Part IB — Electromagnetism Example Sheet 1

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QUESTION 1

Equation for conservation of charge is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Have $\mathbf{J} = C\mathbf{r}e^{-atr^2}$, so

$$\begin{aligned}\nabla \cdot \mathbf{J} &= C\nabla \cdot (e^{-atr^2}\mathbf{r}) \\ &= Ce^{-atr^2}\nabla \cdot \mathbf{r} + C\mathbf{r} \cdot \nabla(e^{-atr^2})\end{aligned}$$

Now $\mathbf{r}_i = x_i$ so $\nabla \cdot \mathbf{r} = \frac{\partial x_j}{\partial x_j} = 3$, and

$$\begin{aligned}\nabla e^{-atr^2} &= \frac{\partial e^{-atr^2}}{\partial r}\hat{\mathbf{r}} \\ &= -2ate^{-atr^2}\mathbf{r}\end{aligned}$$

Hence

$$\nabla \cdot \mathbf{J} = 3Ce^{-atr^2} - 2Cr^2ate^{-atr^2}$$

Suppose that $\rho = (f + tg)e^{-atr^2}$. Then we have

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= (-ar^2 f + g - ar^2 tg)e^{-atr^2} \\ &= (g - ar^2 f)e^{-atr^2} - gtar^2e^{-atr^2}\end{aligned}$$

Hence we conclude that

$$g - ar^2 f = -3C, \qquad g = -2C$$

$$\Rightarrow f = \frac{C}{ar^2}$$

QUESTION 2

Using the continuity equation,

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\nabla \cdot \mathbf{J} \\ &= -\nabla \cdot (-D\nabla\rho) \\ &= D\nabla^2\rho\end{aligned}$$

showing $\rho(\mathbf{x}, t)$ obeys the heat equation with diffusion constant D.
Let $\rho(\mathbf{r}, t)$ be defined as

$$\rho(\mathbf{r}, t) = \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}} \exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)$$

Taking time derivatives,

$$\frac{\partial \rho}{\partial t} = \left(\frac{-6D\rho_0 a^3}{(4D(t-t_0)+a^2)^{5/2}} + \frac{4Dr^2\rho_0 a^3}{(4D(t-t_0)+a^2)^{7/2}}\right) \exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)$$

Now

$$\begin{aligned}\nabla^2 e^{\lambda r^2} &= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r}(e^{\lambda r^2})\right) \\ &= \frac{2\lambda}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 e^{\lambda r^2}\right) \\ &= \frac{2\lambda}{r^2}\left[3r^2 + 2\lambda r^4\right] e^{\lambda r^2} \\ &= \lambda(6+4r^2)e^{\lambda r^2}\end{aligned}$$

Thus with $\lambda = -\frac{1}{4D(t-t_0)+a^2}$, we have

$$\begin{aligned}\nabla^2\rho &= \frac{-\rho_0 a^3}{(4D(t-t_0)+a^2)^{5/2}}\left(6 - \frac{4r^2}{4D(t-t_0)+a^2}\right) \exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right) \\ &= \left(-\frac{6\rho_0 a^3}{(4D(t-t_0)+a^2)^{5/2}} + \frac{4r^2\rho_0 a^3}{(4D(t-t_0)+a^2)^{-7/2}}\right) \exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)\end{aligned}$$

Hence we can see that $\frac{\partial \rho}{\partial t} = D\nabla^2\rho$, as required.

QUESTION 3

Considering the infinite plane $z = 0$, we see this has uniform charge density ρ_0.

By symmetry, the field points vertically, and the field on the bottom is opposite of that on top, we must have

$$\mathbf{E} = E(z)\hat{\mathbf{z}}$$

with

$$E(z) = -E(-z)$$

Consider a vertical cylinder of height $2h$ and cross-sectional area A. Now only the end caps contribute.

First,

$$\begin{aligned}
Q &= \int_V \rho_0 e^{-k|z|}\ \mathrm{d}V \\
&= \int_{-h}^{h}\int_0^{2\pi}\int_0^R \rho_0 e^{-k|z|}\rho\ \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}z \\
&= 2\pi\frac{R^2}{2}\rho_0\int_{-h}^{h} e^{-k|z|}\ \mathrm{d}z \\
&= A\rho_0\int_0^h 2e^{-kz}\ \mathrm{d}z \\
&= 2A\rho_0\left[-\frac{1}{k}e^{-kz}\right]_0^h \\
&= 2A\frac{\rho_0}{k}(1-e^{-kh})
\end{aligned}$$

And

$$\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = E(h)A - E(-h)A = 2AE(h) = 2A\frac{\rho_0}{k\varepsilon_0}(1-e^{-kh})$$

Hence

$$E(z) = \frac{\rho_0}{k\varepsilon_0}(1-e^{-kz})$$

as required.

