

Part IB — Methods Example Sheet 4

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QUESTION 1

(i) Along characteristic curves,

$$\frac{dx}{ds} = 1 \quad \frac{dy}{ds} = y$$

which has general solution $x = s + c$ and $y = Ae^s$ for some constants c, A .
Cauchy data is $B(t) = \{(x = 0, y = t)\}$, intersect B at $s = 0$, thus characteristic curves are

$$x = s \quad y = te^s$$

$\frac{\partial u}{\partial s} \Big|_t = 0$ so that $u = \text{cst}$ along these characteristics.

and the Cauchy data fixes $u(s, t) = t^3$ on the t^{th} curve. Inverting gives

$$s = x \quad t = ye^{-x}$$

and therefore the solution to our problem is

$$u(x, y) = y^3 e^{-3x}$$

throughout \mathbb{R}^2

(ii) Along characteristic curves

$$\frac{dx}{ds} = y \quad \frac{dy}{ds} = x$$

$$\Rightarrow \frac{d^2 x}{ds^2} = x$$

$$x = A \sinh s + B \cosh s$$

Cauchy data is $B(t) = \{(x = 0, y = t)\}$, intersect at $s = 0$ gives $B = 0$, thus

$$x = t \sinh s \quad y = t \cosh s$$

Then $\frac{\partial u}{\partial s} \Big|_t = 0 \Rightarrow u = \text{cst.}$ along these characteristics, and Cauchy data fixes $u(s, t) = e^{-t^2}$, and have that

$$t^2 = (t \cosh s)^2 - (t \sinh s)^2 = y^2 - x^2$$

therefore solution is

$$u(x, y) = e^{x^2 - y^2}$$

which is uniquely defined only in the upper quadrant of the plane $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$

(iii) PDE is $u_x + u_y = e^{x+2y} - u$.

Along characteristic curves

$$\frac{dx}{ds} = 1 \quad \frac{dy}{ds} = 1$$

$$x = s + c \quad y = s + d$$

Cauchy data is $B(t) = \{(x = t, y = 0)\}$, intersect at $s = 0$, thus

$$x = s + t \quad y = s$$

Then $\left. \frac{\partial u}{\partial s} \right|_t = e^{2x+y} - u = e^{3s+t} - u$ along these characteristics, this is an ODE in s so multiplying by the integrating factor e^s gives

$$e^s \frac{du}{ds} + e^s u = e^{4s+t}$$

$$\Rightarrow e^s u = \frac{1}{4} e^{4s+t} + \text{cst.}$$

Cauchy data $u(t, 0) = 0$ fixes $\text{cst} = -\frac{1}{4}e^t$, and have that

$$u(s, t) = \frac{1}{4} e^{t+3s} - \frac{1}{4} e^{t-s}$$

inverting the relations

$$s = y \quad t = x - y$$

the solution is

$$\begin{aligned} u(x, y) &= \frac{1}{4} e^{x+2y} - \frac{1}{4} e^{x-2y} \\ &= \frac{1}{2} e^x \sinh 2y \end{aligned}$$

QUESTION 2

Separation of variables $\Rightarrow u(x, t) = X(x)T(t)$, thus

$$\begin{aligned} X''T + X\dot{T} &= 0 \\ \Rightarrow \frac{X''}{X} &= -\frac{\dot{T}}{T} \end{aligned}$$

LHS independent of x , RHS independent of t , so both sides constant. Setting $\lambda = \dot{T}/T$ we have

$$X'' + \lambda X = 0$$

$$X(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x$$

Boundary conditions $X(0) = X(\pi) = 0$ imply that $A = 0$, $\lambda = n^2$. Then solving $\dot{T} = \lambda T$ gives $T(t) = Ce^{n^2 t}$ for some constant C , so the solution is

$$U(x, t) = \sum_{n=1}^{\infty} D_n e^{n^2 t} \sin nx$$

for some constants D_n , which we can determine by using orthogonality conditions.

