

Part IA — Variational Principles Example Sheet

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Michaelmas 2017

QUESTION 1

$$\begin{aligned}
F[x + \delta x] - F[x] &= \int_{t_1}^{t_2} f(x + \delta x, \dot{x} + \delta \dot{x}, \ddot{x} + \delta \ddot{x}, t) dt - \int_{t_1}^{t_2} f(t, x, \dot{x}, \ddot{x}) dt \\
&= \int_{t_1}^{t_2} \left\{ \delta x \frac{\partial f}{\partial x} + (\delta \dot{x}) \frac{\partial f}{\partial \dot{x}} + (\delta \ddot{x}) \frac{\partial f}{\partial \ddot{x}} \right\} dt + O(t^2)
\end{aligned}$$

Discarding the (small) terms of $O(t^2)$, we call the first order variation $\delta F[x]$ and integrating by parts (twice), we have

$$\begin{aligned}
\delta F[x] &= \int_{t_1}^{t_2} \left\{ \delta x \left[\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) \right] - (\delta \dot{x}) \frac{d}{dt} \left(\frac{\partial f}{\partial \ddot{x}} \right) \right\} dt + \left[\delta x \frac{\partial f}{\partial \dot{x}} + (\delta \dot{x}) \frac{\partial f}{\partial \ddot{x}} \right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \left\{ \delta x \left[\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial f}{\partial \ddot{x}} \right) \right] \right\} dt \\
&\quad + \left[\delta x \left\{ \frac{\partial f}{\partial \dot{x}} - \frac{d}{dt} \left(\frac{\partial f}{\partial \ddot{x}} \right) \right\} + (\delta \dot{x}) \frac{\partial f}{\partial \ddot{x}} \right]_{t_1}^{t_2}
\end{aligned}$$

We have fixed end boundary conditions, so $\delta x(t_1) = \delta x(t_2) = 0$ and also $\delta \dot{x}(t_1) = \delta \dot{x}(t_2) = 0$. Thus the boundary term is zero and we can write $\delta F[x]$ in the form

$$\delta F[x] = \int_{t_1}^{t_2} \left\{ \delta x(t) \frac{\delta F[x]}{\delta x(t)} \right\} dt$$

where the *functional derivative* $\frac{\delta F[x]}{\delta x(t)}$ is defined as

$$\frac{\delta F[x]}{\delta x(t)} := \frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial f}{\partial \ddot{x}} \right)$$

The functional F is stationary when its functional derivative is zero (assuming that this derivative is defined on (t_1, t_2)) and the condition for this to be true is the following Euler-Lagrange equation:

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left(\frac{\partial f}{\partial \ddot{x}} \right) = 0, \quad t_1 < t < t_2$$

Given the functional

$$L[x] = \int_1^2 t^4 [\ddot{x}(t)]^2 dt$$

In this case, $f = t^4 [\ddot{x}(t)]^2$, so $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \dot{x}} = 0$ and the EL equation can be immediately twice integrated to give

$$\frac{\partial}{\partial \ddot{x}} [t^4 [\ddot{x}(t)]^2] = At + B$$

for some constants A and B .

ie.

$$2t^4 (\ddot{x})'(t) = At + B$$

$$\Rightarrow \ddot{x}(t) = At^{-3} + Bt^{-4} \quad t \neq 0$$

$$\Rightarrow \dot{x}(t) = A't^{-2} + B't^{-3} + C$$

Using b.c.s we have

$$\begin{aligned} -2 &= A' + B' + C \\ -\frac{1}{4} &= \frac{1}{4}A' + \frac{1}{8}B' + C \Rightarrow -2 = 2A' + B' + 8C \end{aligned}$$

Subtracting immediately yields $A' = 7C$:

$$-2 = 8C + B'$$

$$-2 = 22C + B'$$

Therefore $C' = 0$, $B' = -2$, and $\dot{x}(n) = -2t^{-3}$.