QUESTION 4

We have that

$$\rho(r) = \begin{cases} 0 & \text{if } r < a \\ \rho & \text{if } a < r < b \\ 0 & \text{if } r > b \end{cases}$$

First consider $r > b$. By symmetry, the force is the same in all directions and points outwards radially. So

$$\mathbf{E} = E(r)\hat{\mathbf{r}}$$

Put S to be a sphere of radius $r > b$. Then the total flux is

$$\begin{aligned} \int_S \mathbf{E} \cdot \, \mathrm{d}S &= \int_S E(r)\hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{S} \\ &= E(r) \int_S \hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{S} \\ &= E(r) \cdot 4\pi r^2 \end{aligned}$$

By Gauss's law, we know this is equal to Q/ε_0, and $Q = \frac{4}{3}\pi(b^3 - a^3)\rho$. Therefore,

$$E(r) = \frac{(b^3 - a^3)\rho}{3\varepsilon_0 r^2}$$

and

$$\mathbf{E}(r) = \frac{(b^3 - a^3)\rho}{3\varepsilon_0 r^2}\hat{\mathbf{r}}$$

Now suppose we are inside the region, $a < r < b$. Then

$$\int_S \mathbf{E} \cdot \, \mathrm{d}S = E(r) 4\pi r^2 = \frac{Q}{\varepsilon_0}\left(\frac{r^3 - a^3}{b^3 - a^3}\right)$$

So

$$\begin{aligned} \mathbf{E}(r) &= \frac{Q(r^3 - a^3)}{4\pi\varepsilon_0(b^3 - a^3)r^2} \\ &= \frac{Q(r^3 - a^3)\rho}{3\varepsilon_0 r^2} \end{aligned}$$

Finally if $r < a$, Gauss' law tells us that the flux depends only on the total charge contained inside the surface, which in this case is none. So $\mathbf{E}(r) = 0$.

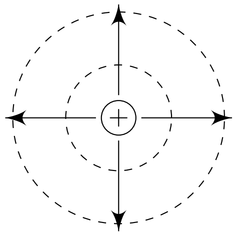
Note that the electric field is discontinuous across the surface. We have

$$\begin{aligned} E(r \to b+) - E(r \to b-) &= \frac{(b-a)(b^2 + 2ab + a^2)\rho}{3\varepsilon_0 b^2} \\ &= \frac{\sigma}{\varepsilon_0} \end{aligned}$$

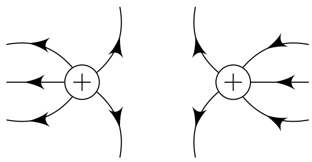
as expected.

QUESTION 5

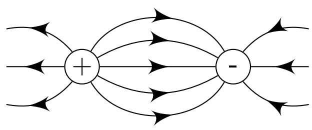
The field lines for a positive charge are:



For two positive charges,



We can also draw field lines for dipoles:



QUESTION 6

The inverse square law, or Coulomb's Law, states that the electric field generated by a particle with total charge Q (at the origin) is given by

$$\mathbf{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

Now consider an infinite line with uniform charge density per unit length η. We use cylindrical polar coordinates. By symmetry, the field is radial, ie.

$$\mathbf{E}(r) = E(r)\hat{\mathbf{r}}$$

Consider an arbitrary point at (r, z_0). We will integrate along the z-axis to find the field at this point.

Here,

$$\begin{aligned}
E(r) &= \int_{-\infty}^{\infty} \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2 + (z - z_0)^2}\, \mathrm{d}z \\
&= \frac{Q}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{1}{r^2 + z^2}\, \mathrm{d}z \\
&= \frac{Q}{4\pi\varepsilon_0} \left[\frac{1}{r} \arctan\left(\frac{z}{r}\right)\right]_{-\infty}^{\infty} \\
&= \frac{Q}{4r\varepsilon_0}
\end{aligned}$$

QUESTION 7

The Green's function for the Laplacian is definied to be the solution to:

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}')$$

We know that

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

We assume all the charge is contained within some compact region V, then

$$\begin{aligned}\phi(\mathbf{r}) &= -\frac{1}{\varepsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\ \mathrm{d}^3\mathbf{r} \\ &= \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\ \mathrm{d}^3\mathbf{r}\end{aligned}$$

So integrating over a circular disk of radius a we using polar coordinates so $\mathbf{r} = (0, 0, z)$, $\mathbf{r}' = (r\cos\phi, r\sin\phi, 0)$ and $|\mathbf{r} - \mathbf{r}'| = r^2 + z^2$, hence

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^a \frac{\sigma}{r^2 + z^2} r\ \mathrm{d}r\ \mathrm{d}\phi \\ &= \frac{\sigma}{4\varepsilon_0}\left[\log(r^2 + z^2)\right]_0^a \\ &= \frac{\sigma}{4\varepsilon_0}\log(1 + \frac{z^2}{a^2})\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\frac{\sigma}{2\varepsilon_0}\frac{1}{r^2 + z^2}\end{aligned}$$

QUESTION 8

From Q7 we have the result that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}^3\mathbf{r}$$

Very far from V, ie. $|\mathbf{r}| \gg |\mathbf{r}'|$, we can use the Taylor expansion

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r} + \mathbf{r}' \cdot \nabla \left(\frac{1}{r} \right) + \cdots \\ &= \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \cdots \end{aligned}$$

Then we get

QUESTION 9

QUESTION 10

QUESTION 11

QUESTION 12