Part IB — Statistics Example Sheet 2

Supervised by Mr. Higson

Examples worked through by Christopher Turnbull

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QUESTION 1

Given $f(x;\theta)$, we calculate the likelihood ratio as

$$\begin{aligned}\Lambda_{\mathbf{x}}(H_0, H_1) &= \frac{L_{\mathbf{x}}(H_1)}{L_{\mathbf{x}}(H_0)} \\ &= \frac{2/(x+2)^2}{1/(x+1)^2} \\ &= 2\left(\frac{x+1}{x+2}\right)^2\end{aligned}$$

For $x > 0$ this is increasing as a function of x, so for any k, $\Lambda_x > k \iff x > c$, for some c.

Hence we reject H_0 if $x > c$, where c is chosen such that $\mathbb{P}(X > c \mid H_0) = \alpha = 0.05$.

Under $H_0, f_X(x|\theta) = \frac{1}{(x+1)^2},\ x > 0$, so

$$\begin{aligned}\mathbb{P}(X > c \mid H_0) &= \int_c^\infty \frac{1}{(x+1)^2}\,\mathrm{d}x \\ &= \frac{1}{c+1}\end{aligned}$$

So for the size 0.05 test, this gives $c = 19$, hence the test rejects H_0 if $x > 19$. Then

$$\begin{aligned}\mathbb{P}(\text{Type II error}) &= \mathbb{P}(X \notin C \mid H_1) \\ &= \int_0^{19} \frac{2}{(x+2)^2}\,\mathrm{d}x \\ &= \left[-2(x+2)^{-1}\right]_0^{19} \\ &= \frac{19}{21}\end{aligned}$$

QUESTION 2

We wish to test $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 \neq \theta_2$. Then

$$\begin{aligned}\Lambda_{\mathbf{x}}(H_0; H_1) &= \frac{L_{\mathbf{x}}(H_1)}{L_{\mathbf{x}}(H_0)} \\ &= \frac{\sup_{\theta_x,\theta_y} \theta_x^n e^{-\theta_x \sum x_i} \theta_y^n e^{-\theta_y \sum y_i}}{\sup_\theta \theta^n e^{-\theta \sum x_i} \theta^n e^{-\theta \sum y_i}}\end{aligned}$$

Under H_1 the MLEs of θ_x and θ_y are $\theta_x = \frac{n}{\sum x_i}$ and $\theta_y = \frac{n}{\sum y_i}$. Under H_0 the MLE of θ is $\hat{\theta} = \frac{2n}{\sum x_i + \sum y_i}$. So

$$\begin{aligned}\Lambda_{\mathbf{x}}(H_0; H_1) &= \frac{L_{\mathbf{x}}(H_1)}{L_{\mathbf{x}}(H_0)} \\ &= \frac{\theta_x^n \theta_y^n e^{-2n}}{\hat{\theta}^{2n} e^{-2n}} \\ &= \left(\frac{n^2}{\sum x_i \sum y_i}\right)^n \left(\frac{\sum x_i + \sum y_i}{2n}\right)^{2n} \\ &= 2^{-2n} \left(\frac{\left(\sum x_i + \sum y_i\right)^2}{\sum x_i \sum y_i}\right)^n\end{aligned}$$

Given

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}$$

We see that

$$\begin{aligned}\Lambda_{\mathbf{x}}(H_0; H_1) &= 2^{-2n} \left(T(1-T)\right)^{-n} \\ &= 2^{-2n} \left(-(T-1/2)^2 + 1/4\right)^{-n}\end{aligned}$$

this is an increasing function of $\left|T - \frac{1}{2}\right|$, so for any k, $\Lambda_x > k \iff \left|T - \frac{1}{2}\right| > c$. for some c

We know that under H_0, $\sum X_i \sim \Gamma(n, \theta)$, $\sum Y_i \sim \Gamma(n, \theta)$ so $X/(X+Y) \sim \mathrm{Beta}(n, n)$ (cf. Example Sheet 1, Question 2,) ie.

$$T \sim \mathrm{Beta}(n, n)$$

So the size α generalised likelihood test rejects H_0 if

$$\left|T - \frac{1}{2}\right| > \beta_{\alpha/2} - 1/2$$

Question: Does $2 \log \Lambda_x(H_0; H_1)$ obviously have a χ_1^2 distribution under H_0 here?

QUESTION 3

The probabilities for a bunch have i defective articles, $i = 0, 1, 2, 3$ are $(1-\theta)^3, 3\theta(1-\theta)^2, 3\theta^2(1-\theta)$ and θ^3 respectively. We wish to test $H_0 : p_i = p_i(\theta)$.

