

Part IB — Complex Methods Example Sheet 3

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QUESTION 1

Know that

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt, \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t} d\omega$$

First, let $f(t) = e^{-a|t|}$. Then

$$\begin{aligned} \tilde{f}(\omega) &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\omega t} dt \\ &= \int_{-\infty}^0 e^{at} e^{-i\omega t} dt + \int_0^{\infty} e^{-at} e^{-i\omega t} dt \\ &= \frac{1}{a - i\omega} [e^{at} e^{-i\omega t}]_{-\infty}^0 + \frac{1}{-a - i\omega} [e^{-at} e^{-i\omega t}]_0^{\infty} \\ &= \frac{1}{a - i\omega} + \frac{1}{a + i\omega} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$

So using the inverse Fourier transform relation we have

$$\begin{aligned} e^{-a|t|} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2} e^{i\omega t} d\omega \\ \iff \frac{\pi}{a} e^{-a|t|} &= \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} e^{i\omega t} d\omega \end{aligned}$$

as required.

Next, let $f(t) = e^{-at} \sin bt H(t)$. Then

$$\begin{aligned} \tilde{f}(\omega) &= \int_{-\infty}^{\infty} e^{-at} \sin bt e^{-i\omega t} H(t) dt \\ &= \int_0^{\infty} e^{-at} e^{-i\omega t} \sin bt dt \\ &= \int_0^{\infty} e^{-at} e^{-i\omega t} \frac{i}{2} (e^{-ibt} - e^{ibt}) dt \\ &= \frac{i}{2} \int_0^{\infty} e^{-at} e^{-i(\omega+b)t} - e^{-at} e^{-i(\omega-b)t} dt \\ &= \frac{i}{2} \left(-\frac{1}{-a - i(\omega+b)} - \frac{-1}{-a - i(\omega-b)} \right) \\ &= \frac{i}{2} \left(\frac{1}{a + i\omega + ib} - \frac{1}{a + i\omega - ib} \right) \\ &= \frac{i(-ib)}{(a + i\omega)^2 + b^2} \\ &= \frac{b}{(a + i\omega)^2 + b^2} \end{aligned}$$

So using the inverse Fourier transform relation we have

$$e^{-at} \sin bt H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{(a + i\omega)^2 + b^2} e^{i\omega t} d\omega$$
$$\iff 2\pi e^{-at} \sin bt H(t) = \int_{-\infty}^{\infty} \frac{b}{(a + i\omega)^2 + b^2} e^{i\omega t} d\omega$$

as required.

Not sure about the $a < 0$, $a = 0$ parts.

QUESTION 2

Given

$$f(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}a \\ 0 & \text{otherwise} \end{cases}$$

We compute the Fourier transform as

$$\begin{aligned} \tilde{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \\ &= \int_{-a/2}^{a/2} e^{-ikx} \, dx \\ &= \left[\frac{1}{-ik} e^{-ikx} \right]_{-a/2}^{a/2} \\ &= \frac{i}{k} \left(e^{-ika/2} - e^{ika/2} \right) \\ &= \frac{2}{k} \sin \left(\frac{ak}{2} \right) \end{aligned}$$

as required. Similarly, given

$$g(x) = \begin{cases} a - |x| & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases}$$

We compute the Fourier transform as

$$\begin{aligned} \tilde{g}(k) &= \int_{-\infty}^{\infty} g(x) e^{-ikx} \, dx \\ &= \int_{-a}^a (a - |x|) e^{-ikx} \, dx \\ &= \underbrace{\int_{-a}^a a e^{-ikx} \, dx}_{(1)} - \left(\underbrace{\int_{-a}^0 -x e^{-ikx} \, dx}_{(2)} + \underbrace{\int_0^a x e^{-ikx} \, dx}_{(3)} \right) \\ &= \frac{2a}{k} \sin ak + \frac{a}{ik} (e^{ika} + e^{-ika}) - \frac{1}{(ik)^2} [1 - e^{ika}] + \frac{1}{(ik)^2} [e^{-ika} - 1] \\ &= \end{aligned}$$

