The role of confidence over timing of investment information*

Christopher Gibson

January 2024

Abstract

I present an investment environment wherein investors demand an asset based on perfectly informative signals, but face uncertainty about the timing of their information acquisition. I show that this uncertainty reduces the demand and price for every period but that in the limit, price converges to the true value of the asset as the number of periods increases. By introducing a concept of confidence over the time in which investors receive a signal, I show that the impact of uncertainty can be exaggerated in either a negative or positive direction, depending on the type of confidence under consideration, with the limit price reflecting the true value of the asset.

1 Introduction

Uncertainty is one of the most widely studied phenomena in all of economics. Without uncertainty, all decisions could be made through a combination of incorporating economically relevant variables and backward induction, yielding definitive answers and leaving economists (and people in general) to dedicate themselves to other pursuits. But uncertainty pervades. Outcomes of investment choices, information quality, and even the preferences of agents all suffer from the whims of uncertainty. As such, in order to accurately capture behavior the field of economics must accommodate and incorporate into models the reality of uncertainty in any form it may take.

One form of uncertainty that has garnered much attention in the realm of financial investment and firm profit maximizing decisions is over the quality of information. The final value of an uncertain decision can be found in the outcome into which uncertainty resolves itself, but when the decision must be made before such resolution the value lies solely in the quality of information over the possible outcomes. It is no wonder then that the quality

*For most recent version visit `www.christopherdgibson.com` of information is of such interest. But a metric over informational quality misses one of uncertainty's most important factors: timing. It is important not only to employ accurate information in making decisions in the face of uncertainty, but it is perhaps equally important to employ this information at the appropriate time.

In this paper I introduce a setting in which profit maximizing agents undertake decisions in the face of uncertainty. However, it is not the quality of information that is uncertain to agents, but rather the timing with which agents receive this information. To emphasize the effect of timing on information driven decisions, multiple agents will receive signals at different times, yet none will be aware of the order in which they receive this profit relevant information.

In order to isolate the role of positional uncertainty, investors will receive a perfectly informative signal about the state of the world, in this case the value of an asset. While the asset's valuation is unambiguous, agents will must determine their investment strategies without knowing their position of movement. That is, they must face the uncertainty of other investors having already made their decisions, incorporating information into the asset price, thereby diminishing the value of the informative signal.

Upon a groundwork of behavior under positional uncertainty I build the notion of confidence. Agents who are equally likely to move in any particular period will be said to suffer from a confidence bias if they place any weight other than the uniform distribution on their beliefs of moving in any period. This notion of confidence encompasses both overconfidence, as is traditionally the focus in the behavioral literature, as well as underconfidence. Overconfidence will manifest in a type of front-loading of beliefs so that the agent believes it is more likely they will move earlier than later, expecting that greater gains to investment are possible than would be so with no such bias. Underconfidence will have the opposite quality, leading agents to place greater weight on the belief that they move in later periods.

The paper will proceed as follows. In section 2 I will discuss the most closely related literature; in section 3 I will introduce a basic model of investment; in section 4 I will introduce uncertainty; in section 5 I develop a notion of confidence that can change based on agents' beliefs in equilibrium; section 6 concludes. All proofs are relegated to the appendix unless they provide useful insight into the decision making process.

2 Related literature

Much work has been done on overconfidence in the trading of financial assets. Perhaps the most closely related work is that of Gervais, Odean (2001) [4]. In this model investors receive either a perfect signal with a fixed probability or a signal that is pure noise, and then must update their belief of receiving the informative signal. Through varying a confidence parameter they show that belief of acquiring an informative signal can either converge to the case of perfect rationality for low levels of overconfidence, or diverge for high levels of confidence.

This work has many related elements including accounting for the confidence of agents and a multi-period investment setting. Among the many departures, however, is that here I investigate the role of confidence over position, not signal acquisition. Agents know they receive a perfect signal about the value of the asset but have imperfect information about the period in which they receive it. In addition in their setting agents receive signals and invest in each period. In order to isolate the role of confidence over positional uncertainty I restrict attention to one signal although the model generalizes to more frequent signals.

In other works Odean (2008b) shows that overconfidence in investors tends to lead to excessive trading and lower expected utility. Overconfident agents tend to overreact to salient information and underreact to trade relevant information, thereby preventing the information of rational agents from being fully reflected in market price [7]. Barber and Odean (2001) also find that men trade stocks 45% more than women, a finding hypothesized to come from overconfidence [1].

This excessive trading and overreaction to salient information is supported by an experiment comparing traders new to online trading to their previous gains (Barber and Odean 2002) [2]. It is found that while phone traders tended to beat the market, upon the switch traders tend to under-perform, a finding unexplained by the reduction in market frictions alone. It is hypothesized that overconfidence coupled with an increased trading speed cause online investors to increase their trading volume and reduce their performance.

Other studies show similar effects of confidence in other settings. Through FMRI scans Peterson (2005) shows that investor overconfidence may be related to reward system activation in the brain [8]. Handy and Underwood (2005) find that overconfidence increases price at which managers repurchase share prices [5], a finding backed up empirically by Shu et.al (2013) [9]. Other studies demonstrate how the salience of news stories can lead to overconfidence and excessive trading (Barber, Odean 2008) [3] and that due to loss aversion traders tend to keep their assets when they suffer large losses disproportionately more often than when they enjoy small gains (Odean 1998) [6].

3 The model

I consider an environment in which agents receive information about the value of a financial asset. The previous value v_0 of the asset is unknown to investors but is assumed to have already been incorporated into the market price. Agents receive a signal η about how the value of the asset changes. Agents receive this signal privately and without distortion but share a prior belief with all market participants that it is drawn from the distribution $\eta \sim N(0, \sigma^2)$.

Agents wish to maximize the difference between the value of the asset and the price they pay. Upon receiving signal η they know the value of the asset is $v_t = \mathbb{E}[v_0] + \eta$, but they are unaware of the prior value v_0. Agents view price p_t and choose their demand for the asset x in order to maximize $\mathbb{E}[x(v_t - p_t)]$. Importantly, there will be no short sale restrictions so that agents can demand a negative amount of the asset.

In addition to not knowing the initial value v_0 of the asset, agents are also unaware of their position of movement. If the agent moves at period t then $t-1$ agents have already had the opportunity to move. In this setting position of movement refers to the time at which the signal is received, which is to say that if an agent moving at period t sees a price p_t, this price already reflects information η incorporated into it by $t-1$ other agents.

Notice that both elements of uncertainty are necessary to capture the idea of positional uncertainty. If the agent knew v_0 they could maximize $x(v_0+\eta-p_t)$ without any information about their position of movement. Likewise if the agent were to know their position, through backward induction the agent could deduce how much information η was incorporated into the price by the previous $t-1$ agents.

In addition to the aforementioned informed traders there is a liquidity trader who demands an amount of the asset every period. This is necessary not only to capture the reality that investors participate in the market for reasons other than price (e.g. to raise capital or they are uninformed) but also to guarantee trade in a market with informed investors who present an information asymmetry for any price setting mechanism. Each period the liquidity trader will demand $z_t \sim N(0, \Omega)$ of the asset, an amount independent of process that yields η and independent of liquidity demands of other periods. All market participants share common knowledge of the i.i.d. z_t and its independence from η.

Finally there is a market maker that sets the price p_t each period. The market maker knows the prior distribution of η, z_t, and their independence from one another. Like the informed agents the market maker does not know the value v_0 of the asset at period 0, but in period 1 the dissemination of information η introduces the informational asymmetry. To combat this asymmetry the market maker sets a price each period in order to match the value v_t as closely as possible given current and historical demands for the asset. That is, $p_t = \mathbb{E}[v_t|\omega_t, h_t]$ where $\omega_t = x_t + z_t$, the sum of demand from the informed and liquidity traders, and $h_t = (w_i)_{i<t}$ is the historical series of market demand for each period.

3.1 The case of no uncertainty

To gain a foothold into the decision making process faced by investors it is useful to start with the case of no uncertainty. Moreover, the case without uncertainty will provide a benchmark against which to compare decision making when agents do not know their position of movement.

Consider the investment setting as described with T periods and one agent moving in each period. Each agent knows their position $t \leq T$ and chooses demand to maximize the difference between the value of the asset and price per share. To describe how agents make this decision, recall that they maximize $\mathbb{E}[x(v_t - p_t)]$. While v_t is perfectly known as agents know their position of movement, there remains uncertainty in the price.

As we will see the linear equilibrium takes the form $p_t = p_{t-1} + \lambda_t \omega_t$. Since demand ω_t includes liquidity traders that behave randomly, agents cannot perfectly predict price movements in period t and must take an expectation. The optimal demand then comes from maximizing $\mathbb{E}[x(v_t - (p_{t-1} + \lambda_t(x + z_t)))] = x(\mathbb{E}[v_0] + \eta - p_{t-1} - \lambda_t x)$, where it is assumed that price information in p_0 already contains v_0; in fact this assumption can (with some error induced by the liquidity trader) be verified by the agent through backward induction. The agent's optimal solution is then $x_t = \frac{\mathbb{E}[v_0]+\eta-p_{t-1}}{2\lambda_t}$.

The market maker sets a price attempting to match the asset's value, taking into account noise from the liquidity investor. Then in a linear equilibrium $p_t = \mathbb{E}[v_0 + \eta | \beta_t x_t + z_t, h_t]$. In equilibrium the value of β_t is known to the market maker so price setting becomes an exercise in signal extraction with noise $z_t \sim N(0, \Omega)$ induced by the liquidity trader and a prior belief $p_{t-1} - p_0$ of the value η. This yields an updated estimated value of the asset $p_t = p_{t-1} + \lambda_t \omega_t$. In this environment the equilibrium values of β_t and λ_t take a simple form.

Proposition 1. *For $T \in \mathbb{N} \cup \{\infty\}$ periods, if each agent knows their position $t \leq T$ then there exists a linear equilibrium of the form $p_t = p_{t-1} + \lambda_t \omega_t$, $x_1 = \beta_1 \eta$, and $x_t = \beta_t \eta + Z_t$ for $t > 1$ with*

$$\lambda_1 = \frac{1}{2}\sqrt{\frac{\sigma^2}{\Omega}}, \quad \beta_1 = \sqrt{\frac{\Omega}{\sigma^2}}, \quad \lambda_t = \frac{1}{\sqrt{8}}, \quad \beta_t = \frac{\sqrt{2}}{2^{t-1}} \text{ for } t > 1, \text{ and } Z_t \sim N(0, V_t)$$

Proposition 2. *In the above linear equilibrium, price can be expressed as $p_t = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right]\eta + Z'_t$ where $Z'_t \sim N(0, V'_t)$*

As expected from the investor's first order condition, the equilibrium demand for the asset more or less halves each period in proportion to the value of the asset. In fact, for periods $t = 2$ and onward demand $x_t = \beta_t \eta$ exactly halves every period. The reason for this is that in equilibrium β is a ratio of the variance of liquidity trading Ω and of the market maker's inferred variance from the procedure of noise signal updating. In the first period the market uses prior belief σ^2 of the asset's variance. But thereafter updated variance of the market maker is constant at $\frac{\Omega}{2}$. This result actually holds in a more general setting.

Lemma 1. *For any T period investment setting as above where agents demand $\beta_t \eta$ and $\beta_t = \frac{y}{\lambda_t}$ is a constant multiple of $\frac{1}{\lambda_t}$ and for any initial asset variance V_0, variance is constant in all periods $t \geq 2$ and takes the form $V_{t-1} = y\Omega$*

A technical detail that explains the constancy of β_t and λ_t for periods $t \geq 2$ to be sure, the instant convergence of inferred variance is also interesting in its own right. Not only does this result apply to the present case where agents are aware of their position, but it also applies when agents face positional uncertainty. This can be seen from the fact that the term y above can be any function of priors over positions of movement, so as long as y is constant so too is the inferred variance v_{t-1}. Another surprising feature of the updated variance is that it is independent of the distribution of signal η, depending only the liquidity trading variance Ω.

In addition to being an expected consequence of the agent's first order condition the result that demand halves in each period also provides insight into the rationality of the market price updating. In equilibrium the change in price can be expressed as $p_t - p_{t-1} = \lambda_t(\beta_t \eta_t + z_t) = \frac{1}{2^t}\eta + \lambda_t z_t$. In each period the market receives half as much information as in the prior period so that the rate of information transmission slows. Price is thus a geometric series save for the error in each period resulting from the presence of liquidity demands. While liquidity traders introduce noise that prevents the market maker from perfectly inferring the value of η, thereby enabling an equilibrium in pure strategies, their presence also hinders the interpretation of price as the true value of the asset even at the limit. However, the fact that liquidity noise has mean zero allows us to at least comment on its expectation.

Corollary 1. $\mathbb{E}[p_t] = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right]\eta$ *and* $\lim_{t\to\infty} \mathbb{E}[p_t] = p_0 + \eta$.

The form of the error Z'_t is not important from the perspective of interpreting the price or its expectation. It will always introduce randomness that prevents the market price from perfectly reflecting the underlying value of the asset, but will always be present for reasons described above. This error does, however, take a convenient form.

Proposition 3. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ periods and each agent knows their position $t \leq T$, the error term Z_t takes the form $Z_t = -\frac{1}{\lambda_t}\sum_{i=1}^{t-1}\left(\frac{1}{2}\right)^{t-i}\lambda_i z_i$ so that equilibrium demand for each period is $x_t = \frac{1}{2^t \lambda_t}\eta - \frac{1}{\lambda_t}\sum_{i=1}^{t-1}\left(\frac{1}{2}\right)^{t-i}\lambda_i z_i$ for $t > 1$. Moreover $p_t = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right]\eta + \sum_{i=1}^{t}\lambda_i z_i\left(\frac{1}{2}\right)^{t-i}$*

As the formulation of Z_t makes clear, each period noise from all previous periods becomes less relevant to price. But even though the Z_t follows a process in proportion to a geometric sum there is always $\lambda_t z_t$ incorporated in price p_t, preventing the price from converging to the true value of the asset. Fortunately a metric that is often referenced as an indication of an asset's value is the moving average, and with good reason.

Proposition 4. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ periods and each agent knows their position $t \leq T$, $\text{plim}_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} p_t = p_0 + \eta$.*

As this proposition shows, the price may not converge to the true value of the asset but the moving average converges in probability. So in a probabilistic sense the market fully incorporates the value η.