But now, looking at what happens for large n , the solution then has oscillations with higher and higher wavenumber and larger and larger (indeed arbitrarily large) amplitude $U(x)e^{n^2 t}$, and so this problem is ill-posed.

QUESTION 3

- (i) The principal part of the symbol of the differential operator is $\mathbf{k}^T \mathbf{A} \mathbf{k}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$$

$\det A$ is product of eigenvalues, therefore we have

$$\begin{array}{ll} \text{elliptic} & x > 0 \\ \text{parabolic} & x = 0 \\ \text{hyperbolic} & x < 0 \end{array}$$

In the hyperbolic region the negative eigenvector points in the y direction. Thus, if $f(x, y) = \text{cst.}$ is to be a characteristic surface, we need $\partial_y f = \pm \sqrt{\partial_x f A_{xx} \partial_x f / x} = \pm \sqrt{x} \partial_x f$. Letting $p = 2x^{1/2}$, this is $(\partial_y \pm \partial_p) f = 0$, so the characteristic surfaces are the two curves of constant $y \pm p$, that is

$$\begin{aligned} u &= y + 2x^{1/2} \\ v &= y - 2x^{1/2} \end{aligned}$$

for u constant and v constant.

- (ii) Here

$$\mathbf{A} = \text{diag}(1, y)$$

so the PDE is hyperbolic in the $y < 0$ region with the negative eigenvector \mathbf{m} points in the y -direction, with $\mathbf{A}\mathbf{m} = y\mathbf{m}$. Thus, if $f(x, y) = \text{cst.}$ is to be a characteristic surface, we need $\partial_y f = \pm \sqrt{\partial_x f A_{xx} \partial_x f / y} = \pm \sqrt{y} \partial_x f$. Letting $\xi = x, \eta = 2y^{1/2}$, this is $(\partial_\xi \pm \partial_\eta) f = 0$, (thus $\partial_{\eta\nu} f = 0$) so the characteristic surfaces are the two curves of constant $\xi \pm \eta$, that is

$$\begin{aligned} \nu_1 &= x + 2y^{1/2} \\ \nu_2 &= x - 2y^{1/2} \end{aligned}$$

for ν_1 constant and ν_2 constant.

(Do we? Not sure about the steps here) have $u_{\xi\eta} = 0 \Rightarrow u(\xi, \eta) = f(\xi + \eta) + g(\xi - \eta)$, so the result follows.

QUESTION 4

Green's second identity is

$$\int_{\Omega} \phi \nabla^2 \psi - \psi \nabla^2 \phi \, dV = \int_{\partial\Omega} \phi(\mathbf{n} \cdot \nabla \psi) - \psi(\mathbf{n} \cdot \nabla \phi) \, dS$$

where $\Omega \subset \mathbb{R}^n$ is a compact set, and $\phi, \psi : \Omega \rightarrow \mathbb{R}$ are a pair of functions on Ω regular throughout Ω .

Green's function for the Laplacian on domain Ω satisfies $\nabla^2 G(\mathbf{r}; \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$, and $G|_{\partial\Omega} = 0$

Consider using this identity with ϕ, ψ being $G(\mathbf{r}'; \mathbf{r}), G(\mathbf{r}'; \mathbf{r}_0)$.

Assume Green's second identity is valid for Green's functions (it isn't).

Then the RHS is zero, as $G|_{\partial\Omega} = 0$. Then we have

$$\begin{aligned} 0 &= \int_{\Omega} G(\mathbf{r}'; \mathbf{r}) \nabla^2 G(\mathbf{r}'; \mathbf{r}_0) - G(\mathbf{r}'; \mathbf{r}_0) \nabla^2 G(\mathbf{r}'; \mathbf{r}) \, dV \\ &= \int_{\Omega} G(\mathbf{r}'; \mathbf{r}) \delta(\mathbf{r}' - \mathbf{r}_0) - G(\mathbf{r}'; \mathbf{r}_0) \delta(\mathbf{r}' - \mathbf{r}) \, dV \\ &= \int_{\Omega} G(\mathbf{r}_0; \mathbf{r}) - G(\mathbf{r}; \mathbf{r}_0) \, dV \end{aligned}$$

Conclude that $G(\mathbf{r}_0; \mathbf{r}) = G(\mathbf{r}; \mathbf{r}_0)$ anywhere in Ω .