Integrating, we get

$$x(t) = t^{-2} + D$$

b.c.s $\Rightarrow D = 0$, so $x(t) = \frac{1}{t^2}$.

This function is a global minimum as it is convex.

Note: $\delta^2 F > 0 \not\Rightarrow$ global min. Need to use convexity. of $\int_1^2 t^4 [\ddot{x}(t)]^2 dt$.
(Clearly convex as we're just doing linear things, then squaring.)

QUESTION 2

Our aim is to maximise $A[x, y]$ subject to the constraint $P[x, y] = L$, where L is the fixed length and $P[x, y] = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2} d\theta$

Using a Lagrange multiplier λ to impose this, we seek to maximize the functional

$$\begin{aligned}\phi_\lambda[x, y] &= A[x, y] - \lambda(P[x, y] - L) \\ &= \int_0^{2\pi} \underbrace{\frac{1}{2}(xy' - yx') - \lambda\sqrt{(x')^2 + (y')^2}}_{=f_\lambda(\mathbf{x}, \mathbf{x}')} d\theta + \lambda L\end{aligned}$$

where $\mathbf{x} = (x, y)$. The boundary conditions imply that $x(0) = x(2\pi), x'(0) = x'(2\pi)$ and similarly for y at the endpoints, so boundary terms vanish and functional is stationary for solutions of the E-L equation. $f_\lambda(\mathbf{x}, \mathbf{x}')$ has no explicit θ dependence, so considering the y E-L equation we have that:

$$f - x' \frac{\partial f_\lambda}{\partial x'} - y' \frac{\partial f_\lambda}{\partial y'} = \text{constant}$$

$$\begin{aligned}\Rightarrow f - y' \left(\frac{1}{2}x - \frac{\lambda y'}{\sqrt{(x')^2 + (y')^2}} \right) - x' \left(-\frac{1}{2}y - \frac{\lambda x'}{\sqrt{(x')^2 + (y')^2}} \right) &= \text{constant} \\ \Rightarrow -\lambda \frac{((x')^2 + (y')^2)}{\sqrt{(x')^2 + (y')^2}} + \lambda \frac{(y')^2}{\sqrt{(x')^2 + (y')^2}} + \lambda \frac{(x')^2}{\sqrt{(x')^2 + (y')^2}} &= \text{constant} \\ \Rightarrow 0 = C\end{aligned}$$

Hmm? Try E-L directly...

QUESTION 3

Using Lagrange multiplier λ , wish to minimize

$$\begin{aligned}\Phi_\lambda[\psi] &= I[\psi] - \lambda \left[\int_{-\infty}^{\infty} \psi^2 \, dx = 1 \right] \\ &= \int_{-\infty}^{\infty} \underbrace{(\psi')^2 + (x^2 - \lambda)\psi^2}_{f_\lambda(\psi, \psi'; x)} \, dx + \lambda\end{aligned}$$

Normalisation condition, can assume $\phi = 0$ at endpoints, so the functional is stationary for solutions of the E-L equation. Euler-Lagrange equations imply:

$$\begin{aligned}2(x^2 - \lambda)\psi - \frac{d}{dx} [2\psi'] &= 0 \\ \Rightarrow \psi'' + (x^2 - \lambda)\psi &= 0\end{aligned}$$

Note that

$$\begin{aligned}I[\psi] &= \int_{-\infty}^{\infty} (\psi' + x^2\psi^2)^2 - 2x\psi\psi' \, dx \\ &= \int_{-\infty}^{\infty} (\psi' + x^2\psi^2)^2 \, dx - \int_{-\infty}^{\infty} x \frac{d}{dx} [\psi^2] \, dx \\ &= \int_{-\infty}^{\infty} (\psi' + x^2\psi^2)^2 \, dx - [x\psi^2]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \psi^2 \, dx \quad \text{by parts} \\ &= \int_{-\infty}^{\infty} (\psi' + x^2\psi^2)^2 \, dx + 1\end{aligned}$$

As $(\psi' + x^2\psi^2)$ is real valued, its square gives a positive function, thus the integral is positive and $I[\psi] \geq 1$. Equality holds for

$$\begin{aligned}(\psi' + x\psi) &= 0 \\ \Rightarrow \psi' + x\psi &= 0 \\ \Rightarrow \frac{d}{dx} (\psi e^{x^2/2}) &= 0 \\ \Rightarrow \psi &= C e^{-x^2/2}\end{aligned}$$

for some constant C (which we recognise as the Gaussian Wave Function). The normalisation condition implies that $C = \left(\frac{1}{\pi}\right)^{1/4}$.