4 Introducing uncertainty

Now suppose agents face uncertainty over their position of movement, but suppose that agents share a common prior belief over the order. We begin with the case of two agents.

4.1 Two agents

As above suppose investors invest in an asset that evolves according to an unobservable process $v_{t+1} = v_t + \eta_{t+1}$ but in different periods each receives the same signal η about the process. Again there is a liquidity investor who demands $z_t \sim N(0, \Omega)$ independent of η_t.

In the case of two agents the common prior assumption provides that if agent 1 has prior belief $\Pr_1(F) = \gamma_1$ of moving first then it anticipates that agent 2 has prior belief $\Pr_2(F) = 1 - \gamma_1$. The agent receives a signal η about the how the value of the asset evolves but does not know the initial valuation and thus cannot infer if this valuation is already incorporated in price. With two agents, each can either be first or second and each observes a price p which may or may not incorporate the information η. Supposing p_0 is the price before information enters the market,

(i) If the agent is first then the observed price is $p_t = p_0$

(ii) If the agent is second then the observed price is $p_t = p_0 + \lambda_t(y_t + z_t)$, where y_t is the demand of the first moving agent and z_t the demand for the liquidity trader.

The difference between the first price p_t and the second is that the second price already incorporates information about the asset's value from the first agent. Thus the remaining profit left to the second mover is less because the price relative the the value of the asset is higher. Given that the agent has no prior information about the value of the asset it must be assumed that $p_0 = \mathbb{E}[v_0]$. Then agents solve

$$\begin{aligned}
&\max_x x\mathbb{E}[v_{t+1} - p_{t+1}] = \max_x x\mathbb{E}[v_{t+1} - (p_t + \lambda_{t+1}\omega_{t+1})] \\
&= \max_x x \cdot \gamma\mathbb{E}[v_t + \eta_{t+1} - (p_0 + \lambda_{t+1}\omega_{t+1})] \\
&\qquad + \max_x x \cdot (1-\gamma)\mathbb{E}[v_t + \eta_{t+1} - (p_0 + \lambda_t(y_t + z_t) + \lambda_{t+1}\omega_{t+1})] \\
&= \max_x x \cdot \gamma(\eta - \lambda_{t+1}x) - x \cdot (1-\gamma)(\eta - \lambda_{t+1}x - \lambda_t\mathbb{E}[y_t])
\end{aligned}$$

where y_t is the expected quantity of the first mover in the event this agent is in fact choosing second. The profit maximization problem then becomes $\max_x x \cdot (\eta - \lambda_{t+1}x - (1-\gamma)\lambda_t\mathbb{E}[y])$.

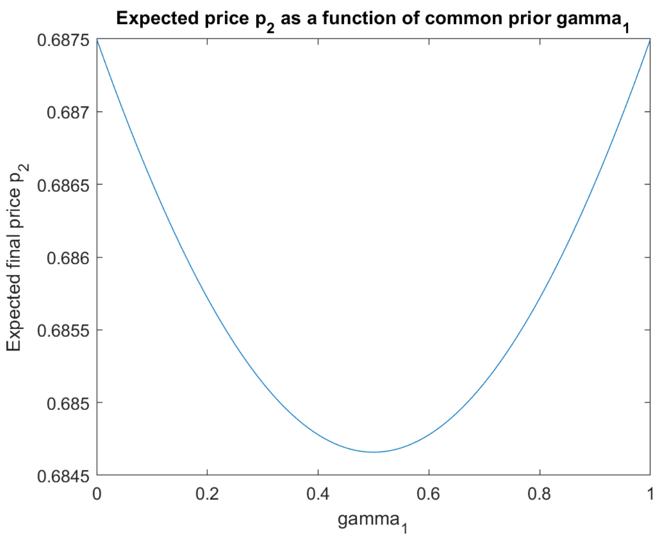
Notice that while $p_t = p_0$ or $p_t = p_0 - \lambda_t y_t$, the maximization function does not contain the term p_t. This is because the agent does not know price p_0 or value v_0, but on the expectation the best guess is that the market sets price $p_0 = \mathbb{E}[v_0]$. Then these two terms cancel and the difference we are left with is that between the future valuation and current price.

Proposition 5. *For $T = 2$ periods and agents who do not know their position but have prior beliefs γ_1, γ_2, and common prior $\gamma_1 = 1 - \gamma_2$ of moving first, there exists a linear equilibrium of the form $p_t = p_0(1 - \varphi_{t-1}) + \varphi_{t-1}p_{t-1} + \lambda_t\omega_t$ where p_0 is the price before information η entered the market, $x_1 = \beta_1(\gamma_1)\eta$ and $x_2 = \beta_2(\gamma_2)\eta$ with*

$$\begin{aligned}
&\beta_1(\gamma_1) = \sqrt{\frac{(1-\varphi_1)\Omega}{\varphi_1\sigma^2}},\ \beta_2(\gamma_2) = \sqrt{\frac{1-\varphi_2}{\varphi_2(1-\varphi_1)}} \\
&\lambda_1 = \sqrt{\frac{\varphi_1(1-\varphi_1)\sigma^2}{\Omega}} \quad \lambda_2 = \sqrt{\varphi_2(1-\varphi_2)(1-\varphi_1)} \\
\textit{and} \quad &\varphi_1\varphi_2\gamma(1-\gamma)(\varphi_1 - 1) = 4\varphi_1 - \varphi_2(1-\gamma) - 2 \\
&\varphi_1\varphi_2\gamma(1-\gamma)(\varphi_2 - 1) = 4\varphi_2 - \varphi_1\gamma - 2
\end{aligned}$$

From this result we can see the manner in which information about the asset's value translates into movements in the price. In equilibrium, $p_2 = p_0 + \lambda_1\varphi_2\omega_1 + \lambda_2\omega_2$. Then

$$\begin{aligned}
\mathbb{E}[p_2] &= p_0 + \lambda_1\varphi_2\beta_1\eta + \lambda_2\beta_2\eta = p_0 + [\varphi_2(1-\varphi_1) + (1-\varphi_2)]\eta \\
&= p_0 + (1 - \varphi_1\varphi_2)\eta
\end{aligned}$$



As the common prior γ_1 increases, each mover becomes less certain who is first until uncertainty is maximized when $\gamma_1 = \gamma_2 = \frac{1}{2}$. This minimum price represents the minimal possible information content resulting from the perfectly informative signal. As γ_1 further increases, uncertainty lessens and the other mover becomes more convinced of being first, resulting in an increased willingness to demand more of the financial asset.

4.2 T identical agents

We can generalize this case to one in which there are T agents, each receiving the signal η in a different period $t \leq T$ and sharing a common prior over their position of movement. A natural prior is uniform, where each agent believes that their probability of moving in period $t \leq T$ is $\Pr(t) = \frac{1}{T}$ for all periods. Furthermore, each agent believes that all other agents share this common prior.

As in the case of no uncertainty we will find a linear equilibrium in demand ω_t. Now, however, since agents do not know if the price they see is the original valuation or the price after $t-1$ periods of agents acting on information η, they will not assign a unit value to p_{t-1}. They weight the previous price based on their beliefs $\Pr(t)$ of moving in every t and their beliefs about other agents' actions. To compensate for this, in equilibrium price at period t will be a weight $\varphi < 1$ of the previous period price and current demand ω_t.

To see why this is, consider the pricing decision of the market maker. As before, each agent demands $x = \frac{\mathbb{E}[v_0]+\eta-p_{t-1}}{2\lambda}$ but now, with equal probability p_{t-1} could have information η incorporated in any number of periods $t \leq T-1$. Thus the agent will shade their demand down by the expected amount of information already incorporated into the price. Each period the market sets p_t in order to estimate $v_0 + \eta$. Then $p_t = \mathbb{E}[v_0 + \eta | \beta x_t + z_t] = p_0 + \frac{1}{\beta}\mathbb{E}[\beta\eta | \beta\eta + z_t] = p_0 + \frac{\Omega(p_{t-1}-p_0)+\beta V \omega_t}{\Omega+\beta^2 V}$ so that $p_t = p_0(1-\varphi) + \varphi p_{t-1} + \lambda\omega_t$ where $\varphi = \frac{\Omega}{\Omega+\beta^2 V}$ and $\lambda = \frac{\beta V}{\Omega+\beta^2 V}$.

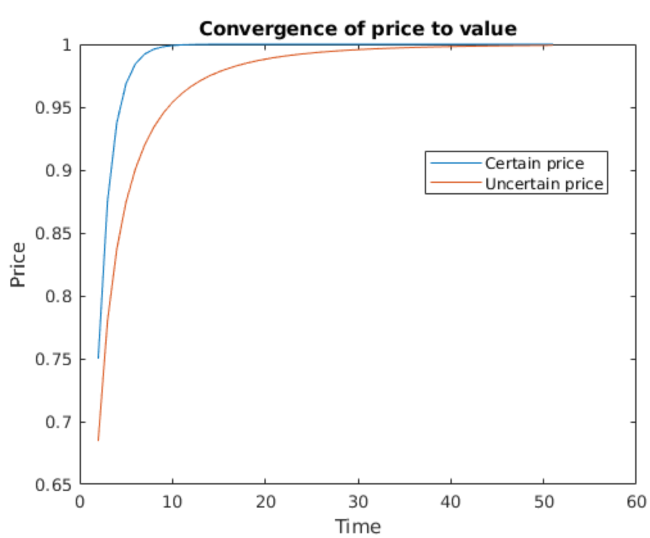
With this formulation price in each period is a φ discounted sum of previous demands plus initial price. If the agent moves in the second period price is $p_1 = p_0 + \lambda\omega_1$. If the agent moves in the third period then price is $p_2 = p_0 + \lambda\omega_2 + \varphi\lambda\omega_1$. Inductively if the agent moves in period t then $t-1$ agents move before and $p_{t-1} = p_0 + \sum_{i=1}^{t-1} \lambda\varphi^{(t-1)-i}\omega_i$. So to the agent, without knowledge of initial value v_0, p_{t-1} is a combination of demand in previous periods, containing $p_0 = \mathbb{E}[v_0]$. This gives rise to a linear equilibrium of the following form.

Proposition 6. *For $T \in \mathbb{N} \cup \{\infty\}$ periods, if agents do not know their position but have a uniform and common prior belief over $t \leq T$ then there exists a linear equilibrium of the form $p_t = p_0(1-\varphi) + \varphi p_{t-1} + \lambda\omega_t$ where p_0 is the price before information η enters the market and $x_t = \beta_t \eta$ with*

$$\beta_t = \frac{1}{\sqrt{\varphi}}, \ \lambda = \sqrt{\varphi}(1-\varphi), \ \text{and}\ \varphi = 1 - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$$

This equilibrium can be solved down to the variable φ which itself cannot be solved for explicitly. Yet it still provides interesting insight. The most obvious result to note is that this equilibrium does not depend on liquidity noise Ω. This comes from the fact that the updated variance of η converges immediately as described above, so $\beta\lambda$ need not include this term. As the market maker gains information from demand each period, since the variance of η does not change, noise introduced by the liquidity traders offers no additional information.

Equilibrium behavior for the informed agents also accords closely to what we would expect. Since agents do not know which of the T positions they occupy when they choose their investment strategies, they tend to behave more cautiously than in the case with no uncertainty.



This figure compares the price for each number of time periods in the certain and uncertain cases, given that the true value of η is 1 and $p_0 = 0$. As we can see comparing the cases of certainty with uncertainty, as the number of periods T increases the information η is more quickly incorporated into the price of the asset in the certain case. Indeed in the certain case information is integrated at the geometric rate $1 - \frac{1}{2}^t$, while in the uncertain case the rate is not quite as fast.

While slower than in the case of certainty, we can say something about the rate of convergence to the true value η as the following proposition describes.

Proposition 7. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ and agents do not know their position but have a uniform and common prior belief over $t \leq T$,* $\mathbb{E}[p_t] = p_0 + \eta(1 - \varphi^t)$

In the case of positional uncertainty, for every number of possible time periods the price is lower than if position of movement were certain, but this price too converges at a (pseudo) geometric rate of $1 - \varphi^t$, with φ as defined above. The difference is that the φ is higher than the $\frac{1}{2}$ of the certain case for all t, and in fact $\lim_{t \to \infty} \varphi = 1$. However, since price depends on φ^t it is this term whose convergence determines the integration of signal η into the price as the number of periods T increases. As the figure makes clear this term indeed does converge to zero.

Proposition 8. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ and agents do not know their position but have a uniform and common prior belief over $t \leq T$,* $\lim_{T \to \infty} \mathbb{E}[p_t] = p_0 + \eta$ *and* $\operatorname{plim}_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} p_t = p_0 + \eta$.

As Proposition 7 describes we have an analogous limit result in the case of positional uncertainty - albeit with a slower rate of convergence. This slower convergence reflects the fact that symmetric agents are more cautious in acting on their signal as there may be up to $T - 1$ periods of signal information already incorporated into the market price, making the gains uncertain. However, as the number of periods increases, the effect each agent has on equilibrium price by placing their optimal demand diminishes, so that demands in the certain and uncertain case merge and information η is fully incorporated.

5 A notion of confidence

Now that we have investigated the informed investing environment with certain and uncertain positions of movement, we can turn attention to how confidence plays a role in investment decisions. In particular, we saw in the environments with and without certainty that as the number of periods T increases price increased to the true value η of the asset. Furthermore we saw that this convergence was slower in the case of positional uncertainty but hardly by much; for $T \geq 40$ the prices were barely distinguishable.

Now we introduce the notion of confidence and attempt to answer the same questions. In particular, we would like to investigate in the presence of confidence over uncertain outcomes:

1. How does equilibrium price with confident agents compare to the case of no uncertainty?

2. How does equilibrium price with confident agents compare to the case of uncertainty with neutral agents possessing uniform priors over positions $t \leq T$?

3. As number of periods T grows large does price reflect the value η of the underlying asset?

In order to begin to answer these questions we will need to introduce a notion of confidence.

Definition 1. *In a T period investment setting, an agent is neutral in terms of confidence if their belief of moving in period t, $\Pr(t) = \mu_t$, is equal for all t so that $\mu_t = \frac{1}{T}$.*

Given this definition, in the uncertain case previously analyzed all agents were neutral. The concept of non-neutrality in terms of confidence takes the obvious definition.

