Part IA — Variational Principles Example Sheet 1

Supervised by Mx Tsang
Examples worked through by Christopher Turnbull

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QUESTION 1

Solution.

$$\nabla\phi = (x_1^3 - x_2 - x_3, x_2^3 - x_3 - x_1, x_3^3 - x_1 - x_2)$$

Setting $\nabla\phi = 0$ yields three equations satisfied by the coordinates $\mathbf{x} = (x_1, x_2, x_3)$ of the stationary points of ϕ. Subtracting the first two of these gives

$$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = -(x_1 - x_2)$$

$$\Rightarrow (x_1 - x_2) = 0 \quad \text{or} \quad (x_1^2 + x_1x_2 + x_2^2 + 1) = 0$$

Treating the second equation as a quadratic in x_1 gives the discriminant as $-3x_2^2 - 4 < 0$ with no real solutions. Hence $x_1 = x_2$, and by symmetry, $x_1 = x_2 = x_3$. Using $[\nabla\phi]_1 = 0$ gives $x_1^3 = 2x_1 \Rightarrow x_1 = \pm\sqrt{2}, 0$. The stationary points of ϕ are:

$$(0,0,0) \qquad \text{and} \qquad (\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{2})$$

For $(\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{2})$, the Hessian is

$$\mathbf{H} = \begin{pmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{pmatrix}$$

with eigenvalues $\lambda_1 = \lambda_2 = 7$, $\lambda_3 = 4$. As all eigenvalues are positive, both of these points are minima. Hence ϕ takes it's minimum value at both of these two points, and this minimum value is:

$$\begin{aligned}\phi(\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{2}) &= \frac{1}{4}(3(\sqrt{2})^4) - 3(2) \\ &= -3\end{aligned}$$

For $(0,0,0)$, the Hessian is

$$\mathbf{H} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

with eigenvalues $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$. As some are positive and the rest negative, $(0,0,0)$ is a saddle point of ϕ.

Now let $x_i = R_{ij}x'_j$ for some rotation matrix R, which we can choose such that the matrix $H' = R^T HR$ is diagonal. Neglecting terms of order x_3 we have

$$\begin{aligned}\phi &= \frac{1}{2}x_i H_{ij} x_j + O(x^3) \\ &= \frac{1}{2}\sum_{i=1}^{3} \lambda_i (x'_i)^2 \\ &= \frac{1}{2}\left(x'_1 + x'_2 - 2x'_3\right)\end{aligned}$$

The surface here becomes $x'_3 = \frac{1}{2}(x'_1 + x'_2)$. Geometrically, we see this is a cone with semi-angle $\arctan\sqrt{2}$

QUESTION 2

Solution. (i) The upper half plane is trivially a convex set, hence $f(x, y) = x^2/y$ is convex on the upper half plane $(x, y) : y > 0$ if and only if for all $(x, y), (x', y')$ in the upper half plane:

$$f\left[(1-t)\begin{pmatrix} x \\ y \end{pmatrix} + t\begin{pmatrix} x' \\ y' \end{pmatrix}\right] \leq (1-t)f\begin{pmatrix} x \\ y \end{pmatrix} + tf\begin{pmatrix} x' \\ y' \end{pmatrix}, \qquad 0 < t < 1$$

This is true if and only if

$$\frac{[(1-t)x_+ tx']^2}{(1-t)y + y'} \leq (1-t)\frac{x^2}{y} + t\frac{(x')^2}{y'}$$

$$\begin{aligned} \iff [(1-t)x_+ tx']^2 yy' &= [(1-t)x^2y' + t(x')^2y][(1-t)y + y'] \\ \iff 2t(1-t)xx'yy' &\leq t(1-t)\left[x^2(y')^2 + (x')^2y^2\right] \\ \iff [xy' + x'y] &\geq 0 \end{aligned}$$

where the last inequality is trivially true. Hence convexity follows.

(ii) Given $F(x, y) = yf(x/y)$, we are trying to show

$$F\left[(1-t)\begin{pmatrix} x \\ y \end{pmatrix} + t\begin{pmatrix} x' \\ y' \end{pmatrix}\right] \leq (1-t)F\begin{pmatrix} x \\ y \end{pmatrix} + tF\begin{pmatrix} x' \\ y' \end{pmatrix}$$

for some $0 < t < 1$, for all $(x, y), (x', y')$ in the upper half plane, given that $f(x)$ is convex. This is true if and only if

$$[(1-t)y + ty']\, f\left(\frac{(1-t)x + tx'}{(1-t)y + ty'}\right) \leq (1-t)f(x/y) + ty'f(x'/y')$$

Using the fact that $f(x/y)$ is convex, for some $0 < t < 1$ we have

$$f\left[(1-t)x/y + tx'/y'\right] \leq (1-t)f(x/y) + tf(x'/y')$$

Upon replacing t with $s = \frac{ty'}{(1-t)y+ty'}$, the result follows immediately.

