Part IB — Numerical Analysis Example Sheet 2

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QUESTION 1

The differential equations, with initial condition $y(0) = 1$ have exact solutions given by

$$y = \frac{1}{1+t} \quad \text{and} \quad y = (1+t)^2, \quad 0 \leq t \leq 1$$

respectively.
Using the Euler method for the first ODE we have $f(t, y) = -\frac{y}{1+t}$.
Here, $y_0 = 1, t_m = mh$. For $n \geq 1$,

$$\begin{aligned} y_n &= y_{n-1} + hf(t_{n-1}, y_{n-1}) \\ &= y_{n-1}\left(1 - \frac{h}{1+(n-1)h}\right) \\ &= y_{n-1} \cdot \frac{1+(n-2)h}{1+(n-1)h} \end{aligned}$$

Have that $y_1 = 1 - h$, thus

$$\begin{aligned} y_n &= 1 \cdot (1-h)\left(\frac{1}{1+h}\right)\left(\frac{1+h}{1+2h}\right)\cdots\left(\frac{1+(n-3)h}{1+(n-2)h}\right)\left(\frac{1+(n-2)h}{1+(n-1)h}\right) \\ &= \frac{1-h}{1+(n-1)h} \end{aligned}$$

As $h \to 0$, $n \to \infty$ in such a way that $nh \to t$. So we deduce

$$\begin{aligned} y_n &= (1-h)(1+t-h)^{-1} \\ &= (1-h)(1+t)^{-1}\left(1 - \frac{h}{1+t}\right)^{-1} \\ &= (1-h)(1+t)^{-1}\left(1 + \frac{h}{1+t} + \cdots\right) \\ &= (1+t)^{-1} + O(h) \end{aligned}$$

which is $y = 1/(1+t)$ as $h \to 0$, as required. Moreover the magnitude of the error is at most $O(h)$. Next question is similar.

$$y_n - y(nh) = \frac{1-h}{1+(n-1)h} - \frac{1}{1+nh}$$

which is clearly $O(h)$.
For the second ODE we have $f(t, y) = \frac{2y}{1+t}$.
Calculating the first few terms we find that

$$\begin{aligned} y_1 &= y_0\left(1 + \frac{2h}{1+t_0}\right) \qquad t_0 = 0 \\ &= (1+2h) \end{aligned}$$

$$
\begin{aligned}
y_2 &= y_1 \left(1 + \frac{2h}{1+t_1}\right) \qquad t_1 = h \\
&= (1+2h) \cdot \left(\frac{1+3h}{1+h}\right)
\end{aligned}
$$

$$
\begin{aligned}
y_3 &= y_2 \left(1 + \frac{2h}{1+t_1}\right) \qquad t_2 = 2h \\
&= (1+2h) \cdot \left(\frac{1+3h}{1+h}\right) \cdot \left(\frac{1+4h}{1+2h}\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
y_n &= (1+2h) \cdot \left(\frac{1+3h}{1+h}\right) \cdot \left(\frac{1+4h}{1+2h}\right) \cdots \left(\frac{1+(n+1)h}{1+(n-1)h}\right) \\
&= \frac{(1+nh)(1+(n+1)h)}{1+h}
\end{aligned}
$$

again $nh \to t$, so we have the result as required.
Here, the error is

$$
\begin{aligned}
y_n - y(nh) &= \frac{(1+nh)(1+(n+1)h)}{1+h} - (1+nh)^2 \\
&= \frac{(1+nh)^2 + h(1+nh) - (1+h)(1+nh)^2}{1+h}
\end{aligned}
$$

which is clearly $O(h)$.

QUESTION 2

The trapezoidal rule states that the numerical solution to the differential equation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}_n) \qquad (2.1)$$

is given by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})] \qquad (2.2)$$

Assuming that $\mathbf{f}$ satisfies the Lipschitz condition: there exists $\lambda \geq 0$ such that

$$||\mathbf{f}(t, \mathbf{v}) - \mathbf{f}(t, \mathbf{w})|| \leq \lambda ||\mathbf{v} - \mathbf{w}||, \quad t \in [0, t^*], \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^N$$

we will prove that the trapezoidal rule converges; ie.

$$\lim_{h \to 0} \max_{n=0,\cdots,\lfloor t^*/h \rfloor} ||\mathbf{y}_n(h) - \mathbf{y}(nh)|| = 0$$

where $\mathbf{y}(nh)$ is the evaluation at time $t = nh$ of the exact solution of (2.1).

