Part IB — Electromagnetism Example Sheet 2

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QUESTION 1

– Given $\mathbf{A} = xB\hat{\mathbf{y}}$,

$$\begin{aligned}\nabla \times \mathbf{A} &= (-\partial_z xB, 0, \partial_x xB) \\ &= (0, 0, B) = B\hat{\mathbf{z}}\end{aligned}$$

– Given $\mathbf{A} = \frac{1}{2}(xB\hat{\mathbf{y}} - yB\hat{\mathbf{x}})$,

$$\begin{aligned}\nabla \times \mathbf{A} &= (-\partial_z \frac{1}{2}xB, -\partial_z \frac{1}{2}yB, \partial_x \frac{1}{2}xB + \partial_y yB) \\ &= (0, 0, B) = B\hat{\mathbf{z}}\end{aligned}$$

– Given $\mathbf{A} = \frac{1}{2}rB\hat{\phi}$, calculating curl in cylindrical polars we have

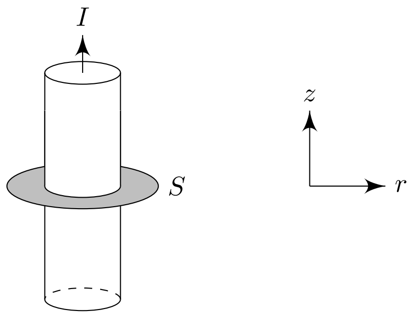
$$\begin{aligned}\nabla \times \mathbf{A} &= \left(0, 0, \frac{1}{r}\partial_r \left(\frac{1}{2}r^2 B\right)\right) \\ &= (0, 0, B) = B\hat{\mathbf{z}}\end{aligned}$$

– Given $\mathbf{A} = \frac{1}{2}r\sin\theta B\hat{\phi}$, calculating curl in spherical polars we have

$$\begin{aligned}\nabla \times \mathbf{A} &= \left(\frac{1}{r^2\sin\theta}\partial_\theta \left(\frac{1}{2}r^2\sin^2\theta B\right), -\frac{1}{r\sin\theta}\partial_r \left(\frac{1}{2}r^2\sin^2\theta B\right), 0\right) \\ &= B(\cos\theta, -\sin\theta, 0) = B\cos\theta\hat{\mathbf{r}} - B\sin\theta\hat{\boldsymbol{\theta}} \\ &= B\cos\theta\left(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\right) \\ &\quad - B\sin\theta\left(\cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}\right) \\ &= B(\cos^2\theta + \sin^2\theta)\mathbf{z} = B\hat{\mathbf{z}}\end{aligned}$$

QUESTION 2

We use cylindrical polar coordinates (r, ϕ, z), where z is along the direction of the current, and r points in the radial direction.



By symmetry, the magnetic field can only depend on the radius, and must lie in the x, y plane. Since we require that $\nabla \cdot \mathbf{B} = 0$, we cannot have a radial component. So the general form is

$$\mathbf{B}(\mathbf{r}) = B(r)\hat{\boldsymbol{\phi}}.$$

To find $B(r)$, we integrate over a disc that cuts through the wire horizontally. In cylindrical polars we have $\mathrm{d}\mathbf{r} = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\phi\hat{\boldsymbol{\phi}} + \mathrm{d}z\hat{\mathbf{z}} = r\mathrm{d}\phi\hat{\boldsymbol{\phi}}$, thus

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = B(r) \int_0^{2\pi} r \,\mathrm{d}\phi = 2\pi r B(r)$$

By Ampere's law, we have

$$2\pi r B(r) = \mu_0 I.$$

where

$$I = \int_S \mathbf{J} \cdot \mathrm{d}S = J\pi r^2$$

So

$$\mathbf{B}(r) = \frac{\mu_0 J r}{2}\hat{\boldsymbol{\phi}}.$$

QUESTION 3

Apply Ampere's Law to a loop of radius r that falls somewhere between the two cylinders. As before we have

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = B(r) \int_0^{2\pi} r \, \mathrm{d}\phi = 2\pi r B(r)$$

Now the current density is uniform, and thus given by $\frac{1}{\pi b^2 - \pi a^2}\hat{\mathbf{z}}$, so the current enclosed inside our loop is given by

$$J(\pi r^2 - \pi a^2) = \frac{r^2 - a^2}{b^2 - a^2}$$

Thus Ampere's Law gives us that

$$B(r) = \frac{\mu_0 I (r^2 - a^2)}{2\pi r (b^2 - a^2)}$$

Not sure about direction, or why it seems that $\frac{r^2 - a^2}{r} = d$

QUESTION 4

Using cylindrical polars we calculate the potential as

We take the wire to point along the $\hat{\mathbf{z}}$ axis and use $r^2 = x^2 + y^2$ as our radial coordinate. This means that the line element along the wire is parametrised by $\mathrm{d}\mathbf{x}' = \hat{\mathbf{z}}\mathrm{d}z$ and, for a point $\mathbf{x}$ away from the wire, the vector $\mathrm{d}\hat{\mathbf{x}}' \times (\mathbf{x} - \mathbf{x}')$ points along the tangent to the circle of radius r,

$$\mathrm{d}\mathbf{x}' = (\mathbf{x} - \mathbf{x}') = r\hat{\boldsymbol{\phi}}\mathrm{d}z$$

So we have

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0 I\hat{\boldsymbol{\phi}}}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}z\frac{r}{(r^2+z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}} \\ &= \end{aligned}$$

which agrees with our work from Question 2.

