Part IA — Markov Chains Example Sheet 1

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QUESTION 1

$X = (X_n)$ is a Markov Chain, and so satisfies the Markov Property

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \cdots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

Given that $Z_k = X_{m+k}$, with $k \geq 0$, want to show that Z_k is a Markov Chain, ie.

$$\mathbb{P}(Z_{n+1} \mid Z_0, \cdots, Z_n) = \mathbb{P}(Z_{n+1} \mid Z_n) \qquad (*)$$

Substituting in our definition of Z_k,

$$\begin{aligned}\mathbb{P}(Z_{n+1} \mid Z_0, \cdots, Z_n) &= \mathbb{P}(X_{m+n+1}|X_m, \cdots, X_{m+n}) \\ &= \mathbb{P}(X_{m+n+1}|X_{m+n}) \qquad \text{as } X \text{ is Markov} \\ &= \mathbb{P}(Z_{n+1} \mid Z_n)\end{aligned}$$

Hence Z_k satisfies the Markov property, and is thus a Markov Chain.
$Z_0 = X_m = i$, so Z_k has starting state i.

QUESTION 2

Let $X_1, \cdots, X_n$ be independent random variables, so $\mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1})$. But then

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \cdots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1})$$

also by independence, so Markov property is satisfied.

Homogeneous if $\mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n)$ does not depend on the value of n, so must have X_n identically distributed.

QUESTION 3

As X_n is the *maximum* reading obtained after n throws, this depends only on the last maximum, so X_n is a Markov chain. In particular, the state transition matrix is

$$\begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 6/6 \end{pmatrix}$$

Now note that

$$p_{i,j}(n) = \begin{cases} 0 & \text{if } j < i \\ \left(\frac{1}{6}\right)^n & \text{if } j = i \\ \left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n & \text{if} j > i \end{cases}$$

QUESTION 4

We have

$$S_{n+1} = \begin{cases} S_n + 1 & \text{with probability } p \\ S_n - 1 & \text{with probability } q \end{cases}$$

and $S_0 = 0$.
To show that X_n is a Markov chain, consider the probability

$$\mathbb{P}(|S_{n+1}| = i + 1 \mid |S_n| = i, H)$$

for some past event H. Conditioning, this is equal to

$$\begin{aligned} &= \underbrace{\mathbb{P}(|S_{n+1}| = i + 1 \mid S_n = i)}_{p} \times \underbrace{\mathbb{P}(S_n = i \mid ; |S_n| = i, H)}_{\pi_1} \\ &\quad + \underbrace{\mathbb{P}(|S_{n+1}| = i + 1 \mid S_n = -i)}_{q} \times \underbrace{\mathbb{P}(S_n = -i \mid ; |S_n| = i, H)}_{\pi_2} \end{aligned}$$

Note that

$$\begin{aligned} \frac{\pi_1}{\pi_2} &= \frac{p^{n/2+i/2} q^{n/2-i/2}}{q^{n/2+i/2} p^{n/2-i/2}} \\ &= \frac{p^i}{q^i} \end{aligned}$$

Hence $\pi_1 = \frac{p^i}{p^i + q^i}$
Define $M_k := \max\{S_k \mid 0 \le k \le n\}$, and $Y_n = M_n - S_n$

S_n M_n

with the distance between S_n and M_n being i, we have:

$$p_{ij} = \begin{cases} i - 1 & \text{with probability } p \\ i + 1 & \text{with probability } q \end{cases}$$

$$p_{0j} = \begin{cases} 0 & \text{with probability } p \\ 1 & \text{with probability } q \end{cases}$$

QUESTION 5

X is a Markov chain, thus

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_0 = i_0, \cdots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

And so

$$\mathbb{P}(Y_{r+1} \mid Y_0, \cdots, Y_r) = \mathbb{P}(X_{n_{r+1}} \mid X_{n_0}, \cdots, X_{n_r})$$

If $n_r = 2r$ and X is a simple random walk on Z st.

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \\ X_n - 1 & \text{with probability } q \end{cases}$$

the allowed states are $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$. Of course, the probability of finding oneself in an odd state is always zero (we are always taking an even number of states).

It is difficult to visualise the state transition matrix as the state space is infinite, but a general row has the form

$$\begin{pmatrix} \cdots & 0 & q^2 & 0 & 2pq & 0 & p^2 & \cdots \end{pmatrix}$$

With probability $2pq$ of returning to the current state, p^2 moving two to the right and q^2 moving two to the left. (Thus rows summing to $(p+q)^2 = 1$)

QUESTION 6

I think the answer should be no. X and Y are both Markov chains, so the value of $X_{n+1} + Y_{n+1}$ only depends on X_n, and Y_n. But we only have the value of $X_n + Y_n$ (loss of information).

For an explicit counterexample, try to smuggle information about X_1 into Y, so that $X_n + Y_n$ depends on X_1 (contradicting the Markov property) but X_n and Y_n are both Markov.

