

Part IB — Linear Algebra

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0 Introduction

Linear algebra is an important component of undergraduate mathematics. At the practical level, matrix theory and the related vector-space concepts provide a language and a powerful computational framework for posing and solving important problems.

Beyond this, elementary linear algebra is a valuable introduction to mathematical abstraction and logical reasoning because the theoretical development is self-contained, consistent, and accessible to most students.

1 Vector Spaces

1.1 Vector Spaces

Definition. An \mathbb{F} -Vector space (a vector space on \mathbb{F}) is an abelian group $(V, +)$ equipped with a function¹ $F \times V \rightarrow V$, $(\lambda, v) \mapsto \lambda v$

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$$

$$(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$$

$$\lambda(\mu v) = \lambda\mu v$$

$$1v = v$$

$$v + \mathbf{0} = v$$

for all $\lambda_i, \lambda, \mu \in F$, $v_i \in V$

Note that we will not be underlining our vectors, as this is cumbersome here. We will however be using $\mathbf{0}$ to denote the zero vector.

Example. For all $n \in \mathbb{N}$, \mathbb{F}^n = space of column vectors of length n , entries in \mathbb{F} . We understand the definition as entry-wise addition, entry-wise scalar multiplication

Example. $M_{m,m}(\mathbb{F})$, the set of $m \times m$ matrices with entries in \mathbb{F}

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

again all operations defined entry-wise

Example. For any set X , $\mathbb{R}^X = \{f : X \rightarrow \mathbb{R}\}$ Addition and scalar multiplication defined pointwise = $f_1(x) + f_2(x)$.

Exercise. Show that the above examples satisfy the axioms

Proposition. $0v = \mathbf{0}$ for all $v \in V$.

Proof. $((0+0)v = 0v \iff 0v + 0v = 0v \iff 0v = \mathbf{0})$ □

Exercise. Show² that $(-1)v = -v$

Definition. Let V be an \mathbb{F} -vector space. A subset U of V is a subspace ($U \leq V$) if:

- (i) $\mathbf{0} \in U$
- (ii) $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$ “ U is closed under addition...”
- (iii) $u \in U$, any $\lambda \in \mathbb{F} \Rightarrow \lambda u \in U$ “...and scalar multiplication”

¹scalar multiplication

²Hint: Use the previous proposition

Exercise. If U is a subspace of V , then U is also an \mathbb{F} -vector space.

Example. Let $V = \mathbb{R}^{\mathbb{R}}$, then $f : \mathbb{R} \rightarrow \mathbb{R}$. The set of all continuous functions $C(\mathbb{R})$ are a subspace. An even smaller subspace is the set of all polynomials.

Exercise. Define $U \subseteq \mathbb{R}^3$ as:

$$\left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mid a_1 + a_2 + a_3 = t \right\}$$

for some constant t . Check that this is a subspace of \mathbb{R}^3 if and only if $t = 0$.

Proposition. Let V be an F -vector space, $U, W \leq V$. Then $U \cap W \leq V$.

Proof. (i) $0 \in U, 0 \in W \Rightarrow 0 \in U \cap W$

(ii) Suppose $u, v \in U \cap W, \lambda, \mu \in F$. U is a subspace $\Rightarrow \lambda u + \mu v \in W$.
Similarly $\lambda u + \mu v \in U \in W$, so it is in the intersection. \square

Example. $V = \mathbb{R}^3, U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0 \right\}, W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = 0 \right\}$ then $U \cap W = U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0, y = 0 \right\}$ (intersect along the z -axis)

Note: union of family of subspaces is almost never a subspace itself.

Definition. Let V be an F -vector space, $U, W \leq V$. The *sum* of U and W is the set:

$$U + W = \{u + w \mid u \in U, w \in W\}$$

Proposition. $U + W \leq V$

Proof. $0 \in U, W \Rightarrow 0 + 0 = 0 \in U + W$

$u_1, u_2 \in U, w_1, w_2 \in W,$

$$(u_1 + w_1) + (u_2 + w_2) = \underbrace{(u_1 + u_2)}_{\in U} + \underbrace{(w_1 + w_2)}_{\in W}$$

Similarly for scalar multiplication (ex.) \square

Note: $U + W$ is the smallest subspace containing both U and W . (This is because all elements of the form $u + w$ are forced to be in such a subspace by the “closed under addition” axiom)

Definition. V is an \mathbb{F} -vector space, $U \leq V$. The quotient space³ V/U is the abelian group V/U equipped with scalar multiplication;

$$F \times V/U \rightarrow V/U$$

$$(\lambda, v + U) \mapsto \lambda v + U$$

³think of this as the collection of cosets of U in V

Proposition. This is well-defined, and V/U is an F -vector space.

Proof. Well-defined: Suppose $v_1 + U = v_2 + U \in V/U$. $\Rightarrow (v_2 - v_1) \in U \Rightarrow (\lambda v_2 - \lambda v_1) \in U \Rightarrow \lambda v_2 + U = \lambda v_1 + U \in V/U$

To show that it is an \mathbb{F} -vector space, we must show that the axioms hold. These follow from the axioms of V . $\lambda(\mu(v + U)) = \lambda(\mu v + U) = \lambda(\mu v) + U = (\lambda\mu)v + U = \lambda\mu(v + U)$ (scalar multiplication on V/U).

Ex. Other axioms follow similarly from using vector space axioms

□

1.2 Bases

Definition. V is an \mathbb{F} -vector space, $S \subset V$. The *span* of S is denoted by

$$\langle S \rangle = \left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{F} \right\}$$

ie. the set of all finite linear combinations, all but finitely many of the λ_s are zero.

Remark: $\langle S \rangle$ is the smallest subspace of V which contains⁴ all of the elements of S

Convention: $\langle \emptyset \rangle = \{\mathbf{0}\}$.

Example. $V = \mathbb{R}^3$,

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \right\}$$

$$\langle S \rangle = \left\{ \begin{pmatrix} a \\ b \\ 2b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

ie. we have took linear combinations of the first two. We don't need the third one.

Example. For X a set, define $\delta_x(y) : X \rightarrow \mathbb{F}$ as

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$\langle \delta_x \mid x \in X \rangle = \{f \in \mathbb{R}^X \mid f \text{ has finite support}\}$$

$$= \langle x \in X \mid f(x) \neq 0 \rangle$$

Definition. S spans V if $\langle S \rangle = V$

Definition. V is *finite dimensional* over \mathbb{F} if it is spanned by a set that is finite.

⁴This is essentially a tautology

Definition. The vectors v_1, \dots, v_n are *linearly independent* over \mathbb{F} if

$$\sum_{i=1}^n \lambda_i v_i = 0 \Rightarrow \lambda_i = 0 \text{ for all } i$$

some coefficients $\lambda_i \in \mathbb{F}$. $S \subset V$ is linearly independent if every finite subset of it is.

Example. The first example, u, v, w are not linearly independent⁵, but the set $\{\delta_X \mid x \in X\}$ is linearly independent.

A lesson to be learnt from our example is that a linearly dependent spanning set contains redundant information. In a sense, a linearly independent spanning set is a minimal spanning set and hence represents the most efficient way of characterizing the subspace. This idea leads to the following definition.

Definition. \mathcal{B} is a *basis* of V if it is linearly independent and spans V

Example. – \mathbb{F}^n standard basis: $\{e_1, e_2, \dots, e_n\}$.

– $V = \mathbb{C}$ over \mathbb{C} has natural basis $\{1\}$, over \mathbb{R} has natural basis $\{1, i\}$

– $V = \mathcal{P}(\mathbb{R})$ space of all polynomials, has natural basis

$$\{1, x, x^2, x^3, \dots\}$$

Exercise. Check this carefully

Lemma. V is an \mathbb{F} -vector space. The vectors v_1, \dots, v_n form a basis of V iff each vector $v \in V$ has a unique expression

$$v = \sum_{i=1}^n \lambda_i v_i, \text{ with } \lambda_i \in \mathbb{F}$$

Proof. (\Rightarrow) Fix $v \in V$. The v_i span, so

$$\exists \lambda_i \in \mathbb{F} \text{ s.t. } v = \sum \lambda_i v_i$$

Suppose also $v = \sum \mu_i v_i$ for some $\mu_i \in \mathbb{F}$. $\sum (\mu_i - \lambda_i) v_i = \mathbf{0}$.

The v_i are linearly independent so $\mu_i - \lambda_i = 0$ for all i , $\lambda_i = \mu_i$

(\Leftarrow) The v_i span V , since any $v \in V$ is a linear combination of them. IF $\sum_{i=1}^n \lambda_i v_i = \mathbf{0}$. Note that $\mathbf{0} = \sum_{i=1}^n 0 v_i$. By uniqueness (applied to $\mathbf{0}$), $\lambda_i = 0$ for all i . \square

Lemma. If v_1, \dots, v_n span V (over \mathbb{F}), then some subset of v_1, \dots, v_n is a basis for V (over \mathbb{F}).

Proof. If v_1, \dots, v_n linearly independent, done. Otherwise for some l , there exist $\alpha_1, \dots, \alpha_{l-1} \in \mathbb{F}$ such that

$$v_l = \alpha_1 v_1 + \dots + \alpha_{l-1} v_{l-1}$$

⁵If *not* linearly independent, say a set is linearly dependent.

(If $\sum \lambda_i v_i = \mathbf{0}$, not all $\lambda_i = 0$. Take l maximaml with $\lambda_i \neq 0$, just $\alpha_i = -\lambda_i/\lambda_l$).

