# Part IB — Linear Algebra

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# 0 Introduction

# 1 Vector Spaces

**Definition.** An $\mathbb{F}$-Vector space (a vector space on $\mathbb{F}$) is an abelian group $(V, +)$ equipped with a function[1] $F \times V \to V$, $(\lambda, v) \mapsto \lambda V$

$$\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$$

$$(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$$

$$\lambda(\mu v) = \lambda\mu v$$

$$1v = v$$

$$v + \mathbf{0} = v$$

for all $\lambda_i, \mu \in F$, $v_i \in V$

Note that we will not be underlining our vectors, as this is cumbersome here. We will however be using $\mathbf{0}$ to denote the zero vector.

**Example.** For all $n \in \mathbb{N}$, $\mathbb{F}^n$ = space of column vectors of length $n$, entries in $\mathbb{F}$. We understand the definition as entry-wise addition, entry-wise scalar multiplication

**Example.** $M_{m,m}(\mathbb{F})$, the set of $m \times m$ matrices with entries in $\mathbb{F}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

again all operations defined entry-wise

**Example.** For any set $X$, $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$ Addition and scalar multiplication defined pointwise $= f_1(x) + f_2(x)$.

**Exercise.** Show that the above examples satisfy the axioms

**Proposition.** $0v = \mathbf{0}$ for all $v \in V$.

*Proof.* ( $(0+0)v = 0v \iff 0v + 0v = 0v \iff 0v = \mathbf{0}$) □

**Exercise.** Show[2] that $(-1)v = -v$

**Definition.** Let V be an $\mathbb{F}$-vector space. A subset $U$ of $V$ is a subspace ( $U \leq V$) if:

(i) $\mathbf{0} \in U$

(ii) $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$ "$U$ is closed under addition..."

(iii) $u \in U$, any $\lambda \in \mathbb{F} \Rightarrow \lambda u \in U$ "...and scalar multiplication"

**Exercise.** If $U$ is a subspace of $V$, then $U$ is also an $\mathbb{F}$-vector space.

[1] scalar multiplication
[2] Hint: Use the previous proposition

**Example.** Let $V = \mathbb{R}^{\mathbb{R}}$, then $f : R \to R$. The set of all continuous functions $C(\mathbb{R})$ are a subspace. An even smaller subspace is the set of all polynomials.

**Exercise.** Define $U \subseteq R^3$ as:

$$\left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad a_1 + a_2 + a_3 = t \right\}$$

for some constant $t$. Check that this is a subspace of $\mathbb{R}^3$ if and only if $t = 0$.

**Proposition.** Let $V$ be an $F$-vector space, $U, W \leq V$. Then $U \cap W \leq V$.

*Proof.* (i) $0 \in U$, $0 \in W \Rightarrow 0 \in U \cap W$

(ii) Suppose $u, v \in U \cap W$, $\lambda, \mu \in F$. $U$ is a subspace $\Rightarrow \lambda u + \mu v \in W$. Similarly $\lambda u + \mu v \in U \in W$, so it is in the intersection.

$\square$

**Example.** $V = \mathbb{R}^3$, $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0 \right\}$, $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid y = 0 \right\}$ then $U \cap W = U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0, y = 0 \right\}$ (intersect along the $z$-axis)

Note: union of family of subspaces is almost never a subspace itself.

**Definition.** Let $V$ be an $F$-vector space, $U, W \leq V$. The *sum* of $U$ and $W$ is the set:

$$U + W = \{u + w \mid u \in U, w \in W\}$$

**Proposition.** $U + W \leq V$

*Proof.* $\mathbf{0} \in U, W \Rightarrow \mathbf{0} + \mathbf{0} = \mathbf{0} \in U + W$
$u_1, u_2 \in U$, $w_1, w_2 \in W$,

$$(u_1 + w_1) + (u_2 + w_2) = \underbrace{(u_1 + u_2)}_{\in U} + \underbrace{(w_1 + w_2)}_{\in W}$$

