

Part IB — Linear Algebra

Based on lectures by Dr. Keating

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0 Introduction

1 Vector Spaces

Definition. An \mathbb{F} -Vector space (a vector space on \mathbb{F}) is an abelian group $(V, +)$ equipped with a function¹ $F \times V \rightarrow V$, $(\lambda, v) \mapsto \lambda v$

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$$

$$(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$$

$$\lambda(\mu v) = \lambda\mu v$$

$$1v = v$$

$$v + \mathbf{0} = v$$

for all $\lambda_i, \mu \in F$, $v_i \in V$

Note that we will not be underlining our vectors, as this is cumbersome here. We will however be using $\mathbf{0}$ to denote the zero vector.

Example. For all $n \in \mathbb{N}$, \mathbb{F}^n = space of column vectors of length n , entries in \mathbb{F} . We understand the definition as entry-wise addition, entry-wise scalar multiplication

Example. $M_{m,m}(\mathbb{F})$, the set of $m \times m$ matrices with entries in \mathbb{F}

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

again all operations defined entry-wise

Example. For any set X , $\mathbb{R}^X = \{f : X \rightarrow \mathbb{R}\}$ Addition and scalar multiplication defined pointwise $= f_1(x) + f_2(x)$.

Exercise. Show that the above examples satisfy the axioms

Proposition. $0v = \mathbf{0}$ for all $v \in V$.

Proof. $((0+0)v = 0v \iff 0v + 0v = 0v \iff 0v = \mathbf{0})$

□

Exercise. Show² that $(-1)v = -v$

Definition. Let V be an \mathbb{F} -vector space. A subset U of V is a subspace ($U \leq V$) if:

- (i) $\mathbf{0} \in U$
- (ii) $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$ “ U is closed under addition...”
- (iii) $u \in U$, any $\lambda \in \mathbb{F} \Rightarrow \lambda u \in U$ “...and scalar multiplication”

Exercise. If U is a subspace of V , then U is also an \mathbb{F} -vector space.

¹scalar multiplication

²Hint: Use the previous proposition

Example. Let $V = \mathbb{R}^{\mathbb{R}}$, then $f : \mathbb{R} \rightarrow \mathbb{R}$. The set of all continuous functions $C(\mathbb{R})$ are a subspace. An even smaller subspace is the set of all polynomials.

Exercise. Define $U \subseteq \mathbb{R}^3$ as:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad a_1 + a_2 + a_3 = t$$

for some constant t . Check that this is a subspace of \mathbb{R}^3 if and only if $t = 0$.

Proposition. Let V be an F -vector space, $U, W \leq V$. Then $U \cap W \leq V$.

Proof. (i) $0 \in U, 0 \in W \Rightarrow 0 \in U \cap W$

(ii) Suppose $u, v \in U \cap W, \lambda, \mu \in F$. U is a subspace $\Rightarrow \lambda u + \mu v \in U$. Similarly $\lambda u + \mu v \in W$, so it is in the intersection. \square

Example. $V = \mathbb{R}^3, U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0 \right\}, W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = 0 \right\}$ then $U \cap W = U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0, y = 0 \right\}$ (intersect along the z -axis)

Note: union of family of subspaces is almost never a subspace itself.

Definition. Let V be an F -vector space, $U, W \leq V$. The *sum* of U and W is the set:

$$U + W = \{u + w \mid u \in U, w \in W\}$$

Proposition. $U + W \leq V$

Proof. $0 \in U, W \Rightarrow 0 + 0 = 0 \in U + W$

$u_1, u_2 \in U, w_1, w_2 \in W,$

$$(u_1 + w_1) + (u_2 + w_2) = \underbrace{(u_1 + u_2)}_{\in U} + \underbrace{(w_1 + w_2)}_{\in W}$$

Similarly for scalar multiplication (ex.) \square

Note: $U + W$ is the smallest subspace containing both U and W . (This is because all elements of the form $u + w$ are forced to be in such a subspace by the “closed under addition” axiom)

Definition. V is an \mathbb{F} -vector space, $U \leq V$. The quotient space³ V/U is the abelian group V/U equipped with scalar multiplication;

$$F \times V/U \rightarrow V/U$$

$$(\lambda, v + U) \mapsto \lambda v + U$$

³think of this as the collection of cosets of U in V

Proposition. This is well-defined, and V/U is an F -vector space.

