Part IB — Methods Example Sheet 3

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Michaelmas 2017

QUESTION 1

- We are given

$$\underbrace{\ddot{\theta} + 2p\dot{\theta} + (p^2 + q^2)}_{\mathcal{L}}\theta = f(t)$$

  with $\theta(0) = \dot{\theta}(0) = 0$, and $p > 0, q \neq 0$.

  Want to find G such that $\mathcal{L}G = \delta(t - \tau)$, so that for each value of τ, the Green's function will solve the homogeneous equation $LG = 0$ whenever $t \neq \tau$.

  We construct G for $0 \leq t < \tau$ as a general solution of the homogeneous equation, so that $G = Ay_1(t) + By_2(t)$. Have

$$G(t, \tau) = \Theta(t - \tau)e^{-p(t-\tau)}[C(\tau)\cos(q(t - \tau)) + D(\tau)\sin(q(t - \tau))]$$

  where Θ is the Heaviside step function. Continuity demands $G(\tau, \tau) = 0$ so $C(\tau) = 0$. The jump condition (with $\alpha(\tau) = 1$) enforces $D(\tau) = \frac{1}{q}$. Therefore the Green's function is

$$G(t, \tau) = \Theta(t - \tau)\frac{1}{q}e^{-p(t-\tau)}\sin(q(t - \tau))$$

  and the general solution to $\mathcal{L}\theta = f(t)$ obeying $\theta(0) = \dot{\theta}(0) = 0$ is

$$\theta(t) = \frac{1}{q}\int_0^t e^{-p(t-\tau)}\sin(q(t - \tau))f(\tau)\,\mathrm{d}\tau$$

- Next we use Fourier Transforms. Taking the Fourier transform of the differential equation gives

$$(i\omega)^2\tilde{\theta} + 2ip\omega\tilde{\theta} + (p^2 + q^2)\tilde{\theta} = \tilde{f}$$

  and so

$$\tilde{\theta} = \frac{\tilde{f}}{-\omega^2 + 2ip\omega + (p^2 + q^2)} =: \tilde{R}(\omega)\tilde{f}(\omega)$$

  which solves the equation algebraically in Fourier space. Note that

$$\begin{aligned}\tilde{R}(\omega) &= \frac{1}{-\omega^2 + 2ip\omega + (p^2 + q^2)} \\ &= \frac{1}{(i\omega + p)^2 - (qi)^2} \\ &= \frac{1}{2qi}\left[\frac{1}{i\omega + p - qi} - \frac{1}{i\omega + p + qi}\right]\end{aligned}$$

To solve for θ in real space we take the inverse Fourier transform to find

$$\begin{aligned}
\theta(t) &= \int_0^t R(t-u)f(u)\,\mathrm{d}u \\
&= \int_0^t \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{R}(\omega)e^{i\omega(t-u)}\,\mathrm{d}\omega\right] f(u)\,\mathrm{d}u \\
&= \frac{1}{q}\int_0^t \underbrace{\left[\frac{1}{4i\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\left[\frac{1}{i\omega+p-qi} - \frac{1}{i\omega+p+qi}\right]\,\mathrm{d}\omega\right]}_{(*)} f(u)\,\mathrm{d}u
\end{aligned}$$

which agrees with the first result, if we can show that

$$(*) = e^{-p(t-u)}\sin[q(t-u)]$$

but not sure how to evaluate the integral.

QUESTION 2

The general homogeneous solution is $c_1 \sinh \lambda x + c_2 \cosh \lambda x$ so we can take $y_1(x) = \sinh \lambda x$ and $y_2(x) = \sinh[\lambda(1-x)]$ as our homogeneous solutions satisfying the boundary conditions at $x = 0$ and $x = 1$ respectively. Then

$$G(x;\varepsilon) = \begin{cases} A(\varepsilon)\sinh(\lambda x) & \text{when } 0 \leq x < \varepsilon \\ B(\varepsilon)\sinh[\lambda(1-x)] & \text{when } \varepsilon < x \leq 1 \end{cases}$$

Applying the continuity condition we get

$$A \sinh \lambda\varepsilon = B \sinh[\lambda(1-\varepsilon)]$$

while the jump condition gives

$$B(-\lambda \cosh[\lambda(1-\varepsilon)]) - A\lambda \cosh(\lambda\varepsilon) = -1$$

Solving the for A and B gives us the Green's function

$$G(x;\varepsilon) = -\frac{1}{\lambda \sinh \lambda}\left[\Theta(\varepsilon - x)\sinh[\lambda(1-\varepsilon)]\sinh \lambda x + \Theta(x-\varepsilon)\sinh(\varepsilon\lambda)\sinh[\lambda(1-x)]\right]$$

