Problem 1

Problem 2

We begin by first finding the Fisher information matrix for $n = 1$:

The log-likelihood for $\theta = (\theta_1, \ldots, \theta_k)$ is

$$l(\theta) = \log f(\underline{x}|\theta) = \log \left[\frac{n!}{N_1! \cdots N_k!} \prod_{j=1}^{k} \theta_j^{N_j} \right] \propto \sum_{j=1}^{k} N_j \log(\theta_j).$$

Taking the first and second partial derivatives with respect to θ_i, we obtain:

$$\frac{\partial}{\partial \theta_i} l(\theta) = \frac{\partial}{\partial \theta_i} \left[\sum_{j=1}^{k-1} N_j \log(\theta_j) + N_k \log \left(1 - \sum_{i=1}^{k-1} \theta_i \right) \right] = \frac{N_i}{\theta_i} - \frac{N_k}{\theta_k},$$

and

$$\frac{\partial^2}{\partial \theta_i \partial \theta_j} l(\theta) = \begin{cases} -\frac{N_i}{\theta_i^2} - \frac{N_k}{\theta_k^2} & i = j \\ -\frac{N_k}{\theta_k^2} & i \neq j \end{cases}.$$

Taking expectation, we find

$$E\left[\frac{\partial^2}{\partial \theta_i \partial \theta_j} l(\theta) \right] = \begin{cases} -\frac{n\theta_i}{\theta_i^2} - \frac{n\theta_k}{\theta_k^2} & i = j \\ -\frac{n\theta_k}{\theta_k^2} & i \neq j \end{cases}$$

Thus, the Fisher information matrix for $n = 1$ has elements

$$I_{i,j}(\theta) = -E\left[\frac{\partial^2}{\partial \theta_i \partial \theta_j} l(\theta) \right] = \begin{cases} \frac{1}{\theta_i} + \frac{1}{\theta_k} & i = j \\ \frac{1}{\theta_k} & i \neq j \end{cases}, \quad i, j = 1, \ldots, k-1.$$

- Additionally, we note that the MLE for $\theta_j, j = 1, \ldots, k-1$ is $\hat{\theta}_j = \frac{N_j}{n}$.

Wald's test states that for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ we shall reject H_0 if

$$W_n(\theta_0) > \chi_r^2(1 - \alpha),$$

where $W_n(\theta_0) = n(\hat{\theta}_n - \theta_0)^T I(\theta_0)(\hat{\theta}_n - \theta_0)$, $\hat{\theta}_n$ is the MLE, $I(\theta_0)$ has dimension $r \times r$, and $\chi_r^2(1 - \alpha)$ is the $(1 - \alpha)^{th}$ quantile of the chi-square distribution with r degrees of freedom. Substituting in the Fisher matrix and the MLE, we arrive at Pearson's χ^2-test:

$$
\begin{aligned}
W_n(\theta_0) &= n \begin{pmatrix} \frac{N_1}{n} - \theta_{0,1} \\ \vdots \\ \frac{N_{k-1}}{n} - \theta_{0,k-1} \end{pmatrix}^T \begin{bmatrix} \frac{1}{\theta_1} + \frac{1}{\theta_{0,k}} & \frac{1}{\theta_{0,k}} & \cdots & \frac{1}{\theta_{0,k}} \\ \frac{1}{\theta_{0,k}} & \frac{1}{\theta_2} + \frac{1}{\theta_{0,k}} & \cdots & \frac{1}{\theta_{0,k}} \\ \vdots & & \ddots & \vdots \\ \frac{1}{\theta_{0,k}} & \frac{1}{\theta_{0,k}} & \cdots & \frac{1}{\theta_{k-1}} + \frac{1}{\theta_{0,k}} \end{bmatrix} \begin{pmatrix} \frac{N_1}{n} - \theta_{0,1} \\ \vdots \\ \frac{N_{k-1}}{n} - \theta_{0,k-1} \end{pmatrix} \\
&= \frac{1}{n} \left[\frac{N_1 - n\theta_{0,1}}{\theta_1} + \sum_{i=i}^{k-1} \frac{(N_i - n\theta_{0,i})}{\theta_{0,k}}, \ldots, \frac{N_{k-1} - n\theta_{0,k-1}}{\theta_{k-1}} + \sum_{i=i}^{k-1} \frac{(N_i - n\theta_{0,i})}{\theta_{0,k}} \right] \\
&\qquad \cdot \begin{pmatrix} N_1 - n\theta_{0,1} \\ \vdots \\ N_{k-1} - n\theta_{0,k-1} \end{pmatrix} \\
&= \frac{1}{n} \left[\frac{N_1 - n\theta_{0,1}}{\theta_1} + \frac{(n - N_k) - n(1 - \theta_{0,k})}{\theta_{0,k}}, \ldots, \frac{N_{k-1} - n\theta_{0,k-1}}{\theta_{k-1}} + \frac{(n - N_k) - n(1 - \theta_{0,k})}{\theta_{0,k}} \right] \\
&\qquad \cdot \begin{pmatrix} N_1 - n\theta_{0,1} \\ \vdots \\ N_{k-1} - n\theta_{0,k-1} \end{pmatrix} \\
&= \frac{1}{n} \left[\frac{N_1 - n\theta_{0,1}}{\theta_1} + \frac{-N_k + n\theta_{0,k}}{\theta_{0,k}}, \ldots, \frac{N_{k-1} - n\theta_{0,k-1}}{\theta_{k-1}} + \frac{-N_k + n\theta_{0,k}}{\theta_{0,k}} \right] \begin{pmatrix} N_1 - n\theta_{0,1} \\ \vdots \\ N_{k-1} - n\theta_{0,k-1} \end{pmatrix} \\
&= \frac{1}{n} \left[\sum_{i=1}^{k-1} \frac{(N_i - n\theta_{0,i})^2}{\theta_i} + \left(\frac{-N_k + n\theta_{0,k}}{\theta_{0,k}} \right) \sum_{i=1}^{k-1} (N_i - n\theta_{0,i}) \right] \\
&= \frac{1}{n} \left[\sum_{i=1}^{k-1} \frac{(N_i - n\theta_{0,i})^2}{\theta_i} + \frac{(-N_k + n\theta_{0,k})^2}{\theta_{0,k}} \right] = \sum_{i=1}^{k} \frac{(N_i - n\theta_{0,i})^2}{n\theta_i} \qquad \square
\end{aligned}
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Problem 3

Problem 4

Problem 5