Proof. Well-defined: Suppose $v_1 + U = v_2 + U \in V/U$. $\Rightarrow (v_2 - v_1) \in U \Rightarrow (\lambda v_2 - \lambda v_1) \in U \Rightarrow \lambda v_2 + U = \lambda v_1 + U \in V/U$

To show that it is an \mathbb{F} -vector space, we must show that the axioms hold. These follow from the axioms of V . $\lambda(\mu(v + U)) = \lambda(\mu v + U) = \lambda(\mu v) + U = (\lambda\mu)v + U = \lambda(\mu v + U)$ (scalar multiplication on V/U).

Ex. Other axioms follow similarly from using vector space axioms

□

Definition. V is an \mathbb{F} -vector space, $S \subset V$. The *span* of S is denoted by

$$\langle S \rangle = \left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{F} \right\}$$

ie. the set of all finite linear combinations, all but finitely many of the λ_s are zero.

Remark: $\langle S \rangle$ is the smallest subspace of V which contains⁴ all of the elements of S

Convention: $\langle \emptyset \rangle = \{\mathbf{0}\}$.

Example. $V = \mathbb{R}^3$,

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \right\}$$

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$$\langle S \rangle = \left\{ \begin{pmatrix} a \\ b \\ 2b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

ie. we have taken linear combinations of the first two. We don't need the third one.

Example. For X a set,

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$\langle \delta_x \mid x \in X \rangle = \{f \in \mathbb{R}^X \mid f \text{ has finite support}\}$$

$$= \langle x \in X \mid f(x) \neq 0 \rangle$$

Definition. S spans V if $\langle S \rangle = V$

Definition. V is *finite dimensional* over \mathbb{F} if it is spanned by a set that is finite.

⁴This is essentially a tautology

Definition. The vectors v_1, \dots, v_n are *linearly independent* over \mathbb{F} if

$$\sum_{i=1}^n \lambda_i v_i = 0 \Rightarrow \lambda_i = 0 \text{ for all } i$$

some coefficients $\lambda_i \in \mathbb{F}$. $S \subset V$ is linearly independent if every finite subset of it is.

Example. The first example, u, v, w are not linearly independent.

Example. The set $\{\delta_X \mid x \in X\}$ is linearly independent.

Definition. If *not* linearly independent, say a set is linearly dependent.

Definition. S is a *basis* of V if it is linearly independent and spans V

Example. \mathbb{F}^n standard basis: e_1, e_2, \dots, e_n .

Example. $V = \mathbb{C}$ over \mathbb{C} has natural basis $\{1\}$, over \mathbb{R} has natural basis $\{1, i\}$

Example. $V = \mathcal{P}(\mathbb{R})$ space of all polynomials, has natural basis

$$\{1, x, x^2, x^3, \dots\}$$

Exercise. Check this carefully

Lemma. V is an \mathbb{F} -vector space. The vectors v_1, \dots, v_n form a basis of V iff each vector $v \in V$ has a unique expression

$$v = \sum_{i=1}^n \lambda_i v_i, \text{ with } \lambda_i \in \mathbb{F}$$

Proof. (\Rightarrow) Fix $v \in V$. The v_i span, so

$$\exists \lambda_i \in \mathbb{F} \text{ s.t. } v = \sum \lambda_i v_i$$

Suppose also $v = \sum \mu_i v_i$ for some $\mu_i \in \mathbb{F}$. $\sum (\mu_i - \lambda_i) v_i = \mathbf{0}$.

The v_i are linearly independent so $\mu_i - \lambda_i = 0$ for all i , $\lambda_i = \mu_i$

(\Leftarrow) The v_i span V , since any $v \in V$ is a linear combination of them. IF $\sum_{i=1}^n \lambda_i v_i = \mathbf{0}$. Note that $\mathbf{0} = \sum_{i=1}^n 0 v_i$. By uniqueness (applied to $\mathbf{0}$), $\lambda_i = 0$ for all i . \square

Lemma. If v_1, \dots, v_n span V (over \mathbb{F}), then some subset of v_1, \dots, v_n is a basis for V (over \mathbb{F}).

Proof. If v_1, \dots, v_n linearly independent, done. Otherwise for some l , there exist $\alpha_1, \dots, \alpha_{l-1} \in \mathbb{F}$ such that

$$v_l = \alpha_1 v_1 + \dots + \alpha_{l-1} v_{l-1}$$

(If $\sum \lambda_i v_i = \mathbf{0}$, not all $\lambda_i = 0$. Take l maximal with $\lambda_i \neq 0$, just $\alpha_i = -\lambda_i / \lambda_l$).

Now $v_2, \dots, v_{l-1}, v_{l+1}, \dots, v_n$ still span V . Continue iteratively until get linear independence. \square

Theorem. (Steinitz exchange lemma) Let V be a finite dimensional vector space over \mathbb{F} . Take v_1, \dots, v_m to be linearly independent w_1, \dots, w_n to span V .