Similarly for scalar multiplication (ex.) $\square$

Note: $U + W$ is the smallest subspace containing both $U$ and $W$. (This is becaues all elements of the form $u + w$ ae forced to be in such a subspace by the "closed under addition" axiom)

**Definition.** $V$ is an $\mathbb{F}$-vector space, $U \leq V$. The quotient space[3] $V/U$ is the abelian group $V/U$ equipped with scalar multiplication;

$$F \times V/U \to V/U$$

$$(\lambda, v + U) \mapsto \lambda v + U$$

[3] think of this as the collection of cosets of $U$ in $V$

**Proposition.** This is well-defined, and $V/U$ is an $F$-vector space.

*Proof.* Well-defined: Suppose $v_1 + U = v_2 + U \in V/U$. $\Rightarrow (v_2 - v_1) \in U \Rightarrow (\lambda v_2 - \lambda v_1) \in U \Rightarrow \lambda v_2 + U = \lambda v_1 + U \in V/U$

To show that it is an $\mathbb{F}$-vector space, we must show that the axioms hold. These follow from the axioms of $V$. $\lambda(\mu(v+U)) = \lambda(\mu v + U) = \lambda(uv) + U = (\lambda u)v + U = \lambda u(v \in U)$ (scalar multiplication on $V/U$).

Ex. Other axioms follow similarly from using vecton space axioms

□

**Definition.** $V$ is an $\mathbb{F}$-vector space, $S \subset V$. The *span* of $S$ is denoted by

$$< S >= \left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{F} \right\}$$

ie. the set of all finite linear combinations, all but finitely many of the $\lambda_s$ are zero.

Remark: $< S >$ is the smallest subspace of $V$ which contains[4] all of the elements of $S$

Convention: $< \emptyset >= \{\mathbf{0}\}$.

**Example.** $V = \mathbb{R}^3$,

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \right\}$$

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$$< S >= \{ \begin{pmatrix} a \\ b \\ 2b \end{pmatrix} \} \mid a, b \in \mathbb{R}$$

ie. we have took linear combinations of the first two. We don't need the third one.

**Example.** For $X$ a set, define $\delta_x(y) : X \to \mathbb{F}$ as

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

$$< \delta_x \mid x \in X >= \{ f \in \mathbb{R}^X \mid f \text{ has finite support} \}$$

$$=< x \in X \mid f(x) \neq 0 >$$

**Definition.** $S$ *spans* $V$ if $< S >= V$

**Definition.** $V$ is *finite dimensiona*l over $\mathbb{F}$ if it is spanned by a set that is finite.

[4]This is essentially a tautology

**Definition.** The vectors $v_1, \cdots, v_n$ are *linearly indepedent* over $\mathbb{F}$ if

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \Rightarrow \lambda_i \text{ for all } i$$

some coefficients $\lambda_i \in \mathbb{F}$. $S \subset V$ is linearly independent if every finite subset of it is.

**Example.** The fist example, $u, v, w$ are not linearly independent.

**Example.** The set $\{\delta_X \mid x \in X\}$ is linearly indepndent.

**Definition.** If *not* linearly indepnedent, say a set is linearly dependent.

**Definition.** $S$ is a *basis* of $V$ if it is linearly indepnedent and spans $V$

**Example.** $\mathbb{F}^n$ standard basis: $e_1, e_2, \cdots, e_n$.

**Example.** $V = \mathbb{C}$ over $\mathbb{C}$ has natural basis $\{1\}$, over $\mathbb{R}$ has natural basis $\{1, i\}$

**Example.** $V = \mathcal{P}(\mathbb{R})$ space of all polynomials, has natural basis

$$\{1, x, x^2, x^3, \cdots\}$$

**Exercise.** Check this carefully

**Lemma.** $V$ is an $\mathbb{F}$-vector space. The vectors $v_1, \cdots, v_n$ form a basis of $V$ iff each vector $v \in V$ has a unique expression

$$v = \sum_{i=1}^{n} \lambda_i v_i, \text{ with } \lambda_i \in \mathbb{F}$$

*Proof.* ($\Rightarrow$) Fix $v \in V$. The $v_i$ span, so

$$\exists \lambda_i \in \mathbb{F} \text{ s.t. } v = \sum \lambda_i v_i$$

Suppose also $v = \sum \mu_i v_i$ for some $\mu_i \in \mathbb{F}$. $\sum (\mu_i - \lambda_i) v_i = \mathbf{0}$.

