Social learning with limited histories[*]

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Abstract

I adapt the standard observational learning environment and introduce a limited history of observation. When agents can only observe the action of the previous agent, complete learning still occurs but with a loss of welfare. When a limited history is coupled with uncertainty over position in the queue of actors, welfare further drops - increasing in uncertainty - but complete learning still occurs in the limit. These results are illustrated with a canonical linear model but learning holds in a more general setting satisfying the usual social learning assumptions.

1 Introduction

The social learning literature has highlighted the tension between the presence of sufficient information for full learning of the true state, and the rationality of agents ignoring their private information and joining a herd, leading to incomplete learning with positive probability. This literature was sparked by the work of Banerjee (1992) [2] and Bikhchandani, Hirshleifer, and, Welch (1992) [3], who introduce a framework of identical agents receiving independent and identically distributed signals. The critical insight is that as a result of observing the full history of previous actions, agents may find it rational to ignore their private signal and infer the state of the world from the actions of previous decision makers. Since agents are identical this implies all future agents face the same decision, and an "information cascade" occurs whereby all agents ignore their private information.

There have been many extensions to this framework that allow for heterogeneous agent types, limited observable histories, or more generally the formation of networks of viewable histories, either exogenously formed or formed endogenously subject to a cost. However, little attention has been given to the social learning problem in which agents do not have full information about their position in the chain of decision makers.

The classic social learning example of deciding whether to eat at a restaurant or stay home suffices in demonstrating how strong are the assumptions of the social learning model. The story goes that a new restaurant opens in town and patrons must decide whether to visit the new eatery or stay home by using their private signal and observing the choices of others. As it goes, the agents later in line for dinner are able to infer the signals of earlier agents through their actions, with such inference either buttressing or altogether overriding their own private signal. But of course this depends on a full observation of the history of actions.

This assumption is actually a composite of two assumptions. The first is that every previous action is observed. This assumption may be reasonable for early diners, but the idea of later diners spending all evening staking out the restaurant in order to make their decision is implausible. Even if such observation were possible indirectly, through word of mouth or consolidated review sources (i.e. Yelp or Google), observations are sure to get lost as the restaurant remains open over a longer period of time. And as time goes on, not only do observations get lost, but the number of choices get lost, highlighting the second assumption in social learning that agents know their position. It may be possible on opening night for a diner to know if they are among the first hundred patrons, but after the restaurant has been open a year diners might not even know if they are among the first hundred thousand!

It is the goal of this paper to investigate a social learning environment in which agents have only a limited history of observable actions. While unbounded signals ensure complete learning in the case of a fully observable history, a limited history leads to faster learning (in terms of convergence of decision thresholds), but a lower expected utility. Complete learning is also shown in the case of positional uncertainty but at the cost of a further decline in expected utility.

1.1 Related literature

The model presented here is most similar to the framework of Bikhchandani, Hirshleifer, and, Welch (1992) [3] (henceforth BHW). BHW present a framework in which agents observe conditionally independent signals, as well as the actions of all previous agents. They show that rational agents enter into a herd, ignoring their private information when deciding on the choice of action, with positive probability.

The work of Banerjee (1992) [2] also helped spark the herding literature. This model differs in that agents face a continuum of choices, with only one (unknown) correct choice for the state of the world. In addition, only some agents receive an informative signal, and it is only known to the agent whether or not she has a signal. As in BHW, Banerjee shows that agents rationally converge to a herd, even if they have an informative signal. This

result is again driven by the fact that signals are not perfectly informative, so it may be more reasonable to discard private information by inferring the state of the world from the actions of others.

Smith and Sørensen (2000) [7] investigate social learning with heterogeneous agents. They discuss the concept of full learning, where the probability of taking the right action tends to 1 as the number of agents increases. They identify the importance of unbounded signals: if signals can be arbitrarily precise, there is always a probability of a strong signal overturning a herd. This differs from BHW in that signal precision is heterogeneous, so even in the presence of a strong herd a well-informed agent can change public opinion.