Now $v_2, \dots, v_{l-1}, v_{l+1}, \dots, v_n$ still span V . Continue iteratively until get linear independence. \square

Theorem. (Steinitz exchange lemma) Let V be a finite dimensional vector space over \mathbb{F} . Take v_1, \dots, v_m to be linearly independent w_1, \dots, w_n to span V .

Then $m \leq n$, and reordering the spanning set if needed,

$$v_1, \dots, v_m, w_{m+1}, \dots, w_n$$

span V .

Proof. (Induction) Suppose that we've replaced $l(\geq 0)$ of the w_i . Reordering the w_i if needed, $v_1, \dots, v_l, w_{l+1}, \dots, w_n$ span V .

If $l = m$, done.

If $l < m$, then

$$v_{l+1} = \sum_{i=1}^l \alpha_i v_i + \sum_{i>l} \beta_i w_i$$

$\alpha_i, \beta_i \in \mathbb{F}$. As the v_i are lin. indep, $\beta_i \neq 0$ for some i . (After reordering, $\beta_{l+1} \neq 0$).

$$w_{l+1} = \frac{1}{\beta_{l+1}} \left(v_{l+1} - \sum_{i \leq l} \alpha_i v_i - \sum_{i>l+1} \beta_i w_i \right)$$

This $v_1, \dots, v_{l+1}, w_{l+2}, \dots, w_n$ also spans V . After m steps, w_i will have replaced m of the w_i by v_i . Thus $m \leq n$. \square

Theorem. If V is a finite dimensional vector space over \mathbb{F} , then any two bases for V have the same number of elements. This is what we call the *dimension* of V , denoted $\dim_{\mathbb{F}} V$.

Proof. If $\{v_1, \dots, v_n\}$ is a basis and w_1, \dots, w_m is another basis, the $\{v_i\}$ span and $\{w_i\}$ is linearly indepndent' so by Steinitz $m \leq n$. Likewise, $n \leq m$. \square

Example. $\dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{R}} \mathbb{C} = 2$

Theorem. V , finite dim, v -space over \mathbb{F} . If w_1, \dots, w_l is a linearly indepndent set of vectors, we can extend it to a basis $w_1, \dots, w_l, v_{l+1}, \dots, v_n$

Proof. Apply Steinitiz to w_1, \dots, w_l (lin indep) and any basis v_1, \dots, v_n .

Or directrly, if $V = \langle w_1, \dots, w_l \rangle$, stop.

Otherwise take $v_{l+1} \in V \setminus \langle w_1, \dots, w_l \rangle$, now w_1, \dots, w_l, v_{l+1} is linearly indep. iterate \square

Corollary. Suppose V is a finite dimensional vector space, with dimension n .

- (i) Any linearly independent set of vectors has at most n elements with equality iff it's a basis

- (ii) Any spanning set of vectors must have at least n elements, with equality if and only if it's a basis.

Slogan "Choose the best basis for the job"

Theorem. Let U, W be subspaces of V . If U, W are finite dim, so is $U + W$ and $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

Proof. Pick basis v_1, \dots, v_l of $U \cap W$. Extend it to basis $v_1, \dots, v_l, u_1, \dots, u_m$ of U . Extend it to basis $v_1, \dots, v_l, w_1, \dots, w_n$ of W .

Claim: $v_1, \dots, v_l, u_1, \dots, u_m, w_1, \dots, w_n$ is a basis for $U + W$.

- (i) Span: $u \in U$, then $u = \sum \alpha_i v_i + \sum \beta_i u_i$, $\alpha_i, \beta_i \in \mathbb{F}$ $w \in W$, then $w = \sum \gamma_i v_i + \sum \delta_i w_i$, $\gamma_i, \delta_i \in \mathbb{F}$

$$u + w = \sum (\alpha_i + \gamma_i) v_i + \sum (\beta_i + \delta_i) u_i$$

- (ii) lin indep: $u = \sum \alpha_i v_i + \sum \beta_i u_i + \sum \gamma_i w_i = \mathbf{0}$

$$\Rightarrow u = \underbrace{\sum \alpha_i v_i + \sum \beta_i u_i}_{\in U} = - \underbrace{\sum \gamma_i w_i}_{\in W} \in U \cap W$$

This is equal to $\sum \delta_i v_i$ for some $\delta_i \in \mathbb{F}$ because v_i are basis for $U \cap W$.

As v_i and w_i are lin indep, $(*) \Rightarrow \gamma_i = \delta_i = 0$ for all i .

$\Rightarrow \sum \alpha_i v_i + \sum \beta_i u_i = 0 \Rightarrow \alpha_i = \beta_i = 0$ because v_i and u_i form a basis for U .

□

Theorem. Let V be a finite dim \mathbb{F} -vector space, $U \leq V$, then U and V/U are also of finite dim, and

$$\dim V = \dim U + \dim V/U$$

Proof.

Exercise. Show that U is finite dim.

Let u_1, \dots, u_l be a basis for U . Extend it to a basis for V . Say $u_1, \dots, u_l, w_{l+1}, \dots, w_n$ of V .

Exercise. Check: $w_{l+1} + U, \dots, w_n + U$ form a basis for V/U .

□

Corollary. If U is a proper subspace of V , V is finite dimensional, $\dim U < \dim V$.

Proof. $V/U \neq \{0\} \Rightarrow \dim V/U > 0 \Rightarrow \dim U < \dim V$

□

Definition. Let V be an \mathbb{F} -vector space, $U, W \leq V$ Then $V = U \oplus W$ (V is an internal direct sum of U and W) if every element of V can be written as $v = u + w, w \in W, u \in U$, uniquely.

W is a *direct complement* of U in V

Lemma. $U, W \leq V$. The following are equivalent

- (i) $V = U \oplus W$, ie. every element of V can be written uniquely as $u + w$, for $u \in U, w \in W$
- (ii) $V = U + W$ and $U \cap W = \{0\}$
- (iii) B_1 any basis of U , B_2 is any basis of W , then $B = B_1 \cup B_2$ is a basis of V .

Proof. (ii) \Rightarrow (i). Any $v \in V$ is $u + w$ for some $u \in U$, $w \in W$.

Suppose that

$$u_1 + w_1 = u_2 + w_2$$

Then

$$\Rightarrow u_1 - u_2 = -w_1 + w_2 \in U \cap W = \{0\} \Rightarrow w_1 = w_2, u_1 = u_2$$

Thus uniqueness of expressions.

(i) \Rightarrow (iii) B spans, any $v \in V$ is $u + w$, for some $u \in U$, $w \in W$, write u in terms of B_1 , w in terms of B_2 , Then $u + w$ is a lin comb. of elements of B .

B indep?

$$\begin{aligned} \sum_{v \in B} \lambda_v v = 0 &= 0_U + 0_W \\ \underbrace{\sum_{v \in B_1} \lambda_v v}_{\in U} + \sum_{v \in B_2} \lambda_v v &= 0 \end{aligned}$$

By uniqueness of expressions,

$$\sum_{v \in B_1} \lambda_v v = 0_U \quad \sum_{v \in B_2} \lambda_v v = 0_W$$

As B_1 and B_2 are basis, all of the λ_v are zero.

$$(iii) \Rightarrow (ii). \text{ If } v \in V, v = \sum_{x \in B} \lambda_x x = \underbrace{\sum_{u \in B_1} \lambda_u u}_{\in U} + \underbrace{\sum_{w \in B_2} \lambda_w w}_{\in W}$$

$$\Rightarrow v \in U + W.$$

If $v \in U \cap W$, $v = \sum_{u \in B_1} \lambda_u u$, $v = \sum_{w \in B_2} \lambda_w w \Rightarrow$ All λ_u, λ_w are zero, because $B_1 \cup B_2$ is lin. indep.

□

Lemma. Let V be an f -dim vector space. $U \leq V$. Then there exists a direct complement to U in V

Proof. Let u_1, \dots, u_l be a basis for U . Extend it to a basis for V ,

$$u_1, \dots, u_l, w_{l+1}, \dots, w_n$$

Then $\langle w_{l+1}, \dots, w_n \rangle$ is a direct complement of U .

□

Note! Direct compliments are not at all unique. In general, if you pick different ways of extending this you will get different direct compliments.

Pick $V = \mathbb{R}^2$. Pick U as the y -axis, then any one of the following green lines are direct compliments.:

Definition. Def $v_1, \dots, v_l \leq V$,

$$\sum V_i = V_1 + \dots + V_l = \{v_1 + \dots + v_l \mid v_i \in V_i\}$$

The sum is direct if

$$v_1 + \dots + v_l = v'_1 + \dots + v'_l \Rightarrow v_i = v'_i \text{ for all } i$$

(“unique expressions”)

Notation:

$$\bigoplus_{i=1}^l V_i$$

Exercise. $V_1, \dots, V_l \leq V$. TFAE

- (i) The sum $\sum V_i$ is direct
- (ii) $V_i \cap \sum_{j \neq i} V_j = \{0\}$ for all i
- (iii) The B_i are pairwise disjoint and their union is a basis for $\sum V_i$

We also discuss *external* direct sums, though will not touch them much in this course. This is simply an internal direct sum $U_1 \oplus U_2$, except now the U_i 's are not subspaces of V , they can be any old vector space.