QUESTION 5

$u(\mathbf{x})$ is harmonic and therefore satisfies $\nabla^2 \mathbf{u} = 0$. Consider a Dirichlet Green's function for the Laplace operator on D ; we have

$$\nabla^2 G(\mathbf{r}; \mathbf{r}_0)$$

It can be shown that

$$G(\mathbf{x}; \mathbf{x}_0) = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0|$$

Now, the result is a direct application of Green's third identity (which I'm guessing the question wants us to derive). Let B_r and B_ε be circles about the point \mathbf{x}_0 , so

$$B_r = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x} - \mathbf{x}_0| \leq r\}$$

$$B_\varepsilon = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x} - \mathbf{x}_0| \leq \varepsilon\}$$

Now consider applying Green's identity on the region $\Omega = B_r - B_\varepsilon$, so $G(\mathbf{x}; \mathbf{x}_0)$ is perfectly regular everywhere within Ω

Using Green's second identity with \mathbf{u} and G we have

$$\begin{aligned} \int_{\Omega} u \nabla^2 G - G \nabla^2 u \, dV &= 0 \\ &= \int_{\delta\Omega} u(\mathbf{n} \cdot \nabla G) - G(\mathbf{n} \cdot \nabla u) dS \\ &= \int_{S_r} u(\mathbf{n} \cdot \nabla G) - G(\mathbf{n} \cdot \nabla u) dS + \int_{S_\varepsilon} u(\mathbf{n} \cdot \nabla G) - G(\mathbf{n} \cdot \nabla u) dS \quad (*) \end{aligned}$$

where the first equality follows since $\nabla^2 G = 0$ in Ω and u is a harmonic function. We've included contributions from both boundary circles in the final line. On the inner boundary we have

$$G|_{\text{inner bdy}} = \frac{1}{2\pi} \log |\varepsilon|$$

$$\mathbf{n} \cdot G|_{\text{inner bdy}} = -\frac{1}{2\pi} \frac{1}{\varepsilon}$$

The final term on the last line of (*) becomes

$$-\int_{S_\varepsilon} G(\mathbf{n} \cdot \nabla u) dS = \oint \frac{1}{2\pi} \log |\varepsilon| \frac{\partial u}{\partial n} \varepsilon d\theta$$

Since u is regular by assumption, the value of this remaining integral is bounded, so this term vanishes as $\varepsilon \rightarrow 0$. On the other hand, the penultimate term in the final line of (*) becomes

$$\int_{S_\varepsilon} u(\mathbf{n} \cdot \nabla G) = -\frac{1}{2\pi} \oint u d\theta = -\tilde{u}$$

where \tilde{u} is the average value of u on the small circle surrounding \mathbf{x}_0 . As the radius of this sphere shrinks to zero we have $\tilde{u} \rightarrow u(\mathbf{x}_0)$, the value of u at the centre of the sphere.

Putting all this together we find

$$u(\mathbf{x}_0) = \oint u \frac{\partial}{\partial n} (\log |\mathbf{x} - \mathbf{x}_0|) - \log |\mathbf{x} - \mathbf{x}_0| \frac{\partial u}{\partial n} dl$$

which is Green's third identity, where we have now taken the boundary of Ω to be just the large circle (the radius of the inner circle having been shrunk to zero).

