Part IB — Linear Algebra Sheet 2

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QUESTION 1

The three types of elementary matrices are:

(i) $$\begin{pmatrix} 1 &&&&&&&&& \\ & \ddots &&&&&&&& \\ && 1 &&&&&&& \\ &&& 0 &&& 1 &&& \\ &&&& 1 &&&&& \\ &&&&& \ddots &&&& \\ &&&&&& 1 &&& \\ &&& 1 &&&& 0 && \\ &&&&&&&& 1 & \\ &&&&&&&&& \ddots \\ &&&&&&&&& 1 \end{pmatrix}$$

The zeros appear in row i, row j. This swaps column i and column j, and is self-inverse.

(ii) $$\begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \lambda &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix}$$

with λ in the i^{th} row. (Multiplies column i by λ) This has inverse

$$\begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& \frac{1}{\lambda} &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix}$$

(iii) $I_n + \lambda E_{ij}$, where E_{ij} is defined as 1 in the (i, j) position and 0 everywhere else. $(i \neq j)$. This has inverse $I_n + \lambda E_{ij}$.

To find inverse of this matrix, we

- add column 1 to column 2
- swap rows 2 and 3
- add row 3 to row 2
- multiply row 2 by $\frac{1}{3}$.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}$$

QUESTION 2

- *Minimality of r*:

  Suppose we have

$$\underbrace{A}_{m\times n} = \underbrace{B}_{m\times k}\underbrace{C}_{k\times n}$$

  wrt. standard basis for $\mathbb{R}^n, \mathbb{R}^k, \mathbb{R}^n$ the matrices A, B, C correspond to lin. maps α, β, γ st. $\alpha = \beta \circ \gamma$

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "\gamma"] \arrow[rr, bend right, "\alpha"'] & \mathbb{R}^k \arrow[r, "\beta"] & \mathbb{R}^n \end{tikzcd}$$

  for some k, $r \leq k \leq n$, and

$$\operatorname{Im}\alpha \leq \operatorname{Im}\beta$$

  since if $v \in \operatorname{Im}\alpha$, then $v = \alpha(\omega)$ for some ω, then $v = \beta(\gamma\omega)$ so $v \in \operatorname{Im}\beta$

  Taking dimensions,

$$r \leq r(\beta) \leq k$$

  where the last inequality follows from Rank Nullity.

- *r is possible*: α has rank r so we seek a map such that

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "\gamma"] \arrow[rr, bend right, "\alpha"'] & \operatorname{im}\alpha \arrow[r, "\beta"] & \mathbb{R}^n \end{tikzcd}$$

  Define $\gamma : \mathbb{R}^m \to \operatorname{Im}\alpha$ by $\gamma(v) = \alpha(v)$. Define $\beta : \operatorname{Im}(\alpha) \to \mathbb{R}^n$ by $\beta(\omega) = \omega$.

  Then, picking bases for $\mathbb{R}$, $\operatorname{Im}(\alpha)$, $\mathbb{R}^m$, we get corresponding matrices A, B, C st.

$$A = \underbrace{B}_{m\times r}\underbrace{C}_{r\times n}$$

For the last part, define $r' =$ col rank of A^T
We know that $A = BC$ with B $m \times r$ ... So

$$A^T = C^T B^T \text{ with } C^T\ n \times r$$

so $r' \leq r$ by previous work
Applying this argument to $A^T, (A^T)^T (= A)$, we also see that $\gamma \leq r'$. So $r = r'$.

QUESTION 3

If V is the vector space with finite basis $\mathcal{B} = \{x_1, x_2, x_3, x_4\}$ then there is a basis for V^*, given by $\mathcal{B}^* = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ where

$$\xi_j \underbrace{\left(\sum_{i=1}^{4} a_i x_i\right)}_{\in V} = a_j \quad 1 \leq j \leq 4 \qquad (*)$$