Definition. Let U, W be \mathbb{F} -vector spaces. External direct sum

$$U \oplus V = \{(u, w) \mid u \in U, w \in W\}$$

$$\begin{aligned} \text{with } (u, w) + (x, y) &= (u + x, w + y), \\ \lambda(u, w) &= (\lambda u, \lambda w) \end{aligned}$$

Note that when we talk about dimension in this course, we have not shown yet that the dimension of an *infinite* vector space is well defined⁶. We will come to this later.

⁶It is!

2 Linear Maps

2.1 Linear Maps

Definition. V, W are \mathbb{F} -vector spaces. A map $\alpha : V \rightarrow W$ is linear if

- (i) $\alpha(v_1 + v_2) = \alpha(v_1) + \alpha(v_2)$
- (ii) $\alpha(\lambda v) = \lambda \alpha(v)$

Can be combined concisely as:

$$\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \alpha(v_1) + \lambda_2 \alpha(v_2) \quad \lambda_i \in \mathbb{F}, v_i \in V$$

Example. A $n \times m$ matrix with coeff in \mathbb{F}

$$\begin{aligned} \alpha : \mathbb{F}^n &\rightarrow \mathbb{F}^m \\ v &\mapsto Av \end{aligned}$$

Example. The set of all polynomials with real coefficients:

$$\mathcal{D} : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$$

$$f \mapsto \frac{df}{dx}$$

Example. The set of continuous functions over $[0, 1]$

$$I : \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1]$$

$$f \mapsto I(f)$$

$$\text{where } I(f)(x) = \int_0^x f(t) \, dt$$

Example. Fix $x \in [0, 1]$

$$\begin{aligned} \mathcal{C}[0, 1] &\rightarrow \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

Notes: If U, V, W are v spaces over \mathbb{F} , then

- (i) The identity map $\text{id} : V \rightarrow V$ is linear
- (ii) If $U \xrightarrow{\alpha} V \xrightarrow{\beta} W$ with α, β both linear, then $\beta \circ \alpha$ is linear.

Lemma. Let V, W be \mathbb{F} -vector spaces, and let \mathcal{B} is a basis for V . If $\alpha_0 : \mathcal{B} \rightarrow W$ is *any* map, then there exists a unique linear map⁷ $\alpha : V \rightarrow W$ extending α_0 , ie.

$$\alpha(v) = \alpha_0(v)$$

for any basis element $v \in \mathcal{B}$.

⁷ie. if I tell you *any* mapping of the basis vectors α_0 (it could be a non-linear mapping), then you have enough information to construct a linear map from this.

Proof. Let $v \in V$. Then $v = \sum \lambda_i v_i$, $v_i \in B$, $\lambda_i \in \mathbb{F}$, unique expression.

Now Linearity forces

$$\begin{aligned}\alpha(v) &= \alpha\left(\sum \lambda_i v_i\right) \\ &= \sum \lambda_i \alpha(v_i) \\ &= \sum \lambda_i \alpha_0(v_i)\end{aligned}$$

linear, exists. expression forced to be unique. \square

Note

- (i) True for infinite dimensional vector space also
- (ii) Very often, to define a linear map, define it on a basis and ‘extend linearly’
- (iii) Let $\alpha_1, \alpha_2 : V \rightarrow W$ be linear maps. If they agree on any basis, then they are equal.

Definition. (Isomorphism)

Let V, W be vector spaces over F . The map $\alpha : V \rightarrow W$ is an *isomorphism* if it is linear and bijective. Notation: $V \simeq W$

Lemma. \simeq is an equivalence notation on the set (score out set and write class) of all vector spaces over \mathbb{F} . That is,

- (i) $i_V : V \rightarrow V$ is an iso
- (ii) If $\alpha : V \rightarrow W$ is an iso, then the inverse map $\alpha^{-1} : W \rightarrow V$ is also linear, hence an iso.
- (iii) If

$$U \xrightarrow{\beta} V \xrightarrow{\alpha} W$$

then

$$U \xrightarrow{\beta \circ \alpha} W$$

is also an iso

Proof. (i) immediate

- (ii) α bijective $\Rightarrow \alpha^{-1}$ exists. Check: linear. $w_2 \in W, w_2 = \alpha(v_2)$, $v_2 \in V$, unique. $\alpha^{-1}(w_1 + w_2) = \alpha^{-1}(\alpha(v_1) + \alpha(v_2)) = \alpha^{-1}(\alpha(v_1 + v_2)) = v_1 + v_2 = \alpha^{-1}(w_1) + \alpha^{-1}(w_2)$.

Similarly, $\lambda \in \mathbb{F}$, $w \in W$,

$$\alpha^{-1}(\lambda w) = \lambda \alpha^{-1}(w)$$

- (iii) immediate \square

Theorem. If V vector space over \mathbb{F} of dimension n , then $V \simeq \mathbb{F}^n$.

Proof. Choose a basis \mathcal{B} for V , say v_1, \dots, v_n

$$V \rightarrow \mathbb{F}^n$$

$$\sum \lambda_i v_i \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \text{ is an iso}$$

□

Remark: Choosing an iso $V \simeq \mathbb{F}^n$ is equivalent to choosing a basis for V .

Theorem. V, W v spaces over \mathbb{F} , finite dim, are isomorphic iff they have the same dimension

Proof. (\Leftarrow) Both V and W are isomorphic

$$\mathbb{F}^{\dim V} = \mathbb{F}^{\dim W}$$

(\Rightarrow) Let $\alpha : V \rightarrow W$ iso, \mathcal{B} a basis for V .

Claim: $\alpha(\mathcal{B})$ is a basis for W .

Exercise. $\alpha(\mathcal{B})$ spans W because of surjectivity of α .

Exercise. $\alpha(\mathcal{B})$ lin indep: follows from injectivity of α .

□

Definition. (Null space/ Kernel of a linear map) Let $\alpha : V \rightarrow W$ be a linear map, the *null space* of α is given by

$$N(\alpha) = \ker \alpha = \{v \in V \mid \alpha(v) = \mathbf{0}\} \leq V$$

Definition. (Image of a linear map) Let $\alpha : V \rightarrow W$ be a linear map, the *image* of α is defined as:

$$\text{Im}(\alpha) = \{w \in W \mid w = \alpha(v), \text{ some } v \in V\} \leq W$$

Definition. (Injective map) α is injective if and only if $N(\alpha) = \{\mathbf{0}\}$

Definition. (Surjective map)⁸ α is surjective if and only if $\text{Im}(\alpha) = W$

Example. Let $\alpha : \mathcal{C}^\infty(\mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R})$ be defined by

$$\alpha(f)(t) = f''(t) + 2f'(t) - 5f$$

$$\ker \alpha \text{ is solns to } f'' + 2f' + 5f = 0$$

$$g \in \text{Im } \alpha \text{ if } \exists \text{ soln } f \text{ to } f'' + 2f' + 5f = g$$

⁸I mean, all definitions are iff statements really. Sometimes we leave it out and just use 'if'

2.2 The First Isomorphism Theorem

Theorem. (First Isomorphism Theorem) Let $\alpha : V \rightarrow W$ be a linear map. It induces an iso :

$$V/\ker \alpha \xrightarrow{\bar{\alpha}} \text{Im}(\alpha)$$

defined by

$$\bar{\alpha}(v + \ker \alpha) = \alpha(v)$$

Proof. (i) $\bar{\alpha}$ is well defined:

$$\begin{aligned} v + \ker \alpha &= v' + \ker \alpha \\ \iff v - v' &\in \ker \alpha \Rightarrow \alpha(v) \\ &= \alpha(v') \end{aligned}$$

(ii) $\bar{\alpha}$ is linear; immediate from linearity of α .

(iii) $\bar{\alpha}$ bijective?

$$\begin{aligned} \bar{\alpha}(v + \ker \alpha) &= \mathbf{0} \\ \Rightarrow \alpha(v) &= 0 \\ \Rightarrow v &\in \ker \alpha \end{aligned}$$

(iv) surjective: by defn of $\text{Im}(\alpha)$. □

Definition. (Rank and Nullity of a linear map) The *rank* of a linear map $r(\alpha) = rk(\alpha)$ is given by $\dim(\text{Im } \alpha)$, and the *nullity* $n(\alpha)$ is likewise given as $\dim(N(\alpha))$

Theorem. (Rank-nullity theorem) Let U, V be vector spaces over \mathbb{F} , $\dim_{\mathbb{F}} U < \infty$. Let $\alpha : U \rightarrow V$ linear. Then:

$$\dim U = r(\alpha) + n(\alpha)$$

Proof.

$$U/\ker \alpha \simeq \text{Im}(\alpha) \Rightarrow \dim(U) - \dim(\ker \alpha) = \dim(\text{Im}(\alpha))$$

□

Lemma. Let V, W be v spaces over \mathbb{F} , of equal finite dim. Let $\alpha : V \rightarrow W$ linear.

TFAE

(i) α injective

(ii) α surjective

(iii) α isomorphism

Definition. The space of linear maps from V to W is denoted by

$$L(V, W) = \{\alpha : V \rightarrow W \text{ linear}\}$$

Proposition. $L(V, W)$ is a v-space over \mathbb{F} under operators

- $(\alpha_1 + \alpha_2)(v) = \alpha_1(v) + \alpha_2(v)$ for all $\alpha_i \in L(V, W)$
- $(\lambda\alpha)(v) = \lambda(\alpha(v))$ for all $v \in V, \lambda \in \mathbb{F}$

If both V and W are finite dim, then so is $L(V, W)$ and $\dim(L(V, W)) = \dim(V) \times \dim(W)$.

