Part IB — Methods

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0 Introduction

I will never say anything that is untrue deliberately...

Self-adjoint ODEs

1 Fourier Series

1.1 Peridoidic Functions

Definition. A function $f(t)$ is *periodic* with period T if $f(t+T) = f(T)$

Fig 1

Example.

$$A \sin \omega t$$

A is the *amplitude*, ω is the *frequency*, $2\pi/\omega$ is the *period*.

Sines and cosines are beautiful because they have an orthogonality property:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\begin{aligned}
\cos A \cos B &= \frac{1}{2} \left[\cos(A-B) + \cos(A+B) \right] \\
\sin A \sin B &= \frac{1}{2} \left[\cos(A-B) - \cos(A+B) \right]
\end{aligned}$$

We want to consider $\sin n\pi x/l$, $\sin m\pi x/l$, where n, m are positive integers. These functions are periodic with period $2l$. [1]

$$\begin{aligned}
SS_{mn} &:= \int_0^{2l} \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \\
&= \frac{1}{2} \int_0^{2l} \cos\left[\frac{(m-n)\pi x}{l}\right] \mathrm{d}x - \frac{1}{2} \cos\left[\frac{(m+n)\pi x}{l}\right] \mathrm{d}x
\end{aligned}$$

if $m \neq n$,

$$SS_{mn} = \frac{l}{2\pi} \left[\frac{\sin(m-n)\pi x/l}{m-n} - \frac{\sin(m+n)\pi x/l}{m+n} \right]_0^{2l} = 0$$

if $m = n$, then $SS_{mn} = 1$ (provided $m \neq 0$, $n \neq 0$). Hence

$$SS_{mn} = \begin{cases} \delta_{mn} & \text{if } m, n \neq 0 \\ 0 & \text{if } m \text{ or } n = 0 \end{cases}$$

Similarly, $CC_{mn} = \int_0^{2l} \cos\left(\frac{m\pi x}{l}\right) \cos\left(\frac{n\pi x}{l}\right) \mathrm{d}x = l\delta_{mn}\ \forall m, n \neq 0$, and $2l$ if $m = n = 0$

Finally,

$$\begin{aligned}
CS_{mn} &= \int_0^{2l} \cos\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}x \\
&= \frac{1}{2} \int_0^{2l} \frac{\sin(m+n)\pi x}{l} \mathrm{d}x + \frac{1}{2} \int_0^{2l} \frac{\sin(m-n)\pi x}{l} \mathrm{d}x = 0
\end{aligned}$$

[1] They have a common period of $2l$, not their smallest period!

By analogy with vectors [these integrals are indeed *inner products*], $\sin n\pi x/l$, $\cos n\pi x/l$ are said to be orthogonal on the interval $[0, 2l]$.

They actually constitute an *orthogonal basis.* ie. it is possible to represent an arbitrary (but sufficiently well behaved[2]) function in terms of an infinite series (Fourier series) formed as a sum of sins and cosines.

1.2 Definition of a Fourier Series

Any well behaved periodic function $f(x)$ with periodic $2L$ can be written as a Fourier Series:

$$\frac{f(x_+) + f(x_-)}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{m\pi x}{L}\right)$$

a_n and b_n are the Fourier Coefficients, $f(x_+)$ and $f(x_-)$ are the right limit approaching form above and the left limit approaching from below respectively

If $f(x)$ is continuous at x_c, then the LHS is just $f(x)$. If $f(x)$ has a bounded discontinuity, at x_d, ie. $f(x_d^-) \neq f(x_d^+)$, but $(f(x_d^-) - f(x_d^+))$ is finite, then the FS tends to the mean value of the two limits.

Coefficient construction: Multiply rhs of (*) by $\sin m\pi x/L$, integrate over 0 to $2L$, assume you can invert order or summation and integration.

$$\int_0^{2L} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{m\pi x}{L}\right)\right] \sin\frac{m\pi x}{2}\,\mathrm{d}x$$

We see that

$$\frac{a_0}{2}\int_0^{2L} \sin\frac{m\pi x}{L}\,\mathrm{d}x = 0$$

$$\sum_{n=1}^{\infty}\int_0^{2L} a_n \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = 0$$

$$\sum_{n=1}^{\infty}\int_0^{2L} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = Lb_n$$

So

$$\text{LHS} = \int_0^{2L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x \Rightarrow b_m = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x$$

Multiply by $\cos\frac{m\pi x}{l}$ and integrate from 0 to $2L$ (inc $m = 0$)

$$\int_0^{2L}\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{m\pi x}{L}\right)\cos\frac{m\pi x}{L}\,\mathrm{d}x$$

Non zero only when $m = 0$

Therefroe

[2] to be definied

$$\frac{a_0}{2}2L = \int_0^{2L} f(x)\ \mathrm{d}x \Rightarrow \frac{a_0}{2} = \frac{1}{2L}\int_0^{2L} f(x)\ \mathrm{d}x$$

$$a_m = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{m\pi x}{L}\right)\ \mathrm{d}x$$

The range of integration is one period so its also permissibel to choose $\int_{-L}^{L}$ a paricularly nice case is when $L = \pi$.

