

Part IB — Methods Example Sheet 3

Supervised by Dr. Saxton
Examples worked through by Christopher Turnbull

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QUESTION 1

- We are given

$$\underbrace{\ddot{\theta} + 2p\dot{\theta} + (p^2 + q^2)}_{\mathcal{L}} \theta = f(t)$$

with $\theta(0) = \dot{\theta}(0) = 0$, and $p > 0, q \neq 0$.

Want to find G such that $\mathcal{L}G = \delta(t - \tau)$, so that for each value of τ , the Green's function will solve the homogeneous equation $LG = 0$ whenever $t \neq \tau$.

We construct G for $0 \leq t < \tau$ as a general solution of the homogeneous equation, so that $G = Ay_1(t) + By_2(t)$. Have

$$G(t, \tau) = \Theta(t - \tau)e^{-p(t-\tau)}[C(\tau)\cos(q(t-\tau)) + D(\tau)\sin(q(t-\tau))]$$

where Θ is the Heaviside step function. Continuity demands $G(\tau, \tau) = 0$ so $C(\tau) = 0$. The jump condition (with $\alpha(\tau) = 1$) enforces $D(\tau) = \frac{1}{q}$. Therefore the Green's function is

$$G(t, \tau) = \Theta(t - \tau)\frac{1}{q}e^{-p(t-\tau)}\sin(q(t-\tau))$$

and the general solution to $\mathcal{L}\theta = f(t)$ obeying $\theta(0) = \dot{\theta}(0) = 0$ is

$$\theta(t) = \frac{1}{q} \int_0^t e^{-p(t-\tau)} \sin(q(t-\tau)) f(\tau) d\tau$$

- Next we use Fourier Transforms. Taking the Fourier transform of the differential equation gives

$$(i\omega)^2 \tilde{\theta} + 2ip\omega \tilde{\theta} + (p^2 + q^2) \tilde{\theta} = \tilde{f}$$

and so

$$\tilde{\theta} = \frac{\tilde{f}}{-\omega^2 + 2ip\omega + (p^2 + q^2)} =: \tilde{R}(\omega) \tilde{f}(\omega)$$

which solves the equation algebraically in Fourier space. Note that

$$\begin{aligned} \tilde{R}(\omega) &= \frac{1}{-\omega^2 + 2ip\omega + (p^2 + q^2)} \\ &= \frac{1}{(i\omega + p)^2 - (qi)^2} \\ &= \frac{1}{2qi} \left[\frac{1}{i\omega + p - qi} - \frac{1}{i\omega + p + qi} \right] \end{aligned}$$

To solve for θ in real space we take the inverse Fourier transform to find

$$\begin{aligned}
 \theta(t) &= \int_0^t R(t-u)f(u) \, du \\
 &= \int_0^t \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{R}(\omega) e^{i\omega(t-u)} \, d\omega \right] f(u) \, du \\
 &= \frac{1}{q} \int_0^t \underbrace{\left[\frac{1}{4i\pi} \int_{-\infty}^{\infty} e^{i\omega(t-u)} \left[\frac{1}{i\omega + p - qi} - \frac{1}{i\omega + p + qi} \right] \, d\omega \right]}_{(*)} f(u) \, du
 \end{aligned}$$

which agrees with the first result, if we can show that

$$(*) = e^{-p(t-u)} \sin[q(t-u)]$$

but not sure how to evaluate the integral.

QUESTION 2

The general homogeneous solution is $c_1 \sinh \lambda x + c_2 \cosh \lambda x$ so we can take $y_1(x) = \sinh \lambda x$ and $y_2(x) = \sinh[\lambda(1-x)]$ as our homogeneous solutions satisfying the boundary conditions at $x = 0$ and $x = 1$ respectively. Then

$$G(x; \varepsilon) = \begin{cases} A(\varepsilon) \sinh(\lambda x) & \text{when } 0 \leq x < \varepsilon \\ B(\varepsilon) \sinh[\lambda(1-x)] & \text{when } \varepsilon < x \leq 1 \end{cases}$$

