Part IB — Methods Example Sheet 1

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QUESTION 1

Let Ω be the region

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\}$$

We have Laplace's equation $\nabla^2 \phi = 0$ inside Ω, with the Dirichlet boundary conditions $\phi = 1$ on the z surface and $\phi = 0$ on all other surfaces:

Assume $\phi(x, y, z) = X(x)Y(y)Z(z)$, so we have

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Moving along the x direction for fixed y, z, $\frac{X''}{X} = -\lambda$ for some constant λ. Solving this with boundary conditions $X(0) = X(a) = 0$ yields

$$\lambda_p = \frac{p^2\pi^2}{a^2}, X_p = \sin\left(\frac{p\pi x}{a}\right), p = 1, 2, 3, \cdots$$

Similarly, solving $Y'' = -\mu Y$ with $Y(0) = Y(b) = 0$ implies that

$$\mu_q = \frac{q^2\pi^2}{b^2}, Y_q = \sin\left(\frac{q\pi x}{b}\right), q = 1, 2, 3, \cdots$$

Lastly, we now have $Z'' = (\lambda + \mu)Z$, which has solutions of the from (using the hint)

$$Z = A\cosh\left[\sqrt{\lambda + \mu}(l - c)z\right] + B\sinh\left[\sqrt{\lambda + \mu}(l - c)z\right]$$

The boundary condition $Z(c) = 0$ gives $A = 0$. (Note that we cannot use the boundary condition on the $z = 0$ surface until we have the general solution).

Hence, we have a family of solutions given by

$$\psi_{p,q}(x, y, z) := A_{p,q}\sin\left(\sqrt{\lambda_p}x\right)\sin\left(\sqrt{\mu_q}y\right)\sinh\left[\sqrt{\lambda + \mu}(c - z)\right]$$

for some constants $A_{p,q}$ to be determined. By linearity, the general solution is

$$\psi(x, y, z) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{p,q}\sin\left(\sqrt{\lambda_p}x\right)\sin\left(\sqrt{\mu_q}y\right)\sinh\left[\sqrt{\lambda + \mu}(c - z)\right]$$

Now using $\phi = 1$ on the surface $z = 0$, we require

$$1 = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{p,q}\sin\left(\sqrt{\lambda_p}x\right)\sin\left(\sqrt{\mu_q}y\right)\sinh\left[\sqrt{\lambda + \mu}c\right]$$

Now using orthogonality relations

$$\int_0^a \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi x}{a}\right)\,\mathrm{d}x = \frac{a}{2}\delta_{p,q}$$

we deduce

$$\begin{aligned}
A_{p,q} &= \frac{4}{ab\sinh\left[\sqrt{\lambda+\mu}c\right]}\int_0^a\int_0^b \sin\left(\frac{q\pi x}{a}\right)\sin\left(\frac{p\pi y}{b}\right)\,\mathrm{d}x\mathrm{d}y \\
&= \frac{4}{ab\sinh\left[\sqrt{\lambda+\mu}c\right]}\left[-\frac{a}{q\pi}\cos\left(\frac{q\pi x}{a}\right)\right]_0^a\left[-\frac{b}{p\pi}\cos\left(\frac{p\pi x}{b}\right)\right]_0^b \\
&= \frac{4}{ab\sinh\left[\sqrt{\lambda+\mu}c\right]}\frac{ab}{\pi^2 pq}\left((-1)^q-1\right)\left((-1)^p-1\right) \\
&= \begin{cases} \frac{16}{\pi^2 pq\sinh\left[\sqrt{\lambda+\mu}c\right]} & \text{if } q \text{ and } p \text{ are both odd} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Noting that now $\lambda_p + \mu_q = (2p+1)^2\pi^2/a^2 + (2q+1)^2\pi^2/b^2 = l^2$
Therefore, the solution satisfying these boundary conditions is

$$\psi(x,y,z) = \frac{16}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\sinh\left[l(c-z)\right]\sin\left((2p+1)\pi x/a\right)\sin\left((2q+1)\pi y/b\right)}{(2p+1)(2q+1)\sinh cl}$$

as required.
As $c \to \infty$, note that

$$\begin{aligned}
\frac{\sinh(L(c-z))}{\sinh(Lc)} &= \frac{\exp[L(c-z)] - \exp[-L(c-z)]}{\exp(Lc)-\exp(-Lc)} \\
&\to \frac{\exp[L(c-z)]}{\exp(Lc)} \\
&\to \exp(-Lz)
\end{aligned}$$

