Part IB — Fluid Dynamics Example Sheet 1

Supervised by Prof. Haynes (P.H.Haynes@damtp.cam.ac.uk)
Examples worked through by Christopher Turnbull

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QUESTION 1

In this example, $\mathbf{u}(x) = \alpha(x, -y)$

(i) Streamlines are curves $\mathbf{X}(s; \mathbf{x}_0)$ with $\mathbf{X}_0 = \mathbf{x}_0$ at $s = 0$.

$$\frac{\partial X}{\partial s} = \alpha X, \quad \frac{\partial Y}{\partial s} = -\alpha Y$$

$X = x_0 e^{\alpha s}$ and $Y = y_0 e^{-\alpha s}$. Eliminating s to get the shape of the streamlines gives $XY = x_0 y_0$. These are hyperbola.

(ii) For steady flow, pathlines and streamlines are coincident. For a particle released at $\mathbf{x}_0 = (x_0, y_0)$, the particle pathline obeys

$$x = x_0 e^{\alpha t}, \quad y = y_0 e^{-\alpha t}$$

as As $x_0^2 + y_0^2 = a^2$ we find that the curve evolves as

$$x^2 e^{-2\alpha t} + y^2 e^{2\alpha t} = a^2$$

ie; this is an ellipse,

$$\frac{x^2}{(ae^{at})^2} + \frac{y^2}{(ae^{-at})^2} = 1$$

(iii) The area of this ellipse does not change in time; it is $\pi \times ae^{at} \times ae^{-at} = \pi a^2$. As we would expect this is the area of the circle $x^2 + y^2 = a^2$ at time $t = 0$.

Intuitively?

QUESTION 2

In this example, $\mathbf{u}(x) = (\gamma y, 0)$

(i) Streamlines are curves $\mathbf{X}(s; \mathbf{x}_0)$ with $\mathbf{X}_0 = \mathbf{x}_0$ at $s = 0$.

$$\frac{\partial X}{\partial s} = \gamma y, \quad \frac{\partial Y}{\partial s} = 0$$

$X = x_0 + \gamma y_0 s$ and $Y = y_0$. Note that we cannot eliminate s to get the shape of the streamlines; they are given by;

$$X = x_0 + \gamma Y s$$

(ii) For steady flow, pathlines and streamlines are coincident. For a particle released at $\mathbf{x}_0 = (x_0, y_0)$, the particle pathline obeys

$$x = x_0 + \gamma y_0 t, \quad y = y_0$$

as As $x_0^2 + y_0^2 = a^2$ we find that the curve evolves as

$$(x - \gamma t y_0)^2 + y^2 = a^2$$

ie. it is just shifted to the right, at a rate of γy_0 units per second.

(iii) Clearly the curve is just translated so the area remains fixed.

At longer times, the ellipse in Question 1 is stretched faster, as this grows exponentially, yet the stretch in this circle is just linear with time.

QUESTION 3

QUESTION 2

In this example, $\mathbf{u}(x) = (\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x})$.

(i) The contours $\psi = c$ have normal

$$\mathbf{n} = \nabla\psi = (\psi_x, \psi_y)$$

We see immediately that

$$\mathbf{u} \cdot \mathbf{n} = \psi_x\psi_y - \psi_y\psi_x = 0$$

So the flow is perpendicular to the normal, ie. tangent to the contours of ψ.

(ii) $|\mathbf{u}| = |(\psi_y, -\psi_x)| = \nabla^2\psi = |(\psi_x, \psi_y)| = |\nabla\psi|$

(iii) The volume flux is

$$q = \int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s.$$

We see that

$$\mathbf{n} \, \mathrm{d}s = (-\mathrm{d}y, \mathrm{d}x).$$

So we can write this as

$$q = \int_{\mathbf{x}_0}^{\mathbf{x}_1} -\frac{\partial\psi}{\partial y} \, \mathrm{d}y - \frac{\partial\psi}{\partial x} \, \mathrm{d}x = \psi(\mathbf{x}_0) - \psi(\mathbf{x}_1).$$

So the flux depends only the difference in the value of ψ. Hence, for closer streamlines, to maintain the same volume flux, we need a higher speed.

(iv) Note that ψ is constant on a stationary rigid boundary, i.e. the boundary is a streamline, since the flow is tangential at the boundary. This is a consequence of $\mathbf{u} \cdot \mathbf{n} = 0$. We often choose $\psi = 0$ as our boundary.

QUESTION 4

$$\nabla \cdot \mathbf{u} = 0 \iff \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

By the quotient rule,

$$\frac{\partial u}{\partial x} = \frac{-(y-b)[2(x-a)]}{((x-a)+(y-b))^2}$$

$$\frac{\partial v}{\partial y} = \frac{-(a-x)[2(y-b)]}{((x-a)^2+(y-b)^2)}$$

So $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\nabla \cdot \mathbf{u} = 0$, our fluid is indeed incompressible.
By inspection, the required streamfunction is

$$\psi(x,y) = \frac{1}{2}\log\left[(x-a)^2 + (y-b)^2\right]$$

QUESTION 5

$$\nabla \cdot \mathbf{u} = 0 \iff \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) = 0$$

$$\begin{aligned}\frac{\partial}{\partial r}(ru_r) &= \frac{\partial}{\partial r}\left[U\left(r - \frac{a^2}{r}\right)\cos\theta\right] \\ &= U\left(1 + \frac{a^2}{r^2}\right)\cos\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial \theta}(u_\theta) &= \frac{\partial}{\partial \theta}\left[-U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right] \\ &= U\left(1 + \frac{a^2}{r^2}\right)\cos\theta\end{aligned}$$