Proof. $\alpha_1 + \alpha_2, \lambda\alpha$ defined above are well-defined linear maps. The v-space axioms are satisfied.

Claim about finite dim: See later □

2.3 Representation of Linear Maps by Matrices

Definition. An $m \times n$ matrix over \mathbb{F} is an array with m rows and n columns, entries in \mathbb{F} .

$$A = (a_{ij}), \quad a_{ij} \in \mathbb{F}, \quad 1 \leq i \leq m, 1 \leq j \leq n$$

$M_{m,n}(\mathbb{F})$ is the set of all such matrices

Proposition. $M_{m,n}(\mathbb{F})$ is an \mathbb{F} vector space, under operations

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

$$\lambda(a_{ij}) = (\lambda a_{ij})$$

$$\text{and } \dim(M_{m,n}(\mathbb{F})) = m \times n$$

Proof. v-space okay, see 1.1. And dim? A standard basis for $M_{m,n}(\mathbb{F})$ is

$$E_{ij} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

(ie a matrix of zeroes, with 1 in i^{th} row and j^{th} column)

$(a_{ij}) = \sum_{i,j} a_{ij} E_{ij}$, from which span and LI follows

This basis has cardinality mn □

Definition. (Coordinate Vectors)

Let V, W be v-spaces over \mathbb{F} , of finite dim, with $\alpha : V \rightarrow W$, linear. Basis \mathcal{B} for V , v_1, \dots, v_n basis \mathcal{C} for W , w_1, \dots, w_n . If $v \in V$, $v = \sum \lambda_i v_i$, write

$$[v]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{F}^n, \text{ called coordinate vector of } v \text{ wrt } \mathcal{B}. \text{ Similarly, } [w]_{\mathcal{C}} \in \mathbb{F}^m.$$

Definition. (Matrix) $[\alpha]_{\mathcal{B},\mathcal{C}}$ matrix of α wrt \mathcal{B} and \mathcal{C}

$$[\alpha]_{\mathcal{B},\mathcal{C}} = ([\alpha(v_1)]_{\mathcal{C}} \mid [\alpha(v_2)]_{\mathcal{C}} \mid \cdots \mid [\alpha(v_n)]_{\mathcal{C}}) \in M_{m,n}(\mathbb{F})$$

$$= (a_{ij})$$

The notation says $\alpha(v_j) = \sum \alpha_{ij} w_i$

Lemma. For any $v \in V$,

$$[\alpha(v)]_{\mathcal{C}} = [\alpha]_{\mathcal{B},\mathcal{C}} \cdot [v]_{\mathcal{B}}$$

where the dot denotes matrix applied to vector

Proof. Fix $v \in V$, $v = \sum_{j=1}^n \lambda_j v_j$, so $[v]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$

$$\begin{aligned} \alpha(v) &= \alpha\left(\sum \lambda_j v_j\right) = \sum \lambda_j \alpha(v_j) = \sum_j \lambda_j \left(\sum_i \alpha_{ij} w_i\right) \\ &= \sum_i \underbrace{\left(\sum_j \alpha_{ij} \lambda_j\right)}_{i^{\text{th}} \text{ entry of } [\alpha]_{\mathcal{B},\mathcal{C}} [v]_{\mathcal{B}}} w_i \end{aligned}$$

□

Lemma. Let α, β be linear maps, with $U \xrightarrow{\beta} V \xrightarrow{\alpha} W$ and $\alpha \circ \beta$ linear. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be basis for U, W, V reps. Then

$$[\alpha \circ \beta]_{\mathcal{A},\mathcal{C}} = \underbrace{[\alpha]_{\mathcal{B},\mathcal{C}}}_{=(a_{ij})} \circ \underbrace{[\beta]_{\mathcal{A},\mathcal{B}}}_{=(b_{ji})}$$

Proof.

$$\begin{aligned} (\alpha \circ \beta) \left(\overbrace{u_i}^{\text{in } \mathcal{A}} \right) &= \alpha(\beta(u_i)) = \alpha\left(\sum_j b_{ji} \overbrace{v_j}^{\text{in } \mathcal{B}}\right) \\ &= \sum_j b_{ji} \alpha(v_j) \\ &= \sum_j b_{ji} \sum_i a_{ij} \overbrace{w_i}^{\text{in } \mathcal{C}} \\ &= \sum_i \underbrace{\left(\sum_j a_{ij} b_{ji}\right)}_{(i,j)^{\text{th}} \text{ entry of } [\alpha]_{\mathcal{B},\mathcal{C}} [\beta]_{\mathcal{A},\mathcal{B}}} w_i \end{aligned}$$

□

Proposition. If V, W are v -spaces over \mathbb{F} with $\dim V = n$, $\dim W = m$, then $L(V, W) \simeq M_{m,n}(\mathbb{F})$

Proof. Fix bases

$$\mathcal{B} \text{ of } V : v_1, v_2, \dots, v_n$$

$$\mathcal{C} \text{ of } W : w_1, w_2, \dots, w_m$$

Claim:

$$L(v, w) \rightarrow M_{m,n}(\mathbb{F})$$

$$\alpha \mapsto [\alpha]_{\mathcal{B}, \mathcal{C}}$$

is an iso.

- θ linear $[\lambda_1 \alpha_1 + \lambda_2 \alpha_2]_{\mathcal{B}, \mathcal{C}} = \lambda_1 [\alpha_1]_{\mathcal{B}, \mathcal{C}} + \lambda_2 [\alpha_2]_{\mathcal{B}, \mathcal{C}}$
- θ surjective: given $A = (a_{ij})$. Let $\alpha : v_j \mapsto \sum_{i=1}^m a_{ij} w_i$, and extend linearly. Then $\alpha \in L(V, W)$, $b\theta(\alpha) = A$.
- θ injective, $[\alpha]_{\mathcal{B}, \mathcal{C}} = 0$ matrix $\Rightarrow \alpha$ is zero-map from V to W .

□

Corollary.

$$\dim(L(V, W)) = (\dim V)(\dim W)$$

Example. $\alpha : V \rightarrow W, Y \leq V, Z \leq W$. Say $\alpha(Y) \subseteq Z$.

Basis of V :

$$\mathcal{B} : \underbrace{v_1, \dots, v_k}_{\text{Basis for } Y, \mathcal{B}'}, v_{k+1}, \dots, v_n$$

Basis of W :

$$\mathcal{C} : \underbrace{w_1, \dots, w_l}_{\text{Basis for } Z, \mathcal{C}'}, w_{l+1}, \dots, w_m$$

Then

$$[\alpha]_{\mathcal{B}, \mathcal{C}} = \begin{pmatrix} A & \cdots & B_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_1 \end{pmatrix}$$

because for $1 \leq j \leq k$, $\alpha(v_j)$ is a lin combo of w_i , where $1 \leq i \leq l$.

And

$$[\alpha|_Y]_{\mathcal{B}', \mathcal{C}'} = A_1$$

Claim: α induces

$$\bar{\alpha} : V/Y \rightarrow W/Z$$

$$v + Y \mapsto \alpha(v) + Z$$

Well defined?

$$\begin{aligned} v_1 + Y = v_2 + Y &\Rightarrow v_1 - v_2 \in Y \\ &\Rightarrow \alpha(v_1 - v_2) \in Z \\ &\Rightarrow \alpha(v_1) + Z = \alpha(v_2) + Z \end{aligned}$$

Exercise. Linear from linearity of α

Basis for Y/V ,

$$\mathcal{B}'' : v_{k+1} + Y, \dots, v_n + Y$$

Basis for W/Z ,

$$\mathcal{B}'' : v_{k+1} + Y, \dots, v_n + Y$$

Exercise. $[\bar{\alpha}]_{\mathcal{B}'', \mathcal{C}''}$

2.4 Change of Basis

Let V and W be v -spaces over \mathbb{F} with the following basis

$$\begin{array}{cc} V & W \\ \mathcal{B} = \{v_1, \dots, v_n\} & \mathcal{C} = \{w_1, \dots, w_m\} \\ \mathcal{B}' = \{v'_1, \dots, v'_n\} & \mathcal{C}' = \{w'_1, \dots, w'_m\} \end{array}$$

Definition. The *change of basis matrix* from \mathcal{B} to \mathcal{B}' is $P = (p_{ij})$ given by $v'_j = \sum p_{ij} v_i$.

Equivalently,

$$P = \left(\begin{array}{ccc|c} [v'_1]_{\mathcal{B}} & [v'_2]_{\mathcal{B}} & \cdots & [v'_n]_{\mathcal{B}} \end{array} \right) = [\text{id}]_{\mathcal{B}', \mathcal{B}}$$

Lemma. $[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}$

Proof.

$$P[v]_{\mathcal{B}'} = [\text{id}]_{\mathcal{B}', \mathcal{B}}[v]_{\mathcal{B}'} = [v]_{\mathcal{B}}$$

□

Lemma. P is an invertible $n \times n$ matrix, and P^{-1} is the change of basis matrix from \mathcal{B} to \mathcal{B}'

Proof.

$$[\text{id}]_{\mathcal{B}, \mathcal{B}'}[\text{id}]_{\mathcal{B}', \mathcal{B}} = [\text{id}]_{\mathcal{B}', \mathcal{B}'} = I_n$$

$$[\text{id}]_{\mathcal{B}', \mathcal{B}}[\text{id}]_{\mathcal{B}, \mathcal{B}'} = [\text{id}]_{\mathcal{B}, \mathcal{B}} = I_n$$

□

Let Q be the change of basis matrix from \mathcal{C}' to \mathcal{C} . Q also invertible $m \times m$.

