Part IA — Variational Principles Example Sheet 2

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Michaelmas 2017

QUESTION 1

$$
\begin{aligned}
F[x+\delta x]-F[x] &= \int_{t_1}^{t_2} f(x+\delta x,\dot{x}+\delta\dot{x},\ddot{x}+\delta\ddot{x},t)\ \mathrm{d}t - \int_{t_1}^{t_2} f(t,x,\dot{x},\ddot{x})\ \mathrm{d}t \\
&= \int_{t_1}^{t_2}\left\{\delta x\frac{\partial f}{\partial x}+(\delta\dot{x})\frac{\partial f}{\partial \dot{x}}+(\delta\ddot{x})\frac{\partial f}{\partial \ddot{x}}\right\}\ \mathrm{d}t+O(t^2)
\end{aligned}
$$

Discarding the (small)terms of $O(t^2)$, we call the first order variation $\delta F[x]$ and integrating by parts (twice), we have

$$
\begin{aligned}
\delta F[x] &= \int_{t_1}^{t_2}\left\{\delta x\left[\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]-(\delta\dot{x})\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \ddot{x}}\right)\right\}\ \mathrm{d}t+\left[\delta x\frac{\partial f}{\partial \dot{x}}+(\delta\dot{x})\frac{\partial f}{\partial \ddot{x}}\right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2}\left\{\delta x\left[\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right)\right]\right\}\ \mathrm{d}t+\left[\delta x\left\{\frac{\partial f}{\partial \dot{x}}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \ddot{x}}\right)\right\}+(\delta\dot{x})\frac{\partial f}{\partial \ddot{x}}\right]_{t_1}^{t_2}
\end{aligned}
$$

We have fixed end boundary conditions, so $\delta x(t_1)=\delta x(t_2)=0$ and also $\delta\dot{x}(t_1)=\delta\dot{x}(t_2)=0$. Thus the boundary term is zero and we can write $\delta F[x]$ in the form

$$
\delta F[x]=\int_{t_1}^{t_2}\left\{\delta x(t)\frac{\delta F[x]}{\delta x(t)}\right\}\ \mathrm{d}t
$$

where the *function derivative* $\frac{\delta F[x]}{\delta x(t)}$ is defined as

$$
\frac{\delta F[x]}{\delta x(t)}:=\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right)
$$

The functional F is stationary when its functional derivative is zero (assuming that the b.c.s are such that this derivative is defined) and the condition for this to be true is the following Euler-Lagrange equation:

$$
\frac{\partial f}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right)=0,\qquad t_1<t<t_2
$$

Given the functional

$$
L[x]=\int_1^2 t^4|\ddot{x}(t)|^2\ \mathrm{d}t
$$

In this case, $f=t^4|\ddot{x}(t)|^2$, so $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial \dot{x}}=0$ and the EL equation can be immediately twice integrated to give

$$
\frac{\partial}{\partial \ddot{x}}\left[t^4|\ddot{x}(t)|^2\right]=At+B
$$

for some constants A and B.

QUESTION 2

Our aim is to maximise $A[x, y]$ subject to the constraint $P[x, y] = L$, where L is the fixed length and $P[x, y] = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, \mathrm{d}\theta$

Using a Lagrange multiplier λ to impose this, we seek to maximize the functional

$$\begin{aligned}\phi_\lambda[x, y] &= A[x, y] - \lambda(P[y] - L)\\ &= \int_0^{2\pi} \underbrace{\frac{1}{2}(xy' - yx') - \lambda\sqrt{(x')^2 + (y')^2}}_{=f_\lambda(\mathbf{x},\mathbf{x}')}\ \mathrm{d}\theta + \lambda L\end{aligned}$$

$f_\lambda(\mathbf{x}, \mathbf{x}')$ has no explicit θ dependence, setting $\frac{\mathrm{d}\phi_\lambda[x,y]}{\mathrm{d}y} = 0$ the E-L equations imply (under the assumption of appropriate boundary conditions)

$$f_\lambda(y, y') - y'\frac{\partial f_\lambda}{\partial y'} = \text{ constant}$$

$$\begin{aligned}&\Rightarrow f - y'\left(\frac{1}{2}x - \frac{\lambda y'}{\sqrt{(x')^2 + (y')^2}}\right) = \text{constant}\\ &\Rightarrow -\frac{1}{2}yx' - \lambda\frac{((x')^2 + (y')^2)}{\sqrt{(x')^2 + (y')^2}} + \lambda\frac{(y')^2}{\sqrt{(x')^2 + (y')^2}} = \text{constant}\\ &\Rightarrow -\frac{1}{2}yx' - \lambda\frac{(x')^2}{\sqrt{(x')^2 + (y')^2}} = \text{constant}\end{aligned}$$

Similarly considering $f_\lambda(x, x')$ we have

$$-\frac{1}{2}xy' - \lambda\frac{(y')^2}{\sqrt{(x')^2 + (y')^2}} = \text{constant}$$

Adding,

$$-\frac{1}{2}(xy' - yx') - \lambda\sqrt{(x')^2 + (y')^2} = \text{ constant}$$

QUESTION 3

Using Lagrange multiplier λ, wish to minimize

$$I[\psi]_\lambda = \int_{-\infty}^{\infty} \underbrace{(\psi')^2 + (x^2 - \lambda\psi^2)}_{f_\lambda(\psi,\psi';x)} \, \mathrm{d}x + \lambda$$

Euler-Lagrange equations imply:

QUESTION 4

QUESTION 5

QUESTION 6

QUESTION 7

QUESTION 8

QUESTION 9

QUESTION 10

QUESTION 11