We observe $N_i = n_i, N = (213, 228, 57, 14)$. Under H_0, the mle $\hat{\theta}$ is found by maximizing

$$\sum n_i \log p_i(\theta) = 3n_1 \log(1-\theta) + 2n_2 \log(3\theta(1-\theta)) + 2n_3 \log(3(1-\theta)\theta) + 3n_4 \log \theta$$

Differentiating the RHS gives

$$\begin{aligned} 0 &= \frac{-3n_1}{1-\theta} + \frac{(2n_2 + 2n_3)(3(1-\theta) - 3\theta)}{3\theta(1-\theta)} + \frac{3n_4}{\theta} \\ &= \frac{-9n_1\theta}{3\theta(1-\theta)} + \frac{(2n_2 + 2n_3)(3(1-\theta) - 3\theta)}{3\theta(1-\theta)} + \frac{9n_4(1-\theta)}{3\theta(1-\theta)} \end{aligned}$$

which gives $\hat{\theta} = \frac{2n_2+2n_3+3n_4}{3n_1+4n_2+4n_3+3n_4}$.
Pearson's chi-squared statistic is

$$\begin{aligned} T &:= \sum_{j=1}^{4} \frac{(o_j - e_j)^2}{e_j} \\ &= \frac{(213 - 512(1-\theta)^3)^2}{512(1-\theta)^3} + \frac{(228 - 15363\theta(1-\theta)^2)^2}{1536\theta(1-\theta)^2} \\ &\quad + \frac{(57 - 1536\theta^2(1-\theta))^2}{1536\theta^2(1-\theta)} + \frac{(14 - 512\theta^3)^2}{512\theta^3} \end{aligned}$$

Also, $|\Theta_0| = 1$ and $|\Theta_1| = 3$, so we refer to χ^2_2.

QUESTION 4

Lemma (Neyman-Pearson lemma for discrete distributions)**.** Suppose $H_0 : f = f_0$, $H_1 : f = f_1$, where f_0 and f_1 are probability mass functions on a countable set $\mathcal{X}$. Then among all tests of size less than or equal to α, the test with the largest power is the likelihood ratio test of size α.

Proof. Under the likelihood ratio test, our critical region is

$$C = \left\{\mathbf{x} : \frac{f_1(\mathbf{x})}{f_0(\mathbf{x})} > k\right\},$$

where k is chosen such that $\alpha = \mathbb{P}(\text{reject } H_0 \mid H_0) = \mathbb{P}(\mathbf{X} \in C \mid H_0) = \sum_{\mathbf{x}_i \in C} f_0(\mathbf{x}_i)$. The probability of Type II error is given by

$$\beta = \mathbb{P}(\mathbf{X} \notin C \mid f_1) = \sum_{\mathbf{x}_i \in \bar{C}} f_1(\mathbf{x}_i).$$

Let C^* be the critical region of any other test with size less than or equal to α. Let $\alpha^* = \mathbb{P}(X \in C^* \mid f_0)$ and $\beta^* = \mathbb{P}(\mathbf{X} \notin C^* \mid f_1)$. We want to show $\beta \leq \beta^*$.

We know $\alpha^* \leq \alpha$, ie

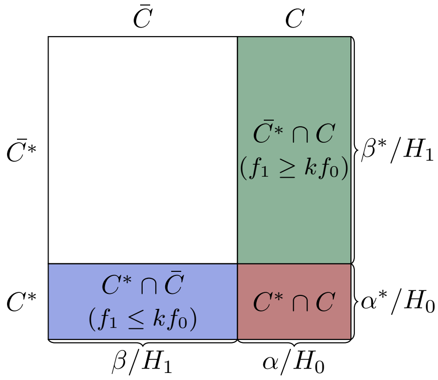
$$\sum_{\mathbf{x}_i \in C^*} f_0(\mathbf{x}_i) \leq \sum_{\mathbf{x}_i \in C} f_0(\mathbf{x}_i).$$

Also, on C, we have $f_1(\mathbf{x}) > k f_0(\mathbf{x})$, while on $\bar{C}$ we have $f_1(\mathbf{x}) \leq k f_0(\mathbf{x})$. So

$$\begin{aligned}
\sum_{\mathbf{x}_i \in \bar{C}^* \cap C} f_1(\mathbf{x}_i) &\geq k \sum_{\mathbf{x}_i \in \bar{C}^* \cap C} f_0(\mathbf{x}_i) \\
\sum_{\mathbf{x}_i \in \bar{C} \cap C^*} f_1(\mathbf{x}_i) &\leq k \sum_{\mathbf{x}_i \in \bar{C} \cap C^*} f_0(\mathbf{x}_i).
\end{aligned}$$

