Sequential quantity setting under positional uncertainty*

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Abstract

In a Stackelberg oligopoly setting two firms set quantity without knowing whether they are the first or second in the market. I find that with a common prior positional uncertainty always leads to a more competitive level of quantity. This finding is exacerbated when firms do not share a common prior and the sum of their prior beliefs of moving first exceeds unity. Even in the presence of a common prior and many identical firms as the number of firms increases the equilibrium quantity in the presence of positional uncertainty can exceed that of perfect competition.

1 Introduction

Sequential models of firms deciding on whether to enter a market and the quantity to produce are as natural as the idea of competition itself. Under the assumption of free entry, firms look at the prevailing price and incumbent firms and enter if there are profitable opportunities. The sequential model has been extensively used to study the behavior of oligopolies, sequential quantity setting à la Stackelberg serving as the workhorse in this area. The standard result is that the leading firm anticipates the reaction of the following firm, enabling it to suppress downstream quantity and produce more than if they moved simultaneously.

The Stackelberg leader has a first-mover advantage because it can commit to a quantity before another firm enters the market. But of course this advantage depends on the leader *knowing* they are the leader. Likewise, the quantity decision of the following firm depends on their awareness that they are the follower. In practice this assumption may not withstand scrutiny, either in the case of duopoly or an arbitrary oligopoly setting. Since minimal effort would be required to determine whether there is an incumbent firm, the scrutiny would not target whether a firm knows if it is a follower. Rather, a firm may not know if it is a leader. That is, a firm deciding on quantity in a certain period may be unsure if a new

*For most recent version visit `www.christopherdgibson.com` entrant will subsequently infuse the market with supply, thereby introducing uncertainty to the profit-maximizing decision of the initial firm.

Notice that the strategic element of the Stackelberg model of oligopolistic competition begins and ends with the leading firm. The following firm merely takes the residual demand and sets quantity q to maximize profit subject to $p(q) = a - bq_1 - bq$. As far as the follower is concerned, they behave as a monopoly facing linear demand with intercept $a' = a - bq_1$. However, if the downstream firm believes there may be yet another follower, the problem becomes game theoretic with the follower responding to linear demand with intercept $a'' = a' - bq_2$ after downstream firm 2 sets quantity q_2. If there is any possibility of another firm entering a market, each firm essentially plays a Stackelberg competition game as a mix of a leader and a follower.

We will use a basic linear demand to model a Stackelberg competition setting. Two firms will be unsure of their position as leader or follower, but will face a prevailing market price p which is either the demand intercept $p = a$ (if they are the Stackelberg leader), or the residual price after the leader $p = a - bq_1$ (if they are the follower). In order that position cannot be perfectly inferred from the prevailing price, demand intercept a will be stochastic. The improper uniform distribution $a \sim U[0, \infty]$ will be the focus of analysis but the results apply to other distributions as well.

1.1 Relevant literature

To our knowledge no work has yet undertaken the study of sequential quantity setting in oligopoly markets with uncertainty over position. However, there has been much work on uncertainty in oligopoly markets with quantity setting firms, mostly focused on uncertainty over demand.

Gal-Or (1985) presents a model of linear demand with a normally distributed intercept, about which each firm receives a private noisy signal [4]. She shows that firms choosing quantity simultaneously after receiving informative signals have no incentive to share their private information about the demand intercept with other firms. Vives (1984) examines the case of heterogenous goods, confirming the result of Gal-Or if goods are complements but shows that information sharing is a dominant strategy if they are substitutes [6].

Other studies consider sequential quantity setting with stochastic demand. De Wolf and Smeers investigate a two period setting in which a Stackelberg leader chooses quantity without knowing the demand intercept, and a group of firms choose quantity simultaneously in the second period after demand has resolved [1]. DeMiguel and Xu generalize this to multiple Stackelberg leaders choosing quantity simultanously in the first period, and they identify conditions under which a unique equilibrium exists [2].

Ferreira and Ferreira (2009) study a two-period stochastic demand environment in which firms have a choice of which period to move. They identify conditions on the resolution of uncertainty in which a sequential decision is preferred to a simultanous decision. If uncertainty is high and it is resolved in the second period, the first-mover advantage reverses, favoring the following firm that faces no uncertainty.

2 The model

We consider a multi-period market in which informed market participants (firms) receive signals and trade according to the information they infer from these signals. Each firm sets quantity in a market with linear demand $p(q) = p_0 - b \cdot q$, with p_0 determined stochastically from some distribution F over $[0, \infty]$ so that $\Pr(p_0 \leq p) = F(p)$ for all $p \in [0, \infty]$. Then given cost of production $c(q)$ and the order in which they move, firms set quantity to maximize profit. We will assume that cost of production takes the form $c(q) = c \cdot q^2$.

Due to the stochastic nature of demand, however, the order in which firms set quantity is unknown. When deciding on the quantity they wish to produce, firms only see the prevailing market price. This price could be the result of the stochastic draw p_0 (in the case that the firm moves first), or could be the residual price after quantity is set by another firm (in the case that the firm moves second).

While the stochastic demand intercept makes it impossible for either firm to perfectly infer their order of play, each has a prior belief $\Pr(First) = \mu$ that they are the first mover. Upon seeing the price p, firms use Bayesian updating to infer their posterior probability $\gamma(p) = \Pr(First|p)$ of being the first mover. In order to calculate this posterior then, they must weight the probability that they are seeing the price $p = p_0$ as the first mover, or if they are seeing the residual price $p = p_0 - b \cdot q_1$ as the second mover.

In order to capture the role position plays in the strategic interaction it is useful to focus on the timing of the game.

t=0: The leading firm observes price $p_0 \sim F$ and decides on q_1.
t=1: (i) The leading firm collects profits $q_1(p_0 - bq_1) - c \cdot q_1^2$.
(ii) The following firm observes price $p_1 = p_0 - bq_1$ and chooses q_2.
t=2: Firm i collects profit $q_i(p_0 - b(q_1 + q_2)) - c \cdot q_i^2$ for $i = 1, 2$.

We will typically be looking at games of two firms, a leader and a follower, but our sequential quantity game in general takes the following form.

Definition 1. *Let $\Lambda_N(F) = (N, F, \mu_i)_{i=1}^N$ denote an N-firm sequential entry oligopoly where* *firms face linear demand $p_0 - bQ$, $p_0 \in [0, \infty)$ given by distribution $F(\cdot)$, and firm i has prior belief μ_i^j of entering the market in position j.*

In the case of $N = 2$, if $q^*(p - b \cdot q)$ is the best response for the follower to quantity q, expected profit is

$$\begin{aligned}\pi(p, q) = \gamma(p) &\left\{q(p - b \cdot q) - c \cdot q^2 + [q(p - b(q + q^*(p - b \cdot q))) - c \cdot q^2]\right\} \\ &+ (1 - \gamma(p))[q(p - b \cdot q) - c \cdot q^2]\end{aligned}$$

The posterior $\gamma(p)$ of setting quantity first can equivalently be viewed as the probability that another period will occur and the following firm will best respond to this quantity setting. With this interpretation, the profit reduces to the intuitive form

$$\begin{aligned}\pi(p, q) &= q(p - b \cdot q) - c \cdot q^2 + \gamma(p) \cdot q[p - b(q + q^*(p - b \cdot q)) - c \cdot q^2] \\ &= (1 + \gamma(p))(q(p - b \cdot q) - c \cdot q^2) - \gamma(p) \cdot bqq^*(p - b \cdot q)\end{aligned}$$

Given the objective function each firm seeks to maximize, the standard notion of equilibrium follows naturally.