Note: We're minimising the energy of a harmonic oscillator. What does λ represent here?

QUESTION 4

Using Lagrange multiplier λ with constraint $|\mathbf{x}| = 1$. We will instead use the equivalent constraint $|\mathbf{x}|^2 = 1$, so we wish to minimize the functional

$$\begin{aligned}\Phi_\lambda[\mathbf{x}] &= I[\mathbf{x}] - \lambda(t)(|\mathbf{x}|^2 - 1) \\ &= \int_{t_1}^{t_2} \underbrace{|\dot{\mathbf{x}}|^2 - \lambda(t)|\mathbf{x}|^2}_{f_\lambda(\mathbf{x}, \dot{\mathbf{x}}; t)} dt + \lambda\end{aligned}$$

The E-L equation for x_i component is $\frac{\partial f}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}_i} \right) = 0$. Have

$$\begin{aligned}\frac{\partial f}{\partial x_i} &= -\lambda(t) \frac{\partial}{\partial x_i} (x_j x_j) \\ &= -2\lambda(t) x_j \underbrace{\frac{\partial x_j}{\partial x_i}}_{\delta_{ij}} \\ &= -2\lambda(t) x_i \\ \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}_i} \right) &= \frac{d}{dt} (2\dot{x}_i) \\ &= 2\ddot{x}_i\end{aligned}$$

So E-L gives

$$\begin{aligned}\ddot{x}_i + \lambda(t)x_i &= 0 \\ \Rightarrow \ddot{\mathbf{x}} + \lambda(t)\mathbf{x} &= \mathbf{0} \quad (*)\end{aligned}$$

Take the constraint $|\mathbf{x}|^2 = 1$ and differentiate to get

$$2\dot{\mathbf{x}} \cdot \mathbf{x} = 0$$

Cancelling the two and differentiating again gives

$$\ddot{\mathbf{x}} \cdot \mathbf{x} + |\dot{\mathbf{x}}|^2 = 0$$

Dotting (*) with \mathbf{x} gives

$$\ddot{\mathbf{x}} \cdot \mathbf{x} + \lambda(t) \underbrace{|\mathbf{x}|^2}_{=1} = 0$$

Hence comparing gives $\lambda(t) = |\dot{\mathbf{x}}|^2$ and thus the desired equation

$$\ddot{\mathbf{x}} + |\dot{\mathbf{x}}|^2 \mathbf{x} = \mathbf{0}$$

QUESTION 5

Have the Lagrangian

$$L = \underbrace{\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\dot{\phi}^2}_T + \underbrace{mga \cos \theta}_{-V}$$

Note that $\frac{\partial L}{\partial \phi} = 0$ so we have the first integral

$$\text{const.} = \frac{\partial L}{\partial \dot{\phi}} = ma^2 \sin^2 \theta$$

Note too that $\frac{\partial L}{\partial t} = 0$ (no t dependence) so we have another first integral

$$\begin{aligned} \text{const.} &= L - \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} \\ &= -T - V \end{aligned}$$

from which we deduce that

$$\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\dot{\phi}^2 - mga \cos \theta = E$$

for the total energy.

The Hamiltonian is defined as the Legendre transform of the Lagrangian with respect to velocity $\mathbf{v} = \dot{\mathbf{x}}$:

$$H(\mathbf{x}, \mathbf{p}; t) = [\mathbf{p} \cdot \mathbf{v} - L(\mathbf{x}, \mathbf{v})]_{\mathbf{v}=\mathbf{v}(\mathbf{p})}$$

where $\mathbf{v}(\mathbf{p})$ is the solution to $\frac{\partial L}{\partial \mathbf{v}} = \mathbf{p}$.

