Part IB — Methods Example Sheet 1

Supervised by ?

Examples worked through by Christopher Turnbull

Michaelmas 2017

QUESTION 1

$$\frac{f(x_+ + f(x_-))}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

For $f(x) = (x-1)^2$ on the interval $-1 \leq x \leq 1$, $f(x)$ is an even function, thus $b_n = 0$. We have $L = 1$, and

$$\begin{aligned}\frac{1}{2}a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\ \mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1} x^4 - 2x^2 + 1\ \mathrm{d}x \\ &= \int_{0}^{1} x^4 - 2x^2 + 1\ \mathrm{d}x \\ &= \frac{8}{15}\end{aligned}$$

and

$$\begin{aligned}a_n &= \frac{1}{L}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \\ &= \int_{-1}^{1} x^4 \cos n\pi x\ \mathrm{d}x - 2\int_{-1}^{1} x^2 \cos n\pi x\ \mathrm{d}x + \int_{-1}^{1} \cos n\pi x\ \mathrm{d}x\end{aligned}$$

Evaluating each integral separately, we have:

(i)

$$\int_{-1}^{1} \cos n\pi x\ \mathrm{d}x = \left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{-1} = 0$$

as $\sin n\pi x = 0\ \forall\ n$

(ii) By parts,

$$\begin{aligned}\int_{-1}^{1} x^2 \cos n\pi x\ \mathrm{d}x &= \left[\frac{x^2 \sin n\pi x}{n\pi} - \frac{1}{n\pi}\int 2x \sin n\pi x\ \mathrm{d}x\right]_{-1}^{1} \\ &= -\frac{2}{n\pi}\int_{-1}^{1} x \sin n\pi x\ \mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{1} x \sin n\pi x\ \mathrm{d}x &= \left[\frac{-x\cos n\pi x}{n\pi} + \frac{1}{n\pi}\int \cos n\pi x\ \mathrm{d}x\right]_{-1}^{1} \\ &= \frac{-2\cos n\pi x}{(n\pi)^2}\end{aligned}$$

Thus the second integral contributes to give

$$-\frac{8cosn\pi x}{(n\pi)^2}$$

(iii)

$$\begin{aligned}\int_{-1}^{1} x^4 \cos n\pi x \, \mathrm{d}x &= \left[\frac{x^4 \sin n\pi x}{n\pi} - \frac{1}{n\pi}\int 4x^3 \sin n\pi x \, \mathrm{d}x\right]_{-1}^{1} \\ &= -\frac{4}{n\pi}\int_{-1}^{1} x^3 \sin n\pi x \, \mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{1} x^3 \sin n\pi x \, \mathrm{d}x &= \left[\frac{-x^3 \cos n\pi x}{n\pi} + \frac{1}{n\pi}\int 3x^2 \cos n\pi x\right]_{-1}^{1} \\ &= \frac{-2\cos n\pi}{n\pi} + \frac{3}{n\pi}\int_{-1}^{1} x^2 \cos n\pi x \, \mathrm{d}x\end{aligned}$$

Whence

$$\begin{aligned}\int_{-1}^{1} x^4 \cos n\pi x \, \mathrm{d}x &= \frac{8\cos n\pi}{n\pi} - \frac{12}{(n\pi)^2}\int_{-1}^{1} x^2 \cos n\pi x \, \mathrm{d}x \\ &= \frac{8\cos n\pi}{n\pi} - \frac{48\cos n\pi}{(n\pi)^4}\end{aligned}$$

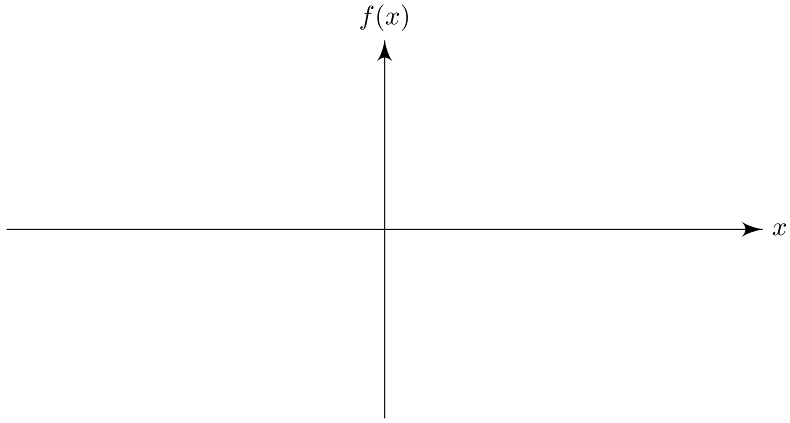
using (ii).

