$$\frac{\partial \Phi}{\partial X} = \lim_{h \to 0} \left(\frac{\Phi((i+1)h, jh) - \Phi(ih, jh)}{h} \right)$$

Part IB — Electromagnetism Example Sheet 1

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Lent 2018

QUESTION 1

Equation for conservation of charge is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Have $\mathbf{J} = C\mathbf{r}e^{-atr^2}$, so

$$\begin{aligned}\nabla \cdot \mathbf{J} &= C\nabla \cdot (e^{-atr^2}\mathbf{r}) \\ &= Ce^{-atr^2}\nabla \cdot \mathbf{r} + C\mathbf{r} \cdot \nabla(e^{-atr^2})\end{aligned}$$

Now $\mathbf{r}_i = x_i$ so $\nabla \cdot \mathbf{r} = \frac{\partial x_j}{\partial x_j} = 3$, and

$$\begin{aligned}\nabla e^{-atr^2} &= \frac{\partial e^{-atr^2}}{\partial r}\hat{\mathbf{r}} \\ &= -2ate^{-atr^2}\mathbf{r}\end{aligned}$$

Hence

$$\nabla \cdot \mathbf{J} = 3Ce^{-atr^2} - 2Cr^2ate^{-atr^2}$$

Suppose that $\rho = (f + tg)e^{-atr^2}$. Then we have

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= (-ar^2 f + g - ar^2 tg)e^{-atr^2} \\ &= (g - ar^2 f)e^{-atr^2} - gtar^2 e^{-atr^2}\end{aligned}$$

Hence we conclude that

$$g - ar^2 f = -3C, \qquad g = -2C$$

$$\Rightarrow f = \frac{C}{ar^2}$$

QUESTION 2

Using the continuity equation,

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\nabla \cdot \mathbf{J} \\ &= -\nabla \cdot (-D\nabla\rho) \\ &= D\nabla^2\rho\end{aligned}$$

showing $\rho(\mathbf{x}, t)$ obeys the heat equation with diffusion constant D.
Let $\rho(\mathbf{r}, t)$ be defined as

$$\rho(\mathbf{r}, t) = \frac{\rho_0 a^3}{(4D(t-t_0)+a^2)^{3/2}} \exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)$$

Taking time derivatives,

$$\frac{\partial \rho}{\partial t} = \left(\frac{-6D\rho_0 a^3}{(4D(t-t_0)+a^2)^{5/2}} + \frac{4Dr^2\rho_0 a^3}{(4D(t-t_0)+a^2)^{7/2}}\right) \exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)$$

Now

$$\begin{aligned}\nabla^2 e^{\lambda r^2} &= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r}(e^{\lambda r^2})\right) \\ &= \frac{2\lambda}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 e^{\lambda r^2}\right) \\ &= \frac{2\lambda}{r^2}\left[3r^2 + 2\lambda r^4\right] e^{\lambda r^2} \\ &= \lambda(6+4r^2)e^{\lambda r^2}\end{aligned}$$

Thus with $\lambda = -\frac{1}{4D(t-t_0)+a^2}$, we have

$$\begin{aligned}\nabla^2\rho &= \frac{-\rho_0 a^3}{(4D(t-t_0)+a^2)^{5/2}}\left(6 - \frac{4r^2}{4D(t-t_0)+a^2}\right)\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right) \\ &= \left(-\frac{6\rho_0 a^3}{(4D(t-t_0)+a^2)^{5/2}} + \frac{4r^2\rho_0 a^3}{(4D(t-t_0)+a^2)^{-7/2}}\right)\exp\left(-\frac{r^2}{4D(t-t_0)+a^2}\right)\end{aligned}$$

Hence we can see that $\frac{\partial\rho}{\partial t} = D\nabla^2\rho$, as required.

QUESTION 3

Considering the infinite plane $z = 0$, we see this has uniform charge density ρ_0.

By symmetry, the field points vertically, and the field on the bottom is opposite of that on top, we must have

$$\mathbf{E} = E(z)\hat{\mathbf{z}}$$

with

$$E(z) = -E(-z)$$

Consider a vertical cylinder of height $2h$ and cross-sectional area A. Now only the end caps contribute.