Definition 2. *In a T period investment setting, an agent is non-neutral in terms of confidence if they are not confidence neutral. That is, if for some t_1, t_2 $\mu_{t_1} \neq \mu_{t_2}$.*

There are infinitely many ways in which an agent can stray from confidence neutrality. In order to narrow the scope of this definition, we will restrict attention to confidence over the first period. An agent will be said to be overconfident if she overweighs the probability of moving in the first period, and underconfident if she underweighs this probability.

Definition 3. *In a T period investment setting, an agent who has beliefs $\mu_1 = \frac{\gamma}{T}$, $\mu_t = \frac{T-\gamma}{T(T-1)}$ for $t \geq 2$ is <u>overconfident</u> if $\gamma > 1$ and <u>underconfident</u> if $\gamma < 1$.*

In the scope of this definition it is the belief of moving first that determines confidence. The probability of receiving the signal η in any other period is then spread uniformly across all other periods.

5.1 Confidence: The mindful investor

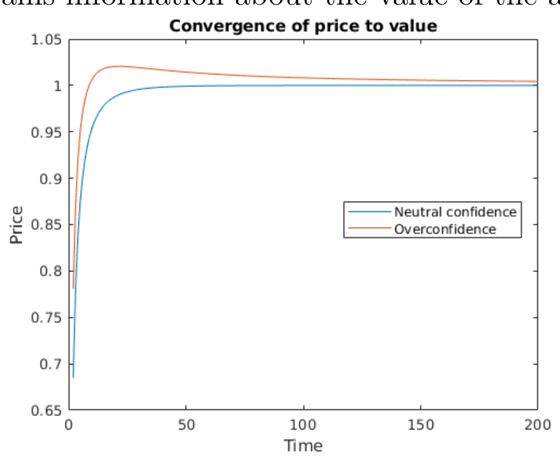
With these definitions regarding the confidence of investors over their uncertain position of movement we can define the equilibrium. Of course equilibrium behavior will depend on beliefs of other agents as well. In particular we begin with agents who are non-neutral ($\gamma \neq 1$) and take into account the non-neutrality of other agents. In this way we can think of these agents as "mindful" of their departure from neutrality and that other agents make the same departure. The market maker, unaware that investors behave anything other than fully rational, will set price exactly as before.

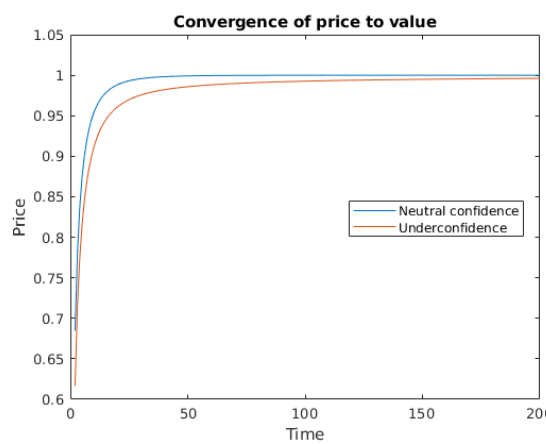
Proposition 9. *In a $T \in \mathbb{N} \cup \{\infty\}$ period investment setting, if informed agents do not know their position but hold a common belief $\mu_1 = \frac{\gamma}{T}$, uniform $\mu_t = \frac{T-\gamma}{T(T-1)}$ for $t > 2$ then* *there exists a linear equilibrium of the form* $p_t = p_0(1-\varphi) + \varphi p_{t-1} + \lambda\omega_t$ *where* p_0 *is the price before information* η *enters the market and* $x_t = \beta\eta$ *with*

$$\beta = \frac{(1-\varphi)}{\frac{\lambda}{T-1}\left[(T-1)-(2\varphi-1)(\gamma-1)\right]},\ \ \lambda = \sqrt{\varphi}(1-\varphi),\ \ \textit{and}\ \varphi = 1-\frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$$

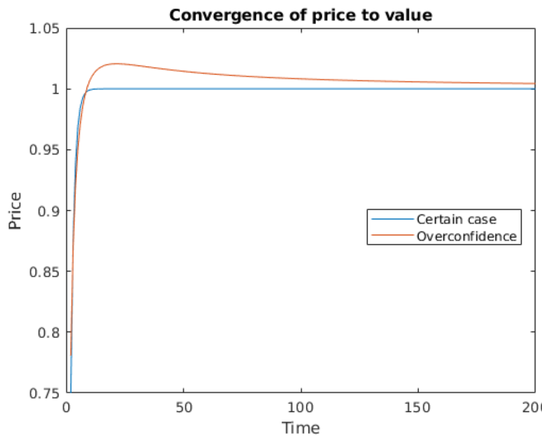
As the parameter β makes clear γ has a predictable effect on demand for the asset. Agents tend to demand more (less) if $\gamma > 1$ ($\gamma < 1$) as is easily seen in the denominator into which γ enters negatively. When $\gamma = 1$ we return to the case of neutral uncertainty described above. Having no way to know or reason to suspect non-neutrality the market maker behaves as in the case of neutral agents. If the market maker were able to compensate for non-neutrality the price would more closely resemble that of the neutral case.

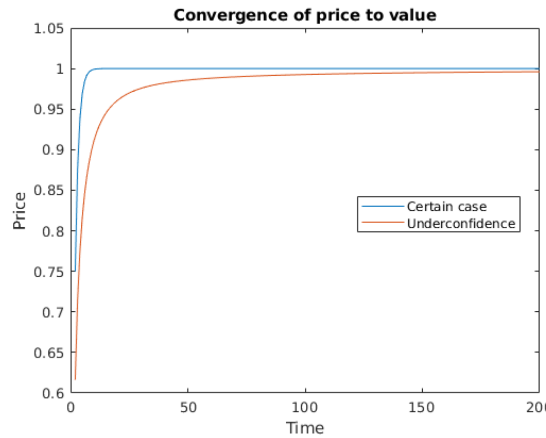
The figures below depict the movements of price as number of periods increases comparing the neutral case to the over/underconfident case when the true value of η is 1 and $p_0 = 1$. The figure on the left shows that in the case of overconfidence ($\gamma = 2$ here) the market price is always higher than in the confidence neutral case. Investors underweight the possibility that the price already contains information about the value η and thus demand more than they otherwise would. In fact as the graph shows, for early periods the price actually exceeds the value of η. With underconfidence we see even more caution than in the case of neutral uncertainty with the underconfident agent even further believing that the price already contains information about the value of the asset.





As the figures below demonstrate, comparing the results to initial market with no uncertainty paints an even more dramatic picture. In the overconfident case with just a few periods the price surpasses the geometric pricing schedule of no uncertainty. The underconfident case takes appreciably longer to integrate information about value into the price.





From these figures it does seem like eventually given enough periods the price does integrate the true value of η; it appears that after 150 periods of investment the value is almost completely incorporated. In fact as with the cases of no uncertainty and neutral uncertainty we can say this unambiguously.

Proposition 10. *For a $T \in \mathbb{N} \cup \{\infty\}$ period investment setting, if informed agents hold a common belief $\Pr(t=1) = \frac{\gamma}{T}$, uniform $\Pr(t) = \frac{T-\gamma}{T(T-1)}$ for $T > 2$, $\lim_{T\to\infty} \mathbb{E}[p_t] = p_0 + \eta$ and $\text{plim}_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} p_t = p_0 + \eta$.*

This proposition confirms that even if over(under)confident agents over(under)shoot the price for small T, for a large enough T all information about the value η is incorporated into the market price.

5.2 Confidence: The myopic investor

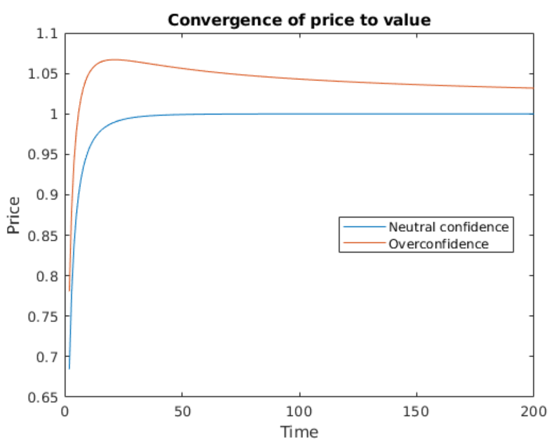
In the previous section we made the assumption that the non-neutral agent was "mindful" in the sense of being aware other agents share the same confidence bias. But it is at least as likely - if not more likely - that the agent is so confident that she believes she is the only agent with the informational advantage that increases (decreases) her likelihood of moving first. This would mean that in solving the maximization problem, it is assumed that other agents behave as if they were neutral investors, and the confident investor would dismiss the possibility of others also biasing their belief of moving first.

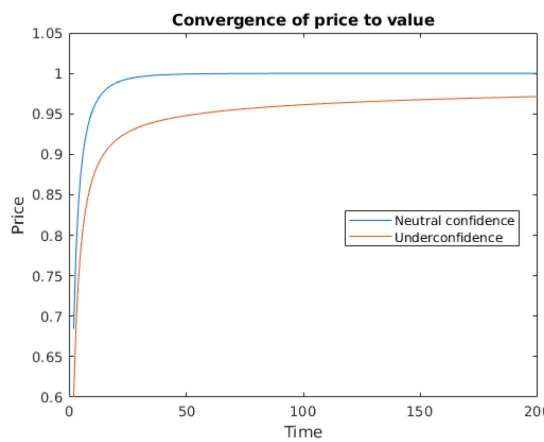
Proposition 11. *In a $T \in \mathbb{N} \cup \{\infty\}$ period investment setting, if informed agents do not know their position but believe $\mu_1 = \frac{\gamma}{T}$, $\mu_t = \frac{T-\gamma}{T(T-1)}$ for $t > 2$ and believe other agents have a uniform prior $\Pr(t) = \frac{1}{T}$ for all $t \leq T$ then there exists a linear equilibrium of the form $p_t = p_0(1-\varphi) + \varphi p_{t-1} + \lambda \omega_t$ where p_0 is the price before information η enters the market* *and* $x_t = \beta_t \eta$ *with*

$$\beta = \frac{1}{\lambda}\left[(1-\varphi) + \frac{(\gamma-1)(2\varphi-1)}{2(T-1)}\right], \ \lambda = \sqrt{\varphi}(1-\varphi), \ \textit{and } \varphi = 1 - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$$

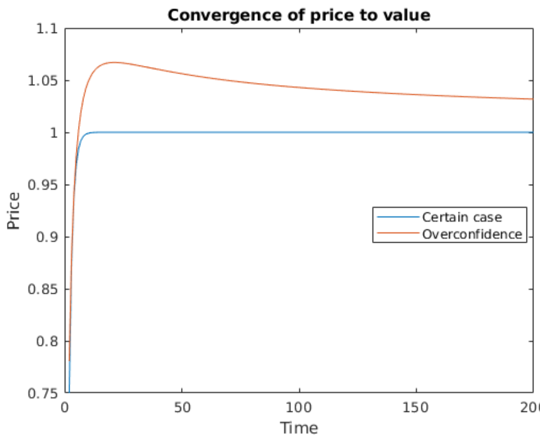
Again the market maker sets price as in the neutral case having no information about the confidence bias of investors. As we have seen, even knowing the existence and magnitude of a bias is insufficient because of they many ways investors can operationalize their bias, mindfully and myopically among them.

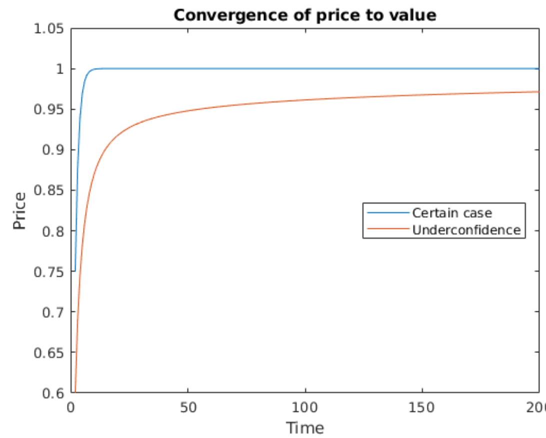
In the following figures we see the comparison of naive confidence and the neutral and certain cases with the true value of $\eta = 1$ and $p_0 = 0$ as in all previous analyses. We see again that demand is increasing in confidence γ which appears positively in both the β and δ terms. Clearly as $\gamma \to 1$ this approaches our previous equilibrium of confidence neutrality. The magnitude of this difference, however, is difficult to interpret from the first order conditions.





As the above figures show, we have the same pattern of the overconfident investor (left) investing so much more than in the neutral case that in very few periods price exceeds the true value of $\eta = 1$. Now, however, convergence of price to the true value of the asset seems questionable. Even after 200 periods the price of the over(under)confident investor over(under) estimates the value by about 3 percent; $p_{200} = 1.032$ ($p_{200} = 0.971$).

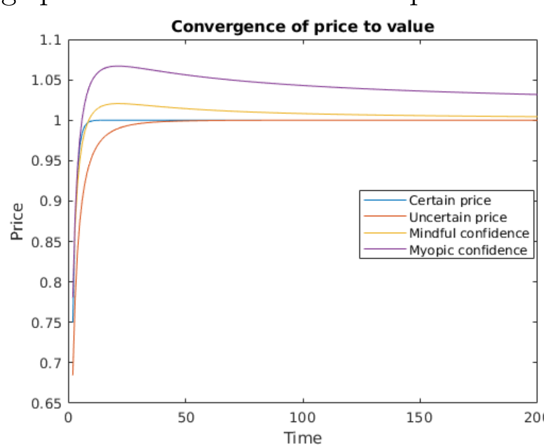


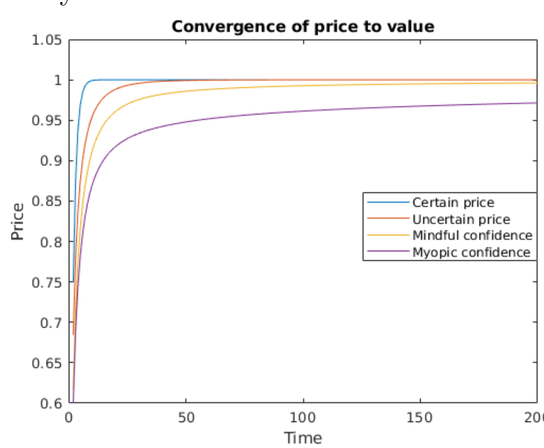


Despite the persistence in price distortion the bias introduces, we can in fact establish a limit result.