QUESTION 3

Solution. The Legrende transform of $f(x) = e^x$ is given by

$$f^*(p) = \sup_x [px - e^x]$$

In this case $p = e^x$, and hence $x = \log p$ at the maximum of $px - e^x$, which is then $f^*(p)$. So

$$f^*(p) = p \log p - p, \qquad p \in \mathbb{R}, p > 0$$

Similarly, the Legrende transform of $f(x) = a^{-1}x^a$, $a > 1, x > 0$ is given by

$$f^*(p) = \sup_x \left[px - a^{-1}x^a\right]$$

In this case $p = x^{a-1}$, and

$$f^*(p) = b^{-1}p^b, \qquad \text{where } b = \frac{a}{a-1}$$

QUESTION 4

Solution. The Hemholtz free energy is defined by

$$F(T,V) = \min_S \left[U(S,V) - TS\right]$$

Differentiating with respect to S gives

$$\begin{aligned} T &= \frac{\partial U}{\partial S} \\ &= T_0 \left(\frac{V_0}{V}\right)^{1/\alpha} \exp\left(\frac{S - S_0}{\alpha n R}\right) \end{aligned}$$

Rearranging, $S = S_0 + \alpha n R \log\left[\frac{T}{T_0}\left(\frac{V_0}{V}\right)^{1/\alpha}\right]$. Hence

$$F(T,V) = U_0 + \alpha n R(T - T_0) - T\left(S_0 + \alpha n R \log\left[\frac{T}{T_0}\left(\frac{V_0}{V}\right)^{1/\alpha}\right]\right)$$

QUESTION 5

Solution. (i) For a triangle of given perimeter $2s$, the area

$$A = \sqrt{s(s-a)(s-b)(a+b-s)}$$

is maximised when $\frac{\partial A}{\partial a} = \frac{\partial A}{\partial b} = 0$.

$$\frac{\partial A}{\partial a} = \frac{-\sqrt{s(s-b)(a+b-s)}}{2\sqrt{(s-a)}} + \frac{\sqrt{s(s-a)(s-b)}}{2\sqrt{(a+b-s)}}$$

Setting $\frac{\partial A}{\partial a} = 0$ we recover $a+b-s = s-a \Rightarrow 2s = 2a+b$. Similarly, $\frac{\partial A}{\partial b} = 0 \Rightarrow 2s = a+2b$. Hence $a = b$, and by symmetry, $a = b = c$ and the triangle is equilateral.

(ii) For a right-angled triangle with sides a, b, c where $c^2 = a^2+b^2$, the perimeter P is given by

$$P = a^2 + b^2 + \sqrt{a^2+b^2}$$

and the area A by

$$A = \frac{ab}{2}$$

Rearranging the first expression for a we find that

$$a = \frac{1}{2}\left(\frac{2bP - P^2}{b-P}\right)$$

Substituting this in, we find

$$A = \frac{1}{4}\left(\frac{2bP - P^2}{b-P}\right) b$$

So setting $\frac{\partial A}{\partial b} = 0$,

$$\frac{1}{4}\left(\frac{2bP - P^2}{b-P}\right) + \frac{1}{4}\left(\frac{(b-P)2P - (2bP - P^2)}{(b-P)^2}\right) b = 0$$
$$\Rightarrow (2bP - P^2)(b-P) + (b-P)2P - (2bP - P^2) = 0$$

QUESTION 6

Solution. The volume of the parallelepiped is $2x \times 2y \times 2z = 8xyz$ Using a Lagrange multiplier λ to impose the constraint, we have

$$\Phi_\lambda[\mathbf{x}] = 8xyz - \lambda \left(\frac{x^2}{a^2} + \frac{x^2}{x^2} + \frac{x^2}{c^2} - 1 \right)$$

$$\frac{\partial \Phi}{\partial x} = 0 \Rightarrow 8yz - \frac{2x\lambda}{a^2} = 0 \Rightarrow \frac{1}{a^2} = \frac{4yz}{x\lambda}$$