QUESTION 6

Following the hint, we set

$$x - x_0 = z_0 s \cos \theta \quad y - y_0 = z_0 s \sin \theta$$

so that $z_0^2 s^2 = (x - x_0)^2 + (y - y_0)^2$. The derivatives change as

$$dx dy = \left| \frac{\partial(x, y)}{\partial(s, \theta)} \right| ds d\theta$$

We calculate the Jacobian as

$$\left| \frac{\partial(x, y)}{\partial(s, \theta)} \right| = z_0^2 s$$

Hence

$$\begin{aligned} u(x_0, y_0, z_0) &= \frac{z_0}{2\pi} \int_0^{2\pi} \int_0^\infty [z_0^2(s^2 + 1)]^{-3/2} h(x, y) z_0^2 s ds d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty s[s^2 + 1]^{-3/2} h(x_0 + z_0 s \cos \theta, y_0 + z_0 s \sin \theta) ds d\theta \end{aligned}$$

First boundary condition,

$$\begin{aligned} u(x_0, y_0, 0) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty s[s^2 + 1]^{-3/2} h(x_0, y_0) ds d\theta \\ &= h(x_0, y_0) \int_0^\infty s[s^2 + 1]^{-3/2} ds \\ &= h(x_0, y_0) \left[-[s^2 + 1]^{-3/2} \right]_0^\infty \\ &= h(x_0, y_0) \end{aligned}$$

as required.

Second boundary condition, as $x^2 + y^2 \rightarrow \infty$, must also have $s^2 \rightarrow \infty$.

So (very loosely), the integral tends to zero.

QUESTION 7

Consider the forced one-dimensional heat equation

$$\partial_t \theta - D \partial_{xx} \theta = f(x, t) \quad 0 < x, t < \infty, \quad \theta(0, t) = h(t), \theta(x, 0) = \Theta(x)$$

Note the initial condition is inhomogeneous. We consider instead that $V(x, t) = \theta(x, t) - h(t)$ solves the equation, since the conditions are now homogeneous.

We seek a Green's function such that

$$\frac{\partial}{\partial t} G(x, t; y, \tau) - D \frac{\partial^2}{\partial x^2} G(x, t; y, \tau) = \delta(t - \tau) \delta(x - y)$$

Taking the Fourier transform wrt x

$$\frac{\partial}{\partial t} \tilde{G}(k, t; y, \tau) + k^2 D \tilde{G}(k, t; y, \tau) = \delta(t - \tau) e^{-iky}$$

Multiplying by $e^{Dk^2 t}$ we obtain

$$\frac{\partial}{\partial t} \left[e^{Dk^2 t} \tilde{G}(k, t; y, \tau) \right] = e^{-iky + Dk^2 t} \delta(t - \tau)$$

subject to the initial condition $\tilde{G}(k, 0; y, \tau) = 0$.

$$\begin{aligned} \tilde{G}(k, t; y, \tau) &= e^{-iky - Dk^2 t} \int_0^t e^{Dk^2 u} \delta(u - \tau) \, du \\ &= \begin{cases} 0 & \text{if } t < \tau \\ e^{-iky - Dk^2(t - \tau)} & \text{if } t > \tau \end{cases} \\ &= \Theta(t - \tau) e^{-iky - Dk^2(t - \tau)} \end{aligned}$$

where $\Theta(t - \tau)$ is the step function.

QUESTION 8

We seek a Green's function such that

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) G(x, t; y, \tau) = V \delta(t - \tau) \delta(x - y)$$

Take the Fourier transform wrt x , obtaining

$$\frac{\partial^2}{\partial t^2} \tilde{G}(k, t; y, \tau) + k^2 c^2 \tilde{G}(k, t; y, \tau) = e^{-iky} V \delta(t - \tau)$$

The solution is

$$\tilde{G}(k, t; y, \tau) = \Theta(t - \tau) V e^{-iky} \frac{\sin(kc(t - \tau))}{ck}$$

To recover the Green's function itself, we must compute the inverse Fourier transform

$$G(x, t; y, \tau) = \frac{\Theta(t - \tau)}{2\pi} \int_{-\infty}^{\infty} V e^{-iky} \frac{\sin(kc(t - \tau))}{ck} dk$$

Have that

$$\mathcal{F}^{-1}[\sin ka] = \frac{i}{2} (\delta(x + a) - \delta(x - a))$$

Alright, I don't know what I'm doing anymore.