Proposition 12. *In the above equilibrium for which $T \in \mathbb{N} \cup \{\infty\}$ and agents hold belief* $\Pr(t = 1) = \frac{\gamma}{T}$, *uniform* $\Pr(t) = \frac{T-\gamma}{T(T-1)}$ *for $T > 2$, and believe other agents hold a uniform prior* $\Pr(t) = \frac{1}{T}$ *for all $t \leq T$,* $\lim_{T\to\infty} \mathbb{E}[p_t] = p_0 + \eta$ *and* $\operatorname{plim}_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} p_t = p_0 + \eta$.

While this limit result confirms that even in the case of myopic confidence we have that the asset price reflects its true value this convergence is extremely slow. This is of course due to the weighting of μ_1 that causes the agent to under/overestimate the probability that price already contains information about the value η from other agents. But even more than in the case of the mindfully confident investor, as more time periods/investors are added, the fact that each investor does not account for others' confidence γ prevents the bias from being spread over more and more periods as efficiently





The above figures show all of the cases together. As can be seen by the comparison, although mindful confidence suffers some pathology for small T, after a relatively short time it converges to the certain and neutral cases. The cases of myopic confidence, however, seem to take their time. While they reach $\eta \pm 3\%$ in relatively short order, with increasing time periods T this difference does not seem to relent. This is due to the slow convergence of $\varphi \to 1$. While all other prices depended on the convergence of $\varphi^T \to 0$, the convergence of this series depends on the convergence of φ. This is, of course, a direct result of agents not considering the confidence biases of other agents.

6 Concluding remarks

In a investment setting with informed investors, liquidity traders, and a market maker seeking to match the unknown value of an asset there are clear predictions in the case of certainty. Agents who face no uncertainty - either about the value of the asset or the number of investors who have acted before them - maximize profit in a linear equilibrium by halving the remaining value, leading to a rapid geometric convergence of the price to the asset's value. A generalization of this model wherein agents do not know the period in which they receive the informative signal, and as such do not know in which period they choose their demand, demonstrates a similar pattern that is slightly blunted by the uncertainty of how many investors had previously incorporated this profit relevant information into the price.

The introduction of confidence into this framework enriched the environment of uncertainty, allowing agents to differ in how they responded to not knowing the period when they receive the signal or how stale the information might be. Overconfident agents overweigh the probability of being first, leading to more demand than is profitable even in the case of certainty. This is reflected in a price that is higher than if the agents were neutral in terms of confidence, and possibly even higher than the value of the asset. Underconfident investors, conversely, tended to demand less of the asset than was profitable, leading to a price that lagged every other case and took longer to converge to the true value.

One operationalization of confidence - "mindful" confidence - led to a higher/lower price than was otherwise profitable, and yet as the number of periods grew large the price converged to the value of the asset rather quickly. This result is appealing in that confidence biases of agents are not too disruptive to the information value of asset price given a suitably large number of periods. And yet, while the concept of mindful confidence allowed for agents' beliefs to take into account that other agents share similar biases, the idea of being concurrently biased about one's own beliefs and mindful of others is in a sense contradictory.

An agent may be overconfident that they are particularly shrewd observers of the financial news, picking up on value-relevant signals before others can catch on. But if they take into account that others act in the same way is it true that they are more adept at interpreting information? They may maintain an edge over some investors, but if they plan investment strategies based on others taking the same factors in mind and undertaking the same line of iterative induction, the belief that these investors are as naïve as all other seems to break down.

Out of this contradiction arose the notion of “myopic” confidence whereby investors are confident that they move first and discount the possibility that other investors share confidence biases. This concept conforms more to our idea of what it means to be too confident. In the setting of myopic confidence we found an even more exaggerated departure in demand behavior and price as a measure of value. Even though the price of the asset in this case converges to the true value in the limit it does so extremely slowly. In fact given a 200 period time horizon we saw the asset price still failed to converge.

In each of these investment environments the asset value was perfectly known (granted, by different investors at different times) and this value never changes. It may be of comfort to the informational value of price that in all but the most extreme case of myopic confidence price converges quickly to the true value. But of course in a more dynamic setting the value is ever changing and signals are constantly being disseminated. If any of the above models were to be repeated every 5-10 periods the informational value at the limit would never have an opportunity to realize, leading to a potentially dramatic departure between the price of an asset and its value. Even if the effects of confidence did not accrue but canceled as a result of value fluctuation this still leaves the market with an undesirable level of volatility that reduces the appeal of investment and the ability of the market operate efficiently.

References

[1] Barber, B. and Odean, T. (2001), "Boys will be boys: Gender, overconfidence, and common stock investment." *The Quarterly Journal of Economics*, 116, pp. 261-292.

[2] Barber, B. and Odean, T. (2002), "Online investors: Do the slow die first?" *The Review of Financial Studies*, 15, pp. 455-487.

[3] Barber, B. and Odean, T. (2008), "All that glitters: The effect of attention and news on the buying behavior of individual and institutional investors." *The Review of Financial Studies*, 21, pp. 785-818.

[4] Gervais, S. and Odean, T. (2001), "Learning to be overconfident." *Review of Financial Studies*, 14, pp. 1-27.

[5] Handy, J. F. and Underwood, S. (2015), "Overconfidence and the Timing of Share Repurchases" (June 23, 2015). Available at SSRN: https://ssrn.com/abstract=2622097 or http://dx.doi.org/10.2139/ssrn.2622097

[6] Odean, T. (1998a), "Are investors reluctant to realize their losses?" *The Journal of Finance*, 53:5, pp. 1775-1798.

[7] Odean, T. (1998b), "Volume, volatility, price, and profit when all traders are above average." *The Journal of Finance*, 53:6, pp. 1887-1934.

[8] Peterson, R.L. (2005) "The neuroscience of investing: fMRI of the reward system." *Brain Research Bulletin*, 67, pp. 391-397.

[9] Shu, P.E., Yeh, Y.H., Chiang, T.L., and Hung, J.Y. (2013). "Managerial overconfidence and share repurchases." *International Review of Finance*, 13:1, pp. 39-65.

A1. Appendix

Proof of Proposition 1. Conjecture a linear price equilibrium of the form $p_t = p_{t-1} + \lambda_t \omega_t$ and consider the first agent's optimization problem. The anticipated market price is $p_1 = p_0 + \lambda_1 \omega_1$ so the agent solves

$$\max_{x_1} x_1 \mathbb{E}_1[v_0 + \eta - p_1] = x_1(\eta - \mathbb{E}_1[\lambda_1 \omega_1]) = x_1(\eta - \lambda_1 x_1)$$

which yields the optimal quantity $x_1 = \frac{\eta}{2\lambda_1}$.

By induction, for a $t > 1$ conjecture that the optimal investment given λ_t is $x_t = \frac{1}{2^t \lambda_t}\eta + Z_t$ with $Z_t \sim N(0, V_t)$. Then the agent in period $t+1$ solves

$$\max_{x_{t+1}} x_{t+1} \mathbb{E}_{t+1}[v_0 + \eta - p_{t+1}] = \max_{x_{t+1}} x_{t+1} \mathbb{E}_{t+1}\left[v_0 + \eta - \left(p_0 + \sum_{i=1}^{t} \lambda_i \omega_i + \lambda_{t+1}\omega_{t+1}\right)\right]$$

since $\mathbb{E}[p_0] = \mathbb{E}[v_0]$. Given the fact that $\mathbb{E}_{t+1}[\omega_{t+1}] = x_{t+1}$, this yields equilibrium $x_{t+1} = \frac{\eta - \sum_{i=1}^{t} \lambda_i \omega_i}{2\lambda_{t+1}}$. By the induction assumption this holds for all preceding t so that $x_t = \frac{\eta - \sum_{i=1}^{t-1} \lambda_i \omega_i}{2\lambda_t}$ and $2\lambda_t x_t = \eta - \sum_{i=1}^{t-1} \lambda_i \omega_i$. Also notice that

$$\begin{aligned} x_{t+1} &= \frac{\eta - \sum_{i=1}^{t} \lambda_i \omega_i}{2\lambda_{t+1}} = \frac{\eta - \sum_{i=1}^{t-1} \lambda_i \omega_i - \lambda_t \omega_t}{2\lambda_{t+1}} = \frac{\eta - \sum_{i=1}^{t-1} \lambda_i \omega_i - \lambda_t x_t - \lambda_t z_t}{2\lambda_{t+1}} = \frac{2\lambda_t x_t - \lambda_t x_t - \lambda_t z_t}{2\lambda_{t+1}} \\ &= \frac{\lambda_t x_t - \lambda_t z_t}{2\lambda_{t+1}} \end{aligned}$$

By the induction assumption, $\lambda_t x_t = \frac{1}{2^t}\eta + \lambda_t Z_t$ and

$$\begin{aligned} x_{t+1} &= \left(\frac{1}{2\lambda_{t+1}}\right)\left[\frac{1}{2^t}\eta + \lambda_t Z_t - \lambda_t z_t\right] \\ &= \left(\frac{1}{\lambda_{t+1}}\right)\left[\frac{1}{2^{t+1}}\eta + \frac{1}{2}\lambda_t Z_t - \frac{1}{2}\lambda_t z_t\right] \end{aligned}$$

so that $x_{t+1} = \frac{1}{2^{t+1}\lambda_{t+1}}\eta + Z_{t+1}$ where $Z_{t+1} = \frac{\lambda_t(Z_t - z_t)}{2\lambda_{t+1}} \sim N(0, \frac{\lambda_t^2(V_t + \Omega)}{4\lambda_{t+1}^2})$. Letting $V_{t+1} = \frac{\lambda_t^2(V_t+\Omega)}{4\lambda_{t+1}^2}$ gives $Z_{t+1} \sim N(0, V_{t+1})$. Then $x_{t+1} = \beta_{t+1}\eta + Z_{t+1}$ where $\beta_{t+1} = \frac{1}{2^{t+1}\lambda_{t+1}}$ Thus by induction this holds for all $t \leq T$.

Now consider the problem of the market maker. In each period the market maker sets the price in order to match the value of the asset. That is $p_t = \mathbb{E}[v_0 + \eta | \omega_t, h_t]$ where again $\omega_t = x_t + z_t$ is market demand, and h_t is the historical series of market demand. Then in period 1

$$\begin{aligned} p_1 &= \mathbb{E}[v_0 + \eta | \omega_1] = p_0 + \mathbb{E}[\eta | \beta_1 \eta + z_1] = p_0 + \frac{1}{\beta_1}\mathbb{E}[\beta_1 \eta | \beta_1 \eta + z_1] \\ &= p_0 + \frac{\beta_1 \sigma^2}{\beta_1^2 \sigma^2 + \Omega}\omega_1 \end{aligned}$$

and so $\lambda_1 = \frac{\beta_1\sigma^2}{\beta_1^2\sigma^2+\Omega}$. Moreover since $\beta_1 = \frac{1}{2\lambda_1}$ then

$$\begin{aligned}
\lambda_1 &= \frac{\frac{1}{2\lambda_1}\sigma^2}{\left(\frac{1}{2\lambda_1}\right)^2\sigma^2+\Omega} = \frac{\sigma^2}{\left(\frac{1}{2\lambda_1}\right)\sigma^2+2\Omega\lambda_1} \\
2\sigma^2 &= \sigma^2 + 4\Omega\lambda_1^2 \\
\lambda_1 &= \sqrt{\frac{\sigma^2}{4\Omega}}
\end{aligned}$$

so that

$$p_1 = \mathbb{E}[v_0+\eta|\omega_1] = p_0 + \left(\sqrt{\frac{\sigma^2}{4\Omega}}\right)\omega_1$$

and the variance of the estimate of η is

$$V_1 = \frac{\beta_1^2\sigma^2\Omega}{\beta_1^2\sigma^2+\Omega} = \frac{(\frac{1}{2\lambda_1})^2\sigma^2\Omega}{(\frac{1}{2\lambda_1})^2\sigma^2+\Omega} = \frac{\sigma^2\Omega}{\sigma^2+4\lambda_1^2\Omega}$$

which reduces to $V_1 = \frac{\Omega}{2}$.

Consider a general $t > 1$. The market maker again sets price to match the expected value of the asset so that

$$\begin{aligned}
p_t &= \mathbb{E}[v_0+\eta|\omega_t,h_t] = \mathbb{E}[v_0+\eta|x_t+z_t,h_t] = \mathbb{E}\left[v_0+\eta\left|\frac{1}{2\lambda_t}\left(\eta-\sum_{i=1}^{t}\lambda_i\omega_i\right)+z_t,h_t\right.\right] \\
&= \mathbb{E}\left[v_0+\eta\left|\frac{1}{2\lambda_t}\left(\eta-(p_{t-1}-p_0)\right)+z_t,h_t\right.\right] = \mathbb{E}\left[v_0+\eta\left|\frac{1}{2\lambda_t}\left(v_0+\eta-p_{t-1}\right)+z_t,h_t\right.\right] \\
&= 2\lambda_t\mathbb{E}\left[\frac{1}{2\lambda_t}(v_0+\eta-p_{t-1})\left|\frac{1}{2\lambda_t}(v_0+\eta-p_{t-1})+z_t,h_t\right.\right]+p_{t-1}
\end{aligned}$$

Since p_{t-t} was the previous expectation of $v_0+\eta$, $\frac{1}{2\lambda_t}(v_0+\eta-p_{t-1}) \sim N(0,\left(\frac{1}{2\lambda_t}\right)^2 V_{t-1})$ where V_{t-1} is the previous variance estimate of η. Suppose that $V_{t-1} = \frac{\Omega}{2}$. If $V_t = \frac{\Omega}{2}$ as well then by induction this is the variance of η for all $t > 1$. Then the above expectation becomes

$$p_t = p_{t-1} + 2\lambda_t\frac{\left(\frac{1}{2\lambda_t}\right)^2\frac{\Omega}{2}}{\left(\frac{1}{2\lambda_t}\right)^2\frac{\Omega}{2}+\Omega}\omega_t = p_{t-1} + \frac{\left(\frac{1}{2\lambda_t}\right)}{\left(\frac{1}{2\lambda_t}\right)^2+2}\omega_t = p_{t-1} + \frac{1}{\left(\frac{1}{2\lambda_t}\right)+4\lambda_t}\omega_t$$

Then

$$\lambda_t = \frac{1}{\left(\frac{1}{2\lambda_t}\right)+4\lambda_t} \Rightarrow 8\lambda_t^2+1 = 2$$

so that $\lambda_t = \frac{1}{\sqrt{8}}$. Also,

$$V_t = \frac{\left(\frac{1}{2\lambda_t}\right)^2 V_{t-1}\Omega}{\left(\frac{1}{2\lambda_t}\right)^2 V_{t-1} + \Omega} = \frac{\left(\frac{1}{2\lambda_t}\right)^2 \frac{\Omega}{2}\Omega}{\left(\frac{1}{2\lambda_t}\right)^2 \frac{\Omega}{2} + \Omega} = \frac{\left(\frac{8}{4}\right)\frac{1}{2}\Omega}{\left(\frac{8}{4}\right)\frac{1}{2} + 1} = \frac{\Omega}{2}$$

and by induction $V_t = \frac{\Omega}{2}$ for all $t > 1$.