QUESTION 2

The potential satisfies

$$\nabla^2\phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

with Dirichlet boundary conditions

$$\phi(r=1,\theta) = \begin{cases} \pi/2 & \text{if } 0 \le \theta < \pi \\ -\pi/2 & \text{if } \pi \le \theta < 2\pi \end{cases}$$

Separating variables by writing $\phi(r,\theta) = R(r)\Theta(\theta)$,

$$\frac{1}{r}\frac{\partial}{\partial r}(rR'\Theta) + \frac{1}{r^2}R\Theta'' = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}(rR') + \frac{\Theta''}{\Theta} = 0$$

Keeping r fixed and varying θ, we see that $\frac{\Theta''}{\Theta} = -\lambda$ constant. Solving $\Theta'' = -\Theta X$, if ϕ single valued, must have $\Theta(\theta+2\pi) = \Theta(\theta)$, thus $\lambda = n^2$ for some integer n, and

$$\Theta_n(\theta) = a_n\cos n\theta + b_n\sin n\theta$$

Next we solve

$$r\frac{\mathrm{d}}{\mathrm{d}r}(rR_n') - n^2R_n = 0$$

For $n \neq 0$, assuming that $R_n \propto r^\beta$, we have

$$\beta^2 - n^2 = 0 \Rightarrow \beta = \pm n$$

Thus

$$R_n(r) = c_nr^n + d_nr^{-n}, \quad n = 1,2,3,\cdots$$

Therefore, the family of particular solutions is

$$\phi_n(r,\theta) = (a_n\cos n\theta + b_n\sin n\theta)(c_nr^n + d_nr^{-n}) \quad n = 1,2,3,\cdots$$

For $n=0$, we solve $r\frac{\mathrm{d}}{\mathrm{d}r}(rR_0') = 0$, thus $rR_0' =$ constant, and we have $R_0 = d_0\log r + c_0$

Hence by linearity, the general solution for Laplace's equation in polar coordinates is

$$\phi(r,\theta) = c_0 + d_0\log r + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)(c_nr^n + d_nr^{-n})$$

Now, requiring regularity at the origin, $d_0 = 0, d_n = 0$, then absorb c_n as a general rescaling, we can write this solution as

$$\phi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) r^n$$

Using boundary conditions,

$$f(\theta) = \phi(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

The function is odd, hence

$$f(\theta) = \phi(1,\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

By orthogonality of sines,

$$\begin{aligned}
b_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(m\theta)\,\mathrm{d}\theta \\
&= \frac{1}{\pi}\left(\int_0^{\pi} \frac{\pi}{2}\sin m\theta\,\mathrm{d}\theta + \int_{\pi}^{2\pi} -\frac{\pi}{2}\sin m\theta\,\mathrm{d}\theta\right) \\
&= \frac{1}{2m}\left([-\cos m\theta]_0^{\pi} + [\cos m\theta]_{\pi}^{2\pi}\right) \\
&= \frac{1}{m}(-1-(-1)^m) \\
&= \begin{cases} \frac{2}{m} & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even} \end{cases}
\end{aligned}$$