Finally,

$$\begin{aligned}a_n &= -\frac{48\cos n\pi}{(n\pi)^4} \\ &= \frac{48(-1)^{n+1}}{(n\pi)^4}\end{aligned}$$

as $\cos n\pi x = (-1)^n$
Hence the Fourier Series is given by

$$\begin{aligned}f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x \\ &= \frac{8}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos n\pi x\end{aligned}$$



$f(x)$ satisfies the Dirichlet conditions. The 1$^{\text{st}}$ derivative is the lowest derivative which is discontinuous (at the endpoints, as $f(x)$ even fn $\Rightarrow$ $f'(x)$ odd), so Fourier coefficients are $\mathcal{O}(\frac{1}{n^2})$ as $n \to \infty$

QUESTION 2

(a)

$$\frac{f(x_+ + f(x_-))}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\pi x^2 \sin nx\,\mathrm{d}x \end{aligned}$$

Integrating by parts,

$$\begin{aligned} \int_0^\pi x^2 \sin nx\,\mathrm{d}x &= \left[\frac{-x^2\cos nx}{n} + \frac{1}{n}\int 2x\cos nx\,\mathrm{d}x\right]_0^\pi \\ &= \frac{-\pi^2\cos n\pi}{n} + \frac{2}{n}\int_0^\pi x\cos nx\,\mathrm{d}x \end{aligned}$$

and once again,

$$\begin{aligned} \int_0^\pi x\cos nx\,\mathrm{d}x &= \left[\frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx\,\mathrm{d}x\right]_0^\pi \\ &= -\frac{1}{n}\int_0^\pi \sin nx\,\mathrm{d}x \\ &= -\frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^\pi \\ &= \frac{1}{n^2}(\cos n\pi - 1) \end{aligned}$$

Back substituting in,

$$\begin{aligned} b_n &= \frac{2}{\pi}\left(\frac{-\pi^2\cos n\pi}{n} + \frac{2}{n^3}(\cos n\pi - 1)\right) \\ &= \frac{2}{\pi n^3}\left(-2 + (2-(\pi n)^2)\cos n\pi\right) \end{aligned}$$

Hence Fourier sine series given by:

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^3}\left(-2 + (2-(\pi n)^2)(-1)^n\right)\sin nx$$

(b) Similarly,

$$\frac{f(x_+ + f(x_-))}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$\begin{aligned}\frac{a_0}{2} &= \frac{1}{L}\int_0^L f(x)\ \mathrm{d}x \\ &= \frac{1}{\pi}\int_0^\pi x^2\ \mathrm{d}x \\ &= \frac{\pi^2}{3}\end{aligned}$$

and

$$\begin{aligned}a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \\ &= \frac{2}{\pi}\int_0^\pi x^2 \cos nx\ \mathrm{d}x\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_0^\pi x^2 \cos nx\ \mathrm{d}x &= \left[\frac{x^2 \sin nx}{n} - \frac{1}{n}\int 2x\sin nx\ \mathrm{d}x\right]_0^\pi \\ &= \frac{-2}{n}\int_0^\pi x \sin nx\ \mathrm{d}x\end{aligned}$$

and once again,

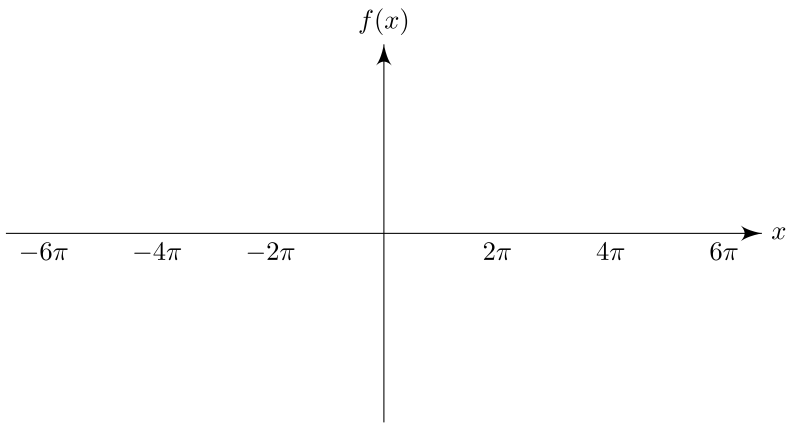
$$\begin{aligned}\int_0^\pi x\sin nx\ \mathrm{d}x &= \left[\frac{-x\cos nx}{n} + \frac{1}{n}\int \cos nx\ \mathrm{d}x\right]_0^\pi \\ &= \frac{-\pi\cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n\pi}\right]_0^\pi \\ &= \frac{-\pi\cos n\pi}{n}\end{aligned}$$

Thus

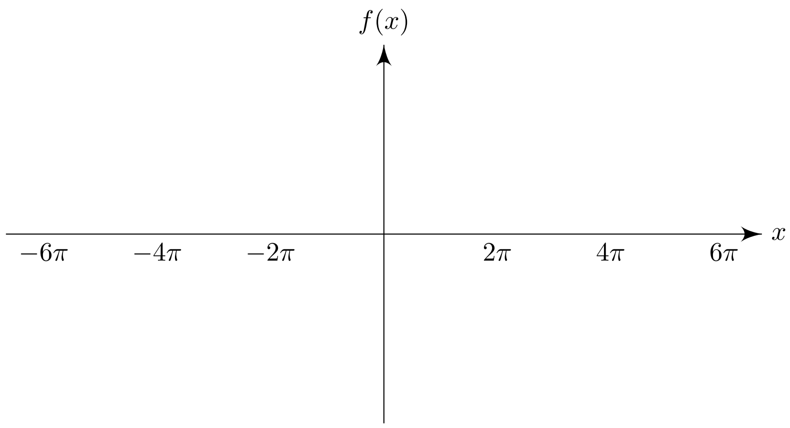
$$a_n = \frac{4}{n^2}\cos n\pi$$

and the Fourier cosine series is given by

$$f(x) = \frac{\pi^2}{3} + \sum_{n=0}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$



(i)



(ii)

QUESTION 3

QUESTION 4

QUESTION 5