Part IB — Electromagnetism Example Sheet 3

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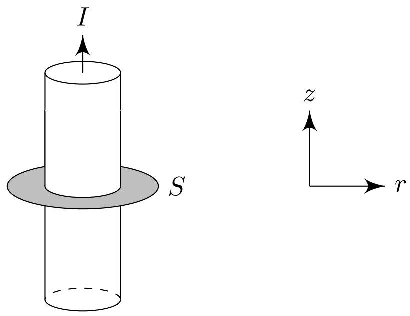
QUESTION 1

We use cylindrical polar coordinates (r, ϕ, z), where z is along the direction of the current, and r points in the radial direction.



We know that inside a conductor, $\mathbf{E} = 0$, and at the surface, $\mathbf{E}_{\parallel} = 0$.

Since

$$\mathbf{J} = \sigma \mathbf{E}$$

we know that inside the conductor, $\mathbf{J} = 0$. Hence the current, given by $I = \int_S \mathbf{J} \cdot \mathrm{d}s$, must be constant across the cylinder.

Using the result

$$\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{outside}} - \hat{\mathbf{n}} \cdot \mathbf{E}_{\text{inside}} = \frac{k}{\varepsilon_0}.$$

where k is the surface current density. Since $\mathbf{E}_{\text{inside}} = 0$, dotting with $\hat{\mathbf{r}} = \hat{\mathbf{n}}$ gives

$$\mathbf{E}_{\text{outside}} = \frac{k}{\varepsilon_0} \hat{\mathbf{r}}$$

Maxwell's equations say that $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, and since $\mathbf{E}$ is constant, we also have that $\mathbf{B}$ is constant.

I think inside the conductor $\mathbf{B} = 0$ but not sure.

Pontying vector: $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$.

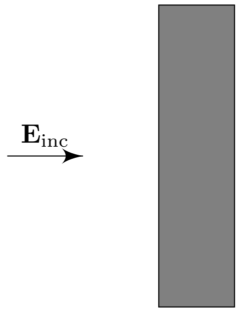
Definition (Joule heating)**.** *Joule heating* is the energy lost in a circuit due to friction. It is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = I^2 R.$$

From the definition of resistivity and conductivity, we have $R = \frac{L}{A\sigma}$, which is simply $1/\sigma$ if we're talking about unit lengths.

Not sure how to finish this off.

QUESTION 2



We have

$$\mathbf{E} = E_0 \hat{\mathbf{x}} e^{i(kz-\omega t)} - E_0 \hat{\mathbf{x}} e^{i(-kz-\omega t)}$$

$$\mathbf{B} = \frac{E_0}{c} \hat{\mathbf{y}} e^{i(kz-\omega t)} - \frac{E_0}{c} \hat{\mathbf{y}} e^{i(-kz-\omega t)}$$

The Maxwell equations in free space $\nabla \cdot \mathbf{E} = 0$,$\nabla \cdot \mathbf{B} = 0$ are trivially satisfied as E_x and B_y have no dependence on x and y respectively. Next we have to satisfy

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Noting that $\nabla \times E(z)\hat{\mathbf{x}} = E'(z)\hat{\mathbf{y}}$ and $\nabla \times B(z)\hat{\mathbf{y}} = -B'(z)\hat{\mathbf{x}}$, we have

$$\nabla \times \mathbf{E} = ikE_0 \hat{\mathbf{y}} e^{i(kz-\omega t)} - (ikE_0 \hat{\mathbf{y}} e^{i(-kz-\omega t)})$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = -i\omega \frac{E_0}{c} \hat{\mathbf{y}} e^{i(kz-\omega t)} - (-i\omega \frac{E_0}{c} \hat{\mathbf{y}} e^{i(-kz-\omega t)})$$

so the the first equation is satisfied if $k = \omega/c$. Next, we have

$$\nabla \times \mathbf{B} = -ik\frac{E_0}{c} \hat{\mathbf{x}} e^{i(kz-\omega t)} - (-ik\frac{E_0}{c} \hat{\mathbf{x}} e^{i(-kz-\omega t)})$$

so can see the next equation is satisfied if $\frac{k}{c} = \omega \mu_0 \varepsilon_0$, which is equivalent to $k = \omega/c$ since $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

We know at the surface of a conductor, $\mathbf{E}_{\parallel} = 0$. Clearly $\mathbf{E} \cdot \hat{y}|_{z=0} = 0$, and

$$\mathbf{E} \cdot \hat{x}|_{z=0} = E_0 e^{-i\omega t} - E_0 e^{-i\omega t} = 0$$

Not sure how to do rest/ ran out of time.

QUESTION 3

Plane wave solutions which propagate in the y direction are independent of x and z. So we can write our electric field as

$$\mathbf{E}(\mathbf{x}) = (E_x(y,t), E_y(y,t), E_z(y,t)).$$

Hence any derivatives wrt x and z are zero. Since we know that $\nabla \cdot \mathbf{E} = 0$, E_y must be constant. We take $E_y = 0$. wlog, assume $E_z = 0$, ie the wave propagate in the y direction and oscillates in the x direction. Then we look for solutions of the form

$$\mathbf{E} = (E(y,t), 0, 0),$$

with

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{\partial^2 \mathbf{E}}{\partial y^2} = 0.$$

The general solution is

$$E(y,t) = f(y - ct) + g(y + ct).$$

The most important solutions are the *monochromatic* waves

$$E(y,t) = E_0 \sin(ky - \omega t).$$

Note that here, $\omega = ck$. The given boundary conditions imposed are that $E(0,t) = E(a,t) = 0$. Thus

$$\sin\left(\frac{a\omega}{c} - \omega t\right) = 0, \quad \sin(-\omega t) = 0$$

from which we deduce that both arguments are integer multiples of π. It then follows that $\frac{a\omega}{c} = n\pi$ for some $n \in \mathbb{Z}$, and hence $\omega = \frac{n\pi c}{a}$.

