Part IB — Quantum Mechanics Example Sheet 1

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QUESTION 1

The first electron has wavelength $\lambda_1 = 3 \times 10^{-7}$ m, and moves at the speed of light, so frequency ν_1 is given by

$$\nu_1 = \frac{c}{\lambda_1} = \frac{3.00 \times 10^8}{3 \times 10^{-7}} = 1 \times 10^{15} \text{ s}^{-1}$$

Similarly $\nu_2 = 0.6 \times 10^{15}$. If W is the minimum energy needed to liberate an electron from Potassium then

$$\begin{aligned} K_1 &= h\nu_1 - W \\ K_2 &= h\nu_2 - W \end{aligned}$$

where K_1, K_2 are the maximum kinetic energy of the liberated electrons. Thus the value of h is given by

$$h = \frac{K_1 - K_2}{\nu_1 - \nu_2} = \frac{1.6 \times (1.60 \times 10^{-19})}{0.4 \times 10^{15}} = 6.4 \times 10^{-34}$$

Thus

$$\begin{aligned} W &= h\nu_1 - K_1 \\ &= 6.4 \times 10^{-19} - 2.1 \times (1.60 \times 10^{-19}) \\ &= 3.04 \times 10^{-19} \text{ J} \\ &= 1.9 \text{ eV} \end{aligned}$$

QUESTION 2

Let the light have energy flux $E = 10^{-10}\text{Jm}^{-2}\text{s}^{-1}$, with the wavelength $\lambda = 5 \times 10^{-7}$. The energy of one photon is given by

$$\begin{aligned} E_p &= \frac{hc}{\lambda} \\ &= \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{5 \times 10^{-7}} \\ &= 3.987 \times 10^{-19} \text{ J} \end{aligned}$$

Take human eye to have area 1 $\text{cm}^2 = 10^{-4}$ m^2, so energy flux E_e entering human eye is $E_e = 10^{-14}\text{Jm}^{-2}\text{s}^{-1}$.

Thus number of photons entering the eye N is given as

$$N = \frac{E_e}{E_p} \approx 2.51 \times 10^5$$

QUESTION 3

Classical equations of motion imply that the total energy E_n for an electron at level n ($n = 1, 2, \cdots$) for the electron must be constant, and is given by

$$E_n = \frac{1}{2}mv_n^2 - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r_n}$$

Resolving radial acceleration gives

$$\frac{mv_n^2}{r_n} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r_n^2} \quad (1)$$

which simplifies are expression for energy levels to

$$E_n = -\frac{1}{2}mv_n^2$$

Next, angular momentum quantisation yields

$$mv_nr_n = n\hbar \quad (2)$$

Rearranging (1) and (2)

$$r_n = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{mv_n^2} \qquad r_n = \frac{n\hbar}{mv_n}$$

Thus setting equal and solving for v_n gives

$$v_n = \frac{1}{n\hbar}\frac{e^2}{4\pi\varepsilon_0} = c\alpha\frac{1}{n}$$

where $\alpha = e^2/4\pi\varepsilon_0\hbar c$ is the fine structure constant.
Substituting this into the expression for energy gives

$$E_n = -\frac{1}{2}mc^2\alpha^2\frac{1}{n^2}$$

(i) It is consistent. $\alpha \approx \frac{1}{137}$, so the highest speed an electron can have is $\frac{1}{137}c$ (when $n = 1$, this decreases for larger n), which is less that 1% of the speed of light.