Hence

$$\begin{aligned}
\beta - \beta^* &= \sum_{\mathbf{x}_i \in \bar{C}} f_1(\mathbf{x}_i) - \sum_{\mathbf{x}_i \in \bar{C}^*} f_1(\mathbf{x}_i) \\
&= \sum_{\mathbf{x}_i \in \bar{C} \cap C^*} f_1(\mathbf{x}_i) + \sum_{\mathbf{x}_i \in \bar{C} \cap \bar{C}^*} f_1(\mathbf{x}_i) \\
&\quad - \sum_{\mathbf{x}_i \in \bar{C}^* \cap C} f_1(\mathbf{x}_i) - \sum_{\mathbf{x}_i \in \bar{C} \cap \bar{C}^*} f_1(\mathbf{x}_i) \\
&= \sum_{\mathbf{x}_i \in \bar{C} \cap C^*} f_1(\mathbf{x}_i) - \sum_{\mathbf{x}_i \in \bar{C}^* \cap C} f_1(\mathbf{x}_i) \\
&\leq k \sum_{\mathbf{x}_i \in \bar{C} \cap C^*} f_0(\mathbf{x}_i) - k \sum_{\mathbf{x}_i \in \bar{C}^* \cap C} f_0(\mathbf{x}_i) \\
&= k \left\{\sum_{\mathbf{x}_i \in \bar{C} \cap C^*} f_0(\mathbf{x}_i) + \sum_{\mathbf{x}_i \in C \cap C^*} f_0(\mathbf{x}_i)\right\} \\
&\quad - k \left\{\sum_{\mathbf{x}_i \in \bar{C}^* \cap C} f_0(\mathbf{x}) + \sum_{\mathbf{x}_i \in C \cap C^*} f_0(\mathbf{x}_i)\right\} \\
&= k(\alpha^* - \alpha) \\
&\leq 0.
\end{aligned}$$



$\square$

So even with non-continuous distributions, the likelihood ratio test is still a good idea; for a discrete distribution, as long as a likelihood ratio test of exactly size α exists, the same result holds.

QUESTION 5

We wish to test H_0: sex and eye colour independent. The actual data is:

		Eye-colour Blue	Eye-colour Brown	**Total**
Sex	Male	19	10	**29**
	Female	9	21	**30**
	Total	*28*	*31*	***59***

while the expected values given by H_0 is

		Eye-colour Blue	Eye-colour Brown	**Total**
Sex	Male	$\frac{812}{59}$	$\frac{812}{59}$	**29**
	Female	$\frac{840}{59}$	$\frac{930}{59}$	**30**
	Total	*28*	*31*	***59***

It is not quite clear that they do not match well, so we can find the p value to be sure.

$\sum\sum \frac{(o_{ij}-e_{ij})^2}{e_{ij}} = 7.46$, and the degrees of freedom is $(2-1)(2-1) = 1$.

From the tables, $\chi_1^2(0.05) = 3.841$ and $\chi_4^2(0.01) = 6.63$.

So our observed value of 7.46 is significant at the 1% level, i.e. there is strong evidence against H_0.

Next we wish to test H_0' : all cell probabilities are equal to 1/4.

$\sum\sum \frac{(o_{ij}-e_{ij})^2}{e_{ij}} = 7.64$, also $|\Theta_0| = 0$, $|\Theta_0| = 4-1 = 3$. From the tables, $\chi_3^2(0.05) = 7.82$. Hence we do not reject H_0.

QUESTION 6

In general, we have independent observations from r multinomial distributions, each of which has c categories, i.e. we observe an $r \times c$ table (n_{ij}), for $i = 1, \cdots, r$ and $j = 1, \cdots, c$, where

$$(N_{i1}, \cdots, N_{ic}) \sim \text{multinomial}(n_{i+}, p_{i1}, \cdots, p_{ic})$$

independently for each $i = 1, \cdots, r$. We want to test

$$H_0 : p_{1j} = p_{2j} = \cdots = p_{rj} = p_j,$$

for $j = 1, \cdots, c$ (ie. homogeneity down the rows), and

$$H_1 : p_{ij} \text{ are unrestricted.}$$

Using H_1,

$$\text{like}(p_{ij}) = \prod_{i=1}^{r} \frac{n_{i+}!}{n_{i1}! \cdots n_{ic}!} p_{i1}^{n_{i1}} \cdots p_{ic}^{n_{ic}},$$

and

$$\log \text{like} = \text{constant} + \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij} \log p_{ij}.$$

Using Lagrangian methods, we find that $\hat{p}_{ij} = \frac{n_{ij}}{n_{i+}}$.