Since it has been shown that $\beta_t = \frac{1}{2^t \lambda_t}$, then by the formulation of λ_t it holds that $\beta_1 = \sqrt{\frac{\Omega}{\sigma^2}}$ and $\beta_t = \frac{\sqrt{2}}{2^{t-1}}$ for $t > 1$ as desired. □

Proof of Proposition 2. Lastly, $p_t = p_{t-1} + \lambda_t \omega_t$ so inductively

$$\begin{aligned} p_t &= p_0 + \sum_{i=1}^{t} \lambda_i \omega_i = p_0 + \sum_{i=1}^{t} \lambda_i (x_i + z_i) = p_0 + \sum_{i=1}^{t} \lambda_i \left(\frac{1}{2^i \lambda_i} \eta + Z_i \right) + \lambda_i z_i \\ &= p_0 + \sum_{i=1}^{t} \left(\frac{1}{2^i} \right) \eta + \sum_{i=1}^{t} \lambda_i (Z_i + z_i) = p_0 + \left[1 - \left(\frac{1}{2} \right)^t \right] \eta + Z'_t \end{aligned}$$

where Z'_t is a linear combination of independently normally distributed random variables with mean zero so $Z'_t \sim N(0, V'_t)$ □

Proof of Lemma 1. Suppose the agent demands $x_t = \beta_t \eta$ where $\beta_t = \frac{y}{\lambda_t}$. The market maker sets price so that $p_t = \mathbb{E}[v_t | \beta_t \eta + z_t] = \left(\frac{V_{t-1}\beta_t}{V_{t-1}\beta_t^2 + \Omega} \right) \omega_t$, where V_{t-1} is the market maker's prior belief of the informative signal's variance. Then

$$\beta_t \lambda_t = \frac{\beta_t^2 V_{t-1}}{\beta_t^2 V_{t-1} + \Omega} = y$$

This yields $\beta_t^2 V_{t-1} = \frac{y\Omega}{1-y}$. When the market maker updates variance of the agent's signal given that liquidity noise $z_t \sim N(0, \Omega)$,

$$V_t = \frac{\beta_t^2 V_{t-1}\Omega}{\beta_t^2 V_{t-1} + \Omega} = \frac{\frac{y\Omega}{1-y}\Omega}{\frac{y\Omega}{1-y} + \Omega} = \frac{y\Omega^2}{y\Omega + (1-y)\Omega} = y\Omega$$

Since this was independent of the value V_{t-1}, variance will be $V_t = y\Omega$ for every period with only the possible exception of V_0 before variance can be updated from the prior belief. □

Proof of Proposition 3. The previous proof shows that $x_1 = \frac{\eta}{2\lambda_1}$ and the optimal quantity for the agent in period t is $x_t = \frac{\lambda_{t-1}x_{t-1} - \lambda_{t-1}z_{t-1}}{2\lambda_t}$. Then for $t = 2$, $x_2 = \frac{\lambda_1 x_1 - \lambda_1 z_1}{2\lambda_2} = \frac{\eta}{2^2 \lambda_2} - \frac{1}{\lambda_2}\left(\frac{1}{2}\right)\lambda_1 z_t$ so that $\beta_2 = \frac{1}{4\lambda_2}$ and $Z_2 = -\frac{1}{\lambda_2}\sum_{i=1}^{2-1}\left(\frac{1}{2}\right)^{t-i}\lambda_i z_i$ so the result holds for $t = 2$.

By induction, suppose that $x_t = \frac{1}{2^t \lambda_t}\eta - \frac{1}{\lambda_t}\sum_{i=1}^{t-1}\left(\frac{1}{2}\right)^{t-i}\lambda_i z_i$ for $t \geq 2$. Then $\lambda_t x_t = \frac{1}{2^t}\eta - \sum_{i=1}^{t-1}\left(\frac{1}{2}\right)^{t-i}\lambda_i z_i$ and given that $x_{t+1} = \frac{\lambda_t x_t - \lambda_t z_t}{2\lambda_{t+1}}$,

$$\begin{aligned}
x_{t+1} &= \left(\frac{1}{2\lambda_{t+1}}\right)\left[\left(\frac{1}{2^t}\eta - \sum_{i=1}^{t-1}\left(\frac{1}{2}\right)^{t-i}\lambda_i z_i\right) - \lambda_t z_t\right] \\
&= \left(\frac{1}{\lambda_{t+1}}\right)\left[\frac{1}{2^{t+1}}\eta - \sum_{i=1}^{t-1}\left(\frac{1}{2}\right)^{t+1-i}\lambda_i z_i - \frac{1}{2}\lambda_t z_t\right] \\
&= \left(\frac{1}{\lambda_{t+1}}\right)\left[\frac{1}{2^{t+1}}\eta - \sum_{i=1}^{t}\left(\frac{1}{2}\right)^{t+1-i}\lambda_i z_i\right] \\
&= \left(\frac{1}{\lambda_{t+1}}\right)\left[\frac{1}{2^{t+1}}\eta - \sum_{i=1}^{(t+1)-1}\left(\frac{1}{2}\right)^{(t+1)-i}\lambda_i z_i\right]
\end{aligned}$$

so that $x_{t+1} = \frac{1}{2^{t+1}\lambda_{t+1}}\eta - \frac{1}{\lambda_{t+1}}\sum_{i=1}^{(t+1)-1}\left(\frac{1}{2}\right)^{(t+1)-i}\lambda_i z_i$. Thus $Z_{t+1} = -\frac{1}{\lambda_{t+1}}\sum_{i=1}^{(t+1)-1}\left(\frac{1}{2}\right)^{(t+1)-i}\lambda_i z_i$ and by induction the result holds for all $t \leq T$.

As noted in proposition 2,

$$p_t = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right]\eta + Z'_t$$

where $Z_t\prime = \sum_{i=2}^{t}\lambda_i Z_i + \sum_{i=1}^{t}\lambda_i z_i$. From the above $Z_i = -\frac{1}{\lambda_i}\sum_{j=1}^{i-1}\left(\frac{1}{2}\right)^{i-j}\lambda_j z_j$ so that

$$\begin{aligned}
\sum_{i=1}^{t}\lambda_i Z_i &= -\sum_{i=2}^{t}\sum_{j=1}^{i-1}\left(\frac{1}{2}\right)^{i-j}\lambda_j z_j = -\sum_{j=1}^{t-1}\sum_{i=j+1}^{t}\left(\frac{1}{2}\right)^{i-j}\lambda_j z_j = -\sum_{j=1}^{t-1}\lambda_j z_j\sum_{i=1}^{t-j}\left(\frac{1}{2}\right)^{i} \\
&= -\sum_{j=1}^{t-1}\lambda_j z_j\left[1 - \left(\frac{1}{2}\right)^{t-j}\right]
\end{aligned}$$

Together,

$$\begin{aligned}
\sum_{i=2}^{t}\lambda_i Z_i + \sum_{i=1}^{t}\lambda_i z_i &= \sum_{i=1}^{t}\lambda_i z_i - \sum_{j=1}^{t-1}\lambda_j z_j\left[1 - \left(\frac{1}{2}\right)^{t-j}\right] = \sum_{i=1}^{t}\lambda_i z_i - \sum_{i=1}^{t-1}\lambda_i z_i\left[1 - \left(\frac{1}{2}\right)^{t-i}\right] \\
&= \lambda_t z_t + \sum_{i=1}^{t-1}\lambda_i z_i\left(\frac{1}{2}\right)^{t-i} = \sum_{i=1}^{t}\lambda_i z_i\left(\frac{1}{2}\right)^{t-i}
\end{aligned}$$

Then $Z'_t = \sum_{i=1}^{t}\lambda_i z_i\left(\frac{1}{2}\right)^{t-i}$ and $p_t = p_0 + \left[1 - \left(\frac{1}{2}\right)^t\right]\eta + \sum_{i=1}^{t}\lambda_i z_i\left(\frac{1}{2}\right)^{t-i}$ $\square$

Proof of Proposition 4. Define the partial series $X_T = \frac{1}{T}\sum_{t=1}^{T} p_t$.

$$X_T = \frac{1}{T}\sum_{t=1}^{T} p_t = \frac{1}{T}\sum_{t=1}^{T}\left[p_0 + \left(1-\left(\frac{1}{2}\right)^t\right)\eta + \sum_{i=1}^{t}\lambda_i z_i \left(\frac{1}{2}\right)^{t-i}\right]$$
$$= p_0 + \eta - \frac{\eta}{T}\left(1-\left(\frac{1}{2}\right)^T\right) + \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\lambda_i z_i \left(\frac{1}{2}\right)^{t-i}$$

For each T variance of X_T (given that the z_i are independent) is

$$Var(X_T) = \mathbb{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\lambda_i z_i\left(\frac{1}{2}\right)^{t-i}\right)^2\right] = \frac{1}{T^2}\frac{\Omega}{8}\sum_{t=1}^{T}\sum_{i=1}^{t}\left(\frac{1}{2}\right)^{2(t-i)}$$
$$= \frac{\Omega}{8T^2}\sum_{t=1}^{T}\sum_{j=0}^{t-1}\left(\frac{1}{4}\right)^j = \frac{\Omega}{8T^2}\sum_{t=1}^{T}\left(\frac{\left(1-\left(\frac{1}{4}\right)^t\right)}{1-\frac{1}{4}}\right)$$
$$= \frac{\Omega}{6T^2}\sum_{t=1}^{T}\left(1-\left(\frac{1}{4}\right)^t\right) = \frac{\Omega}{6T} - \frac{\Omega}{6T^2}\left(\left(\frac{1}{4}\right)\frac{\left(1-\left(\frac{1}{4}\right)^t\right)}{1-\frac{1}{4}}\right)$$
$$= \frac{\Omega}{6T} - \frac{\Omega}{18T^2}\left(1-\left(\frac{1}{4}\right)^T\right) = \frac{\Omega}{6T} - \frac{\Omega}{18T^2} + \frac{\Omega}{18T^2}\left(\frac{1}{4}\right)^T$$

Let $\varepsilon > 0$. By Markov's inequality,

$$\Pr(|X_T - (p_0+\eta)| \geq \varepsilon) \leq \frac{\mathbb{E}\left[\left(p_0+\eta-\frac{\eta}{T}\left(1-\left(\frac{1}{2}\right)^T\right)+\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\lambda_i z_i\left(\frac{1}{2}\right)^{t-i}-(p_0+\eta)\right)^2\right]}{\varepsilon^2}$$
$$= \frac{1}{\varepsilon^2}\left\{\left[\frac{\eta}{T}\left(1-\left(\frac{1}{2}\right)^T\right)\right]^2 + \mathbb{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\lambda_i z_i\left(\frac{1}{2}\right)^{t-i}\right)^2\right]\right\}$$
$$= \frac{\left(\frac{\eta}{T}\right)^2\left(1-\left(\frac{1}{2}\right)^T\right)^2 + \frac{\Omega}{6T} - \frac{\Omega}{18T^2} + \frac{\Omega}{18T^2}\left(\frac{1}{4}\right)^T}{\varepsilon^2}$$

so that

$$\Pr(|X_T - (p_0+\eta)| < \varepsilon) \leq 1 - \frac{\left(\frac{\eta}{T}\right)^2\left(1-\left(\frac{1}{2}\right)^T\right)^2 + \frac{\Omega}{6T} - \frac{\Omega}{18T^2} + \frac{\Omega}{18T^2}\left(\frac{1}{4}\right)^T}{\varepsilon^2} \longrightarrow 1 \text{ as } T \to \infty$$

so that $\text{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} p_t = p_0 + \eta$.