Idea: Let X be a random walk on the integers, so $X_1 = \pm 1$ with some probabilities. Define Y_n as a symmetric 'boosted' random walk:

$$Y_n = \begin{cases} Y_{n-1} + 1 + 3X_1 & \text{with probability } 1/2 \\ Y_{n-1} - 1 + 3X_1 & \text{with probability } 1/2 \end{cases}$$

Then $X_n + Y_n$ depends on X_1, not just X_{n-1}, Y_{n-1}

Note that we can also make use of M_n as defined in question 4.

$$\mathbb{P}(M_3 = 2 \mid M_2 = 1, M_1 = 1) = 0$$
$$\mathbb{P}(M_3 = 2 \mid M_2 = 1, M_1 = 0) = \frac{1}{2}$$

So M_n not M.C.

QUESTION 7

(i) Let the probability of the flea being on the original vertex after n hops be p_n. Then

$$p_n = \begin{cases} 0 & \text{if flea on first vertex after } n-1 \text{ hops} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Hence we set up the recurrence, with $p_0 = 1$

$$\begin{aligned} p_n &= 0 \cdot p_{n-1} + \frac{1}{2}(1 - p_{n-1}) \\ &= \frac{1}{2}(1 - p_{n-1}) \end{aligned}$$

Solving this difference equation gives general solution $p_n = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$.

(ii) Let p_n be defined as before. This can be though of as a Markov chain on the state space of triangle vertices $\{1, 2, 3\}$, with transition matrix

$$P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Given $p_{1,1}(0) = 1$ (the flea starts on vertex 1) we want to find $p_{1,1}(n)$ The eigenvalues κ of P are the roots of the equation $\det(P - \kappa I)$, with solutions

$$\kappa_1 = 1, \quad \kappa_2, \kappa_3 = -\frac{1}{2} \pm \frac{i}{2\sqrt{3}}$$

Therefore

$$P = U^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix} U$$

for some invertible matrix U. It follows that

$$P^n = U^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa_2^n & 0 \\ 0 & 0 & \kappa_3^n \end{pmatrix} U$$

Now note that $\kappa_2, \kappa_3 = \frac{1}{\sqrt{3}} e^{\pm i\pi/6}$, so

$$\begin{aligned} p_{1,1}(n) &= A + Be^{i\pi n/6} + Ce^{-i\pi n/6} \\ &= A + B'\left(3^{-1/2}\right)^n \cos(\pi n/6) + C'\left(3^{-1/2}\right)^n \sin(\pi n/6) \end{aligned}$$

Using the b.c.s,
$p_{1,1}(0) = 1, p_{1,1}(1) = 0, p_{1,1}(2) = 4/9$ and hence

$$p_{1,1}(n) = \frac{1}{3} + \frac{2}{3}\left(3^{-1/2}\right)^n \cos(\pi n/6)$$

QUESTION 8

Note that

$$p_{66}(n-1) = \frac{1}{5}(1 - p_{66}(n-2))$$

Hence,

$$p = \frac{1}{6} + \frac{5}{6}\left(-\frac{1}{2}\right)^{n-1}$$

For the next part, let Y_n denote the die now, and this is related to the previous die by

$$Y_n - Y_{n-1} = X_n - X_{n-1} \pmod 6$$

Thus

$$Y_n = X_n + (n-1) \mod 6$$

Thus $n-1 = 0 \mod 6 \Rightarrow p_{66}(n-1) = p$, and $n-1 \neq 0 \mod 6 \Rightarrow \frac{1}{5}(1-p)$

QUESTION 9

(a) For $p = 1/16$, the characteristic equation $\det(P - \kappa\iota) = 0$ gives roots $\kappa_1 = 1$, $\kappa_2 = -1/4$, $\kappa_3 = -1/12$.

Thus

$$P^n = U^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1/4)^n & 0 \\ 0 & 0 & (-1/12)^n \end{pmatrix} U$$

for some invertible matrix U, and so

$$p_{1,1}(n) = A + B\left(-\frac{1}{4}\right)^n + C\left(-\frac{1}{12}\right)^n$$

Using $p_{1,1}(0) = 1$, $p_{1,1}(1) = 0$, $p_{1,1}(2) = 0$

This gives $A = 1/65$, $B = -2/5$ and $C = 18/13$

(b) (Check, this is incorrect) For $p = 1/6$, the characteristic equation $\det(P - \kappa\iota) = 0$ gives roots $\kappa_1 = 1$, $\kappa_2, \kappa_3 = -1/6 \pm i/6 = \frac{1}{\sqrt{18}}e^{\pm i\pi/4}$, so similarly,

$$p_{1,1}(n) = A + \left(18^{-1/2}\right)^n B\cos(\pi n/4) + \left(18^{-1/2}\right)^n C\sin(\pi n/4)$$

yielding $A = 1/13, B = 12/13$, $C = -18/13$

(c)

$$P^n = U^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1/6)^n & 0 \\ 0 & 0 & (-1/6)^n \end{pmatrix} U$$