Similarly

$$\frac{\partial \Phi}{\partial y} = 0 \Rightarrow \frac{1}{b^2} = \frac{4zx}{y\lambda}, \quad \frac{\partial \Phi}{\partial z} = 0 \Rightarrow \frac{1}{c^2} = \frac{4xy}{z\lambda}$$

Whence,

$$\begin{aligned} \frac{\partial \Phi}{\partial \lambda} = 0 \Rightarrow \frac{x^2}{a^2} + \frac{x^2}{x^2} + \frac{x^2}{c^2} = 1 \\ \Rightarrow \lambda = 12xyz \end{aligned}$$

Substituting our value of λ into the $\frac{\partial \Phi}{\partial x} = 0$ equation gives

$$\frac{1}{a^2} = \frac{4yz}{12x^2yz} \Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly, $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$ and

$$\begin{aligned} V &= 8xyz \\ &= \frac{8abc}{3\sqrt{3}} \end{aligned}$$

QUESTION 7

Solution. In spherical coordinates the distance functional is given by

$$F[r,\theta,\phi] = \int_t^{t+\delta t} \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2}\ \mathrm{d}t$$

On the unit sphere, $r = 1$, and this becomes:

$$F[\theta,\phi] = \int_t^{t+\delta t} \sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}\ \mathrm{d}t$$

Equivalently, if θ is a good parameter for the curve, we can consider the functional obtained from a change of variables

$$F[\phi] = \int_\theta^{\theta+\delta\theta} \sqrt{1 + \sin^2\theta(\phi')^2}\ \mathrm{d}\theta$$

where the curve is now specified by the function $\phi(\theta)$.

The functional for the total path length between any two points on the unit sphere is given by:

$$L[\phi] = \int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2\theta(\phi')^2}\ \mathrm{d}\theta$$

In this case with $f = \sqrt{1 + \sin^2\theta(\phi')^2}$, $\frac{\partial f}{\partial \phi} = 0$ and the Euler Lagrange equation can be immediately once integrated to give the first integral:

$$\frac{\phi' \sin^2\theta}{\sqrt{1 + \sin^2\theta(\phi')^2}} = c$$

for some constant c. Hence

$$\phi' = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}$$

Substitute $u = \cot\theta$ so that $\mathrm{d}u = -\operatorname{cosec}^2\theta\mathrm{d}\theta$. Then

$$\begin{aligned}
\phi &= \int \frac{-c\ \mathrm{d}u}{\sqrt{1 - c^2\operatorname{cosec}^2\theta}} \\
&= \int \frac{-c\mathrm{d}u}{\sqrt{1 - c^2(1+u^2)}} \\
&= \frac{-\mathrm{d}u}{\sqrt{a^2 - u^2}} \qquad \text{where } \ a = \sqrt{1-c^2}/c \\
&= \cos^{-1}(u/a) + \phi_0
\end{aligned}$$

where ϕ_0 is a constant of integration. Hence, the path is given by

$$\cot\theta = a\cos(\phi - \phi_0)$$

which is the path of a great circle.

QUESTION 8

Solution. Assuming z is a good parameter for the curve, it can be written as $r(z)$, and we are trying to maximize the functional for the surface area

$$S[r] = \int_{-b}^{b} 2\pi r\sqrt{1+(r')^2}\,\mathrm{d}z \quad \Rightarrow \quad f = 2\pi r\sqrt{1+(r')^2}$$

As f has no explicit z-dependence, we have the first integral

$$\text{constant } = f - r'\frac{\partial f}{\partial r'} = \frac{2\pi r}{\sqrt{1+(r')^2}} \quad \Rightarrow r = c\cosh(z/c)$$

Using the boundary conditions $r = a \Rightarrow z = \pm b$, we have $a/c = \cosh(b/c)$

QUESTION 9

Solution. Let Stratford be the origin of coordinates in the vertical plane with x being horizontal distance from the origin and y being the vertical distance below the origin. The train depart with zero velocity so conservation of energy implies that its speed $v$at any later time is given by

$$\frac{1}{2}mv^2 = mgy \;\Rightarrow\; v = \sqrt{2gy}$$

We have to find the path that minimizes the travel time when the speed depends on position, ie. we have to minimize:

$$T = \int_A^B \frac{\mathrm{d}l}{v} = \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{\sqrt{y}}$$

Assuming x is a good coordinate for the curve,

$$T[y] \propto \int_0^{x_B} \sqrt{\frac{1 + (y')^2}{y}}\,\mathrm{d}x, \quad \Rightarrow \quad f = \sqrt{\frac{1 + (y')^2}{y}}$$

