

# Part IB — Electromagnetism Example Sheet 1

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## QUESTION 1

Equation for conservation of charge is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Have  $\mathbf{J} = C\mathbf{r}e^{-atr^2}$ , so

$$\begin{aligned}\nabla \cdot \mathbf{J} &= C\nabla \cdot (e^{-atr^2}\mathbf{r}) \\ &= Ce^{-atr^2}\nabla \cdot \mathbf{r} + C\mathbf{r} \cdot \nabla(e^{-atr^2})\end{aligned}$$

Now  $\mathbf{r}_i = x_i$  so  $\nabla \cdot \mathbf{r} = \frac{\partial x_j}{\partial x_j} = 3$ , and

$$\begin{aligned}\nabla e^{-atr^2} &= \frac{\partial e^{-atr^2}}{\partial r}\hat{\mathbf{r}} \\ &= -2ate^{-atr^2}\mathbf{r}\end{aligned}$$

Hence

$$\nabla \cdot \mathbf{J} = 3Ce^{-atr^2} - 2Cr^2ate^{-atr^2}$$

Suppose that  $\rho = (f + tg)e^{-atr^2}$ . Then we have

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= (-ar^2f + g - ar^2tg)e^{-atr^2} \\ &= (g - ar^2f)e^{-atr^2} - gtar^2e^{-atr^2}\end{aligned}$$

Hence we conclude that

$$g - ar^2f = -3C, \quad g = -2C$$

$$\Rightarrow f = \frac{C}{ar^2}$$

## QUESTION 2

Using the continuity equation,

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\nabla \cdot \mathbf{J} \\ &= -\nabla \cdot (-D\nabla \rho) \\ &= D\nabla^2 \rho\end{aligned}$$

showing  $\rho(\mathbf{x}, t)$  obeys the heat equation with diffusion constant  $D$ .

Let  $\rho(\mathbf{r}, t)$  be defined as

$$\rho(\mathbf{r}, t) = \frac{\rho_0 a^3}{(4D(t-t_0) + a^2)^{3/2}} \exp\left(-\frac{r^2}{4D(t-t_0) + a^2}\right)$$

Taking time derivatives,

$$\frac{\partial \rho}{\partial t} = \left( \frac{-6D\rho_0 a^3}{(4D(t-t_0) + a^2)^{5/2}} + \frac{4Dr^2 \rho_0 a^3}{(4D(t-t_0) + a^2)^{7/2}} \right) \exp\left(-\frac{r^2}{4D(t-t_0) + a^2}\right)$$

Now

$$\begin{aligned}\nabla^2 e^{\lambda r^2} &= \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} (e^{\lambda r^2}) \right) \\ &= \frac{2\lambda}{r^2} \frac{d}{dr} (r^3 e^{\lambda r^2}) \\ &= \frac{2\lambda}{r^2} [3r^2 + 2\lambda r^4] e^{\lambda r^2} \\ &= \lambda(6 + 4r^2) e^{\lambda r^2}\end{aligned}$$

Thus with  $\lambda = -\frac{1}{4D(t-t_0) + a^2}$ , we have

$$\begin{aligned}\nabla^2 \rho &= \frac{-\rho_0 a^3}{(4D(t-t_0) + a^2)^{5/2}} \left( 6 - \frac{4r^2}{4D(t-t_0) + a^2} \right) \exp\left(-\frac{r^2}{4D(t-t_0) + a^2}\right) \\ &= \left( -\frac{6\rho_0 a^3}{(4D(t-t_0) + a^2)^{5/2}} + \frac{4r^2 \rho_0 a^3}{(4D(t-t_0) + a^2)^{7/2}} \right) \exp\left(-\frac{r^2}{4D(t-t_0) + a^2}\right)\end{aligned}$$

Hence we can see that  $\frac{\partial \rho}{\partial t} = D\nabla^2 \rho$ , as required.

### QUESTION 3

Considering the infinite plane  $z = 0$ , we see this has uniform charge density  $\rho_0$ .

By symmetry, the field points vertically, and the field on the bottom is opposite of that on top, we must have

$$\mathbf{E} = E(z)\hat{\mathbf{z}}$$

with

$$E(z) = -E(-z)$$

Consider a vertical cylinder of height  $2h$  and cross-sectional area  $A$ . Now only the end caps contribute.

First,

$$\begin{aligned} Q &= \int_V \rho_0 e^{-k|z|} dV \\ &= \int_{-h}^h \int_0^{2\pi} \int_0^R \rho_0 e^{-k|z|} \rho d\rho d\phi dz \\ &= 2\pi \frac{R^2}{2} \rho_0 \int_{-h}^h e^{-k|z|} dz \\ &= A\rho_0 \int_0^h 2e^{-kz} dz \\ &= 2A\rho_0 \left[ -\frac{1}{k} e^{-kz} \right]_0^h \\ &= 2A \frac{\rho_0}{k} (1 - e^{-kh}) \end{aligned}$$

And

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(h)A - E(-h)A = 2AE(h) = 2A \frac{\rho_0}{k\varepsilon_0} (1 - e^{-kh})$$

Hence

$$E(z) = \frac{\rho_0}{k\varepsilon_0} (1 - e^{-kz})$$

as required.