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\ \mathrm{d}x \qquad m \geq 0$$

$$b_m = \frac{1}{\pi}\int_{\pi}^{\pi} f(x)\sin mx\ \mathrm{d}x \qquad m \geq 1$$

1.3 Dirichlet Conditions

If $f(x)$ is a periodic function with period $2l$ st.

(i) it is absolutely integrable [3]

(ii) it has a finite number of extrema (ie maxs and mins) in $[0, 2l]$

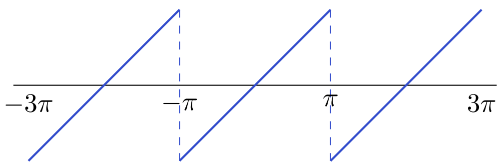
(iii) it has a finite number of bounded discontinuities in $[0, 2l]$

then the FS representation converges to $f(x)$ for all points where $f(x)$ is cts, and at points x_d where $f(x)$ is discontinuous, the series coverges to the avg value of the left and right limits, ie. to $\frac{1}{2}\left(f(x_{d_+}) + f(x_{d_-})\right)$. These conditions are satisfied if the function is of 'bounded variation'

1.4 Smoothness and order of Fourier coefficients

If the p^{th} derivative is the lowest derivative which is discontinuous somewhere (inc at the endpoints), then the F.C. are $\mathcal{O}[n^{-(p+1)}]$ as $n \to \infty$, eg. if a function has a bounded discontinuity, zeroth derivative is discontinuous: coefficients are of order $\frac{1}{n}$ as $n \to \infty$

Example. The sawtooth function, $f(x) = x$ on $-L \leq x \leq L$



Function is odd, so

$$a_m = \frac{1}{L}\int_{L}^{-L} xcos\left(\frac{m\pi x}{L}\right)\mathrm{d}x = 0$$

,

[3] ie. $\int_0^{2l} |f(x)|\ \mathrm{d}x$ is well defined

$$\begin{aligned} b_m = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x &= \frac{1}{L}\left(\left[-\frac{xL}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\right]_{-L}^{L} - \int_{-L}^{L}\frac{-L}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x\right) \\ &= \frac{1}{m\pi}\left(-2Lcos(m\pi) + \left[\sin\left(\frac{m\pi x}{L}\right)\frac{L}{m\pi}\right]_{-L}^{L}\right) \\ &= \frac{2L}{m\pi}(-1)^{m+1} \end{aligned}$$

So

$$\frac{f(x_+)+f(x_-)}{2} = \frac{2L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \cdots\right]$$

(i) $f_N(x) := \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right) \to f(x)$ almost everywhere, but the convergence is non-uniform.

(ii) Persistence overshoot @ $x = L$: 'Gibbs phenomenon'

(iii) $f(L) = 0$ average of right and left limits

(iv) Coefficients are $\mathcal{O}(\frac{1}{n})$ as $n \to \infty$

Example. The integral of the sawtooth function, $f(x) = \frac{1}{2}x^2$, $-L \leq x \leq L$

Exercise.

$$f(x) = L^2\left[\frac{1}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]$$

Note at $x = 0$,

$$0 = L^2\left[\frac{1}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^2}{(n\pi)^2}\right] \Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

2 Properties of the Fourier Series

2.1 Integration and Differentiation

2.1.1 Integration: Always works!

FS. can be integrated term by term:

$f(x)$ periodic with period $2L$ and has a FS (so it satisfies Dirichlet conditions)[4]:

$$\frac{f(x_+)+f(x_-)}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{m\pi x}{L}\right)$$

$$\begin{aligned}
F(x) = \int_{-L}^{x} f(x')\,\mathrm{d}x' &= \frac{a_0(x+L)}{2} + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{b_n L}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi x}{L}\right)\right] \\
&= \frac{a_0 L}{2} + L\sum_{n=1}^{\infty}(-1)^n \frac{b_n}{n\pi} \\
&\quad - L\sum_{n=1}^{\infty} \frac{b_n}{n\pi}\cos\left(\frac{n\pi x}{L}\right) \\
&\quad + L\sum_{n=1}^{\infty}\left(\frac{a_n - (-1)^n a_0}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

If a_n and b_n are FC then the series involving $\frac{a_n}{n}$ and $\frac{b_n}{n}$ (multipled by cos or sin) must also converge

The Fourier series of $f(x)$ exists, so b_n is at least of $\mathcal{O}(\frac{1}{n})$ as $n \to \infty \Rightarrow \frac{b_n}{n}$ is at least $\mathcal{O}(\frac{1}{n^2})$ as $n \to \infty$, and so by the comparison test with $\sum_{n=1}^{\infty} \frac{M}{n^2}$, the second term on the RHS converges $\Rightarrow F(x)$ has a FS.