Definition 2. *An equilibrium of the game $\Lambda_2(F)$ is a function $q_i^*(\cdot), i = 1, 2$ such that for all p, $q_i^*(p)$ solves*

$$q_i = \arg\max_q \quad q(p - b \cdot q) - c \cdot q^2 + \gamma(p) \cdot q[p - b(q + q^*(p - b \cdot q)) - c \cdot q^2]$$

2.1 The case of no uncertainty

To fix ideas we can look no further than the extreme cases where there is no uncertainty ($\gamma(p) = 0$ or $\gamma(p) = 1$). This is the reduced game form $\Lambda_2(\delta_p(p_0))$, where $\delta_p(p_0)$ is the Dirac measure that has mass only on $p = p_0$. In this extreme, each firm knows which is the quantity leader and which is the follower, so that the market reduces to the familiar Stackelberg oligopoly setting. Through backward induction, the first mover solves

$$\max_{q_1} \quad q(p - b \cdot q) - c \cdot q^2 + q[p - b(q + q^*(p - b \cdot q)) - c \cdot q^2]$$

where $q_2(p)$ maximizes $q_2((p_0 - b \cdot q_1) - q_2(b + c))$. Solving this system of equations the equilibrium in this case is

$$q_1 = p_0 \left(\frac{3b + 4c}{2(3b^2 + 8bc + 4c^2)} \right) \quad \text{and} \quad q_2 = p_0 \left(\frac{3b^2 + 12bc + 8c^2}{4(b + c)(3b^2 + 8bc + 4c^2)} \right)$$

3 Introducing uncertainty: A uniform intercept

Now return to the case of an uncertain linear demand, and therefore an uncertain order of quantity setting. The linear demand intercept p_0 is distributed over $[0, \infty)$ according to the distribution F. Now, however, suppose that every demand intercept $p_0 \in [0, \infty]$ is equally likely, so that F is the improper uniform distribution.

Apart from the technical tractability the improper uniform distribution offers, we will see that there are compelling reasons to analyze this case. Not least of these reasons is that without additional information about the linear demand, each firm has no reason to believe any initial price p_0 to be more likely than any other. Moreover, while this particular distribution over p_0 sacrifices some generality, we will see later that the loss of generality is actually minimal. Not only is the uniform distribution the limiting case of many distributions for p_0 that may be of interest, but the cases short of the limits are locally well approximated by the normal distribution with little error as is demonstrated in section 4.

In order to determine the equilibrium in the uniform case it is necessary to first characterize the posterior $\gamma(p)$ of being the first mover under this distribution.

Lemma 1. *In the Stackelberg game $\Lambda_2(U[0, \infty))$, let $\gamma(p) = \Pr(First|p)$ be the posterior probability of being first upon observing price p and $\Pr(First) = \mu$ the prior probability. Then in a pure strategy equilibrium, $\gamma(p) \in \{0, \mu, 1\}$.*

This lemma shows that if any point is equally likely to be the initial price p_0 and if the observed p is a possible residual price from some initial p_0, then it is equally likely that price p is observed by a first mover or a second mover, so the posterior collapses to the prior μ. The only cases in which the posterior will not be the prior is if either p cannot be the residual price from any initial p_0, so price p is a "hole," or if p is a "mass point" and as such is the residual price of a non-zero mass of initial p_0.

In this case that p is a hole and there is no possible initial price such that $p_0 - b \cdot q^*(p_0) = p$, then the probability of being a follower is zero so a firm observing the price p knows they must be the first mover and $\gamma(p) = 1$. In the case that p is a mass point, there are uncountably infinitely many initial prices p_0 such that $p_0 - b \cdot q^*(p_0) = p$, and only one possible way for p to be the initial price. As such this mass overwhelms the probability of being the first mover, so that the posterior is $\gamma(p) = 0$.

These equilibrium pathologies involving holes or mass points in the support of $p - b \cdot q^*(p)$ both complicate equilibrium analysis and detract from its interest in describing reasonable market behavior. As such we will focus on an equilibrium devoid of such cases in order to focus on firms' reactions to market variables instead of abstract equilibrium considerations. In doing so we will highlight the interactions that result from a multi-period market with positional uncertainty, and how firms respond to beliefs of their own position as well as their beliefs over other firms' beliefs.

Given the absence of holes or mass points in the distribution, the posterior $\gamma(p) = \mu$ for all values of $p > 0$. This leads to a natural equilibrium result.

Proposition 1. *In the Stackelberg game $\Lambda_2(U[0,\infty))$ such that p_0 is distributed according to the improper uniform distribution, a linear equilibrium exists.*

Proposition 1 states that for a given firm i there is a constant k_i such that $q_i^*(p) = k_i p$. But to determine exactly what form this constant takes more needs to be said of the beliefs of each firm. In particular, we know each firm has some prior μ, but we have not yet considered how these priors may relate to one another. If both firms have identical structures and information it may be natural to assume they have identical priors as well. If firms' structures are not identical but their information over this asymmetry is shared, then they may not have identical priors but instead a common prior in that the sum of prior beliefs still sum to one. In fact the equilibrium composition and comparative statics will differ depending on how firms form these beliefs, differences that will be revealed in turn.

3.1 Identical priors

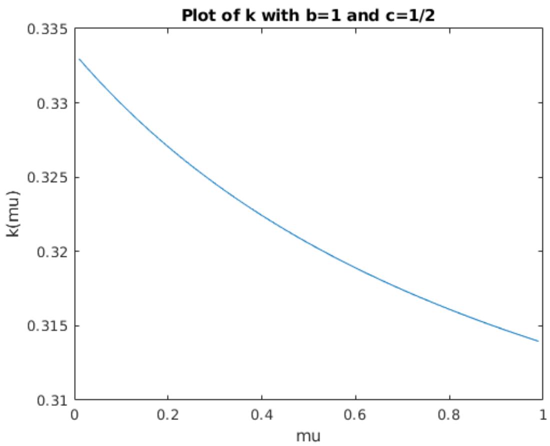
In the case of identical priors if we denote $\Pr_i(First) = \mu_i$, then $\mu_i = \mu_j = \mu$ and the linear equilibrium takes simple form.