Applying the continuity condition we get

$$A \sinh \lambda \varepsilon = B \sinh[\lambda(1-\varepsilon)]$$

while the jump condition gives

$$B(-\lambda \cosh[\lambda(1-\varepsilon)]) - A\lambda \cosh(\lambda \varepsilon) = -1$$

Solving the for A and B gives us the Green's function

$$G(x; \varepsilon) = -\frac{1}{\lambda \sinh \lambda} [\Theta(\varepsilon - x) \sinh[\lambda(1-\varepsilon)] \sinh \lambda x + \Theta(x - \varepsilon) \sinh(\varepsilon \lambda) \sinh[\lambda(1-x)]]$$

Using this Green's function we are immediately able to write down the complete solution as

$$y = -\frac{1}{\lambda \sinh \lambda} \left\{ \sinh \lambda x \int_x^1 f(\varepsilon) \sinh \lambda(1-\varepsilon) \, d\varepsilon + \sinh \lambda(1-x) \int_0^x f(\varepsilon) \sinh \lambda \varepsilon \, d\varepsilon \right\}$$

QUESTION 3

Use the substitution $y = z/x$, (quotient rule speeds thing up)

$$L_x y = -\frac{1}{x} \frac{d^2 z}{dx^2} + \frac{z}{x}$$

So

$$y = \frac{1}{x} (c_1 \cosh x + c_2 \sinh x)$$

$$y = \frac{1}{x} (c'_1 e^x + c'_2 e^{-x})$$

For solutions that are (a) bounded as $x \rightarrow 0$, we have $y = A \frac{1}{x} \sinh x$. For solutions that are (b) bounded as $x \rightarrow \infty$, have $y = B \frac{1}{x} e^{-x}$.

Green's function satisfying $L_x G = \delta(x - \xi)$ and both conditions (a), (b) is given by

$$G(x; xi) = \begin{cases} A(\xi) \frac{\sinh x}{x} & \text{if } 0 < x < \xi \\ B(\xi) \frac{e^{-x}}{x} & \text{if } \xi < x \end{cases}$$

Continuity \Rightarrow

$$A \sinh \xi = B e^{-\xi}$$

Jump ($\alpha = -1$) \Rightarrow

$$B e^{\xi} (\xi + 1) + A (\xi \cosh \xi - \sinh \xi) = \xi^2$$

Solving,

$$A = \xi e^{-\xi} \quad B = \xi \sinh \xi$$

Next, to solve

$$L_x y(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq R \\ 0 & \text{if } x > R \end{cases}$$

Using this G ,

$$y = x^{-1} e^{-x} \int_0^x \xi \sinh \xi \, d\xi + x^{-1} \sinh x \int_x^R \xi e^{-\xi} \, d\xi$$

For $0 \leq x \leq R$, solve. (things combine into a \sinh) For $x > R$ note the RHS integral vanishes, so solve.

QUESTION 4

Can use a similar argument to the one in lectures to show that $G(t, \tau) = 0$ whenever $t \in [0, \tau]$. Following the usual procedure we get

$$G(t, \tau) = \Theta(t - \tau)[A(\tau) \cos[k(t - \tau)] + B(\tau) \sin[k(t - \tau)] + C(\tau)(t - \tau) + D(\tau)]$$

Continuity demands $G(\tau, \tau) = G'(\tau, \tau) = G''(\tau, \tau) = 0$, yielding

$$A(\tau) + D(\tau) = 0$$

$$kB(\tau) + C(\tau) = 0$$

$$A(\tau) = 0$$

respectively. The third equation also implies $D(\tau) = 0$, and the jump condition on G''' gives

$$-k^3 B(\tau) = 1$$

$\Rightarrow B(\tau) = -k^{-3} \Rightarrow C(\tau) = k^{-2}$, showing the Green's function is indeed what is required.