## QUESTION 4

We have that

$$\rho(r) = \begin{cases} 0 & \text{if } r < a \\ \rho & \text{if } a < r < b \\ 0 & \text{if } r > b \end{cases}$$

First consider  $r > b$ . By symmetry, the force is the same in all directions and points outwards radially. So

$$\mathbf{E} = E(r)\hat{\mathbf{r}}$$

Put  $S$  to be a sphere of radius  $r > b$ . Then the total flux is

$$\begin{aligned} \int_S \mathbf{E} \cdot d\mathbf{S} &= \int_S E(r)\hat{\mathbf{r}} \cdot d\mathbf{S} \\ &= E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} \\ &= E(r) \cdot 4\pi r^2 \end{aligned}$$

By Gauss's law, we know this is equal to  $Q/\varepsilon_0$ , and  $Q = \frac{4}{3}\pi(b^3 - a^3)\rho$ . Therefore,

$$E(r) = \frac{(b^3 - a^3)\rho}{3\varepsilon_0 r^2}$$

and

$$\mathbf{E}(r) = \frac{(b^3 - a^3)\rho}{3\varepsilon_0 r^2} \hat{\mathbf{r}}$$

Now suppose we are inside the region,  $a < r < b$ . Then

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r)4\pi r^2 = \frac{Q}{\varepsilon_0} \left( \frac{r^3 - a^3}{b^3 - a^3} \right)$$

So

$$\begin{aligned} \mathbf{E}(r) &= \frac{Q(r^3 - a^3)}{4\pi\varepsilon_0(b^3 - a^3)r^2} \\ &= \frac{Q(r^3 - a^3)\rho}{3\varepsilon_0 r^2} \end{aligned}$$

Finally if  $r < a$ , Gauss' law tells us that the flux depends only on the total charge contained inside the surface, which in this case is none. So  $\mathbf{E}(r) = 0$ .

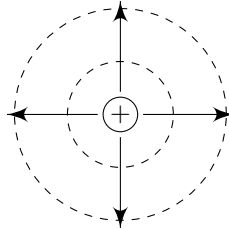
Note that the electric field is discontinuous across the surface. We have

$$\begin{aligned} E(r \rightarrow b+) - E(r \rightarrow b-) &= \frac{(b-a)(b^2 + 2ab + a^2)\rho}{3\varepsilon_0 b^2} \\ &= \frac{\sigma}{\varepsilon_0} \end{aligned}$$

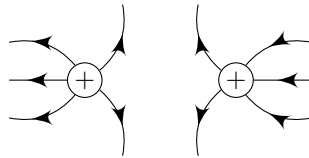
as expected.

**QUESTION 5**

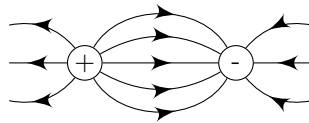
The field lines for a positive charge are:



For two positive charges,



We can also draw field lines for dipoles:



**QUESTION 6**

The inverse square law, or Coulomb's Law, states that the electric field generated by a particle with total charge  $Q$  (at the origin) is given by

$$\mathbf{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

Now consider an infinite line with uniform charge density per unit length  $\eta$ . We use cylindrical polar coordinates. By symmetry, the field is radial, ie.

$$\mathbf{E}(r) = E(r) \hat{\mathbf{r}}$$

Consider an arbitrary point at  $(r, z_0)$ . We will integrate along the  $z$ -axis to find the field at this point.

Here,

$$\begin{aligned} E(r) &= \int_{-\infty}^{\infty} \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2 + (z - z_0)^2} dz \\ &= \frac{Q}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{1}{r^2 + z^2} dz \\ &= \frac{Q}{4\pi\epsilon_0} \left[ \frac{1}{r} \arctan\left(\frac{z}{r}\right) \right]_{-\infty}^{\infty} \\ &= \frac{Q}{4r\epsilon_0} \end{aligned}$$

## QUESTION 7

The Green's function for the Laplacian is defined to be the solution to:

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}')$$

We know that

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

We assume all the charge is contained within some compact region  $V$ , then

$$\begin{aligned} \phi(\mathbf{r}) &= -\frac{1}{\varepsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \, d^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \end{aligned}$$

So integrating over a circular disk of radius  $a$  we use polar coordinates so  $\mathbf{r} = (0, 0, z)$ ,  $\mathbf{r}' = (r \cos \phi, r \sin \phi, 0)$  and  $|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + z^2}$ , hence

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^a \frac{\sigma}{r^2 + z^2} r \, dr \, d\phi \\ &= \frac{\sigma}{4\varepsilon_0} [\log(r^2 + z^2)]_0^a \\ &= \frac{\sigma}{4\varepsilon_0} \log\left(1 + \frac{a^2}{z^2}\right) \end{aligned}$$

Then

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\frac{\sigma}{2\varepsilon_0} \frac{1}{r^2 + z^2} \end{aligned}$$



**QUESTION 8**

From Q7 we have the result that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

Very far from  $V$ , ie.  $|\mathbf{r}| \gg |\mathbf{r}'|$ , we can use the Taylor expansion

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r} + \mathbf{r}' \cdot \nabla \left( \frac{1}{r} \right) + \dots \\ &= \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \dots \end{aligned}$$

Then we get

## **QUESTION 9**

## **QUESTION 10**

## **QUESTION 11**

## **QUESTION 12**