Proof. Let $\mathbf{e}_n = \mathbf{y}_n - \mathbf{y}(t_n)$, the error at step n, where $0 \leq n \leq t^*/h$, $t_n := nh$. Thus,

$$\begin{aligned}
\mathbf{e}_{n+1} &= \mathbf{y}_{n+1} - \mathbf{y}(t_{n+1}) \qquad \text{(Taylor expand about } t = nh) \\
&= [\mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) + \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})]] - [\mathbf{y}(t_n) + h\mathbf{y}'(t_n) + \frac{h^2}{2!}\mathbf{y}''(t_n) + O(h^3)] \\
&= [\mathbf{y}_n - \mathbf{y}(t_n)] + \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) - \mathbf{f}(t_n, \mathbf{y}(t_n))] \\
&\quad + \frac{1}{2}h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) \underbrace{- \frac{1}{2}h\mathbf{y}'(t_n) - \frac{1}{2}h^2\mathbf{y}''(t_n)}_{(*)} + O(h^3)
\end{aligned}$$

$$\begin{aligned}
(*) &= -\frac{1}{2}h\,[\mathbf{y}'(t_n) + h\mathbf{y}''(t_n)] \\
&= -\frac{1}{2}h\,\left[\mathbf{y}'(t_{n+1}) + O(h^2)\right] \\
&= -\frac{1}{2}h\mathbf{f}(t_{n+1}, \mathbf{y}(t_{n+1})) + O(h^3)
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbf{e}_{n+1} = \mathbf{e}_n &+ \frac{1}{2}h[\mathbf{f}(t_n, \mathbf{y}_n) - \mathbf{f}(t_n, \mathbf{y}(t_n))] \\
&+ \frac{1}{2}h[\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - \mathbf{f}(t_{n+1}, \mathbf{y}(t_{n+1}))] + O(h^3)
\end{aligned}$$

By the Taylor theorem, the $O(h^3)$ term can be bounded uniformly for all $[0, t^*]$ by ch^3, where $c > 0$. Thus, using Lipshitz and the triangle inequality,

$$||\mathbf{e}_{n+1}|| \leq ||\mathbf{e}_n|| + \frac{1}{2}h\lambda||\mathbf{e}_n|| + \frac{1}{2}h\lambda||\mathbf{e}_{n+1}|| + ch^3$$

Therefore we can say that

$$\begin{aligned}
||\mathbf{e}_{n+1}|| &= \frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}||\mathbf{e}_n|| + \frac{ch^3}{1-\frac{1}{2}h\lambda} \\
&= \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^2 ||\mathbf{e}_{n-1}|| + \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)\frac{ch^3}{1-\frac{1}{2}h\lambda} + \frac{ch^3}{1-\frac{1}{2}h\lambda} \\
&= \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^{n+1} ||\mathbf{e}_0|| + \frac{ch^3}{1-\frac{1}{2}h\lambda}\sum_{k=0}^{n}\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^k \\
&= \frac{ch^3}{1-\frac{1}{2}h\lambda}\left\{1-\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^{n+1}\right\}\frac{1}{1-\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)} \\
&= \frac{ch^2}{\lambda}\left\{\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^{n+1} - 1\right\}
\end{aligned}$$

□

eg.

$$\begin{cases} 1+x < e^x & \text{for } x > 0 \\ 1-x > e^{-2x} & \text{for } x < \frac{1}{2} \end{cases} \Rightarrow \frac{1+x}{1-x} < e^{3x} \qquad \text{for } 0 < x < 1/2$$

$$\Rightarrow \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^{n+1} < e^{\frac{3}{2}h\lambda(n+1)}$$

$$\Rightarrow ||\mathbf{e}_n|| \leq Ah^2, \qquad A = \frac{c}{\lambda}e^{\frac{3}{2}\lambda t^*}$$

QUESTION 3

The s-step Adams-Bashforth method is of order s and has the form

$$\mathbf{y}_{n+s} - \mathbf{y}_{n+s-1} = h \sum_{j=0}^{s-1} \sigma_j \mathbf{f}(t_{n+j}, \mathbf{y}_{n+j})$$

For $s = 3$ we have $\rho(w) = w^2(w-1)$. To maximize order, we let σ be the 2 degree polynomial ($\sigma_3 = 0$) arising from the truncation of the Taylor expanison of

$$\frac{\rho(w)}{\log w}$$

Fix this from here, you should have $\rho(w) = w^2(w+1)$; ie the ROOTs of this thing are the coeffs.