Not sure about the next two bits.

QUESTION 5

A current $\mathbf{J}$, localized on some closed curve C, is in an external magnetic field $\mathbf{B}(r)$, then this field causes the current to experience the Lorentz force

$$\mathbf{F} = \int \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) \, \mathrm{d}V.$$

While we are integrating over all of space, the current is localized at a curve C. So

$$\mathbf{F} = I \oint_C \mathrm{d}\mathbf{r} \times \mathbf{B}(\mathbf{r}).$$

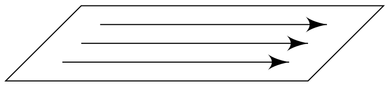
Similarly the torque is

$$\begin{aligned} \mathbf{F} &= \int \mathbf{r} \times (\mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})) \, \mathrm{d}V \\ &= I \oint_C \mathbf{r} \times (\mathrm{d}\mathbf{r} \times \mathbf{B}(\mathbf{r})) \end{aligned}$$

as required.

QUESTION 6

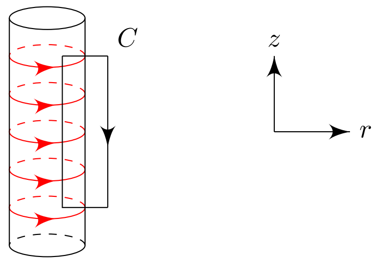
Consider the plane $z = 0$ with *surface current density* $\mathbf{k}$ (i.e. current per unit length).



Across any surface we have

$$\hat{\mathbf{n}} \times \mathbf{B}_+ - \hat{\mathbf{n}} \times \mathbf{B}_- = \mu_0 \mathbf{k} \qquad (*)$$

Consider the solenoid:



We use cylindrical polar coordinates with z in the direction of the extension of the cylinder. By symmetry, $\mathbf{B} = B(r)\hat{\mathbf{z}}$.

Away from the cylinder, $\nabla \times \mathbf{B} = 0$. So $\frac{\partial B}{\partial r} = 0$, which means that $B(r)$ is constant outside. Since we know that $\mathbf{B} = \mathbf{0}$ at infinity, $\mathbf{B} = \mathbf{0}$ everywhere outside the cylinder.

To compute $\mathbf{B}$ inside, use Ampere's law with a curve C. Note that only the vertical part (say of length L) inside the cylinder contributes to the integral. Then

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = BL = \mu_o INL.$$

where N is the number of wires per unit length and I is the current in each wire (so INL is the total amount of current through the wires).

So

$$B = \mu_0 IN\hat{\boldsymbol{z}}$$

Note that since $K = IN$ this is consistent with (*). The average field is thus $\frac{1}{2}\mu_0 IN\hat{\boldsymbol{z}}$, and the Lorentz force is:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q\mathbf{v} \times \frac{1}{2}\mu_0 NI\hat{\mathbf{z}}$$

For a wire with cross-sectional area A, the total current is just $I = JA$. We want to compute the force on the wire per unit length, $\mathbf{f}$. Since the number of charges per unit area is nAN and F is the force on each charge, we have

$$\mathbf{f} = nAN\mathbf{F} = \frac{1}{2}\mu_0 I^2 N^2 \hat{\mathbf{n}}$$

where

QUESTION 7

A current $\mathbf{J}_1$, localized on some closed curve C_1, sets up a magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \mathrm{d}\mathbf{r}_1 \times \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3}.$$

A second current $\mathbf{J}_2$ on C_2 experiences the Lorentz force

$$\mathbf{F}_{12} = \int \mathbf{J}_2(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\, \mathrm{d}V.$$

While we are integrating over all of space, the current is localized at a curve C_2. So

$$\mathbf{F}_{12} = I_2 \oint_{C_2} \mathrm{d}\mathbf{r}_2 \times \mathbf{B}(\mathbf{r}_2).$$

Hence

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \mathrm{d}\mathbf{r}_2 \times \left(\mathrm{d}\mathbf{r}_1 \times \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \right).$$

Manipulating the integrand,

$$\frac{\mathrm{d}\mathbf{r}_2 \times \mathrm{d}\mathbf{r}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = \frac{\mathrm{d}\mathbf{r}_2 \cdot (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \mathrm{d}\mathbf{r}_1 - \frac{\mathrm{d}\mathbf{r}_2 \cdot \mathrm{d}\mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} (\mathbf{r}_2 - \mathbf{r}_1)$$

it is now in a from that is symmetric in $\mathrm{d}\mathbf{r}_1$, $\mathrm{d}\mathbf{r}_2$ and thus explicitly satisfies Newton's third law.