Proposition. Let $\alpha : V \rightarrow W$ linear, $A = [\alpha]_{\mathcal{B}, \mathcal{C}}$, $A' = [\alpha]_{\mathcal{B}', \mathcal{C}'}$. Then

$$A' = Q^{-1}AP$$

Proof.

$$\begin{aligned} Q^{-1}AP &= [\text{id}]_{\mathcal{C}, \mathcal{C}'} [\alpha]_{\mathcal{B}, \mathcal{C}} [\text{id}]_{\mathcal{B}', \mathcal{B}} \\ &= [\text{id} \circ \alpha \circ \text{id}]_{\mathcal{B}', \mathcal{C}'} \\ &= A' \end{aligned}$$

□

Definition. $A, A' \in M_{m,n}(\mathbb{F})$ are *equivalent* if $A' = Q^{-1}AP$ for some invertible $P \in M_{n,n}(\mathbb{F})$, $Q \in M_{m,m}(\mathbb{F})$

Note: this defines an equivalence relation on $M_{m,n}(\mathbb{F})$, eg. $A' = Q^{-1}AP$, $A'' = (Q^{-1})^{-1}A'P' \Rightarrow A'' = (QQ^{-1})^{-1}APP'$

Proposition. Let V, W be \mathbb{F} -vector spaces of dim n and m resp. Let $\alpha : V \rightarrow W$ be a linear map. Then there exists bases \mathcal{B} of V and \mathcal{C} of W , and some $r \leq m, n$ st.

$$[\alpha]_{\mathcal{B}, \mathcal{C}} = \begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

where I_r is the identity matrix.

Note: $r = \text{rank}(\alpha) = r(\alpha)$

Proof. Fix r st. $N(\alpha)$ has dim $n - r$. Fix a basis for $N(\alpha)$, say $v_{r+1}, v_{r+2}, \dots, v_n$. Extend this to a basis for V , say $\underbrace{v_1, \dots, v_r}_{\mathcal{B}}, v_{r+1}, \dots, v_n$. Now $\alpha(v_1), \dots, \alpha(v_r)$

is a basis for $\text{im}(\alpha)$.

– span: $\alpha(v_1), \dots, \alpha(v_r), \underbrace{\alpha(v_{r+1})}_{=0}, \dots, \underbrace{\alpha(v_n)}_{=0}$ certainly span $\text{im}(\alpha)$

– LI:

$$\begin{aligned} \sum_{i=1}^r \lambda_i \alpha(v_i) = \mathbf{0} &\Rightarrow \alpha \left(\underbrace{\sum_{i=1}^r \lambda_i v_i}_{\in \ker(\alpha)} \right) = \mathbf{0} \\ &\Rightarrow \sum_{i=1}^r \lambda_i v_i = \sum_{j=r+1}^n \mu_j v_j \text{ some } \mu_j \in \mathbb{F} \\ &\Rightarrow \text{as } v_1, \dots, v_n \text{ LI, } \lambda_i = \mu_j = 0 \forall i, j \end{aligned}$$

Extend $\alpha(v_1), \dots, \alpha(v_r)$ to a basis of W , say \mathcal{C} . By construction,

$$[\alpha]_{\mathcal{B}, \mathcal{C}} = \begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Remark: didn't need to assume in the proof that $r = r(\alpha)$. Can think of this as giving a different proof of the r-n theorem. \square

Corollary. Any $m \times n$ matrix is equivalent to $\begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ for some r .

Definition. Let $A \in M_{m,n}(\mathbb{F})$. The *column rank* of A is the dimension of the subspace of \mathbb{F}^m spanned by the columns of A . The *row rank* of A is the column rank of A^T (the dimension of the subspace of \mathbb{F}^n spanned by the row vectors of A).

Note: if α is a linear map represented by A wrt. any choice of basis, then $r(\alpha) = r(A)$, ie column rank = rank.

Proposition. Two $m \times n$ matrices A, A' are equivalent iff $r(A') = r(A)$.

Proof. (\Leftarrow) Both A and A' are equivalent to $\begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$, $r = r(A') = r(A)$

(this is a transitive relation)

(\Rightarrow) Let α be the linear map: $\alpha : \mathbb{F}^n \rightarrow \mathbb{F}^m$ represented by A wrt. the standard basis $A' = Q^{-1}AP$. P and Q invertible, so A' represents α wrt. two other bases. $r(\alpha)$ is defined in a basis invariant way, so $r = r(\alpha) = r(A) = r(A')$ \square

Theorem. $r(A) = r(A^T)$ ("row rank = column rank").

Proof. $Q^{-1}AP = \begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{m,n}$ where Q, P invertible

Take transpose of whole equation:

$$\begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n,m} = (Q^{-1}AP)^T = P^T A^T (Q^T)^{-1}$$

so A^T equiv to $\begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n,m}$. Thus $r(A) = r(A^T)$. \square

$V = W$, $\mathcal{C} = \mathcal{B}$, other basis \mathcal{B}' . P change of basis matrix from \mathcal{B} to \mathcal{B}' , $\alpha \in L(V, V)$.

$$[\alpha]_{\mathcal{B}', \mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}, \mathcal{B}}P$$

Definition. $A, A' \in M_{n,n}(\mathbb{F})$, A, A' are *similar* (or *conjugate*) if $A' = P^{-1}AP$ for some invertible P .

2.5 Elementary Matrices and Operations

Definition. *Elementary column operators* on an $m \times n$ matrix A :

- (i) swap columns i and j (wlog $i \neq j$)
- (ii) replace column i by λ (column i), $\lambda \neq 0$
- (iii) add λ (column i) to column j , $i \neq j$, $\lambda \neq 0$.

Elementary row operators analogous (replace ‘column’ by ‘row’)

Note: all of these operations are reversible.

Corresponding elementary matrices: effect of performing the column operations on $I_n = n \times n$ id. For row operations, I_m .

$$(i) \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & 1 & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & 0 \\ & & 1 & & & & & 1 \\ & & & & & & 0 & 1 \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix}$$

The zeros appear in row i , row j .

$$(ii) \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \lambda & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 \end{pmatrix}$$

with λ in the i^{th} row

- (iii) $I_n + \lambda E_{ij}$, where E_{ij} is defined as 1 in the (i, j) position and 0 everywhere else.

An elementary column operation on $A \in M_{m,n}(\mathbb{F})$ can be performed by multiplying A by the corresponding elementary matrix on the right.

Exercise.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

For row operations, multiply on the left

Exercise.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

Theorem. Any $m \times n$ matrix is equivalent to

$$\begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

for some r

Proof. Start with A . If all entries of A are 0, we're done ($r = 0$). If not, some $a_{ij} = \lambda \neq 0$.

- swap rows $1, i$
- swap columns $1, j$
- multiply column 1 by $\frac{1}{\lambda}$

to get 1 in position $(1, 1)$. Now

- add $(-a_{12})(\text{column } 1)$ to column 2.
- Similarly clear out all other entries in row 1.
- Also use row operations to clear out all other entries in column 1

Upshot: get

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \tilde{A} & & \\ 0 & & & \end{pmatrix}, \tilde{A} \in M_{m-1, n-1}(\mathbb{F})$$

$$\text{Now iterate, to get } \begin{pmatrix} I_r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \underbrace{E'_1, \dots, E'_k}_{\text{elem row}} \overset{Q^{-1}}{A} \underbrace{E_1, \dots, E_k}_P$$

Row/column ops are reversible \Rightarrow elem matrices are invertible. $Q : m \times m$ invertible, $P : n \times n$ invertible.

□

Variations:

If you use elementary row operations, can get the row echelon form of a matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & a \\ & & 1 & 0 & b \\ & & & 1 & c \end{pmatrix}$$

How? Assume $a'_{i1} = \lambda \neq 0$ some i .

- swap rows 1 and $i \Rightarrow$ get λ in $(1, 1)$
- divide row 1 by $\lambda \Rightarrow 1$ in $(1, 1)$
- use (iii)-type operation to clear out rest of column 1, then move on to second column etc.

Lemma. If A is $n \times n$ invertible, we can obtain I_n by using only elementary row operations (or elementary column operations).

Proof. Induction on number of rows

Suppose we have

$$\begin{pmatrix} 1 & 0 & 0 & & \\ & & 1 & & 0 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

There exists $j > k$ with $a_{k+1,j} = \lambda \neq 0$

If not,

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

with 1 in the $(k+1)$ th entry would not lie in the span of the column vectors, which would contradict invertability

- Swap columns $k+1$ and j
- Divide column $k+1$ by λ
- Use type 3 operators to clear the other entries of the $(k+1)$ th row.
- now proceed inductively

□

Upshot

$$AE_1E_2 \cdots E_l = I_n \Rightarrow A^{-1} = E_1E_2 \cdots E_l$$

one recipe for inverses.

Proposition. Any invertible matrix can be written as a product of elementary matrices.

3 Dual Spaces and Dual Maps

3.1 Dual Vector Spaces

Definition. Let V be a vector space over \mathbb{F} . The *dual vector space* V^* of V

$$V^* = L(V, \mathbb{F}) = \{ \alpha : V \rightarrow \mathbb{F} \text{ linear} \}$$

V^* is a vector space over \mathbb{F} . Its elements are sometimes called linear functionals.