Letting $\xi = w - 1$ and expanding,

$$\begin{aligned}
\frac{w^2(w-1)}{\log w} &= \frac{(\xi+1)^2\xi}{\log(1+\xi)} = \frac{\xi + 2\xi^2 + \xi^3}{\xi - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3 - \cdots} \\
&= \frac{1 + 2\xi + \xi^2}{1 - \frac{1}{2}\xi + \frac{1}{3}\xi^2 - \cdots} \\
&= [1 + 2\xi + \xi^2][1 + (\frac{1}{2}\xi - \frac{1}{3}\xi^2) + (\frac{1}{2}\xi - \frac{1}{3}\xi^2)^2 + O(\xi^3)] \\
&= 1 + \frac{5}{2}\xi + \frac{5}{3}\xi^2 + O(\xi^3) \\
&= 1 + \frac{5}{2}(w-1) + \frac{5}{3}(w-1)^2 + O(|w-1|^3) \\
&= \frac{1}{6} - \frac{5}{3}w + \frac{5}{3}w^2 + O(|w-1|^3)
\end{aligned}$$

Therefore $\sigma_0 = \frac{1}{6}, \sigma_1 = -\frac{5}{3}, \sigma_2 = \frac{5}{3}, \sigma_3 = 0$

QUESTION 4

Applying the *explicit midpoint rule*

$$\mathbf{y}_{n+2} = \mathbf{y}_n + 2h\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1})$$

to the ODE $y' = -y$, we have

$$y_{n+2} = y_n - 2hy_{n+1}$$

Making the ansatz $y_n = k^n$ gives

$$k^2 + 2hk - 1 = 0$$

and hence

$$k = -h \pm \sqrt{h^2 + 1}$$

giving

$$y_n = A\left(-h - \sqrt{h^2+1}\right)^n + B\left(-h + \sqrt{h^2+1}\right)^n$$

Now $y_0 = 1 \Rightarrow A + B = 1$, and $y_1 = 1 - h \Rightarrow 1 = (A - B)\sqrt{h^2+1}$, thus (FIX)

$$A = \frac{1}{2\sqrt{h^2-1}} + \frac{1}{2}$$
$$B = \frac{1}{2\sqrt{h^2-1}} - \frac{1}{2}$$

Now as $n \to \infty$, we wish to show that y_n diverges, ie. one of the terms blow up, and we want to show this happens for all $h > 0$. Can see that if $h > 1$, the $A\left(-h - \sqrt{h^2+1}\right)^n$ explodes as $|-h - \sqrt{h^2-1}| = |h + \sqrt{h^2+1}| > 1$. What if $h < 0$? (Does it make sense that $h > 0$?).

Next, note that $\rho_0 = -1$, $\rho_1/0$, $\rho_2 = 1$, thus $|\rho_k| \leq 1$ and when $|\rho_k| = 1$, zero is simple. Therefore the root condition is obeyed.

Is it clear that order ≥ 1, do we need to show? Consider

$$\begin{aligned} -h + \sqrt{1+h^2} &= -h + 1 + \frac{1}{2}h^2 + O(h^3) \\ &= e^{-h} + O(h^3) \end{aligned}$$

and

$$\begin{aligned} -h - \sqrt{1+h^2} &= -h - 1 - \frac{1}{2}h^2 + O(h^3) \\ &= -e^{-h} + O(h^3) \end{aligned}$$

should find coeffs such that

$$
\begin{aligned}
y_n &= \frac{1}{2\sqrt{1+h^2}}\left\{(1+\sqrt{1+h^2})(-h+\sqrt{1+h^2})^n + (1-\sqrt{1+h^2})(-h-\sqrt{1+h^2})^n\right\} \\
&= \frac{1}{2\sqrt{1+h^2}}\left\{(1+\sqrt{1+h^2})e^{-nh}(1+O(e^h h^3))^n + (1-\sqrt{1+h^2})(1+O(e^h h^3))^n\right\} \\
&\to \frac{1}{2}2e^{-t} = e^{-t} \qquad \text{as } h \to 0 \text{ with } nh = t = O(1)
\end{aligned}
$$

Thus convergence in a finite interval, or whatever.