The notion of endogenous timing in social learning was explored by Gul and Lundholm (1995) [5]. They investigate a setting in which agents receive payoff-relevant signals, and attempt to guess the sum of these signals. Agents choose when to make their prediction, with an associated cost to waiting. They show that since agents with higher signals perceive a higher opportunity cost to waiting, they will act sooner. Given that higher signals convey more information, endogenizing the timing actually results in the efficient ordering of agents' actions.

Limited observable histories has been investigated through the idea of a "network" which describes the set of actions a given agent can observe. Acemoglu et al. (2011) [1] identify the unboundedness of networks as the condition that guarantees full learning. That is, as long as the size of a given agent's network is not bounded by some integer, convergence to the correct action occurs at the limit, so there is full learning. Song (2014) [8] arrives at a similar finding in a setting where networks are formed endogenously subject to a cost.

An early theoretical work addressing social learning with limited histories is Çelen, B. and Kariv, S. (2004) [4], wherein agents decide between actions sequentially with the aid of a private signal and observation of the previous action. This model, however, featured a payoff as the sum of signals, a departure from the traditional framework of a correct action for each state. In fact in their model the state itself changes as the sum of signals oscillates between negative and positive. The present work attempts to apply the traditional social learning framework to a setting of limited observation, showing that complete learning still holds under the usual assumptions.

Perhaps the most closely related work is Monzón and Rapp (2014). In this model agents receive private signals but their observational history is limited to a random sampling of previous decision makers. They show that social learning persists under positional uncertainty, provided that action samples satisfy a stationarity assumption whereby they cannot be from the too distant past. This work also demonstrates the welfare loss of positional uncertainty. The focus of this work is a stationarity assumption, whereas at present we focus on the

role of beliefs over position and how changes in these beliefs affects learning and welfare in equilibrium.

2 Model

Suppose N agents decide sequentially between one of two actions, $a \in \mathcal{A} = \{0,1\}$ in an exogenously determined order. There are two states of the world, $\Omega = \{l, h\}$, and all agents agree that it is preferable to take action $a = 1$ in state h and $a = 0$ in state l. As such, agents share common risk-neutral vN-M utilities $u_n(1,h) = u_n(0,l) = 1$ and $u_n(0,h) = u_n(1,l) = 0$. In addition agents share a common flat prior $\Pr(h) = \Pr(l) = \frac{1}{2}$.

2.1 Private signals

Before deciding on an action each agent receives a private signal $\theta \in [-1, 1]$ about the state of the world. The signal θ is distributed according to $F = \{F_\omega(\theta)\}_{\omega \in \Omega}$, and conditional on the state ω signals are drawn independently. We will assume the signal distributions are continuous and admit density functions.

Assumption 1 ($\mathcal{C}^1$). *Signal distributions F_l and F_h are continuously differentiable. Denote their densities as f_l and f_h, respectively.*

We will require the usual (strict) monotone likelihood ratio property suggesting it is more likely to receive high values of signal θ in state h and low values in state l.

Assumption 2 (**MLRP**). *The distribution functions f_l and f_h satisfy the (strict) monotone likelihood ratio property in the sense that $\frac{f_h(\theta)}{f_l(\theta)}$ is strictly increasing in θ.*

We will assume that $F_h(\theta)$ and $F_l(\theta)$ are mutually absolutely continuous on the interval $[-1, 1]$. While this rules out any signal being perfectly informative of the state, we will assume that signals can come pretty close in the sense of an unbounded likelihood ratio.

Assumption 3 (**Unbounded signal strength**). *The informativeness of signal θ is unbounded in the sense that*

$$\lim_{\theta \to -1} \frac{f_h(\theta)}{f_l(\theta)} = 0 \quad and \quad \lim_{\theta \to 1} \frac{f_h(\theta)}{f_l(\theta)} = \infty$$

And finally, to avoid diverting the analysis from the implications of the learning environment through unnecessary complication, we will assume that the state dependent distributions are mutually symmetric about zero, though the results hold in the absence of this assumption.

