Part IB — Complex Methods Example Sheet 3

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QUESTION 1

Know that

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\ \mathrm{d}t, \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t}\ \mathrm{d}\omega$$

First, let $f(t) = e^{-a|t|}$. Then

$$\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^{\infty} e^{-a|t|}e^{-i\omega t}\ \mathrm{d}t \\
&= \int_{-\infty}^{0} e^{at}e^{-i\omega t}\ \mathrm{d}t + \int_{0}^{\infty} e^{-at}e^{-i\omega t}\ \mathrm{d}t \\
&= \frac{1}{a-i\omega}\left[e^{at}e^{-i\omega t}\right]_{-\infty}^{0} + \frac{1}{-a-i\omega}\left[e^{-at}e^{-i\omega t}\right]_{0}^{\infty} \\
&= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} \\
&= \frac{2a}{a^2+\omega^2}
\end{aligned}$$

So using the inverse Fourier transform relation we have

$$\begin{aligned}
e^{-a|t|} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2}e^{i\omega t}\ \mathrm{d}\omega \\
\iff \frac{\pi}{a}e^{-a|t|} &= \int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2}e^{i\omega t}\ \mathrm{d}\omega
\end{aligned}$$

as required.
Next, let $f(t) = e^{-at}\sin btH(t)$. Then

$$\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^{\infty} e^{-at}\sin bte^{-i\omega t}H(t)\ \mathrm{d}t \\
&= \int_{0}^{\infty} e^{-at}e^{-i\omega t}\sin bt\ \mathrm{d}t \\
&= \int_{0}^{\infty} e^{-at}e^{-i\omega t}\frac{i}{2}\left(e^{-ibt} - e^{ibt}\right)\ \mathrm{d}t \\
&= \frac{i}{2}\int_{0}^{\infty} e^{-at}e^{-i(\omega+b)t} - e^{-at}e^{-i(\omega-b)t}\ \mathrm{d}t \\
&= \frac{i}{2}\left(-\frac{1}{-a-i(\omega+b)} - \frac{-1}{-a-i(\omega-b)}\right) \\
&= \frac{i}{2}\left(\frac{1}{a+i\omega+ib} - \frac{1}{a+i\omega-ib}\right) \\
&= \frac{i(-ib)}{(a+i\omega)^2+b^2} \\
&= \frac{b}{(a+i\omega)^2+b^2}
\end{aligned}$$

So using the inverse Fourier transform relation we have

$$
\begin{aligned}
e^{-at}\sin bt H(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b}{(a+i\omega)^2+b^2}e^{i\omega t}\,\mathrm{d}\omega \\
\iff 2\pi e^{-at}\sin bt H(t) &= \int_{-\infty}^{\infty}\frac{b}{(a+i\omega)^2+b^2}e^{i\omega t}\,\mathrm{d}\omega
\end{aligned}
$$

as required.
Not sure about the $a<0$, $a=0$ parts.

QUESTION 2

Given

$$f(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}a \\ 0 & \text{otherwise} \end{cases}$$

We compute the Fourier transform as

$$\begin{aligned} \tilde{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x \\ &= \int_{-a/2}^{a/2} e^{-ikx} \, \mathrm{d}x \\ &= \left[\frac{1}{-ik} e^{-ikx} \right]_{-a/2}^{a/2} \\ &= \frac{i}{k} \left(e^{-ika/2} - e^{ika/2} \right) \\ &= \frac{2}{k} \sin\left(\frac{ak}{2} \right) \end{aligned}$$

as required. Similarly, given

$$g(x) = \begin{cases} a - |x| & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases}$$

We compute the Fourier transform as

$$\begin{aligned} \tilde{g}(k) &= \int_{-\infty}^{\infty} g(x) e^{-ikx} \, \mathrm{d}x \\ &= \int_{-a}^{a} (a - |x|) e^{-ikx} \, \mathrm{d}x \\ &= \underbrace{\int_{-a}^{a} a e^{-ikx} \, \mathrm{d}x}_{(1)} - \left(\underbrace{\int_{-a}^{0} -x e^{-ikx} \, \mathrm{d}x}_{(2)} + \underbrace{\int_{0}^{a} x e^{-ikx} \, \mathrm{d}x}_{(3)} \right) \\ &= \frac{2a}{k} \sin ak + \frac{a}{ik}(e^{ika} + e^{-ika}) - \frac{1}{(ik)^2} \left[1 - e^{ika} \right] + \frac{1}{(ik)^2} \left[e^{-ika} - 1 \right] \\ &= \end{aligned}$$

