Part IA — Markov Chains Example Sheet 2

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QUESTION 1

Suppose there exists some state $i \in S, i \neq s$, st. i is recurrent. Then

$$\mathbb{P}_i(T_i < \infty) = 1 \text{ where } T_i := \min\{n \geq 1 : X_n = i\}$$

But $i \to s$, so by definition there exists some $r > 0$ st. $p_{i,s}(r) > 0$, and as s is absorbing, $p_{s,i}(n) = 0 \ \forall \ n \geq 0$. Hence there is a non-zero probability of being 'trapped' in s, ie. a non-zero probability that $T_i = \infty$. Hence, for all $i \in S$, $\mathbb{P}_i(T_i < \infty) < 1$, ie. i is transient.

QUESTION 2

Let P denote the transition matrix in question. The characteristic equation $\det(P - \kappa\iota) = 0$ gives roots $\kappa_1 = 1$, $\kappa_2 = 1 - 2p$, $\kappa_3 = 1 - 4p$.

Thus

$$P^n = U^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-2p)^n & 0 \\ 0 & 0 & (1-4p)^n \end{pmatrix} U$$

for some invertible matrix U, and so

$$p_{1,1}(n) = A + B(1-2p)^n + C(1-4p)^n$$

Using $p_{1,1}(0) = 1, p_{1,1}(1) = 1 - 2p, p_{1,1}(2) = (1-2p)^2 + 2p^2$ gives

$$\begin{aligned} A + B + C &= 1 \\ A + B(1-2p) + C(1-4p) &= 1 - 2p \\ A + B(1-2p)^2 + C(1-4p)^2 &= 1 - 4p + 6p^2 \end{aligned}$$

Setting $p = \frac{1}{2}$ in the second equation, $p = \frac{1}{4}$ in the third yields

$$\begin{aligned} A + B + C &= 1 \\ A - C &= 0 \\ A + B/4 &= 3/8 \end{aligned}$$

which gives $A = C = 1/4$, $B = 1/2$, thus

$$p_{1,1}(n) = \frac{1}{4} + \frac{1}{2}(1-2p)^n + \frac{1}{4}(1-4p)^n$$

We have $1 \leftrightarrow 2 \leftrightarrow 3$ so the chain is irreducible (all states recurrent of all states transient). Thus as $n \to \infty$, $p_{1,1} \to 1/4$, so state 1 is recurrent. hence all states are recurrent.

QUESTION 3

Using the symmetry of the problem, the invariant probabilities are $\pi_i = 1/2^3$ for all vertices x. For a finite irreducible MC, the mean recurrent time μ_i to state i is

$$\mu_i = \frac{1}{\pi_i}$$

Thus the expected number of steps until the particle first returns to v is $2^3 = 8$.

Should be a way to do the next two using invariant distributions but I can't see it. Let e_i be the expected number of steps to reach w given we are i steps away from w. Hence

$$\begin{aligned}
e_0 &= 0 \\
e_1 &= 1 + \frac{1}{4}e_1 + \frac{1}{2}e_2 \\
e_2 &= 1 + \frac{1}{2}e_1 + \frac{1}{4}e_2 + \frac{1}{4}e_3 \\
e_3 &= 1 + \frac{3}{4}e_2 + \frac{1}{4}e_3
\end{aligned}$$

Solving gives $e_3 = 40/3$, the expected number of steps to reach v starting in v.

For the last part: as the chain is irreducible and positive recurrent, the mean number of visits to state w between two consecutive visits to state v equals $\pi_w/\pi_v = 8/8 = 1$.

QUESTION 4

Suppose after one step the particle is in state j, $a_j > 0$. Now the particle can only travel to state j or $j-1$; once it is in state $j-1$, it can only travel to state $j-1$ or $j-2$, and so on... eventually it returns to state 0, where it is then sent to some other state k st. $a_k > 0$. In other words, we can only travel to a higher state by doing so from the origin. Motivated by this, we define

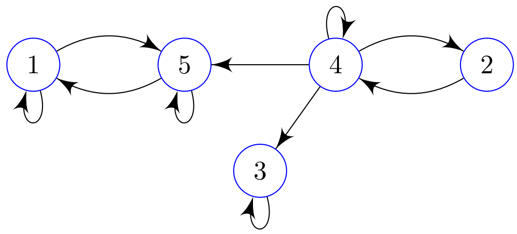
$$J = \sup\{j \mid a_j > 0\}$$

It is now easy to see that all states $\geq J$ are transient; after we leave them (which we will do eventually), we can never return there.

Not sure how to find the mean recurrent times.