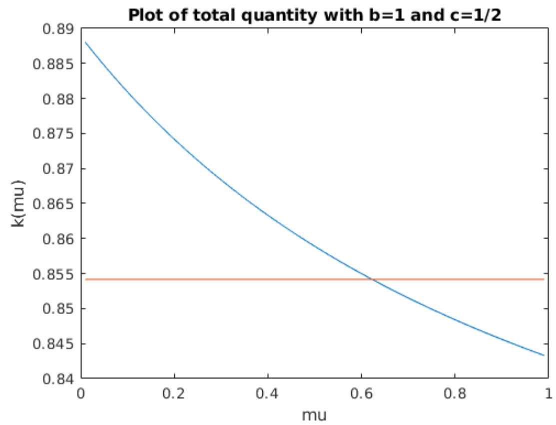
Proposition 2. *If priors are identical so that $\mu_i = \mu_j = \mu$, then a linear equilibrium of $\Lambda_2(U[0,\infty))$ exists and takes the form $q^*(p) = k \cdot p$ for all $p > 0$ with*

$$k = \frac{b(2+3\mu) + 2c(1+\mu) - \sqrt{(b(\mu+2) + 2c(\mu+1))^2 - 8bc\mu(\mu+1)}}{4b^2\mu}$$

It can be shown that for all values of parameters b and c, equilibrium parameter k is decreasing in prior belief μ. This is to be expected, as the prior μ is also the posterior $\gamma(p)$ of being the first mover. For a given price p, as the probability of being first increases, by definition the probability of a subsequent firm setting quantity in the market increases. Just as in the Stackelberg case the leading firm must reduce the production quantity in anticipation of the following firm's quantity reducing price even further, so too does the increase in the probability μ of a follower add weight to the trade-off between maximizing profit under current demand versus final demand if another firm were to enter.

Considering the parameters $b = 1$ and $c = 1/2$, the linear equilibrium constant takes the simplified form $k = \frac{3+4\mu-\sqrt{8\mu^2+16\mu+9}}{4\mu}$. Total quantity for initial price p_0 is then $kp_0 + k(p_0 - kp_0) = p_0(2k - k^2)$.





The plot on the left shows the relationship between prior μ and the linear parameter k. As described above k is a decreasing function of μ. The figure on the right plots total quantity as a proportion of initial price (q/p_0). As total quantity is a decreasing function of k it is also to be expected that it too would be a decreasing function of prior belief μ for the exact same reason. In fact, as the belief of each firm that they move first increases and each becomes more certain that their production will be followed with an additional infusion of quantity from the following firm, total quantity in the case of uncertainty actually drops below the Stackelberg equilibrium with no uncertainty.

Recalling from above that in the Stackelberg equilibrium $q_1 = p_0 \cdot \frac{b+2c}{6b^2+8bc+4c^2}$ and $q_2 = p_0 \cdot \frac{(3b^2+6bc+4c^2)}{4(b+c)(3b^2+4bc+2c^2)}$, with the parameters $b = 1$ and $c = 1/2$ total quantity $q_1 + q_2 = 0.5238 \cdot p_0$. This total quantity is shown by the horizontal line in the right graph. As can be seen from this comparison, for low values of μ the total quantity in the uncertain case is higher than in the case of certainty but for high values of μ this relationship reverses. In fact this pattern holds for all values of b and c.

While we have no cause to question this pattern at the moment, we will see later there is indeed plenty of reason to expect that the introduction of uncertainty will lead to a strictly higher quantity. The Stackelberg case is that in which the order of quantity setting is commonly known: one agent has prior $\mu = 1$ of moving first and the other has prior $\mu = 0$. But if the quantity leader had even a little uncertainty of their position, the firm would have an incentive to increase quantity as the trade-off between maximizing current and final demand has shifted toward current. If at the same time the quantity follower had an equal amount of uncertainty in their position but in the reverse direction, this firm would have an incentive to decrease quantity as their quantity setting trade-off has shifted toward maximizing final demand. If the leading firm were able to anticipate the quantity reduction of the follower, the leading firm would be incentivized to even further increase quantity. And so forth.

While the end result of this iterative loop of backward induction is unclear in terms of how total quantity is effected, it is at least clear that being able to anticipate the rival firm's response to a new prior will mitigate the declining total quantity as μ increases. In the case of identical priors this anticipation fails because firms have − and expect the other firm to have − the exact same prior. Thus when their own prior changes each firm expects the prior of the other to change in exactly the same way. Barring the case where $\mu = 1/2$, the identical prior assumption comes with it the untenable shared belief that the total probability of moving first could exceed or fall short of unity, and moreover that firms are aware that they share this belief.

3.2 A common prior

The case of identical priors provided a simple solution characterizing the linear equilibrium that allowed for the analysis of firm behavior in the presence of uncertainty and how behavior changes with beliefs about their position of quantity setting. But valuable as a foothold into the problem at hand, the assumption that firms have exactly the same belief of moving first and are aware of this shared contradiction of probability theory seems an unlikely reality in which otherwise rational firms might operate.

In light of this incongruity a more fitting environment might be one in which firms correctly anticipate the rival firm's prior belief in relation to their own. The assumption that gives us this belief congruity is the common prior assumption. Defined in the usual way the common prior imposes the following structure on how the prior belief of each firm relate to one another.

Definition 3. *Firms i and j share a* ***common*** *prior if $\mu_i + \mu_j = 1$.*

The common prior assumption is useful because not only do priors beliefs accord with probability theory under this structure but also it introduces a consistency of beliefs that would be expected of rational profit-maximizing firms. From a technical standpoint a market in which firms may have different beliefs μ introduces a layer of complication to firms' interaction, but we can still find a linear equilibrium, one with notably more desirable and realistic properties.

Proposition 3. *If firms have a common prior so that $\mu_i + \mu_j = 1$, then a linear equilibrium of $\Lambda_2(U[0,\infty))$ exists and takes the form $q^*(p) = k(\mu) \cdot p$ for all $p > 0$ with a linear parameter $k(\mu)$ of the form*

$$k(\mu) = \frac{b^2(3\mu^2+\mu-10)+8bc(\mu+1)(\mu-2)+4c^2(\mu+1)(\mu-2)+\sqrt{A(b,c,\mu)}}{4b^2(1-\mu)(b(2+\mu)+2c(1+\mu))}$$

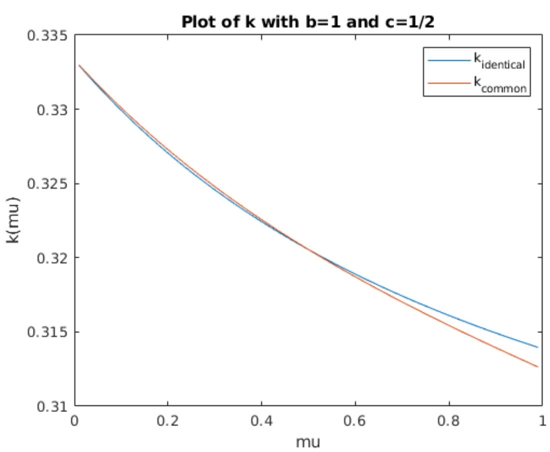
where

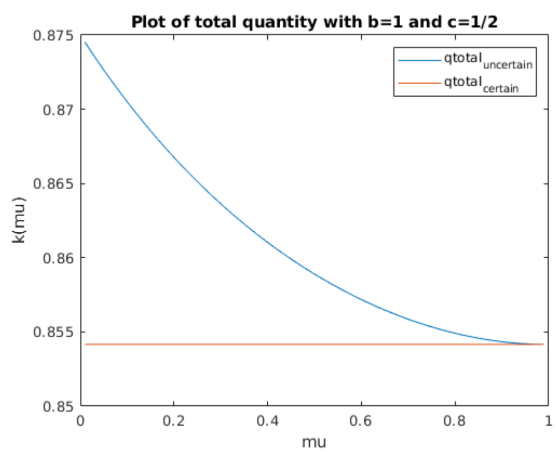
$$\begin{aligned} A(b,c,\mu) &= b^4(\mu^2-\mu-6)^2+16b^3c(\mu^4-2\mu^3-7\mu^2+8\mu+12) \\ &\quad +8b^2c^2(7\mu^4-14\mu^3-29\mu^2+36\mu+44)+16c^3(4b+c)(\mu^2-\mu-2)^2 \end{aligned}$$