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) \implies \nabla \cdot \mathbf{u} = 0$$

By inspection the streamfunction is

$$\psi(r, \theta) = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

QUESTION 6

$$\nabla \cdot \mathbf{u} = 0 \iff \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0$$

$$\begin{aligned}\frac{\partial}{\partial r}(ru_r) &= \frac{\partial}{\partial r}\left[-\frac{1}{2}\alpha r^2\right] \\ &= -r\alpha\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial z}(u_z) &= \frac{\partial}{\partial z}\left[\alpha z\right] \\ &= \alpha\end{aligned}$$

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{\partial u_z}{\partial z} \implies \nabla \cdot \mathbf{u} = 0$$

By inspection the streamfunction is

$$\Psi(r, z) = \frac{1}{2}\alpha r^2 z$$

QUESTION 7

In this example, $\mathbf{u}(x) = (1/(1+t), 1)$

(i) Streamlines are curves $\mathbf{X}(s; \mathbf{x}_0)$ with $\mathbf{X}_0 = \mathbf{x}_0$ at $s = 0$.

$$\frac{\partial X}{\partial s} = \frac{1}{1+t}, \quad \frac{\partial Y}{\partial s} = 1$$

$X = \frac{s}{1+t} + x_0$ and $Y = s + y_0$. Eliminating s to get the shape of the streamlines gives $X = \frac{Y - y_0}{1+t} + x_0$. At time $t = 0$ this gives the streamline as

$$Y = X + y_0 - x_0$$

(ii) The pathline of the particle is given by

$$\frac{\partial X}{\partial t} = \frac{1}{1+t}, \quad \frac{\partial Y}{\partial t} = 1$$

with $\mathbf{X} = \mathbf{x}_0$ at $t = 0$. Thus $X = \log(1+t) + x_0$ and $Y = t + y_0$. Eliminating t we get the shape of the pathline as

$$X = \log(1 + Y - y_0) + x_0$$

(iii) The area of this ellipse does not change in time; it is $\pi \times ae^{at} \times ae^{-at} = \pi a^2$. As we would expect this is the area of the circle $x^2 + y^2 = a^2$ at time $t = 0$.

Intuitively?

(i) At $t = 1$, the velocity makes an angle of $45°$ angle with the horizontal , and the streamlines are slanted.

(ii) For a particle released at $(1, 1)$ we get

$$\dot{x}(t) = \frac{1}{1+t}, \qquad \dot{y}(t) = 1$$

Hence we get

$$x = \log(1+t) + 1, \qquad y = t + 1$$

Eliminating t, we get that the path is given by

$$x = \log y + 1 \implies y = e^{x-1}$$

Consider the fluid flow $\mathbf{u} = (1/(1+t), 1, 0)$ for $t > 0$. Supposing our fluid is incompressible, $\nabla \cdot \mathbf{u} = 0$ and there exists some vector potential $\mathbf{A}$ such that $\mathbf{u} = \nabla \times \mathbf{A}$. As $\mathbf{u}$ is two dimensional, we know $\mathbf{A}$ is of the from

$$\mathbf{A} = (0, 0, \psi(x, y, t))$$

And taking the curl of this,

$$\mathbf{u} = (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0)$$

This ψ is our streamfunction.

We define streamlines to be contours of our stream function. These contours have normal $\mathbf{n} = (\psi_x, \psi_y, 0)$, this normal is obviously perpendicular to the flow $\mathbf{u}$ ($\mathbf{u} \cdot \mathbf{n} = 0$). ie. the flow is tangent to the contours of ψ.

QUESTION 8

The integral from of the momentum equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\ \mathrm{d}V = \int_V \rho\mathbf{g}\ \mathrm{d}V - \int_S p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\ \mathrm{d}S$$

For steady flow, the LHS is zero. Neglecting gravity, we have the total force on the wall as

$$\begin{aligned} &= \int_S p\mathbf{n}\ \mathrm{d}S \\ &= -\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\ \mathrm{d}S \end{aligned}$$

So the force on the wall is equal to the surface integral, which gives $-\rho AU\mathbf{n}$. Assuming $\rho = 10^3\text{kg m}^{-3}$; $\Rightarrow$ 0.6 N force.

QUESTION 9

Proposition (Euler momentum equation)**.**

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mathbf{f}.$$

For conservative forces, we can write $\mathbf{f} = -\nabla \chi$, where χ is a scalar potential. We notice the vector identity

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \cdot \nabla \mathbf{u}.$$

We use this to rewrite the Euler momentum equation as

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) - \rho \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla p - \nabla \chi.$$

Dotting with $\mathbf{u}$, the last term on the left vanishes, and we get

Proposition (Bernoulli's equation)**.**

$$\frac{1}{2} \rho \frac{\partial |\mathbf{u}|^2}{\partial t} = -\mathbf{u} \cdot \nabla \left(\frac{1}{2} \rho |\mathbf{u}|^2 + p + \chi \right).$$

Note also that this tells us that high velocity goes with low pressure; low pressure goes with high velocity.

In the case where we have a steady flow, we know

$$H = \frac{1}{2} \rho |\mathbf{u}|^2 + p + \chi$$

is constant along streamlines.

Even if the flow is not steady, we can still define the value H, and then we can integrate Bernoulli's equation over a volume $\mathcal{D}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{D}} \frac{1}{2} \rho |\mathbf{u}|^2 \, \mathrm{d}V + \int_{\partial \mathcal{D}} H \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S. = 0$$

So H is the transportable energy of the flow.

QUESTION 10

QUESTION 11

QUESTION 12