This gives $b_m = 2/m$, hence

$$\phi(r,\theta) = 2\sum_{n \text{ odd}} \frac{r^n \sin n\theta}{n}$$

Next, under the substitution $z = re^{i\theta}$, we have $z^n = r^n \cos n\theta + i \sin n\theta$

$$\begin{aligned}
\phi(r,\theta) &= 2\sum_{n \text{ odd}} \frac{r^n \sin n\theta}{n} \\
&= 2\,\mathrm{Im}\left(\sum_{n \text{ odd}} \frac{z^n}{n}\right) \\
&= 2\,\mathrm{Im}\left(\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}\right) \\
&= 2\,\mathrm{Im}\left(\sum_{n=0}^{\infty} \int_0^z t^{2n}\,\mathrm{d}t\right)
\end{aligned}$$

and, assuming we can swap the order of summation and integration, we have

$$
\begin{aligned}
&= 2\,\mathrm{Im}\left(\int_0^z \sum_{n=0}^{\infty} t^{2n}\ \mathrm{d}t\right) \\
&= 2\,\mathrm{Im}\int_0^z \frac{1}{1-t^2}\ \mathrm{d}t \\
&= \mathrm{Im}\int_0^z \left(\frac{1}{1+t}+\frac{1}{1-t}\right)\ \mathrm{d}t \\
&= \mathrm{Im}\left[\log(1+t)-\log(1-t)\right]_0^z \\
&= \mathrm{Im}\left[\log(1+z)-\log(1-z)\right] \\
&= \arg(1+z)-\arg(1-z)
\end{aligned}
$$

which is some angle, see $w^2 = \frac{1+z}{1-z}$, V and M sheet 1

QUESTION 3

The potential satisfies

$$\nabla^2\psi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$

and $\psi(r,\theta)$ satisfies the Dirichlet boundary conditions (hence solution existence and uniqueness)

$$\psi(r,\theta) = \begin{cases} V & \text{if } 0 \le \theta < \frac{\pi}{2} \\ -V & \text{if } \frac{\pi}{2} < \theta \le \pi \end{cases}$$

Separating variables by writing $\phi(r,\theta) = R(r)\Theta(\theta)$, we have the two ODEs

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \lambda\sin\theta\,\Theta = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \lambda R = 0$$

with $\lambda \in \mathbb{R}$ separation constant.

Making the substitution $x = \cos\theta$ in the angular equation yields $\frac{\mathrm{d}}{\mathrm{d}\theta} = -\sin\theta\frac{\mathrm{d}}{\mathrm{d}x}$, and

$$-\sin\theta\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin\theta\left(-\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)\right] + \lambda\sin\theta\,\Theta = 0$$

which becomes Legendre's equation:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\Theta\right] = \lambda\Theta$$

substituting $\Theta = \sum_{n=0}^{\infty} a_n x^n$ (only non-negative powers as we want solution to be regular at origin) yields

$$(1-x^2)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} a_n n x^n + \lambda\sum_{n=0}^{\infty} x^n = 0$$

which must hold for each power separately, thus we obtain the recursion relation:

$$0 = a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2a_n n + \lambda a_n$$

yielding the recursion relation

$$a_{n+2} = \left[\frac{n(n+1)-\lambda}{(n+1)(n+2)}\right]a_n$$

Thus we find two linearly independent (even and odd) solutions

$$\Theta_e = a_0\left[1 + \frac{(-\lambda)x^2}{2!}\cdots\right]$$
$$\Theta_o = a_1\left[x + \frac{(2-\lambda)x^3}{3!} + \cdots\right]$$

Solution must remain bounded at $x = \pm 1$, so $\lambda = m(m+1)$ for some integer m. These $\Theta_n(\theta) = P_n(x) = P_n(\cos\theta)$, with $\lambda = n(n+1)$, are Legendre polynomials of order n, with the orthogonal property $\int_{-1}^{1} P_m(z)P_n(z)\,\mathrm{d}z = \frac{2}{2n+1}\delta_{mn}$. Then the DE for R becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R_n}{\mathrm{d}r}\right) - n(n+1)R_n = 0$$

Assuming that $R_n \propto r^{\beta}$,

$$\begin{aligned}\beta(\beta+1) &= n(n+1)\\ \Rightarrow \beta &= n \text{ or } -(n+1)\end{aligned}$$