Using this Green's function we are immediately able to write down the complete solution as

$$y = -\frac{1}{\lambda \sinh \lambda}\left\{\sinh \lambda x \int_x^1 f(\varepsilon)\sinh \lambda(1-\varepsilon)\,\mathrm{d}\varepsilon + \sinh \lambda(1-x)\int_0^x f(\varepsilon)\sinh \lambda\varepsilon\,\mathrm{d}\varepsilon\right\}$$

QUESTION 3

Use the substitution $y = z/x$, (quotient rule speeds thing up)

$$L_x y = -\frac{1}{x}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \frac{z}{x}$$

So

$$y = \frac{1}{x}\left(c_1 \cosh x + c_2 \sinh x\right)$$
$$y = \frac{1}{x}\left(c_1' e^{x} + c_2' e^{-x}\right)$$

For solutions that are (a) bounded as $x \to 0$, we have $y = A\frac{1}{x}\sinh x$. For solutions that are (b) bounded as $x \to \infty$, have $y = B\frac{1}{x}e^{-x}$.

Green's function satisfying $L_x G = \delta(x - \xi)$ and both conditions (a), (b) is given by

$$G(x; xi) = \begin{cases} A(\xi)\frac{\sinh x}{x} & \text{if } 0 < x < \xi \\ B(\xi)\frac{e^{-x}}{x} & \text{if } \xi < x \end{cases}$$

Continuity $\Rightarrow$

$$A \sinh \xi = B e^{-\xi}$$

Jump ($\alpha = -1$) $\Rightarrow$

$$B e^{\xi}(\xi + 1) + A(\xi \cosh \xi - \sinh \xi) = \xi^2$$

Solving,

$$A = \xi e^{-\xi} \qquad B = \xi \sinh \xi$$

Next, to solve

$$L_x y(x) = \begin{cases} 1 & \text{if } 0 \le x \le R \\ 0 & \text{if } x > R \end{cases}$$

Using this G,

$$y = x^{-1} e^{-x} \int_0^x \xi \sinh \xi \,\mathrm{d}\xi + x^{-1} \sinh x \int_x^R \xi e^{-xi} \,\mathrm{d}\xi$$

QUESTION 4

Can use a similar argument to the one in lectures to show that $G(t,\tau) = 0$ whenever $t \in [0,\tau]$. Following the usual procedure we get

$$G(t,\tau) = \Theta(t-\tau)[A(\tau)\cos[k(t-\tau)] + B(\tau)\sin[k(t-\tau)] + C(\tau)(t-\tau) + D(\tau)]$$

Continuity demands $G(\tau,\tau) = G'(\tau,\tau) = G''(\tau,\tau) = 0$, yielding

$$\begin{aligned} &A(\tau) + D(\tau) = 0 \\ &kB(\tau) + C(\tau) = 0 \\ &A(\tau) = 0 \end{aligned}$$

respectively. The third equation also implies $D(\tau) = 0$, and the jump condition on G''' gives

$$-k^3 B(\tau) = 1$$

$\Rightarrow B(\tau) = -k^{-3} \Rightarrow C(\tau) = k^{-2}$, showing the Green's function is indeed what is required.

Solution to $\mathcal{L}y = e^{-t}$ is given by

$$y(t) = \int_0^t [k^{-2}(t-\tau) - k^{-3}\sin k(t-\tau)]e^{-\tau}\,\mathrm{d}\tau$$

QUESTION 5

Under the substitution $y = \phi[x]$, hence $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{|\phi'(x)|}$ (monotone increasing) the result becomes easy to show

$$\begin{aligned}\int_a^b f(x)\delta[\phi(x)]\,\mathrm{d}x &= \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))\frac{1}{|\phi'(\phi^{-1}(y))|}\delta(y)\mathrm{d}y \\ &= f(\phi^{-1}(0))\frac{1}{|\phi'(\phi^{-1}(0))|} \\ &= \frac{f(c)}{|\phi'(c)|} \qquad \text{as } \phi^{-1}(0) = c\end{aligned}$$

If monotone decreasing, $\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{1}{|\phi'(x)|}$, but $\phi(b) < \phi(a)$, so $\int_{\phi(a)}^{\phi(b)} = -\int_{\phi(b)}^{\phi(a)}$ and the result holds.

Hence for a general $\phi(x)$ with simple zeros at $c_1, c_2, c_3 \cdots, c_N$ we have

$$\int_a^b f(x)\delta[\phi(x)]\,\mathrm{d}x = \sum_{i=1}^{N} \frac{f(c_i)}{|\phi'(c_i)|} \qquad (*)$$

Next for any $a \in \mathbb{R} \setminus \{0\}$

$$\int_{\mathbb{R}} \delta(at)\phi(t)\,\mathrm{d}t = \frac{1}{|a|}\int_{\mathbb{R}} \delta(y)\phi(y\ a)\,\mathrm{d}y = \frac{1}{|a|}\phi(0)$$

so we may write $\delta(at) = \delta(t)/|a|$.