Assumption 4 (**Mutual Symmetry**). *Signal distributions F_l and F_h are mutually symmetric in the sense that for all $\theta \in supp(F)$, $f_l(\theta) = f_h(-\theta)$.*

2.2 Observable histories

In addition to receiving the conditionally independent private signal θ, agents observe a history $H_n \subseteq \{a_1, a_2, \ldots, a_{n-1}\}$ of actions of preceding agents. Action profiles of the $n-1$ agents who have moved by the start of period n take realizations $A_{n-1} \in \mathcal{A}^{n-1}$. Letting $H_n = A_{n-1}$ collapses the problem to the traditional sequential learning framework a la Smith and Sørensen. While our focus is social learning settings with limited observable histories, we will be interested in the traditional framework of fully observable histories as a baseline for comparison.

2.3 Equilibrium

The preliminaries above define a social learning game.

Definition 1. *Let* $\Gamma(H_n) = \{F, u_n, a_n, H_n\}_{n=1}^{N}$ *denote a social learning game satisfying assumptions (A1)-(A4) with history* $H_n \subset A_{n-1}$.

In equilibrium each agent chooses $a_n \in \mathcal{A}$ to maximize expected utility $\mathbb{E}[u(a_n, \omega)|\theta_n, H_n]$. Given the assumption of monotonicity on the likelihood ratio, a natural notion of equilibrium is that of a threshold $\hat{\theta}$ which if exceeded will induce an agent to take action $a_n = 1$.

Definition 2. *Agent n follows a threshold strategy if*

$$a_n = \begin{cases} 1 & \text{if } \theta_n \geq \hat{\theta}_n \\ 0 & \text{if } \theta_n < \hat{\theta}_n \end{cases}$$

for some $\hat{\theta}_n \in supp(F)$.

Since the probability distribution has no masses, the tie breaking rule for $\theta_n = \hat{\theta}_n$ will play no role in the analysis. Under the above assumptions, an equilibrium in which players utilize threshold strategies always exists.

Proposition 1. *For social learning game* $\Gamma(H_n)$ *with* $H_n \subset A_{n-1}$, *a threshold strategy equilibrium exists.*

The existence of a threshold strategy equilibrium follows easily from the monotone likelihood ratio property. All proofs are relegated to the appendix.

2.4 Social learning

Finally, we will examine the information aggregation properties of any equilibrium. In particular it will be of interest whether given a large enough game of social learning, agents tend to take the right action. For this we introduce a natural definition of learning.
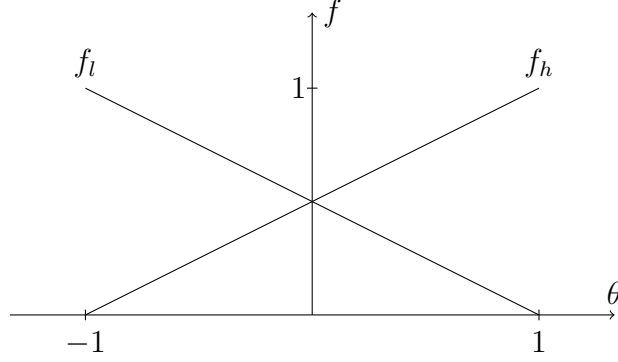
Figure 1: Final market price for certain and uncertain position

Definition 3. *For a social learning game $\Gamma(H_n)$, we will say that complete learning occurs if $\lim_{n\to\infty} \Pr(a_n = 1|h) = 1$ and $\lim_{n\to\infty} \Pr(a_n = 0|l) = 1$.*

3 The linear case

To fix ideas consider the distribution functions $F_h(\theta) = \frac{1}{4}(1+\theta)^2$ and $F_l(\theta) = 1 - \frac{1}{4}(1-\theta)^2$ which admit linear densities $f_h(\theta) = \frac{1}{2}(1+\theta)$ and $f_l(\theta) = \frac{1}{2}(1-\theta)$.

The probability densities depicted above demonstrate the linear manner in which higher signals becoming more likely than low signals in state $\omega = h$. It can easily be verified that the densities f_l and f_h satisfy assumptions **(A1)-(A4)**.

