Part IB — Methods Example Sheet 1

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QUESTION 1

$$\frac{f(x_+ + f(x_-))}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

For $f(x) = (x-1)^2$ on the interval $-1 \leq x \leq 1$, $f(x)$ is an even function, thus $b_n = 0$. We have $L = 1$, and

$$\begin{aligned}\frac{1}{2}a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\ \mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1} x^4 - 2x^2 + 1\ \mathrm{d}x \\ &= \int_{0}^{1} x^4 - 2x^2 + 1\ \mathrm{d}x \\ &= \frac{8}{15}\end{aligned}$$

and

$$\begin{aligned}a_n &= \frac{1}{L}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \\ &= \int_{-1}^{1} x^4 \cos n\pi x\ \mathrm{d}x - 2\int_{-1}^{1} x^2 \cos n\pi x\ \mathrm{d}x + \int_{-1}^{1} \cos n\pi x\ \mathrm{d}x\end{aligned}$$

Evaluating each integral separately, we have:

(i)

$$\int_{-1}^{1} \cos n\pi x\ \mathrm{d}x = \left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{-1} = 0$$

as $\sin n\pi x = 0\ \forall\ n$

(ii) By parts,

$$\begin{aligned}\int_{-1}^{1} x^2 \cos n\pi x\ \mathrm{d}x &= \left[\frac{x^2 \sin n\pi x}{n\pi} - \frac{1}{n\pi}\int 2x \sin n\pi x\ \mathrm{d}x\right]_{-1}^{1} \\ &= -\frac{2}{n\pi}\int_{-1}^{1} x \sin n\pi x\ \mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{1} x \sin n\pi x\ \mathrm{d}x &= \left[\frac{-x\cos n\pi x}{n\pi} + \frac{1}{n\pi}\int \cos n\pi x\ \mathrm{d}x\right]_{-1}^{1} \\ &= \frac{-2\cos n\pi x}{(n\pi)^2}\end{aligned}$$

Thus the second integral contributes to give

$$-\frac{8cosn\pi x}{(n\pi)^2}$$

(iii)

$$\begin{aligned}\int_{-1}^{1} x^4 \cos n\pi x \,\mathrm{d}x &= \left[\frac{x^4 \sin n\pi x}{n\pi} - \frac{1}{n\pi}\int 4x^3 \sin n\pi x \,\mathrm{d}x\right]_{-1}^{1} \\ &= -\frac{4}{n\pi}\int_{-1}^{1} x^3 \sin n\pi x \,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{1} x^3 \sin n\pi x \,\mathrm{d}x &= \left[\frac{-x^3 \cos n\pi x}{n\pi} + \frac{1}{n\pi}\int 3x^2 \cos n\pi x\right]_{-1}^{1} \\ &= \frac{-2\cos n\pi}{n\pi} + \frac{3}{n\pi}\int_{-1}^{1} x^2 \cos n\pi x \,\mathrm{d}x\end{aligned}$$

Whence

$$\begin{aligned}\int_{-1}^{1} x^4 \cos n\pi x \,\mathrm{d}x &= \frac{8\cos n\pi}{n\pi} - \frac{12}{(n\pi)^2}\int_{-1}^{1} x^2 \cos n\pi x \,\mathrm{d}x \\ &= \frac{8\cos n\pi}{n\pi} - \frac{48\cos n\pi}{(n\pi)^4}\end{aligned}$$

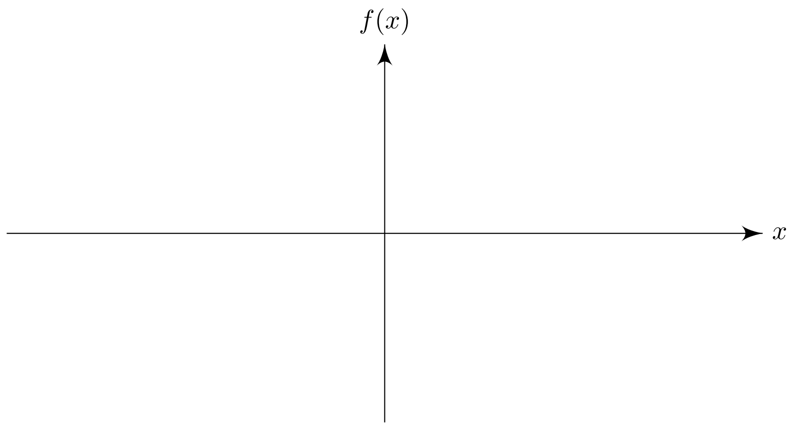
using (ii).