QUESTION 5

The multistep method

$$\sum_{j=0}^{3} \rho_j \mathbf{y}_{n+j} = h \sum_{j=0}^{2} \sigma_j \mathbf{f}(t_{n+j}, \mathbf{y}_{n+j}), \quad \rho_3 = 1$$

is of order 4 iff

$$\rho(e^z) - z\sigma(e^z) = O(z^5), \quad z \to 0$$

Expanding into Taylor series,

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + O(z^5)$$

$$e^{2z} = 1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{5}{8}z^4 + O(z^5)$$

$$e^{3z} = 1 + 3z + \frac{9}{2}z^2 + \frac{9}{2}z^3 + \frac{10}{3}z^4 + O(z^5)$$

$$\begin{aligned}\rho(e^z) - z\sigma(e^z) = {} & [1 + 3z + \frac{9}{2}z^2 + \frac{9}{2}z^3 + \frac{10}{3}z^4] + \rho_2[1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{5}{8}z^4] \\ & + \rho_1[1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4] + \rho_0 - z\sigma_2\left[1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{5}{8}z^4\right] \\ & - z\sigma_1\left[1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4\right] - z\sigma_0\end{aligned}$$

For this expression to be $O(z^5)$, looking at first order terms we deduce that $\rho_1 + \rho_2 + \rho_3 = -1$.

Similarly, we get

$$\rho_1 + 4\rho_2 + 9 - 2\sigma_1 - 4\sigma_2 = 0$$

$$\rho_1 + 8\rho_2 + 27 - 3\sigma_1 - 12\sigma_2 = 0$$

$$\rho_1 + 16\rho_2 + 81 - 4\sigma_1 - 36\sigma_2 = 0$$

$3 \times (2) - (3) \Rightarrow$

$$2\rho_1 + 4\rho_2 - 3\sigma_1 = 0$$

$8 \times (2) - (4)$

$$7\rho_1 + 16\rho_2 - 9 - 12\sigma_1 = 0$$

So we get $\rho_1 = -9$, sub into first to get $\rho_0 + \rho_2 = 8$ as required.

So we have

$$\begin{aligned}\rho(w) &= w^3 + \rho w^2 - 9w + (8-\rho) \\ &= (w-1)\underbrace{(w^2 + (p+1)w + p - 8)}_{(*)}\end{aligned}$$

Now the roots of $(*)$ are

$$= \frac{-(p+1) \pm \sqrt{p^2 - 2p + 33}}{2}$$

now $\min(p^2 - 2p + 33) = 32$, so $|w| > 1$ and it cannot satisfy the root condition.

QUESTION 6

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QUESTION 7

Consider the ODE $y' = y$ with $y(0) = 1$ whose solution is $y(t) = e^t$. For this ODE we can write the local error explicitly: indeed we have

$$
\begin{aligned}
k_1 &= f(t_n, y(t_n)) = e^{t_n} \\
k_2 &= y(t_n) + \frac{1}{3}hk_1 = e^{t_n}(1 + \frac{1}{3}h) \\
k_3 &= y(t_n) - \frac{1}{3}hk_1 + hk_2 = e^{t_n}\left(1 + \frac{2}{3}h + \frac{1}{3}h^2\right) \\
k_4 &= y(t_n) + hk_1 - hk_2 + hk_3 = e^{t_n}\left(1 + h + \frac{1}{3}h^2 + \frac{1}{3}h^3\right)
\end{aligned}
$$

Then the local error is $y(t_{n+1}) - y_{n+1}$, ie:

$$
\begin{aligned}
y(t_{n+1}) - (y(t_n) + \frac{1}{8}hk_1 + \frac{3}{8}hk_2 + \frac{3}{8}hk_3 + \frac{1}{8}hk_4) &= e^{t_n}\left[e^h - 1 - h\ \frac{h^2}{2} - \frac{h^3}{6} - \frac{h^4}{24}\right] \\
&= e^{th}\left[O(h^5)\right]
\end{aligned}
$$

Thus the method is at most order 4.
For f independent of y we have

$$
\begin{aligned}
y_{n+1} = y_n + h\frac{f}{8} + h\frac{3}{8}&\left(f' + \frac{h}{3}f'' + \frac{h^2}{8}f''' + \frac{h^3}{162}f''''\right) \\
&+ h\frac{3}{8}\left(f' + \frac{2h}{3}f'' + \frac{2h^2}{18}f''' + \frac{4h^3}{81}f''''\right) \\
&+ h\frac{1}{8}\left(f' + hf'' + \frac{h^2}{2}f''' + \frac{h^3}{6}f''''\right) + O(h^4)
\end{aligned}
$$