QUESTION 8

Given

$$\mathbf{E} = e^{-t}\hat{\boldsymbol{\phi}}, \qquad \mathbf{B} = \frac{e^{-t}}{r}\hat{\mathbf{z}}$$

by considering divergence and curl in cylindrical polars, we have

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{1}{r}\frac{\partial}{\partial \phi}(e^{-t}) \\ &= 0\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{B} &= \frac{\partial}{\partial z}\frac{e^{-t}}{r} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\nabla \times \mathbf{E} &= \left(0, 0, \frac{1}{r}\frac{\partial}{\partial r}(re^{-t})\right) \\ &= \frac{e^{-t}}{r}\hat{\mathbf{z}} \\ &= -\frac{\partial \mathbf{B}}{\partial t}\end{aligned}$$

thus these three Maxwell equations are satisfied.
Next we seek to verify the general from of Faraday's Law:

$$\oint_{C(t)} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \quad (*)$$

where $C(t)$ is the circle with radius $1 + t$. Note any point on this curve will have velocity $\mathbf{v} = \hat{\mathbf{r}}$, so $\mathbf{v} \times \mathbf{B} = -\frac{e^{-t}}{r}\hat{\boldsymbol{\phi}}$ Noting in cylindricals, we parametrise $C(t)$ by θ, so

$$\mathbf{r} = (1 + t, \theta, 0), \quad 0 \leq \theta < 2\pi$$

$\mathrm{d}\mathbf{r} = (\mathrm{d}r, r\mathrm{d}\phi, \mathrm{d}z)$, the LHS of (*) is

$$\begin{aligned}\oint_{C(t)} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} e^{-t}\left(1 - \frac{1}{r}\right)\hat{\boldsymbol{\phi}} \cdot r\hat{\boldsymbol{\phi}}\,\mathrm{d}\theta \\ &= 2\pi(r-1)e^{-t} \\ &= 2\pi t e^{-t} \qquad (r = t + 1)\end{aligned}$$

The surface integral has $\mathbf{n} = (1, 0, 0)$ so $\mathbf{B} \cdot \mathbf{n} = \frac{e^{-t}}{r}$. We have

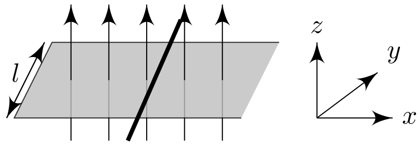
$$\begin{aligned}\int_{S(t)} \mathbf{B} \cdot \mathrm{d}\mathbf{S} &= \int_0^{1+t} \int_0^{2\pi} \frac{e^{-t}}{r} r \mathrm{d}\theta \mathrm{d}r \\ &= 2\pi(1+t)e^{-t}\end{aligned}$$

Thus RHS of (*) is (product rule)

$$2\pi t e^{-t}$$

in agreement with the LHS, thus Faraday's Law holds.

QUESTION 9



If a current I flows, the force on a small segment of the bar is

$$\mathbf{F} = IB\hat{\mathbf{y}} \times \hat{\mathbf{z}}$$

So the total force on a bar is

$$\mathbf{F} = I\frac{\alpha}{t}\ell\hat{\mathbf{x}}.$$

So

$$m\ddot{x} = I\frac{\alpha}{t}\ell.$$

We can compute the emf as

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(B\ell x) = \frac{\alpha}{t^2}\ell x - \frac{\alpha}{t}\ell\dot{x}.$$

So Ohm's law gives

$$IR = \frac{\alpha}{t^2}\ell x - \frac{\alpha}{t}\ell\dot{x}$$

Hence

$$m\ddot{x} = \frac{\ell^2\alpha^2}{Rt^2}\left(\frac{1}{t}x - \dot{x}\right)$$

Not sure how to solve this.

QUESTION 10

Computing $\mathbf{B} = \nabla \times \mathbf{A}$ in cylindricals gives

$$\mathbf{B} = -\frac{1}{2}Br^2\hat{\mathbf{r}} + Brz\hat{\mathbf{z}}$$

Next, we have the induced emf given by $\mathcal{E} = -\frac{\partial \phi}{\partial t}$, where

$$\begin{aligned}\phi &= \int_{S(t)} \mathbf{B}(t) \cdot \mathrm{d}\mathbf{S} \\ &= \pi a^2 Brz(t)\end{aligned}$$

Hence $\mathcal{E} = -\pi a^2 Br\dot{z}(t)$. According to Ohm's Law

$$\mathcal{E} = IR$$

the induced current is given by

$$I = -\frac{1}{R}\pi a^2 Br\dot{z}(t)$$

Next, *Joule heating* is the energy lost in a circuit due to friction. It is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = I^2 R.$$

Not sure how to calculate the force on the loop.