1. Consider kernel density estimation based on $X_1, \ldots, X_n \sim_{iid} U(0,1)$ and a kernel K. Let $0 < x < 1$.
   (a) Find a (simple) expression for bias and variance of the kernel density estimator $\widehat{f}_n(x)$.
   (b) Show that if K has compact support, then the bias of $\widehat{f}_n(x)$ is exactly zero if h_n is small enough.
   (c) In the case of a uniform kernel $K(x) = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)$ show that the variance of $\widehat{f}_n(x)$ is exactly zero if h_n is large enough. Explain this fact.
   (d) Discuss the properties from (b) and (c). (From a more practical point of view, how useful are these properties? Do these properties mean that the kernel estimator is performing wonderfully? etc.)

2. Consider kernel density estimation based on $X_1, \ldots, X_n \sim_{iid} f$ where f has support $[a,b] \subset \mathbb{R}$. Assume that (i) f is continuous on $[a,b]$, that (ii) K is a symmetric pdf with mean zero, and that (iii) $nh \to \infty$ and $h \to 0$ as $n \to \infty$. Study the behavior of the kernel density estimator $\widehat{f}_n(x)$, $x \in \mathbb{R}$ at $x = a$ and $x = b$ by deriving the behavior of bias and variance.

   HINT: Notice that we do not necessarily assume that $f(a) = 0$ or $f(b) = 0$. For studying the bias consider the three cases separately: (i) $x \in (a,b)$, (ii) $x \notin [a,b]$ and (iii) $x = a$ or $x = b$.

3. Consider the same set-up as in problem 2 but assume now that (i) K is a differentiable pdf of bounded variation, symmetric about 0 with $\int_{-\infty}^{\infty} u^3 K(u)\, du < \infty$, $\mathbb{K}_2' := \int_{-\infty}^{\infty} \big(K'(v)\big)^2 dv < \infty$ and $\int_{-\infty}^{\infty} v^2 \big(K'(v)\big)^2 dv < \infty$. Also assume that (ii) f is three times continuously differentiable with all the derivatives bounded, and that (iii) $h_n \to 0$. Show that
   (a) $\mathrm{bias}\big[\widehat{f}_n'(x)\big] = \frac{\sigma_K^2}{2} f'''(x) h_n^2 + o(h_n^2)$ with $\sigma_K^2 = \int_{-\infty}^{\infty} u^2 K(u)\, du$.
   (b) $\mathrm{Var}\big[\widehat{f}_n'(x)\big] = \mathbb{K}_2' \, f(x) \, \frac{1}{nh_n^3} + o(\frac{1}{nh_n^3})$.