Solution to $\mathcal{L}y = e^{-t}$ is given by

$$y(t) = \int_0^t [k^{-2}(t - \tau) - k^{-3} \sin k(t - \tau)] e^{-\tau} \, d\tau$$

QUESTION 5

Under the substitution $y = \phi(x)$, hence $\frac{dx}{dy} = \frac{1}{|\phi'(x)|}$ (monotone increasing) the result becomes easy to show

$$\begin{aligned} \int_a^b f(x) \delta[\phi(x)] \, dx &= \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y)) \frac{1}{|\phi'(\phi^{-1}(y))|} \delta(y) \, dy \\ &= f(\phi^{-1}(0)) \frac{1}{|\phi'(\phi^{-1}(0))|} \\ &= \frac{f(c)}{|\phi'(c)|} \quad \text{as } \phi^{-1}(0) = c \end{aligned}$$

If monotone decreasing, $\frac{dx}{dy} = -\frac{1}{|\phi'(x)|}$, but $\phi(b) < \phi(a)$, so $\int_{\phi(a)}^{\phi(b)} = -\int_{\phi(b)}^{\phi(a)}$ and the result holds.

Hence for a general $\phi(x)$ with simple zeros at $c_1, c_2, c_3 \dots, c_N$ we have

$$\int_a^b f(x) \delta[\phi(x)] \, dx = \sum_{i=1}^N \frac{f(c_i)}{|\phi'(c_i)|} \quad (*)$$

Next for any $a \in \mathbb{R} \setminus \{0\}$

$$\int_{\mathbb{R}} \delta(at) \phi(t) \, dt = \frac{1}{|a|} \int_{\mathbb{R}} \delta(y) \phi(y/a) \, dy = \frac{1}{|a|} \phi(0)$$

so we may write $\delta(at) = \delta(t)/|a|$.

Lastly, we apply (*) for $f(x) = |x|$ and $\phi(x) = x^2 - a^2$, which has simple zeros at $x = a$ and $x = -a$. Get

$$\begin{aligned} \int_{-\infty}^{\infty} |x| \delta(x^2 - a^2) \, dx &= \frac{f(a)}{|\phi'(a)|} + \frac{f(-a)}{|\phi'(-a)|} \\ &= \frac{|a|}{2|a|} + \frac{|a|}{2|a|} \\ &= 1 \end{aligned}$$

as required

QUESTION 6

QUESTION 7

- (i)
- $f(x) = 1, |x| < c$
- .

$$\begin{aligned}
\tilde{f}(k) &= \int_{-c}^c e^{-ikx} dx \\
&= \frac{i}{k} [e^{-ikx}]_{-c}^c \\
&= \frac{i}{k} (-2i \sin kc) \\
&= \frac{2 \sin kc}{k}
\end{aligned}$$

- (ii)

$$\mathcal{F}[e^{iax} f(x)] = \tilde{f}(k - a)$$

we have for $f(x) = e^{iax}$ and using our previous answer,

$$\tilde{f}(k) = \frac{2 \sin[(k - a)c]}{k - a}$$

- (iii)
- $f(x) = \sin(ax)$

Noting that this is the imaginary part of what we've just done (but not seeing how to make use of that... so stuck with calculating) :

No idiot, use $\sin(ax) = \frac{1}{2i(e^{iax} - e^{-iax})}$ and linearity of Fourier Transform...

$$\begin{aligned}
\tilde{f}(k) &= \int_{-c}^c \sin(ax) e^{-ikx} dx \\
&= \frac{i}{k} [\sin(ax) e^{-ikx}]_{-c}^c - \frac{ia}{k} \int_{-c}^c \cos(ax) e^{-ikx} dx \\
&= \frac{i}{k} 2 \sin(ac) \cos(kc) - \frac{ia}{k} \left(\frac{i}{k} [\cos(ax) e^{-ikx}]_{-c}^c + \frac{ia}{k} \int_{-c}^c \sin(ax) e^{-ikx} dx \right) \\
&= \frac{i}{k} 2 \sin(ac) \cos(kc) + \frac{a}{k^2} (-2 \cos(ax) \sin(kc) + a \tilde{f}(k))
\end{aligned}$$

Thus multiplying upon rearranging, we get

$$\tilde{f}(k) = \frac{2i(-k \cos[kc] \sin[ac] + a \cos[ac] \sin[kc])}{a^2 - k^2}$$