(a) By (*), the dual basis is

$$\{\xi_2, \xi_1, \xi_4, \xi_3\}$$

(b) we have $\xi_2\left(\sum_{i=1}^{4} a_i x_i\right) = a_2 \Rightarrow \xi_2(a_2 x_2) = a_2$. Hence clear to see dual basis is

$$\{\xi_1, \frac{1}{2}\xi_2, 2\xi_3, \xi_4\}$$

(c) Call the new dual basis $\{\eta_1, \eta_2, \eta_3, \eta_4\}$. It is clear that $\eta_1 = \xi_1$. To find η_2, we aim to solve the system of linear equations

$$\begin{aligned}
\eta_2(x_1 + x_2) &= 0 \\
\eta_2(x_2 + x_3) &= 1 \\
\eta_2(x_3 + x_4) &= 0 \\
\eta_2(x_4) &= 0
\end{aligned}$$

and we deduce that $\eta_2 = \xi_2 - \xi_1$. Similarly, $\eta_3 = \xi_3 - \xi_2$, $\eta_4 = \xi_4 - \xi_3$.

$$\{\xi_1, \xi_2 - \xi_1, \xi_3 - \xi_2, \xi_4 - \xi_3\}$$

(d) Similar method to (c), the dual basis is:

$$\{\xi_1 + \xi_2, \xi_2 + \xi_3, \xi_3 + \xi_4, \xi_4\}$$

QUESTION 4

We have that $\tau_A(B) = \sum_i \sum_j a_{ij} b_{ji}$, so linearity follows immediately by the definition of the sum.

Next, want to show that

$$\mathrm{Mat}_{m,n(\mathbb{F})} \xrightarrow{\phi} \mathrm{Mat}^*_{m,n(\mathbb{F})}$$

defined by

$$A \mapsto \tau_A$$

defines an iso. Have already show linearity. Easy to see this is well defined.

- Injective: Suppose $\phi(A) = 0$. Then $\tau_A(B) = 0 \;\forall\; B$, ie. $\mathrm{tr}(AB) = 0 \;\forall\; B$.

  In particular, for each i, j, we have that

  $$\mathrm{tr}(AE_{ij}) = 0$$

  where E_{ij} is the matrix with 1 in the i, j position and zeroes everywhere else.

  Hence by definition of trace, $\sum_{k,l} A_{k,l}(e_{ij})_{k,l} = A_{ji}$. So $A = 0$.

- $\dim(\mathrm{Mat}_{m,n}(\mathbb{F})^*) = \dim(\mathrm{Mat}_{m,n}(\mathbb{F})) = mn$, and $\dim(\mathrm{Mat}_{m,n}(\mathbb{F})) = mn$. So isomorphism.

QUESTION 5

(a) Suppose two such endomorphisms exists, with matrices A, respectively. Take the trace of both sides of the equation. As $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, clearly the LHS is zero, but the RHS is $\dim V$. Contradiction.

(b) Define

$$\begin{aligned} \alpha : V &\to V & \beta : V &\to V \\ f(x) &\mapsto xf(x) & f(x) &\mapsto f'(x) \end{aligned}$$

Then

$$\begin{aligned} (\alpha\beta - \beta\alpha)(f) &= (xf)' - xf' \\ &= f \end{aligned}$$

That is, $\alpha\beta - \beta\alpha = \mathrm{id}_V$

QUESTION 6

Let $\psi : U \times V \to \mathbb{F}$ represent our bilinear form. Pick any bases,

$$e'_1, \cdots, e'_m \text{ for } U$$
$$f'_1, \cdots, f'_n \text{ for } V$$

If $\psi(e'_i, f'_j) = 0 \; \forall \, i, j$ then $\psi = 0$ and we're done. Otherwise pick some

$$e'_i, f'_j \text{s.t. } \psi(e'_i, f'_j) \neq 0$$

(After rescaling, assume $\psi(e'_i, f'_j) = 1$).

Set $e_1 = e'_i$, $f_1 = f'_j$. Pick a basis $\{f_2, \cdots, f_n\}$ for $\ker(\psi_L(e_1))$.

