General Poisson's Equation Using Boundary Element Method

General Poisson's Equation with known spatial function b_o:

$$\boxed{\nabla^2 \Phi = b_o} \tag{1}$$

Taking dot product with arbitrary weight function w_o and integrating over the domain Ω to create the weak formulation of the equation (1):

$$\int\int_{\Omega} w_o \nabla^2 \Phi \, d\xi \, d\eta = \int\int_{\Omega} w_o b_o d\xi \, d\eta \tag{2}$$

Using gauss second identity on the LHS (left hand side) of equation (2):

$$\int\int_{\Omega} w_o \nabla^2 \Phi \, d\xi \, d\eta = \int\int_{\Omega} \Phi \nabla^2 \mathrm{w_o} \, d\xi \, d\eta + \oint_{\partial\Omega} \left(w_o \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial \mathrm{w_o}}{\partial n} \right) ds \tag{3}$$

We choose w so that $\nabla^2 w_o = \delta(\xi - x, \eta - y)$. The fundamental solution for this is:

$$\boxed{w_o = \frac{1}{4\pi} \ln(r^2), \qquad r^2 = (\xi - x)^2 + (\eta - y)^2} \tag{3}$$

Therefore,

$$\int\int_{\Omega} \Phi \nabla^2 \mathrm{w_o} \, d\xi \, d\eta = \Phi \tag{4}$$

Assuming (ξ, η) is inside the domain (Ω) and not on the boundary $(\partial\Omega)$,

$$\int\int_{\Omega} w_o \nabla^2 \Phi \, d\xi \, d\eta = \Phi + \oint_{\partial\Omega} \left(w_o \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial \mathrm{w_o}}{\partial n} \right) ds \tag{5}$$

Substituting equation (2) to equation (5):

$$\int\int_{\Omega} w_o b_o d\xi \, d\eta = \Phi + \oint_{\partial\Omega} \left(w_o \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial \mathrm{w_o}}{\partial n} \right) ds$$

$$\Phi = \int\int_{\Omega} w_o b_o d\xi \, d\eta - \oint_{\partial\Omega} \left(w_o \frac{\partial \Phi}{\partial n} - \Phi \frac{\partial \mathrm{w_o}}{\partial n} \right) ds$$

$$\boxed{\begin{gathered} \Phi = \oint_{\partial\Omega} \left(\Phi \frac{\partial \mathrm{w_o}}{\partial n} - w_o \frac{\partial \Phi}{\partial n} \right) ds + \int\int_{\Omega} w_o b_o d\xi \, d\eta \\ for\ (\xi, \eta) \text{ is inside the domain } \Omega \end{gathered}} \tag{6}$$

Similarly, for (ξ, η) on the boundary $(\partial\Omega)$,

$$\boxed{\begin{gathered}\frac{1}{2}\Phi = \oint_{\partial\Omega}\left(\Phi\frac{\partial \mathrm{w_o}}{\partial n} - w_o\frac{\partial\Phi}{\partial n}\right)ds + \int\int_{\Omega} w_o b_o d\xi\, d\eta \\ for\ (\xi, \eta) \text{ in boundary } \partial\Omega\end{gathered}} \tag{7}$$

Now, equation (6) and (7) still consists of the domain integral $\int\int_{\Omega} w_o b_o d\xi\, d\eta$ which needs to be changed to boundary integral form to be solved with boundary element method. For this, we define:

$$w_o = \nabla^2 w_1 \tag{8}$$

Realize that equation (8) is a Poisson's equation. Multiplying equation (8) with b_o and integrating over the domain Ω, we get:

$$\int\int_{\Omega} w_o b_o d\xi\, d\eta = \int\int_{\Omega} b_o \nabla^2 w_1 d\xi\, d\eta \tag{9}$$