Finally,

$$\begin{aligned}a_n &= -\frac{48\cos n\pi}{(n\pi)^4} \\ &= \frac{48(-1)^{n+1}}{(n\pi)^4}\end{aligned}$$

as $\cos n\pi x = (-1)^n$
Hence the Fourier Series is given by

$$\begin{aligned}f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \\ &= \frac{8}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos n\pi x\end{aligned}$$



$f(x)$ satisfies the Dirichlet conditions. The 1st derivative is the lowest derivative which is discontinuous (at the endpoints, as $f(x)$ even fn $\Rightarrow$ $f'(x)$ odd), so Fourier coefficients are $\mathcal{O}(\frac{1}{n^2})$ as $n \to \infty$

QUESTION 2

Extending on range $(-\pi, \pi)$ so $L = \pi$ and

(a)
$$\frac{f(x_+ + f(x_-))}{2} = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\pi x^2 \sin nx\,\mathrm{d}x \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int_0^\pi x^2 \sin nx\,\mathrm{d}x &= \left[\frac{-x^2\cos nx}{n} + \frac{1}{n}\int 2x\cos nx\,\mathrm{d}x\right]_0^\pi \\ &= \frac{-\pi^2\cos n\pi}{n} + \frac{2}{n}\int_0^\pi x\cos nx\,\mathrm{d}x \end{aligned}$$

and once again,

$$\begin{aligned} \int_0^\pi x\cos nx\,\mathrm{d}x &= \left[\frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx\,\mathrm{d}x\right]_0^\pi \\ &= -\frac{1}{n}\int_0^\pi \sin nx\,\mathrm{d}x \\ &= -\frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^\pi \\ &= \frac{1}{n^2}(\cos n\pi - 1) \end{aligned}$$

Back substituting in,

$$\begin{aligned} b_n &= \frac{2}{\pi}\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2}{n^3}(\cos n\pi - 1)\right) \\ &= \frac{2}{\pi n^3}\left(-2 + (2 - (\pi n)^2)\cos n\pi\right) \end{aligned}$$

Hence Fourier sine series given by:

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^3}\left(-2 + (2-(\pi n)^2)(-1)^n\right)\sin nx$$

(b) Similarly,

$$\frac{f(x_+ + f(x_-))}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$\begin{aligned}\frac{a_0}{2} &= \frac{1}{L}\int_0^L f(x)\ \mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x^2\ \mathrm{d}x \\ &= \frac{\pi^2}{3}\end{aligned}$$

and

$$\begin{aligned}a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\ \mathrm{d}x\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_0^{\pi} x^2 \cos nx\ \mathrm{d}x &= \left[\frac{x^2 \sin nx}{n} - \frac{1}{n}\int 2x \sin nx\ \mathrm{d}x\right]_0^{\pi} \\ &= \frac{-2}{n}\int_0^{\pi} x \sin nx\ \mathrm{d}x\end{aligned}$$