Then $m \leq n$, and reordering the spanning set if needed,

$$v_1, \dots, v_m, w_{m+1}, \dots, w_n$$

span V .

Proof. (Induction) Suppose that we've replaced $l(\geq 0)$ of the w_i . Reordering the w_i if needed, $v_1, \dots, v_l, w_{l+1}, \dots, w_n$ span V .

If $l = m$, done.

If $l < m$, then

$$v_{l+1} = \sum_{i=1}^l \alpha_i v_i + \sum_{i>l} \beta_i w_i$$

$\alpha_i, \beta_i \in \mathbb{F}$. As the v_i are lin. indep, $\beta_i \neq 0$ for some i . (After reordering, $\beta_{l+1} \neq 0$).

$$w_{l+1} = \frac{1}{\beta_{l+1}} \left(v_{l+1} - \sum_{i \leq l} \alpha_i v_i - \sum_{i>l+1} \beta_i w_i \right)$$

This $v_1, \dots, v_{l+1}, w_{l+2}, \dots, w_n$ also spans V . After m steps, w_i will have replaced m of the w_i by v_i . Thus $m \leq n$. \square

Theorem. If V is a finite dimensional vector space over \mathbb{F} , then any two bases for V have the same number of elements. This is what we call the *dimension* of V , denoted $\dim_{\mathbb{F}} V$.

Proof. If $\{v_1, \dots, v_n\}$ is a basis and w_1, \dots, w_m is another basis, the $\{v_i\}$ span and $\{w_i\}$ is linearly independent' so by Steinitz $m \leq n$. Likewise, $n \leq m$. \square

Example. $\dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{R}} \mathbb{C} = 2$

Theorem. V , finite dim, v -space over \mathbb{F} . If w_1, \dots, w_l is a linearly independent set of vectors, we can extend it to a basis $w_1, \dots, w_l, v_{l+1}, \dots, v_n$

Proof. Apply Steinitz to w_1, \dots, w_l (lin indep) and any basis v_1, \dots, v_n .

Or directly, if $V = \langle w_1, \dots, w_l \rangle$, stop.

Otherwise take $v_{l+1} \in V \setminus \langle w_1, \dots, w_l \rangle$, now w_1, \dots, w_l, v_{l+1} is linearly indep. iterate \square

Corollary. Suppose V is a finite dimensional vector space, with dimension n .

- (i) Any linearly independent set of vectors has at most n elements with equality iff it's a basis
- (ii) Any spanning set of vectors must have at least n elements, with equality if and only if it's a basis.

Slogan "Choose the best basis for the job"

Theorem. Let U, W be subspaces of V . If U, W are finite dim, so is $U + W$ and $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

Proof. Pick basis v_1, \dots, v_l of $U \cap W$. Extend it to basis $v_1, \dots, v_l, u_1, \dots, u_m$ of U . Extend it to basis $v_1, \dots, v_l, w_1, \dots, w_n$ of W .

Claim: $v_1, \dots, v_l, u_1, \dots, u_m, w_1, \dots, w_n$ is a basis for $U + W$.

- (i) Span: $u \in U$, then $u = \sum \alpha_i v_i + \sum \beta_i u_i$, $\alpha_i, \beta_i \in \mathbb{F}$ $w \in W$, then $w = \sum \gamma_i v_i + \sum \delta_i w_i$, $\gamma_i, \delta_i \in \mathbb{F}$

$$u + w = \sum (\alpha_i + \gamma_i) v_i + \sum (\beta_i + \delta_i) u_i$$

- (ii) lin indep: $u = \sum \alpha_i v_i + \sum \beta_i u_i + \sum \gamma_i w_i = \mathbf{0}$

$$\Rightarrow u = \underbrace{\sum \alpha_i v_i + \sum \beta_i u_i}_{\in U} = - \underbrace{\sum \gamma_i w_i}_{\in W} \in U \cap W$$

This is equal to $\sum \delta_i v_i$ for some $\delta_i \in \mathbb{F}$ because v_i are basis for $U \cap W$.

As v_i and w_i are lin indep, $(*) \Rightarrow \gamma_i = \delta_i = 0$ for all i .

$\Rightarrow \sum \alpha_i v_i + \sum \beta_i u_i = 0 \Rightarrow \alpha_i = \beta_i = 0$ because v_i and u_i form a basis for U .

□

Theorem. Let V be a finite dim \mathbb{F} -vector space, $U \leq V$, then U and V/U are also of finite dim, and

$$\dim V = \dim U + \dim V/U$$

Proof.