Note: integration smooths. Proof relies on discontinuity being bdd, ($f(x)$ satisfies Dirichlet condition).

2.1.2 Differentiation: Doesn't always work!

Let $f(x)$ be a periodic function with period 2, st.

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

Odd function $\Rightarrow a_m = 0$.

[4] pay attention to the limits here

$$\begin{aligned} b_m &= -\int_{-1}^{0} \sin(m\pi x)\ \mathrm{d}x + \int_0^1 \sin(m\pi x)\ \mathrm{d}x \\ &= \left[\frac{\cos(m\pi x)}{m\pi}\right]_{-1}^{0} - \left[\frac{\cos(m\pi x)}{m\pi}\right]_0^1 \\ &= \frac{1}{m\pi}\left[1 - (-1)^m - (-1)^m + 1\right] \\ &= \frac{4}{\pi x} \text{ if } m \text{ odd, or } 0 \text{ if } m \text{ even} \end{aligned}$$

$$\frac{f(x_+) + f(x_-)}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left((2n-1)\pi x\right)}{2n-1}$$

Apply diff rules:

$$f'(x) = 4\sum_{n=1}^{\infty} \cos\left((2n-1)\pi x\right)$$

This is clearly divergent, even though $f(x) = 0$ for all $x \neq 0$.

The extra factor of $2n-1$ is the problem. It's related to the discontinuity, $f'(x)$ does not satisfy the Dirichlet condition

Differentiation can be done under certain circumstances.

Example. Assume the function $f(x)$ is continuous and is extended as a $2L$-periodic function, piece-wise continuously differentiable on $(0, 2L)$. Let $g(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$ such that $g(x)$ satisfies D.C.[5]

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{m\pi x}{L}\right)$$

$$\frac{g(x_+) + g(x_-)}{2} = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{L}\int_0^{2L} g(x)\ \mathrm{d}x = \frac{f(2L) - f(0)}{L} = 0 \qquad \text{by periodicity}$$

$$\begin{aligned} A_n &= \frac{1}{L}\int_0^{2L} \frac{\mathrm{d}f}{\mathrm{d}x} \cos\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \\ &= \frac{1}{L}\left[f(x)\cos\left(\frac{n\pi x}{L}\right)\right]_0^{2L} + \frac{n\pi}{L^2}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \\ &= 0 + \frac{n\pi b_n}{L} \end{aligned}$$

Exercise. $B_n = \frac{-n\pi a_n}{L}$

We see differentiation reduces to multiplying by $\pm\frac{n\pi}{L}$

[5] $g(x)$ has at worst a finite number of bounded discontinuitites.

2.2 Alternate representation: complex form

Remember

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left(e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}}\right)$$

$$\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2i}\left(e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}\right)$$

$$\frac{f(x_+)+f(x_-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}}\right) - \frac{b_{ni}}{2}\left(e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}\right)$$

$$= \frac{a_0}{2} = \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{\frac{in\pi x}{L}} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{\frac{-in\pi x}{L}}$$

$$= \sum_{-\infty}^{\infty} c_n e^{\frac{i\pi n x}{L}} \quad c_0 = \frac{a_0}{2};\ \underbrace{c_n}_{(n>0)} = \frac{a_n - ib_n}{2};\ \underbrace{c_n}_{(n<0)} = \frac{a_n + ib_n}{2}$$

Note that

$$c_m^* = c_{-m}$$

complex exponentials are orthogonal

$$\int_0^{2L} e^{\frac{in\pi x}{L}} e^{\frac{-in\pi x}{L}}\ \mathrm{d}x = \int_0^{2L} \cos\left(\frac{(n-m)\pi x}{L}\right)\ \mathrm{d}x + i\int_0^{2L} \underbrace{\sin(n-m)}_{0\text{ by periodicity}}\ \frac{\pi x}{L}\ \mathrm{d}x = 2L\delta_{nm}$$

$$c_m = \frac{1}{2L}\int_0^{2L} f(x)e^{\frac{-im\pi x}{L}}\ \mathrm{d}x = \frac{1}{2L}\int_0^{2L} \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} e^{\frac{-in\pi x}{L}}\ \mathrm{d}x$$