In order to highlight the differences between the equilibrium under a common prior and that under an identical prior a sketch of the proof is useful. Each firm i solves for an equilibrium under the assumption that the other firm plays a linear strategy. However now the linear parameter depends on prior μ_i, as the prior is no longer the same for both firms. Firm i solves $\max_{k_i(\mu_i)} pk_i(\mu_i)[p-pk_i(\mu_i)p(b+c)]-bpk_i(\mu_i)\mu_i k_j(1-\mu_i)$. This yields for each firm a first order condition for $k_i(\mu_i)$ and an inferred condition for $k_j(1-\mu_i)$, from which the constants $k_i(\mu_i)$ and $k_j(\mu_j)$ can be solved.

As in the previous case – and for the same reason – it can be shown that k is decreasing in the prior belief μ. As the probability of being first increases, the trade-off between maximizing current and final demand shifts toward final and quantity is decreased. However unlike in the previous case, this decrease in quantity is amplified by the common prior realization that at the same time the prior belief of the other firm decreases, leading to an increase in quantity in the case of a following quantity setter.

For the parameters $b=1$ and $c=1/2$ the linear equilibrium parameter simplifies to $k(\mu) = \frac{20+4\mu-8\mu^2-\sqrt{32\mu^4-64\mu^3-152\mu^2+184\mu+256}}{4(2\mu+3)(1-\mu)}$. If initial price is p_0 and μ_1 is the prior belief of the leading firm, total quantity is $k(\mu_1)p_0+k(1-\mu_1)(p_0-bk(\mu_1)p_0) = p_0[k(\mu_1)+k(1-\mu_1)-bk(\mu_1)k(1-\mu_1)]$





The plot on the left shows the inverse relationship of linear parameter k with μ. Moreover, this graph highlights the different behavior of the parameter k in the case of a common prior as compared to an identical prior. The two values of k meet at $\mu = 0$ and $\mu = 1/2$. when the prior is zero, in both cases the firm acts as a stand alone entity given the price, maximizing profit by equating marginal revenue and marginal cost, ruling out the possibility of a following quantity setter. When the prior $\mu = 1/2$ the priors are both identical and common so the cases overlap.

An increase in the prior will lead to a decrease in quantity as the firm becomes more confident that there is a follower. But now when $\mu \in (0, 1/2)$ the parameter k is higher than in the case of an identical prior. In this region, while k still decreases with μ, this decrease is mitigated by the awareness given by the common prior assumption that the rival firm's prior is in the higher region $(1 - \mu > 1/2)$, so that while the rival's declining belief of having a follower will lead to a quantity increase, this increase will be much lower than if their prior were less than 1/2. As a result, under a common prior a firm with $\mu \in (0, 1/2)$ can afford less of a decrease in quantity in response to an increase in μ than if their rival shared the same prior $\mu \in (0, 1/2)$.

For $\mu > 1/2$ this logic reverses, and now any increase in prior μ is met with an equal and yet more formidable change in behavior from the rival firm. Equal in the sense that the rival's prior will decrease by the same magnitude with which μ increases, but more formidable in that the rival is in the more quantity-responsive region where $1 - \mu \in (0, 1/2)$. Thus increases to the prior $\mu > 1/2$ are exacerbated by the common prior assumption as compared to the case of identical priors.

The figure on the right shows the total quantity summed across the two periods $(2q_1 + q_2)$ in proportion to initial price (q/p_0) in the case of a common prior compared with the Stackelberg case of no uncertainty. Unsurprisingly we see that these cases intersect when $\mu = 1$, when order is known. More interestingly, it is clear that in the case of uncertainty with a common prior, the total quantity in the market lies above the certain case for all values of μ.

As described, this is due to the inverse relationship between prior μ and linear parameter k, and the joint awareness of how this is influenced by the common prior. For prior $\mu < 1/2$, an increase in μ causes a decrease in k and the resultant linear quantity. But if at the same time the rival firm sees an decrease in its own μ, this and a higher residual price left by the leading firm causes an increase in quantity. These two factors, coupled with the more precipitous slope of the rival's parameter k in the high μ region more than compensate for the initial quantity drop, leading to a total quantity increase. The case of a decrease of $\mu > 1/2$ is symmetric from the other firm's perspective, leading to a peak quantity at the neutral prior $\mu = 1/2$.

The common prior case characterizes a richer environment in which firms interact in the case of positional uncertainty. The mutual awareness that the change in a firm's own prior must be met with an equal change of the rival's prior restores a consistency to this interaction and a strengthening of the explanatory power of the model. The common prior solves the violations of probability theory suffered by the previously identical prior μ, and highlights a key result. **Introducing uncertainty in the position of quantity setting leads to a higher total level of production in equilibrium.**

This result is intuitive, as a movement away from certainty introduces to the following firm possibility being the leader, and to the leading firm the possibility of not having a follower. As we was the decrease in quantity of the former is more than compensated for by the increase in quantity from the latter, leading to a net increase in total quantity over the certain case.

For the remainder of the analysis we will focus on the case of the common prior. In many instances this will not matter as the cases intersect with a neutral prior, a natural assumption for otherwise identical firms. This assumption becomes even more important in the case of $n > 2$ identical firms, where both intuition and tractability call upon the neutral prior.