Example. $V = \mathbb{R}^3$,

$$\theta : V \rightarrow \mathbb{R}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow a - c \quad \theta \in V^*$$

Example.

$$t_n : M_{m,n}(\mathbb{F}) \rightarrow \mathbb{F}$$

$$A \mapsto \sum_i A_{ii}, \quad t_n \in (M_{m,n}(\mathbb{F}))^*$$

Lemma. Let V be a vector space over \mathbb{F} with finite basis $\mathcal{B} = \{e_1, \dots, e_n\}$. Then there is a basis for V^* , given by $\mathcal{B}^* = \{\varepsilon_1, \dots, \varepsilon_n\}$ where

$$\varepsilon_j \left(\underbrace{\sum_{i=1}^m a_i e_i}_{\in V} \right) = a_j \quad 1 \leq j \leq n$$

\mathcal{B}^* is called the dual basis to \mathcal{B}

Proof. – LI

$$\begin{aligned} \sum_{j=1}^n \lambda_j \varepsilon_j = \mathbf{0} &\Rightarrow \left(\sum_{j=1}^n \lambda_j \varepsilon_j \right) e_i = \mathbf{0} \\ &= \sum_j \lambda_j \underbrace{\varepsilon_j(e_i)}_{\delta_{ij}} \end{aligned}$$

$$\Rightarrow \lambda_i = 0 \quad \forall i = 1, \dots, n$$

– Span: If $\alpha \in V^*$, then $\alpha = \sum_{i=1}^n \alpha(e_i) \varepsilon_i$
 (“linear maps are determined by their action on a basis”)

□

Corollary. If V is finite dim, then $\dim V = \dim V^*$

Remark: Sometimes useful to think about $(\mathbb{F}^n)^*$ as the space of row vectors of length n over \mathbb{F} . Suppose

V basis e_1, \dots, e_n

V^* dual basis $\varepsilon_1, \dots, \varepsilon_n$

$$x = \sum x_i e_i \in V$$

$$\alpha = \sum a_i \varepsilon_i \in V^*$$

$$\alpha(x) = \sum_{i=1}^n \alpha_i x_i = (a_1 \quad \dots \quad a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Definition. If $U \subseteq V$,

$$U^0 = \{\alpha \in V^* \text{ st. } \alpha(u) = 0 \text{ for all } u \in U\}$$

is the *annihilator* of U

Lemma. (i) $U^0 \leq V^*$

(ii) If $U \leq V$ and $\dim V = n < \infty$, then

$$\dim V = \dim U + \dim U^0$$

Proof. (i) $0 \in U^0$. If $\alpha, \alpha' \in U^0$, then $(\alpha + \alpha')(u) = \alpha(u) + \alpha'(u) = 0 + 0 = 0$, for $u \in U$ thus $\alpha + \alpha' \in U^0$

Similarly, $\lambda\alpha \in U^0$ for any $\lambda \in \mathbb{F}$

(ii) Let e_1, \dots, e_k be a basis for U . Extend to a basis for V . $e_1, \dots, e_k, e_{k+1}, \dots, e_n$.

Let \mathcal{B}^* be the dual basis to this. $\varepsilon_1, \dots, \varepsilon_n$

Claim: $\varepsilon_{k+1}, \varepsilon_{k+2}, \dots, \varepsilon_n$ is a basis for U^0

- If $i > k$, $\varepsilon_i(e_j) = 0$ where $j \leq k$, so ε_i (for $i > k$) is in U^0 .
- LI comes from the fact that \mathcal{B}^* is a basis. (So any subset of it is LI).
- Span? If $\alpha \in U^0$, then $\sum_{i=1}^n \alpha_i \varepsilon_i$, some $a_i \in \mathbb{F}$.

$$\left(\sum_{i=1}^n a_i \varepsilon_i \right) (e_j) = 0 \Rightarrow a_j = 0, \text{ any } j \leq k$$

where e_j is a basis element for U , for $j \leq k$

$$\Rightarrow \alpha \in \langle \varepsilon_{k+1}, \dots, \varepsilon_n \rangle$$

□

3.2 Dual Maps

Lemma. Let V, W be vector spaces over \mathbb{F} . Let $\alpha \in L(V, W)$. Then the map

$$\alpha^* : W^* \rightarrow V^*$$

$$\varepsilon \mapsto \varepsilon \circ \alpha \text{ is linear}$$

$$V \xrightarrow{\alpha} W \xrightarrow{\varepsilon} F$$

We'll call α^* the *dual* of α .

Proof. – $\varepsilon \circ \alpha$ is linear, so in V^* .

– α^* linear? Fix $\theta_1, \theta_2 \in W^*$

$$\begin{aligned} \alpha^*(\theta_1 + \theta_2) &= (\theta_1 + \theta_2) \circ \alpha \\ &= \theta_1 \circ \alpha + \theta_2 \circ \alpha \\ &= \alpha^* \theta_1 + \alpha^* \theta_2 \end{aligned}$$

Similarly, $\alpha^*(\lambda\theta) = \lambda\alpha^*\theta$ □

Proposition. Let V, W be v-spaces over \mathbb{F} , with basis \mathcal{B}, \mathcal{C} respectively. Let $\mathcal{B}^*, \mathcal{C}^*$ be the dual basis. Consider $\alpha \in L(V, W)$ with dual α^* .

$$[\alpha^*]_{\mathcal{C}^*, \mathcal{B}^*} = [\alpha]_{\mathcal{B}, \mathcal{C}}^T$$

Proof. Say $\mathcal{B} = \{b_1, \dots, b_n\}$, $\mathcal{C} = \{c_1, \dots, c_n\}$

$$\mathcal{B}^* = \{\beta_1, \dots, \beta_n\}, \mathcal{C}^* = \{\gamma_1, \dots, \gamma_n\}$$

$$\text{and } [\alpha]_{\mathcal{B}, \mathcal{C}} = (a_{ij}) \quad m \times n$$

$$\begin{aligned} \alpha^*(\gamma_r)(b_s) &= \gamma_r \circ \alpha(b_s) \\ &= \gamma_r(\alpha(b_s)) \\ &= \gamma_r\left(\sum_t a_{ts} c_t\right) \\ &= \sum_t a_{ts} \gamma_r(c_t) \\ &= a_{rs} \\ &= \left(\sum_i a_{ri} \beta_i\right)(b_s) \end{aligned}$$

$$\Rightarrow \alpha^*(\gamma_r) = \sum_i a_{ri} \beta_i$$

$$\Rightarrow [\alpha^*]_{\mathcal{C}^*, \mathcal{B}^*} = [\alpha]_{\mathcal{B}, \mathcal{C}}^T$$

□

Let V be a finite dim \mathbb{F} vector space.

Bases $\varepsilon = \{e_1, \dots, e_n\}$, $F = \{f_1, \dots, f_n\}$

Dual bases $\varepsilon^* = \{\varepsilon_1, \dots, \varepsilon_n\}$, $\mathcal{F} = \{\eta_1, \dots, \eta_n\}$

And let us consider $P = [\text{id}]_{\mathcal{F}\mathcal{E}}$

Lemma. Change of basis matrix from \mathcal{F}^* to \mathcal{E}^* is $(P^{-1})^T$

Proof.

$$[\text{id}]_{\mathcal{F}^*, \mathcal{E}^*} = [\text{id}]_{\mathcal{E}\mathcal{F}}^T = ([\text{id}]_{\mathcal{F}\mathcal{E}}^{-1})^T$$

□

CAUTION: $V \simeq V^*$ only if V is finite dimensional.

Let $V = \mathcal{P}$, the space of all real polynomials, with basis

$$p_j, j = 0, 1, 2, \dots \quad p_j(t) = t^j$$

Ex sheet 2 Q 9:

$$P^* \simeq \mathbb{R}^{\mathbb{N}}$$

$$\varepsilon \mapsto (\varepsilon(p_0), \varepsilon(p_1), \dots)$$

Ex sheet 1, Q3 g) $P \not\simeq \mathbb{R}^{\mathbb{N}}$ does NOT have a countable basis

Lemma. Let V, W be vector spaces over \mathbb{F} . Fix $\alpha \in L(V, W)$, let $\alpha^* \in L(W^*, V^*)$ be the dual map. Then

- (i) $N(\alpha^*) = (\text{Im}(\alpha))^0$ ie. α^* injective iff α is surjective
- (ii) $\text{Im}(\alpha^*) \leq (N(\alpha))^0$, with equality if V and W are finite dimensional. ie. α^* surjective iff α is injective

Proof. (i) Let $\varepsilon \in W^*$. Then

$$\begin{aligned} \varepsilon \in N(\alpha^*) &\iff \alpha^* \varepsilon = 0 \\ &\iff \varepsilon \circ \alpha = 0 \\ &\iff \varepsilon(u) = 0 \text{ for all } u \in \text{Im } \alpha \\ &\iff \varepsilon \in (\text{Im}(\alpha))^0 \end{aligned}$$

- (ii) Let $\varepsilon \in \text{Im } \alpha^*$. Then $\varepsilon = \alpha^* \varphi$, for some $\varphi \in W^*$.

