\
& & & & & \ddots & \\
& & & & & & 1
\end{pmatrix}$$

with λ in the i^{th} row. (Multiplies column i by λ) This has inverse

$$\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & & \frac{1}{\lambda} & & & \\
& & & & 1 & & \\
& & & & & \ddots & \\
& & & & & & 1
\end{pmatrix}$$

(iii) $I_n + \lambda E_{ij}$, where E_{ij} is defined as 1 in the (i, j) position and 0 everywhere else. $(i \neq j)$. This has inverse $I_n + \lambda E_{ij}$.

To find inverse of this matrix, we

– add column 1 to column 2

– swap rows 2 and 3

– add row 3 to row 2

– multiply row 2 by $\frac{1}{3}$.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}$$

QUESTION 2

Theorem. Any $m \times n$ matrix is equivalent to

$$\begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

for some r, where r is the (column) rank of the matrix.

Therefore,
??????

QUESTION 3

If V is the vector space with finite basis $\mathcal{B} = \{x_1, x_2, x_3, x_4\}$ then there is a basis for V^*, given by $\mathcal{B}^* = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ where

$$\xi_j \underbrace{\left(\sum_{i=1}^{4} a_i x_i\right)}_{\in V} = a_j \quad 1 \leq j \leq 4 \qquad (*)$$

(a) By (*), the dual basis is

$$\{\xi_2, \xi_1, \xi_4, \xi_3\}$$

(b) we have $\xi_2\left(\sum_{i=1}^4 a_i x_i\right) = a_2 \Rightarrow \xi_2(a_2 x_2) = a_2$. Hence clear to see dual basis is

$$\{\xi_1, \frac{1}{2}\xi_2, 2\xi_3, \xi_4\}$$

(c) Call the new dual basis $\{\eta_1, \eta_2, \eta_3, \eta_4\}$. It is clear that $\eta_1 = \xi_1$. To find η_2, we aim to solve the system of linear equations

$$\begin{aligned} \eta_2(x_1 + x_2) &= 0 \\ \eta_2(x_2 + x_3) &= 1 \\ \eta_2(x_3 + x_4) &= 0 \\ \eta_2(x_4) &= 0 \end{aligned}$$

and we deduce that $\eta_2 = \xi_2 - \xi_1$. Similarly, $\eta_3 = \xi_3 - \xi_2$, $\eta_4 = \xi_4 - \xi_3$.

$$\{\xi_1, \xi_2 - \xi_1, \xi_3 - \xi_2, \xi_4 - \xi_3\}$$

(d) Similar method to (c), the dual basis is:

$$\{\xi_1 + \xi_2, \xi_2 + \xi_3, \xi_3 + \xi_4, \xi_4\}$$

QUESTION 4

We have that $\tau_A(B) = \sum_i \sum_j a_{ij}b_{ji}$, so linearity follows immediately by the definition of the sum.

Next, want to show that $\tau_A(A)$ defines an iso from $\mathrm{Mat}_{m,n}(\mathbb{F})$ to $\mathrm{Mat}_{m,n}(\mathbb{F})^*$, ie. $L(\mathrm{Mat}_{m,n}(\mathbb{F}), \mathbb{F})$ Have already show linearity. Easy to see this is well defined.

– Injective: (Not sure)

– Surjective: Can we just pick a matrix such that the trace gives us any scalar in $\mathbb{F}$ to show surjectivity?

QUESTION 5

(a) Suppose two such endomorphisms exists, with matrices A, respectively. Take the trace of both sides of the equation. As $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, clearly the LHS is zero, but the RHS is $\dim V$. Contradiction.

(b) Define

$$\begin{aligned} \alpha : V &\to V & \beta : V &\to V \\ f(x) &\mapsto xf(x) & f(x) &\mapsto f'(x) \end{aligned}$$

Then

$$\begin{aligned} (\alpha\beta - \beta\alpha)(f) &= (xf)' - xf' \\ &= f \end{aligned}$$

That is, $\alpha\beta - \beta\alpha = \mathrm{id}_V$

QUESTION 6

Say $u = \underbrace{\sum x_i e_i}_{\in U}$, $v = \underbrace{\sum y_j f_j}_{\in V}$.

$$\begin{aligned}
\psi(u,v) &= \psi\left(\sum x_i e_i, \sum y_j f_j\right) \\
&= \sum_i x_i \psi\left(e_i, \sum_j y_j f_j\right) \\
&= \sum_{i,j} x_i \psi(e_i, f_j) y_j
\end{aligned}$$

So in some basis where $\psi(e_i, f_j) = \delta_{ij}$, this is $\sum_i x_i y_i$
(not sure why this exists?)
The left and right maps are determined by

$$\varphi_L : U \to V^* \qquad \text{and} \qquad \varphi_R : V \to U^*$$

$$\varphi_L(u)(v) = \varphi(u,v) \qquad \text{and} \qquad \varphi_R(v)(u) = \varphi(u,v)$$

QUESTION 7

(a) Since a_i distinct, all columns of A are linearly independent, so the matrix is of full rank. Thus $n(A) = 0$ by rank nulity, and $\det A \neq 0$

QUESTION 8

(i)

$$\begin{aligned}
\text{adj }(AB) &= \det(AB)(AB)^{-1} \\
&= \det(A)\det(B)B^{-1}A^{-1} \\
&= \det(B)B^{-1}\det(A)A^{-1} \\
&= \text{adj }(B)\text{adj }(A)
\end{aligned}$$

(ii)

$$\begin{aligned}
\det(\text{adj } A) &= \\
&= \det(\det(A)A^{-1}) \\
&= \det(\det(A)I)\det(A^{-1}) \qquad = (\det A)^n(\det A)^{-1} \\
&= (\det A)^{n-1}
\end{aligned}$$

(iii)

$$\begin{aligned}
\text{adj }(\text{adj } A) &= \text{adj }(\det(A)A^{-1}) \\
&= \det(\det(A)A^{-1})(\det(A)A^{-1})^{-1} \\
&= (\det A)^{n-1}A(\det A)^{-1} \\
&= (\det A)^{n-2}A
\end{aligned}$$

QUESTION 9

QUESTION 10

QUESTION 11