(ii) Suppose th electron makes a transition between levels n' and n, (with $n' > n$ say), accompanied by emission or a photon of frequency ν. Then

$$h\nu = E_{n'} - E_n = \frac{1}{2}mc^2\alpha^2\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)$$

$E = \frac{hc}{\lambda}$, so smallest wavelength $\Rightarrow$ most amount of energy, which is emitted when the electron falls from 'infinity' to level 1, ie. $(1/n^2 - 1/n'^2) = 1$. ie.

$$
\begin{aligned}
\lambda &= \frac{hc}{E} \\
&= \frac{2hc}{mc^2\alpha^2} \\
&= \frac{4\pi\hbar}{mc\alpha^2} \\
&= \frac{e^2}{\varepsilon_0 c}\frac{4\pi\varepsilon_0\hbar c}{e^2}\frac{1}{mc\alpha^2} \\
&= \frac{e^2}{\varepsilon_0 mc^2\alpha}
\end{aligned}
$$

Bohr radius is given by:

$$r_1 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$$

QUESTION 4

Bohr radius r_1 and corresponding $(n = 1)$ radius r_1' of muon are given by

$$r_1 = \frac{\hbar}{m_e c\alpha} \qquad r_1' = \frac{\hbar}{m_m c\alpha}$$

where m_e is the mass of the electron, and $m_m = 207m_e$ is the mass of the muon.

So the radius the $n = 1$ state of muonic Hydrogen is 207 times smaller than normal Hydrogen.

QUESTION 5

Given $\psi_0(x) = C_0 e^{-x^2/2\alpha}$, we calculate

$$\psi_0'(x) = -\frac{x}{\alpha}\psi_0(x) \text{ and } \psi_0''(x) = -\frac{1}{\alpha}\psi_0(x) + \frac{x^2}{\alpha^2}\psi_0(x)$$

Substituting into the time-indep SE gives

$$-\frac{\hbar^2}{2m}\left(-\frac{1}{\alpha} + \frac{x^2}{\alpha^2}\right)\psi_0 + \frac{1}{2}Kx^2\psi_0 = E_0\psi_0$$

Comparing constants and x^2 coefficients respectively

$$\frac{\hbar^2}{2m\alpha} = E_0 \quad \text{and} \quad \frac{\hbar^2}{2m}\frac{1}{\alpha^2} = \frac{1}{2}K$$

Thus $\alpha = \sqrt{\frac{\hbar^2}{Km}}$ and hence energy eigenvalue $E_0 = \frac{\hbar}{2}\sqrt{\frac{K}{m}}$.
Similarly, given $\psi_1(x) = C_1 x e^{-x^2/2\alpha} = x\phi(x)$, where $\phi(x) = C_1 e^{-x^2/2\alpha}$,

$$\psi_1'(x) = \phi(x) - \frac{x^2}{\alpha}\phi(x) \quad \text{and} \quad \psi_1''(x) = \frac{-x}{\alpha}\phi(x) - \frac{2x}{\alpha}\phi(x) + \frac{x^3}{\alpha^2}\phi(x)$$

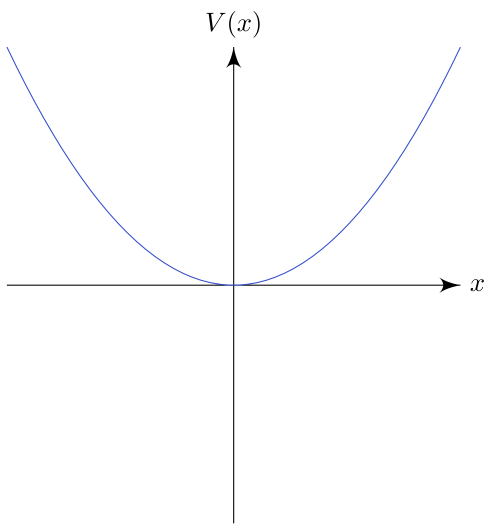
Substituting into time-indep SE yields,

$$-\frac{\hbar^2}{2m}\left(-\frac{3x}{\alpha} + \frac{x^3}{\alpha^2}\right)\phi(x) + \frac{1}{2}Kx^2 x\phi(x) = E_1 x\phi(x)$$

Comparing x and x^3 coefficients respectively,

$$\frac{3\hbar^2}{2m\alpha} = E_1 \quad \text{and} \quad \frac{\hbar^2}{2m}\frac{1}{\alpha^2} = \frac{1}{2}K$$