Thus our general solution takes the from

$$\psi(r,\theta) = \sum_{n=0}^{\infty}(a_n r^n + b_n r^{-(n+1)})P_n(\cos\theta)$$

for solution to be regular at origin must have $b_n = 0$, so

$$\psi(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)$$

We can apply our boundary condition easiest by setting $r = 1$, then

$$\begin{aligned}
f(\theta) := \psi(r=1,\theta) &= \sum_{n=0}^{\infty} a_n P_n\cos(\theta), \quad 0 \le \theta \le \pi\\
F(x) &:= \sum_{n=0}^{\infty} a_n P_n(x), \qquad x = \cos\theta, -1 \le x \le 1\\
\text{where } F(x) &= \begin{cases} V & \text{if } 0 \le x < 1\\ -V & \text{if } -1 \le x < 0\end{cases}\\
\Rightarrow a_n &= \frac{(2n+1)}{2}\int_{-1}^{1} F(x)P_n(x)\,\mathrm{d}x\\
&= \frac{(2n+1)}{2}\int_{0}^{1} VP_n(x)\,\mathrm{d}x + \frac{(2n+1)}{2}\int_{-1}^{0} -VP_n(x)\,\mathrm{d}x\\
&= \frac{V}{2}\int_0^1 P'_{n+1}(x) - P'_{n-1}(x)\,\mathrm{d}x - \frac{V}{2}\int_{-1}^{0} P'_{n+1}(x) - P'_{n-1}(x)\,\mathrm{d}x
\end{aligned}$$

Note that for n even, the integrals cancel out and we have $a_n = 0$. Otherwise,

$$\begin{aligned}
a_n &= V\int_0^1 P'_{n+1}(x) - P'_{n-1}(x)\,\mathrm{d}x \qquad \text{as } n \text{ odd}\\
&= V[P_{n+1}(z) - P_{n-1}(z)]_0^1\\
&= V(P_{n-1}(0) - P_{n+1}(0)) \quad \text{as } P_n(1) = 1\ \forall\, n
\end{aligned}$$

Hence the potential inside the region is given by

$$\psi(r,\theta) = V\sum_{n=0}^{\infty} r^n (P_{n-1}(0) - P_{n+1}(0))P_n(\cos\theta)$$

QUESTION 4

y_m is an eigenfunction, hence satisfies the Sturm-Liouville equation, so we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(py_m'\right) = -(\lambda_m - q)y_m$$

Starting with the left hand side

$$\int_a^b (py_m'y_n' + qy_my_n)\ \mathrm{d}x = \int_a^b py_m'y_n'\ \mathrm{d}x + \int_a^b qy_my_n\ \mathrm{d}x$$

and integrating by parts

$$\int_a^b py_m'y_n'\ \mathrm{d}x = \left[py_m'y_n\right]_a^b - \int_a^b -(\lambda_m - q)y_my_n\ \mathrm{d}x$$

Choosing suitable boundary conditions such that

$$\left[py_m'y_n\right]_a^b = 0$$

we have

$$\begin{aligned}\int_a^b (py_m'y_n' + qy_my_n)\ \mathrm{d}x &= \int_a^b (\lambda_m - q)y_my_n\ \mathrm{d}x + \int_a^b qy_my_n\ \mathrm{d}x \\ &= \int_a^b \lambda_m y_my_n\ \mathrm{d}x \\ &= \lambda_m\delta_{mn} \quad \text{by S-L orthogonality}\end{aligned}$$

Last part? Get solution of Zimaras. Note that

$$y_n = P_n \not\Longrightarrow \int y_n^2\ \mathrm{d}x = 1$$

QUESTION 5

(a)

(i)

$$q_n = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\underbrace{\left(x^{2n} - \binom{n}{2}x^{2n-2} + \cdots\right)}_{(*)}$$

Differentiating (*) n times produces a polynomial with highest power $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^{2n}) = x^n$. Hence $q_n(x)$ is of degree n.