Now

$$\begin{aligned} (1) &= a \left[\frac{1}{-ik} e^{-ikx} \right]_{-a}^{a} \\ &= a \frac{i}{k} \left(e^{-ika} - e^{ika} \right) \\ &= \frac{2a}{k} \sin ak, \end{aligned}$$

$$
\begin{aligned}
(2) &= \left[-x\left(-\frac{1}{ik}e^{-ikx}\right) - \int -\frac{1}{-ik}e^{-ikx}\right]_{-a}^{0} \\
&= \left[-\frac{a}{ik}e^{ika} + \frac{1}{(ik)^2}\left[e^{-ikx}\right]_{-a}^{0}\right] \\
&= -\frac{a}{ik}e^{ika} - \frac{1}{k^2}\left(1 - e^{ika}\right)
\end{aligned}
$$

$$
\begin{aligned}
(3) &= \left[x\left(-\frac{1}{ik}e^{-ikx}\right) - \int -\frac{1}{ik}e^{-ikx}\right]_{0}^{a} \\
&= \left[-\frac{a}{ik}e^{-ika} - \frac{1}{(ik)^2}\left[e^{-ikx}\right]_{0}^{a}\right] \\
&= -\frac{a}{ik}e^{-ika} + \frac{1}{k^2}\left(e^{-ika} - 1\right)
\end{aligned}
$$

QUESTION 3

The convolution of $g(x) = e^{-|x|}$ with itself is given by

$$g * g(x) = \int_{-\infty}^{\infty} e^{-|x-y|}e^{-|y|}\ \mathrm{d}y$$

If $x > 0$ then we can split up the integral as

$$\begin{aligned} g * g(x) &= \int_{-\infty}^{0} e^{x-y}e^{y}\ \mathrm{d}y + \int_{0}^{x} e^{x-y}e^{-y}\ \mathrm{d}y + \int_{x}^{\infty} e^{-(x-y)}e^{-y}\ \mathrm{d}y \\ &= \end{aligned}$$

Similarly if $x < 0$ then
If $x > 0$ then we can split up the integral as

$$\begin{aligned} g * g(x) &= \int_{-\infty}^{x} e^{x-y}e^{y}\ \mathrm{d}y + \int_{x}^{0} e^{-(x-y)}e^{y}\ \mathrm{d}y + \int_{0}^{\infty} e^{-(x-y)}e^{-y}\ \mathrm{d}y \\ &= \end{aligned}$$

Next, the convolution theorem for Fourier transforms states that $\mathcal{F}[g*g(x)] = \mathcal{F}[g]\mathcal{F}[g]$. Applied here, we have

$$\int_{-\infty}^{\infty} (1+|x|)e^{-|x|}e^{-ikx}\ \mathrm{d}x = \left[\int_{-\infty}^{\infty} e^{-|x|}e^{ikx}\ \mathrm{d}x\right]^2$$

QUESTION 4

QUESTION 5

Starting with

$$\mathcal{L}(1) = \frac{1}{s}$$

(i) By shifting,

$$\mathcal{L}(e^{-2t}) = \frac{1}{s+2}$$

(ii) Starting with shifting,

$$\mathcal{L}(e^{-3t}) = \frac{1}{s+3}$$

Then

$$\begin{aligned}\mathcal{L}(t^3 e^{-3t}) &= -\frac{\mathrm{d}^3}{\mathrm{d}s^3}\mathcal{L}(e^{-3t}) \\ &= -\frac{\mathrm{d}^3}{\mathrm{d}s^3}\frac{1}{s+3} \\ &= \frac{6}{(s+3)^4}\end{aligned}$$

(iii) Write $\sin 4t = \frac{1}{2i}(e^{4it} - e^{-4it})$. Then

$$\begin{aligned}\mathcal{L}(e^{3t}\sin 4t) &= \frac{1}{2i}\mathcal{L}(e^{(3+4i)t}) - \frac{1}{2i}\mathcal{L}(e^{(3-4i)t}) \\ &= \frac{1}{2i}\left[\frac{1}{s-(3+4i)} - \frac{1}{s-(3-4i)}\right] \\ &= \frac{4}{(s-3)^2+16}\end{aligned}$$