Applying green's formula as done previously, we get:

$$\int\int_{\Omega} w_o b_o d\xi\, d\eta = \oint_{\partial\Omega}\left(b_0\frac{\partial w_1}{\partial n} - w_1\frac{\partial b_o}{\partial n}\right)ds + \int\int_{\Omega} w_1 \nabla^2 b_o d\xi\, d\eta \tag{10}$$

Now, we can define $b_1 = \nabla^2 b_o$ and the redo the previous process to change the new additional domain integral $\int\int_{\Omega} w_1 \nabla^2 b_o d\xi\, d\eta$ to boundary integral form and it will introduce another new boundary integral term to the equation. For arbitrary b_o function, $b_i \neq 0$ for $i = 1,2,\ldots,n-1$ and this whole process can be done multiple times until $b_n = 0$. Therefore, rewriting equation (6) and (7), we get:

$$\boxed{\begin{gathered}\Phi = \oint_{\partial\Omega}\left(\Phi\frac{\partial \mathrm{w_o}}{\partial n} - w_o\frac{\partial\Phi}{\partial n}\right)ds + \sum_{i=0}^{n}\oint_{\partial\Omega}\left(b_i\frac{\partial w_{i+1}}{\partial n} - w_{i+1}\frac{\partial b_i}{\partial n}\right)ds \\ for\ (\xi, \eta) \text{is inside the domain } \Omega\end{gathered}} \tag{11}$$

$$\boxed{\begin{gathered}\frac{1}{2}\Phi = \oint_{\partial\Omega}\left(\Phi\frac{\partial \mathrm{w_o}}{\partial n} - w_o\frac{\partial\Phi}{\partial n}\right)ds + \sum_{i=0}^{n}\oint_{\partial\Omega}\left(b_i\frac{\partial w_{i+1}}{\partial n} - w_{i+1}\frac{\partial b_i}{\partial n}\right)ds \\ for\ (\xi, \eta) \text{ in boundary } \partial\Omega\end{gathered}} \tag{12}$$

w_i can be determined through some rigorous mathematics process as done by [2] which is not detailed here. w_i is given as follows:

$$w_i = \frac{1}{4\pi} C_i r^{2i} (\ln r^2 - D_i), \qquad for\ i = 0,1,2,\dots,n \tag{13}$$

$$C_i = \frac{1}{4^i . i!}, \qquad for\ i = 0,1,2,\dots,n \tag{14}$$

$$D_i = 0, \qquad for\ i = 0 \tag{15}$$

$$D_i = 2\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i}\right), \qquad for\ i = 1,2,3,\dots,n \tag{16}$$

Therefore, from equation (11) and (12), we can solve the Poisson's equation by discretizing only the boundary of the domain. First, we solve for the value of Φ or ${}^{\partial\Phi}/_{\partial n}$ depending on the given boundary condition on the boundary $\partial\Omega$. This can be solved by constructing the system of linear equation matrix $\boldsymbol{Ax} = \boldsymbol{b}$ from the integral equation (12). After solving for the value of either Φ or ${}^{\partial\Phi}/_{\partial n}$ in the boundary with the other being supplemented through boundary conditions, the inner value of Φ can be computed using equation (11).

Torsion Problem Using Boundary Element Method

Torsion problem in terms of Prandtl stress function (Φ) is a Poisson's equation with $b_0 = -2$ and the boundary conditions are given as $\Phi = 0$ at $\partial\Omega$.

$$\nabla^2\Phi = -2 \tag{17}$$

$$\Phi = 0, \qquad \text{at } \partial\Omega$$

From equation (17), we can see that $b_o = -2$ and $b_j = 0$ for $j = 1,2,\dots,m$. Therefore, we only need to evaluate the right summation terms in equation (11) and (12) at $i = 0$. Rewriting equation (12) for torsion problem:

$$\frac{1}{2}\Phi = \oint_{\partial\Omega}\left(\overset{\Phi = 0 \text{ at } \partial\Omega}{\cancel{\Phi\frac{\partial w_o}{\partial n}}} - w_o\frac{\partial\Phi}{\partial n}\right)ds + \oint_{\partial\Omega}\left(b_o\frac{\partial w_1}{\partial n} - \overset{b_o\ is\ constant}{\cancel{w_1\frac{\partial b_o}{\partial n}}}\right)ds$$

$$\frac{1}{2}\Phi = \oint_{\partial\Omega} b_o\frac{\partial w_1}{\partial n}ds - \oint_{\partial\Omega} w_o\frac{\partial\Phi}{\partial n}ds \tag{18}$$