(ii) By induction:

$$\begin{aligned} q_1(1) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\Big|_{x=1} \\ &= \frac{1}{2}2x\Big|_{x=1} \\ &= 1 \end{aligned}$$

True for $n = 1$. Now suppose $q_k(1) = 1$ for some $k > 0$.

$$\begin{aligned} q_{k+1}(x) &= \frac{1}{2^{k+1}(k+1)!}\frac{\mathrm{d}^n}{\mathrm{d}x^{k+1}}(x^2-1)^{k+1} \\ &= \frac{1}{2(k+1)}\left[\frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(2(k+1)x(x^2-1)^k\right)\right] \\ &= \frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x(x^2-1)^k\right) \end{aligned}$$

(b)

(i) P_n are polynomial solutions to Legendre's equation with $\lambda_n = n(n+1)$, and q_n are polynomial solutions to Legendre's equation with $\lambda_n = n(n+1)$, we have $P_n \propto q_n$. But $P(1) = q(1) = 1$, so $P_n = q_n$ (are there non poly solutions to worry about? See Skinners notes)

(ii) Hence we see that

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^2-1)^n$$

Using this result, by parts,

$$\begin{aligned} \int_{-1}^{1}[P_n(x)]^2\,\mathrm{d}x &= \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n(x^2-1)^n\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n(x^2-1)^n\,\mathrm{d}x \\ &= \frac{1}{2^{2n}(n!)^2}\left(\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}(x^2-1)^n\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n(x^2-1)^n\right]_{-1}^{1}\right) \\ &\quad - \frac{1}{2^{2n}(n!)^2}\left(\int_{-1}^{1}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}(x^2-1)^n\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n+1}(x^2-1)^n\,\mathrm{d}x\right) \end{aligned}$$

But we know that the boundary term vanishes since

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m (x^2-1)\big|_{x=\pm 1} = 0 \text{ for } m < n$$

Thus, we integrate by parts iteratively, until

$$\begin{aligned}\int_{-1}^{1} [P_n(x)]^2 \, \mathrm{d}x = \cdots \\ &= \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2n} (x^2-1)^n \, \mathrm{d}x \\ &= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \, \mathrm{d}x\end{aligned}$$

Finally,

$$\begin{aligned}I_n &= \int_{-1}^{1} (x^2-1)^n \, \mathrm{d}x \qquad \text{(by parts with 1)} \\ &= \left[x(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} 2x^2 n (x^2-1)^{n-1} \, \mathrm{d}x \\ &= -2n \int_{-1}^{1} (x^2-1)(x^2-1)^{n-1} + (x^2-1)^{n-1} \, \mathrm{d}x \\ &= -2n(I_n + I_{n-1})\end{aligned}$$

$$\Rightarrow I_n = \frac{2}{(2n+1)!!}$$

Arrive at (?) result.

QUESTION 6

$y(x,t)$ satisfies the 1D wave equation

$$\frac{\partial^2}{\partial t^2}y(x,t) = c^2\frac{\partial^2}{\partial x^2}y(x,t) \qquad c^2 = \frac{T}{\mu}$$

Assume $y(x,t) = X(x)T(t)$ and separate variables:

$$\begin{aligned} X\ddot{T} &= c^2 X''T \\ \Rightarrow \frac{1}{c^2}\frac{\ddot{T}}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

for some $\lambda > 0$, so we have

$$\begin{aligned} X'' + \lambda X &= 0 \\ \ddot{T} + \lambda c^2 T &= 0 \end{aligned}$$

Solving the spatial equation first,

$$X = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$$

Applying initial conditions

- $X(0) = 0 \Rightarrow \alpha = 0$
- $X(l) = 0 \Rightarrow \beta\sin(\sqrt{\lambda}x) = 0 \Rightarrow \lambda = n^2\pi^2/l^2$, for integer n

The normal modes are the associated eigenfunctions given by

$$X_n(x) = \beta_n \sin\left(\frac{n\pi x}{l}\right)$$

The associated $T_n(t)$ is given by

$$\begin{aligned} \ddot{T}_n + \frac{n^2\pi^2c^2}{l^2}T_n &= 0 \\ \Rightarrow T_n(t) &= \gamma_n\cos\left(\frac{n\pi ct}{l}\right) + \delta_n\sin\left(\frac{n\pi ct}{l}\right) \end{aligned}$$

