Part IB — Fluid Dynamics Example Sheet 1

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Lent 2018

QUESTION 1

In this example, $\mathbf{u}(x) = (\alpha x, -\alpha y)$

(i) Streamlines are curves $\mathbf{X}(s; \mathbf{x}_0)$ with $\mathbf{X} = \mathbf{x}_0$ at $s = 0$.

$$\frac{\partial X}{\partial s} = \alpha X, \quad \frac{\partial Y}{\partial s} = -\alpha Y$$

$X = x_0 e^{\alpha s}$ and $Y = y_0 e^{-\alpha s}$. Eliminating s to get the shape of the streamlines gives $XY = x_0 y_0$. These are hyperbola.

(ii) For steady flow, pathlines and streamlines are coincident. For a particle released at $\mathbf{x}_0 = (x_0, y_0)$, the particle pathline obeys

$$x = x_0 e^{\alpha t}, \quad y = y_0 e^{-\alpha t}$$

as As $x_0^2 + y_0^2 = a^2$ we find that the curve evolves as

$$x^2 e^{-2\alpha t} + y^2 e^{2\alpha t} = a^2$$

ie; this is an ellipse,

$$\frac{x^2}{(ae^{\alpha t})^2} + \frac{y^2}{(ae^{-\alpha t})^2} = 1$$

(iii) The area of this ellipse does not change in time; it is $\pi \times ae^{\alpha t} \times ae^{-\alpha t} = \pi a^2$. As we would expect this is the area of the circle $x^2 + y^2 = a^2$ at time $t = 0$.

Intuitively?

QUESTION 2

In this example, $\mathbf{u}(x) = (\gamma y, 0)$

(i) Streamlines are curves $\mathbf{X}(s; \mathbf{x}_0)$ with $\mathbf{X}_0 = \mathbf{x}_0$ at $s = 0$.

$$\frac{\partial X}{\partial s} = \gamma y, \quad \frac{\partial Y}{\partial s} = 0$$

$X = x_0 + \gamma y_0 s$ and $Y = y_0$. Note that we cannot eliminate s to get the shape of the streamlines; they are given by;

$$X = x_0 + \gamma Y s$$

(ii) For steady flow, pathlines and streamlines are coincident. For a particle released at $\mathbf{x}_0 = (x_0, y_0)$, the particle pathline obeys

$$x = x_0 + \gamma y_0 t, \quad y = y_0$$

as As $x_0^2 + y_0^2 = a^2$ we find that the curve evolves as

$$(x - \gamma t y_0)^2 + y^2 = a^2$$

ie. the circle is just shifted to the right, at a rate of γy_0 units per second.

(iii) The curve is just translated so the area remains fixed.

At longer times, the ellipse in Question 1 is 'stretched' faster, as this grows exponentially, yet the ~~stretch~~ translation in this circle is just linear with time.

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QUESTION 3

In this example, $\mathbf{u}(x) = (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x})$.

(i) The contours $\psi = c$ have normal

$$\mathbf{n} = \nabla \psi = (\psi_x, \psi_y)$$

We see immediately that

$$\mathbf{u} \cdot \mathbf{n} = \psi_x \psi_y - \psi_y \psi_x = 0$$

So the flow is perpendicular to the normal, ie. tangent to the contours of ψ.

(ii) $|\mathbf{u}| = |(\psi_y, -\psi_x)| = |(\psi_x, \psi_y)| = |\nabla \psi|$

Hence flow is faster when streamlines are closer together. (Why does $|\nabla \psi|$ increasing mean that they are closer together?) Can also see this is true intuitively, as the fluid between any two streamlines must be between the streamlines.

(iii) The volume flux is
$$q = \int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s.$$

We see that
$$\mathbf{n} \, \mathrm{d}s = (-\mathrm{d}y, \mathrm{d}x).$$

So we can write this as

$$q = \int_{\mathbf{x}_0}^{\mathbf{x}_1} -\frac{\partial \psi}{\partial y} \, \mathrm{d}y - \frac{\partial \psi}{\partial x} \, \mathrm{d}x = \psi(\mathbf{x}_0) - \psi(\mathbf{x}_1).$$

So the flux depends only the difference in the value of ψ. Hence, for closer streamlines, to maintain the same volume flux, we need a higher speed.

