Part IB — Numerical Analysis Example Sheet 1

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QUESTION 1

We seek some polynomial interpolant $p \in \mathbb{P}_3[x]$. Using the Lagrange formula we have that

$$p(x) = \sum_{k=0}^{3} f(k) l_k$$

where

$$l_k = \prod_{i=0, i \neq k}^{3} \frac{x-i}{k-i}$$

that is,

$$p(x) = f(0)\frac{(x-1)(x-2)(x-3)}{-6} + f(1)\frac{x(x-2)(x-3)}{2} + f(2)\frac{x(x-1)(x-3)}{-2} + f(3)\frac{x(x-1)(x-2)}{6}$$

(i) The approximant $p(6)$

We have

$$\begin{aligned} p(6) &= f(0)\frac{5 \cdot 4 \cdot 3}{-6} + f(1)\frac{6 \cdot 4 \cdot 3}{2} + f(2)\frac{6 \cdot 5 \cdot 3}{-2} + f(3)\frac{6 \cdot 5 \cdot 4}{6} \\ &= -10f(0) + 36f(1) + -45f(2) + 20f(3) \end{aligned}$$

(ii) The approximant $p'(0)$

Taking the derivative of each term individually, we then plug in $x = 0$. We deduce that

$$p'(0) = -\frac{11}{6}f(0) + 3f(1) - \frac{3}{2}f(2) + \frac{1}{3}f(3)$$

(iii) The approximant $\int_0^3 p(x)\, \mathrm{d}x$

Expanding each term and integrating (I can't see a shorter way) we have that

$$p(x) = f(0)\frac{x^3 - 6x^2 + 11x - 6}{-6} + f(1)\frac{x^3 - 5x^2 + 6x}{2} + f(2)\frac{x^3 - 4x^2 + 3x}{-2} + f(3)\frac{x^3 - 3x^2 + 2x}{6}$$

and thus

$$\int_0^3 p(x)\, \mathrm{d}x = \frac{3}{8}f(0) + \frac{9}{8}f(1) + \frac{9}{8}f(2) + \frac{3}{8}f(3)$$

We can check this by supposing $f(x) = x$, so that $f(k) = k$ for each k. Indeed, $p(6) = 6, p'(0) = 1$, and $\int_0^3 p(x)\, \mathrm{d}x = 9/2$,

QUESTION 2

The formula is true when $x = 0, 1$ since both sides of the equation vanish. Let $x \in (0, 1)$ be any other point and define (for x fixed).

$$\phi(t) := [f(t) - p(t)] \prod_{i=0}^{3}(x - x_i) - [f(x) - p(x)] \prod_{i=0}^{3}(t - x_i), \quad t \in (0, 1)$$

where $x_0 = x_1 = 0, x_2 = x_3 = 1$. Note $\phi(0) = \phi(1) = 0$, and also $\phi(x) = 0$. Hence, ϕ has at least 3 zeroes. Applying Rolle's theorem, and using the condition that $f'(0) = f'(1) = 0$, we deduce that $\phi'(t)$ has at least 4 zeroes: one at $x_0 = 0$, one at $x_2 = 1$, and two more: one in the interval $(0, x)$, and the other in $(x, 1)$.

Then $\phi''(t)$ has at least 3 zeroes in $(0, 1)$, and... $\phi^{(4)}(x)$ has at least one zero in $(0, 1)$; call it ξ. Then

$$0 = \phi^{(4)}(\xi) = \left[f^{(n+1)}(\xi) - p^{(n+1)}(\xi)\right] \prod_{i=0}^{3}(x - x_i) - [f(x) - p(x)] \left.\frac{\mathrm{d}^4}{\mathrm{d}t^4}\right|_{t=\xi} \prod_{i=0}^{3}(t - x_i)$$

Since $p^{(4)} \equiv 0$, and $\left.\frac{\mathrm{d}^4}{\mathrm{d}t^4}\right|_{t=\xi} \prod_{i=0}^{3}(t - x_i) = 4!$, we obtain

$$\begin{aligned} f(x) - p(x) &= \frac{1}{4!} f^{(4)}(\xi) \prod_{i=0}^{3}(x - x_i) \\ &= \frac{1}{24} x^2 (1 - x)^2 f^{(4)}(\xi) \end{aligned}$$

QUESTION 3

Seeking a contradiction we suppose there exists some nonzero polynomial $p \in \mathbb{P}_4[x]$ st.