The momentum p_θ is given by

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} \\ &= ma^2 \dot{\theta} \end{aligned}$$

Similarly,

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial \dot{\phi}} \\ &= ma^2 (\sin^2 \theta) \dot{\phi} \end{aligned}$$

Thus $\mathbf{v} = (\dot{\theta}, \dot{\phi}) = \left(\frac{p_\theta}{ma^2}, \frac{p_\phi}{ma^2 \sin^2 \theta} \right)$ and

$$\begin{aligned} H &= \frac{p_\theta^2}{ma^2} + \frac{p_\phi^2}{ma^2 \sin^2 \theta} - \left(\frac{1}{2}ma^2 \left(\frac{p_\theta}{ma^2} \right)^2 + \frac{1}{2}ma^2 \left(\frac{p_\phi}{ma^2 \sin^2 \theta} \right)^2 + mga \cos \theta \right) \\ &= \frac{1}{2} \frac{p_\theta^2}{ma^2} + \frac{1}{2} \frac{p_\phi^2}{ma^2 \sin^2 \theta} - mga \cos \theta \end{aligned}$$

Hamilton's equations are given by

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}$$

ie.

$$\dot{\theta} = \frac{p_{\theta}}{ma^2}$$

$$\dot{\phi} = \frac{p_{\phi}}{ma^2 \sin^2 \theta}$$

$$\dot{p}_{\theta} = \frac{-p_{\phi}^2 \sin(2\theta)}{2ma^2 \sin^4 \theta} - mga \cos \theta$$

$$\dot{p}_{\phi} = 0$$

QUESTION 6

- (i) Consider the variation directly, $u_t \rightarrow u_t + (\delta u)_t$ and $u_x \rightarrow u_x + (\delta u)_x$.
Then

$$\begin{aligned}
 \delta I[u] &= I[u + \delta u] - I[u] \\
 &= \int \left[\frac{1}{2} (u_t + (\delta u)_t)^2 - F(u_x + (\delta u)_x) \right] dx dt - \int \left[\frac{1}{2} u_t^2 - F(u_x) \right] dx dt \\
 &= \int u_t (\delta u)_t - (\delta u)_x \frac{\partial F(u_x)}{\partial u_x} dx dt + O(t^2) \\
 &= \int u_{tt} (\delta u) - (\delta u) \frac{d}{dx} \left[\frac{\partial F(u_x)}{\partial u_x} \right] dx dt + \text{boundary terms} \quad (\text{by parts})
 \end{aligned}$$

ignoring boundary terms

So the Euler-Lagrange equation is

$$\frac{\partial^2 u}{\partial t^2} - \frac{d}{dx} \left[\frac{\partial F(u_x)}{\partial u_x} \right] = 0$$

$$\text{ie } u_{tt} = u_{xx} \frac{\partial^2 F}{\partial u_x^2}$$

- (ii) Discarding second order terms throughout:

$$\begin{aligned}
 \delta I[u] &= \int \left[(u_x + (\delta u)_x)^2 + (u_y + (\delta u)_y)^2 + e^{2(u+\delta u)} \right] dx dy - \int u_x^2 + u_y^2 + e^{2u} dx dy \\
 &= \int 2u_x (\delta u)_x + 2u_y (\delta u)_y + \left(e^{2(u+\delta u)} - e^{2u} \right) dx dy + O(u^2) \\
 &= \int 2u_x (\delta u)_x + 2u_y (\delta u)_y + e^{2u} (1 + 2\delta u) - e^{2u} dx dy \quad \text{using } e^x \approx 1 + x \\
 &= 2 \int (\delta u) \{ u_{xx} + u_{yy} + 2e^{2u} \} dx dy
 \end{aligned}$$