3.3 The case of N firms

In a market of $N > 2$ firms, while tractability concerns impede an explicit solution for the linear equation parameter k the logic is very much the same. In such an environment we will assume the firms are identical and as such the trivial prior of $\mu = 1/N$ is assumed. To illustrate, if there are three firms then profit for any given firm (say firm 1 without loss) is

$$\frac{1}{3}\left(q_1(p - bq_1) - cq_1^2\right) + \frac{1}{3}\left(q_1(p - b(q_1 + q_2)) - cq_1^2\right) + \frac{1}{3}\left(q_1(p - b(q_1 + q_2 + q_3)) - cq_1^2\right)$$

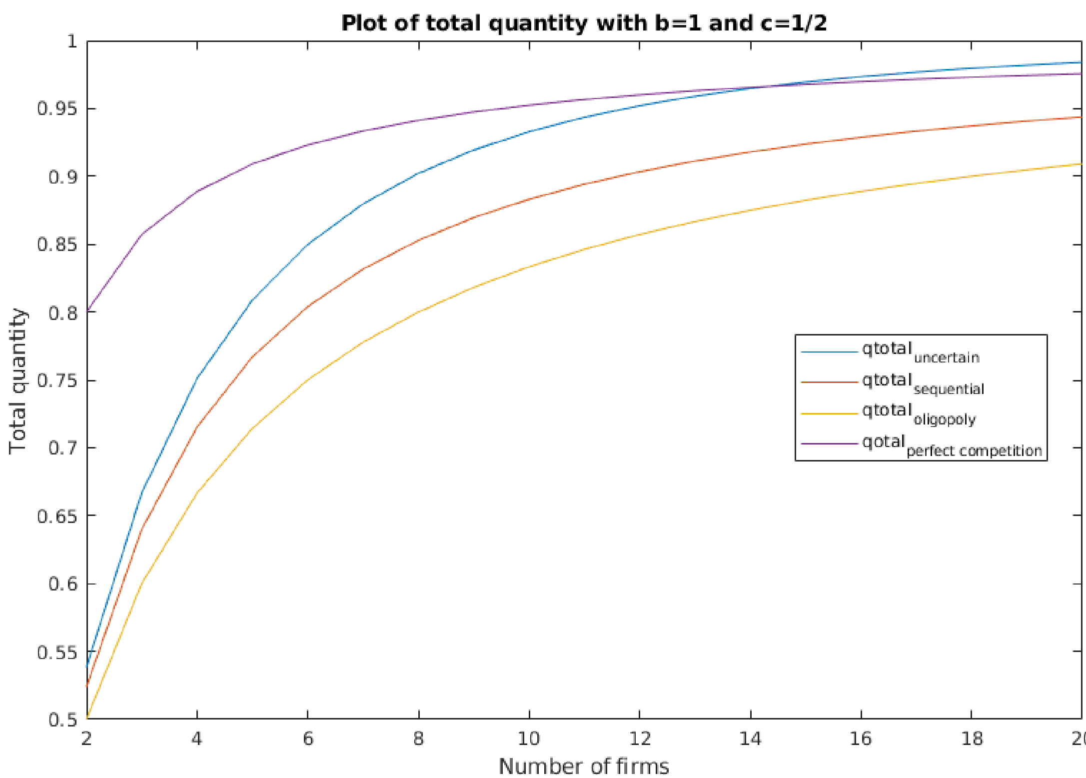
This can be simplified to $q_1(p - q_1(b + c)) - \frac{1}{3}bq_1(2q_2 + q_3)$. The technical complication arises from the observation that in a linear equilibrium, $q_3 = k(p - b(pk + k(p - bpk)))$, and iteratively when the market reaches N firms the parameter k must be solved from a polynomial equation of order N. While an explicit solution is no longer guaranteed the general case can still be solved implicitly for any N.

Proposition 4. *For the game $\Lambda_N(U[0, \infty))$ with $N \geq 2$ firms and a shared uniform prior $\mu_j^i = 1/N$ that firm i chooses quantity after $j - 1$ predecessors, a linear equilibrium $q = kp$ is defined implicitly by*

$$(1 - 2bk)(1 - (1 - bk)^N) = 2bcNk^2$$

We saw in the case of two firms that the introduction of position uncertainty resulted in a higher total quantity than if positions are certain. In fact, this is a result that generalizes to the case of N firms. Moreover since sequential quantity setting always results in a higher level of production than Cournot oligopoly, sequential quantity setting with positional uncertainty is too bounded below by Cournot.

The figure below shows the total output in the case of Cournot oligopoly, sequential quantity with positional uncertainty, and the perfectly competitive outcome (where firms make zero profit). As expected the case of sequential quantity setting with uncertainty lies above the simultaneous quantity setting of Cournot. But the surprising result is the relationship with the perfectly competitive quantity. It is known that Cournot converges to the perfectly competitive case as the number of firms goes to infinity, but it is striking how quickly the uncertain case converges. In fact, with the parameters $b = 1$ and $c = 1/2$, when the number of firms is more than 14 the quantity actually surpasses the perfectly competitive outcome.



That the sequential quantity case with uncertainty exceeds the perfectly competitive outcome gives pause as it must imply that some firms make negative profit. While this is true, it is not true that all firms make losses, nor is it true than any firm expects losses ex-ante. It is only the leading firms who make losses, as they produce the most given the initial price p_0, and as such are shouldered with consequences of this low probability event of being an early quantity setter.

Given the uncertainty in this environment, it is understandable that in the case of low probability events, an a priori optimal strategy yields losses a posteriori. The possibility of losses highlights another departure from certainty in quantity setting. Not only does the case of sequential quantity setting with positional uncertainty converge quickly (even surpassing) the perfectly competitive outcome, but this convergence comes at the expense of leading firms' profits. While for a low number of firms there is a leading advantage in terms of profits, when the number of profits grows this advantage switches as the gains from being a quantity leader are outweighed by proximity to the final price. A leading firm can inject more quantity into the market than can a following firm, but if the price drops too much (if too many firms follow), the leader's high quantity turns out to be too high.

4 A generalization: A normal intercept

Previously the intercept p_0 that determined the linear demand curve was distributed over $[0, \infty)$ according to the improper uniform distribution so that any initial p_0 was equally likely. As a generalization, suppose now that demand intercept p_0 is distributed over $[0, \infty)$ according to the truncated normal distribution $N(\mu, \sigma^2)$. For ease of exposition suppose that $\mu = 0$ so that the truncated distribution is given by the density function $f(x, \sigma^2) = \sqrt{\frac{2}{\pi\sigma^2}} \exp\{-\frac{x^2}{2\sigma^2}\}$ with mean $\sqrt{\frac{2\sigma^2}{\pi}}$.

Notice that finding the equilibrium $q^*(p_0)$ which maps p_0 into residual price p according to $p = p_0 - bq^*(p_0)$ is equivalent to finding the inverse mapping of p to initial price p_0 according to $p_0^*(p) = p + bq^*(p_0^*(p))$. This latter mapping is solved as a fixed point problem but a unique solution will exist as long the mapping $p_0 \mapsto p$ is injective. Then upon seeing price p, the initial price, or demand intercept, if the firm is the first mover is p, while the initial price if second is $p_0^*(p)$. Then the probability of being first given the observed price p can be found through Bayesian updating as follows:

$$\gamma(p) = \frac{\Pr(F)\Pr(p|F)}{\Pr(F)\Pr(p|F) + \Pr(S)\Pr(p|S)} = \frac{\mu \exp\{-\frac{p^2}{2\sigma^2}\}}{\mu \exp\{-\frac{p^2}{2\sigma^2}\} + (1-\mu)\exp\{-\frac{p_0^*(p)^2}{2\sigma^2}\}}$$

The assumption of a normal distribution is a generalization in the sense that as variance σ^2 increases, the distribution of p_0 increasingly resembles the improper uniform distribution, converging to it in the limit. However, the updated probability $\gamma(p)$ of being the first hints at the complication the normal distribution introduces. This posterior can be reduced to $\mu/[\mu+(1-\mu)\exp\{-\frac{p_0^*(p)^2-p^2}{2\sigma^2}\}]$, which makes clear its dependency on the difference $p_0^*(p)^2-p^2$. However, as price increases the optimal quantity will increase, leading to an increase in the difference between initial and residual prices $p_0^*(p)-p$. This alone is not unique to the normal case, as we saw the same in the uniform case - constant in the case of a uniform intercept was not the difference between initial and residual prices but the ratio between them.