The $v_i$ are linearly indepnedent so $\mu_i - \lambda_i = 0$ for all $i$, $\lambda_i = \mu_i$

($\Leftarrow$) The $v_i$ span $V$, since any $v \in V$ is a linear combination of them. IF $\sum_{i=1}^{n} \lambda_i v_i = \mathbf{0}$. Note that $\mathbf{0} = \sum_{i=1}^{n} 0 v_i$. By uniqueness (applied to $\mathbf{0}$), $\lambda_i = 0$ for all $i$. □

**Lemma.** If $v_1, \cdots, v_n$ span $V$ (over $\mathbb{F}$), then some subset of $v_1, \cdots, v_n$ is a basis for $V$ (over $\mathbb{F}$).

*Proof.* If $v_1, \cdots, v_n$ ilnearly indepnedent, done. Otherwise for some $l$, there exist $\alpha_1, \cdots, \alpha_{l-1} \in \mathbb{F}$ such that

$$v_l = \alpha_1 v_1 + \cdots + \alpha_{l-1} v_{l-1}$$

( If $\sum \lambda_i v_i = \mathbf{0}$, not all $\lambda_i = 0$. Take $l$ maximaml with $\lambda_i \neq 0$, just $\alpha_i = -\lambda_i / \lambda_l$ ).

Now $v_2, \cdots, v_{l-1}, v_{l+1}, \cdots, v_n$ still span $V$. Continue interatively until get linear independence.

□

**Theorem.** (Steinitz exchange lemma) Let $V$ be a finite dimensional vector space over $\mathbb{F}$. Take $v_1, \cdots, v_m$ to be linearly independent $w_1, \cdots, w_n$ to span $V$.

Then $m \leq n$, and reordering the spanning set if needed,

$$v_1, \cdots, v_m, w_{m+1}, \cdots, w_n$$

span $V$.

*Proof.* (Induction) Suppose that we've replaced $l (\geq 0)$ of the $w_i$. Reordering the $w_i$ if needed, $v_1, \cdots, v_l, w_{l+1}, \cdots, w_n$ span $V$.

If $l = m$, done.

If $l < m$, then

$$v_{l+1} = \sum_{i=1}^{l} \alpha_i v_i + \sum_{i>l} \beta_i w_i$$

$\alpha_i, \beta_i \in \mathbb{F}$. As the $v_i$ are lin. indep, $\beta_i \neq 0$ for some $i$. (After reordering, $\beta_{l+1} \neq 0$).

$$w_{l+1} = \frac{1}{\beta_{l+1}} \left( v_{l+1} - \sum_{i \leq l} \alpha_i v_i - \sum_{i > l+1} \beta_i w_i \right)$$

This $v_1, \cdots, v_{l+1}, w_{l+2}, \cdots, w_n$ also spans $V$. After $m$ steps, $w_i$ will have replaced $m$ of the $w_i$ by $v_i$. Thus $m \leq n$. $\square$

**Theorem.** If $V$ is a finite dimensional vector space over $\mathbb{F}$, then any two bases for $V$ have the same number of elements. This is what we call the *dimension* of $V$, denoted $\dim_{\mathbb{F}} V$.

*Proof.* If $\{v_1, \cdots, v_n\}$ is a basis and $w_1, \cdots, w_m$ is another basis, the $\{v_i\}$ span and $\{w_i\}$ is linearly indepnedent' so by Steinitz $m \leq n$. Likewise, $n \leq m$. $\square$

**Example.** $\dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{R}} \mathbb{C} = 2$

**Theorem.** $V$, finite dim, $v$-space over $\mathbb{F}$. If $w_1, \cdots, w_l$ is a linearly indepnedent set of vectors, we can extend it to a basis $w_1, \cdots, w_l, v_{l+1}, \cdots, v_n$

*Proof.* Apply Steinitiz to $w_1, \cdots, w_l$ (lin indep) and any basis $v_1, \cdots, v_n$.