Hence as before $\alpha = \sqrt{\frac{\hbar^2}{Km}}$ and $E_1 = 3E_0 = \frac{3\hbar}{2}\sqrt{\frac{K}{m}}$



QUESTION 6

Wavefunction $\Psi(x,t)$ under one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}Kx^2$ has time-dependent Schrödinger equation:

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{2}Kx^2\Psi$$

For separable $\Psi(x,t) = \psi(x)f(t)$, have solutions of type

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar} \quad \text{with} \quad H\psi = E\psi$$

(i) $\Psi(x,0) = \psi_0(x) \Rightarrow \psi(x) = \psi_0(x)$ and $E = \frac{\hbar}{2}\sqrt{\frac{K}{m}}$ from question 5, so we have

$$\Psi(x,t) = C_0 e^{-x^2/2\alpha}\exp\left(-\frac{i}{2}\sqrt{\frac{K}{m}}t\right)$$

For some normalisation constant C_0

(ii) Similarly $\psi(x) = \psi_1(x)$ and again using results for E from question 5,

$$\Psi(x,t) = C_1 x e^{-x^2/2\alpha}\exp\left(-\frac{3i}{2}\sqrt{\frac{K}{m}}t\right)$$

For some normalisation constant C_1

(iii) For $\Psi(x,0) = \frac{1}{2}(\sqrt{3}\psi_0(x) - i\psi_1(x)) = \psi(x)$, and solving $H\psi = E\psi$ for E, ψ_0 part only has constant and x^2 terms, whereas ψ_1 only contains x and x^3 terms. Thus from parts (i) and (ii) we have

$$\begin{aligned} E &= \frac{1}{2}(\sqrt{3}E_0 - iE_1) \\ &= \frac{\sqrt{3}-3i}{2}\left(\frac{\hbar}{2}\sqrt{\frac{K}{m}}\right) \end{aligned}$$

Thus

$$\Psi(x,t) = \frac{1}{2}(\sqrt{3}\psi_0(x) - i\psi_1(x))\exp\left(-\frac{3-i\sqrt{3}}{4}\sqrt{\frac{K}{m}}t\right)$$

QUESTION 7

Note that $\Psi(x,t) = C\gamma(t)^{-1/2}\exp(-x^2/2\gamma(t))$ is not separable, $V = 0$.
t -dep SE

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}Kx^2\Psi$$

Calculating the partial derivatives

$$\begin{aligned}\frac{\partial\Psi}{\partial t} &= -\frac{1}{2}\gamma' C\gamma^{-3/2}\exp(-x^2/2\gamma) + C\gamma^{-1/2}\left(\frac{x^2}{2\gamma^2}\gamma'\exp(-x^2/2\gamma)\right)\\ &= \left(-\frac{1}{2}\frac{\gamma'}{\gamma} + \gamma'\frac{x^2}{2\gamma^2}\right)C\gamma^{-1/2}\exp(-x^2/2\gamma)\\ &= \left(-\frac{1}{2}\frac{\gamma'}{\gamma} + \gamma'\frac{x^2}{2\gamma^2}\right)\Psi\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\Psi}{\partial x^2} &= \frac{\partial}{\partial x}\left[-\frac{x}{\gamma}\Psi\right]\\ &= -\frac{1}{\gamma}\Psi + \frac{x^2}{\gamma^2}\Psi\end{aligned}$$

So t-dep SE becomes

$$i\hbar\left(-\frac{1}{2}\frac{\gamma'}{\gamma} + \gamma'\frac{x^2}{2\gamma^2}\right)\Psi = -\frac{\hbar^2}{2m}\left(-\frac{1}{\gamma} + \frac{x^2}{\gamma^2}\right)\Psi$$

Comparing constant and x^2 coefficients repectively,

$$-i\hbar\frac{1}{2}\frac{\gamma'}{\gamma} = \frac{\hbar^2}{2m}\frac{1}{\gamma} \quad \text{and} \quad i\hbar\frac{\gamma'}{2\gamma^2} = -\frac{\hbar^2}{2m}\frac{1}{\gamma^2}$$

