Part IB — Quantum Mechanics

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0 Introduction

0.1 Discrete and Continuous Spectra

In statement measurement axioms we assumed spectrum of Q discrete. For a particle in 1d, spectra of $\hat{p}$ and H may be continuous, but can be made discrete as in section 5.1 : impose periodic boundary conditions ($\psi(x) = \psi(x+l)$ and confine to box $-l/2 \leq x \leq 1/2$)

Example. $Q = \hat{p} = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$ (particle in 1d)
eigenstates $\chi_n(x) = \frac{1}{\sqrt{l}}e^{ik_nx}$ with $k_n = \frac{2\pi n}{l}$, $n \in \mathbb{Z}$, and eigenvalues $\lambda_n = \hbar k_n$. These states are indeed orthonormal.

$$(\chi_m, \chi_n) = \int_{-l/2}^{l/2} \chi_m(x)^* \chi_n(x) \,\mathrm{d}x = \delta_{mn}$$

Expansion of a general state in terms of eigenstates:

$$\psi(x) = \sum_n \alpha_n \chi_n(x) \text{ with } \alpha_n(\chi_n, \psi) = \int_{-l/2}^{l/2} \frac{1}{\sqrt{l}} e^{-ik_nx}\psi(x) \,\mathrm{d}x$$

ie a complex Fourier series with Fourier coefficients.

0.1.1 Alternative Approach (Non Examiniable)

Extend from discrete to continuous spectra by replacing discrete label n with continuous label ξ and $\sum_n \to \int \,\mathrm{d}\xi$, $\delta_{mn} \to \delta(\xi - \nu)$.

Then orthonormal eigenstates of Q satisfy

$$Q\chi_\xi = \lambda_\xi \chi_\xi \quad \text{with} \quad (\chi_\xi, \chi_\iota) = \delta(\xi - \eta)$$

Expansion in terms of eigenstates becomes

$$\psi = \int \alpha_\xi \chi_\xi \,\mathrm{d}\xi \quad \text{with} \quad \alpha_\xi = (\chi_\xi, \psi)$$

$P_\xi = |\alpha_\xi|^2$ is now probability density with

$$\int_a^b P_\xi \,\mathrm{d}\xi \text{ probability that result corresponds } a \leq \xi \leq b$$

Example. (i) $Q = \hat{p}$, have $\xi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$, $-\infty < p < \infty$. ($\lambda_p = p$).
Note

$$(\chi_p, \chi_q) = \frac{1}{2\pi\hbar}\int e^{i(p-q)x/\hbar} \,\mathrm{d}x = \delta(p-q)$$

Now

$$\psi(x) = \int_{-\infty}^{\infty} \alpha_p \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \,\mathrm{d}p$$

with

$$\alpha_p = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x)\ \mathrm{d}x$$

Fourier representation transform

(ii) $Q = \hat{x}$ have eigenstates $\chi_\xi(x) = \delta(x-\xi)$ with eigenvalue $\lambda_\xi = \xi$ since $\hat{x}X_\xi(x) = x\delta(x-\xi) = \xi\delta(x-\xi) = \xi\chi_\xi(x)$

Then

$$\begin{aligned}\psi(x) &= \int_{-\infty}^{\infty} \alpha_\xi \chi_\xi(x)\ \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \alpha_\xi \delta(x-\xi)\ \mathrm{d}\xi \\ &= \alpha_x\end{aligned}$$

So $\int_a^b P_\xi\ \mathrm{d}\xi = \int_a^b |\alpha_\xi|^2\ \mathrm{d}\xi = \int_a^b |\psi(\xi)|^2\ \mathrm{d}\xi$

probability of measuring position to be in range $a \leq x \leq b$

Hence we recover (P1)

0.2 Evolution in Time

0.2.1 Stationary States

Consider energy eigenstates ψ_n with

$$H\psi_n = E_n\psi_n \text{ and } (\psi_m, \psi_n) = \delta_{mn}$$

Then $\psi_n e^{iE_nt/\hbar}$ is a solution of t-dep SE for any n. This, or ψ_n, called a stationary state. Given any initial state (not necc stationary) we can expand

$$\Psi(0) = \sum_n \alpha_n\psi_n \qquad \text{with } \alpha_n = (\psi_n, \Psi(0))$$

By linearity, the solution of the SE is

$$\Psi(t) = \sum_n \alpha_n\psi_n e^{-iE_nt/\hbar}$$

0.2.2 General Form of Ehrenfest's Theorem

If Q is any observable (with no explicit time dependence) for a quantum system with state $\Psi(t)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q\rangle_{\Psi(t)} = \langle\ \frac{1}{i\hbar}[Q,H]\ \rangle_{\Psi(t)}$$

where $[Q,H] = QH - HQ$, the commutator.

Proof. Use SE and follow the same steps as for particle in 1d. □

This closely resembles equation of motion in Hamiltonian formalism

0.3 Degeneracy and Simultaneous Measurements

0.3.1 Degeneracy

For an observable Q, the number of linearly independent states with eigenvalue λ is called the *degeneracy* of λ; this is the dimension of the *eigenspace* $V_\lambda = \{\psi : Q\psi = \lambda\psi\}$. Also say eigenvalue λ is non-degenerate iff the degeneracy $= 1$. The eigenvalues are degenerate iff the degeneracy > 1.

Also say *states* are degenerate if they have the same eigenvalue.

Physically - cannot distinguish degenerate states by measuring Q alone. We are free to chose any orthonormal basis of states for each V_λ.

Consider discussion leading to (M1), (M2), (M3) and use same notation. If there is degeneracy then:

(M2)' Probability of measuring λ is $\sum_{n:\lambda_n=\lambda} P_n = \sum_{n:\lambda_n=\lambda} |\alpha_n|^2$.

(M3)' State after such a measurement is

$$(\text{ const }) \sum_{n:\lambda_n=\lambda} \alpha_n \chi_n$$

Measurement projects onto the eigenspace V_λ