QUESTION 9

QUESTION 10

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and suppose $\psi : \Omega \rightarrow \mathbb{R}$ solves Laplace's equation $\nabla^2 \psi = 0$ inside Ω , subject to

$$\psi(x, 0) = f(x) \quad \text{and} \quad \lim_{|\mathbf{x}| \rightarrow \infty} \psi = 0$$

- (i) We must construct a Green's function that vanishes on $\delta\Omega$. As well as vanishing on the x -axis, we also require G vanishes as $|\mathbf{x}| \rightarrow \infty$. We'll set $\mathbf{x} = (x, y)$ and $\mathbf{y} := \mathbf{x}_0^+ = (x_0, y_0)$ in terms of Cartesian coordinates, with $y_0 > 0$. We know that the free-space Green's function

$$G_2(\mathbf{x}, \mathbf{x}_0^+) = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0^+|$$

satisfies all conditions except that

$$G_2(\mathbf{x}, \mathbf{x}_0^+)|_{y=0} = \frac{1}{2\pi} \log |(x - x_0)^2 + y_0^2|^{1/2} \neq 0 \quad (*)$$

We need to cancel the nonzero boundary value of G_2 by adding on some function.

Let \mathbf{x}_0^- be the point $(x_0, -y_0)$. The location $\mathbf{x}_0^- \notin \Omega$, so the Green's function $G_2(\mathbf{x}, \mathbf{x}_0^-)$ is regular everywhere within Ω , and so obeys Laplace's equation everywhere in the upper half-space. Also,

$$G_2(\mathbf{x}, \mathbf{x}_0^-)|_{y=0} = G_2(\mathbf{x}, \mathbf{x}_0^+)|_{y=0}$$

Thus the Dirichlet Green's function we seek is

$$\begin{aligned} G(\mathbf{x}; \mathbf{x}_0) &= G_2(\mathbf{x}; \mathbf{x}_0^+) - G_2(\mathbf{x}; \mathbf{x}_0^-) \\ &= \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0^+| - \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0^-| \end{aligned}$$

Note this satisfies both conditions,

- (ii) Know that given a Green's function, the solution to our Laplace's equation problem is

$$\psi(\mathbf{y}) = \int_{\partial\Omega} f(\mathbf{x}) \mathbf{n} \cdot \nabla G(\mathbf{x}; \mathbf{y}) \, dS$$

Note that there is no contribution from the far field since $\psi \rightarrow 0$ asymptotically by our boundary condition. The *outward* normal at $y = 0$ points in the negative y -direction, so the only contribution to the formula above comes from the lower boundary:

$$\begin{aligned}
(\mathbf{n} \cdot \nabla G)|_{y=0} &= -\frac{\partial G}{\partial y}\bigg|_{y=0} \\
&= \frac{1}{2\pi} \left(\frac{y+y_0}{|\mathbf{x}-\mathbf{x}_0^-|^2} - \frac{y-y_0}{|\mathbf{x}-\mathbf{x}_0^-|^2} \right) \bigg|_{y=0} \\
&= \frac{y_0}{\pi} [(x-x_0)^2 + y_0^2]^{-1}
\end{aligned}$$

At last the solution is given by

$$\psi(x_0, y_0) = \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{(x-x_0)^2 + y^2} dx$$

(iii) Next, take the Fourier transform of $\nabla^2(x, y) = 0$ with respect to x ;

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) e^{-ikx} dx = 0 \\
\Rightarrow 0 &= \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial x^2} e^{-ikx} dx + \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y^2} e^{-ikx} dx \\
&= -k^2 \tilde{\phi}(k) + \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} \phi e^{-ikx} dx \\
&= -k^2 \tilde{\phi}(k) + \frac{\partial^2 \tilde{\phi}(k)}{\partial y^2}
\end{aligned}$$