Hence the general solution is

$$y(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\left(A_n\cos\left(\frac{n\pi ct}{l}\right) + B_n\sin\left(\frac{n\pi ct}{l}\right)\right)$$

Using BCs, $y(x,0) = 0 \Rightarrow A_n = 0\ \forall\ n$.
Next, have $y_t(x,0) = \frac{4V}{l^2}x(l-x)$, so

$$\frac{4V}{l^2}x(l-x) = \sum_{n=1}^{\infty}B_n\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$\begin{aligned}
\Rightarrow B_n &= \frac{4V}{l^2}\left(\frac{l}{n\pi c}\right)\frac{2}{l}\int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x \\
&= \frac{8V}{n\pi c l^2}\int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right)\,\mathrm{d}x \\
&= \cdots \\
&= \frac{16lV^2}{c\pi^4 n^4}\left(1-(-1)^n\right)
\end{aligned}$$

(Think should be V, and not V^2, do check later.)
Hence

$$y(x,t) = \frac{32lV^2}{c\pi^4}\sum_{n\text{ odd}}\frac{1}{n^4}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

Next, kinetic energy is given by

$$K = \int_0^L \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2\,\mathrm{d}x$$

Have

$$y_t(x,t) = \frac{32V^2}{\pi^3}\sum_{n\text{ odd}}\frac{1}{n^3}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

Thus

$$\begin{aligned}
K &= \frac{512V^4}{\pi^6}\mu\int_0^l\sum_{n,m\text{ odd}}\frac{1}{n^3m^3}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)\,\mathrm{d}x \\
&= \frac{512V^4}{\pi^6}\mu\sum_{n,m\text{ odd}}\frac{1}{n^3m^3}\cos\left(\frac{n\pi ct}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)\underbrace{\int_0^l\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\,\mathrm{d}x}_{=\frac{l}{2}\delta_{mn}} \\
&= \frac{256V^4l}{\pi^6}\mu\sum_{n\text{ odd}}\frac{1}{n^6}\cos^2\left(\frac{n\pi ct}{l}\right)
\end{aligned}$$

Similarly, PE given by

$$V = \int_0^l\frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2\,\mathrm{d}x$$

Have

$$y_x(x,t) = \frac{32V^2}{c\pi^3}\sum_{n\text{ odd}}\frac{1}{n^3}\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

Thus

$$
\begin{aligned}
V &= \frac{512V^4}{c^2\pi^6}T\int_0^l \sum_{n,m \text{ odd}} \frac{1}{n^3m^3}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{m\pi ct}{l}\right)\,\mathrm{d}x \\
&= \frac{512V^4}{c^2\pi^6}T\sum_{n,m \text{ odd}} \frac{1}{n^3m^3}\sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{m\pi ct}{l}\right)\int_0^l \underbrace{\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)\,\mathrm{d}x}_{\frac{l}{2}\delta_{mn}} \\
&= \frac{256V^4 l}{c^2\pi^6}T\sum_{n \text{ odd}} \frac{1}{n^6}\sin^2\left(\frac{n\pi ct}{l}\right)
\end{aligned}
$$

Compare this with the initial energy

$$
\begin{aligned}
K &= \int_0^l \frac{1}{2}\mu\left(\left.\frac{\partial y}{\partial t}\right|_{t=0}\right)^2\,\mathrm{d}x \\
&= \int_0^l \frac{1}{2}\mu\left(\frac{4V}{l^2}x(l-x)\right)^2\,\mathrm{d}x \\
&= \frac{8V^2\mu}{l^4}\int_0^l x^2(l-x)^2\,\mathrm{d}x \\
&= \frac{8V^2\mu}{l^4}\left(\frac{l^5}{30}\right) \\
&= \frac{4V^2\mu l}{15}
\end{aligned}
$$