Note that $\lambda_1^2 = \frac{\sigma^2}{4\Omega}$. This was ignored for expositional clarity because $Var(\frac{1}{T}\lambda_1 z_1) = \frac{\sigma^2}{4T^2} \to 0 \iff \frac{\Omega}{8T^2} \to 0$. $\square$

Proof of Proposition 5. Conjecture a linear price equilibrium of the form $p_t = p_0(1 - \varphi_{t-1}) + \varphi_{t-1}p_{t-1} + \lambda_t\omega_t$ for $t > 1$ where p_0 is the price before information η entered the market. The agent seeks to maximize

$$\begin{aligned} x_i\mathbb{E}[v_t - p_t] &= x_i\mathbb{E}[v_0 + \eta - p_0(1-\varphi_t) - \varphi_t p_{t-1} - \lambda_i\omega_t] \\ &= x_i\left(\mathbb{E}[v_0] + \eta - p_0(1-\varphi_t) - \varphi_t\mathbb{E}[p_t] - \lambda_i x_i\right) \end{aligned}$$

Which is maximized for $x_i = \frac{\varphi_t p_0 + \eta - \varphi_t\mathbb{E}[p_{t-1}]}{2\lambda_i}$ since $p_0 = \mathbb{E}[v_0]$. If the agent moves in the first period then price is p_0 while if in the second period price is $p_1 = p_0 + \lambda_1\omega_1$. With a prior belief of moving first $\Pr_i(F) = \gamma_i$, the expected price is $p_{t-1} = \gamma_i p_0 + (1-\gamma_i)p_1 = p_0 + (1-\gamma_i)\lambda\omega_j$. For agent i, the expected demand of any other period is $\mathbb{E}[\omega_j] = \mathbb{E}[x_j]$, so that demand can be written as

$$\begin{aligned} x_i &= \frac{\eta + \varphi_j(p_0 - p_{t-1})}{2\lambda_i} = \frac{\eta - \varphi_j(1-\gamma_i)\lambda_j\mathbb{E}[x_j]}{2\lambda_i} \\ &= \frac{\eta}{2\lambda_i} - \frac{\varphi_j(1-\gamma_i)\lambda_j\mathbb{E}[x_j]}{2\lambda_i} \end{aligned}$$

and as a result of the common prior belief expected quantity $\mathbb{E}[x_j]$ is

$$\mathbb{E}[x_j] = \frac{\eta}{2\lambda_j} - \frac{\varphi_i\gamma_i\lambda_i}{2\lambda_j}x_i$$

Thus in equilibrium

$$x_i = \frac{2 - \varphi_j(1-\gamma_i)}{\lambda_i[4 - \varphi_i\varphi_j\gamma_i(1-\gamma_i)]}\eta \quad \text{and} \quad \mathbb{E}[x_j] = \frac{2 - \varphi_i\gamma_i}{\lambda_j[4 - \varphi_i\varphi_j\gamma_i(1-\gamma_i)]}\eta$$

so that

$$\beta_i(\gamma_i) = \frac{2 - \varphi_j(1-\gamma_i)}{\lambda_i[4 - \varphi_i\varphi_j\gamma_i(1-\gamma_i)]} \quad \text{and} \quad \mathbb{E}_i[\beta_j(\gamma_j)] = \frac{2 - \varphi_i\gamma_i}{\lambda_j[4 - \varphi_i\varphi_j\gamma_i(1-\gamma_i)]}$$

The market maker sets price so that $p_t = \mathbb{E}[v_t|\omega_t, h_t]$ so

$$p_t = \mathbb{E}[v_t|\omega_t, h_t] = \mathbb{E}[v_0 + \eta|\omega_t, h_t] = p_0 + \mathbb{E}[\eta|\omega_t, h_t] = p_0 + \frac{1}{\beta_t}\mathbb{E}[\beta_t\eta|\beta_t\eta + z_t, h_t]$$

Recall that $z_t \sim N(0, \Omega)$ and the prior belief on the value is $\eta \sim N(p_{t-1} - p_0, V_{t-1})$. Then $\beta_t\eta \sim N(\beta_t(p_{t-1} - p_0), \beta_t^2 V_{t-1})$ and

$$p_t = p_0 + \frac{\Omega\beta_t(p_{t-1} - p_0) + \beta_t^2 V_{t-1}\omega_t}{\beta_t(\Omega + \beta_t^2 V_{t-1})} = p_0 + \frac{\Omega(p_{t-1} - p_0) + \beta_t V_{t-1}\omega_t}{\Omega + \beta_t^2 V_{t-1}}$$

so that

$$\lambda_t = \frac{\beta_t V_{t-1}}{\Omega + \beta_t^2 V_{t-1}} \text{ and } \varphi_t = \frac{\Omega}{\Omega + \beta_t^2 V_{t-1}}$$

Together these imply that $\beta_t \lambda_t = 1 - \varphi_t = \frac{\beta_t^2 V_{t-1}}{\Omega + \beta_t^2 V_{t-1}}$, this solves to $\beta_t^2 = \frac{(1-\varphi_t)\Omega}{V_{t-1}\varphi_t}$.

Since the prior belief of the market maker is determined by the distribution of η, for $t = 1$ $V_{t-1} = \sigma^2$. Notice also that for any V_{t-1} updated variance is

$$V_t = \frac{\beta_t^2 V_{t-1}\Omega}{\beta_t^2 V_{t-1} + \Omega} = \frac{\left(\frac{(1-\varphi_t)\Omega}{\varphi_t}\right)\Omega}{\frac{(1-\varphi_t)\Omega}{\varphi_t} + \Omega} = \frac{(1-\varphi_t)\Omega}{(1-\varphi_t) + \varphi_t} = (1-\varphi_t)\Omega$$

Then given that $\beta_t^2 = \frac{(1-\varphi_t)\Omega}{V_{t-1}\varphi_t}$, substitution for $V_0 = \sigma^2$ and $V_1 = (1-\varphi_1)\Omega$ gives that

$$\beta_1 = \sqrt{\frac{(1-\varphi_1)\Omega}{\varphi_1 \sigma^2}} \quad \text{and} \quad \beta_2 = \sqrt{\frac{(1-\varphi_2)}{\varphi_2(1-\varphi_1)}}$$

Substitution into the relationship $\lambda_t = \frac{\beta_t V_{t-1}}{\Omega + \beta_t^2 V_{t-1}}$ gives

$$\lambda_1 = \sqrt{\frac{\varphi_1(1-\varphi_1)\sigma^2}{\Omega}} \quad \lambda_2 = \sqrt{\varphi_2(1-\varphi_2)(1-\varphi_1)}$$

From the definition of β_t above for i, j, we can also express this as $\beta_i \lambda_i = \frac{2-\varphi(1-\gamma_i)}{[4-\varphi^2\gamma_i(1-\gamma_i)]}$

$$1 - \varphi_i = \frac{2 - \varphi_j(1-\gamma_i)}{[4 - \varphi_i\varphi_j\gamma_i(1-\gamma_i)]}$$

Given the common prior assumption that $\gamma_1 = 1 - \gamma_2 = \gamma$, we conclude with the two remaining equilibrium conditions

$$\varphi_1\varphi_2\gamma(1-\gamma)(\varphi_1 - 1) = 4\varphi_1 - \varphi_2(1-\gamma) - 2$$
$$\varphi_1\varphi_2\gamma(1-\gamma)(\varphi_2 - 1) = 4\varphi_2 - \varphi_1\gamma - 2$$

$\square$

Proof of Proposition 6. Conjecture a linear price equilibrium of the form $p_t = p_0(1 - \varphi_{t-1}) + \varphi_{t-1}p_{t-1} + \lambda_t\omega_t$ where p_0 is the price before information η entered the market. The market maker sets price such that $p_t = \mathbb{E}[v_t|\omega_t]$. Then

$$p_t = \mathbb{E}[v_t|\omega_t] = \mathbb{E}[v_0 + \eta|x_t + z_t] = p_0 + \mathbb{E}[\eta|\beta_t\eta + z_t] = p_0 + \frac{1}{\beta_t}\mathbb{E}[\beta_t\eta|\beta_t\eta + z_t]$$

Recall that $z_t \sim N(0, \Omega)$ and the prior belief on the value is $\eta \sim N(p_{t-1} - p_0, V_{t-1})$. Then $\beta_t \eta \sim N(\beta_t(p_{t-1} - p_0), \beta_t^2 V_{t-1})$ and

$$p_t = p_0 + \frac{\Omega\beta_t(p_{t-1} - p_0) + \beta_t^2 V_{t-1}\omega_t}{\beta_t(\Omega + \beta_t^2 V_{t-1})} = p_0 + \frac{\Omega(p_{t-1} - p_0) + \beta_t V_{t-1}\omega_t}{\Omega + \beta_t^2 V_{t-1}}$$

so that

$$\lambda_t = \frac{\beta_t V_{t-1}}{\Omega + \beta_t^2 V_{t-1}} \text{ and } \varphi_t = \frac{\Omega}{\Omega + \beta_t^2 V_{t-1}}$$

Together these imply that $\beta_t \lambda_t = 1 - \varphi_t = \frac{\beta_t^2 V_{t-1}}{\Omega + \beta_t^2 V_{t-1}}$, this solves to $\beta_t^2 = \frac{(1-\varphi_t)\Omega}{V_{t-1}\varphi_t}$. Since $\beta_t^2 V_{t-1} = \frac{(1-\varphi_t)\Omega}{\varphi_t}$, then for any V_{t-1} updated variance is

$$V_t = \frac{\beta_t^2 V_{t-1}\Omega}{\beta_t^2 V_{t-1} + \Omega} = \frac{\left(\frac{(1-\varphi_t)\Omega}{\varphi_t}\right)\Omega}{\frac{(1-\varphi_t)\Omega}{\varphi_t} + \Omega} = \frac{(1-\varphi_t)\Omega}{(1-\varphi_t) + \varphi_t} = (1-\varphi_t)\Omega$$

Then given that $\beta_t^2 = \frac{(1-\varphi_t)\Omega}{V_{t-1}\varphi_t}$ and $V_{t-1} = (1 - \varphi_{t-1})\Omega$ we have the following.

$$\beta_t = \sqrt{\frac{(1-\varphi_t)}{\varphi_t(1-\varphi_{t-1})}}$$

Given the form of β_t

$$\lambda_t = \frac{\sqrt{\frac{(1-\varphi_t)}{\varphi_t(1-\varphi_{t-1})}}\Omega(1-\varphi_{t-1})}{\Omega + \left(\frac{(1-\varphi_t)}{\varphi_t(1-\varphi_{t-1})}\right)\Omega(1-\varphi_{t-1})} = \frac{\sqrt{\varphi_t(1-\varphi_t)(1-\varphi_{t-1})}}{\varphi_t + 1 - \varphi_t} = \sqrt{\varphi_t(1-\varphi_t)(1-\varphi_{t-1})}$$

The agent seeks to maximize

$$\begin{aligned} x\mathbb{E}[v_t - p_t] &= x\mathbb{E}[v_0 + \eta - p_0(1-\varphi_t) - \varphi_t p_{t-1} - \lambda_t\omega_t] \\ &= x\left(\mathbb{E}[v_0] + \eta - \mathbb{E}[p_0(1-\varphi_t) - \varphi_t p_{t-1}] - \lambda_t x\right) \end{aligned}$$

Which is maximized for $x = \frac{\eta - \mathbb{E}[\varphi_t(p_{t-1} - p_0)]}{2\lambda_t}$ since $\mathbb{E}[p_0] = \mathbb{E}[v_0]$. If the agent moves in the first period the prevailing price is p_0. If the agent moves in the second period the prevailing price is $p_1 = p_0 + \lambda_1\omega_1$. If the agent moves in the third period the prevailing price is $p_2 = p_0 + \varphi_2\lambda_1\omega_1 + \lambda_2\omega_2$. Inductively if the agent moves in period t then $t-1$ agents move before and

$$p_{t-1} = p_0 + \lambda_{t-1}\omega_{t-1} + \sum_{i=2}^{t-1} \lambda_{t-i}\omega_{t-i} \prod_{j=1}^{i-1} \varphi_{t-j}$$

If there are T periods and each agent has the belief μ_t that they are moving in period t and since $\omega_t = x_t + z_t$ and z_t are independently distributed with zero mean, $\omega_t = x_t$ is the expectation for each period. Then

$$\mathbb{E}[\varphi_t(p_{t-1} - p_0)] = \sum_{t=1}^{T} \mu_t \mathbb{E}_t[\varphi_t(p_{t-1} - p_0)] = \sum_{t=2}^{T} \mu_t \left(\sum_{i=1}^{t-1} \lambda_{t-i} x_{t-i} \prod_{j=0}^{i-1} \varphi_{t-j} \right)$$

where the outer summation starts from $t = 2$ because when $t = 1$ the agent moves in the first period and there is no previous demand (e.g. $p_{t-1} = p_0$). Demand can then be expressed as

$$x_t = \frac{1}{2\lambda_t} \left[\eta - \sum_{t=2}^{T} \mu_t \left(\sum_{i=1}^{t-1} \lambda_{t-i} x_{t-i} \prod_{j=0}^{i-1} \varphi_{t-j} \right) \right]$$

Imposing agents' uniform belief over their period of movement, $\mu_t = \frac{1}{T}$ for all t and demand reduces to

$$x_t = \frac{1}{2\lambda_t} \left[\eta - \sum_{t=2}^{T} \frac{1}{T} \left(\sum_{i=1}^{t-1} \lambda_{t-i} x \prod_{j=0}^{i-1} \varphi_{t-j} \right) \right]$$

Given the conjecture of a linear equilibrium, $x_t = \beta_t \eta$ and since $\sum_{t=2}^{T} \frac{1}{T} \left(\sum_{i=1}^{t-1} \lambda_{t-i} x \prod_{j=0}^{i-1} \varphi_{t-j} \right)$ is the same for all x_t, the above implies that

$$\beta_t \lambda_t = \frac{1}{2\eta} \left[\eta - \sum_{t=2}^{T} \frac{1}{T} \left(\sum_{i=1}^{t-1} \lambda_{t-i} x \prod_{j=0}^{i-1} \varphi_{t-j} \right) \right]$$

so that $\beta_t \lambda_t = \beta_{t+1} \lambda_{t+1}$ for all t. Given that $\beta_t \lambda_t = 1 - \varphi_t$, this also implies that $\varphi_t = \varphi_{t+1} = \varphi$ for all t. Demand then becomes

$$x = \frac{1}{2\lambda} \left[\eta - \sum_{t=2}^{T} \frac{1}{T} \sum_{i=1}^{t-1} \lambda \varphi^{t-i} x \right] = \frac{1}{2\lambda} \left[\eta - \frac{\lambda x}{T} \sum_{t=2}^{T} \sum_{i=1}^{t-1} \varphi^{t-i} \right]$$

Since $\Omega > 0$, by the definition above $\varphi \in (0, 1]$. Suppose that $\varphi = 1$. Demand reduces to

$$\begin{aligned} x = \frac{1}{2\lambda} \left[\eta - \frac{\lambda x}{T} \sum_{t=2}^{T} \sum_{i=1}^{t-1} 1 \right] &= \frac{1}{2\lambda} \left[\eta - \frac{\lambda x}{T} \left(\frac{1}{2} T(T-2) \right) \right] \\ = \frac{1}{2\lambda} \left[\eta - \frac{\lambda x}{2}(T-2) \right] \end{aligned}$$

which simplifies to

$$x = \frac{2\eta}{\lambda[T+2]}$$

Then by the definition of λ above,

$$\beta\lambda = \frac{2}{T+2} = \frac{\beta^2 V}{\Omega + \beta^2 V}$$

which implies $\beta^2 V = \frac{2\Omega}{T}$. Then from the definition of φ,

$$\varphi = \frac{\Omega}{\Omega + \beta^2 V} = \frac{\Omega}{\Omega + \frac{2\Omega}{T}} = \frac{T}{T+2} < 1$$