First,

$$\begin{aligned}
Q &= \int_V \rho_0 e^{-k|z|} \, \mathrm{d}V \\
&= \int_{-h}^{h} \int_0^{2\pi} \int_0^R \rho_0 e^{-k|z|} \rho \, \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z \\
&= 2\pi \frac{R^2}{2} \rho_0 \int_{-h}^{h} e^{-k|z|} \, \mathrm{d}z \\
&= A\rho_0 \int_0^h 2e^{-kz} \, \mathrm{d}z \\
&= 2A\rho_0 \left[-\frac{1}{k} e^{-kz} \right]_0^h \\
&= 2A \frac{\rho_0}{k} (1 - e^{-kh})
\end{aligned}$$

And

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E(h)A - E(-h)A = 2AE(h) = 2A \frac{\rho_0}{k\varepsilon_0}(1 - e^{-kh})$$

Hence

$$E(z) = \frac{\rho_0}{k\varepsilon_0}(1 - e^{-kz})$$

as required.

QUESTION 4

We have that

$$\rho(r) = \begin{cases} 0 & \text{if } r < a \\ \rho & \text{if } a < r < b \\ 0 & \text{if } r > b \end{cases}$$

Note that if $r < a$ there is no field; Gauss' law tells us that the flux only depends on the total charge contained inside the surface. Now consider $r > b$. By symmetry, the force is the same in all directions and points outwards radially. So

$$\mathbf{E} = E(r)\hat{\mathbf{r}}$$

Put S to be a sphere of radius $r > b$. Then the total flux is

$$\begin{aligned} \int_S \mathbf{E} \cdot \, \mathrm{d}S &= \int_S E(r)\hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{S} \\ &= E(r) \int_S \hat{\mathbf{r}} \cdot \mathrm{d}\mathbf{S} \\ &= E(r) \cdot 4\pi r^2 \end{aligned}$$

By Gauss's law, we know this is equal to Q/ε_0, and $Q = \frac{4}{3}\pi(b^3 - a^3)\rho$. Therefore,

$$E(r) = \frac{(b^3 - a^3)\rho}{3\varepsilon_0 r^2}$$

and

$$\mathbf{E}(r) = \frac{(b^3 - a^3)\rho}{3\varepsilon_0 r^2}\hat{\mathbf{r}}$$

Now suppose we are inside the region, $a < r < b$. Then

$$\int_S \mathbf{E} \cdot \, \mathrm{d}S = E(r) 4\pi r^2 = \frac{Q}{\varepsilon_0}\left(\frac{r^3 - a^3}{b^3 - a^3}\right)$$

So

$$\begin{aligned} \mathbf{E}(r) &= \frac{Q(r^3 - a^3)}{4\pi\varepsilon_0 (b^3 - a^3) r^2} \\ &= \frac{Q(r^3 - a^3)\rho}{3\varepsilon_0 r^2} \end{aligned}$$

Finally if $r < a$, Gauss' law tells us that the flux depends only on the total charge contained inside the surface, which in this case is none. So $\mathbf{E}(r) = 0$.

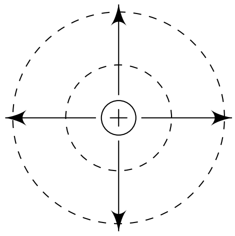
Note that the electric field is discontinuous across the surface. We have

$$
\begin{aligned}
E(r \to b+) - E(r \to b-) &= \frac{(b-a)(b^2+2ab+a^2)\rho}{3\varepsilon_0 b^2} \\
&= \frac{\sigma}{\varepsilon_0}
\end{aligned}
$$

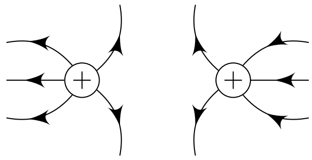
as expected.

QUESTION 5

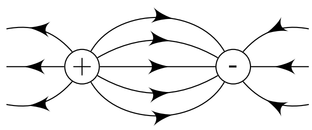
The field lines for a positive charge are:



For two positive charges,



We can also draw field lines for dipoles:



QUESTION 6

The inverse square law, or Coulomb's Law, states that the electric field generated by a particle with total charge Q (at the origin) is given by

$$\mathbf{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

Now consider an infinite line with uniform charge density per unit length η. We use cylindrical polar coordinates. By symmetry, the field is radial, ie.

$$\mathbf{E}(r) = E(r)\hat{\mathbf{r}}$$

Consider an arbitrary point at (r, z_0). We will integrate along the z-axis to find the field at this point; summing the contributions from the changes at $(0, z)$ as z goes from $-\infty$ to ∞. By Coloumb's law;

Here,

$$\begin{aligned}
E(r) &= \int_{-\infty}^{\infty} \frac{\eta}{4\pi\varepsilon_0} \frac{1}{r^2 + (z - z_0)^2} \, \mathrm{d}z \\
&= \frac{\eta}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{1}{r^2 + z^2} \, \mathrm{d}z \\
&= \frac{\eta}{4\pi\varepsilon_0} \left[\frac{1}{r} \arctan\left(\frac{z}{r}\right) \right]_{-\infty}^{\infty} \\
&= \frac{\eta}{4\varepsilon_0 r}
\end{aligned}$$

But this is a different result than what we want... Can't spot my error.