$$p(a) = p(b) = p'(a) = p'(b) = p'(c) = 0 \qquad (*)$$

Suppose that $q_1 \in \mathbb{P}_4[x]$ and $q_2 \in \mathbb{P}_4[x]$ both interpolate the data, then $q_1 - q_2$ vanishes at these points. Hence, we have

$$q_1 = q_2 + kp$$

for some $k \in \mathbb{R}$, so the solution of this interpolation problem is not unique.
To pick a value of c that satisfies $(*)$, try

$$p(x) = (x-a)(x-b) + (x-a)(x-b)(x-c)$$

Immediately we have $p(a) = p(b) = 0$. Now,

$$p'(x) = (x-a) + (x-b)(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$$

QUESTION 4

The Newton interpolation formula states that a polynomial intepolating f at pairwise distinct points $x_0, \cdots, x_n$ is given by

$$p_n(x) := f[x_0] + f[x_0, x_1](x_1 - x_0) + \cdots + f[x_0, \cdots, x_n] \prod_{i=0}^{n-1} (x - x_i)$$

In particular,

$$p_{n+1}(x) - p_n(x) = f[x_0, \cdots, x_{n+1}] \prod_{i=0}^{n} (x - x_i) \qquad (*)$$

To deduce the identity in question 4, we think of x as a new interpolation point (the $n+1^{\text{th}}$). As $x \neq x_i$ for any i, we can now apply (*), which gives

$$p_{n+1}(t) - p_n(t) = f[x_0, \cdots, x_n, x] \prod_{i=0}^{n} (t - x_i)$$

for all $t \in \mathbb{R}$. In particular, setting $t = x$, we have $p_{n+1}(x) = f(x)$, which is the identity as required.

QUESTION 5

The Newton divided difference table for Question 5 is shown below, where arithmetic has been rounded to 4 decimal places at each step.

x_i	f_i	$f[*,*]$	$f[*,*,*]$	$f[*,*,*,*]$
0	$f[0]=0$			
		$f[0,0.1]=0.9980$		
0.1	$f[0.1]=0.9980$		$f[0,0.1,0.4]$	
		$f[0.1,0.4]=0.9653$	$=-0.0817$	$f[0,0.1,0.4,0.7]$
0.4	$f[0.4]=0.3894$		$f[0.1,0.4,0.7]$	$=-0.1680$
		$f[0.4,0.7]=0.8493$	$=-0.1993$	
0.7	$f[0.7]=0.6442$			

Using Newton's formula, the polynomial interpolating these points is given as

$$\begin{aligned}
p(x) &= f[0] + f[0,0.1]x + f[0,0.1,0.4]x(x-0.1) + f[0,0.1,0.4,0.7]x(x-0.1)(x-0.4) \\
&= 0.9980x - 0.0817(x^2 - 0.1x) - 0.1680(x^3 - 0.5x^2 + 0.04x) \\
&= 0.9995x + 0.0023x^2 - 0.1680x^3
\end{aligned}$$

As we have rounded erroneously this is indeed different from $\sin x$.

QUESTION 6

The condition

$$\int_0^1 [f(x) - p(x)]^2 \, \mathrm{d}x < 10^{-4}$$

is equivalent to

$$\frac{1}{3} - 2\int_0^1 f(x)p(x) \, \mathrm{d}x + \int_0^1 p(x)^2 \, \mathrm{d}x < 10^{-4}$$

Fourier series?

QUESTION 7

We will first prove that, under the substitution $x = \cos\theta$, $p_n(x) = \sin(n+1)\theta/\sin\theta$. We will use induction with two base cases;

For $n = 0$, we have $p_0(x) = \sin\theta/\sin\theta = 1$ as required, and for $n = 1$ we have $p_0(x) = \sin 2\theta/\sin\theta = 2\cos\theta = 2x$, as required (as $x = \cos\theta$).

Now assuming true for $p_{n-1}(x)$ and $p_n(x)$ yields:

$$\begin{aligned} p_{n+1}(x) &= 2xp_n(x) - p_{n-1}(x) \\ &= (2\cos\theta\sin(n+1)\theta - \sin n\theta)/\sin\theta \qquad (x = \cos\theta) \end{aligned}$$

Fiddling around with the numerator;

$$\begin{aligned} 2\cos\theta\sin(n+1)\theta - \sin n\theta &= (\sin(n+1)\theta\cos\theta + \cos(n+1)\theta\sin\theta) \\ &\quad + (\sin(n+1)\theta\cos\theta - \cos(n+1)\theta\sin\theta) - \sin n\theta \\ &= \sin(n+2)\theta + \sin n\theta - \sin n\theta = \sin(n+2)\theta \end{aligned}$$

Hence $p_{n-1}(x) = \sin(n)\theta/\sin\theta$, $p_n(x) = \sin(n+1)\theta/\sin\theta$ together imply that $p_{n+1}(x) = \sin(n+2)\theta/\sin\theta$, which completes our inductive proof.