We know that $y(t_n + h) = y(t_n) + hf + \frac{h^2 f''}{2} + h^3\frac{f'''}{6} + \frac{h^4 f^4}{24} + O(h^5)$.
Hence $y(t_n + h) - y_{n+1} = O(h^5)$ and so the method is at least order 4.
For f independent of t we get

$$
\begin{aligned}
k_1 &= f(y_n) \\
k_2 &= f(y_n) + \frac{1}{3}hf(y_n)f'(y_n) + \frac{1}{18}h^2 f(y_n)f''(y_n) + O(h^3) \\
k_3 &= f(y_n) + h\left[-\frac{1}{3}f(y_n) + f(y_n) - \frac{1}{3}hf(y_n)f'(y_n)\right]f' + O(h^5) \\
k_4 &= f(y_n) + h\left[f(y_n) - f(y_n) - \frac{1}{3}hf(y_n)f'(y_n) + f(y_n) + \frac{2h}{3}f(y_n)\right]f' + O(h^5)
\end{aligned}
$$

Hence

$$
\begin{aligned}
y_{n+1} = y_n &+ h\frac{1}{8}f \\
&+ h\frac{3}{8}\left[f + \frac{h}{3}ff' + \frac{h^2}{18}f^2f\right] \\
&+ h\frac{3}{8}\left[f + \frac{2h}{3}ff' + \frac{h^2}{3}f(f')^2\right] \\
&+ h\frac{1}{8}\left[f + hff' - \frac{h^2}{3} + \frac{h^2}{3}f(f')^2 + \frac{2h^2}{3}ff'\right] + O(h^4)
\end{aligned}
$$

Now $y(t_{n+1}) = y(t_n) + hf + \frac{h^2}{2}f'f + \frac{h^3}{6}\left(f''f^2 + (f')^2f\right) + O(h^4)$
Hence $y_{n+1} - y(t_{n+1}) = O(h^4)$. So the method is at least order 3.

QUESTION 8

Consider the linear scalar system

$$\begin{cases} y' = \lambda y \\ y(0) = 1 \end{cases}$$

where $\lambda < 0$. The solution is $y(t) = e^{\lambda t}$ which decays to 0 as $t \to \infty$.

(i) For the explicit Euler method we get $y_{n+1} = y_n + h\lambda y_n$ whose solution is $y_n = (1+h\lambda)^n$, so $y_n \to 0$ iff $|1+h\lambda| < 1$, therefore $\mathcal{D} = \{z \in \mathbb{C}\ ;\ |1+z| < 1\}$, and $\mathcal{D} \cap \mathbb{R} = \{z \in \mathbb{R} \mid -2 < z < 0\}$.

(ii) Considering now the trapezoidal rule we get $y_{n+1} = [(1+\frac{1}{2}h\lambda)/(1-\frac{1}{2}h\lambda)]y_n$, and thus by induction, $y_n = [(1+\frac{1}{2}h\lambda)/(1-\frac{1}{2}h\lambda)]^n y_0$. Therefore

$$z \in \mathcal{D} \iff \left|\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}\right| < 1 \iff \operatorname{Re} z < 0$$

and we deduce that $\mathcal{D} = \mathbb{C}^-$. Hence the method is A-stable, and $\mathcal{D} \cap \mathbb{R} = (-\infty, 0)$.

(iii) Solving should give

$$\lambda = z \pm \sqrt{z^2+1}$$

So we require $|z + \sqrt{z^2+1}| < 1$ and $|z - \sqrt{z^2+1}| < 1$. Everything fails (you might think $z = 0$ is fine, but inequalities are strict.) and $\mathcal{D} = \emptyset$.

(iv) Give it a go! Hint: Consider the borderline cases where $|\lambda_+| = 1$, or $|\lambda_-| = 1$.