Pick a basis $\{e_2, \cdots, e_m\}$ for $\ker(\psi_R)(f_1)$. Then, wrt. $\{e_1, \cdots, e_m\}, \{f_1, \cdots, f_n\}$, have matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \text{something} & \\ 0 & & & \end{pmatrix}$$

Continue inductively, end up with

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$\begin{aligned} \psi(\sum_{i=1}^{m} x_i e_i, \sum_{j=1}^{n} y_j f_j) &= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j \psi(e_i, f_j) \text{ by linearity of } \psi \\ &= \sum_{k=1}^{r} x_k y_k \end{aligned}$$

The dimensions of the left and right kernels are $m - r$ and $n - r$ respectively, by R-N.

QUESTION 7

(a) We show the rows are linearly independent: suppose

$$\lambda_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} a_0^n \\ a_1^n \\ \vdots \\ a_n^n \end{pmatrix} = 0$$

This says that the polynomial

$$f(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n$$

has roots $a_0, a_1, \cdots, a_n$. f of degree n has $n+1$ distinct roots; but this can only be the case if f is the zero polynomial. So $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Rows are linearly independent, so the matrix is of full rank. Thus $n(A) = 0$ by rank nulity, and $\det A \neq 0$

(b) $e_x \in P_n^*$ with $e_x(p) = p(x)$. Want to show that with respect to the standard basis, $\{e_0, \cdots, e_n\}$ is linearly independent.

Proof. Suppose $\lambda_0 e_0 + \cdots + \lambda_n e_n = 0$.

Then

$$\begin{aligned} \lambda_0 + \cdots + \lambda_n e_n &= 0 \\ 0 \cdot \lambda_0 + \cdots + n \cdot \lambda_n &= 0 \\ &\vdots \\ 0^n \cdot \lambda_0 + \cdots + n^n \cdot \lambda_n &= 0 \end{aligned}$$

ie.

$$\lambda_0 \begin{pmatrix} 1 \\ 0^1 \\ \vdots \\ 0^n \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} 1 \\ n^1 \\ \vdots \\ n^n \end{pmatrix} = 0$$

So by (a), $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

□

Now, $\dim P_n^* = n + 1$, and $\{e_0, \cdots, e_{n+1}\}$ is a linearly independent set. Hence it is a basis for P_n^*

(c) For the basis of P_n for which $(e_0, \cdots, e_n)$ is dual, we want $\{p_0, \cdots, p_n\}$ with $e_i(p_j) = \delta_{ij}$. So this polynomial is zero for all $i \in \{0, 1, \cdots, i-1, i+1, n\}$. Hence $p_i \propto (x-1)(x-2)\cdots(x-(i-1))(x-(i+1))\cdots(x-n)$

As $p_i(i) = 1$,

$$p_i = \frac{(x-1)(x-2)\cdots(x-(i-1))(x-(i+1))\cdots(x-n)}{(i-1)(i-2)\cdots(i-(i-1))(i-(i+1))\cdots(i-n)}$$

QUESTION 8

(i)

$$\begin{aligned}\text{adj }(AB) &= \det(AB)(AB)^{-1}\\ &= \det(A)\det(B)B^{-1}A^{-1}\\ &= \det(B)B^{-1}\det(A)A^{-1}\\ &= \text{adj }(B)\text{adj }(A)\end{aligned}$$

(ii)

$$\begin{aligned}\det(\text{adj }A) &=\\ &= \det(\det(A)A^{-1})\\ &= \det(\det(A)I)\det(A^{-1}) \qquad = (\det A)^n(\det A)^{-1}\\ &= (\det A)^{n-1}\end{aligned}$$

(iii)

$$\begin{aligned}\text{adj }(\text{adj }A) &= \text{adj }(\det(A)A^{-1})\\ &= \det(\det(A)A^{-1})(\det(A)A^{-1})^{-1}\\ &= (\det A)^{n-1}A(\det A)^{-1}\\ &= (\det A)^{n-2}A\end{aligned}$$

- If $r(A) = n$, then
$$\text{adj }A = \underbrace{\det A}_{\neq 0}\ \underbrace{A^{-1}}_{\text{invertible}}$$
So adj$A$ invertible so $r(\text{adj }A) = n$
- If $r(A) = n - 1$, recall that
$$(\text{adj }A)A = 0 \text{ in this case}$$
So adj$A$ maps $\text{Im}\,A$ to 0

  ie. $\ker(\text{adj }A)$ has dimension at least $\dim(\text{Im}\,A) = n - 1$. So $r(\text{adj }A) = 0$ or 1.