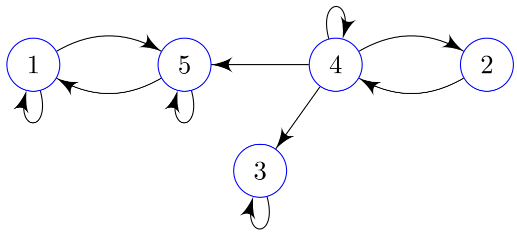
Hence, $p_{1,1}(n) = A + B\left(-\frac{1}{6}\right)^n + Cn\left(-\frac{1}{6}\right)^n$

Check, should get

$$\frac{1}{49} + \left(-\frac{1}{6}\right)^n \left(\frac{48}{49} - \frac{6}{7}n\right)$$

QUESTION 10

Possible transitions of the chain are illustrated below:



The communicating classes are $C_1 = \{1, 5\}, C_2 = \{3\}$ and $C_3 = \{2, 4\}$. The classes C_1 and C_3 are not closed, but C_2 is closed.

QUESTION 11

Suppose S contains no closed communicating classes.

Pick some $i_0 \in S$. If i is a singleton communicating class, then $\{i_0\}$ is closed. So $\exists\, i_1 \in S$ st. $i_0 \to i_1$. Similarly, $\exists\, i_2 \in S$ st. $i_1 \to i_2$. ($i_2 \neq i_0$, or $\{i_0, i_1\}$ is a closed communicating class). Continuing in this fashion, we eventually hit the 'last' element in the state space i_n, as it is finite. Then $S = \{i_0, \cdots, i_n\}$ itself is a closed communicating class.

For a transition matrix with no communicating classes a simple example is:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & & \vdots & & \ddots \end{pmatrix}$$

ie. $i_0 \to i_1 \to i_2 \to \cdots$

QUESTION 12

See Weber's Right Hand Equations
$\alpha = P\alpha$, $k = 1 + Pk$

$$\begin{aligned}
\alpha_2 &= \frac{1}{2} \cdot 0 + \frac{1}{2}\alpha_4 \\
\alpha_4 &= \frac{1}{2} \cdot 0 + \frac{1}{2}\alpha_8 \\
\alpha_6 &= \frac{1}{2}\alpha_6 + \frac{1}{2}1 \\
\alpha_8 &= \frac{1}{2}\alpha_2 + \frac{1}{2}1 \\
\alpha_2 &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\alpha_2\right)\right)\right) \\
&= \frac{1}{8} + \frac{1}{16} + \frac{1}{16}\alpha_2
\end{aligned}$$

Hence

$$\frac{15}{16}\alpha_2 = \frac{3}{16} \qquad \alpha_2 = \frac{1}{5}$$

Could also argue it has to be 1/5 since it's a fair game, and the placement of 2 between 0 and 10.

Similarly,

$$\begin{aligned}
k_2 &= 1 + \frac{1}{2} \cdot 0 + \frac{1}{2}k_4 \\
k_4 &= 1 + \frac{1}{2} \cdot 0 + \frac{1}{2}k_8
\end{aligned}$$

But since $k_6 = k_4$, and $k_8 = k_2$, we have

$$k_2 = 1 + \frac{1}{2}\left(1 + \frac{1}{2}k_2\right)$$

and we recover $k_2 = 2$, as expected.

QUESTION 13

$\alpha_0 = 1$

$$\alpha_i = \frac{(i+1)^2}{i^2+(i+1)^2}\alpha_{i+1} + \frac{i^2}{i^2+(i+1)^2}\alpha_{i-1}$$

This is equivalent to

$$(i+1)^2(\alpha_{i+1} - a_i) = i^2(\alpha_i - \alpha_{i-1})$$

Iterating down,

$$\cdots = 1^2(\alpha_1 - \alpha_0) = \alpha_1 - 1$$

ie.

$$\alpha_{i+1} - \alpha_i = \frac{1}{(i+1)^2}(\alpha_1 - 1)$$

Summing,

$$\alpha_{k+1} - \alpha_0 = \left[\frac{1}{1^2} + \cdots ? \frac{1}{(k+1)^2}\right](\alpha_1 - 1)$$

The result follows, taking $k \to \infty$.

QUESTION 14

Again, expand on the right side,

$$\begin{aligned}\phi(s) &= \mathbb{E}s^{H_0} \\ &= \frac{1}{2}s^1 + \frac{1}{2}s^1 \underbrace{\mathbb{E}s^{H_0'+H_0''}}_{=\phi(s)^2}\end{aligned}$$

where H_0', H_0'' are independent variables.

$$\phi(s) = \frac{1 \pm \sqrt{1-s^2}}{s}$$

where we pick the $-$ from the $\pm$, as terms in the expansion need to be positive.

Thus

$$\phi(s) = \frac{s}{2} + \frac{s^3}{8} + \frac{s^5}{16} + O\left(s^6\right)$$

Next we have the same idea,

$$\phi(s) = \frac{1}{2}s^1 + \frac{1}{2}s^1\phi(s)^3$$

Can use mathematica to solve this. The smallest root of this equation, interestingly, is the golden mean!