Doing the same to equation (11) and rewriting it:

$$\Phi = \oint_{\partial\Omega} b_o \frac{\partial w_1}{\partial n} ds - \oint_{\partial\Omega} w_o \frac{\partial \Phi}{\partial n} ds \tag{19}$$

Now, we want to construct the system of linear equation matrix $\boldsymbol{Ax} = \boldsymbol{b}$ from equation (18). Because $\Phi = 0$ is given at $\partial\Omega$, we write the equation matrix in terms of ${\partial\Phi}/{\partial n}$.

$$\oint_{\partial\Omega} b_o \frac{\partial w_1}{\partial n} ds = \oint_{\partial\Omega} w_o \frac{\partial \Phi}{\partial n} ds$$

Discretizing the integral to N number of panels for the equation matrix and taking out b_o from the integral,

$$b_o \sum_{k=1}^{N} \oint_{\partial\Omega_{\mathrm{ik}}} \frac{\partial w_1}{\partial n} ds = \sum_{k=1}^{N} \left(\frac{\partial \Phi}{\partial n}\right)_k \oint_{\partial\Omega_{\mathrm{ik}}} w_o ds \tag{20}$$

Writing equation (20) in matrix:

$$\boxed{\begin{bmatrix} \oint_{\partial\Omega_{11}} w_o ds & \cdots & \oint_{\partial\Omega_{1\mathrm{N}}} w_o ds \\ \vdots & \ddots & \vdots \\ \oint_{\partial\Omega_{\mathrm{N}1}} w_o ds & \cdots & \oint_{\partial\Omega_{\mathrm{NN}}} w_o ds \end{bmatrix} \begin{bmatrix} \left(\frac{\partial \Phi}{\partial n}\right)_1 \\ \vdots \\ \left(\frac{\partial \Phi}{\partial n}\right)_N \end{bmatrix} = b_o \begin{bmatrix} \sum_{k=1}^{N} \oint_{\partial\Omega_{1\mathrm{k}}} \frac{\partial w_1}{\partial n} ds \\ \vdots \\ \sum_{k=1}^{N} \oint_{\partial\Omega_{\mathrm{Nk}}} \frac{\partial w_1}{\partial n} ds \end{bmatrix}} \tag{21}$$

The integral in equation (21) are improper when $i = k$, but they can be evaluated to give:

$$\oint_{\partial\Omega_{\mathrm{kk}}} w_o ds = \frac{S_k}{2\pi}\left(\ln\left(\frac{S_k}{2}\right) - 1\right) \tag{22}$$

$$\oint_{\partial\Omega_{\mathrm{kk}}} \frac{\partial w_1}{\partial n} ds = 0 \tag{23}$$

Therefore, the equation matrix in equation (21) can be solved easily using methods such as LU decomposition or gauss elimination and ${\partial\Phi}/{\partial n}$ at every panel in $\partial\Omega$ can be calculated. After calculating ${\partial\Phi}/{\partial n}$ at every panel in $\partial\Omega$, we can proceed to calculate Φ anywhere inside Ω using equation (19). Rewriting equation (19) in summation form:

$$\boxed{\Phi = b_o \sum_{k=1}^{N} \oint_{\partial\Omega_{\mathrm{k}}} \frac{\partial w_1}{\partial n} ds - \sum_{k=1}^{N} \left(\frac{\partial \Phi}{\partial n}\right)_k \oint_{\partial\Omega_{\mathrm{k}}} w_o ds} \tag{24}$$