(v) Applying the RK method to $y' = \lambda y$ we have

$$\begin{aligned} hk_1 &= h\lambda y_n \\ hk_2 &= h\lambda(y_n + hk_1) \end{aligned}$$

therefore

$$y_{n+1} = y_n + \frac{1}{2}hk_1 + \frac{1}{2}hk_2 = (1 + h\lambda + \frac{1}{2}h^2\lambda^2)y_n$$

Let

$$r(z) = 1 + z + \frac{1}{2}z^2$$

Then $y_{n+1} = r(h\lambda)y_n$, therefore, by induction, $y_n = [r(h\lambda)]^n y_0$ and we deduce that

$$\mathcal{D} = \{z \in \mathbb{C}\ ;\ |r(z)| < 1\}$$

r is analytic in $\mathcal{V} = \{z \in \mathbb{C} \ ; \ \mathrm{Re}\, z <\leq 0\}$. Therefore it attains its maximum on $\partial\mathcal{V} = i\mathbb{R}$.

Which $x \in \mathbb{R}$ give $|1 + x + \frac{1}{2}x^2| < 1$?

QUESTION 9

Consider the two-step BDF method: $\mathbf{y}_{n+2} - \frac{4}{3}\mathbf{y}_{n+1} + \frac{1}{3}\mathbf{y}_n = \frac{2}{3}hf(t_{n+2}, \mathbf{y}_{n+2})$. Applied to $y' = \lambda y$ we get

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}h\lambda y_{n+2}$$

If $z \in \partial\mathcal{D}$, $y_n = e^{in\theta}$. So,

$$e^{2i\theta} - \frac{4}{3}e^{i\theta} + \frac{1}{3} = \frac{2}{3}ze^{2i\theta}$$

$$\Rightarrow z = \frac{1}{2}\left(3 - 4e^{i\theta} + e^{2i\theta}\right)$$

$$\begin{aligned}\Rightarrow \operatorname{Re} z &= \frac{1}{2}(3 - 4\cos\theta + \cos 2\theta) \\ &= \frac{1}{2}\left(3 - 4\cos\theta + 2\cos^2\theta - 1\right) \\ &= (1-\cos\theta)^2 \\ &\geq 0\end{aligned}$$

$$(3 - 2h\lambda)k^2 - 4k + 1 = 0$$

A-stable $\iff$ $\{\operatorname{Re} z < 0\} \subset \mathcal{D}$.

We have deduced ∂D lies completely to the right of the imaginary axis. Now we just need to check one point to determine which side of ∂D we are on, eg. $z = -1$.

QUESTION 10

Given that $|y_n - y(t_n)| \leq 10^{-6}$, with Euler's method, setting $h = 2 \times 10^{-4}$, we have

For backward Euler we have:

$$y_{n+1} = y_n + h_n \left[-10^4 \left(y_{n+1} - t_{n+1}^{-1}\right) - t_{n+1}^{-2}\right]$$

$$\Rightarrow (1 + 10^4 h_n) y_{n+1} = y_n + 10^4 h_n t_{n+1}^{-1} - h_n t_{n+1}^{-2}$$

$$\begin{aligned} Rightarrow (1 + 10^4 h_n)(y_{n+1} - t_{n+1}^{-1}) &= y_n + 10^4 h_n t_{n+1}^{-1} - h_n t_{n+1}^{-2} - (1 + 10^4 h_n) t_{n+1}^{-1} \\ &= y_n - h_n t_{n-1}^{-2} - t_{n+1}^{-1} \\ &= (y_n - t_n^{-1}) - h_n t_{n+1}^{-2} - t_{n+1}^{-1} + t_n^{-1} \end{aligned}$$

Therefore

$$\begin{aligned} (1 + 10^4 h_n) e_{n+1} &= e_n - h_n t_{n+1}^{-2} - t_{n+1}^{-1} + t_n^{-1} \\ &= e_n - \frac{h_n}{t_{n+1}^2} + \frac{t_{n+1} - t_n}{t_n t_{n+1}} \\ &= e_n - \frac{h_n}{t_{n+1}^2} + \frac{h_n}{t_n t_{n+1}} \\ &= e_n + \frac{h_n (t_{n+1} - t_n)}{t_n t_{n+1}^2} \\ &= e^n + \frac{h_n^2}{t_n t_{n+1}^2} \end{aligned}$$

Now suppose $||e_n|| \leq 10^{-6}$.
Then

$$\begin{aligned} ||e_{n+1}|| &= \frac{10^{-6}}{1 + 10^4 h_n} + \frac{h_n^2}{(1 + 10^4 h_n) t_n t_{n+1}^2} \\ &\leq \frac{10^{-6} + h_n 10^{-2}}{1 + 10^4 h_n} \qquad \text{if } h \leq 10^{-2} t_n t_{n+1}^2 \\ &= \frac{10^{-6}(1 + h_n 10^4)}{1 + 10^4 h_n} \\ &= 10^{-6} \end{aligned}$$