For any $u \in N(\alpha)$,

$$\begin{aligned} \varepsilon(u) &= (\alpha^* \varphi)(u) \\ &= (\varphi \circ \alpha)(u) \\ &= \varphi(\alpha(u)) \\ &= \varphi(0) \\ &= 0 \end{aligned}$$

So $\varepsilon \in N(\alpha^0)$

Now use the fact that $\dim V, \dim W$ are finite.

$$\begin{aligned}
 \dim(\text{Im}(\alpha^*)) &= r(\alpha^*) \\
 &= r(\alpha) \quad \text{as } r(A) = r(A^T) \\
 &= \dim V - \dim N(\alpha) \quad \text{by R-N} \\
 &= \dim(N(\alpha))^0
 \end{aligned}$$

□

3.3 Double Duals

Definition. Let V be an \mathbb{F} vector space, $V^* = L(V, \mathbb{F})$ dual of V . Then the *double dual* of V is the dual of V^* , given by

$$V^{**} = L(V^*, \mathbb{F})$$

Theorem. If V is a finite dimensional vector space over \mathbb{F} , then the map

$$\begin{aligned}
 \hat{} : V &\rightarrow V^{**} \\
 v &\mapsto \hat{v}, \quad \hat{v}(\varepsilon) = \varepsilon(v)
 \end{aligned}$$

is an isomorphism

Proof. Firstly, for $v \in V$, the map $\hat{v} : V^* \rightarrow \mathbb{F}$ is linear, so $\hat{}$ does indeed give a map from V to V^{**}

– $\hat{}$ is linear. If $v_1, v_2 \in V, \lambda_1, \lambda_2 \in \mathbb{F}, \varepsilon \in V^*$.

$$\begin{aligned}
 \widehat{(\lambda_1 v_1 + \lambda_2 v_2)}(\varepsilon) &= \varepsilon(\lambda_1 v_1 + \lambda_2 v_2) \\
 &= \lambda_1 \varepsilon(v_1) + \lambda_2 \varepsilon(v_2) \\
 &= \lambda_1 \hat{v}_1(\varepsilon) + \lambda_2 \hat{v}_2(\varepsilon)
 \end{aligned}$$

– $\hat{}$ is injective: Let $e \in V \setminus \{\mathbf{0}\}$. Extend it to a basis of V , say e_1, e_2, \dots, e_n . Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be the dual basis for V^* .

$\hat{e}(\varepsilon) = \varepsilon(e) = 1$. So $\hat{e} \neq 0$.

Thus $N(\hat{}) = \{\mathbf{0}\}$, so $\hat{}$ is injective.

– V is finite dim, so $\dim V = \dim V^* = \dim V^{**}$.

Thus $\hat{}$ is an isomorphism

□

Lemma. Let V be a finite dim vector space over \mathbb{F} and $U \leq V$.

Then $\hat{U} = U^{00}$, so after identification of V with V^{**} , we have that $U^{00} = U$.

Proof. – First show $\hat{U} \leq U^{00}$.

$$\begin{aligned} u \in U &\Rightarrow \varepsilon(u) = 0 \quad \forall \varepsilon \in U^0 \\ &= \hat{u}(\varepsilon) = 0 \\ &\Rightarrow \hat{u} \in (U^0)^0 = U^{00} \end{aligned}$$

$$\begin{aligned} \dim U^{00} &= \dim V^* - \dim U^0 \\ &= \dim V - \dim U^0 \\ &= \dim U \end{aligned}$$

Thus $\hat{U} = U^{00}$

□

Lemma. Let V be a finite dim vector space of \mathbb{F} , Let $U_1, U_2 \leq V$. Then

$$(i) \quad (U_1 + U_2)^0 = U_1^0 \cap U_2^0$$

$$(ii) \quad (U_1 \cap U_2)^0 = U_1^0 + U_2^0$$

Proof. (i) Let $\theta \in V^*$

$$\begin{aligned} \theta \in (U_1 + U_2)^0 &\iff \theta(u_1 + u_2) = 0 \text{ for all } u_1 \in U_1, u_2 \in U_2 \\ &= \theta(u) = 0 \text{ for all } u \in U_1 \cap U_2 \\ &\theta \in U_1^0 \cap U_2^0 \end{aligned}$$

(ii) Apply annihilator to (i).

$$w_i = U_i^0 \quad u_i = W_i^0$$

$$\begin{aligned} (W_1^0 + W_2^0)^0 &= W_1 \cap W_2 \\ W_1^0 + W_2^0 &= (W_1 \cap W_2)^0 \end{aligned}$$

□

4 Bilinear Forms I

Definition. Let U, V be vector spaces over \mathbb{F} .

$$\varphi : U \times V \rightarrow \mathbb{F}$$

is *bilinear* or a *bilinear form* if its linear in both arguments

$$\varphi(u, -) : V \rightarrow \mathbb{F} \in V^* \quad \forall u \in U$$

$$\varphi(-, v) : U \rightarrow \mathbb{F} \in U^* \quad \forall v \in V$$

Example. (i) $V \times V^* \rightarrow \mathbb{F}$ with $(v, \theta) \mapsto \theta(v)$

(ii) $U = V = \mathbb{R}^n$ with $\varphi(x, y) = \sum_{i=1}^n x_i y_i$ for $x \in U, y \in V$

(iii) $A \in M_{m,n}(\mathbb{F})$ with $\varphi : \mathbb{F}^m \times \mathbb{F}^n \rightarrow \mathbb{F}, (u, v) \mapsto u^T A v$

(iv) (infinite dim) $U = V = C([0, 1], \mathbb{R})$ with $\varphi(f, g) = \int_0^1 f(t)g(t) dt$ for $f \in U, g \in V$

Definition. $\mathcal{B} = \{e_1, \dots, e_m\}$ basis for U ,

$\mathcal{C} = \{f_1, \dots, f_n\}$ basis for V

$\varphi : U \times V \rightarrow \mathbb{F}$ bilinear,

The matrix of φ wrt \mathcal{B} and \mathcal{C}

$$[\varphi]_{\mathcal{B}, \mathcal{C}} = (\varphi(e_i, f_j))$$

$m \times n$, i, j th entry

Lemma.

$$\varphi(u, v) = [u]_{\mathcal{B}}^T [\varphi]_{\mathcal{B}, \mathcal{C}} [v]_{\mathcal{C}}$$

Proof. Say $u = \sum \lambda_i e_i, v = \sum \mu_j f_j$

$$\begin{aligned} \varphi(u, v) &= \varphi\left(\sum \lambda_i e_i, \sum \mu_j f_j\right) \\ &= \sum_i \lambda_i \varphi(e_i, \sum_j \mu_j f_j) \\ &= \sum_{i,j} \lambda_i \varphi(e_i, f_j) \mu_j \end{aligned}$$

□

Note: $[\varphi]_{\mathcal{B}, \mathcal{C}}$ is the unique representation with this property

Note: $\varphi : U \times V \rightarrow \mathbb{F}$ bilinear, determines linear maps

$$\varphi_L U \rightarrow V^* \quad \text{and} \quad \varphi_R : V \rightarrow U^*$$

$$\varphi_L(u)(v) = \varphi(u, v) \quad \text{and} \quad \varphi_R(v)(u) = \varphi(u, v)$$

Lemma. $\mathcal{B} = \{e_1, \dots, e_m\}$ basis for U ,

dual $\mathcal{B}^* = \{\varepsilon_1, \dots, \varepsilon_m\}$ basis for U^* .

Similarly, $\mathcal{C} = \{f_1, \dots, f_n\}$ for V , $\mathcal{C}^* = \{\eta_1, \dots, \eta_n\}$ for V^*

If $[\varphi]_{\mathcal{B}, \mathcal{C}} = A$, then $[\varphi_R]_{\mathcal{C}, \mathcal{B}^*} = A, [\varphi_L]_{\mathcal{B}, \mathcal{C}^*} = A^T$

Proof.

$$\begin{aligned}\varphi_L(e_i)(f_j) &= A_{ij} \Rightarrow \varphi_L(e_i) = \sum_j A_{ij} \eta_j \\ \varphi_R(f_j)(e_i) &= A_{ij} \Rightarrow \varphi_R(f_j) = \sum_i A_{ij} \varepsilon_i\end{aligned}$$

□

Definition. Left kernel of $\varphi = \ker \varphi_L$, Right kernel of $\varphi = \ker \varphi_R$

Definition. φ is *non-degenerate* if $\ker \varphi_L = 0$ and $\ker \varphi_R = 0$. Otherwise φ is *degenerate*

Lemma. Let $U, \mathcal{B}, V, \mathcal{C}$ as before,

$$\varphi : U \times V \rightarrow \mathbb{F}$$

$$A = [\varphi]_{\mathcal{B}, \mathcal{C}}$$

assume $\dim U, \dim V$ finite. Then

$$\varphi \text{ non-degenerate} \iff A \text{ invertible}$$

Proof.

$$\begin{aligned}\varphi \text{ non-degenerate} &\iff \ker \varphi_L = \mathbf{0} \text{ and } \ker \varphi_R = \{\mathbf{0}\} \\ &\iff n(A^T) = 0 \text{ and } n(A) = 0 \\ &\iff r(A^T) = \dim V \text{ and } r(A) = \dim U \\ &\iff A \text{ invertible} \quad (\text{and necessarily } \dim U = \dim V)\end{aligned}$$

□

Corollary. If φ is non-degenerate and U and V are finite, then

$$\dim U = \dim V$$

Corollary. When U and V are finite dim, choosing a non-degenerate bilinear form $\varphi : U \times V \rightarrow \mathbb{F}$ is equivalent to picking an isomorphism $\varphi_L : U \rightarrow V^*$

Definition. For $T \subset U$, $T^\perp = \{v \in V \mid \varphi(t, v) = 0 \forall t \in T\} \leq V$
For $S \subset T$, ${}^\perp S = \{u \in U \mid \varphi(u, s) = 0 \forall s \in S\} \leq U$
(Generalisation of annihilators)

Proposition. U bases $\mathcal{B}, \mathcal{B}'$, $P = [\text{id}]_{\mathcal{B}', \mathcal{B}}$
 V bases $\mathcal{C}, \mathcal{C}'$ with $Q = [\text{id}]_{\mathcal{C}', \mathcal{C}}$
Let $\varphi : U \times V \rightarrow \mathbb{F}$ bilinear. Then

$$[\varphi]_{\mathcal{B}', \mathcal{C}'} = P^T [\varphi]_{\mathcal{B}, \mathcal{C}} Q$$

Proof.

$$\begin{aligned}
 \varphi_{u,v} &= [u]_{\mathcal{B}}^T [\varphi]_{\mathcal{B},\mathcal{C}} [v]_{\mathcal{C}} \\
 &= (P[u]_{\mathcal{B}'})^T [\varphi]_{\mathcal{B},\mathcal{C}} (Q[v]_{\mathcal{C}'}) \\
 &= [u]_{\mathcal{B}'}^T [\varphi]_{\mathcal{B},\mathcal{C}} [v]_{\mathcal{C}'}
 \end{aligned}$$

□

Definition. The rank of φ , $r(\varphi)$ is the rank of any matrix representing it (well-def) by prev thm.