Under H_0,

$$\log \text{like} = \text{constant} + \sum_{j=1}^{c} n_{+j} \log p_j.$$

By Lagrangian methods, we have $\hat{p}_j = \frac{n_{+j}}{n_{++}}$.

Hence

$$2 \log \Lambda = \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij} \log \left(\frac{\hat{p}_{ij}}{\hat{p}_j} \right) = 2 \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij} \log \left(\frac{n_{ij}}{n_{i+} n_{+j} / n_{++}} \right),$$

which is the same as what we had last time, when the row totals are unrestricted!

We have $|\Theta_1| = r(c-1)$ and $|\Theta_0| = c - 1$. So the degrees of freedom is $r(c-1) - (c-1) = (r-1)(c-1)$, and under H_0, $2 \log \Lambda$ is approximately $\chi^2_{(r-1)(c-1)}$. Again, it is exactly the same as what we had last time!

We reject H_0 if $2 \log \Lambda > \chi^2_{(r-1)(c-1)}(\alpha)$ for an approximate size α test.

If we let $o_{ij} = n_{ij}, e_{ij} = \frac{n_{i+} n_{+j}}{n_{++}}$, and $\delta_{ij} = o_{ij} - e_{ij}$, using the same approximating steps as for Pearson's Chi-squared, we obtain

$$2 \log \Lambda \approx \sum \frac{(o_{ij} - e_{ij})^2}{e_{ij}}.$$

Continuing our previous example, our data is

	Improved	No difference	Worse	**Total**
Placebo	18	17	15	**50**
Half dose	20	10	20	**50**
Full dose	25	13	12	**50**
Total	*63*	*40*	*47*	***150***

The expected under H_0 is

	Improved	No difference	Worse	**Total**
Placebo	21	13.3	15.7	**50**
Half dose	21	13.3	15.7	**50**
Full dose	21	13.3	15.7	**50**
Total	*63*	*40*	*47*	***150***

We find $2 \log \Lambda = 5.129$, and we refer this to χ_4^2. Clearly this is not significant, as the mean of χ_4^2 is 4, and is something we would expect to happen solely by chance.

We can calculate the p-value: from tables, $\chi_4^2(0.05) = 9.488$, so our observed value is not significant at 5%, and the data are consistent with H_0.

We conclude that there is no evidence for a difference between the drug at the given doses and the placebo.

For interest,

$$\sum \frac{(o_{ij} - e_{ij})^2}{e_{ij}} = 5.173,$$

giving the same conclusion.

QUESTION 7

Let $X_1, \cdots, X_n \sim \text{Exp}(\theta)$. We want to find the best size α test of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where θ_0 and θ_1 are known fixed values with $\theta_1 > \theta_0$. Then

$$\begin{aligned}\Lambda_{\mathbf{x}}(H_0; H_1) &= \frac{\theta_0^n e^{-\theta_0 \sum x_i}}{\theta_1^n e^{-\theta_1 \sum x_i}} \\ &= \left(\frac{\theta_1}{\theta_0}\right)^n \exp\left[-(\theta_1 - \theta_0)\sum x_i\right]\end{aligned}$$

This is an increasing function of $\sum x_i$, so for any k, $\Lambda_x > k \Leftrightarrow \sum x_i > c$ for some c. Hence we reject H_0 if $\sum x_i > c$, where c is chosen such that $\mathbb{P}(\sum X_i > c \mid H_0) = \alpha$.

Under H_0, $\sum X_i \sim \Gamma(n, \theta_0)$,

Note that

$$C = \left\{x \ : \ \sum x_i > c\right\}$$

For $\theta \in \mathbb{R}$, the power function is

$$\begin{aligned}W(\theta) &= \mathbb{P}_\theta(\text{reject } H_0) \\ &= \mathbb{P}_\theta\left(\sum X_i > c\right) \\ &= 1 - G_\theta(c)\end{aligned}$$

To show this is UMP, we know that $W(\theta_0) = c$ (by plugging in). $W(\mu)$ is an increasing function of θ. So

$$\sup_{\theta \le \theta_0} W(\theta) = \alpha.$$

So the first condition is satisfied.

