Part IB — Methods Example Sheet 1

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QUESTION 1

We have $\nabla^2\phi = 0$ on $0 < x < a$, $0 < y < b$, $0 < z < c$ with $\phi = 1$ on the z surface and $\phi = 0$ on all other surfaces:

Assume $\phi(x, y, z) = X(x)Y(y)Z(z)$, so we have

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Solving $X'' = -\lambda_p X$ such that $X(0) = X(a) = 0$ implies that

$$\lambda_p = \frac{p^2\pi^2}{a^2}, X_l = \sqrt{\frac{2}{a}}\sin\left(\frac{p\pi x}{a}\right), l = 1, 2, 3, \cdots$$

Similarly, solving $Y'' = -\mu_q Y$, such that $Y(0) = Y(b) = 0$ implies that

$$\mu_q = \frac{q^2\pi^2}{b^2}, Y_q = \sqrt{\frac{2}{b}}\sin\left(\frac{q\pi x}{b}\right), m = 1, 2, 3, \cdots$$

Now solving for Z using the eigenvalues:

$$Z'' = \left(\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2}\right)Z,$$

$$Z = \alpha\cosh\left[\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)^{1/2}\pi z\right] + \beta\sinh\left[\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)^{1/2}\pi z\right]$$

Therefore, the general solution is

$$\psi(x, y, z) = \frac{2}{\sqrt{ab}}\sum_{p=0}\sum_{q=0} a_{pq}\sin\left(\frac{p\pi}{a}x\right)\sin\left(\frac{q\pi}{b}y\right)\sinh\left[\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)^{1/2}\pi z\right]$$

QUESTION 2

QUESTION 3

QUESTION 4

QUESTION 5

QUESTION 6

QUESTION 7

QUESTION 8

QUESTION 9

QUESTION 10