Now

$$\begin{aligned} (1) &= a \left[\frac{1}{-ik} e^{-ikx} \right]_{-a}^a \\ &= a \frac{i}{k} (e^{-ika} - e^{ika}) \\ &= \frac{2a}{k} \sin ak, \end{aligned}$$

$$\begin{aligned}
(2) &= \left[-x \left(-\frac{1}{ik} e^{-ikx} \right) - \int -\frac{1}{ik} e^{-ikx} \right]_{-a}^0 \\
&= \left[-\frac{a}{ik} e^{ika} + \frac{1}{(ik)^2} [e^{-ikx}]_{-a}^0 \right] \\
&= -\frac{a}{ik} e^{ika} - \frac{1}{k^2} (1 - e^{ika})
\end{aligned}$$

$$\begin{aligned}
(3) &= \left[x \left(-\frac{1}{ik} e^{-ikx} \right) - \int -\frac{1}{ik} e^{-ikx} \right]_0^a \\
&= \left[-\frac{a}{ik} e^{-ika} - \frac{1}{(ik)^2} [e^{-ikx}]_0^a \right] \\
&= -\frac{a}{ik} e^{-ika} + \frac{1}{k^2} (e^{-ika} - 1)
\end{aligned}$$

Not sure where error is.

The inversion formula for f is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk$$

ie.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{k} \sin \frac{ak}{2} e^{ikx} dk = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}a \\ 0 & \text{otherwise} \end{cases}$$

We see that $\tilde{f}(k)$ has simple poles at $2n\pi/a$. Not sure how to verify.

The convolution of $f(x)$ with itself is given by

$$f * f(x) = \int_{-\infty}^{\infty} f(x-y) f(y) dy$$

Have

$$\begin{aligned}
f(y) &= \begin{cases} 1 & \text{if } |y| < \frac{1}{2}a \\ 0 & \text{otherwise} \end{cases} \\
f(x-y) &= \begin{cases} 1 & \text{if } |x-y| < \frac{1}{2}a \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

So we have

$$f * f(x) = \int_{-a/2}^{a/2} f(x-y) dy$$

Note that $|x - y| < a/2 \iff -a/2 + x < y < a/2 + x$.
Hence for $-a/2 < -a/2 + x < a/2$ (ie $0 < x < a$)

$$\begin{aligned} f * f(x) &= \int_{-a/2+x}^{a/2} dy \\ &= a - x \end{aligned}$$

Similarly, if $-a/2 < a/2 + x < a/2$ (ie. $-a < x < 0$) have

$$\begin{aligned} f * f(x) &= \int_{-a/2}^{a/2+x} dy \\ &= a + x \end{aligned}$$

And the integral is zero otherwise.

QUESTION 3

The convolution of $g(x) = e^{-|x|}$ with itself is given by

$$g * g(x) = \int_{-\infty}^{\infty} e^{-|x-y|} e^{-|y|} \, dy$$

If $x > 0$ then we can split up the integral as

$$\begin{aligned} g * g(x) &= \int_{-\infty}^0 e^{x-y} e^y \, dy + \int_0^x e^{x-y} e^{-y} \, dy + \int_x^{\infty} e^{-(x-y)} e^{-y} \, dy \\ &= 0 - \frac{1}{2} [e^{-x} - e^x] + 0 \end{aligned}$$

Similarly if $x < 0$ then

$$\begin{aligned} g * g(x) &= \int_{-\infty}^x e^{x-y} e^y \, dy + \int_x^0 e^{-(x-y)} e^y \, dy + \int_0^{\infty} e^{-(x-y)} e^{-y} \, dy \\ &= \end{aligned}$$

Next, the convolution theorem for Fourier transforms states that $\mathcal{F}[g * g(x)] = \mathcal{F}[g]\mathcal{F}[g]$. Applied here, we have

$$\int_{-\infty}^{\infty} (1 + |x|) e^{-|x|} e^{-ikx} \, dx = \left[\int_{-\infty}^{\infty} e^{-|x|} e^{ikx} \, dx \right]^2$$

QUESTION 4

QUESTION 5

Starting with

$$\mathcal{L}(1) = \frac{1}{s}$$

(i) By shifting,

$$\mathcal{L}(e^{-2t}) = \frac{1}{s+2}$$

(ii) Starting with shifting,

$$\mathcal{L}(e^{-3t}) = \frac{1}{s+3}$$

Then

$$\begin{aligned}\mathcal{L}(t^3 e^{-3t}) &= -\frac{d^3}{ds^3} \mathcal{L}(e^{-3t}) \\ &= -\frac{d^3}{ds^3} \frac{1}{s+3} \\ &= \frac{6}{(s+3)^4}\end{aligned}$$

