# Part IB — Methods

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# 0 Introduction

I will never say anything that is untrue deliberately...

Self-adjoint ODEs

# 1 Fourier Series

## 1.1 Peridoidic Functions

**Definition.** A function $f(t)$ is *periodic* with period $T$ if $f(t+T) = f(T)$

Fig 1

**Example.**

$$A \sin \omega t$$

$A$ is the *amplitude*, $\omega$ is the *frequency*, $2\pi/\omega$ is the *period*.

Sines and cosines are beautiful because they have an orthogonality property:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$
$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

We want to consider $\sin n\pi x/l$, $\sin m\pi x/l$, where $n, m$ are positive integers. These functions are periodic with period $2l$. [1]

$$\begin{aligned} SS_{mn} &:= \int_0^{2l} \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \\ &= \frac{1}{2}\int_0^{2l} \cos\left[\frac{(m-n)\pi x}{l}\right] \mathrm{d}x - \frac{1}{2}\cos\left[\frac{(m+n)\pi x}{l}\right] \mathrm{d}x \end{aligned}$$

if $m \neq n$,

$$SS_{mn} = \frac{l}{2\pi}\left[\frac{\sin(m-n)\pi x/l}{m-n} - \frac{\sin(m+n)\pi x/l}{m+n}\right]_0^{2l} = 0$$

if $m = n$, then $SS_{mn} = 1$ (provided $m \neq 0$, $n \neq 0$). Hence

$$SS_{mn} = \begin{cases} \delta_{mn} & \text{if } m, n \neq 0 \\ 0 & \text{if } m \text{ or } n = 0 \end{cases}$$

Similarly, $CC_{mn} = \int_0^{2l} \cos\left(\frac{m\pi x}{l}\right) \cos\left(\frac{n\pi x}{l}\right) \mathrm{d}x = l\delta_{mn}\ \forall m, n \neq 0$, and $2l$ if $m = n = 0$

Finally,

$$\begin{aligned} CS_{mn} &= \int_0^{2l} \cos\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \\ &= \frac{1}{2}\int_0^{2l} \frac{\sin(m+n)\pi x}{l} \mathrm{d}x + \frac{1}{2}\int_0^{2l} \frac{\sin(m-n)\pi x}{l} \mathrm{d}x = 0 \end{aligned}$$

[1] They have a common period of $2l$, not their smallest period!

By analogy with vectors [these integrals are indeed *inner products*], $\sin n\pi x/l$, $\cos n\pi x/l$ are said to be orthogonal on the interval $[0, 2l]$.

They actually constitute an *orthogonal basis.* ie. it is possible to represent an arbitrary (but sufficiently well behaved[2]) function in terms of an infinite series (Fourier series) formed as a sum of sins and cosines.

## 1.2 Definition of a Fourier Series

Any well behaved periodic function $f(x)$ with periodic $2L$ can be written as a Fourier Series:

$$\frac{f(x_+) + f(x_-)}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{m\pi x}{L}\right)$$

$a_n$ and $b_n$ are the Fourier Coefficients, $f(x_+)$ and $f(x_-)$ are the right limit approaching form above and the left limit approaching from below respectively

If $f(x)$ is continuous at $x_c$, then the LHS is just $f(x)$. If $f(x)$ has a bounded discontinuity, at $x_d$, ie. $f(x_d^-) \neq f(x_d^+)$, but $(f(x_d^-) - f(x_d^+))$ is finite, then the FS tends to the mean value of the two limits.

Coefficient construction: Multiply rhs of (*) by $\sin m\pi x/L$, integrate over 0 to $2L$, assume you can invert order or summation and integration.

$$\int_0^{2L} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{m\pi x}{L}\right)\right] \sin\frac{m\pi x}{2}\,\mathrm{d}x$$

We see that

$$\frac{a_0}{2}\int_0^{2L} \sin\frac{m\pi x}{L}\,\mathrm{d}x = 0$$

$$\sum_{n=1}^{\infty}\int_0^{2L} a_n \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = 0$$

$$\sum_{n=1}^{\infty}\int_0^{2L} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = Lb_n$$

So

$$\text{LHS} = \int_0^{2L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x \Rightarrow b_m = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x$$

Multiply by $\cos\frac{m\pi x}{l}$ and integrate from 0 to $2L$ (inc $m = 0$)

$$\int_0^{2L}\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{m\pi x}{L}\right)\cos\frac{m\pi x}{L}\,\mathrm{d}x$$

Non zero only when $m = 0$
Therefroe

[2] to be definied

$$\frac{a_0}{2}2L = \int_0^{2L} f(x)\ \mathrm{d}x \Rightarrow \frac{a_0}{2} = \frac{1}{2L}\int_0^{2L} f(x)\ \mathrm{d}x$$

$$a_m = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{m\pi x}{L}\right)\ \mathrm{d}x$$

The range of integration is one period so its also permissibel to choose $\int_{-L}^{L}$ a paricularly nice case is when $L = \pi$.