3.1 Fully observable history

Consider first the case where the complete history of actions taken by preceding agents is observable (e.g. $H_n = A_{n-1}$). The first agent has no predecessor and thus observes the history $A_0 = \varnothing$. Given that $a_1 = 1$ is preferred in state $\omega = h$ and $a_1 = 0$ is preferred in state $\omega = l$, with utilities u_n the agent will choose $a_1 = 1$ if and only if

$$\Pr(h|\theta_1) \geq \Pr(l|\theta_1) \iff \Pr(\theta_1|h)\Pr(h) \geq \Pr(\theta_1|l)\Pr(l) \iff f_h(\theta_1) \geq f_l(\theta_1)$$

and with flat prior $\Pr(h) = \frac{1}{2}$

$$\frac{1}{2}f_h(\theta) \geq \frac{1}{2}f_l(\theta) \iff \frac{1}{2}(1+\theta) \geq \frac{1}{2}(1-\theta)$$

which reduces to $\theta_1 \geq 0$ so that $\hat{\theta}_1 = 0$.

Having observed a_1, the second agent will choose $a_2 = 1$ if and only if

$$\Pr(h|\theta_2, a_1) \geq \Pr(l|\theta_2, a_1) \iff \Pr(\theta_2, a_1|h) \geq \Pr(\theta_2, a_1|l)$$

The threshold $\hat{\theta}_2$ will depend on a_1, with $\Pr(a_1 = 1|\omega) = \Pr(\theta \geq \hat{\theta}_1|\omega) = 1 - F_\omega(\theta_1)$ and $\Pr(a_1 = 0|\omega) = F_\omega(\hat{\theta}_1)$. If $a_1 = 1$, then since $\hat{\theta}_1 = 0$ agent 2 will choose $a_2 = 1$ if

$$\Pr(\theta_2, \theta_1 > 0|h)\Pr(h) \geq \Pr(\theta_2, \theta_1 > 0|l)\Pr(l)$$
$$\iff f_h(\theta_2)(1 - F_h(0))\Pr(h) \geq f_l(\theta_2)(1 - F_l(0))\Pr(l)$$
$$\iff \frac{1}{2}(1 + \theta_2)(1 - \frac{1}{4}(1+0)^2)\frac{1}{2} \geq \frac{1}{2}(1 - \theta_2)(\frac{1}{4}(1-0)^2)\frac{1}{2}$$

where the second inequality comes from the conditional independence of the signal. The threshold then reduces to $\theta_2 \geq -\frac{1}{2}$. A similar calculation shows that agent 2 chooses the high action after $a_1 = 0$ if $\theta \geq \frac{1}{2}$, yielding the conditional threshold

$$\hat{\theta}_2 = \begin{cases} -\frac{1}{2} & \text{if } a_1 = 1 \\ \frac{1}{2} & \text{if } a_1 = 0 \end{cases}$$

The equilibrium threshold for an arbitrary agent n is solved in much the same way, taking into account the entire history A_{n-1} leading up to the decision to act. As above, agent n will choose $a_n = 1$ if and only if

$$f_h(\theta_n)\Pr(A_{n-1}|h)\Pr(h) \geq f_l(\theta_n)\Pr(A_{n-1}|l)\Pr(l)$$

so that the threshold is defined by the likelihood ratio

$$\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} = \frac{\Pr(A_{n-1}|l)\Pr(l)}{\Pr(A_{n-1}|h)\Pr(h)}$$

While each threshold strategy $\hat{\theta}_n$ depends on the entire history of actions A_{n-1}, in comparing thresholds $\hat{\theta}_n$ and $\hat{\theta}_{n-1}$, the only informational asymmetry between agents n and $(n-1)$ is in the realization of θ_{n-1}, known only to $(n-1)$. Since the threshold $\hat{\theta}_{n-1}$ already contains information about the full history A_{n-2} up to the decision a_{n-1}, this suggests the possibility of a direct relationship between adjacent thresholds, enabling a recursive formulation of $\hat{\theta}_n$. Indeed this is the case.