(iii) Write $\sin 4t = \frac{1}{2i}(e^{4it} - e^{-4it})$. Then

$$\begin{aligned}\mathcal{L}(e^{3t} \sin 4t) &= \frac{1}{2i} \mathcal{L}(e^{(3+4i)t}) - \frac{1}{2i} \mathcal{L}(e^{(3-4i)t}) \\ &= \frac{1}{2i} \left[\frac{1}{s - (3+4i)} - \frac{1}{s - (3-4i)} \right] \\ &= \frac{4}{(s-3)^2 + 16}\end{aligned}$$

(iv) Writing $\cosh 4t = \frac{1}{2}(e^{4t} + e^{-4t})$, we have

$$\begin{aligned}\mathcal{L}(e^{-4t} \cosh 4t) &= \frac{1}{2} \mathcal{L}(1) + \frac{1}{2} \mathcal{L}(e^{-8t}) \\ &= \frac{1}{2} \left[\frac{1}{s} + \frac{1}{s+8} \right] \\ &= \frac{s+4}{s(s+8)}\end{aligned}$$

(v) First by shifting,

$$\mathcal{L}(e^{-(t+1)}) = e^{-1} \mathcal{L}(e^{-t}) = \frac{e^{-1}}{s+1}$$

Then by translation,

$$\mathcal{L}(e^{-t}H(t-1)) = \frac{e^{-(s+1)}}{s+1}$$

QUESTION 6

By partial fractions we have

$$\hat{f}(s) = \frac{s+3}{(s-2)(s^2+1)} = \frac{1}{s-2} - \frac{s+1}{s^2+1}$$

Have

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2}, \quad \mathcal{L}(\cos t) = \frac{s}{s^2+1}, \quad \mathcal{L}(\sin t) = \frac{1}{s^2+1}$$

Hence by linearity the inverse Laplace transform of $\hat{f}(s)$ is given by

$$f(t) = e^{2t} - \cos t - \sin t$$

Alternatively we can use the Bromwich inversion formula,

$$f(t) = \sum_{k=1}^n \operatorname{res}_{p=p_k} (\hat{f}(p)e^{pt}),$$

as $\hat{f}(s)$ has only a finite number of singularities, namely $s = 2, i, -i$, and $\hat{f}(s) \rightarrow 0$ as $|s| \rightarrow \infty$. First,

$$\operatorname{res}_{s=2} \left(\frac{s+3}{(s-2)(s^2+1)} e^{st} \right) = \lim_{s \rightarrow 2} \left(\frac{s+3}{(s^2+1)} e^{st} \right) = e^{2t}$$

Next,

$$\begin{aligned} \operatorname{res}_{s=i} \left(\frac{s+3}{(s^2+1)} e^{st} \right) &= \lim_{s \rightarrow i} \left(\frac{s+3}{(s-2)(s+i)} e^{st} \right) \\ &= \frac{i+3}{2i(i-2)} e^{it} \\ &= -\frac{i+3}{4i+2} e^{it} \\ &= \frac{-10+10i}{20} e^{it} = \frac{-1+i}{2} (\cos t + i \sin t) \end{aligned}$$

Similarly,

$$\begin{aligned} \operatorname{res}_{s=-i} \left(\frac{s+3}{(s^2+1)} e^{st} \right) &= \lim_{s \rightarrow -i} \left(\frac{s+3}{(s-2)(s-i)} e^{st} \right) \\ &= \frac{-i+3}{2i(i+2)} e^{-it} \\ &= \frac{-10-10i}{20} e^{-it} = \frac{-1-i}{2} (\cos t - i \sin t) \end{aligned}$$

Hence adding the residues we achieve

$$f(t) = e^{2t} - \cos t - \sin t$$

in agreement with what we had above.

QUESTION 7

Consider the differential equation

$$\frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - y = t^2 e^t$$

$$y(0) = 1, \dot{y}(0) = 0, \ddot{y}(0) = -2$$

Taking the Laplace transform of this equation, where $\mathcal{L}(y) = \hat{y}$, we have

$$\begin{aligned} \mathcal{L}(\dot{y}) &= p\hat{y} + y(0) \\ &= p\hat{y} + 1 \end{aligned}$$