- (iv) Next, by the differentiating property of Fourier Transforms, know that

$$\mathcal{F}[a \cos(ax)] = ik \tilde{f}(k)$$

where $\tilde{f}(k)$ is the FT of $f(x) = \sin(ax)$. Then, by scaling,

$$\mathcal{F}[\cos(ax)] = |a|ik\tilde{f}(k)$$

Thus the Fourier transform of $\cos(ax)$ is given by

$$\tilde{f}(k) = \frac{-2ka(-k\cos[kc]\sin[ac] + a\cos[ac]\sin[kc])}{a^2 - k^2}$$

QUESTION 8

Have that

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x > 0 \end{cases}$$

The Fourier transform is given by

$$\begin{aligned} \tilde{f}(k) &= \int_0^{\infty} e^{-(ik+1)x} \, dx \\ &= \frac{1}{-ik-1} \left[e^{-(ik+1)x} \right]_0^{\infty} \\ &= \frac{1}{1+ik} \\ &= \frac{1-ik}{1+k^2} \end{aligned}$$

The inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k) \, dk$$

Thus

$$\begin{aligned} f(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1-ik}{1+k^2} \, dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+k^2} \, dk - \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{k}{1+k^2} \, dk \\ &= \frac{1}{2\pi} [\arctan k]_{-\infty}^{\infty} - \frac{i}{4\pi} \underbrace{[\log(1+k^2)]_{-\infty}^{\infty}}_{=0} \\ &= \frac{1}{2\pi} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\ &= \frac{1}{2} \end{aligned}$$

Wait. Note that

QUESTION 9

This is easy. Define $g(x) = e^{-x^2}$, calculate $\tilde{g}(k)$, then do rephasing and scaling in any order. Is good practice. Should get

$$\tilde{f}(k) = e^{-ik\mu} \frac{1}{n} \sqrt{\pi} e^{-k^2/4n^2}$$

$$f(x) = e^{-n^2(x-\mu)^2}$$

$$f(x + \mu) = e^{-n^2x^2}$$

$$f'(x + \mu) = -2n^2xe^{-n^2x^2}$$

Calculating the Fourier Transform of the last function gives

$$\begin{aligned} \tilde{f}(k) &= -2n^2 \int_{-\infty}^{\infty} xe^{-n^2x^2} e^{-ikx} dx \\ &= \end{aligned}$$

QUESTION 10

Let us suppose we have N measurements of a function $h(t)$, where N is even, with constant sampling interval Δ , ie. we have the set of measurements

$$h_m = h(t_m), \quad t_m = m\Delta, \quad m = 0, 1, \dots, N-1$$

Parseval's theorem for DFT is

$$\sum_{m=0}^{N-1} |h(t_m)|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{h}_d(f_n)|^2 \quad (*)$$

where \tilde{h}_d is the *Discrete Fourier transform*

$$\tilde{h}_d(f_n) = \sum_{m=0}^{N-1} h_m \exp \left[\frac{-2\pi i}{N} mn \right]$$

and

$$f_n = \frac{n}{N\Delta}, \quad n = -N/2, \dots, N/2,$$

To prove (*), consider a fixed $h(t_m)$ on the LSH. Applying the inversion formula, and making a Riemann approximation to the integral, we obtain

$$\begin{aligned} h(t_m) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t_m} \tilde{h}(\omega) \, d\omega \\ &= \int_{-\infty}^{\infty} e^{2\pi i f t_m} \tilde{h}(f) \, df \quad (2\pi f = \omega) \\ &\approx \frac{\Delta}{\Delta N} \sum_{n=0}^{N-1} \tilde{h}_d(f_n) e^{2\pi i f_n t_m} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{h}_d(f_n) \exp \left[\frac{2\pi i}{N} mn \right] \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{h}_d(f_n) \kappa^{mn} \end{aligned}$$

where $\kappa = e^{2\pi i/N}$ is an N th root of unity.
Now

$$\begin{aligned}
|h(t_m)|^2 &= (h(t_m))^* h(t_m) = \frac{1}{N^2} \sum_{p,q=0}^{N-1} \tilde{h}_d^*(f_p) \kappa^{-mp} \tilde{h}_d(f_q) \kappa^{mq} \\
&= \frac{1}{N^2} \sum_{p,q=0}^{N-1} \tilde{h}_d^*(f_p) \tilde{h}_d(f_q) \delta_{pq} \\
&= \frac{1}{N^2} \sum_{p=0}^{N-1} \tilde{h}_d^*(f_p) \tilde{h}_d(f_p) \\
&= \frac{1}{N} \sum_{p=0}^{N-1} |h_d(f_p)|^2
\end{aligned}$$

noting the independence from m . Hence, summing over N on both sides gives (*) as required, as the right hand side just gains a factor of N .

