Part IB — Complex Methods Example Sheet 1

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Lent 2018

QUESTION 1

(i) *[For each of the following we let $f(z) = u(x,y) + iv(x,y)$ and check the Cauchy-Riemann equations.]*

- $f(z) = \operatorname{Im} z$. This has $u = y$, $v = 0$. But

$$\frac{\partial u}{\partial y} = 1 \neq 0 = -\frac{\partial u}{\partial x}$$

So $\operatorname{Im} z$ is nowhere differentiable, and hence nowhere analytic.

- $f(z) = |z|^2 = x^2 + y^2$. This has $u = x^2 + y^2$, $y = 0$. Have

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = 0$$

Hence the Cauchy-Riemann equations are only satisfied at the origin. So f in only differentiable at $z = 0$, however it is not analytic since there is no neighbourhood of 0 throughout which f is differentiable.

- $f(z) = \operatorname{sech} z$. First note that if $f(z) = u + iv \neq 0$, then

$$\frac{1}{f(z)} = \frac{u}{u^2 + v^2} - \frac{iv}{u^2 + v^2}$$

So if $f(z)$ is analytic, then $\frac{1}{f(z)}$ is analytic provided $f(z) \neq 0$.
$g(z) := \cosh(z) = \frac{1}{2}(e^z + e^{-z})$ is entire since e^z is entire (from lectures). Checking when g is zero gives us $z = \frac{1}{2}\log(-1) = \frac{1}{2}\left[\log(1) + (2n+1)i\pi\right]$ for integer n.
Hence $\operatorname{sech}(z)$ is differentiable at all points expect those at $(0, (n+\frac{1}{2})\pi)$ for integer n, and hence also analytic everywhere but these points.

(ii) Writing $z = r(\cos\theta + i\sin\theta)$, we obtain

$$u = r\cos 5\theta \quad v = r\sin 5\theta$$

Using the chain rule with $r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}$,
~~This is going to be messy~~ First note that

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$$

and

$$\frac{\partial r}{\partial x} = x(x^2 + y^2)^{-1/2}, \qquad \frac{\partial r}{\partial y} y(x^2 + y^2)^{-1/2}$$

The first Cauchy Riemann equation is

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\
&= \cos(5\theta)x(x^2+y^2)^{-1/2} - r\sin 5\theta \frac{-y}{x^2+y^2} \\
&= \cos(5\theta)r\cos\theta r^{-1} - r\sin 5\theta \frac{-r\sin\theta}{r^2} \\
&= \cos 4\theta
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} \\
&= \sin(5\theta)y(x^2+y^2)^{-1/2} + r\cos 5\theta \frac{x}{x^2+y^2} \\
&= \sin(5\theta)r\sin\theta r^{-1} + r\cos 5\theta \frac{r\cos\theta}{r^2} \\
&= \cos 4\theta
\end{aligned}$$

For second CR equation,

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} \\
&= \sin(5\theta)x(x^2+y^2)^{-1/2} + r\cos 5\theta \frac{-y}{x^2+y^2} \\
&= \sin(5\theta)r\cos\theta r^{-1} + r\cos 5\theta \frac{-r\sin\theta}{r^2} \\
&= \sin 4\theta
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \\
&= \cos(5\theta)y(x^2+y^2)^{-1/2} + -\sin 5\theta \frac{x}{x^2+y^2} \\
&= \cos(5\theta)r\sin\theta r^{-1} - r\sin 5\theta \frac{r\cos\theta}{r^2} \\
&= -\sin 4\theta
\end{aligned}$$

We conclude in fact that the Cauchy-Riemann equations are satisfied everywhere.

Now, looking closer at $\frac{\partial u}{\partial x}$, we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \cos 4\theta \\
&= \cos^2 2\theta - \sin^2 2\theta \\
&= (\cos^2\theta - \sin^2\theta)^2 - 4\sin^2\theta\cos^2\theta \\
&= (\cos^2\theta + \sin^2\theta)^2 - 8\sin^2\theta\cos^2\theta \\
&= 1 - \frac{8x^2y^2}{r^4} \\
&= 1 - \frac{8x^2y^2}{(x^2+y^2)^2}
\end{aligned}$$

Now we see that when $x = y = 0$, $\frac{\partial u}{\partial x}$ is not defined, which is enough to show that f is not differentiable at the origin.

QUESTION 2

[Each of the following analytical functions will be of the form $f(z) = u(x, y) + iv(x, y)$. Given u, we find v using the Cauchy-Riemann equations, and thus f.]

(i) The first Cauchy Riemann equation determines

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 1 \implies v = y + g(x)$$

The other Cauchy Riemann equation gives

$$0 = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = g'(x)$$

So g must be a constant, say α. Wlog set it to zero. The corresponding analytic function is therefore

$$f(z) = x + iy = z$$

(ii) $u = xy$, so the first Cauchy Riemann equation determines

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = y \implies v = \frac{1}{2}y^2 + g(x)$$

The other Cauchy Riemann equation gives

$$-x = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = g'(x)$$

So $g'(x) = -x$, giving us $g(x) = -\frac{1}{2}x^2 + \alpha$ for some constant α, wlog 0. The corresponding analytic function is therefore

$$\begin{aligned} f(z) &= xy + \frac{1}{2}i(y^2 - x^2) \\ &= \frac{1}{2}i\left(y^2 - 2ixy - x^2\right) \\ &= -\frac{1}{2}i\left(x^2 + 2ixy - y^2\right) \\ &= -\frac{1}{2}i\left(x + iy\right)^2 \\ &= -\frac{1}{2}iz^2 \end{aligned}$$