Similarly

$$\begin{aligned} \mathcal{L}(\ddot{y}) &= p\mathcal{L}(\dot{y}) + \dot{y}(0) \\ &= p^2 \hat{y} + py(0) + \dot{y}(0) \\ &= p^2 \hat{y} + p \end{aligned}$$

and,

$$\begin{aligned} \mathcal{L}\left(\frac{d^3 y}{dt^3}\right) &= p\mathcal{L}(\ddot{y}) + \ddot{y}(0) \\ &= p^3 \hat{y} + p^2 y(0) + p\dot{y}(0) + \ddot{y}(0) \\ &= p^3 \hat{y} + p^2 + -2 \end{aligned}$$

The term on the RHS gives

$$\begin{aligned} \mathcal{L}(t^2 e^t) &= \frac{d^2}{dp^2} \mathcal{L}(e^t) \\ &= \frac{d^2}{dp^2} \left(\frac{1}{p-1} \right) \\ &= \frac{2}{(p-1)^3} \end{aligned}$$

Thus substituting in gives:

$$(p^3 - 3p^2 + 3p - 1)\hat{y} + p^2 - 3p + 1 = \frac{2}{(p-1)^3}$$

Hence

$$\hat{y} = \frac{2}{(p-1)^6} - \frac{1}{(p-1)} + \frac{p}{(p-1)^3}$$

We know $\mathcal{L}(e^t) = 1/(p-1)$. For the other two terms, we will use the Bromwich inversion formula from the previous question, noting both tend to zero as $|p| \rightarrow \infty$.

$$f(t) = \sum_{k=1}^n \operatorname{res}_{p=p_k} (\hat{f}(p)e^{pt}),$$

For the first term, the singularity $p = 1$ has a pole of order 6. Hence the residue here is given by

$$\begin{aligned} \lim_{p \rightarrow 1} \frac{1}{5!} \frac{d^5}{dp^5} (p-1)^5 \left(\frac{2}{(p-1)^6} e^{pt} \right) &= \lim_{p \rightarrow 1} \frac{1}{5!} \frac{d^5}{dp^5} (-2e^{pt}(1-p)^{-1}) \\ &= \lim_{p \rightarrow 1} \frac{1}{5!} \frac{d^5}{dp^5} [-2e^{pt}(1+p+p^2+p^3+\dots)] \\ &= \dots \end{aligned}$$

The last term has a singularity at $p = 3$ with a pole of order 3 there; the residue is given by

$$\begin{aligned} \lim_{p \rightarrow 1} \frac{1}{2!} \frac{d^2}{dp^2} (p-1)^2 \left(\frac{p}{(p-1)^3} e^{pt} \right) &= \lim_{p \rightarrow 1} \frac{1}{2!} \frac{d^2}{dp^2} \left(\frac{p}{1-p} e^{pt} \right) \\ &= \lim_{p \rightarrow 1} \frac{1}{2!} \frac{d}{dp} \left(\frac{1}{(1-p)^2} e^{pt} + \frac{p^2}{(1-p)} e^{pt} \right) \\ &= \lim_{p \rightarrow 1} \frac{1}{2!} \left(\frac{2}{(1-p)^3} e^{pt} + \frac{p}{(1-p)^2} e^{pt} + \frac{1-p^2}{(1-p)^2} e^{pt} + \frac{p^3}{(1-p)} e^{pt} \right) \end{aligned}$$

Not sure how to take limit.

QUESTION 8

The differential equation we are investigating is

$$\ddot{y} - 2\dot{y} - 2y = \delta(t) - \delta(t - t_0)$$

Note that

$$\mathcal{L}(\delta(t)) = \int_0^\infty \delta(t)e^{pt} dt = 1, \quad \mathcal{L}(\delta(t - t_0)) = e^{-pt_0}$$

Taking the Laplace transform of the equation gives

$$p^2\hat{y} + 2(p\hat{y}) + 2\hat{y} = 1 + e^{-pt_0}$$

which gives

$$\hat{y} = \frac{1 + e^{-pt_0}}{(p^2 + 2p + 2)}$$

QUESTION 9

Taking the Laplace transform of the equation gives

$$\hat{f} + 4\frac{1}{p}\hat{f} = \frac{1}{p^2}$$

Thus

$$\hat{f} = \frac{1}{p^2 + 4p} = \frac{1}{4p} - \frac{1}{4(p + 4)}$$

By inspection,

$$f = \frac{1}{4} - \frac{1}{4}e^{-4t}$$

QUESTION 10

QUESTION 11

(whenever I can be bothered) <http://www.robots.ox.ac.uk/~jmb/lectures/pdelecture4.pdf>

QUESTION 12