QUESTION 11

Thus the Fourier transform of $\cos(x)$ (Q7 (iv)) is given by

$$\begin{aligned}\tilde{f}(k) &= \frac{-2k \left(-k \cos\left[k\frac{\pi}{2}\right] \sin\left[\frac{\pi}{2}\right] + \cos\left[\frac{\pi}{2}\right] \sin\left[k\frac{\pi}{2}\right] \right)}{1 - k^2} \\ &= \frac{2}{1 - k^2} \cos\left(\frac{k\pi}{2}\right)\end{aligned}$$

And the Fourier transform of the derivative $-\sin(x)$ is given by

$$\tilde{f}(k) = -i \frac{2k}{1 - k^2} \cos\left(\frac{k\pi}{2}\right)$$

QUESTION 12

Inverse Fourier transform of $\tilde{f}(k)$ is

$$\begin{aligned}
 f(x) &= \frac{1}{2\pi} \int_{-1}^1 e^{ikx} e^k - e^{-k} dk \\
 &= \frac{1}{\pi} \int_{-1}^1 e^{ikx} \sinh k dk \\
 &= \frac{1}{\pi} \left[-\frac{i}{x} e^{ikx} \cosh k \right]_{-1}^1 + \frac{i}{x\pi} \int_{-1}^1 e^{ikx} \sinh k dk \\
 &= \frac{-i}{\pi x} (e^{ix} \cosh 1 + e^{-ix} \cosh 1) + \frac{i}{x\pi} \left(\left[-\frac{i}{x} e^{ikx} \cosh k \right]_{-1}^1 + \frac{i}{x} \int_{-1}^1 e^{ikx} \sinh k dk \right) \\
 &= \frac{-2i}{\pi x} (\cos x \cosh 1) + \frac{i}{x\pi} \left(-\frac{2i}{x} \sin x \cosh 1 + \frac{i}{x} f(x) \right) \\
 \Rightarrow \pi x^2 f(x) &= -2ix \cos x \cosh 1 + 2i \sin x \cosh 1 - f(x)
 \end{aligned}$$

Hence rearranging gives

$$f(x) = \frac{2i}{\pi(1+x^2)} (\cosh 1 \sin x - x \cos x \cosh 1)$$

(a long algebraic exercise. good for me?)

Next, take the Fourier transform of $\nabla^2(x, y) = 0$ with respect to x ;

$$\begin{aligned}
 \int_{-\infty}^{\infty} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) e^{-ikx} dx &= 0 \\
 \Rightarrow 0 &= \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{-ikx} dx + \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y^2} e^{-ikx} dx \\
 &= -k^2 \tilde{\phi}(k) + \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} \phi e^{-ikx} dx \\
 &= -k^2 \tilde{\phi}(k) + \frac{\partial^2 \tilde{\phi}(k)}{\partial y^2}
 \end{aligned}$$

Solving this second order ODE in k gives

$$\tilde{\phi}(k, y) = A(k)e^{ky} + B(k)e^{-ky}$$

We have the boundary conditions $\tilde{\phi}(k, 0) = \tilde{f}(k)$, and $\tilde{\phi}(k, 1) = 0$. This gives

$$\begin{aligned}
 A + B &= e^k - e^{-k} \\
 Ae^k + Be^{-k} &= 0
 \end{aligned}$$

respectively. Solving gives $A = -e^{-k}$, $B = e^k$. Thus, we take the inverse FT to finish.

$$\phi(x, y) = -\frac{1}{2\pi} \int_{-1}^1 e^{k(y-1)} - e^{-k(y-1)} \, dk$$