This problem becomes more complicated because now the rival firm's optimal response $q^*(p-bq)$ changes not just on the residual price $p-bq$, but also with the induced posterior $\gamma(p-bq)$. Moreover, since for any quantity q_1 the responding firm maximizes $q(p-bq_1)-q^2(b+c)-q\cdot b\gamma(p-bq_1)q_1$, varying q_1 will affect the responding firm's first order condition with respect to q both linearly and exponentially, so there is no closed form solution to the original first order condition with respect to q_1. This points to an numeric solution.

The positional uncertainty and infinite state space of this problem join to present another complication: the problem is infinitely recursive. This is in itself is not new but the constant posterior of the uniform case allowed us to conjecture a linear equilibrium; the dynamic posterior here suggests a dynamic programming solution. However, the infinite recursiveness on both sides of the distribution - since for any price $p>0$ it is always possible that there was a leader facing initial price $p_0^*(p)$ or will be a follower facing residual price $p-bq$ - leaves a dynamic programming problem with no initial point. Fortunately under the normal distribution we can find a "good enough" starting point in the following sense.

Lemma 2. *In the game $\Lambda_2(N)$ where demand intercept p_0 is distributed according to the truncated normal distribution on the interval $[0,\infty)$, then $\lim_{p\to 0}\gamma(p)=\mu$.*

The essence of this result is that since the posterior takes the form

$$\gamma(p)=\frac{\mu}{\mu+(1-\mu)\exp\left\{-\frac{p_0^*(p)^2-p^2}{2\sigma^2}\right\}}$$

then if $p_0^*(p)\to 0$ as $p\to 0$ then $\gamma(p)\to\mu$. If this were not the case some $\varepsilon_2>\varepsilon_1>0$ could be found such that $p<\varepsilon_1$ and $p_0>\varepsilon_2$ for $p_0-bq^*(p_0)=p$. But this induces a hole in the range $p\in(\varepsilon_1,\varepsilon_2)$ so that $\gamma(p)=0$ which by assumption does not exist.

Lemma 3. *In the game $\Lambda_2(N)$ where demand intercept p_0 is distributed according to the truncated normal distribution on the interval $[0,\infty)$ and $q^*(\cdot)$ is an equilibrium, then $\lim_{p\to\infty}\gamma(p)=1$.*

This result relies on the fact that

$$\gamma(p)=\frac{\mu\exp\left\{\frac{p_0^*(p)^2-p^2}{2\sigma^2}\right\}}{\mu\exp\left\{\frac{p_0^*(p)^2-p^2}{2\sigma^2}\right\}+(1-\mu)}$$

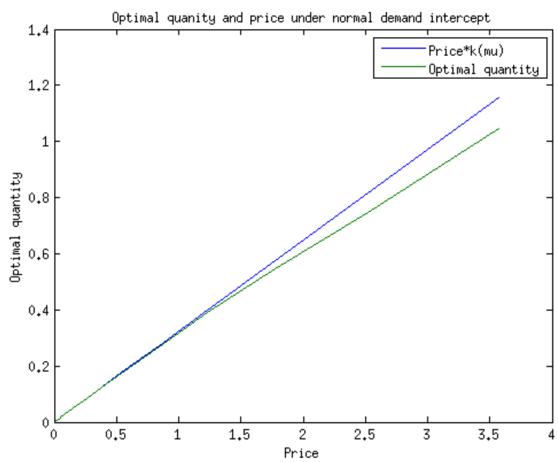
and that as $p \to \infty$ the inducing $p_0^*(p)$ must increase in distance so $p_0^*(p)^2 - p^2$ diverges. If this were not true then

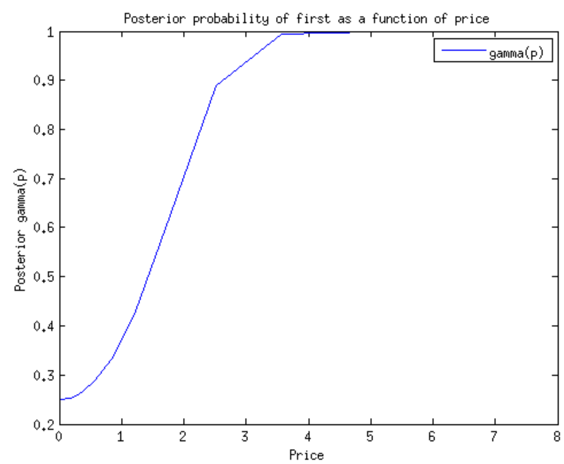
Then for a small enough initial price p_0 it can be assumed with little error that the a firm will behave as in the naïve case of a uniform prior, assuming the posterior of itself and any potential follower to be μ and setting the quantity $q(p) = k(\mu) \cdot p$ as above. From here, the best response $q^*(p)$ to any initial p can be determined recursively by choosing a suitably small starting point $\underline{p}$ so $\gamma(\underline{p}) \approx \mu$ and iterating a finite number of steps.

Consider the case where each firm has prior $\Pr(F) = \mu$. Then for our canonical case of $b = 1$ and $c = 1/2$ let variance $\sigma^2 = 1$ to start and consider the lower bound for our numerical approximation of $\underline{p} = 1/100,000$.

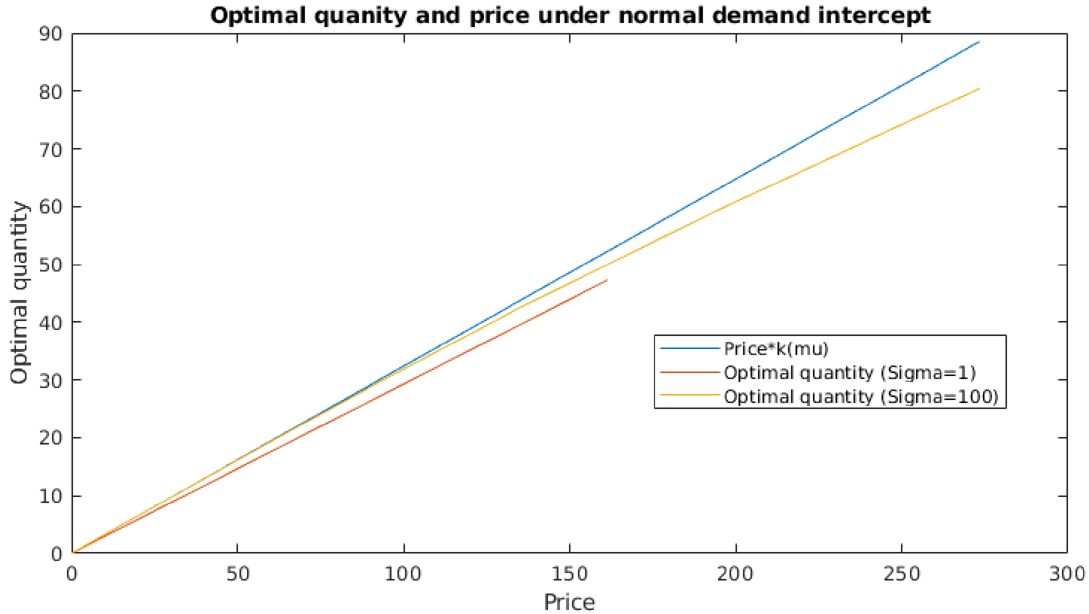
For such a small lower bound $\underline{\text{p}}$ we would expect that $\gamma(p) \approx \mu$ for p near $\underline{p}$, and that the optimal quantities for such prices would approximate $k(\mu) \cdot p$ as in the uniform case. Since the lowest price player has posterior $\gamma(p) = \mu$ and assumes any follower will have the same, then by design this holds.