QUESTION 7

The Green's function for the Laplacian is definied to be the solution to:

$$\nabla^2 G(\mathbf{r};\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}')$$

The Green's function in three dimensions is:

$$G(\mathbf{r};\mathbf{r}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

We assume all the charge is contained within some compact region V, then

$$\begin{aligned}\phi(\mathbf{r}) &= -\frac{1}{\varepsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\ \mathrm{d}^3\mathbf{r} \\ &= \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ \mathrm{d}^3\mathbf{r}\end{aligned}$$

Here, the charge is contained in a circular disk of radius a, uniform charge density σ. Using cylindrical polars, the charge at $\mathbf{r} = (0,0,z)$, due to $\mathbf{r}' = (r\cos\phi, r\sin\phi, 0)$, we have $|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2+z^2}$ and hence

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^a \frac{\sigma}{\sqrt{r^2+z^2}} r\ \mathrm{d}r\ \mathrm{d}\phi \\ &= \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r^2+z^2}\right]_{r=0}^a \\ &= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+z^2}-|z|\right)\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{a^2+z^2}} - \mathrm{sgn}(z)\right)\end{aligned}$$

Thus again, with $\mathbf{E} = E(z)\hat{\mathbf{z}}$, we see we have the expected discontinuity

$$E(z\to 0+) - E(z\to 0-) = \frac{\sigma}{\varepsilon_0}$$

As $z\to\infty$,

$$\begin{aligned}\frac{z}{\sqrt{a^2+z^2}} &= \left(1+\frac{z^2}{a^2}\right)^{-1/2} \\ &= 1 + -\frac{z^2}{2a^2} + \cdots\end{aligned}$$

$$\mathbf{E} \approx \frac{\sigma a^2}{4\pi\varepsilon_0 z^2}\hat{\mathbf{z}}$$

which is Coloumb's Law for a particle due to charge $Q = \sigma a^2$

QUESTION 8

From Q7 we have the result that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}$$

Very far from V, ie. $|\mathbf{r}| \gg |\mathbf{r}'|$, we can use the Taylor expansion

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r} + \mathbf{r}'\cdot\nabla\left(\frac{1}{r}\right) - \frac{1}{2}(\mathbf{r}'\cdot\nabla)^2\left(\frac{1}{r}\right) + \cdots \\ &= \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} - \frac{1}{2}\left(\frac{\mathbf{r}'\cdot\mathbf{r}'}{r^3} - \frac{3(\mathbf{r}'\cdot\mathbf{r})^2}{r^5}\right) + \cdots\end{aligned}$$

Then we get

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{r}')\left\{\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} - \frac{1}{2}\left(\frac{\mathbf{r}'\cdot\mathbf{r}'}{r^3} - \frac{3(\mathbf{r}'\cdot\mathbf{r})^2}{r^5}\right) + \cdots\right\}\mathrm{d}^3\mathbf{r}' \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \frac{1}{2}\frac{\mathbb{Q}_{ij}r_ir_j}{r^5} + \cdots\right)\end{aligned}$$

where

$$\begin{aligned}Q &= \int_V \rho(\mathbf{r}')\,\mathrm{d}V' \\ \mathbf{p} &= \int_V \mathbf{r}'\rho(\mathbf{r}')\,\mathrm{d}V' \\ \hat{\mathbf{r}} &= \frac{\mathbf{r}}{\|\mathbf{r}\|}. \\ \mathbb{Q}_{ij} &= \int_V \mathrm{d}^3r'(3r_i'r_j' - \delta_{ij}r'^2)\rho(\mathbf{r}')\end{aligned}$$

– For the first two charges we have

$$\rho(\mathbf{r}') = q\delta(\mathbf{r}') - q\delta(\mathbf{r}'-\mathbf{d})$$

Then

$$\begin{aligned}Q &= \int_V q\delta(\mathbf{r}') - q\delta(\mathbf{r}'-\mathbf{d})\,\mathrm{d}V' \\ &= 1 - 1 = 0,\end{aligned}$$

$$\begin{aligned}\mathbf{p} &= \int_V q\delta(\mathbf{r}') - q\delta(\mathbf{r}'-\mathbf{d})\mathbf{r}'\,\mathrm{d}V' \\ &= q(0-\mathbf{d}) \\ &= -q\mathbf{d}\end{aligned}$$

and

$$\begin{aligned}
\mathbb{Q}_{ij} &= \int_V (3r_i'r_j' - \delta_{ij}r'^2)\left[q\delta(\mathbf{r}') - q\delta(\mathbf{r}' - \mathbf{d})\right]\ \mathrm{d}^3r' \\
&= -q\int_V (3r_i'r_j' - \delta_{ij}r'^2)\delta(\mathbf{r}' - \mathbf{d})\ \mathrm{d}^3r' \\
&= -q\left(3d_id_j - \delta_{ij}|d|^2\right)
\end{aligned}$$