QUESTION 5

Possible transitions of the chain are illustrated below:



The communicating classes are $C_1 = \{1,5\}, C_2 = \{3\}$ and $C_3 = \{2,4\}$. The classes C_1 and C_3 are not closed, but C_2 is closed. We know that inside communicating classes, every state is recurrent or every state is transient. It is easy to see from the diagram that

$$C_1 \text{ recurrent }, C_2 \text{ recurrent}, C_3 \text{ transient}$$

QUESTION 6

- There is a non zero probability that a MC beginning in a transient state will never return to that state
- There is a guarantee that a process beginning in a recurrent state will return to that state.

QUESTION 7

Collapsing the tree into a random walk on $\mathbb{Z}^+$, where the root of the tree R is represented by 0, with two branches extending to 1, four extending to 2 etc.

The walker moves rightwards with probability 2/3 and leftwards with probability 1/3, at at R moves rightwards with probability 1.

It is now intuitively obvious that this random walk is transient. For a concrete proof, we will argue that the state 0 is transient, and as the MC is irreducible, all states are transient. By the usual arguments, the probability of return in $2n$ steps is given by

$$p_0(2n) = \binom{2n}{n}\left(\frac{2}{3}\right)^n\left(\frac{1}{3}\right)^n$$

Using Stirling's formula: $n! \sim (n/e)^n\sqrt{2\pi n}$ as $n \to \infty$, we have

$$\begin{aligned} p_0(2n) &= \frac{(2n)!}{(n!)^2}\left(\frac{2}{3}\right)^n\left(\frac{1}{3}\right)^n \\ &\sim \frac{2^{3n}}{3^{2n}\sqrt{n\pi}} \\ &= \left(\frac{8}{9}\right)^n\frac{1}{\sqrt{n\pi}} \end{aligned}$$

We can conclude $\sum_{n=0}^{\infty} p_0(2n) < \infty$ as this series can be shown to converge using the ratio test.

QUESTION 8

The walk is at the origin $\mathbf{0} = (0, 0, 0, 0)$ at time $2n$ if and only if it has taken equal number of steps negative and positive in each dimension (ie. in 1D, equal number right and left). Therefore,

$$p_{\mathbf{0},\mathbf{0}}(2n) = \left(\frac{1}{8}\right)^{2n} \sum_{i_1,\cdots,i_4} \frac{(2n)!}{(i_1!i_2!i_3!i_4!)^2}$$

where $i_1 + i_2 + i_3 + i_4 = n$
Thus

$$\begin{aligned} p_{\mathbf{0},\mathbf{0}}(2n) &= \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \sum_{i_1,\cdots,i_4} \left(\frac{n!}{4^n i_1!i_2!i_3!i_4!}\right)^2 \\ &\leq \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} M \sum_{i_1,\cdots,i_4} \frac{n!}{4^n i_1!i_2!i_3!i_4!} \qquad (*) \end{aligned}$$

where

$$M = \max\left\{\frac{n!}{4^n i_1!i_2!i_3!i_4!} \ : \ i_1 \cdots, i_4 \geq 0, \sum_{r=1}^{4} i_r = n\right\}$$

It is not difficult to see that the maximum is attained when i, j and k are all closest to $\frac{1}{3}n$, so that

$$M \leq \frac{n!}{4^n(\lfloor \frac{1}{4}n \rfloor!)^4}$$

Furthermore, the final summation in (*) equals 1, since the summand is the probability that, in allocating n balls randomly to four urns, the urns contain $i_1, \cdots, i_4$ balls respectively. It follows that

$$p_{\mathbf{0},\mathbf{0}}(2n) \leq \frac{(2n)!}{16^n n!(\lfloor \frac{1}{4}n \rfloor!)^4}$$

which, by Stirling's formula, is strictly no bigger than Cn^{-2}, for some constant C. Therefore:

$$\sum_{n=0}^{\infty} p_{\mathbf{0},\mathbf{0}}(2n) < \infty$$

implying that the origin $\mathbf{0}$ is transient.

Shorter way: project onto $\mathbb{Z}^3$ by discarding all coordinates except the first 3. Now we have a new possibility of the random walk X_n^{proj} staying where it is with probability $\frac{4-3}{4} = \frac{1}{4}$ (when the original walk jumps in one of the discarded directions), and when it jumps,

$$\mathbb{P}(X_n^{\text{proj}} = \mathbf{i} + \mathbf{e}_\alpha \mid X_n^{\text{proj}} = \mathbf{i}) = \frac{1/8}{1 - 1/4} = \frac{1}{6} \quad \alpha = 1, 2, 3$$

where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$.

Clearly, if the original walk is recurrent then the projected walk is too. But we see when the chain jumps, it behaves like the simple symmetric random walk on $\mathbb{Z}^3$, which we know is transient. Hence the projected walk is transient, and so must the original chain be.