As the graph below shows, the relationship between price and quantity is roughly linear but not quite. As described, the best response to a low price is approximately $p \cdot k(\mu)$, but as p increases the posterior moves away from $\gamma(p) \approx \mu$ and approaches $\gamma(p) = 1$.

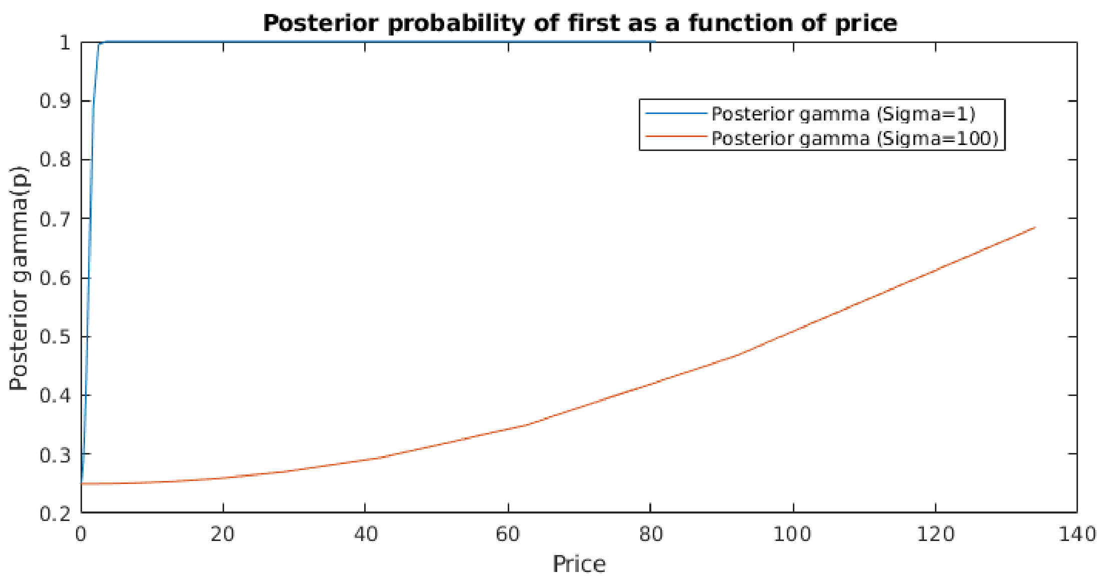




However, the speed of this movement depends on the variance of the signal σ^2. As the variance increases, high prices become less informative of position and the posterior does not update as much. This leads to the optimal q and $k^*(\mu) \cdot p$ coinciding for a larger number of prices



As the figure above shows, while in the case of $\sigma^2 = 1$ the linear equilibrium with posterior $\gamma = \mu$ and the optimal quantity under the normal distribution diverged around $p = 1$, for $\sigma^2 = 100$ this difference is only perceptible near $p = 100$. This corresponds to the slowing of the posterior $\gamma(p)$ to update away from μ, as the following graph shows.



Beginning with a prior of $\mu = \frac{1}{4}$ the speed at which the posterior updates slows significantly. This is no surprise given that $\gamma(p) = \frac{\mu}{\mu + (1-\mu)\exp\left\{-\frac{p_0^*(p)^2 - p^2}{2\sigma^2}\right\}}$, so that $\lim_{\sigma^2 \to \infty} \gamma(p) = \mu$.

Given the stubbornness of $\gamma(p)$, the assumption of the previous section that price is distributed uniformly is even more appealing. A uniform price intercept is a good approximation for small p and large variance σ^2, offering credence to the improper uniform distribution as more than just a tractable choice.

5 Conclusion

We have introduced a model of Stackelberg competition in which firms are unsure of their position as leader or follower. As a result, the probability of a competitor subsequently responding to quantity causes the downstream firm to reduce output, while the nonzero probabiltiy of being the follower causes the upstream firm to produce more than if position were perfectly known. Of these two opposing effects the incentive to increase production in response to the chance of being the downstream firm outweighs the incentive of the true follower to restrict quantity.

As a result of this interplay of incentives, uncertainty over position ultimately leads to a higher level of output in the market and a more competitive outcome for consumers. As the number of firms increases this difference widens, with the total quantity under positional uncertainty approaching − in some cases surpassing − quantity under perfect competition.

While the focus of this study was a model in which the stochastic demand intercept was uniformly distributed. The key feature of the uniform distribution that lends so much tractability is that since every price is equally likely, no price gives agents any more information about their position and the belief of being first remains as the prior. The results presented hold under other distributions, including the truncated normal. Moreover, as the variance of the intercept increases and the signal becomes less informative, the posterior belief remains close to the prior and the truncated normal case is well approximated by the uniform distribution.

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A1. Appendix

Proof of Lemma 1. A firm observes p which induces a belief $\gamma(p)$. By Bayesian updating with prior $\Pr(First) = \mu$ and given $p_0 \sim U[0, a_0]$

$$
\begin{aligned}
\gamma(p) &= \frac{\Pr(p|F)\Pr(F)}{\Pr(p|F)\Pr(F) + \Pr(S)\Pr(p|S)} \\
&= \frac{\mu \Pr(p|F)}{\mu \Pr(p|F) + (1-\mu)\Pr(p|S)} \\
&= \frac{\mu \lim_{\varepsilon\to 0} \int_{p-\varepsilon}^{p+\varepsilon} f(s)ds}{\mu \lim_{\varepsilon\to 0} \int_{p-\varepsilon}^{p+\varepsilon} f(s)ds + (1-\mu)\lim_{\varepsilon\to 0} \int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0 - s}{b}|s) f(s) dp_0 \cdot ds}
\end{aligned}
$$

Where because initial price is distributed uniformly $f(s) = \frac{1}{a_0}$. The value of the posterior hinges on the integral $\int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b}|s) f(s) dp_0 \cdot ds$. There are three possibilities

(i) There is no initial p_0 such that $q^*(p_0)$ satisfies $p = p_0 - b \cdot q^*(p_0)$ in which case we will say p is a hole in the support. If p is a hole then $f(q = \frac{p_0-p}{b}|p) = 0$ for all p_0, so that $\lim_{\varepsilon\to 0} \int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b}|s) f(s) dp_0 \cdot ds = 0$. Thus if p is a hole in the distribution $\gamma(p) = 1$.