Both reveal that

$$\gamma' = \frac{i\hbar}{m} \quad \text{and hence} \quad \gamma = \alpha + \frac{i\hbar}{m}t$$

where $\alpha = \gamma(0)$
Probability density given as

$$|\Psi(x,t)|^2 = \frac{|C|^2}{|\gamma(t)|}e^{-\alpha x^2/|\gamma(t)|^2}$$

localised around $x = 0$ on scale

$$\frac{|\gamma(t)|}{\sqrt{\alpha}} \qquad \text{solution "diffuses"}$$

how quickly does solution "diffuse"?
Time scale $m\alpha/\hbar$
For $m = m_e$, and $\sqrt{\alpha} = 10^{-12}$ m $\Rightarrow$ time scale $\sim 10^{-20}$ s
For $m = 10^{-6}$ kg, $\sqrt{\alpha} = 10^{-6}$ m, $\Rightarrow$ time scale $\sim 10^{16}$ s.

QUESTION 8

(i)

$$H\psi_1 = E\psi_1$$
$$H\psi_2 = E\psi_2$$

Time indep SE with $V(x) \to 0$ rapidly, is

Time-indep SE is

$$-\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$$

Note that

$$\det \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{pmatrix} = \psi_1\psi_2' - \psi_2\psi_1'$$

(ii)

(iii)

QUESTION 9

Time dependent SE is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + -U\delta(x)\Psi$$

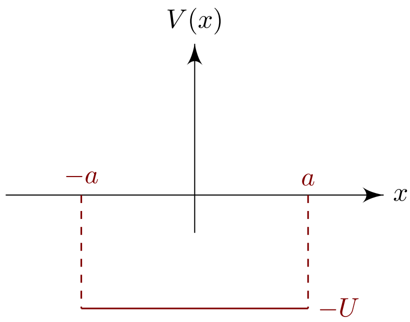
Integrating from $x-\varepsilon$ to $x+\varepsilon$ gives

$$\int_{x-\varepsilon}^{x+\varepsilon} i\hbar\frac{\partial\Psi}{\partial t}\,\mathrm{d}x = -\frac{\hbar^2}{2m}\left[\frac{\partial\Psi}{\partial x}\right]_{x-\varepsilon}^{x+\varepsilon} - U\Psi$$

Taking $\frac{\partial\Psi}{\partial t}$ to be sufficiently smooth, LHS $=0$ and we have

$$\left[\frac{\partial\Psi}{\partial x}\right]_{x-\varepsilon}^{x+\varepsilon} = -\frac{2mU}{\hbar^2}\Psi$$

QUESTION 10



Seek energy functions and eigenvalues given by

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$$

with energies in range $-U < E < 0$
SE becomes

$$\underbrace{-\frac{\hbar^2}{2m}\psi'' = (E+U)\psi}_{|x|<a} \text{ and } \underbrace{-\frac{\hbar^2}{2m}\psi'' = E\psi}_{|x|>a}$$

Set

$$U + E = \frac{\hbar^2 k^2}{2m} \text{ and } E = \frac{-\hbar^2\kappa^2}{2m}$$

$$k > 0 \qquad \text{and} \qquad \kappa > 0$$

then SE is

$$\psi'' + k^2\psi = 0 \text{ and } \psi'' - \kappa^2\psi = 0$$

$$|x| < a \qquad \text{and} \qquad |x| > a$$

At $x = \pm a$, ψ, ψ' continuous (ψ'' discontinuous, matching step in $V(x)$)
[Integrate SE from $a-\varepsilon$ to $a+\varepsilon$, then provided U, ψ bounded, find $[\psi']_{a-\varepsilon}^{a+\varepsilon} \to 0$ as $\varepsilon \to 0^+$]
Consider *odd parity* solutions, ie. those with $\psi(-x) = -\psi(x)$,

$$\psi = \begin{cases} A\sin kx & \text{if } |x| < a \\ Be^{-\kappa x} & \text{if } x > a \end{cases}$$