Now we can show orthogonality:

$$\begin{aligned} \langle p_n, p_m\rangle &= \int_{-1}^{1} p_n(x)p_m(x)\sqrt{1-x^2}\ dx \\ &= \int_{\pi}^{0} \frac{\sin(n+1)\theta}{\sin\theta}\frac{\sin(m+1)\theta}{\sin\theta}\sqrt{1-\cos^2\theta}(-\sin\theta)\ \mathrm{d}\theta \qquad (x = \cos\theta) \\ &= \int_0^{\pi} \sin(n+1)\theta\sin(m+1)\theta\ \mathrm{d}\theta \\ &= \frac{\pi}{2}\delta_{mn} \end{aligned}$$

Thus these polynomials are orthogonal with respect to the defined inner product, and $\langle p_n, p_n\rangle = \pi/2$

QUESTION 8

We apply Gaussian quadrature with Legendre polynomials, and pick c_1 and c_2 as the zeros of $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, so

$$c_1 = -\frac{\sqrt{3}}{3}, \qquad c_2 = \frac{\sqrt{3}}{3}$$

Using the formula for the choice of weights given in lectures, we have

$$b_1 = \int_0^1 \frac{x - c_2}{c_1 - c_2}\,\mathrm{d}x, \quad b_2 = \int_0^1 \frac{x - c_1}{c_2 - c_1}\,\mathrm{d}x$$

Thus $b_1 = \frac{1}{2} - \frac{\sqrt{3}}{4}$, $b_2 = \frac{1}{2} + \frac{\sqrt{3}}{4}$, and the exact approximate when f is cubic is:

$$\int_0^1 f(x)\,\mathrm{d}x \approx \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right) f\left(-\frac{\sqrt{3}}{3}\right) + \left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right) f\left(\frac{\sqrt{3}}{3}\right)$$

Note $\int_0^1 1\,\mathrm{d}x = 1$, $\int_0^1 x\,\mathrm{d}x = \frac{1}{2}$, $\int_0^1 x^2\,\mathrm{d}x = 1/3$, $\int_0^1 x^3\,\mathrm{d}x = 1/4$.

QUESTION 9

We have that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}(e^{-x}) = (-1)^k e^{-x}$, thus using the Leibniz rule:

$$\begin{aligned}
p_n(x) &= e^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x}) \\
&= e^x \sum_{r=0}^{n} \binom{n}{r} \frac{\mathrm{d}^r}{\mathrm{d}x^r} x^n \frac{\mathrm{d}^{n-r}}{\mathrm{d}x^{n-r}} e^{-x} \\
&= \sum_{r=0}^{n} \binom{n}{r} \frac{\mathrm{d}^r}{\mathrm{d}x^r}(x^n)(-1)^{n-r} \\
&= \sum_{r=0}^{n} r! \binom{n}{r}^2 (-x)^{n-r} (*)
\end{aligned}$$

so $p_n(x)$ indeed a polynomial. Next, with respect to the defined scalar product, we have

$$\begin{aligned}
\langle p_n, p \rangle &= \int_0^\infty e^{-x} p_n(x) p(x) \, \mathrm{d}x \\
&= \int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x}) p(x) \, \mathrm{d}x \\
&= \left[p(x) \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^n e^{-x}) \right]_0^\infty - \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x} p(x) \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^n e^{-x}) \, \mathrm{d}x \\
&= -\int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}x^n} p(x)(x^n e^{-x}) \, \mathrm{d}x \\
&= 0
\end{aligned}$$

and in going to the final line we have used the fact that $p(x)$ is a polynomial of degree $n-1$.

The boundary terms vanish by inspection; each term in the expression $\frac{\mathrm{d}^r}{\mathrm{d}x^r}(x^l e^{-x})$ contains an e^{-x}, and if $r < l$, each term is a multiple of x, thus the expression is zero at ∞ and 0 respectively.

To evaluate p_3, p_4 and p_5 using the Rodrigues formula we use $(*)$:

$$\begin{aligned}
p_3(x) &= -x^3 + 9x^2 - 18x + 6 \\
p_4(x) &= x^4 - 16x^3 + 72x^2 - 96x + 24 \\
p_5(x) &= -x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120
\end{aligned}$$

If these are to obey the relation

$$p_5(x) = (\gamma x - \alpha) p_4(x) - \beta p_3(x), \quad x \in \mathbb{R}$$

Then comparing coefficients of x^5, x^4 and x^3 respectively give, $\gamma = -1$, $\alpha = -9, \beta = -200$.