Setting $t = 0$ in our previous result gives

$$
K = \frac{256V^4 l}{\pi^6}\mu\sum_{n \text{ odd}} \frac{1}{n^6}
$$

Thus by comparison

$$
\begin{aligned}
\frac{256}{\pi^6}\sum_{n \text{ odd}} \frac{1}{n^6} &= \frac{4}{15} \\
\Rightarrow \sum_{n \text{ odd}} \frac{1}{n^6} &= \frac{\pi^6}{960}
\end{aligned}
$$

QUESTION 7

(i)
- Assume all displacements are sufficiently small ($y \ll l$)
- Assume all displacements are vertical
- Consider two points x and $x + \delta x$. The angle of the string to the horizontal at x is θ_1, and the angle at $x + \delta x$ is θ_2.
- Resolving horizontally shows that $T(x)\cos\theta_1 = T(x+\delta x)\cos\theta_2$, since $\theta \ll 1$, the tension is approximately constant.
- Resolving vertically

$$T\sin\theta_2 - T\sin\theta_1 - \mu g\delta x - 2k\mu\delta x\frac{\partial}{\partial t}y = \mu\delta x\frac{\partial^2}{\partial t^2}y \qquad (*)$$

- Assume angles are small

$$\sin\theta_2 \approx tan\theta_2 = \frac{\partial y}{\partial x}\Big|_{x+\delta x} \approx \frac{\partial y}{\partial x}\Big|_x + \delta x\frac{\partial^2 y}{\partial x^2}\Big|_x$$

$$\sin\theta_1 \approx \tan\theta_1 = \frac{\partial y}{\partial x}\Big|_x$$

- (*) becomes

$$\begin{aligned} T\delta x\frac{\partial^2 y}{\partial x^2} - \mu g\delta x - 2k\mu\delta x\frac{\partial}{\partial t}y &= \mu\delta x\frac{\partial^2}{\partial t^2}y \\ \Rightarrow \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2}{\partial t^2}y + 2k\frac{\partial}{\partial t}y + g \end{aligned}$$

- Further assume the weight is insignificant ($g \to 0$)
- Hence arrive at the equation of motion

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial t^2}y + 2k\frac{\partial}{\partial t}y$$

where $c^2 = \frac{T}{\mu}$

Assume $y(x,t) = X(x)T(t)$ and separating variables gives

$$\begin{aligned} c^2\frac{X''}{X} &= \frac{\ddot{T}}{T} + 2k\frac{\dot{T}}{T} \\ \Rightarrow \frac{X''}{X} &= \frac{\ddot{T} + 2k\dot{T}}{T} = -\lambda \end{aligned}$$

for some $\lambda > 0$, so we have

$$X'' + \lambda X = 0$$
$$\ddot{T} + 2k\dot{T} + \lambda c^2 T = 0$$

Solving the spatial equation first,

$$X = \alpha \cos(\sqrt{\lambda} x) + \beta \sin(\sqrt{\lambda} x)$$

Applying initial conditions

- $X(0) = 0 \Rightarrow \alpha = 0$
- $X(l) = 0 \Rightarrow \beta \sin(\sqrt{\lambda} x) = 0 \Rightarrow \lambda = n^2\pi^2/l^2$, for integer n

These λ are eigenvalues, with associated eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

The associated $T_n(t)$ is given by

$$\begin{aligned} &\ddot{T}_n + 2k\dot{T}_n + \frac{n^2\pi^2c^2}{L^2}T_n = 0 \qquad k = \frac{\pi c}{l} \\ \Rightarrow\ &T_n(t) = e^{-kt}\left(\gamma_n \cos\left(\sqrt{n^2-1}kt\right) + \delta_n \sin\left(\sqrt{n^2-1}kt\right)\right) \qquad (n \geq 2) \end{aligned}$$

Being careful with the $n = 1$ case, must have

$$T_1(t) = e^{-kt}\left(\gamma_1 + \delta_1 t\right)$$

Hence the general solution is

$$\begin{aligned} y(x,t) =& e^{-kt}\sin\left(\frac{\pi x}{l}\right)(A_1 + B_1 t) \\ &+ \sum_{n=2}^{\infty} e^{-kt}\sin\left(\frac{n\pi x}{l}\right)\left(A_n \cos\left(\sqrt{n^2-1}kt\right) + B_n \sin\left(\sqrt{n^2-1}kt\right)\right) \end{aligned}$$