Or directrly, if $V = < w_1, \cdots, w_l >$, stop.

Otherwise take $v_{l+1} \in V \setminus < w_1, \cdots, w_l >$, now $w_1, \cdots, w_l, v_{l+1}$ is linearly indep. iterate $\square$

**Corollary.** Suppose $V$ is a finite dimensional vector space, with dimension $n$.

(i) Any linearly independent set of vectors has at most $n$ elements with equality iff it's a basis

(ii) Any spanning set of vectors must have at least $n$ elements, with equality if and only if it's a basis.

Slogan "Choose the best basis for the job"

**Theorem.** Let $U, W$ be subspaces of $V$. If $U$, $W$ are finite dim, so is $U + W$ and $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

*Proof.* Pick basis basis $v_1, \cdots, v_l$ of $U \cap W$. Extend it to basis $v_1, \cdots, v_l, u_1, \cdots, u_m$ of $U$. Extend it to basis $v_1, \cdots, v_l, w_1, \cdots, w_n$ of $W$.

Claim: $v_1, \cdots, v_l, u_1, \cdots, u_m, w_1, \cdots, w_n$ is a basis for $U + W$.

(i) Span: $u \in U$, then $u = \sum \alpha_i v_i + \sum_{\beta_i u_i}$, $\alpha_i, \beta_i \in \mathbb{F}$ $w \in W$, then $w = \sum \gamma_i v_i + \sum_{\delta_i w_i}$, $\gamma_i, \delta_i \in \mathbb{F}$

$$u + w = \sum (\alpha_i + \gamma_i) v_i + \sum (\beta_i + \delta_i) u_i$$

(ii) lin indep: $u = \sum \alpha_i v_i + \sum_{\beta_i u_i} + \sum \gamma_i w_i = \mathbf{0}$

$$\Rightarrow u = \underbrace{\sum \alpha_i v_i + \sum \beta_i u_i}_{\in U} = \underbrace{-\sum \gamma_i w_i}_{\in W} \in U \cap W$$

This is equal to $\sum \delta_i v_i$ for some $\delta_i \in \mathbb{F}$ because $v_i$ are basis for $U \cap W$.

AS $v_i$ and $w_i$ are lin indep, $(*) \Rightarrow \gamma_i = \delta_i = 0$ for all $i$.

$\Rightarrow \sum \alpha_i v_i + \sum \beta_i u_i = 0 \Rightarrow \alpha_i = \beta_i = 0$ because $v_i$ and $u_i$ rom a basis for $U$.

□

**Theorem.** Let $V$ be a finite dim $\mathbb{F}$-vector space, $U \leq V$, then $U$ and $V/U$ are also of finite dim, and

$$\dim V = \dim U + \dim V/U$$

*Proof.*

**Exercise.** Show that $U$ is finite dim.

Let $u_1, \cdots, u_l$ be a basis for $U$. Extend it to a basis for $V$. Say $u_1, \cdots, u_l, w_{l+1}, \cdots, w_n$ of $V$.

**Exercise.** Check: $w_{l+1} + U, \cdots, w_m + U$ form a basis for $V/U$.

□

**Corollary.** If $U$ is a proper subspace of $V$, $V$ is finite dimensional, $\dim U < \dim V$.

*Proof.* $V/U \neq \{\mathbf{0}\} \Rightarrow \dim V/U > 0 \Rightarrow \dim U < \dim V$ □

**Definition.** Let $V$ be an $\mathbb{F}$-vector space, $U, W \leq V$ Then $V = U + \oplus W$ ($V$ is an internal direct sum of $U$ and $W$) if every element of $V$ can be written as $v = u + w, w \in W, u \in U$, uniquely.