QUESTION 10

For the $k = 0$ case: want to choose the least c_0 such that

$$\left| f(\frac{1}{2}) - \frac{1}{2}\left(f(0) + f(1)\right) \right| \leq c_0 ||f||_\infty$$

In the extreme case where $f(\frac{1}{2}) \approx ||f||_\infty$ and $f(0) \approx f(1) \approx -||f||_\infty$, we see that $c_0 = 2$.

I'm not sure how to do the $p = 1$ case without the Peano Kernel theorem.

$k = 2$: Consider $f(\frac{1}{2}) \approx \frac{1}{2}(f(0) + f(1))$.

Let $L(f) = f(\frac{1}{2}) - \frac{1}{2}(f(0) + f(1))$. $L(f) = 0$ for all $f \in \mathcal{C}[0, 1]$ since $L(f) = 0$ when $f(x) = 1, x$, and using linearity. Peano Kernel theorem tells us that

$$L(f) = \int_0^1 K(\theta) f''(\theta)\, \mathrm{d}\theta$$

where $K(\theta) = L(x \mapsto (x - \theta)_+)$. For fixed θ, let $g(x) := (x - \theta)_+$. We have

$$\begin{aligned}
K(\theta) = L(g) &= g(\frac{1}{2}) - \frac{1}{2}(g(0) + g(1)) \\
&= (1/2 - \theta)_+ - \frac{1}{2}\left((0 - \theta)_+ + (1 - \theta)_+\right) \\
&= \begin{cases} -\frac{1}{2}(1 - \theta) & \text{if } 0 \leq \theta \leq 1/2 \\ -\frac{1}{2}\theta & \text{if } 1/2 \leq \theta \leq 1 \end{cases}
\end{aligned}$$

Now

$$\begin{aligned}
\int_0^1 |K(\theta)|\, \mathrm{d}\theta &= \int_0^{1/2} \frac{1}{2}(1 - \theta)\, \mathrm{d}\theta + \int_{1/2}^1 \frac{1}{2}\theta\, \mathrm{d}\theta \\
&= \left(\frac{1}{4} - \frac{1}{16}\right) + \left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{8}
\end{aligned}$$

This allows us to bound the approximation error, for any $f \in \mathcal{C}^2[0, 1]$ we get

$$|L(f)| \leq \int_0^1 |K(\theta) f''(\theta)|\, \mathrm{d}\theta \leq ||f''||_\infty \int_0^1 |K(\theta)|\, \mathrm{d}\theta \leq \frac{3}{8}|f''||_\infty$$

We can try the Peano kernel theorem in the $k = 1$ ($n = 0$) case, which tells us

$$L(f) = \int_0^1 K(\theta) f'(\theta)\, \mathrm{d}\theta$$

where $K(\theta) = L(x \mapsto (x - \theta)_+^0)$. For fixed θ, let $g(x) := (x - \theta)_+^0$. Similar to before, we have

$$\begin{aligned}
K(\theta) = L(g) &= g(\frac{1}{2}) - \frac{1}{2}(g(0) + g(1)) \\
&= (1/2 - \theta)^0_+ - \frac{1}{2}\left((0-\theta)^0_+ + (1-\theta)^0_+\right) \\
&= \begin{cases} -\frac{1}{2} & \text{if } 0 \leq \theta \leq 1/2 \\ \frac{1}{2} & \text{if } 1/2 \leq \theta \leq 1 \end{cases}
\end{aligned}$$

We can verify that $\int_0^1 |K(\theta)| \, \mathrm{d}\theta = \frac{1}{2}$. This allows us to bound the approximation error, for any $f \in \mathcal{C}[0,1]$ we get

$$|L(f)| \leq \int_0^1 |K(\theta) f'(\theta)| \, \mathrm{d}\theta \leq ||f'||_\infty \int_0^1 |K(\theta)| \, \mathrm{d}\theta \leq \frac{1}{2}|f'||_\infty$$