(ii) p is a mass point so that there is a set P_0 with non-zero measure such that for all $p_0 \in P_0$, $q^*(p_0) = \frac{p_0-p}{b}$. Suppose the measure of $P_0 = \lambda > 0$. Also, since we are considering only full strategies, $f(q = \frac{p_0-p}{b}|p) = 1$ for all $p_0 \in P_0$. Then $\lim_{\varepsilon\to 0} \int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b}|s) f(s) dp_0 \cdot ds \geq \int_{P_0} f(s)ds = \lambda \cdot \frac{1}{a}$. And since the numerator of $\gamma(p)$ becomes arbitrarily small as $\varepsilon \to 0$, $\gamma(p) = 0$.

(iii) If p is neither a mass point or a hole then for the set P_0 such that for all $p_0 \in P_0$, $q^*(p_0) = \frac{p_0-p}{b}$, then the measure of P_0 is zero since it can contain at most countably many points. Then $\int_{p-\varepsilon}^{p+\varepsilon} \int f(q = \frac{p_0-s}{b}|s) f(s) dp_0 \cdot ds = 2\varepsilon$ and $\gamma(p) = \mu$

Since this holds for all values of a_0, it holds for the limiting case of the improper uniform distribution as $a_0 \to \infty$. □

Proof of Proposition 2. Since we are looking for a pure strategy equilibrium with an infinite state space, according to Lemma 1 the posterior $\gamma(p) = \mu$. Moreover since firms have an identical prior $\Pr(F) = \mu$, each solves the problem

$$\max_q \quad (1+\mu)(q \cdot p - q^2(b+c)) - \mu q b q^*(p - b \cdot q)$$

Conjecture a linear equilibrium so $q^*(p) = k \cdot p$. Then the maximization problem becomes

$$\max_q \quad (1+\mu)(q \cdot p - q^2(b+c)) - \mu q b k(p - b \cdot q)$$

Solving for the optimal quantity and isolating the first order condition for q gives $q = p\left(\frac{1+\mu(1-bk)}{2b(1+\mu(1-bk))+2c(1+\mu)}\right)$ so that $k = \left(\frac{1+\mu(1-bk)}{2b(1+\mu(1-bk))+2c(1+\mu)}\right)$. Isolating k yields

$$k = \frac{b(2+3\mu) + 2c(1+\mu) - \sqrt{(b(\mu+2) + 2c(\mu+1))^2 - 8bc\mu(\mu+1)}}{4b^2\mu}$$

$\square$

Proof of Proposition 3. As in the previous proposition firms have the posterior $\gamma_i(p) = \mu_i$, however now $\mu_i = 1 - \mu_j$, where the priors are not necessarily the same. Each firm solves

$$\max_q \quad (1+\mu)(q \cdot p - q^2(b+c)) - \mu_i q \cdot bk_j(p - b \cdot q)$$

Where k_j is the assumed constant of the rival firm as a function of their prior. As above this yields the equation

$$k_i = \frac{1 + \mu_i(1 - bk_j)}{2b(1 + \mu_i(1 - bk_j)) + 2c(1 + \mu_i)}$$

However, unlike the previous case k_j is not necessarily equal to k_i because they firms might have different priors. But due to the common prior assumption firm one solves for k_1 under the assumption that

$$k_j = \frac{1 + \mu_j(1 - bk_i)}{2b(1 + \mu_j(1 - bk_i)) + 2c(1 + \mu_j)}$$

Solving for $k(\mu)$ yields

$$k(\mu) = \frac{b^2(3\mu^2 + \mu - 10) + 8bc(\mu+1)(\mu-2) + 4c^2(\mu+1)(\mu-2) + \sqrt{A(b,c,\mu)}}{4b^2(1-\mu)(b(2+\mu) + 2c(1+\mu))}$$

where

$$\begin{aligned} A(b,c,\mu) = {} & b^4(\mu^2 - \mu - 6)^2 + 16b^3c(\mu^4 - 2\mu^3 - 7\mu^2 + 8\mu + 12) \\ & + 8b^2c^2(7\mu^4 - 14\mu^3 - 29\mu^2 + 36\mu + 44) + 16c^3(4b + c)(\mu^2 - \mu - 2)^2 \end{aligned}$$

$\square$

Proof of Proposition 4. Suppose all other firms play a linear strategy $q = k_1 p$. Then firm i will choose quantity $q_i = kp_i$, where p_i is the residual price after $i-1$ firms set quantities $q_1, \ldots, q_{i-1}$. Then $q_{i+1} = k_1 p_{i+1} = k_1(p_i - bkp_i) = k_1 p_i(1 - bk)$, $q_{i+2} = k_1 p_{i+2} =$

$k_1(p_{i+1} - bk_1 p_{i+1}) = k_1 p_{i+1}(1 - bk_1) = k_1 p_i(1 - bk_1)(1 - bk)$, and inductively, $q_{i+j} = p_i(1 - bk)k_1(1 - bk_1)^{j-1}$. If μ_i is the probability for firm i can be written as

$$\pi = q(p - q(b + c)) - qb \sum_{m=1}^{n} \mu_m \sum_{j=1}^{n-m} q_{i+j}$$

If we assume a uniform prior so that $\mu_m = \frac{1}{n}$ for all m

$$\begin{aligned}
\pi &= q(p - q(b + c)) - qb \sum_{m=1}^{n} \frac{1}{n} \sum_{j=1}^{n-m} p(1 - bk)k_1(1 - bk_1)^{j-1} \\
&= q(p - q(b + c)) - qbp(1 - bk)k_1 \frac{1}{n} \sum_{m=1}^{n} (n - m)(1 - bk_1)^{m-1}
\end{aligned}$$

and imposing that $q = kp$

$$= p^2 \left\{ k - k^2(b + c) - kb(1 - kb)k_1 \frac{1}{n} \sum_{m=1}^{n} (n - m)(1 - bk_1)^{m-1} \right\}$$

Solving for the optimal k gives first order condition

$$p^2 \left\{ 1 - 2k(b + c) - b(1 - 2kb)k_1 \frac{1}{n} \sum_{m=1}^{n} (n - m)(1 - bk_1)^{m-1} \right\} = 0$$

Using properties of geometric sums it can be shown that $\sum_{m=1}^{n} (n - m)r^{m-1} = \frac{n(1-r)-1+r^n}{(1-r)^2}$, so that replacing $r = 1 - bk_1$,

$$\sum_{m=1}^{n} (n - m)(1 - bk_1)^{m-1} = \frac{nbk_1 - 1 + (1 - bk_1)^n}{(bk_1)^2}$$

so that the first order condition becomes

$$p^2 \left\{ 1 - 2k(b + c) - b(1 - 2kb)k_1 \left(\frac{nbk_1 - 1 + (1 - bk_1)^n}{n(bk_1)^2} \right) \right\} = 0$$

Imposing symmetry in the equilibrium so that $k = k_1$ this reduces to

$$(1 - 2kb)(1 - (1 - kb)^n) = 2bcnk^2$$

□