Solving this second order ODE in k gives

$$\tilde{\phi}(k, y) = A(k)e^{ky} + B(k)e^{-ky}$$

We have the boundary conditions $\tilde{\phi}(k, 0) = \tilde{f}(k)$, and $\tilde{\phi} \rightarrow 0$ as $y \rightarrow \infty$. Thus

$$\tilde{\phi}(k, y) = \tilde{f}(k)e^{-ky}$$

respectively. Solving gives $A = -e^{-k}$, $B = e^k$. Thus, we take the inverse FT to finish.

$$\phi(x, y) = -\frac{1}{2\pi} \int_{-1}^1 e^{k(y-1)} - e^{-k(y-1)} dk$$

QUESTION 11

QUESTION 12

Consider $u_1 = 1/(R^2 + z^2)^{1/2} = (x^2 + y^2 + z^2)^{-1/2}$. This satisfies the boundary conditions. Also

$$\begin{aligned}\frac{\partial^2}{\partial x^2}(u_1) &= \frac{\partial}{\partial x} \left[-x(x^2 + y^2 + z^2)^{-3/2} \right] \\ &= 3x^2(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}\end{aligned}$$

Hence

$$\begin{aligned}\nabla^2 u_1 &= (3x^2 + 3y^2 + 3z^2)(x^2 + y^2 + z^2)^{-5/2} - 3(x^2 + y^2 + z^2)^{-3/2} \\ &= 0\end{aligned}$$

So u_1 is a solution for $r = 0$.

Now suppose

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial u_2}{\partial R} \right) + \frac{\partial^2 Z}{\partial z^2} = 0$$

Let $u_2 = \rho(R)Z(z)$. Separating variables shows that

$$\left(\frac{\rho''}{\rho} + \frac{1}{R} \frac{\rho'}{\rho} \right) + \frac{Z''}{Z} = 0$$

$\frac{Z''}{Z}$ cannot vary as R held fixed, so it must be equal to some constant, λ^2 . Solving gives $Z(z) = Ce^{-\lambda|z|}$ (want solution to decay as $z \rightarrow \infty$).

The radial equation, upon multiplying by R^2 becomes

$$R^2 \rho'' + R \rho' + \lambda^2 R^2 \rho = 0$$

We cancel the factor of λ^2 with the substitution $x = \lambda R$, and obtain

$$x^2 \rho'' + x \rho' + x^2 \rho = 0$$

which we recognise as Bessel's equation of order zero, having solutions $J_0(x), Y_0(x)$ which are regular and singular at the origin respectively. We want our solutions to be regular at the origin, hence we take $\rho(R) = \text{const. } J_0(R)$

Hence the solution is $u_\lambda = f(\lambda)e^{-\lambda|z|}J_0(\lambda R)$. There where no restrictions on $\lambda \in Z^+$, so this must hold for all possible values of λ , so (can we say?) our solution is given by

$$u = \int_0^\infty f(\lambda)e^{-\lambda|z|}J_0(\lambda R)d\lambda$$

Now, setting $u_1 = u_2$ on $R = 0$, and noting $J_0(0) = 1$ gives

$$\frac{1}{|z|} = \int_0^\infty f(\lambda)e^{-\lambda|z|}d\lambda$$

Multiplying by $|z|$ and integrating by parts,

$$\begin{aligned}1 &= \int_0^\infty f(\lambda)|z|e^{-\lambda|z|}d\lambda \\&= \left[-f(\lambda)e^{-\lambda|z|}\right]_0^\infty - \int_0^\infty f'(\lambda)e^{-\lambda|z|}d\lambda \\&= f(0) - \int_0^\infty f'(\lambda)e^{-\lambda|z|}d\lambda\end{aligned}$$

ie.

$$\int_0^\infty f'(\lambda)e^{-\lambda|z|}d\lambda = f(0) - 1$$

Not sure about last bit.