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\ \mathrm{d}x \qquad m \geq 0$$

$$b_m = \frac{1}{\pi}\int_{\pi}^{\pi} f(x)\sin mx\ \mathrm{d}x \qquad m \geq 1$$

## 1.3 Dirichlet Conditions

If $f(x)$ is a periodic function with period $2l$ st.

(i) it is absolutely integrable [3]

(ii) it has a finite number of extrema (ie maxs and mins) in $[0, 2l]$

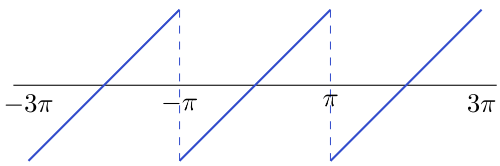
(iii) it has a finite number of bounded discontinuities in $[0, 2l]$

then the FS representation converges to $f(x)$ for all points where $f(x)$ is cts, and at points $x_d$ where $f(x)$ is discontinuous, the series coverges to the avg value of the left and right limits, ie. to $\frac{1}{2}\left(f(x_{d_+}) + f(x_{d_-})\right)$. These conditions are satisfied if the function is of 'bounded variation'

## 1.4 Smoothness and order of Fourier coefficients

If the $p^{\text{th}}$ derivative is the lowest derivative which is discontinuous somewhere (inc at the endpoints), then the F.C. are $\mathcal{O}[n^{-(p+1)}]$ as $n \to \infty$, eg. if a function has a bounded discontinuity, zeroth derivative is discontinuous: coefficients are of order $\frac{1}{n}$ as $n \to \infty$

**Example.** The sawtooth function, $f(x) = x$ on $-L \leq x \leq L$




Function is odd, so

$$a_m = \frac{1}{L}\int_{L}^{-L} xcos\left(\frac{m\pi x}{L}\right)\mathrm{d}x = 0$$

,

[3] ie. $\int_0^{2l} |f(x)|\ \mathrm{d}x$ is well defined

$$b_m = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x = \frac{1}{L}\left(\left[-\frac{xL}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\right]_{-L}^{L} - \int_{-L}^{L}\frac{-L}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x\right)$$
$$= \frac{1}{m\pi}\left(-2Lcos(m\pi) + \left[\sin\left(\frac{m\pi x}{L}\right)\frac{L}{m\pi}\right]_{-L}^{L}\right)$$
$$= \frac{2L}{m\pi}(-1)^{m+1}$$

So

$$\frac{f(x_+)+f(x_-)}{2} = \frac{2L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \cdots\right]$$

(i) $f_N(x) := \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right) \to f(x)$ almost everywhere, but the convergence is non-uniform.

(ii) Persistence overshoot @ $x = L$: 'Gibbs phenomenon'

(iii) $f(L) = 0$ average of right and left limits

(iv) Coefficients are $\mathcal{O}(\frac{1}{n})$ as $n \to \infty$

**Example.** The integral of the sawtooth function, $f(x) = \frac{1}{2}x^2$, $-L \le x \le L$

**Exercise.**

$$f(x) = L^2\left[\frac{1}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]$$

Note at $x = 0$,

$$0 = L^2\left[\frac{1}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^2}{(n\pi)^2}\right] \Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

# 2 Properties of the Fourier Series

## 2.1 Integration and Differentiation

### 2.1.1 Integration: Always works!

FS. can be integrated term by term:

$f(x)$ periodic with period $2L$ and has a FS (so it satisfies Dirichlet conditions)[4]:

$$\frac{f(x_+)+f(x_-)}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{m\pi x}{L}\right)$$

$$\begin{aligned} F(x) = \int_{-L}^{x} f(x')\,\mathrm{d}x' &= \frac{a_0(x+L)}{2} + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \\ &\quad + \sum_{n=1}^{\infty} \frac{b_n L}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi x}{L}\right)\right] \\ &= \frac{a_0 L}{2} + L\sum_{n=1}^{\infty}(-1)^n \frac{b_n}{n\pi} \\ &\quad - L\sum_{n=1}^{\infty} \frac{b_n}{n\pi}\cos\left(\frac{n\pi x}{L}\right) \\ &\quad + L\sum_{n=1}^{\infty}\left(\frac{a_n - (-1)^n a_0}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

If $a_n$ and $b_n$ are FC then the series involving $\frac{a_n}{n}$ and $\frac{b_n}{n}$ (multipled by cos or sin) must also converge

### 2.1.2 Differentiation: Doesn't always work!

Let $f(x)$ be a periodic function with period 2, st.

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

$$\frac{f(x_+)+f(x_-)}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left((2n-1)\pi x\right)}{2n-1}$$

Apply diff rules:

$$f'(x) = 4\sum_{n=1}^{\infty} \cos\left((2n-1)\pi x\right)$$

This is clearly divergent, even though $f(x) = 0$ for al $x \neq 0$.

The extra factor of $2n-1$ is the problem. It's related to the discontinuity, $f'(x)$ does not satisfy the Dirichlet condition

[4] pay attention to the limits here