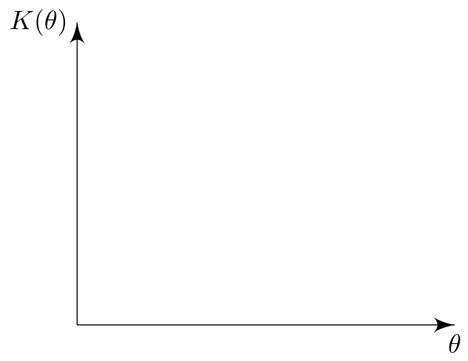
QUESTION 11

Define $L(f) := f[0,1,2,4]$. Easy to check that $L(f) = 0$ for $f \in \mathbb{P}_2[x]$. Thus for $f \in C^3[0,4]$, we have

$$L(f) = \frac{1}{6}\int_0^4 K(\theta) f'''(\theta)\,\mathrm{d}\theta$$

with $K(\theta) = L(x \mapsto (x-\theta)_+^2)$. For fixed θ, let $g(x) := (x-\theta)_+^2$. Then using the Lagrange formula, noting that $g(0) = 0$

$$\begin{aligned} K(\theta) &= L(g) = g[0,1,2,4] \\ &= g(1)\frac{1}{1-0}\frac{1}{1-2}\frac{1}{1-4} + g(2)\frac{1}{2-0}\frac{1}{2-1}\frac{1}{2-4} + g(4)\frac{1}{4-0}\frac{1}{4-1}\frac{1}{4-2} \\ &= \frac{1}{3}(1-\theta)_+^2 - \frac{1}{4}(2-\theta)_+^2 + \frac{1}{24}(4-\theta)_+^2 \\ &= \begin{cases} \frac{1}{8}\theta^2 & \text{if } 0 \le \theta \le 1 \\ -\frac{1}{4}(2-\theta)^2 + \frac{1}{24}(4-\theta)^2 & \text{if } 1 \le \theta \le 2 \\ \frac{1}{24}(4-\theta)^2 & \text{if } 2 \le \theta \le 4 \end{cases} \end{aligned}$$



QUESTION 12

For $f \in \mathbb{P}_3[x]$, consider the approximant

$$f'''(\xi) \approx \alpha f(0) + \beta f(1) + \gamma f'(0) + \delta f'(1) \qquad (*)$$

Note that $f'''(\xi) = 3! \; \forall \, \xi \in \mathbb{R}$. Requiring (*) is exact for $f(x) = 1, x, x^2$ and x^3 respectively yields

$$\begin{aligned} 0 &= \alpha + \beta \\ 0 &= \beta + \gamma + \delta \\ 0 &= \beta + 2\delta \\ 6 &= \beta + 3\delta \end{aligned}$$

which gives

$$f'''(\xi) \approx 12f(0) - 12f(1) + 6f'(0) + 6f'(1)$$

Let $L(f) = f'''(\xi) - [12f(0) - 12f(1) + 6f'(0) + 6f'(1)]$. $L(f) = 0$ for all $f \in C^4[0,1]$, so Peano Kernel theorem tells us that

$$L(f) = \frac{1}{3!} \int_0^1 K(\theta) f^{(4)}(\theta) \, \mathrm{d}\theta$$

where $K(\theta) = L(x \mapsto (x-\theta)_+^3)$. For fixed θ, let $g(x) := (x-\theta)_+^2$. We have

$$\begin{aligned} K(\theta) = L(g) &= g'''(\xi) - [12g(0) - 12g(1) + 6g'(0) + 6g'(1)] \\ &= 6(\xi - \theta)_+^0 - [12(0-\theta)_+^3 - 12(1-\theta)_+^3 + 18(0-\theta)_+^2 + 18(1-\theta)_+^2] \\ &= \begin{cases} 12(1-\theta)^3 - 18(1-\theta)^2 & \text{if } 0 \le \theta \le \xi \\ 6 + 12(1-\theta)^3 - 18(1-\theta)^2 & \text{if } \xi \le \theta \le 1 \end{cases} \end{aligned}$$

Consequently for any $f \in C^4[0,1]$ we have

$$|L(f)| \le \frac{1}{3!} \int_0^1 |K(\theta) f^{(4)}(\theta)| \, \mathrm{d}\theta \le \frac{1}{6} ||f^{(4)}||_\infty \int_0^1 |K(\theta)| \, \mathrm{d}\theta$$

Now,

$$\begin{aligned} \frac{1}{6} \int_0^1 |K(\theta)| \, \mathrm{d}\theta &= \int_0^\xi 2(1-\theta)^3 - 3(1-\theta)^2 \, \mathrm{d}\theta + \int_\xi^1 1 + 2(1-\theta)^3 - 3(1-\theta)^2 \, \mathrm{d}\theta \\ &= \frac{1}{4}(1-\xi)^4 - (1-\xi)^3 + (1-\xi) + \end{aligned}$$