So E-L equation is

$$\nabla^2 u = -2e^u$$

QUESTION 7

The E-L equations are of the form $\frac{\partial L}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) = 0$

We have

$$\frac{\partial L}{\partial x_i} = -q \nabla_i \phi + q \frac{\partial}{\partial x_i} (\mathbf{v} \cdot \mathbf{A})$$

and

$$\frac{\partial L}{\partial \dot{x}_i} = m \gamma v_i + q A_i$$

E-L equations imply

$$\begin{aligned} -q \nabla_i \phi + q \frac{\partial}{\partial x_i} (\mathbf{v} \cdot \mathbf{A}) - \frac{d}{dt} (m \gamma v_i + q A_i) &= 0 \\ \Rightarrow \frac{d}{dt} (m \gamma v_i) = -q \nabla_i \phi + q \frac{\partial}{\partial x_i} (\mathbf{v} \cdot \mathbf{A}) - q \frac{d}{dt} (A_i) \end{aligned}$$

Using the chain rule:

$$\begin{aligned} \frac{dA_i}{dt} &= \frac{\partial A_i}{\partial t} \frac{dt}{dt} + \frac{\partial A_i}{\partial x_j} \frac{dx_j}{dt} \\ &= \frac{\partial A_i}{\partial t} + v_j \frac{\partial A_i}{\partial x_j} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x} (\mathbf{v} \cdot \mathbf{A}) &= v_j \frac{\partial A_j}{\partial x_i} \\ \Rightarrow \frac{d}{dt} (m \gamma v_i) &= -q \nabla_i \phi + q v_j \frac{\partial A_j}{\partial x_i} - q \left[\frac{\partial A_i}{\partial t} + v_j \frac{\partial A_i}{\partial x_j} \right] \\ &= q \left(\underbrace{\nabla_i \phi - \frac{\partial A_i}{\partial t}}_{E_i} + \underbrace{\left[v_j \frac{\partial A_j}{\partial x_i} - v_j \frac{\partial A_i}{\partial x_j} \right]}_{(*)} \right) \end{aligned}$$

Lastly, want to show (*) is equal to $[\mathbf{v} \times \mathbf{B}]_i = [\mathbf{v} \times \nabla \times \mathbf{A}]_i$

$$\begin{aligned} [\mathbf{v} \times \nabla \times \mathbf{A}]_i &= \varepsilon_{ijk} v_j [\nabla \times \mathbf{A}]_k \\ &= \varepsilon_{kij} \varepsilon_{kpq} v_j \frac{\partial A_q}{\partial x_p} \\ &= \delta_{ip} \delta_{jq} v_j \frac{\partial A_q}{\partial x_p} - \delta_{iq} \delta_{jp} v_j \frac{\partial A_q}{\partial x_p} \\ &= \frac{\partial A_i}{\partial t} + v_j \frac{\partial A_i}{\partial x_j} \end{aligned}$$

as required.

QUESTION 8

$$S[\rho, \mathbf{v}, \phi] = \int dt \int d^3x \left\{ \frac{1}{2} \rho |\mathbf{v}|^2 - u(\rho) + \phi [\dot{\rho} + \nabla \cdot (\rho \mathbf{v})] \right\}$$

Note we need a Lagrange multiplier field $\phi(t, \mathbf{x})$ to impose the condition at an infinite number of points. This looks like an action for sure (KE + PE). We will have 2 E-L equations, one from varying ρ and the other from varying \mathbf{v} .

Taking variations directly,

$$S[\rho + \delta\rho, \mathbf{v}, \phi] = S[\rho, \mathbf{v}, \phi] + \int dt \int d^3x \left\{ \frac{1}{2} \delta\rho |\mathbf{v}|^2 - \delta\rho \frac{\partial u}{\partial \rho} + \phi [\delta\dot{\rho} + \nabla \cdot (\delta\rho \mathbf{v})] \right\}$$

$$=$$

QUESTION 9

QUESTION 10

QUESTION 11