For the second condition, observe that for any $\theta > \theta_0$, the Neyman Pearson size α test of $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ has critical region C. Let C^* and W^* belong to any other test of H_0 vs H_1 of size $\le \alpha$. Then C^* can be regarded as a test of H_0 vs H_1 of size $\le \alpha$, and the Neyman-Pearson lemma says that $W^*(\theta_1) \le W(\theta_1)$. This holds for all $\theta_1 > \theta_0$. So the condition is satisfied and it is UMP.

Not sure about last bit; I think not true, but need some explanation.

QUESTION 8

Suppose $X \sim N(0,1)$, $Y \sim \chi^2_n$. We want to find the joint PDF of

$$X/\sqrt{Y/n}$$

Consider the map

$$S : (x,y) \mapsto (u,v), \quad \text{where } \ u = x, \ v = \frac{x}{\sqrt{y/n}}$$

where $x, y, u \geq 0$, $0 \leq v \leq 1$ The inverse map T^{-1} acts by

$$S^{-1} : (u,v) \mapsto (x,y), \ \text{where } x = u, \ y = nu^2/v^2$$

and has the Jacobian

$$\begin{aligned} J(u,v) &= \det \begin{pmatrix} 1 & 0 \\ 2nu/v^2 & -2nu^2/v^3 \end{pmatrix} \\ &= -2nu^2/v^3 \end{aligned}$$

Then the joint PDF, by independence, is

$$f_{U,V}(u,v) = f_{X,Y}(u, nu^2/v^2) \left| -2nu^2/v^3 \right|$$

Substituting in $f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{y^{n/2-1} e^{-y/2}}{2^{n/2}\Gamma(n/2)}$, yields

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2^{n/2}\Gamma(n/2)} e^{-u^2/2} (nu^2/v^2)^{n/2-1} e^{-nu^2/2v^2} 2nu^2/v^3$$

Here, we integrate over the u variables. The pdf of T is given by $\int_{-\infty}^{\infty} f_{U,V}(u,v) \, du$. Can't get the result out; ~~this is gruesome.~~

QUESTION 9

Single sample: testing a given mean, known variance (z-test). Suppose that $X_1, \cdots, X_n$ are iid $N(\mu, \sigma^2)$, with σ^2 unknown, and we wish to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ (for given constant μ_0).

Here $\Theta_0 = \{\mu_0\}$ and $\Theta = \mathbb{R}$.

For the denominator, we have $\sup_\Theta f(\mathbf{x} \mid \mu) = f(\mathbf{x} \mid \hat{\mu})$, where $\hat{\mu}$ is the mle. We know that $\hat{\mu} = \bar{x}$. Hence

$$\Lambda_{\mathbf{x}}(H_0; H_1) = \frac{(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum (x_i - \bar{x})^2\right)}{(2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum (x_i - \mu_0)^2\right)}.$$

Then H_0 is rejected if Λ_x is large.

To make our lives easier, we can use the logarithm instead:

$$2 \log \Lambda(H_0; H_1) = \frac{1}{\sigma^2}\left[\sum (x_i - \mu_0)^2 - \sum (x_i - \bar{x})^2\right] = \frac{n}{\sigma^2}(\bar{x} - \mu_0)^2.$$

So we can reject H_0 if we have

$$\left|\frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma}\right| > c$$

for some c.

We know that under H_0, $Z = \dfrac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma} \sim N(0, 1)$. So the size α generalised likelihood test rejects H_0 if

$$\left|\frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma}\right| > z_{\alpha/2}.$$

Alternatively, since $\dfrac{n(\bar{X} - \mu_0)}{\sigma^2} \sim \chi_1^2$, we reject H_0 if

$$\frac{n(\bar{X} - \mu_0)^2}{\sigma^2} > \chi_1^2(\alpha),$$

(check that $z_{\alpha/2}^2 = \chi_1^2(\alpha)$).

Note that this is a two-tailed test — i.e. we reject H_0 both for high and low values of $\bar{x}$.

Also $S_{XX}/\sigma^2 \sim \chi_{n-1}^2$ and is independent of $\bar{X}$, and hence Z. So, from the previous question, know that

$$\frac{\sqrt{n}(\bar{X} - \mu)/\sigma}{\sqrt{S_{XX}/((n-1)\sigma^2)}} \sim t_{n-1},$$

or

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sqrt{S_{XX}/(n-1)}} \sim t_{n-1}.$$

QUESTION 10

QUESTION 11

QUESTION 12