  However, adj $A \neq 0$: We can remove some column i st. the remaining cols are L.I ($r(A) = n - 1$). This gives us an $n \times (n - 1)$ matrix with row rank $n - 1$ (since row rank = column rank.) So we can remove a row j st. remaining rows are LI.

  Then $A_{\hat{i}\hat{j}} \neq 0$ ($(n-1)\times(n-1)$ matrix left over has full rank). So adj $A \neq 0$. So $r(\text{adj }A) = 1$.
- If $r(A) = n - 2$

  If we remove a col, the remaining cols are L.D (still). This does not change if we further remove a row. So adj $A = 0$.

Singular case:

(i) Define $f(\lambda) = \text{adj}((A + \lambda I)B) - \text{adj}(B)\text{adj}(A + \lambda I)$. We know that $f(\lambda)$ is zero whenever λ is st.

$$\det(A + \lambda I) \neq 0$$

ie. $f(\lambda)$ is a poly with infinitely many roots.

(since $\det(A + \lambda I)$ is a non-zero poly so has at most n roots). So f is the zero polynomial, and (i) holds in general.

QUESTION 9

Let α be the map $\alpha : P^* \to \mathbb{R}^{\mathbb{N}}$ defined by

$$[\xi : P \to \mathbb{R}] \mapsto (\xi(1), \xi(t), \xi(t^2), \cdots)$$

$$\begin{aligned}\alpha(\lambda\xi_1 + \mu\xi_2) &= ((\lambda\xi_1 + \mu\xi_2)(1), (\lambda\xi_1 + \mu\xi_2)(t), (\lambda\xi_1 + \mu\xi_2)(t^2), \cdots) \\ &= (\lambda\xi_1(1) + \mu\xi_2(1), \lambda\xi_1(t) + \mu\xi_2(t), \lambda\xi_1(t^2) + \mu\xi_2(t^2), \cdots) \\ &= \lambda(\xi_1(1), \xi_1(t), \xi_1(t^2), \cdots) + \mu(\xi_2(1), \xi_2(t), \xi_2(t^2), \cdots)\end{aligned}$$

Hence α is linear.
Let the map $B : \mathbb{R}^{\mathbb{N}} \to P^*$ be defined as

$$(a_0, a_1, a_2, \cdots) \mapsto [\xi : P \to \mathbb{R} \mid \xi(t^n) = a_n]$$

As this defines ξ on the basis of P, it fully defines ξ so this is well defined. This is an inverse to α so α is a bijective map. Hence α is an isomorphism and $P^* \simeq \mathbb{R}^{\mathbb{N}}$

Let the sequence $(a_0, a_1, a_2, \cdots)$ correspond to the linear map $\xi : P \to \mathbb{R}$ with $\xi(t^n) = a_n$. For $\alpha \in L(P, P)$

α^* dual to α is defined by $\alpha^* : P^* \to P^*$, $\varepsilon \mapsto \varepsilon \circ \alpha$.

(a) $D^*(\xi) \mapsto \xi \circ D$ where $\xi \circ D : P \to \mathbb{R}$ with $\xi \circ D(t^n) = \xi(nt^{n-1}) = na_{n-1}$

(b) $S^*(\xi) \mapsto \xi \circ S$ where $\xi \circ S : P \to \mathbb{R}$ with $\xi \circ S(t^n) = \xi(t^{2n}) = a_{2n}$

(c) $(DS)^*(\xi) \mapsto \xi \circ DS$ where $\xi \circ DS : P \to \mathbb{R}$ with $\xi \circ DS(t^n) = \xi(2nt^{2n-1}) = 2na_{2n-1}$

(d) $(SD)^*(\xi) \mapsto \xi \circ SD$ where $\xi \circ SD : P \to \mathbb{R}$ with $\xi \circ SD(t^n) = \xi(nt^{2(n-1)}) = na_{2(n-1)}$

QUESTION 10

QUESTION 11