(iv) Note that ψ is constant on a stationary rigid boundary, i.e. the boundary is a streamline, since the flow is tangential at the boundary. This is a consequence of $\mathbf{u} \cdot \mathbf{n} = 0$. We often choose $\psi = 0$ as our boundary.

QUESTION 4

$$\nabla \cdot \mathbf{u} = 0 \iff \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

By the quotient rule,

$$\frac{\partial u}{\partial x} = \frac{-(y-b)[2(x-a)]}{((x-a)+(y-b))^2}$$

$$\frac{\partial v}{\partial y} = \frac{-(a-x)[2(y-b)]}{((x-a)^2+(y-b)^2)}$$

So $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\nabla \cdot \mathbf{u} = 0$, our fluid is indeed incompressible.
By inspection, the required streamfunction is

$$\psi(x,y) = \frac{1}{2}\log\left[(x-a)^2 + (y-b)^2\right]$$

QUESTION 5

$$\nabla \cdot \mathbf{u} = 0 \iff \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) = 0$$

$$\frac{\partial}{\partial r}(ru_r) = \frac{\partial}{\partial r}\left[U\left(r - \frac{a^2}{r}\right)\cos\theta\right]$$
$$= U\left(1 + \frac{a^2}{r^2}\right)\cos\theta$$

$$\frac{\partial}{\partial \theta}(u_\theta) = \frac{\partial}{\partial \theta}\left[-U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right]$$
$$= U\left(1 + \frac{a^2}{r^2}\right)\cos\theta$$

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) \implies \nabla \cdot \mathbf{u} = 0$$

By inspection the streamfunction is

$$\psi(r,\theta) = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

QUESTION 6

$$\nabla \cdot \mathbf{u} = 0 \iff \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial}{\partial r}(ru_r) = \frac{\partial}{\partial r}\left[-\frac{1}{2}\alpha r^2\right]$$
$$= -r\alpha$$

$$\frac{\partial}{\partial z}(u_z) = \frac{\partial}{\partial z}\left[\alpha z\right]$$
$$= \alpha$$

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{\partial u_z}{\partial z} \implies \nabla \cdot \mathbf{u} = 0$$

By inspection the streamfunction is

$$\Psi(r,z) = \frac{1}{2}\alpha r^2 z$$

QUESTION 7

In this example, $\mathbf{u}(x) = (1/(1+t), 1)$

(i) Streamlines are curves $\mathbf{X}(s; \mathbf{x}_0)$ with $\mathbf{X}_0 = \mathbf{x}_0$ at $s = 0$.

$$\frac{\partial X}{\partial s} = \frac{1}{1+t}, \quad \frac{\partial Y}{\partial s} = 1$$

$X = \frac{s}{1+t} + x_0$ and $Y = s + y_0$. Eliminating s to get the shape of the streamlines gives $X = \frac{Y - y_0}{1+t} + x_0$. At time $t = 0$, with $\mathbf{x}_0 = (1, 1)$ this gives the streamline as

$$Y = X$$

(ii) The pathline of the particle is given by

$$\frac{\partial X}{\partial t} = \frac{1}{1+t}, \quad \frac{\partial Y}{\partial t} = 1$$

with $\mathbf{X} = \mathbf{x}_0$ at $t = 0$. Thus $X = \log(1+t) + x_0$ and $Y = t + y_0$. Eliminating t we get the shape of the pathline as

$$X = \log(1 + Y - y_0) + x_0$$

At $\mathbf{x}_0 = (1, 1)$ we have

$$X = \log Y + 1$$

or $Y = e^{X-1}$

(iii) Streaklines are curves $\mathbf{X}(s; \mathbf{x}_0)$ with $\mathbf{X}_0 = \mathbf{x}_0$ at $t = s$.

$$\frac{\partial X}{\partial t} = \frac{1}{1+t}, \quad \frac{\partial Y}{\partial t} = 1$$

This gives $X = \log\left(\frac{1+t}{1+s}\right) + 1$, $Y = t + 1 - s$.