$W$ is a *direct compliment* of $U$ in $V$

**Lemma.** $U, W \leq V$. The following are equivalent

(i) $V = U \oplus W$

(ii) $V = U + W$ and $U \cap W = \{\mathbf{0}\}$

(iii) $B_1$ any basis of $U$, $B_2$ is any basis of $W$, then $B = B_1 \cup B_2$ is a basis of $V$.

*Proof.* (ii) $\Rightarrow$ (i). Any $v \in V$ is $u + w$ for some $u \in U$, $w in W$.

$u_1 + w_1 = u_2 + w_2 \Rightarrow u_1 - u_2 = -w_1 + w_2 \in U \cap W = \{\mathbf{0}\} \Rightarrow w_1 = w_2, u_1 = u_2$

(i) $\Rightarrow$ (iii) $B$ spans, any $v \in V$ is $u + w$, for some $u \in U$, $w \in W$, write $u$ in terms of $B_1$, $w$ in terms of $B_2$, Then $u + w$ is a lin comb. of elements of $B$.

$B$ indep?

$$\sum_{v \in B} \lambda_v v = \mathbf{0} = \mathbf{0}_v + \mathbf{0}_w$$

$$\underbrace{\sum_{v \in B_1} \lambda_v v}_{\in U} + \sum_{v \in B_2} \lambda_v v$$

By uniqueness of expressions,

$$\sum_{v \in B_1} \lambda_v v = \mathbf{0}_U \qquad \sum_{v \in B_2} \lambda_v v = \mathbf{0}_W$$

AS $B_1$ and $B_2$ are basis, all of the $\lambda_v$ are zero.

(iii) $\Rightarrow$ (ii). If $v \in V$, $v = \sum_{x \in B} \lambda_x x = \underbrace{\sum_{u \in B} \lambda_u u}_{\in U} + \underbrace{\sum_{w \in B_2} \lambda_w w}_{\in W}$

$\Rightarrow v \in U + W$.

If $v \in U \cap W$, $v = \sum_{u \in B_1} \lambda_u u = \sum_{w \in B_2} \lambda_w w \Rightarrow$ All $\lambda_u, \lambda_w$ are zero, because $B_1 \cup B_2$ is lin. indep.

□

**Lemma.** Let $V$ be an f-dim vector space. $U \leq V$. Then there exists a direct compliment to $U$ in $V$

*Proof.* Let $u_1, \cdots, u_l$ be a basis for $U$. Extend it to a basis for $V$,

$$u_1, \cdots, u_l, w_{l+1}, \cdots, w_n$$

Then $< w_{l+1}, \cdots, w_n >$ is a direct compliment of $U$. □

Note! Direct compliments are not at all unique. In general, if you pick different ways of extending this you will get different direct compliments.

Pick $V = \mathbb{R}^2$. Pick $U$ as the $y$-axis, then any one of the following green lines are direct compliments.:

**Definition.** Def $v_1, \cdots, v_l \leq V$,

$$\sum V_i = V_1 + \cdots + V_l = \{v_1 + \cdots + v_l \mid v_i \in V_i\}$$

The sum is direct if

$$v_1 + \cdots + v_l = v_1' + \cdots + v_l' \Rightarrow v_i = v_i' \text{ for all} l$$

("unique expressions")

Notation:

$$\bigoplus_{i=1}^{l} V_i$$

**Exercise.** $V_1, \cdot, V_l \leq V$. TFAE

(i) The sum $\sum V_i$ is direct

(ii) $V_i \cap \sum_{j \neq i} V_j = \{\mathbf{0}\}$ for all $i$

(iii) For any basis $B_i$ of $V_2$, the union $B = \bigcup_{i=1}^{l} B_i$ is a basis for $\sum V_i$

**Definition.** Let $U, W$ eb $\mathbb{F}$-vector spaces. External direct sum

$$U \oplus V = \{(u, w) \mid u \in U, w \in W\}$$

with $(u, w) + (x, y) = (u + x, w + y)$,
$\lambda(u, w) = (\lambda u, \lambda w)$