and once again,

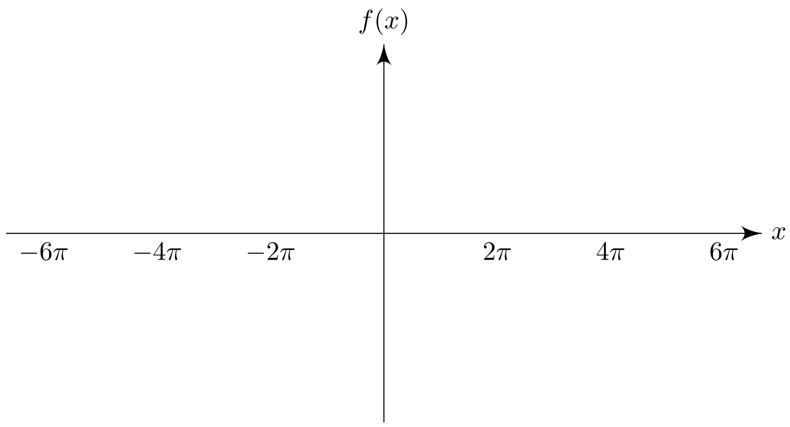
$$\begin{aligned}\int_0^{\pi} x \sin nx\ \mathrm{d}x &= \left[\frac{-x\cos nx}{n} + \frac{1}{n}\int \cos nx\ \mathrm{d}x\right]_0^{\pi} \\ &= \frac{-\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n\pi}\right]_0^{\pi} \\ &= \frac{-\pi\cos n\pi}{n}\end{aligned}$$

Thus

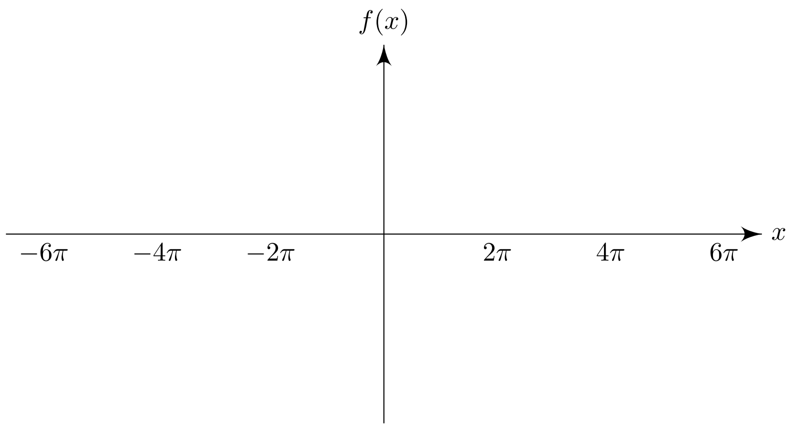
$$a_n = \frac{4}{n^2}\cos n\pi$$

and the Fourier cosine series is given by

$$f(x) = \frac{\pi^2}{3} + \sum_{n=0}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$



(i)



(ii)

Fourier series for $g(x) = 2x$ (odd function) in the range $(-\pi, \pi)$ given by

$$\frac{f(x_+ + f(x_-))}{2} = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \,\mathrm{d}x \\ &= \frac{2}{\pi} \int_{-\pi}^{\pi} x \sin nx \,\mathrm{d}x \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int_{-\pi}^{\pi} x \sin nx \,\mathrm{d}x &= \left[\frac{-x\cos nx}{n} + \frac{1}{n}\int \cos nx \,\mathrm{d}x\right]_{-\pi}^{\pi} \\ &= \frac{-2\pi \cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n\pi}\right]_{-\pi}^{\pi} \\ &= \frac{-2\pi(-1)^n}{n} \end{aligned}$$

Whence

$$g(x) = \sum_{n=1}^{\infty} \frac{4\pi^2(-1)^{n+1}}{n^2} \sin nx$$

Fourier series for $h(x) = 2|x|$ (even function) in the range $(-\pi, \pi)$ given by

$$\frac{f(x_+ + f(x_-))}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$\begin{aligned}
\frac{1}{2}a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\, \mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2|x|\, \mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\, \mathrm{d}x \\
&= \pi
\end{aligned}$$

and

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, \mathrm{d}x \\
&= \frac{2}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, \mathrm{d}x
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
\int_{-\pi}^{\pi} |x| \cos nx\, \mathrm{d}x &= 2\int_{0}^{\pi} x \cos nx\, \mathrm{d}x \\
&= 2\left[\frac{x \sin nx}{n} - \frac{1}{n}\int \sin nx\, \mathrm{d}x\right]_0^{\pi} \\
&= -\frac{2}{n}\int_{0}^{\pi} \sin nx\, \mathrm{d}x \\
&= -\frac{2}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi} \\
&= \frac{2}{n^2}(\cos n\pi - 1)
\end{aligned}$$

