Part IB — Linear Algebra Sheet 3

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QUESTION 1

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

This matrix has characteristic polynomial

$$\chi_{A_1}(\lambda) = (\lambda - 1)^2(\lambda - 2)$$

For $\lambda = 2$ eigenvectors satisfy

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$$

So $\mathbf{v} = (2, 2, 1)$, and we take this as a basis for the $\lambda = 2$ eigenspace. Similarly for $\lambda = 1$ we have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$$

This implies $v_2 = v_3 = 0$, so eigenvector must be of the form $(1, 0, 0)$, again a basis with one element.

A_2: Next, we note that $\chi_{A_1}(\lambda) = \chi_{A_2}(\lambda)$ as the determinant calculation expanding down the first column will remain unchanged, so same eigenvalues. We can see that for $\lambda = 2$, $\mathbf{v} = (1, 2, 1)$ is a basis. For $\lambda = 1$ we have $v_2 = v_3$, so an eigenvector basis is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Next, $\chi_{A_3}(\lambda) = (\lambda - 1)(\lambda - 2)^2$. For $\lambda = 1$ the eigenspace basis is $\{(1, 1, 1)\}$, for $\lambda = 2$ it is $\{(1, -2, 1)\}$.

QUESTION 2

Consider $\det(A - \kappa\iota)$.

Add all the columns to the first column; it becomes a column where all entries are equal to $\lambda - \mu + (n-1)$. Now subtract first row from all others. We are left with $\det(A - \kappa\iota) = (\lambda - \mu + (n-1))\det(M)$ where M is an $n-1 \times n-1$ lower triangular matrix with $\lambda - \mu - 1$ in every diagonal entry.

Hence

$$\det(A - \kappa\iota) = (\lambda - \mu + (n-1))(\lambda - \mu - 1)^{n-1}$$

So n eigenvalues are given as

$$\mu = \lambda + n - 1, \underbrace{\lambda - 1, \cdots, \lambda - 1}_{n-1 \text{ times}}$$

QUESTION 3

Define $\pi_j = q_j(\alpha) : V \to V$ by

$$q_j(\alpha) = \prod_{i \neq j}^{k} \frac{\alpha - \lambda_i}{\lambda_j - \lambda_i}$$

QUESTION 4

QUESTION 5

QUESTION 6

QUESTION 7

QUESTION 8

QUESTION 9

QUESTION 10

QUESTION 11

QUESTION 12