This contradiction shows that $\varphi \in (0,1)$. Then demand from above reduces to

$$x = \frac{1}{2\lambda}\left[\eta - \left(\frac{\lambda x \varphi}{T}\right)\frac{T(1-\varphi)-(1-\varphi^T)}{(1-\varphi)^2}\right] = \frac{\eta}{2\lambda} - \frac{x\varphi}{2}\left(\frac{1}{1-\varphi} - \frac{(1-\varphi^T)}{T(1-\varphi)^2}\right)$$

which simplifies to

$$x = \frac{(1-\varphi)\eta}{\lambda\left[(2-\varphi) - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}\right]}$$

For notational convenience let $\varepsilon_T = \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$. Then $\beta = \frac{1-\varphi}{\lambda[(2-\varphi)-\varepsilon_T]}$ and

$$\beta\lambda = \frac{1-\varphi}{[(2-\varphi)-\varepsilon_T]} = 1-\varphi$$

which implies $\varepsilon_T = 1-\varphi$. Then given that φ is constant, β and λ above can be reduced and the equilibrium can be characterized as

$$\beta_t = \frac{1}{\sqrt{\varphi}},\ \lambda = \sqrt{\varphi}(1-\varphi),\ \text{and}\ \varphi = 1 - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$$

□

Proof of Proposition 7. From the above the price can be expanded as

$$\begin{aligned}
p_t &= (1-\varphi)p_0 + \lambda\omega_t + \varphi p_{t-1} = (1-\varphi)p_0 + \varphi p_0 + \sum_{i=1}^{t}\varphi^{t-i}\lambda\omega_t = p_0 + \sum_{i=1}^{t}\varphi^{t-i}\lambda(x+z_t)\\
&= p_0 + \sum_{i=1}^{t}\varphi^{t-i}\lambda x + \sum_{i=1}^{t}\varphi^{t-i}\lambda z_t = p_0 + \lambda x\sum_{j=0}^{t-1}\varphi^j + \sum_{i=1}^{t}\varphi^{t-i}\lambda z_t\\
&= p_0 + \lambda x\frac{1-\varphi^t}{1-\varphi} + \sum_{i=1}^{t}\varphi^{t-i}\lambda z_t = p_0 + \lambda\beta\eta\frac{1-\varphi^t}{1-\varphi} + \sum_{i=1}^{t}\varphi^{t-i}\lambda z_t\\
&= p_0 + (1-\varphi)\eta\frac{1-\varphi^t}{1-\varphi} + \sum_{i=1}^{t}\varphi^{t-i}\lambda z_t = p_0 + (1-\varphi^T)\eta + \sum_{i=1}^{t}\varphi^{t-i}\lambda z_t
\end{aligned}$$

and so $\mathbb{E}[p_t] = p_0 + (1-\varphi^T)\eta$. □

Lemma 2. *If $x, \varphi \in (0,1)$ and $\lim_{t\to\infty} x^t = \lim_{t\to\infty} \varphi^t = p$ for some $p \in (0,1)$, then $\lim_{t\to\infty} t(1-\varphi)^2 = \lim_{t\to\infty} t(1-x)^2$ if such a limit exists.*

Proof. Let $\varepsilon > 0$ small enough so $0 < p-\varepsilon < p+\varepsilon < 1$ and choose $T \in \mathbb{N}$ such that $t \geq T$ implies both $p-\varepsilon < x^t < p+\varepsilon$ and $p-\varepsilon < x^t < p+\varepsilon$. Then $(p-\varepsilon)^{1/t} < x < (p+\varepsilon)^{1/t}, (p-\varepsilon)^{1/t} < \varphi < (p+\varepsilon)^{1/t}$, and moreover $|\varphi - x| < |(p+\varepsilon)^{1/t} - (p-\varepsilon)^{1/t}|$. Then

$$\begin{aligned} |t(1-\varphi)^2 - t(1-x)^2| &= t|\varphi^2 - x^2 - 2(\varphi - x)| = t|(\varphi - x)((\varphi + x) - 2)| \\ &< 4t|\varphi - x| < 4t|(p+\varepsilon)^{1/t} - (p-\varepsilon)^{1/t}| \end{aligned}$$

Since this applies for all $t \geq T$ it will also apply in the limit. Note that

$$\begin{aligned} \frac{d}{dt}k^{c/t} &= \frac{d}{dt}\exp\left\{\frac{c}{t}\ln(k)\right\} = \exp\left\{\frac{c}{t}\ln(k)\right\}\frac{-c}{t^2}\ln(k) \\ &= -\frac{c}{t^2}\ln(k)k^{c/t} \end{aligned}$$

Then using L'Hôpital's Rule

$$\begin{aligned} |t(1-\varphi)^2 - t(1-x)^2| &< \lim_{t\to\infty} 4t|(p+\varepsilon)^{1/t} - (p-\varepsilon)^{1/t}| = \lim_{t\to\infty} \frac{4|(p+\varepsilon)^{1/t} - (p-\varepsilon)^{1/t}|}{\frac{1}{t}} \\ &= \lim_{t\to\infty} \frac{4|-\frac{1}{t^2}\ln(p+\varepsilon)(p+\varepsilon)^{1/t} + \frac{1}{t^2}\ln(p-\varepsilon)(p-\varepsilon)^{1/t}|}{\frac{-1}{t^2}} \\ &= \lim_{t\to\infty} 4|\ln(p+\varepsilon)(p+\varepsilon)^{1/t} - \ln(p-\varepsilon)(p-\varepsilon)^{1/t}| \\ &= 4\ln\left(\frac{p+\varepsilon}{p-\varepsilon}\right) \end{aligned}$$

since both $(p-\varepsilon)^t$ and $(p+\varepsilon)^t$ converge to 1. For any $\delta > 0$ letting $\varepsilon < \frac{p(\exp\{\delta/4\}-1)}{\exp\{\delta/4\}+1}$ yields the result that $|t(1-\varphi)^2 - t(1-x)^2| < \delta$ and thus $\lim_{t\to\infty} t(1-\varphi)^2 = \lim_{t\to\infty} t(1-x)^2$. $\square$

Lemma 3. *In equilibrium, φ implicitly defined by $1 - \varphi = \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$ must converge to 1.*

Proof. In equilibrium $\varphi = \frac{\Omega}{\Omega + \beta^2 V}$ so $\varphi \in (0,1)$. If $\lim_{T\to\infty} \varphi = p \in [0,1)$ then $\varphi^T \to 0$ and $1 - \varphi = \frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \to 0$ so $\varphi \to 1$. Thus it must be that $\varphi \to 1$. $\square$

Corollary 2. *In equilibrium, $\varepsilon_T = \frac{\varphi(1-\varphi)}{T(1-\varphi)}$ must converge to 0.*

Lemma 4. *For the above where $1 - \varphi = \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$, $\lim_{T\to\infty} \varphi^T = 0$.*

Proof. Since $\varphi \in (0,1)$, $\lim_{T\to\infty} \varphi^T \in [0,1]$. Suppose the series converges to some number inside the interval so $\lim_{T\to\infty} \varphi^T = p \in (0,1)$. By definition $(1-\varphi) = \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$ so that

$$\lim_{T\to\infty} T(1-\varphi)^2 = \lim_{T\to\infty} \varphi(1-\varphi^T) = 1 - p$$

Consider $x = p^{1/T}$. Clearly x^T converges to p and since $p \in (0,1)$ $\lim_{T\to\infty} x = 1$. Then by the above lemma since the limit exists $\lim_{T\to\infty} T(1-x)^2 = 1-p$. However,

$$\begin{aligned}\lim_{T\to\infty} T(1-x)^2 &= \lim_{T\to\infty} \frac{1-2p^{1/T}+p^{2/T}}{\frac{1}{T}} = \lim_{T\to\infty} \frac{2\frac{1}{T^2}\ln(p)p^{1/T} - \frac{2}{T^2}\ln(p)p^{2/T}}{\frac{-1}{T^2}} \\ &= \lim_{T\to\infty} 2\ln(p)(p^{2/T}-p^{1/T}) = 0\end{aligned}$$

since both $p^{1/T} \to 1$ and $p^{2/T} \to 1$. This contradiction shows that p cannot be interior so that $\lim_{T\to\infty} \varphi^T \in \{0,1\}$.

Suppose then that $\lim_{T\to\infty} \varphi^T = 1$. Recall that given the definition of ε_T,

$$\begin{aligned}\lim_{T\to\infty}(1-\varphi) &= \lim_{T\to\infty} \frac{\varphi(1-\varphi^T)}{T(1-\varphi)} = \lim_{T\to\infty} \frac{\varphi'(T) - \varphi^{T+1}[\ln(\varphi) + (T+1)\varphi'(T)/\varphi]}{(1-\varphi) - T\varphi'(T)} \\ &= \lim_{T\to\infty} \frac{\varphi'(T) - \varphi^{T+1}\ln(\varphi) - (T+1)\varphi'(T)\varphi^T}{(1-\varphi) - T\varphi'(T)} \\ &= \lim_{T\to\infty} \frac{\varphi'(T)(1-\varphi^T) - \varphi^{T+1}\ln(\varphi) - T\varphi'(T)\varphi^T}{(1-\varphi) - T\varphi'(T)} = 1\end{aligned}$$

since $\varphi^T \to 1, \varphi \to 1$, and the convergence of φ implies that its derivative $\varphi'(T)$ converges as well. Thus $1-\varphi \to 1$ so $\varphi \to 0$. Then it must be that φ converges to something less than 1, but if this is so then $\frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \to 0$ which contradicts that $\lim_{T\to\infty} \varphi < 1$. The only remaining possibility is that $\lim_{T\to\infty} \varphi^T = 0$. ☐

Lemma 5. *For the price series defined in Proposition 7* $p_t = p_0 + (1-\varphi^t)\eta + \sum_{i=1}^t \lambda\varphi^{t-i}z_t$, *define the partial series* $X_T = \frac{1}{T}\sum_{t=1}^T p_t$. *This series can be expressed as*

$$X_T = p_0 + \eta - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}\eta + \frac{1}{T}\sum_{t=1}^T \sum_{i=1}^t \lambda\varphi^{t-i}z_t$$

Moreover, the variance of the unknown part of this series (to the agent who knows η*) takes the form*

$$Var\left(\frac{1}{T}\sum_{t=1}^T\sum_{i=1}^t \lambda\varphi^{t-i}z_i\right) = \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega$$

Proof. Define the partial series $X_T = \frac{1}{T}\sum_{t=1}^T p_t$. Then

$$\begin{aligned}X_T = \frac{1}{T}\sum_{t=1}^T p_t &= \frac{1}{T}\sum_{t=1}^T \left[p_0 + (1-\varphi^t)\eta + \sum_{i=1}^t \varphi^{t-i}\lambda z_t\right] \\ &= p_0 + \eta - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}\eta + \frac{1}{T}\sum_{t=1}^T\sum_{i=1}^t \varphi^{t-i}\lambda z_t\end{aligned}$$

For each T variance of X_T (given that the z_i are independent) is

$$
\begin{aligned}
Var\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\lambda\varphi^{t-i}z_i\right) &= \mathbb{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\lambda_i z_i\varphi^{t-i}\right)^2\right] = \frac{1}{T^2}\lambda^2\Omega\sum_{t=1}^{T}\sum_{i=1}^{t}\varphi^{2(t-i)} \\
&= \frac{\varphi(1-\varphi)^2\Omega}{T^2}\sum_{t=1}^{T}\sum_{j=0}^{t-1}\left(\varphi^2\right)^j = \frac{\varphi(1-\varphi)^2\Omega}{T^2}\sum_{t=1}^{T}\left(\frac{\left(1-\left(\varphi^2\right)^t\right)}{1-\varphi^2}\right) \\
&= \frac{\varphi(1-\varphi)^2\Omega}{T^2(1-\varphi^2)}\sum_{t=1}^{T}\left(1-\left(\varphi^2\right)^t\right) = \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \frac{\varphi(1-\varphi)\Omega}{T^2(1+\varphi)}\sum_{t=1}^{T}\left(\varphi^2\right)^t \\
&= \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \frac{\varphi(1-\varphi)\Omega}{T^2(1+\varphi)}\left(\frac{\varphi^2(1-\varphi^{2T})}{1-\varphi^2}\right) \\
&= \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \frac{\varphi^3(1-\varphi^{2T})\Omega}{T^2(1+\varphi)^2} \\
&= \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega
\end{aligned}
$$

□

Proof of Proposition 8. Combining Proposition 7 and lemmas 2 - 4

$$\lim_{t\to\infty}\mathbb{E}[p_t] = \lim_{t\to\infty} p_0 + (1-\varphi^T)\eta = p_0 + \eta.$$

Define the partial series $X_T = \frac{1}{T}\sum_{t=1}^{T} p_t$. Then

Let $\varepsilon > 0$. By Markov's inequality,

$$
\begin{aligned}
\Pr(|X_T - (p_0+\eta)| \geq \varepsilon) &\leq \frac{\mathbb{E}\left[\left(p_0+\eta-\frac{\varphi(1-\varphi^T)}{T(1-\varphi)}\eta+\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\varphi^{t-i}\lambda z_t-(p_0+\eta)\right)^2\right]}{\varepsilon^2} \\
&= \frac{1}{\varepsilon^2}\left\{\left[\frac{\varphi(1-\varphi^T)}{T(1-\varphi)}\eta\right]^2 + \mathbb{E}\left[\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\varphi^{t-i}\lambda z_t\right)^2\right]\right\} \\
&= \frac{\left(\frac{\varphi(1-\varphi^T)}{T(1-\varphi)}\right)^2\eta^2 + \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2}
\end{aligned}
$$

where the last equality comes from Lemma 5. Then

$$\Pr(|X_T-(p_0+\eta)| < \varepsilon) \leq 1 - \frac{\left(\frac{\varphi(1-\varphi^T)}{T(1-\varphi)}\right)^2\eta^2 + \frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \longrightarrow 1$$

as $T\to\infty$ since $\varphi\to 1$ by Lemma 3, $\varphi^T\to 0$ by Lemma 4, and $0 \leq \frac{\varphi(1-\varphi^T)\Omega}{T(1+\varphi)} \leq \frac{\varphi(1-\varphi^T)\Omega}{T(1-\varphi)} \to 0$ by Lemma 3 which implies $\frac{\varphi(1-\varphi^T)\Omega}{T(1+\varphi)} \to 0$. Therefore $\text{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}p_t = p_0+\eta$. □