Lastly, we apply $(*)$ for $f(x) = |x|$ and $\phi(x) = x^2 - a^2$, which has simple zeros at $x = a$ and $x = -a$. Get

$$\begin{aligned}\int_{-\infty}^{\infty} |x|\delta(x^2 - a^2)\,\mathrm{d}x &= \frac{f(a)}{|\phi'(a)|} + \frac{f(-a)}{|\phi'(-a)|} \\ &= \frac{|a|}{2|a|} + \frac{|a|}{2|a|} \\ &= 1\end{aligned}$$

as required

QUESTION 6

QUESTION 7

(i) $f(x) = 1$, $|x| < c$.

$$\begin{aligned}\tilde{f}(k) &= \int_{-c}^{c} e^{-ikx}\,\mathrm{d}x \\ &= \frac{i}{k}\left[e^{-ikx}\right]_{-c}^{c} \\ &= \frac{i}{k}(-2i\sin kc) \\ &= \frac{2\sin kc}{k}\end{aligned}$$

(ii) Notation: $\tilde{f} = \mathcal{F}[f]$ "Taking the Fourier transform of f. By the re-phasing property

$$\mathcal{F}[e^{iax}f(x)] = \tilde{f}(k-a)$$

we have for $f(x) = e^{iax}$ and using our previous answer,

$$\tilde{f}(k) = \frac{2\sin[(k-a)c]}{k-a}$$

(iii) $f(x) = \sin(ax)$

Noting that this is the imaginary part of what we've just done (but not seeing how to make use of that... so stuck with calculating) :

$$\begin{aligned}\tilde{f}(k) &= \int_{-c}^{c} \sin(ax)e^{-ikx}\,\mathrm{d}x \\ &= \frac{i}{k}\left[\sin(ax)e^{-ikx}\right]_{-c}^{c} - \frac{ia}{k}\int_{-c}^{c}\cos(ax)e^{-ikx}\,\mathrm{d}x \\ &= \frac{i}{k}2\sin(ax)\cos(kc) - \frac{ia}{k}\left(\frac{i}{k}\left[\cos(ax)e^{-ikx}\right]_{-c}^{c} + \frac{ia}{k}\int_{-c}^{c}\sin(ax)e^{-ikx}\,\mathrm{d}x\right) \\ &= \frac{i}{k}2\sin(ax)\cos(kc) + \frac{a}{k^2}\left(-2\cos(ax)\sin(kc) + a\tilde{f}(k)\right)\end{aligned}$$

Thus multiplying upon rearranging, we get

$$\tilde{f}(k) = \frac{2i\left(-k\cos[kc]\sin[ac] + a\cos[ac]\sin[kc]\right)}{a^2-k^2}$$

(iv) Next, by the differentiating property of Fourier Transforms, know that

$$\mathcal{F}[a\cos(ax)] = ik\tilde{f}(k)$$

where $\tilde{f}(k)$ is the FT of $f(x) = \sin(ax)$. Then, by scaling,

$$\mathcal{F}[\cos(ax)] = |a| ik \tilde{f}(k)$$

Thus the Fourier transform of $\cos(ax)$ is given by

$$\tilde{f}(k) = \frac{-2ka\left(-k\cos[kc]\sin[ac] + a\cos[ac]\sin[kc]\right)}{a^2 - k^2}$$

QUESTION 8

Have that

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x > 0 \end{cases}$$

The Fourier transform is given by

$$\begin{aligned} \tilde{f}(k) &= \int_0^\infty e^{-(ik+1)x} \, \mathrm{d}x \\ &= \frac{1}{-ik-1} \left[e^{-(ik+1)x} \right]_0^\infty \\ &= \frac{1}{1+ik} \\ &= \frac{1-ik}{1+k^2} \end{aligned}$$

The inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k) \, \mathrm{d}k$$

Thus

$$\begin{aligned} f(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1-ik}{1+k^2} \, \mathrm{d}k \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+k^2} \, \mathrm{d}k - \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{k}{1+k^2} \, \mathrm{d}k \\ &= \frac{1}{2\pi} [\arctan k]_{-\infty}^{\infty} - \frac{i}{4\pi} \underbrace{\left[\log(1+k^2) \right]_{-\infty}^{\infty}}_{=0} \\ &= \frac{1}{2\pi} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\ &= \frac{1}{2} \end{aligned}$$

QUESTION 9

We have

$$\begin{aligned} f(x) &= e^{-n^2(x-\mu)^2} \\ f(x+\mu) &= e^{-n^2x^2} \\ f'(x+\mu) &= -2n^2xe^{-n^2x^2} \end{aligned}$$