Note the solution for $x < -a$ fixed by parity. Matching at $x = a$,

$$\psi \text{ cts } : A\sin ka = Be^{-\kappa a}$$

$$\psi' \text{ cts } : kA\cos ka = -B\kappa e^{-\kappa a}$$

These equations give same solution for A or B iff:

$$\kappa \tan ka = -k$$

To find when solutions exists it is convenient to set

$$\xi = ak, \quad \eta = a\kappa \quad \text{dimensionless and positive}$$

So

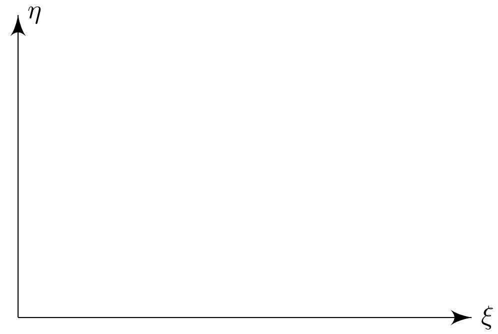
$$\eta = -\frac{\xi}{\tan \xi}$$

but also

$$\xi^2 + \eta^2 = \frac{2ma^2U}{\hbar^2} \quad \text{from definitions of } k \text{ and } \kappa$$

Intersection of $\nu = \xi \tan \xi$ with circle of $(\text{radius})^2 = \frac{2ma^2U}{\hbar^2}$. Have energy eigenstate for each point of intersection (a, U fixed parameters, determining ξ, ν determines E)

We can look for solutions by plotting these two equations. We first plot the curve $\eta = -\frac{\xi}{\tan \xi}$:



The other equation is the equation of a circle. Depending on the size of the constant $2ma^2U/\hbar^2$, there will be a different number of points of intersections.

Can see that circle must have radius $\geq \frac{\pi}{2}$ for intersection; anything smaller will produce no intersections, ie. no intersections if

$$2ma^2U/\hbar^2 < \left(\frac{\pi}{2}\right)^2$$

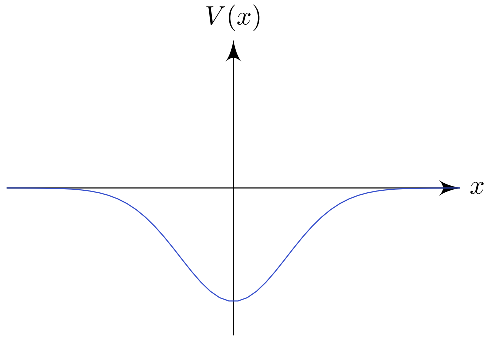
which rearranges to

$$aU^2 < (\pi\hbar)^2/8m$$

as required

QUESTION 11

Potential $V(x) = -\frac{\hbar^2}{2m}\operatorname{sech}^2 x$



Time-indep SE is

$$-\frac{\hbar^2}{2m}\psi''(x) - \frac{\hbar^2}{m}\operatorname{sech}^2(x)\psi(x) = E\psi(x)$$

which is

$$-\psi''(x) - 2\operatorname{sech}^2(x)\psi(x) = \varepsilon\psi(x) \quad (*)$$

where $\varepsilon = 2mE/\hbar^2$.

$$\begin{aligned}
A^\dagger A\psi &= A^\dagger\left[\psi'(x) + \tanh(x)\psi(x)\right] \\
&= -\psi''(x) - \operatorname{sech}^2(x)\psi(x) - \tanh(x)\psi'(x) + \tanh(x)\psi'(x) + \tanh^2(x)\psi(x) \\
&= -\psi'' - 2\operatorname{sech}^2(x)\psi + \psi \qquad \text{using } \tanh^2(x) = 1 - \operatorname{sech}^2(x)
\end{aligned}$$

Hence, adding $\psi(x)$ to both sides of $(*)$ it is rewritten as

$$A^\dagger A\psi = (\varepsilon + 1)\psi$$