Proof of Proposition 9. Since the market maker is not aware of the confidence bias it still sets $\lambda = \sqrt{\varphi}(1-\varphi)$ such that $\varphi = 1 - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}$. From the proof of Proposition 6 we was that the agent's optimal demand for the asset is

$$x_t = \frac{1}{2\lambda}\left[\eta - \sum_{t=2}^{T} \mu_t \left(\sum_{i=1}^{t-1} \lambda \varphi^{t-i} x_i\right)\right]$$

Now with a weight γ put on being a first mover, $\mu_1 = \frac{\gamma}{T}$ and all other beliefs $\mu_t = \frac{T-\gamma}{T(T-1)}$, and supposing that all other x_i are symmetric,

$$\begin{aligned} x &= \frac{1}{2\lambda}\left[\eta - \sum_{t=2}^{T} \frac{T-\gamma}{T(T-1)} \sum_{i=1}^{t-1} \lambda \varphi^{t-i} x_i\right] = \frac{\eta}{2\lambda} - \frac{T-\gamma}{T(T-1)} \frac{x_i}{2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \varphi^j \\ &= \frac{\eta}{2\lambda} - \frac{T-\gamma}{T(T-1)} \frac{x_i}{2} \sum_{t=2}^{T} \frac{\varphi - \varphi^T}{1-\varphi} = \frac{\eta}{2\lambda} - \frac{T-\gamma}{T(T-1)} \frac{x_i \varphi}{2(1-\varphi)} \left(T - \frac{1-\varphi^T}{1-\varphi}\right) \end{aligned}$$

Imposing symmetry of $x = x_i$ demand becomes

$$2x(1-\varphi) = \frac{(1-\varphi)\eta}{\lambda} - x\frac{T-\gamma}{T-1}\left(\varphi - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)}\right)$$

which simplifies to

$$x = \frac{(1-\varphi)\eta}{\frac{\lambda}{T-1}\left[(T-1) - (2\varphi-1)(\gamma-1)\right]}$$

□

Proof of Proposition 10. From the proof of Proposition 7 we determined that

$$p_t = p_0 + \lambda x \frac{1-\varphi^t}{1-\varphi} + \sum_{i=1}^{t} \varphi^{t-i} \lambda z_t$$

So with demand $x = \frac{(1-\varphi)\eta}{\frac{\lambda}{t-1}[(t-1)-(2\varphi-1)(\gamma-1)]}$ expected price in time t is

$$\mathbb{E}[p_t] = p_0 + \frac{(1-\varphi)\eta}{\frac{1}{t-1}\left[(t-1) - (2\varphi-1)(\gamma-1)\right]}\left(\frac{1-\varphi^t}{1-\varphi}\right)$$

Then

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}[p_t] &= \lim_{t\to\infty}\left\{p_0 + \frac{(1-\varphi^t)\eta}{\frac{1}{t-1}\left[(t-1) - (2\varphi-1)(\gamma-1)\right]}\right\} \\ &= p_0 + \frac{\lim_{t\to\infty}(1-\varphi^t)\eta}{\lim_{t\to\infty}\frac{1}{t-1}\left[(t-1) - (2\varphi-1)(\gamma-1)\right]} = p_0 + \eta \end{aligned}$$

since $\varphi \to 1$, and $\varphi^t \to 0$. Thus $\lim_{t\to\infty} \mathbb{E}[p_t] = p_0 + \eta$.

Define the partial series $X_T = \frac{1}{T}\sum_{t=1}^{T} p_t$. Then

$$\begin{aligned} X_T = \frac{1}{T}\sum_{t=1}^{T} p_t &= \frac{1}{T}\sum_{t=1}^{T}\left[p_0 + \frac{(1-\varphi^t)\eta}{\frac{1}{t-1}\left[(t-1)-(2\varphi-1)(\gamma-1)\right]} + \sum_{i=1}^{t}\varphi^{t-i}\lambda z_t\right] \\ &= p_0 + \frac{1}{T}\sum_{t=1}^{T}\left[\frac{(1-\varphi^t)\eta}{\frac{1}{t-1}\left[(t-1)-(2\varphi-1)(\gamma-1)\right]}\right] + \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\varphi^{t-i}\lambda z_t \end{aligned}$$

Let $\varepsilon > 0$ and $\delta > 0$. By Markov's inequality,

$$\begin{aligned} \Pr(|X_T - (p_0+\eta)| \geq \varepsilon) &\leq \frac{\mathbb{E}[(X_T-(p_0+\eta))^2}{\varepsilon^2} \\ &= \frac{\mathbb{E}\left[\left(p_0 + \frac{1}{T}\sum_{t=1}^{T}\left[\frac{(1-\varphi^t)\eta}{\frac{1}{t-1}[(t-1)-(2\varphi-1)(\gamma-1)]}\right] + \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\varphi^{t-i}\lambda z_t - (p_0+\eta)\right)^2\right]}{\varepsilon^2} \\ &= \frac{\left(\frac{1}{T}\sum_{t=2}^{T}\left[\frac{(t-1)(1-\varphi^t)-[(t-1)-(2\varphi-1)(\gamma-1)]}{[(t-1)-(2\varphi-1)(\gamma-1)]}\right]\eta\right)^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \\ &= \frac{\left(\frac{1}{T}\sum_{t=2}^{T}\left[\frac{-[(2\varphi-1)(1-\gamma)+\varphi^t(t-1)]}{[(t-1)+(2\varphi-1)(1-\gamma)]}\right]\eta\right)^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \\ &< \frac{\left(\frac{1}{T}\sum_{t=2}^{T}\left[\frac{(2\varphi-1)(1-\gamma)+\varphi^t(t-1)}{(t-1)}\right]\eta\right)^2}{\varepsilon^2} \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \\ &= \frac{\left(\frac{1}{T}\sum_{t=2}^{T}\frac{(2\varphi-1)(1-\gamma)}{(t-1)} + \frac{1}{T}\sum_{t=1}^{T}\varphi^t\right)^2\eta^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \end{aligned}$$

Choose $T\prime$ such that $T > T\prime \implies \frac{1}{T}\sum_{t=2}^{T}\frac{(2\varphi-1)(1-\gamma)}{(t-1)} < \left(\frac{\varepsilon}{\eta}\sqrt{\frac{\delta}{6}}\right)$, $\frac{1}{T}\sum_{t=1}^{T}\varphi^t < \left(\frac{\varepsilon}{\eta}\sqrt{\frac{\delta}{6}}\right)$, and $\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega < \frac{\varepsilon^2\delta}{3}$, where the last is assured by lemmas 2 - 4. Then for $T > T\prime$,

$$\begin{aligned} &\Pr(|X_T - (p_0+\eta)| \geq \varepsilon) \\ &\quad< \frac{\left(\frac{1}{T}\sum_{t=2}^{T}\frac{(2\varphi-1)(1-\gamma)}{(t-1)} + \frac{1}{T}\sum_{t=1}^{T}\varphi^t\right)^2\eta^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \\ &\quad< \frac{\left(\left(\frac{\varepsilon}{\eta}\sqrt{\frac{\delta}{6}}\right) + \left(\frac{\varepsilon}{\eta}\sqrt{\frac{\delta}{6}}\right)\right)^2\eta^2}{\varepsilon^2} + \frac{\frac{\varepsilon^2\delta}{3}}{\varepsilon^2} = \delta \end{aligned}$$

Since the choice of ε and δ was arbitrary, $\lim_{T\to\infty}\Pr(|X_T - (p_0+\eta)| \leq \varepsilon) = 1$ and $\text{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} p_t = p_0 + \eta$. □

Proof of Proposition 11. From the proof of Proposition 9 we saw that

$$x = \frac{\eta}{2\lambda} - \frac{T-\gamma}{T(T-1)} \frac{x_i \varphi}{2(1-\varphi)} \left(T - \frac{1-\varphi^T}{1-\varphi} \right)$$

If the agent believes all others act as though they have a uniform distribution over position $t \leq T$, then $x_i = \frac{(1-\varphi)\eta}{\lambda}$ and

$$\begin{aligned} x &= \frac{\eta}{2\lambda} - \left(\frac{T-\gamma}{T-1} \right) \frac{(1-\varphi)\eta}{2(1-\varphi)\lambda} \left(\varphi - \frac{\varphi(1-\varphi^T)}{T(1-\varphi)} \right) \\ &= \frac{\eta}{2\lambda} \left[\frac{2(T-\gamma)(1-\varphi)}{T-1} + \frac{\gamma - 1}{T-1} \right] \\ &= \frac{\eta}{\lambda} \left[(1-\varphi) + \frac{(\gamma-1)(2\varphi-1)}{2(T-1)} \right] \end{aligned}$$

□

Proof of Proposition 12. From the proof of Proposition 7 we determined that

$$p_t = p_0 + \lambda x \frac{1-\varphi^t}{1-\varphi} + \sum_{i=1}^{t} \varphi^{t-i} \lambda z_t$$

Given that demand is $x = \frac{(1-\varphi)\eta}{\lambda} + \frac{(\gamma-1)(2\varphi-1)\eta}{2\lambda(t-1)}$ from Proposition 11, the expected price in time t becomes

$$\mathbb{E}[p_t] = p_0 + (1-\varphi^t)\eta + \left(\frac{(1-\varphi^t)}{(t-1)(1-\varphi)} \right) \frac{(\gamma-1)(2\varphi-1)}{2} \eta$$

so that

$$\lim_{t\to\infty} \mathbb{E}[p_t] = p_0 + \eta$$

since $\varphi^t \to 0$ by Lemma 4, and $\left(\frac{1-\varphi^t}{(t-1)(1-\varphi)} \right) \to 0$ by Lemma 3. Thus $\lim_{t\to\infty} \mathbb{E}[p_t] = p_0 + \eta$.

Define the partial series $X_T = \frac{1}{T} \sum_{t=1}^{T} p_t$. Then

$$\begin{aligned} X_T &= \frac{1}{T} \sum_{t=1}^{T} \left[p_0 + (1-\varphi^t)\eta + \left(\frac{(1-\varphi^t)}{(t-1)(1-\varphi)} \right) \frac{(\gamma-1)(2\varphi-1)}{2} \eta \right] + \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{t} \varphi^{t-i} \lambda z_t \\ &= p_0 + \frac{1}{T} \sum_{t=2}^{T} \left[(1-\varphi^t)\eta + \left(\frac{(1-\varphi^t)(\gamma-1)(2\varphi-1)}{2(t-1)(1-\varphi)} \right) \eta \right] + \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{t} \varphi^{t-i} \lambda z_t \end{aligned}$$

Let $\varepsilon > 0$ and $\delta > 0$. By Markov's inequality,

$$\begin{aligned}
\Pr(|X_T - (p_0+\eta)| \geq \varepsilon) &\leq \frac{\mathbb{E}[(X_T-(p_0+\eta))^2}{\varepsilon^2} \\
&= \frac{\mathbb{E}\left[\left(p_0 + \frac{1}{T}\sum_{t=1}^{T}\left[(1-\varphi^t)\eta + \left(\frac{(1-\varphi^t)(\gamma-1)(2\varphi-1)}{2(t-1)(1-\varphi)}\right)\eta\right] + \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{t}\varphi^{t-i}\lambda z_t - (p_0+\eta)\right)^2\right]}{\varepsilon^2} \\
&= \frac{\left(\frac{1}{T}\sum_{t=2}^{T}\left[\frac{(1-\varphi^t)(\gamma-1)(2\varphi-1)}{2(t-1)(1-\varphi)} - \varphi^t\right]\eta\right)^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \\
&< \frac{\left(\frac{1}{T}\sum_{t=2}^{T} - \left[\frac{[\varphi(1-\varphi^t)(1-\gamma)(2\varphi-1)]}{2\varphi t(1-\varphi)} + \varphi^t\right]\eta\right)^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \\
&= \frac{\left(\frac{1}{T}\sum_{t=2}^{T}\frac{[(1-\gamma)(1-\varphi)(2\varphi-1)]}{2\varphi} + \varphi^t\right)^2\eta^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \\
&= \frac{\left(\left(\frac{T-1}{T}\right)\frac{[(1-\gamma)(1-\varphi)(2\varphi-1)]}{2\varphi} + \frac{1}{T}\sum_{t=2}^{T}\varphi^t\right)^2\eta^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2}
\end{aligned}$$

Choose $T\prime$ such that $T > T\prime \implies \left(\frac{T-1}{T}\right)\frac{[(1-\gamma)(1-\varphi)(2\varphi-1)]}{2\varphi} < \left(\frac{\varepsilon}{\eta}\sqrt{\frac{\delta}{6}}\right)$, $\frac{1}{T}\sum_{t=1}^{T}\varphi^t < \left(\frac{\varepsilon}{\eta}\sqrt{\frac{\delta}{6}}\right)$, and $\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega < \frac{\varepsilon^2\delta}{3}$, which is assured by lemmas 2 - 4. Then for $T > T\prime$,

$$\begin{aligned}
&\Pr(|X_T - (p_0+\eta)| \geq \varepsilon) \\
&\quad < \frac{\left(\left(\frac{T-1}{T}\right)\frac{[(1-\gamma)(1-\varphi)(2\varphi-1)]}{2\varphi} + \frac{1}{T}\sum_{t=1}^{T}\varphi^t\right)^2\eta^2}{\varepsilon^2} + \frac{\frac{\varphi(1-\varphi)\Omega}{T(1+\varphi)} - \left(\frac{(1-\varphi^T)}{T(1+\varphi)}\right)\left(\frac{(1+\varphi^T)}{T(1+\varphi)}\right)\varphi^3\Omega}{\varepsilon^2} \\
&\quad < \frac{\left(\left(\frac{\varepsilon}{\eta}\sqrt{\frac{\delta}{6}}\right) + \left(\frac{\varepsilon}{\eta}\sqrt{\frac{\delta}{6}}\right)\right)^2\eta^2}{\varepsilon^2} + \frac{\frac{\varepsilon^2\delta}{3}}{\varepsilon^2} = \delta
\end{aligned}$$

Since the choice of ε and δ was arbitrary, $\lim_{T\to\infty}\Pr(|X_T - (p_0+\eta)| \leq \varepsilon) = 1$ and $\text{plim}_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}p_t = p_0+\eta$. □