(iii) $u = \sin x \cosh y$, so the first Cauchy Riemann equation determines

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \cos x \cosh y \implies v = \cos x \sinh y + g(x)$$

The other Cauchy Riemann equation gives

$$\sin x \sinh y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \sin x \sinh y + g'(x)$$

So $g'(x) = 0$, giving us $v(x) = \cos x \sinh y + \alpha$ for some constant α (wlog set it to zero). The corresponding analytic function is therefore

$$\begin{aligned} f(z) &= \sin x \cosh y + i \cos x \sinh y \\ &= \frac{1}{2}e^{y}(\sin x + i\cos x) + \frac{1}{2}e^{-y}(\sin x - i \cos x) \\ &= \frac{1}{2}i[e^{y-ix} - e^{ix-y}] \\ &= i\sinh(z^*) \end{aligned}$$

(iv) $u = \log(x^2 + y^2)$, so Cauchy Riemann determine that
Recall $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan(x/a)$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} \implies y = 2\arctan(y/x) + g(x)$$

Next,

$$\frac{2y}{x^2+y^2} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{2y}{x^2+y^2} + g'(x)$$

Hence $g'(x) = 0$, set $g(x) = 0$ wlog, have that

$$\begin{aligned} f(z) &= \log(x^2+y^2) + i2\arctan(y/x) \\ &= \log(|z|^2) + 2i\mathrm{sgn}(x)arg(z) \end{aligned}$$

(v) $u = \frac{y}{(x+1)^2+y^2}$, so Cauchy Riemann determine that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{-2y(x+1)}{[(x+1)^2+y^2]^2}$$

$$\begin{aligned} \implies v &= -(x+1)\int 2y[(x+1)^2+y^2]^{-2}\,\mathrm{d}y \\ &= \frac{-(x+1)}{(x+1)^2+y^2} + g(x) \end{aligned}$$

CBA checking the next one (is this necessary?)

The corresponding analytic function is therefore

$$\begin{aligned}
f(z) &= \frac{y}{(x+1)^2+y^2} + -i\frac{(x+1)}{(x+1)^2+y^2} \\
&= y + i(x+1) \\
&= i(x-iy) + i \\
&= iz^* + i
\end{aligned}$$

(vi) $u = \arctan\left(\frac{2xy}{x^2-y^2}\right)$, so Cauchy Riemann determine that

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} &= \frac{(x^2-y^2)2y - 2xy(2x)}{(x^2-y^2)^2+(2xy)^2} \\
&= \frac{2y[(x^2-y^2)-2x^2]}{(x^2+y^2)^2} \\
&= \frac{-2y(x^2+y^2)}{(x^2+y^2)^2} \\
&= \frac{-2y}{x^2+y^2}
\end{aligned}$$

$$\begin{aligned}
\implies v &= \int \frac{-2y}{x^2+y^2}\,\mathrm{d}y \\
&= -\log(x^2+y^2) + g(x)
\end{aligned}$$

Hmm, deduce that $g'(x) = 0$, set the constant to zero, so we have

$$\begin{aligned}
f(z) &= \arctan\left(\frac{2xy}{x^2-y^2}\right) - i\log(x^2+y^2) \\
&=
\end{aligned}$$

Now, if these $f = u + iv$ are analytic, (and therefore satisfy the Cauchy-Riemann equations) we can compute

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \\
&= -\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Therefore, when the CR equations are satisfied, the function u is harmonic. Hence, for the above questions, we have u harmonic on

(i) $\mathbb{R}^2$

(ii) $\mathbb{R}^2$

(iii) $\mathbb{R}^2$

(iv) $\mathbb{R}^2$

(v) $\mathbb{R}^2$

(vi) $\mathbb{R}^2$

QUESTION 3

QUESTION 4

$$\phi(x, y) = e^x(x \cos y - y \sin y)$$

Calculating the partial derivatives,

$$\begin{aligned}
\partial_x \phi &= \phi + e^x \cos y \\
\partial_{xx} \phi &= \partial_x \phi + e^x \cos y \\
&= \phi + 2e^x \cos y
\end{aligned}$$

$$\begin{aligned}
\partial_y \phi &= e^x(-x \sin y - \sin y - y \cos y) \\
\partial_{yy} \phi &= e^x(-x \cos y - 2 \cos y + y \sin y) \\
&= -\phi - 2e^x \cos y
\end{aligned}$$

Hence $\partial_{xx}\phi + \partial_{yy}\phi = 0$ and the function is indeed harmonic.
The harmonic conjugate $\psi(x, y)$ satisfies the Cauchy Riemann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

The first of these gives

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = e^x(x \cos y - y \sin y) + e^x \cos y$$

Noting $\int y \sin y \mathrm{d}y = -y \cos y + \sin y$, we must have $\psi = e^x(x \sin y + y \cos y) + g(x)$. The other Cauchy Riemann equation gives

$$e^x(x \sin y + \sin y + y \cos y) = -\frac{\partial \phi}{\partial y} = \frac{\partial v}{\partial x} = e^x(x \sin y + \sin y + y \cos y) + g'(x)$$

So g must be a constant, say 0, so the harmonic conjugate of ϕ is

$$\psi(x, y) = e^x(x \sin y + y \cos y)$$

Can now show that $\nabla\phi \cdot \nabla\psi = 0$ (by the CR equations), ie. contours of harmonic conjugate function are perpendicular (in 2D).

Recall that a gradient of a function is perpendicular to its contours.

QUESTION 5

QUESTION 6

QUESTION 7

QUESTION 8

QUESTION 9

QUESTION 10

QUESTION 11

QUESTION 12