Part IB — Fluid Dynamics Example Sheet 1

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QUESTION 1

QUESTION 2

QUESTION 3

QUESTION 4

QUESTION 5

QUESTION 6

(i) At $t = 1$, the velocity makes an angle of $45°$ angle with the horizontal , and the streamlines are slanted.

(ii) For a particle released at $(1, 1)$ we get

$$\dot{x}(t) = \frac{1}{1+t}, \qquad \dot{y}(t) = 1$$

Hence we get

$$x = \log(1+t) + 1, \qquad y = t + 1$$

Eliminating t, we get that the path is given by

$$x = \log y + 1 \implies y = e^{x-1}$$

QUESTION 7

Consider the fluid flow $\mathbf{u} = (1/(1+t), 1, 0)$ for $t > 0$. Supposing our fluid is incompressible, $\nabla \cdot \mathbf{u} = 0$ and there exists some vector potential $\mathbf{A}$ such that $\mathbf{u} = \nabla \times \mathbf{A}$. As $\mathbf{u}$ is two dimensional, we know $\mathbf{A}$ is of the from

$$\mathbf{A} = (0, 0, \psi(x, y, t))$$

And taking the curl of this,

$$\mathbf{u} = (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0)$$

This ψ is our streamfunction.

We define streamlines to be contours of our stream function. These contours have normal $\mathbf{n} = (\psi_x, \psi_y, 0)$, this normal is obviously perpendicular to the flow $\mathbf{u}$ ($\mathbf{u} \cdot \mathbf{n} = 0$). ie. the flow is tangent to the contours of ψ.

QUESTION 8

$$\nabla \cdot \mathbf{u} = 0 \iff \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

By the quotient rule,

$$\frac{\partial u}{\partial x} = \frac{-(y-b)[2(x-a)]}{((x-a)+(y-b))^2}$$

$$\frac{\partial v}{\partial y} = \frac{-(a-x)[2(y-b)]}{((x-a)^2+(y-b)^2)}$$

So $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\nabla \cdot \mathbf{u} = 0$, our fluid is indeed incompressible.

QUESTION 9

$$\nabla \cdot \mathbf{u} = 0 \iff \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) = 0$$

$$\begin{aligned}\frac{\partial}{\partial r}(ru_r) &= \frac{\partial}{\partial r}\left[U\left(r - \frac{a^2}{r}\right)\cos\theta\right] \\ &= U\left(1 + \frac{a^2}{r^2}\right)\cos\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial \theta}(u_\theta) &= \frac{\partial}{\partial \theta}\left[-U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right] \\ &= U\left(1 + \frac{a^2}{r^2}\right)\cos\theta\end{aligned}$$

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) \implies \nabla \cdot \mathbf{u} = 0$$

QUESTION 10

QUESTION 11

QUESTION 12