Using the boundary condition $y(x,0) = A\sin(\pi x/l)$, we have

$$A\sin(\pi x/l) = A_1 \sin\left(\frac{\pi x}{l}\right) + \sum_{n=2}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

Ah. So we conclude that $A_1 = A$, and $A_n = 0$ for $n \geq 2$, thus

$$y(x,t) = (A + B_1 t)e^{-kt}\sin\left(\frac{\pi x}{l}\right) + \sum_{n=2}^{\infty} e^{-kt}\sin\left(\frac{n\pi x}{l}\right) B_n \sin\left(\sqrt{n^2-1}kt\right)$$

Next, use the boundary condition $y_t(x,0) = 0$. We have that

$$\begin{aligned} y_t(x,t) &= (-kA + B_1 - kB_1 t)e^{-kt} \sin\left(\frac{\pi x}{l}\right) \\ &\quad + \sum_{n=2}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left[-kte^{-kt} B_n \sin\left(\sqrt{n^2-1}kt\right) + (\sqrt{n^2-1}k)e^{-kt} B_n \cos\left(\sqrt{n^2-1}kt\right)\right] \end{aligned}$$

Thus

$$0 = y_t(x,0) = (-kA + B_1)\sin\left(\frac{\pi x}{l}\right) + \sum_{n=2}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)(\sqrt{n^2-1}k)$$

Thus we conclude that $B_1 = kA$ and $B_n = 0$ for $n \geq 2$.
Thus the general solution is given by

$$y(x,t) = A(1+kt)e^{-kt}\sin\left(\frac{\pi x}{l}\right)$$

(ii)

QUESTION 8

(i)
- Assume all displacements are sufficiently small ($y \ll l$)
- Assume all displacements are vertical
- With mass M at $x = 0$, consider the tension of the string acting on the two points $-\varepsilon$ and ε either side, with the angle the string makes with the horizontal denoted by $\theta_{-\varepsilon}$ and $\theta_{+\varepsilon}$
- Resolving horizontally shows that the tension is constant.
- Using Newton's Second Law vertically, we have that

$$M \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\Big|_{x=0} = T \sin\theta_\varepsilon - T \sin\theta_{-\varepsilon}$$

- Assume angles are small

$$\sin\theta_\varepsilon \approx \tan\theta_\varepsilon = \frac{\partial y}{\partial x}\Big|_{x=\varepsilon}$$

- Hence

$$\begin{aligned} M \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\Big|_{x=0} &= T \frac{\partial y}{\partial x}\Big|_{x=\varepsilon} - T \frac{\partial y}{\partial x}\Big|_{x=-\varepsilon} \\ &= T \left[\frac{\partial y}{\partial x}\right]_{x=0_-}^{x=0_+} \qquad \text{as } \varepsilon \to 0 \end{aligned}$$

(ii) The incident wave $W_I = \exp(i\omega(t - x/c))$ will produce a transmitted wave $W_T = T\exp(i\omega(t - x/c))$ and a reflected wave $W_R = R\exp(i\omega(t + x/c))$. Know that

$$y(x,t) = \begin{cases} W_I + W_R & \text{if } x < 0 \\ W_T & \text{if } x > 0 \end{cases}$$

with boundary conditions continuity at zero ($[y(0,t)]_{x=0_-}^{x=0_+} = 0$) and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\Big|_{x=0} = \frac{T}{M}\left[\frac{\partial y}{\partial x}\right]_{x=0_-}^{x=0_+}$$

QUESTION 9

$y(x,t)$ define on $0 \leq x \leq l$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with $y(0,t) = y(l,t) = 0$ (fixed at endpoints). We can find the solution $y(x,t)$ for $t < 0$, given $y(x,0) = 0$, and

$$\left[\frac{\partial y}{\partial t}\right]_{t=0_-}^{t=0_+} = \lambda \delta \left(x - \frac{l}{2}\right)$$

QUESTION 10

Thus, the solution is