# 2 Linear Maps

**Definition.** $V, W$ are $\mathbb{F}$-vector spaces. A map $\alpha : V \to W$ is linear if

(i) $\alpha(v_1 + v_2) = \alpha(v_1) + \alpha(v_2)$

(ii) $\alpha(\lambda v) = \lambda\alpha(v)$

Can be combined as: $\alpha(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1\alpha(v_1) + \lambda_2\alpha(v_2)$

**Example.** A $n \times m$ matrix with coeff in $\mathbb{F}$

$$\begin{aligned} \alpha : \mathbb{F}^n &\to \mathbb{F}^w \\ v &\mapsto Av \end{aligned}$$

**Example.**

$$\mathcal{D} : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$$

$$f \mapsto \frac{\mathrm{d}f}{\mathrm{d}x}$$

**Example.**

$$\begin{aligned} I : \mathcal{C}[0,1] &\to \mathcal{C}[0,1] \\ f &\mapsto I(f) \end{aligned}$$

where $I(f)(x) = \int_0^x f(t)\,\mathrm{d}t$

**Example.** Fix $x \in [0,1]$

$$\begin{aligned} \mathcal{C}[0,1] &\to \mathbb{R} \\ f &\mapsto f(x) \end{aligned}$$

Notes: $U, V, W$ are v spaces over $\mathbb{F}$.

(i) id: $V \to V$ linear

(ii) $U \to V \to W$

$U \xrightarrow{\alpha} V \xrightarrow{\beta} W$

$\alpha, \beta$ both linear, then $\beta \circ \alpha$ is linear.

**Lemma.** $V, W$ are $\mathbb{F}$-vector spaces, $B$ is a basis for $V$. If $\alpha_0 : B \to W$ is *any* map, then there exits a unique linear map $\alpha : V \to W$ extending $\alpha_0$

$$\alpha(v) = \alpha_0(v)$$

for any basis element $v$.

*Proof.* Let $v \in V$. Then $v = \sum \lambda_i v_i$, $v_i \in B$, $\lambda_i \in \mathbb{F}$, unique expression.
Linear forces

$$\alpha(v) = \alpha\left(\sum \lambda_i v_i\right) = \sum \lambda_i \alpha(v_i) = \sum \lambda_i \alpha_0(v_i)$$

linear, exists. expression forced to be unique. □

Note

(i) True for infinite dimensional vector space also

(ii) Very often, to define a linear map, define it on a basis.

(iii) $\alpha_1, \alpha_2 : V \to W$ linear maps. If they agree on a basis, then they are equal.

**Definition.** $V, W$ over $F$. The map $\alpha : V \to W$ is an *isomorphism* if it is linear and if it is bijective. Notation: $V \simeq W$

**Lemma.** $\simeq$ is an equivalence notation on the set (score out set and write class) of all vector spaces over $\mathbb{F}$.

(i) $i_V : V \to V$ is an iso

(ii) If $\alpha : V \to W$ is an iso, then the inverse map $\alpha^{-1} : W \to V$ is also linear, hence an iso.

(iii) If

$U \xrightarrow{\beta} V \xrightarrow{\alpha} W$

then

$U \xrightarrow{\beta \circ \alpha} W$

is also an iso

*Proof.* (i) immediate

(ii) $\alpha$ bijective $\Rightarrow \alpha^{-1}$ exists. Check: linear. $w_2 \in W, w_2 = \alpha(v_2)$, $v_2 \in V$, unique. $\alpha^{-1}(w_1 + w_2) = \alpha^{-1}(\alpha(v_1) + \alpha(v_2) = \alpha^{-1}(\alpha(v_1 + v_2)) = v_1 + v_2 = \alpha^{-1}(w_1) + \alpha^{-1}(w_2)$.