Eliminating s to get the shape at time $t > s$:

$$X = \log\left(\frac{1+t}{1+t-Y}\right)$$

QUESTION 8

The integral from of the momentum equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\ \mathrm{d}V = \int_V \rho\mathbf{g}\ \mathrm{d}V - \int_S p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\ \mathrm{d}S$$

For steady flow, the LHS is zero. Neglecting gravity, we have the total force on the wall as

$$= \int_S p\mathbf{n}\ \mathrm{d}S$$
$$= -\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\ \mathrm{d}S$$

So the force on the wall is equal to the surface integral, which gives $-\rho A U^2\mathbf{n}$. Assuming $\rho = 10^3\mathrm{kg}\ \mathrm{m}^{-3}$; $\Rightarrow 0.6$ N force.

QUESTION 9

Proposition (Euler momentum equation)**.**

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mathbf{f}.$$

For conservative forces, we can write $\mathbf{f} = -\nabla\chi$, where χ is a scalar potential. We notice the vector identity

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) - \mathbf{u} \cdot \nabla\mathbf{u}.$$

We use this to rewrite the Euler momentum equation as

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) - \rho\mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla p - \nabla\chi.$$

Dotting with $\mathbf{u}$, the last term on the left vanishes, and we get

Proposition (Bernoulli's equation)**.**

$$\frac{1}{2}\rho\frac{\partial|\mathbf{u}|^2}{\partial t} = -\mathbf{u} \cdot \nabla\left(\frac{1}{2}\rho|\mathbf{u}|^2 + p + \chi\right).$$

Note also that this tells us that high velocity goes with low pressure; low pressure goes with high velocity.

In the case where we have a steady flow, we know

$$H = \frac{1}{2}\rho|\mathbf{u}|^2 + p + \chi$$

is constant along streamlines.

Even if the flow is not steady, we can still define the value H, and then we can integrate Bernoulli's equation over a volume $\mathcal{D}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}}\frac{1}{2}\rho|\mathbf{u}|^2 \,\mathrm{d}V + \int_{\partial\mathcal{D}}H\mathbf{u} \cdot \mathbf{n} \,\mathrm{d}S. = 0$$

So H is the transportable energy of the flow.

QUESTION 10

Bernoulli says the quantity H is constant along streamlines:

$$H = \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \chi$$

Applying Bernoulli along surface streamline gives

$$\frac{p_1}{\rho} + \frac{1}{2}U_1^2 = \frac{p_2}{\rho} + \frac{1}{2}U_2^2$$

The water has zero velocity when it hits the arm, and 1 before. Thus we have $U_1 = 1$ and $U_2 = 0$.

Then

$$\frac{p_1}{\rho} + \frac{1}{2} = \frac{p_2}{\rho}$$

$$\Rightarrow (p_2 - p_1) = \frac{\rho}{2}$$

Use $p_1 - p_{\text{atm}} = \rho g h_1$, $p_2 - p_{\text{atm}} = \rho g h_2$

$$h_2 - h_1 = \frac{1}{2g}$$

ie, water can rise about 5 cm.

QUESTION 11

Euler momentum equation states

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \rho \mathbf{g}$$

Have $\rho \mathbf{g} = -\nabla \chi$, where $\chi = g\rho z$ as $\mathbf{g} = (0, 0, -z)$.
Now if $\rho = $ constant, and $\mathbf{u} = (-\omega y, \omega x, 0)$.
Note completely sure how to integrate this...

QUESTION 12

$$\frac{p_1}{\rho} + \frac{1}{2}U_1^2 + gh_1 = \frac{p_2}{\rho} + \frac{1}{2}U_2^2 + gh_2 \qquad \text{(Bernoulli)}$$

$$A_1U_1 = A_2U_2 = 10^{-4} \text{ m}^3\text{s}^{-1} \qquad \text{(mass conservation)}$$

$$A_2 = 4 \times 10^{-5} \text{ m}^2 \Rightarrow U_2 = 5 \text{ ms}^{-1}$$

Use $p_1 - p_\text{atm} = \rho gh_1$, $p_2 - p_\text{atm} = \rho gh_2$
But not sure what U_1, A_1 are?

QUESTION 13

Bernoulli:

$$\frac{U^2}{2} + \frac{p}{\rho} + gh = \text{constant}$$

Apply at the opening, where $h = 0$, $p = p_{\text{atm}}$. Apply again at the top of the vessel, where we can assume $U = 0$, $h = h$, $p = p_{atm}$. Comparing we find that

$$\frac{U^2}{2} = gh$$

$$\Rightarrow U = \sqrt{2gh}$$

So the shape is...?