Now assume $g(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$

$$\begin{aligned} c_n &= \frac{1}{2L}\int_0^{2L} \frac{\mathrm{d}f}{\mathrm{d}x} e^{\frac{-in\pi x}{L}}\ \mathrm{d}x \\ &= \frac{1}{2L}\left[f(x)e^{\frac{-in\pi x}{L}}\right]_0^{2L} + \frac{in\pi}{2L^2}\int_0^{2L} f(x)e^{\frac{-in\pi x}{L}}\ \mathrm{d}x \\ &= \frac{in\pi}{L}c_n \quad \text{by periodicity} \end{aligned}$$

2.3 Half-range series

Consider a functon defined only on $0 \leq x \leq L$.

There are two possible ways to extend it to a $2L$-periodic function that can be represented as a FS. [6]

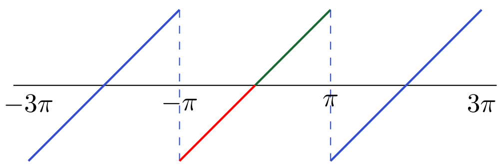
[6] It would be a dumbass thing to extend it as an odd function; you'll have a discontinuity!

2.3.1 Fourier sine series: odd function

$f(x)$ can be extended as an *odd* function $f(x) = -f(-x)$ on $-L \leq x \leq L$ and then extended as a $2L$-periodic function. In this case $a_n = 0$ and we can define the *Fourier sine series*

$$\frac{f(x_+) + f(x_-)}{2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

Example. Sawtooth function



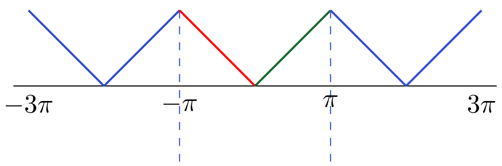
FS describes the $2L$ periodic fn. Sine seires still describes the fn on $[0, L]$

2.3.2 Even functions: fourier cosine series

$f(x)$ can also be extended as an even fn on $-L \leq x \leq L$ ie. $f(x) = f(-x)$ and then extended as a $2L$-periodic fn: $\Rightarrow b_n = 0\ \forall n$.

Fourier cosine series:

$$\frac{f(x_+) + f(x_-)}{2} = \frac{a_0}{2}\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$



2.4 Parserals Theorem

'Energy' of a periodic signal is often of interest, ie

$$E = \int_0^{2L} f^2(x)\,\mathrm{d}x$$

Consider the general case

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where

$$g(x) = \sum_{m=-\infty}^{\infty} d_m e^{\frac{im\pi x}{L}}$$

$$\int_0^{2L} f(x)g(x)\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n d_m \int_0^{2L} \exp\left[\frac{i\pi x}{L}(n+m)\right]\,\mathrm{d}x$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n d_m \left(2L\delta_{n[-m]}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n d_{-n} = 2L \sum_{n=-\infty}^{\infty} c_n d_n^*$$

So if $g(x) = f(x)$

$$\int_0^{2L} [f(x)]^2\,\mathrm{d}x = 2L\sum_{n=-\infty}^{\infty} |c_n|^2 = L\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

Example. Remember $f(x) = x$ for $-L \leq x \leq L$

$$b_n = \frac{2L}{m\pi}(-1)^{m+1}$$

$$\int_{-L}^{L} x^2\,\mathrm{d}x = \frac{2L^3}{3} = L\sum_{m=1}^{\infty}\frac{4L^2}{m^2\pi^2}$$

$$\Rightarrow \sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}$$

Exercise. From the FS of $\frac{x^2}{2}$ show that

$$\sum_{m=1}^{\infty}\frac{1}{m^4} = \frac{\pi^4}{90}$$

3 Strum-Liouville Theory Motivation

3.1 Second order ODEs

$$\mathcal{L}y(x) = \alpha(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}y + \beta(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \gamma(x)y = f(x)$$

α, β, γ continuous, α non zero except perhaps at a finite number of isolated points,

$f(x)$ is bounded, defined on $a \leq x \leq b$ (a or b may be $\pm\infty$).

HOMOGENOUS: eg

$$\mathcal{L}y = 0$$

has two linearly independent solutions y_1, y_2 complementary function $y_c = Ay_1 + By_2$

INHOMOGENEOUS OR FORCED EQUATION.

$$\mathcal{L}y = f(x)$$

F is the *forcing* and has a particular integral $y_p(x)$. G.S. is $y = y_c(x) + y_p(x)$ where $A + B$ are determined in a 'PROBLEM' by applying conditions