Similarly, $\lambda \in \mathbb{F}$, $w \in W$,

$$\alpha^{-1}(\lambda w) = \lambda \alpha^{-1}(w)$$

□

**Theorem.** If $V$ vector space over $\mathbb{F}$ of dimension $n$, then $V \simeq \mathbb{F}^n$.

*Proof.* Choose a basis $B$ for $V$, say $v_1, \cdots, v_n$

$$V \to \mathbb{F}^n$$

$$\sum \lambda_i v_i \mapsto \begin{pmatrix} \lambda_1 \\ \cdots \\ \lambda_n \end{pmatrix} \text{ is an iso}$$

□

Remark: Choosing an iso $V \simeq F^n$ is equivalent to choosing a basis for $V$.

**Theorem.** $V, W$ $v$ spaces over $\mathbb{F}$, finite dim, are isomorphic iff they have the same dimension

*Proof.* ($\Leftarrow$) Both $V$ and $W$ are isomorphic

$$\mathbb{F}^{\dim V} = \mathbb{F}^{\dim W}$$

($\Rightarrow$) Let $\alpha : V \to W$ iso, $B$ a basis for $V$.
Claim: $\alpha(B)$ is a basis for $W$.
Check: $\alpha(B)$ spans $W$ because of surjectivity of $\alpha$.

**Exercise.** $\alpha(B)$ lin indep: follows from injectivity of $\alpha$.

$\square$

**Definition.** $\alpha : V \to W$ linear, $N(\alpha) = \ker\alpha = \{v \in V \mid \alpha(v) = \mathbf{0}\} \leq V$. $Im(\alpha) = \{w \in W \mid w\ \alpha(v),\ \text{some } v \in V\} \leq W$

$$\alpha \text{ injective} \iff N(\alpha) = \{\mathbf{0}\}$$

$$\alpha \text{ surjective } \iff Im(\alpha) = W$$

**Example.**

$$\alpha : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$$

$\alpha(f)(t) = f''(t) + 2f'(t) - 5f,$ is solutions to

$$\ker\alpha \text{ is solut to } f'' + 2f' + 5f = 0$$

$$g \in \operatorname{Im}\alpha \text{ if } \exists \text{ solut to } f'' + 2f' + 5f = g$$

**Theorem.** (First Isomorphism Theorem) Let $\alpha : V \to W$ linear map. It induces an iso :

$$V/\ker\alpha \to \operatorname{Im}\alpha$$

$$\overline{\alpha}(v + \ker\alpha) = \alpha(v)$$

*Proof.* (i) $\overline{\alpha}$ is well defined:

$$v + \ker\alpha = v' + \ker\alpha$$

the

$$\iff v - v' \in \ker\alpha \Rightarrow \alpha(v) = \alpha(v')$$

(ii) $\overline{\alpha}$ linear is immediate from linearly of $\alpha$.

(iii) $\overline{\alpha}$ bijective?

$$\overline{\alpha}(v + \ker\alpha) = \mathbf{0} \Rightarrow \alpha(v) = 0 \Rightarrow v \in \ker\alpha$$

(iv) surjectie: by def of $\operatorname{Im}(\alpha)$.

$\square$

**Definition.**

$$r(\alpha) = rk(\alpha) = \dim(\operatorname{Im} \alpha)$$

$$n(\alpha) = \dim(N(\alpha))$$

rank, nullity

**Theorem.** (Rank-nullity theorem) Let $U, V$ be vector spaces over $\mathbb{F}$, $\dim_{\mathbb{F}} U < \infty$.

Let $\alpha : U \to V$ linear.

Then

$$\dim U = r(\alpha) + n(\alpha)$$

*Proof.*

$$U/\ker \alpha \simeq \operatorname{Im}(\alpha) \Rightarrow \dim(U) - \dim \ker \alpha = \dim \operatorname{Im}(\alpha)$$

□

**Lemma.** Let $V, W$ be v spaces over $\mathbb{F}$, of equal finite dim. Let $\alpha : V \to W$ linear.

TFAE

(i) $\alpha$ injective

(ii) $\alpha$ surjective

(iii) $\alpha$ isomorphism