(iv) Writing $\cosh 4t = \frac{1}{2}(e^{4t} + e^{-4t})$, we have

$$\begin{aligned}\mathcal{L}(e^{-4t}\cosh 4t) &= \frac{1}{2}\mathcal{L}(1) + \frac{1}{2}\mathcal{L}(e^{-8t}) \\ &= \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+8}\right] \\ &= \frac{s+4}{s(s+8)}\end{aligned}$$

(v) First by shifting,

$$\mathcal{L}(e^{-(t+1)}) = e^{-1}\mathcal{L}(e^{-t}) = \frac{e^{-1}}{s+1}$$

Then by translation,

$$\mathcal{L}(e^{-t}H(t-1)) = \frac{e^{-(s+1)}}{s+1}$$

QUESTION 6

By partial fractions we have

$$\hat{f}(s) = \frac{s+3}{(s-2)(s^2+1)} = \frac{1}{s-2} - \frac{s+1}{s^2+1}$$

Have

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2}, \quad \mathcal{L}(\cos t) = \frac{s}{s^2+1}, \quad \mathcal{L}(sint) = \frac{1}{s^2+1}$$

Hence by linearity the inverse Laplace transform of $\hat{f}(s)$ is given by

$$f(t) = e^{2t} - \cos t - \sin t$$

Alternatively we can use the Bromwich inversion formula,

$$f(t) = \sum_{k=1}^{n} \operatorname*{res}_{p=p_k} (\hat{f}(p)e^{pt}),$$

as $\hat{f}(s)$ has only a finite number of singularities, namely $s = 2, i, -i$, and $\hat{f}(s) \to 0$ as $|s| \to \infty$. First,

$$\begin{aligned} \operatorname*{res}_{s=2} \left(\frac{s+3}{(s-2)(s^2+1)} e^{st} \right) &= \lim_{s\to 2} \left(\frac{s+3}{(s^2+1)} e^{st} \right) \\ &= e^{2t} \end{aligned}$$

Next,

$$\begin{aligned} \operatorname*{res}_{s=i} \left(\frac{s+3}{(s^2+1)} e^{st} \right) &= \lim_{s\to i} \left(\frac{s+3}{(s-2)(s+i)} e^{st} \right) \\ &= \frac{i+3}{2i(i-2)} e^{it} \\ &= -\frac{i+3}{4i+2)} e^{it} \\ &= \frac{-10+10i}{20} e^{it} = \frac{-1+i}{2} (\cos t + i \sin t) \end{aligned}$$

Similarly,

$$\begin{aligned} \operatorname*{res}_{s=-i} \left(\frac{s+3}{(s^2+1)} e^{st} \right) &= \lim_{s\to -i} \left(\frac{s+3}{(s-2)(s-i)} e^{st} \right) \\ &= \frac{-i+3}{2i(i+2)} e^{-it} \\ &= \frac{-10-10i}{20} e^{-it} = \frac{-1-i}{2} (\cos t - i \sin t) \end{aligned}$$

Hence adding the residues we achieve

$$f(t) = e^{2t} - \cos t - \sin t$$

in agreement with what we had above.

QUESTION 7

Consider the differential equation

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} - 3\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} - y = t^2 e^t$$

$$y(0) = 1, \dot{y}(0) = 0, \ddot{y}(0) = -2$$

Taking the Laplace transform of this equation, where $\mathcal{L}(y) = \hat{y}$, we have

$$\begin{aligned}\mathcal{L}(\dot{y}) &= p\hat{y} + y(0)\\ &= p\hat{y} + 1\end{aligned}$$

Similarly

$$\begin{aligned}\mathcal{L}(\ddot{y}) &= p\mathcal{L}(\dot{y}) + \dot{y}(0)\\ &= p^2\hat{y} + py(0) + \dot{y}(0)\\ &= p^2\hat{y} + p\end{aligned}$$

and,

$$\begin{aligned}\mathcal{L}\left(\frac{\mathrm{d}^3 y}{\mathrm{d}t^3}\right) &= p\mathcal{L}(\ddot{y}) + \ddot{y}(0)\\ &= p^3\hat{y} + p^2 y(0) + p\dot{y}(0) + \ddot{y}(0)\\ &= p^3\hat{y} + p^2 + -2\end{aligned}$$