QUESTION 9

Setting $\pi = \pi P$ reveals

$$\begin{aligned} \frac{1}{2}\pi_1 + \frac{1}{2}\pi_5 &= \pi_1 \\ \frac{1}{2}\pi_2 + \frac{1}{4}\pi_4 &= \pi_2 \\ \pi_3 + \frac{1}{4}\pi_4 &= \pi_3 \\ \frac{1}{2}\pi_2 + \frac{1}{4}\pi_4 &= \pi_4 \\ \frac{1}{2}\pi_1 + \frac{1}{4}\pi_4 + \frac{1}{2}\pi_5 &= \pi_5 \end{aligned}$$

which gives immediately $\pi_4 = 0 = \pi_2$, $\pi_1 = \pi_5$. Thus any vector of the form $\pi = (\pi_1, 0, \pi_3, 0, \pi_1)$ st. $2\pi_1 + \pi_3 = 1$ is an invariant distribution

QUESTION 10

Let X_n be the number of molecules in A after n epoch of time. $X = \{X_n \mid n \geq 0\}$ as a Markov chain, owing to the independence of the choice for the passing molecule from previous events. We have $p_{i,i+1} = (N-i)/N$, $p_{i,i-1} = i/N$.

The detailed balance equations

$$\pi_i p_{i,i+1} = \pi_{i+1} p_{i+1,i}$$

are solved by $\pi_{i+1} = \pi_i (N-i)/(i+1)$, thus

$$\pi_N = \frac{1 \times 2 \times \cdots \times (N-1) \times N}{N \times (N-1) \times \cdots \times 2 \times 1} \pi_0 = \frac{N!}{N!} \pi_0 = \pi_0$$

So for some $0 < M < N$,

$$\begin{aligned} \pi_M = \frac{N-(M-1)}{M} \pi_{M-1} &= \frac{(N-(M-1)) \times (N-(M-2)) \times \cdots \times N}{M \times (M-1) \times \cdots \times 1} \pi_0 \\ &= \frac{N!}{(N-M)!M!} \pi_0 = \binom{N}{M} \pi_0 \end{aligned}$$

As π_i is a distribution, must have $\sum_i \pi_i = 1$, ie.

$$\pi_0 \sum_{i=0}^{N} \binom{N}{i} = 1$$

Thus $\pi_0 = 2^{-N}$, and

$$\pi_i = 2^{-N} \binom{N}{i}, \qquad i = 0, 1, \cdots, N$$

ie $\pi_i \sim \text{Bin}\ (N, 1/2)$

QUESTION 11

QUESTION 12

Equations yield

$$\begin{aligned}\pi_1 &= p\pi_3 \\ \pi_2 &= \pi_1 + \frac{2}{3}\pi_2 + (1-p)\pi_3 \\ \pi_3 &= \frac{1}{3}\pi_2\end{aligned}$$

Hence the invariant distribution is of the form $C(p/3, 1, 1/3)$ for different normalisation constants C depending on the value of p. For $p = 1/16$, we obtain

$$\pi = \left(\frac{1}{65}, \frac{48}{65}, \frac{16}{65}\right)$$

For $p = 1/6$,

$$\pi = \left(\frac{1}{25}, \frac{18}{25}, \frac{6}{25}\right)$$

and for $p = 1/12$,

$$\pi = \left(\frac{1}{49}, \frac{36}{49}, \frac{12}{49}\right)$$

We can see that the first entry of the invariant distribution in each case is exactly what we would obtain letting $n \to \infty$ in the calculation of of $p_{1,1}(n)$ in the previous example sheet.

QUESTION 13

QUESTION 14

(a) Detailed balance equations are

$$\pi_1(1-p) = \pi_2 q$$

Hence reversible.

(b) Detailed balance equations are

$$\begin{aligned}\pi_1 p &= \pi_2(1-p)\\ \pi_2 p &= \pi_3(1-p)\\ \pi_3 p &= \pi_1(1-p)\end{aligned}$$

Hence reversible, except when $p = \frac{1}{2}$

(c) Have

$$\pi_i = \pi_{i+1}(1-p) \Rightarrow \pi_i = \frac{p^{i+1}}{(1-p)^i}\pi_0$$

Hence reversible.

(d) Detailed balance equations give $\pi_i = \frac{1}{n}$ for each i. Thus reversible.

QUESTION 15

Detailed balance equations give

$$\pi_i \lambda_i = \pi_{i+1}\mu_{i+1} \Rightarrow \pi_i = \frac{\lambda_{i-1}\cdots\lambda_0}{\mu_i\cdots\mu_1}\pi_0$$

Hence reversible.