As f has no explicit x-dependence, we have the first integral

$$\text{constant } = f - y'\frac{\partial f}{\partial y'} = \frac{1}{y[1 + (y')^2]} \quad \Rightarrow \quad y[1 + (y')^2] = 2c$$

for positive constant c. The solution of this first-order ODE with $y(0) = 0$ is given parametrically by

$$x = c(\theta - \sin\theta), \qquad y = c(1 - \cos\theta)$$

which is an inverted cycloid. The origin (Stratford) corresponds to $\theta = 0$. Requiring the cycloid passes through $(l, 0)$ (Acton) gives $\theta = 2\pi$, $c = \frac{l}{2\pi}$. Hence, the time taken is given by

$$\begin{aligned}
T &= \frac{1}{\sqrt{2g}} \int_A^B \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{\sqrt{y}} \\
&= \frac{1}{\sqrt{2g}} \int_0^{2\pi} \sqrt{\frac{c^2(1 - \cos\theta)^2 + \theta^2 \sin^2\theta}{c(1 - \cos\theta)}}\,\mathrm{d}\theta \\
&= \sqrt{\frac{c}{2g}} \int_0^{2\pi} \sqrt{2}\,\mathrm{d}\theta \\
&= \sqrt{\frac{2\pi l}{g}}
\end{aligned}$$

QUESTION 10

Solution. Fermat's principle states that light takes the path of least time. We wish to minimize

$$\begin{aligned}T[y] =& \int \frac{\mathrm{d}l}{v} \\ =\propto& \int \sqrt{(1-ky)(1+(y')^2}\,\mathrm{d}x\end{aligned}$$

Notice that

$$f = \sqrt{(1-ky)(1+(y')^2} \quad \Rightarrow \quad \frac{\partial f}{\partial x} = 0$$

so we have the first integral

$$\text{constant } = f - y'\frac{\partial f}{\partial y'} = \sqrt{\frac{1-ky}{1+(y')^2}}$$

Squaring, we deduce that

$$(y')^2 = (k/c^2)(y_0 - y), \qquad y_0 = \frac{1-c^2}{k}$$

Taking the square root we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{y-y_0} \pm \frac{k}{2\sqrt{c}}x\right] = 0 \Rightarrow y = y_0 - \frac{k^2}{4c^2}(x-x_0)^2$$

where x_0 is another integration constant. This is a parabola, with maximum height $y = y_0$. If the ray enters the medium at $(-x_0, 0)$ and leaves at $(x_0, 0)$, maximum height is reached at $x = 0$. Substituting in y_0 for c gives:

$$y = y_0 - \frac{(kx_0)^2}{4(1-ky_0)^2}$$

QUESTION 11

QUESTION 12

Solution. Writing the area of the enclosed region as an integral over x of area elements of vertical strips, the total area is:

$$A[y] = \int_0^a y(x)\,\mathrm{d}x$$

We must maximize A subject to the condition that $P[y] = L$, where

$$P[y] = \int \sqrt{1+(y')^2}\,\mathrm{d}x$$

Using a Lagrange multiplier to impose the constraint, we have

$$\Phi_\lambda[y] = \int f_\lambda(y, y')\,\mathrm{d}x + \lambda L, \qquad f(y, y') = y - \lambda\sqrt{1+(y')^2}$$

$f_\lambda(y, y')$ has no explicit x-dependence, so the EL equations imply that

$$\text{constant } = f_\lambda - y'\frac{\partial f_\lambda}{\partial y'} = y - \frac{\lambda}{\sqrt{1+(y')^2}}$$

This is equivalent to

$$(y')^2 = \frac{\lambda^2}{(y-y_0)^2} - 1$$

for some constant y_0. This ODE has the solution $y = y_0 \pm \sqrt{\lambda^2 - (x-x_0)^2}$ for some constant x_0, so

$$(x-x_0)^2 + (y-y_0)^2 = \lambda^2$$

QUESTION 13

Solution. Functional for the total length is

$$P[y] = \int_{-a}^{a} y\sqrt{1+(y')^2}\,\mathrm{d}x$$

Using a Lagrange multiplier to impose the constraint, we have

$$\Phi_\lambda[y] = \int f_\lambda(y, y')\,\mathrm{d}x + \lambda P, \qquad f(y, y') = (y-\lambda)\sqrt{1+(y')^2}$$

$f_\lambda(y, y')$ has no explicit x-dependence, so the EL equations imply that

$$\text{constant } = f_\lambda - y'\frac{\partial f_\lambda}{\partial y'} =$$