The term on the RHS gives

$$\begin{aligned}\mathcal{L}(t^2 e^t) &= \frac{\mathrm{d}^2}{\mathrm{d}p^2}\mathcal{L}(e^t)\\ &= \frac{\mathrm{d}^2}{\mathrm{d}p^2}\left(\frac{1}{p-1}\right)\\ &= \frac{2}{(p-1)^3}\end{aligned}$$

Thus substituting in gives:

$$(p^3 - 3p^2 + 3p - 1)\hat{y} + p^2 - 3p + 1 = \frac{2}{(p-1)^3}$$

Hence

$$\hat{y} = \frac{2}{(p-1)^6} - \frac{1}{(p-1)} + \frac{p}{(p-1)^3}$$

We know $\mathcal{L}(e^t) = 1/(p-1)$. For the other two terms, we will use the Bromwich inversion formula from the previous question, noting both tend to zero as $|p| \to \infty$.

$$f(t) = \sum_{k=1}^{n} \operatorname*{res}_{p=p_k} (\hat{f}(p)e^{pt}),$$

For the first term, the singularity $p = 1$ has a pole of order 6. Hence the residue here is given by

$$\begin{aligned}
\lim_{p\to 1} \frac{1}{5!}\frac{\mathrm{d}^5}{\mathrm{d}p^5}(p-1)^5\left(\frac{2}{(p-1)^6}e^{pt}\right) &= \lim_{p\to 1} \frac{1}{5!}\frac{\mathrm{d}^5}{\mathrm{d}p^5}\left(-2e^{pt}(1-p)^{-1}\right) \\
&= \lim_{p\to 1} \frac{1}{5!}\frac{\mathrm{d}^5}{\mathrm{d}p^5}\left[-2e^{pt}\left(1+p+p^2+p^3+\cdots\right)\right] \\
&= \cdots
\end{aligned}$$

The last term has a singularity at $p = 3$ with a pole of order 3 there; the residue is given by

$$\begin{aligned}
\lim_{p\to 1} \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}p^2}(p-1)^2\left(\frac{p}{(p-1)^3}e^{pt}\right) &= \lim_{p\to 1} \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}p^2}\left(\frac{p}{1-p}e^{pt}\right) \\
&= \lim_{p\to 1} \frac{1}{2!}\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{(1-p)^2}e^{pt} + \frac{p^2}{(1-p)}e^{pt}\right) \\
&= \lim_{p\to 1} \frac{1}{2!}\left(\frac{2}{(1-p)^3}e^{pt} + \frac{p}{(1-p)^2}e^{pt} + \frac{1-p^2}{(1-p)^2}e^{pt} + \frac{p^3}{(1-p)}e^{pt}\right)
\end{aligned}$$

Not sure how to take limit.

QUESTION 8

The differential equation we are investigating is

$$\ddot{y} - 2\dot{y} - 2y = \delta(t) - \delta(t - t_0)$$

Note that

$$\mathcal{L}(\delta(t)) = \int_0^\infty \delta(t)e^{\text{pt}}\,\text{d}t = 1, \quad \mathcal{L}(\delta(t - t_0)) = e^{-pt_0}$$

Taking the Laplace transform of the equation gives

$$p^2\hat{y} + 2(p\hat{y}) + 2\hat{y} = 1 + e^{-pt_0}$$

which gives

$$\hat{y} = \frac{1 + e^{-pt_0}}{(p^2 + 2p + 2)}$$

QUESTION 9

Taking the Laplace transform of the equation gives

$$\hat{f} + 4\frac{1}{p}\hat{f} = \frac{1}{p^2}$$

Thus

$$\hat{f} = \frac{1}{p^2 + 4p} = \frac{1}{(p+2)^2 - 4}$$

QUESTION 10

QUESTION 11

(whenever I can be bothered) http://www.robots.ox.ac.uk/ jmb/lectures/pdelecture4.pdf

QUESTION 12