Part IB — Linear Algebra Sheet 3

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Michaelmas 2017

QUESTION 1

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

This matrix has characteristic polynomial

$$\chi_{A_1}(\lambda) = (\lambda - 1)^2(\lambda - 2)$$

For $\lambda = 2$ eigenvectors satisfy

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$$

So $\mathbf{v} = (2, 2, 1)$, and we take this as a basis for the $\lambda = 2$ eigenspace. Similarly for $\lambda = 1$ we have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$$

This implies $v_2 = v_3 = 0$, so eigenvector must be of the form $(1, 0, 0)$, again a basis with one element.

A_2: Next, we note that $\chi_{A_1}(\lambda) = \chi_{A_2}(\lambda)$ as the determinant calculation expanding down the first column will remain unchanged, so same eigenvalues. We can see that for $\lambda = 2$, $\mathbf{v} = (1, 2, 1)$ is a basis. For $\lambda = 1$ we have $v_2 = v_3$, so an eigenvector basis is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Next, $\chi_{A_3}(\lambda) = (\lambda - 1)(\lambda - 2)^2$. For $\lambda = 1$ the eigenspace basis is $\{(1, 1, 1)\}$, for $\lambda = 2$ it is $\{(1, -2, 1)\}$.

QUESTION 2

Consider $\det(A - \kappa\iota)$.

Add all the columns to the first column; it becomes a column where all entries are equal to $\lambda - \mu + (n-1)$. Now subtract first row from all others. We are left with $\det(A - \kappa\iota) = (\lambda - \mu + (n-1))\det(M)$ where M is an $n-1 \times n-1$ lower triangular matrix with $\lambda - \mu - 1$ in every diagonal entry.

Hence

$$\det(A - \kappa\iota) = (\lambda - \mu + (n-1))(\lambda - \mu - 1)^{n-1}$$

So n eigenvalues are given as

$$\mu = \lambda + n - 1, \underbrace{\lambda - 1, \cdots, \lambda - 1}_{n-1 \text{ times}}$$

QUESTION 3

Define $\pi_j = q_j(\alpha) : V \to V$ by

$$q_j(\alpha) = \prod_{i \neq j}^{k} \frac{\alpha - \lambda_i}{\lambda_j - \lambda_i}$$

There's a bit about this in the notes, but I don't understand the proof and am having difficulty recreating it here.

QUESTION 4

Let $\alpha : V \to V$, complex finite dimensional vector space. λ eigenvalue for α $\Rightarrow$ $\alpha v = \lambda v$. Then

$$\begin{aligned} \alpha^2(v) &= \alpha(\alpha(v)) \\ &= \alpha(\lambda v) \\ &= \lambda(\alpha v) \quad \text{by linearity} \\ &= \lambda^2 v \end{aligned}$$

Thus λ^2 is an eigenvalue of α^2.

QUESTION 5

Suppose $\alpha_1, \alpha_2 : V \to V$, and $\dim V = n$.

Let $\tilde{\alpha}_2$ be the restriction of α_2 to $\mathrm{Im}(\alpha_1)$. We have

$$\mathrm{Im}(\tilde{\alpha}_2) = \mathrm{Im}(\alpha_2\alpha_1)$$
$$\ker(\tilde{\alpha}_2) = \ker(\alpha_2) \cap \mathrm{Im}\,\alpha_1$$

By rank-nullity (note the domain of $\tilde{\alpha}_2$ is $\mathrm{Im}(\alpha_1)$).

$$\begin{aligned} r(\alpha_2\alpha_1) &= r(\alpha_1) - \dim(\ker(\alpha_2) \cap \mathrm{Im}\,\alpha_1) \\ &\geq r(\alpha_1) - n(\alpha_2) \\ &= (n - n(\alpha_1)) - n(\alpha_2) \end{aligned}$$

Thus

$$n - n(\alpha_2\alpha_1) \geq (n - n(\alpha_1) - n(\alpha_2)$$

ie.

$$n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$$

Not sure about the last bit.

QUESTION 6

Clearly the identity matrix is only similar to itself, as $P^{-1}IP = P^{-1}P = I$ for all invertible matrices P.

Not sure how to show dissimilarity of the first two. They both have characteristic polynomial $\det(M - \lambda\iota) = (\lambda - 1)^3$.

QUESTION 7

QUESTION 8

Set of $n = \dim V$ vectors, only need to check for linear independence to see this is a basis. Suppose

$$A_0 y + A_1 \alpha(y) + A_2 \alpha^2(y) + \cdots + A_{n-1}\alpha^{n-1}(y) = 0$$

Taking α of both sides repeatedly

$$A_0 \alpha(y) + A_1 \alpha^2(y) + \cdots + A_{n-2}\alpha^{n-1}(y) + \underbrace{A_{n-1}\alpha^{n}(y)}_{=0} = 0$$

$$\vdots$$

$$A_0 \alpha^{n-2}(y) + A_1 \alpha^{n-1}(y) = 0$$

$$A_0 \alpha^{n-1}(y) = 0$$

But $\alpha^{n-1}(y) \neq 0$, so $A_0 = 0$.

Similarly, following backwards we see that $A_1 \neq 0$, and so on until $A_i = 0 \ \forall \ i \in \{0, 1, 2, \cdots, n{-}1\}$. Thus linear independence is achieved, and $\{y, \alpha(y), \alpha^2(y), \cdots, \alpha^{n-1}(y)\}$ is indeed a basis.

QUESTION 9

(a) Consider $Av = \lambda v$, A invertible thus

$$A^{-1}(\lambda v) = v$$

So A has eigenvalue $\lambda \Rightarrow A^{-1}$ has eigenvalue $\frac{1}{\lambda}$

QUESTION 10

Define f as $f(\lambda) = \det(A - \lambda B)$. $C = A + iB$ invertible $\Rightarrow f(i) \neq 0$.

Therefore a polynomial of degree n does not have i as a root. Somehow, but I cannot see how, this must mean that there exists some $\lambda \in \mathbb{R}$ that is also not a root, ie. $f(\lambda) \neq 0$, thus $\det(A - \lambda B) \neq 0$ and $A - \lambda B$ is invertible.

Next, suppose that $P = C^{-1}QC$ for some complex invertible matrix $C = A + iB$, so

$$P = (A + iB)^{-1}Q(A + iB)$$

Know $(A + \lambda B)$ invertible for some λ, but not sure how this helps.

QUESTION 11

QUESTION 12

(i) If λ is an eigenvalue, then $f' = \lambda f$. But we can always find a differentiable function (namely $f(x) = e^{\lambda x}$) st. $f'(x)$ is $\lambda f(x)$. Hence every real number is an eigenvalue of f.

$\ker(\alpha - \lambda\iota)$ is just the span of the λ eigenspace, ie. $< e^{\lambda x} >$, thus has dimension 1.

(ii) To show surjectivity of $(\alpha - \lambda\iota)$, must show that there exists an f st. $f' - \lambda f$.

$$(\ \forall\, g \in V)(\ \exists\, f \in V \text{ s.t. } g = f' - \lambda f)$$

Try $f = e^{\lambda x} \int g e^{-\lambda x}$. Then

$$\begin{aligned} f' - \lambda f &= \lambda e^{\lambda x} \int g e^{-\lambda x} + e^{\lambda x}(g e^{-\lambda x}) - \lambda e^{\lambda x} \int g e^{-\lambda x} \\ &= e^{\lambda x}(g e^{-\lambda x} \\ &= g \end{aligned}$$