Note: $r(\varphi) = r(\varphi_L) = r(\varphi_R)$

5 Determinant and Trace

5.1 Trace

Definition. For $A \in M_n(\mathbb{F})$ (this is $M_{n,n}(\mathbb{F})$), then

$$\text{tr}(A) = \sum_i A_{ii}$$

is the *trace* of A . This is a linear map.

Lemma. For $A, B \in M_n(\mathbb{F})$,

$$\text{tr}(AB) = \text{tr}(BA)$$

Proof.

$$\begin{aligned} \text{tr}(AB) &= \sum_i \sum_j a_{ij} b_{ji} \\ &= \sum_j \sum_i b_{ji} a_{ij} \\ &= \text{tr}(BA) \end{aligned}$$

□

Lemma. Similar (= conjugate) matrices have the same trace.

Proof. $B = P^{-1}AP$, $A, B \in M_n(\mathbb{F})$.

$$\begin{aligned} \text{tr}(B) &= \text{tr}(P^{-1}AP) \\ &= \text{tr}(APP^{-1}) \\ &= \text{tr}A \end{aligned}$$

□

Definition. If $\alpha : V \rightarrow V$ linear, define $\text{tr } \alpha = \text{tr}[\alpha]_{\mathcal{B}, \mathcal{B}}$. By the above, this is well defined.

Lemma. Let $\alpha : V \rightarrow V$ linear, and $\alpha^* : V^* \rightarrow V^*$ its dual. Then

$$\text{tr } \alpha = \text{tr } \alpha^*$$

Proof.

$$\begin{aligned} \text{tr } \alpha &= \text{tr}[\alpha]_{\mathcal{B}, \mathcal{B}} \\ &= \text{tr}[\alpha]_{\mathcal{B}, \mathcal{B}}^T \\ &= \text{tr}[\alpha^*]_{\mathcal{B}^*, \mathcal{B}^*} \\ &= \text{tr } \alpha^* \end{aligned}$$

□

5.2 Determinants

S_n = group of permutations of $\{1, \dots, n\}$

Define $\varepsilon_n : S_n \rightarrow \{-1, 1\}$ as

$$\varepsilon(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ product of even no. of transposes} \\ -1 & \text{if } \sigma \text{ product of odd no. of transposes} \end{cases}$$

Definition. Let $A \in M_n(\mathbb{F})$, $A = (a_{ij})$. Then

$$\det(A) = \sum_{\sigma \in S} \varepsilon(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

There are $n!$ summands, each is sign \times product of n elements (one for each row and each column).

Eg $n = 2$,

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \underbrace{a_{11}a_{22}}_{\sigma=\text{id}} - \underbrace{a_{12}a_{21}}_{\sigma=(12)}$$

Lemma. If $A = (a_{ij})$ is an upper triangular matrix (ie. $a_{ij} = 0$ if $i > j$) then $\det A = a_{11}a_{22} \cdots a_{nn}$. Similar for lower triangular matrices (ie. $a_{ij} = 0$ if $i < j$).

Proof.

$$\det(A) = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

For a summand to be non-zero, need $\sigma(j) \leq j \forall j$. Thus $\sigma = \text{id}$ □

Lemma.

$$\det(A) = \det(A^T)$$

Proof.

$$\begin{aligned} \det(A) &= \sum_{\sigma \in S_n} \varepsilon(\sigma) \times \prod_{i=1}^n a_{\sigma(i)i} \\ &= \sum_{\sigma \in S_n} \underbrace{\varepsilon(\sigma)}_{=\varepsilon^{-1}} \prod_{i=1}^n a_{i\sigma^{-1}(i)} \\ &= \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{i=1}^n a_{i\tau(i)} \quad (\sigma^{-1} = \tau) \\ &= \det(A^T) \end{aligned}$$

□

Definition. A *volume form* on \mathbb{F}^n is a function:

$$d : \underbrace{\mathbb{F}^n \times \mathbb{F}^n \times \cdots \times \mathbb{F}^n}_{n \text{ copies}} \rightarrow \mathbb{F}$$

such that

(i) d is *multilinear*: for any i and $v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n \in \mathbb{F}^n$,

$$d(v_1, v_2, \dots, v_{i-1}, -, v_{i+1}, \dots, v_n) \in (\mathbb{F}^n)^*$$

(ii) d is *alternating*: if $v_i = v_j$ for $i \neq j$, then $d(v_1, \dots, v_n) = 0$

Note that the notation we will use will look like

$$A = (a_{ij}) = (A^{(1)} \mid A^{(2)} \mid \dots \mid A^{(n)})$$

If $\{e_i\}$ is the standard basis for \mathbb{F}^n then

$$I = (e_1 \mid \dots \mid e_n)$$

Lemma.

$$\det : \mathbb{F}^n \times \dots \times \mathbb{F}^n$$

$$(A^{(1)}, \dots, A^{(n)}) \mapsto \det(A) \text{ is a volume form}$$

Proof. (i) Multilinear: For any fixed $\sigma \in S_n$, $\prod_{i=1}^n a_{\sigma(i)}$ contains exactly one term from each column, and so is multilinear. Now use the fact that the sum of multilinear functions is multilinear.

(ii) Alternating: Suppose $A^{(k)} = A^{(j)}$, for $j \neq k$. Let $\tau = (kj)$ transposition. $a_{ij} = a_{i\tau(j)} \forall i, j \in \{1, \dots, n\}$, $S_n = A_n \sqcup \tau A_n$, where \sqcup is disjoint union.

$$\begin{aligned} \det(A) &= \sum_{\sigma \in A_n} \prod a_{i\sigma(i)} - \sum_{\sigma \in \tau A_n} \prod a_{i\sigma(i)} \\ &= \sum_{\sigma \in A_n} \prod a_{i\sigma(i)} - \sum_{\sigma \in A_n} \prod a_{i\sigma(i)} \\ &= 0 \end{aligned}$$

□

Lemma. Let d be a volume form. Then swapping two entries changes the sign.

$$d(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -d(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

Proof.

$$\begin{aligned} 0 &= d(v_1, \dots, v_{i-1}, v_i + v_j, v_{i+1}, \dots, v_{j-1}, v_j + v_i, v_{j+1}, \dots, v_n) \\ &= d(v_1, \dots, v_i, \dots, v_i, \dots, v_n) + d(v_1, \dots, v_j, \dots, v_i, \dots, v_n) \\ &\quad + d(v_1, \dots, v_i, \dots, v_j, \dots, v_n) + d(v_1, \dots, v_j, \dots, v_j, \dots, v_n) \\ &= d(v_1, \dots, v_j, \dots, v_i, \dots, v_n) + d(v_1, \dots, v_i, \dots, v_j, \dots, v_n) \end{aligned}$$

□

Corollary. If $\sigma \in S_n$, $d(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \varepsilon(\sigma)d(v_1, \dots, v_n)$

Theorem. Let d be a volume form on \mathbb{F}^n . $A = (A^{(1)} \mid \dots \mid A^{(n)})$. Then

$$d(A^{(1)}, \dots, A^{(n)}) = \det(A) \times d(e_1, \dots, e_n)$$

Proof.

$$\begin{aligned}
d(A_1, \dots, A^n) &= d\left(\sum_{i=1}^n a_{ij} e_i, A^{(2)}, \dots, A^{(n)}\right) \\
&= \sum_{i=1}^n a_{i1} d(e_i, A^{(2)}, \dots, A^{(n)}) \\
&= \sum_i \sum_j a_{i1} a_{j2} d(e_i, e_j, A^{(3)}, \dots, A^{(n)}) \\
&= \sum_{i_1, \dots, i_n} \prod_{k=1}^n a_{ik} \underbrace{d(e_{i_1}, e_{i_2}, \dots, e_{i_n})}_{\substack{0 \text{ unless all of } i_k \text{ are distinct}^9}} \\
&= \sum_{i_1, \dots, i_n} \prod_{k=1}^n A_{\sigma(k)k} \underbrace{d(e_{\sigma(1)}, \dots, e_{\sigma(n)})}_{= \varepsilon(\sigma) d(e_1, \dots, e_n)}
\end{aligned}$$

□

Corollary. \det is the unique volume form s.t.

$$d(e_1, \dots, e_n) = 1$$