QUESTION 4

We check that the given electric and magnetic fields satisfy the equations:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

First,

$$\begin{aligned}\nabla^2 \mathbf{E} &= \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} \\ &= \left[-\left(\frac{n\pi}{a}\right)^2 - k^2\right] \omega A \sin\left(\frac{n\pi y}{a}\right) \sin(kz - \omega t)\end{aligned}$$

Next,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{\omega^3}{c^2} A \sin\left(\frac{n\pi y}{a}\right) \sin(kz - \omega t)$$

Thus the first equation reduces to

$$\omega^2 = \left(\frac{n\pi c}{a}\right)^2 + c^2 k^2 \quad (*)$$

which is the criteria for the wave to propagate between the plates.
Next, we have

$$\nabla \times \mathbf{E} = \partial_z E_x \hat{\mathbf{y}} - \partial_y E_x \hat{\mathbf{z}}$$

which is trivial to show, matches the negative first time derivative of B_y and B_z.
Next, we have $\lambda = 2\pi/k$ and

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{n\pi}{a}\right)^2$$

from (*). Hence

$$\begin{aligned}\frac{1}{\lambda^2} &= k^2/4\pi^2 \\ &= \left(\frac{\omega}{2\pi c}\right)^2 - \frac{n^2}{4a^2}\end{aligned}$$

This suggests that $\lambda_\infty = \frac{2\pi c}{\omega}$ but I'm not sure why this would be true.
Also not sure if (*) is true; don't we have $\omega^2 = c^2 k^2$ for monochromatic waves?

QUESTION 5

Recall we define **E** and **B** in terms of ϕ and **A** as follows:

$$\begin{aligned}\mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A}.\end{aligned}$$

Since this is an electromagnetic wave, it must satisfy $\nabla \cdot \mathbf{E} = 0$. Now

$$\begin{aligned}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} &= \partial_j[-\omega \mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]_j \\ &= -\omega[\mathbf{A}_0]_j k_j e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ &= -\omega \mathbf{A}_0 \cdot \mathbf{k} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\end{aligned}$$

and

$$\begin{aligned}\nabla \cdot \nabla\phi &= \nabla^2\phi \\ &= k^2\phi_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\end{aligned}$$

Therefore we must have

$$\omega \mathbf{A}_0 \cdot \mathbf{k} = k^2\phi_0$$

QUESTION 6

Under a boost by v in the x-direction , we have

$$\begin{aligned}
E'_x &= E_x \\
E'_y &= \gamma(E_y - vB_z) \\
E'_z &= \gamma(E_z + vB_y) \\
B'_x &= B_x \\
B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\
B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right)
\end{aligned}$$

The conditions are:

$$E_xB_x + E_yB_y + E_zB_z = 0$$

and

$$E_x^2 + E_y^2 + E_z^2 - c^2\left(B_x^2 + B_y^2 + B_z^2\right) \neq 0$$

Not sure what I'm doing here.

QUESTION 7

We know that inside a conductor, $\mathbf{E} = 0$, and at the surface, $\mathbf{E}_{\parallel} = 0$. So $\mathbf{E}_0 \cdot \hat{\mathbf{y}}|_{x=0} = 0$. This is clearly satisfied by our field, as it just becomes $f(t) - f(t) = 0$.

Maxwell's equations says $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. So

$$\begin{aligned} \nabla \times \mathbf{E} &= \hat{\mathbf{z}} \left[\frac{\partial f(t_-)}{\partial x} - \frac{\partial f(t_+)}{\partial x} \right] \\ &= \end{aligned}$$

Note sure how to link in the time derivatives of f.

Under a boost by v in the x-direction , we have

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma(E_y - vB_z) \\ E'_z &= \gamma(E_z + vB_y) \\ B'_x &= B_x \\ B'_y &= \gamma\left(B_y + \frac{v}{c^2} E_z\right) \\ B'_z &= \gamma\left(B_z - \frac{v}{c^2} E_y\right) \end{aligned}$$

QUESTION 8

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Since this is antisymmetric, the diagonals are all 0, and $A_{\mu\nu} = -A_{\nu\mu}$. So this thing has $(d \times d - d)/2 =$ independent components (for $d \geq 3$).

QUESTION 9

The Lorentz force law in relativistic from is

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}U_\nu$$

Here $U_\nu = \gamma(c, 0, -p_0/m, 0)$, and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E/c & 0 & 0 \\ E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So here, the RHS is $(0, qE, 0, 0)$.
Then

$$m\frac{\mathrm{d}\gamma u}{\mathrm{d}t} = qE$$

which gives

$$m\gamma u = qEt$$

where

$$\gamma(\mathbf{u}) = \frac{1}{\sqrt{1 - \frac{u^2+v^2}{c^2}}}$$

QUESTION 10