Exercise. Show that U is finite dim.

Let u_1, \dots, u_l be a basis for U . Extend it to a basis for V . Say $u_1, \dots, u_l, w_{l+1}, \dots, w_n$ of V .

Exercise. Check: $w_{l+1} + U, \dots, w_n + U$ form a basis for V/U .

□

Corollary. If U is a proper subspace of V , V is finite dimensional, $\dim U < \dim V$.

Proof. $V/U \neq \{\mathbf{0}\} \Rightarrow \dim V/U > 0 \Rightarrow \dim U < \dim V$

□

Definition. Let V be an \mathbb{F} -vector space, $U, W \leq V$. Then $V = U + \oplus W$ (V is an internal direct sum of U and W) if every element of V can be written as $v = u + w$, $w \in W, u \in U$, uniquely.

W is a *direct complement* of U in V

Lemma. $U, W \leq V$. The following are equivalent

- (i) $V = U \oplus W$
- (ii) $V = U + W$ and $U \cap W = \{\mathbf{0}\}$
- (iii) B_1 any basis of U , B_2 is any basis of W , then $B = B_1 \cup B_2$ is a basis of V .

Proof. (ii) \Rightarrow (i). Any $v \in V$ is $u + w$ for some $u \in U$, $w \in W$.

$$u_1 + w_1 = u_2 + w_2 \Rightarrow u_1 - u_2 = -w_1 + w_2 \in U \cap W = \{\mathbf{0}\} \Rightarrow w_1 = w_2, u_1 = u_2$$

(i) \Rightarrow (iii) B spans, any $v \in V$ is $u + w$, for some $u \in U$, $w \in W$, write u in terms of B_1 , w in terms of B_2 , Then $u + w$ is a lin comb. of elements of B .

B indep?

$$\begin{aligned} \sum_{v \in B} \lambda_v v &= \mathbf{0} = \mathbf{0}_U + \mathbf{0}_W \\ \underbrace{\sum_{v \in B_1} \lambda_v v}_{\in U} + \sum_{v \in B_2} \lambda_v v & \end{aligned}$$

By uniqueness of expressions,

$$\sum_{v \in B_1} \lambda_v v = \mathbf{0}_U \quad \sum_{v \in B_2} \lambda_v v = \mathbf{0}_W$$

As B_1 and B_2 are basis, all of the λ_v are zero.

$$(iii) \Rightarrow (ii). \text{ If } v \in V, v = \sum_{x \in B} \lambda_x x = \underbrace{\sum_{u \in B_1} \lambda_u u}_{\in U} + \underbrace{\sum_{w \in B_2} \lambda_w w}_{\in W}$$

$$\Rightarrow v \in U + W.$$

If $v \in U \cap W$, $v = \sum_{u \in B_1} \lambda_u u = \sum_{w \in B_2} \lambda_w w \Rightarrow$ All λ_u, λ_w are zero, because $B_1 \cup B_2$ is lin. indep.

□

Lemma. Let V be an n -dim vector space. $U \leq V$. Then there exists a direct complement to U in V

Proof. Let u_1, \dots, u_l be a basis for U . Extend it to a basis for V ,

$$u_1, \dots, u_l, w_{l+1}, \dots, w_n$$

Then $\langle w_{l+1}, \dots, w_n \rangle$ is a direct complement of U .

□

Note! Direct complements are not at all unique. In general, if you pick different ways of extending this you will get different direct complements.

Pick $V = \mathbb{R}^2$. Pick U as the y -axis, then any one of the following green lines are direct complements.:

Definition. Def $v_1, \dots, v_l \leq V$,

$$\sum V_i = V_1 + \dots + V_l = \{v_1 + \dots + v_l \mid v_i \in V_i\}$$

The sum is direct if

$$v_1 + \dots + v_l = v'_1 + \dots + v'_l \Rightarrow v_i = v'_i \text{ for all } i$$

(“unique expressions”)

Notation:

$$\bigoplus_{i=1}^l V_i$$

Exercise. $V_1, \dots, V_l \leq V$. TFAE

- (i) The sum $\sum V_i$ is direct
- (ii) $V_i \cap \sum_{j \neq i} V_j = \{\mathbf{0}\}$ for all i
- (iii) For any basis B_i of V_i , the union $B = \bigcup_{i=1}^l B_i$ is a basis for $\sum V_i$

Definition. Let U, W be \mathbb{F} -vector spaces. External direct sum

$$U \oplus W = \{(u, w) \mid u \in U, w \in W\}$$

with $(u, w) + (x, y) = (u + x, w + y)$,
 $\lambda(u, w) = (\lambda u, \lambda w)$