Part IB — Fluid Dynamics Example Sheet 2

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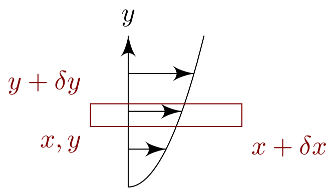
QUESTION 1

To derive the equations of motion, we can consider a small box in the fluid.



We know that this block of fluid accelerates in the x direction, so the total forces here should equal $\rho\frac{\partial u}{\partial t}\delta x\delta y$; and in the y direction, the total forces of the surrounding environment on the box should vanish.

We first consider the x direction. There are normal stresses at the sides, and tangential stresses at the top and bottom, plus body forces per unit volume. The sum of forces in the x-direction (per unit transverse width) gives

$$p(x)\delta y - p(x+\delta x)\delta y + \tau_s(y+\delta y)\delta x + \tau_s(y)\delta x + f_x\delta x\delta y = \rho\frac{\partial u}{\partial t}\delta x\delta y$$

By the definition of τ_s, we can write

$$\tau_s(y+\delta y) = \mu\frac{\partial u}{\partial y}(y+\delta y), \quad \tau_s(y) = -\mu\frac{\partial u}{\partial y}(y),$$

where the different signs come from the different normals (for a normal pointing downwards, $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial(-y)}$).

Dividing by $\delta x\delta y$, we get

$$\frac{1}{\delta x}(p(x) - p(x+\delta x)) + \mu\frac{1}{\delta y}\left(\frac{\partial u}{\partial y}(y+\delta y) - \frac{\partial u}{\partial y}(y)\right) + f_x = \rho\frac{\partial u}{\partial t}$$

Taking the limit as $\delta x, \delta y \to 0$, we end up with the equation of motion

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + f_x = \rho\frac{\partial u}{\partial t}.$$

Performing similar calculations in the y direction, we obtain

$$-\frac{\partial p}{\partial y} + f_y = 0.$$

QUESTION 2

Using the derived equations in Question 1 (switch y and x), we assume that this is a steady flow, but we will also include gravity. First, in the x direction, we have

$$\frac{\partial p}{\partial x} = 0$$

Using the fact that $p = p_0$ at the boundary, we get simply

$$p = p_0$$

In particular, p is independent of y. In the y component, we get

$$\mu \frac{\partial^2 u}{\partial x^2} = -g\rho$$

The no slip condition gives $u = 0$ when $x = 0$. The other condition is that there is no stress at $x = h$. So we get $\frac{\partial u}{\partial x} = 0$ when $x = h$.

The solution is thus

$$u = \frac{g\rho}{2\mu} x(2h - x).$$

The *volume flux* is the volume of fluid traversing a cross-section per unit time. This is given by

$$q = \int_0^h u(x)\, \mathrm{d}x$$

per unit transverse width.

We calculate this as

$$\begin{aligned} q &= \int_0^h \frac{g\rho}{2\mu} x(2h - x) \mathrm{d}x \\ &= \frac{g\rho}{3\mu} h^3 \end{aligned}$$

QUESTION 3

QUESTION 4

QUESTION 5

QUESTION 6

QUESTION 7

We calculate the vorticity as

$$\begin{aligned}\omega &= \nabla \times \mathbf{u} \\ &= 3rf(t)\hat{\mathbf{z}}\end{aligned}$$

Will try next bit before supo.

QUESTION 8

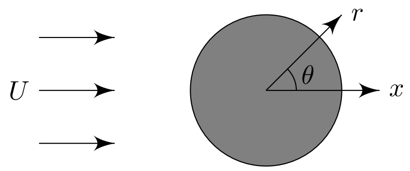
Have $\mathbf{\Omega}$ a constant, so $\nabla\mathbf{\Omega} = \nabla \cdot \mathbf{\Omega} = 0$, so that

$$\begin{aligned}
\omega &= \nabla \times (\mathbf{\Omega} \times \mathbf{x}) \\
&= \mathbf{\Omega}(\nabla \cdot \mathbf{x}) + \mathbf{x} \cdot \nabla\mathbf{\Omega} - \mathbf{\Omega} \cdot \nabla\mathbf{x} - \mathbf{x}(\nabla \cdot \mathbf{\Omega}) \\
&= 3\mathbf{\Omega} - \mathbf{\Omega} = 2\mathbf{\Omega}
\end{aligned}$$

Not sure about this question

QUESTION 9

We can change our frame of reference, and suppose the sphere is stationary and the fluid is moving past it at U. Solving this, we then translate the velocities back by U to get the solution.



We suppose the upstream flow is $\mathbf{u} = U\hat{\mathbf{x}}$. So

$$\phi = Ux = Ur\cos\theta.$$

So we need to solve

$$\begin{aligned} \nabla^2\phi &= 0 && r > a \\ \phi &\to Ur\cos\theta && r \to \infty \\ \frac{\partial\phi}{\partial r} &= 0 && r = a. \end{aligned}$$

The last condition is there to ensure no fluid flows into the sphere, i.e. $\mathbf{u}\cdot\mathbf{n} = 0$, for $\mathbf{n}$ the outward normal.

We can use Legendre polynomials to write the solution as

$$\phi = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})P_n(\cos\theta).$$

We then have

$$\mathbf{u} = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, 0\right).$$

Since $P_1(\cos\theta) = \cos\theta$, and the P_n are orthogonal, our boundary conditions at infinity require ϕ to be of the form

$$\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta.$$

We now just apply the two boundary conditions. The condition that $\phi \to Ur\cos\theta$ tells us $A = u$, and the other condition tells us

$$A - \frac{2B}{a^3} = 0.$$

So we get

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta.$$

We can compute the velocity to be

$$\begin{aligned} u_r &= \frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta \\ u_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta. \end{aligned}$$

Finally we subtract $U\hat{\mathbf{x}} = U(\sin\theta, \cos\theta, 0)$

QUESTION 10

The general result is that given a point source of strength q placed at the origin, in spherical polars

$$\nabla^2\phi = q\delta(r)$$

We get

$$\phi = \frac{q}{2\pi}\log r$$

Thus, for a point source of strength m located at the origin we have

$$\phi = \frac{m}{2\pi}\log\sqrt{x^2+y^2}$$

Not sure about this question