Calculating the Fourier Transform of the last function gives

$$\begin{aligned} \tilde{f}(k) &= -2n^2 \int_{-\infty}^{\infty} xe^{-n^2x^2} e^{-ikx} \, \mathrm{d}x \\ &= \end{aligned}$$

QUESTION 10

Let us suppose we have N measurements of a function $h(t)$, where N is even, with constant sampling interval Δ, ie. we have the set of measurements

$$h_m = h(t_m), \quad t_m = m\Delta,\ m = 0, 1, \cdots, N-1$$

Parseval's theorem for DFT is

$$\sum_{m=0}^{N-1} |h(t_m)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{h}_d(f_n)|^2 \qquad (*)$$

where h_d is the *Discrete Fourier transform*

$$\tilde{h}_d(f_n) = \sum_{n=0}^{N-1} h_m \exp\left[\frac{-2\pi i}{N} mn\right]$$

and

$$f_n = \frac{n}{N\Delta}, n = -N/2, \cdots, N/2,$$

To prove (*), consider a fixed $h(t_m)$ on the LSH. Appyling the inversion formula, and making a Riemann approximation to the integral, we obtain

$$\begin{aligned}
h(t_m) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t_m} \tilde{h}(\omega)\, \mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} e^{2\pi i f t_m} \hbar(f)\, \mathrm{d}f \qquad (2\pi f = \omega) \\
&\approx \frac{\Delta}{\Delta N} \sum_{n=0}^{N-1} \tilde{h}_d(f_n) e^{2\pi i f_n t_m} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{h}_d(f_n) \exp\left[\frac{2\pi i}{N} mn\right] \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{h}_d(f_n) \kappa^{mn}
\end{aligned}$$

where $\kappa = e^{2\pi i/N}$ is an Nth root of unity.

Now

$$\begin{aligned}
|h(t_m)|^2 = (h(t_m))^* h(t_m) &= \frac{1}{N^2} \sum_{p,q=0}^{N-1} \tilde{h}_d^*(f_p) \kappa^{-mp} \tilde{h}_d(f_q) \kappa^{mq} \\
&= \frac{1}{N^2} \sum_{p,q=0}^{N-1} \tilde{h}_d^*(f_p) \tilde{h}_d(f_q) \delta_{pq} \\
&= \frac{1}{N^2} \sum_{p=0}^{N-1} \tilde{h}_d^*(f_p) \tilde{h}_d(f_p) \\
&= \frac{1}{N} \sum_{p=0}^{N-1} |h_d(f_p)|^2
\end{aligned}$$

noting the independence from m. Hence, summing over N on both sides gives $(*)$ as required, as the right hand side just gains a factor of N.

QUESTION 11

Thus the Fourier transform of $\cos(x)$ (Q7 (iv)) is given by

$$\begin{aligned}\tilde{f}(k) &= \frac{-2k\left(-k\cos[k\frac{\pi}{2}]\sin[\frac{\pi}{2}] + \cos[\frac{\pi}{2}]\sin[k\frac{\pi}{2}]\right)}{1-k^2} \\ &= \frac{2}{1-k^2}\cos\left(\frac{k\pi}{2}\right)\end{aligned}$$

And the Fourier transform of the derivative $-\sin(x)$ is given by

$$\tilde{f}(k) = -i\frac{2k}{1-k^2}\cos\left(\frac{k\pi}{2}\right)$$

QUESTION 12

Inverse Fourier transform of $\tilde{f}(k)$ is

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-1}^{1} e^{ikx}e^{k} - e^{-k}\ \mathrm{d}k \\
&= \frac{1}{\pi}\int_{-1}^{1} e^{ikx}\sinh k\ \mathrm{d}k \\
&= \frac{1}{\pi}\left[-\frac{i}{x}e^{ikx}\sinh k\right]_{-1}^{1} + \frac{i}{x\pi}\int_{-1}^{1} e^{ikx}\cosh k\ \mathrm{d}k \\
&= \frac{-i}{\pi x}\left(e^{ix}\sinh 1 + e^{-ix}\sinh 1\right) + \frac{i}{x\pi}\left(\left[-\frac{i}{x}e^{ikx}\cosh k\right]_{-1}^{1} + \frac{i}{x}\int_{-1}^{1} e^{ikx}\sinh k\ \mathrm{d}k\right) \\
&= \frac{-2i}{\pi x}(\cos x\sinh 1) + \frac{i}{x\pi}\left(-\frac{2i}{x}i\sin x\cosh 1 + \frac{i}{x}f(x)\right)
\end{aligned}$$

$$\Rightarrow \pi x^2 f(x) = -2ix\cos x\sinh 1 + 2i\sin x\cosh 1 - f(x)$$

Hence rearranging gives

$$f(x) = \frac{2i}{\pi(1+x^2)}(\cosh 1\sin x - x\cos x\sinh 1)$$