Whence

$$h(x) = \pi + \sum_{n=1}^{\infty} \frac{2}{n^2}((-1)^n - 1) \cos nx$$

QUESTION 3

$f(x) = e^x$ on $(-\pi, \pi)$ has Fourier series given by

$$\frac{f(x_+ + f(x_-))}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$\begin{aligned} \frac{1}{2}a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \, \mathrm{d}x \\ &= \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) \\ &= \frac{1}{\pi}\sinh \pi \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} e^x \cos nx \, \mathrm{d}x}_{I_a} \\ I_a &= \left[e^x \cos nx + \int e^x n \sin nx \, \mathrm{d}x\right]_{-\pi}^{\pi} \\ &= (e^{\pi} - e^{-\pi})\cos n\pi + n\int_{-\pi}^{\pi} e^x \sin nx \, \mathrm{d}x \\ &= 2\sinh \pi \, (-1)^n + n\left[e^x \sin nx - \int e^x n \cos x \, \mathrm{d}x\right]_{-\pi}^{\pi} \\ &= 2\sinh \pi \, (-1)^n + -n^2 \int_{-\pi}^{\pi} e^x \cos nx \, \mathrm{d}x \\ &= 2\sinh \pi \, (-1)^n + -n^2 I_a \end{aligned}$$

Hence

$$a_n = \frac{2}{\pi}\frac{1}{1+n^2}\sinh \pi \, (-1)^n$$

QUESTION 4

(i) Reposing the Fourier Series of $f(t)$ using complex variables,

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} \left(e^{\frac{in\pi t}{L}} + e^{\frac{-in\pi t}{L}} \right) + \frac{b_n}{2i} \left(e^{\frac{in\pi t}{L}} - e^{\frac{-in\pi t}{L}} \right) \right] \\
&= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi t}{L}}, \\
c_n &= \frac{a_n - ib_n}{2} \; n > 0; \\
c_{-n} &= \frac{a_n + ib_n}{2} \; n > 0; \\
c_0 &= \frac{a_0}{2}
\end{aligned}
$$

Using the orthogonality of complex exponentials and the properties of complex Fourier coefficients, we deduce that

$$
\begin{aligned}
\int_{-L}^{L} [f(t)]^2 \; \mathrm{d}t &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \int_{-T}^{T} \exp \left[\frac{i\pi t(n+m)}{L} \right] \; \mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m 2T \delta_{n[-m]} \\
&= 2T \sum_{n=-\infty}^{\infty} c_n c_{-n} \\
&= 2T \sum_{n=-\infty}^{\infty} c_n c_n^* \\
&= 2T \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}
$$

This can be then re-expressed in terms of the a_n and b_n as

$$
\int_{-L}^{L} [f(t)]^2 \; \mathrm{d}t = L \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]
$$

as required.

(ii)

QUESTION 5

QUESTION 6

Call $-\mathrm{d}^2/\mathrm{d}x^2 = \mathcal{L}$, and search for solutions to:

$$\mathcal{L}y_n = \lambda_n y_n, \quad y_n(0) = 0,\ y_n(1) + y_n'(1) = 0$$

Then

$$y_n'' + \lambda_n y_n = 0 \Rightarrow y_n = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$y_n(0) = 0 \Rightarrow A = 0$, and the second condition implies

$$B\sin\sqrt{\lambda_n} + B\sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0$$

$B \neq 0$, so we have $\sqrt{\lambda_n} = -\tan\sqrt{\lambda_n}$,
so eigenvalues are given by the squares of the solutions to $\xi = -\tan\xi$, $\xi > 0$

QUESTION 7

QUESTION 8

QUESTION 9