QUESTION 11

First consider the predictor; substituting the true solution

$$\mathbf{y}(t_{n+3}) - \{-\frac{1}{2}\mathbf{y}(t_n) + 3\mathbf{y}(t_{n+1}) - \frac{3}{2}\mathbf{y}(t_{n+2}) + 3h\mathbf{y}'(t_{n+2})\} \quad (*)$$

Performing Taylor expansions:

$$\mathbf{y}(t_{n+3}) = \mathbf{y}(t_n) + 3h\mathbf{y}'(t_n) + \frac{9}{2}h^2\mathbf{y}''(t_n) + \frac{9}{2}h^3\mathbf{y}'''(t_n) + \frac{27}{8}h^4\mathbf{y}''''(t_n) + O(h^5)$$

$$\mathbf{y}(t_{n+1}) = \mathbf{y}(t_n) + h\mathbf{y}'(t_n) + \frac{1}{2}h^2\mathbf{y}''(t_n) + \frac{1}{6}h^3\mathbf{y}'''(t_n) + \frac{1}{24}h^4\mathbf{y}''''(t_n) + O(h^5)$$

$$\mathbf{y}(t_{n+2}) = \mathbf{y}(t_n) + 2h\mathbf{y}'(t_n) + 2h^2\mathbf{y}''(t_n) + \frac{4}{3}h^3\mathbf{y}'''(t_n) + \frac{2}{3}h^4\mathbf{y}''''(t_n) + O(h^5)$$

$$h\mathbf{y}'(t_{n+2}) = h\mathbf{y}'(t_n) + 2h^2\mathbf{y}''(t_n) + 2h^3\mathbf{y}'''(t_n) + \frac{4}{3}h^4\mathbf{y}''''(t_n) + O(h^5)$$

Substituting these into $(*)$ it is clear that the predictor method is third order; moreover we deduce that

$$\mathbf{y}(t_{n+3}) - \{-\frac{1}{2}\mathbf{y}(t_n) + 3\mathbf{y}(t_{n+1}) - \frac{3}{2}\mathbf{y}(t_{n+2}) + 3h\mathbf{y}'(t_{n+2})\} = \frac{1}{4}h^4\mathbf{y}''''(t_n) + O(h^5)$$

and thus

$$\mathbf{y}^P_{n+3} \approx \mathbf{y}(t_{n+3}) - \frac{1}{4}h^4\mathbf{y}''''(t_n)$$

Similarly for the corrector; substituting the true solution

$$\mathbf{y}(t_{n+3}) - \frac{1}{11}\{2\mathbf{y}(t_n) - 9\mathbf{y}(t_{n+1}) + 18\mathbf{y}(t_{n+2}) + 6h\mathbf{y}'(t_{n+3})\} \quad (**)$$

Noting that

$$h\mathbf{y}'(t_{n+3}) = h\mathbf{y}'(t_n) + 3h^2\mathbf{y}''(t_n) + \frac{9}{2}h^3\mathbf{y}'''(t_n) + 9h^4\mathbf{y}''''(t_n) + O(h^5)$$

We again see this method is third order, and that

$$\mathbf{y}(t_{n+3}) - \frac{1}{11}\{2\mathbf{y}(t_n) - 9\mathbf{y}(t_{n+1}) + 18\mathbf{y}(t_{n+2}) + 6h\mathbf{y}'(t_{n+3})\} =$$

Would finish but numbers aren't looking right.

Thus SHOULD GET

$$\mathbf{y}_{n+3}^{C} \approx \mathbf{y}(t_{n+3}) - \frac{3}{22} h^4 \mathbf{y}''''(t_n)$$

So

$$\mathbf{y}_{n+3}^{P} - \mathbf{y}_{n+3}^{C} = \frac{17}{44} h^4 \mathbf{y}''''(t_n)$$

$$\Rightarrow y_{n+3}^{C} - y(t_{n+1}) \approx \frac{3}{22}\frac{44}{17}(y_{n+3}^{P} - y_{n+3}^{C})$$

$$= \frac{6}{17}\left(y_{n+3}^{P} - y_{n+3}^{C}\right)$$

QUESTION 12

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