Now $\mathbf{d} = (d, 0, 0)$ so $\mathbb{Q}_{11} = -q(3d^2 - d^2) = -2d^2$, and $\mathbb{Q}_{ij} = 0$ for $i, j \neq 1$

– Next, similarly we have

$$\rho(\mathbf{r}') = q\delta(\mathbf{r}') - q\delta(\mathbf{r}' - \mathbf{d}_1) - q\delta(\mathbf{r}' - \mathbf{d}_2) + -q\delta(\mathbf{r}' - \mathbf{d}_3)$$

where $\mathbf{d}_1 = (d, 0, 0), \mathbf{d}_2 = (0, d, 0)$, and $\mathbf{d}_3 = (d, d, 0)$.
Again it can be easily verified that $Q = 0$. The dipole this times gives

$$\begin{aligned}
p &= q(0 - \mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3) \\
&= 0
\end{aligned}$$

And the Quadrupole:

$$\mathbb{Q}_{ij} = -q\int_V (3r_i'r_j' - \delta_{ij}r'^2)\left[\delta(\mathbf{r}' - \mathbf{d}_1) + \delta(\mathbf{r}' - \mathbf{d}_2) - \delta(\mathbf{r}' - \mathbf{d}_3)\right]\ \mathrm{d}^3r'$$

$$\begin{aligned}
\mathbb{Q}_{11} &= -q((3d^2 - d^2) + (-d^2) - (3d^2 - 2d^2)) \\
&= 0
\end{aligned}$$

Similarly $\mathbb{Q}_{22} = 0$. Also have

$$\begin{aligned}
\mathbb{Q}_{12} &= -q\left((0 - 0) + (0 - 0) - (3d^2 - 0\right) \\
&= 3qd^2 = \mathbb{Q}_{21}
\end{aligned}$$

– Now,

QUESTION 9

We define the electric dipole moment to be $\mathbf{p} = Q\mathbf{d}$. Now

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \mathbf{d}\frac{\mathrm{d}Q}{\mathrm{d}t} \\
&= \mathbf{d}\int_V \frac{\partial}{\partial t}\rho \, \mathrm{d}V \quad V \text{ fixed} \\
&= \mathbf{d}\int_V -\nabla \cdot \mathbf{J} \, \mathrm{d}V \quad \text{ by continuity equation} \\
&= -\mathbf{d}\int_S \mathbf{J}\mathrm{d}\mathbf{S}
\end{aligned}$$

QUESTION 10

$$U = \frac{1}{2}\int \rho(\mathbf{r})\phi(\mathbf{r})\ \mathrm{d}^3\mathbf{r}.$$

Hence we obtain

$$\begin{aligned} U &= \frac{\varepsilon_0}{2}\int (\nabla \cdot \mathbf{E})\phi\ \mathrm{d}^3\mathbf{r} \\ &= \frac{\varepsilon_0}{2}\int [\nabla \cdot (\mathbf{E}\phi) - \mathbf{E} \cdot \nabla\phi]\ \mathrm{d}^3\mathbf{r}. \end{aligned}$$

The first term is a total derivative and vanishes. In the second term, we use the definition $\mathbf{E} = -\nabla\phi$ and obtain

$$U = \frac{\varepsilon_0}{2}\int \mathbf{E} \cdot \mathbf{E}\ \mathrm{d}^3\mathbf{r}.$$

This result shows that the potential energy depends only on the field itself, and not the charges.

Next, considering a charge Q contained within some compact region V; we have

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\ \mathrm{d}^3\mathbf{r}$$

Here, $\rho(\mathbf{r}') = Q$, and using spherical polars, we model $\mathbf{r}(r, \theta, \phi)$ in the usual way; take our point $\mathbf{r} = (a, 0, 0)$, $\mathbf{r}' = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, so $|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2\sin^2\theta + (a - r\cos\theta)^2} = \sqrt{a^2 - 2ar\cos\theta + r^2}$

$$\phi(\mathbf{r})$$

Did not finish due to time constraints.

QUESTION 11