Proposition 2. *For the social learning game with fully observable histories $\Gamma(A_{n-1})$ and canonical signal structure defined by $F_h(\theta) = \frac{1}{4}(1+\theta)^2$ and $F_l(\theta) = 1 - \frac{1}{4}(1-\theta)^2$, $\hat{\theta}_1 = 0$ and decision thresholds for $n \geq 2$ can be expressed recursively as*

$$\hat{\theta}_n = \begin{cases} \frac{-1}{2 + \hat{\theta}_{n-1}} & \text{if } a_{n-1} = 1 \\ \frac{1}{2 - \hat{\theta}_{n-1}} & \text{if } a_{n-1} = 0 \end{cases}$$

Since the canonical case satisfies all of the traditional social learning assumptions that guarantee complete learning (e.g. MLRP, unbounded signals), it should be no surprise that the thresholds $\hat{\theta}_n$ converge and that complete learning is indeed achieved with a linear signal structure. Given the form of the decision thresholds, the conditional expectation is easily calculated as

$$\mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, h] - \hat{\theta}_{n-1} = \frac{(\hat{\theta}_{n-1} + 1)^2(\hat{\theta}_{n-1} - 1)}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

and

$$\mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, l] - \hat{\theta}_{n-1} = \frac{(\hat{\theta}_{n-1} + 1)(\hat{\theta}_{n-1} - 1)^2}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

enabling application of the Martingale Convergence Theorem to yield the result.

Proposition 3. *For the social learning game with fully observable histories $\Gamma(A_{n-1})$ and the canonical signal structure, state dependent thresholds $\hat{\theta}_n(\omega)$ converge with $\lim_{n\to\infty} \hat{\theta}_n(l) = 1$, $\lim_{n\to\infty} \hat{\theta}_n(h) = -1$, and complete learning occurs.*

3.2 Limited histories

Suppose now that instead of observing the entire history of preceding agents A_{n-1}, histories are limited in that each agent can only observe the predecessor's action. The viewable history is then $H_n = a_{n-1}$. The first two movers will behave the same way because they observe the histories $A_0 = \varnothing$ and $A_1 = a_1$, respectively, exactly as before. Then $\hat{\theta}_1 = 0$; $\hat{\theta}_2 = -\frac{1}{2}$ if $a_1 = 1$ and $\hat{\theta}_2 = \frac{1}{2}$ if $a_1 = 0$. Now, however, $H_n \subsetneq A_{n-1}$ for $n \geq 2$ so agents will have less information with which to decide on an action a_n. With observable histories $H_n = a_{n-1}$, the thresholds are now defined by the likelihood ratio

$$\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} = \frac{\Pr(a_{n-1}|l)\Pr(l)}{\Pr(a_{n-1}|h)\Pr(h)}$$

Given that for each n there are only two possible histories H_n (with the exception of $n = 1$), we can reduce the decision of agent n to two thresholds

$$\hat{\theta}_n = \begin{cases} \bar{\theta}_n & \text{if } a_{n-1} = 1 \\ \underline{\theta}_n & \text{if } a_{n-1} = 0 \end{cases}$$

In the case of $n = 2$, $\bar{\theta}_2 = -\frac{1}{2}$ and $\underline{\theta}_2 = \frac{1}{2}$. Notice that $\bar{\theta}_n + \underline{\theta}_n = 0$ for $n = 2$. In fact this will be true for all n. Given the symmetry of the payoff function in states $\omega = \{l, h\}$ this result makes sense.

The departure from the case of fully observable histories begins with $n = 3$. Now agent n observes history $H_n = a_{n-1}$ but does not observe the action a_{n-2}. But the probability of a_{n-1} for a given state will depend on action a_{n-2}, which itself will depend on a_{n-3} and so on. The probability of observing an action then is

$$\Pr(a_{n-1}|\omega) = \sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1}|A_{n-2}, \omega) \Pr(A_{n-2}|\omega)$$

Given that the updated probability of a state depends on $n-2$ unobservable previous actions the agent must account for 2^{n-2} possible history profiles, and the decision thresholds take the cumbersome form

$$\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} = \frac{\sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1}|A_{n-2}, l) \Pr(A_{n-2}|l) \Pr(l)}{\sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1}|A_{n-2}, h) \Pr(A_{n-2}|h) \Pr(h)}$$

In the case of fully observable histories it was possible to solve for thresholds $\hat{\theta}_n$ recursively because both agents n and $(n-1)$ condition on A_{n-2}. But now agent $(n-1)$ conditions action a_{n-2} which is unobservable to agent n. Notice, however, that the thresholds $\bar{\theta}_{n-1}$ and $\underline{\theta}_{n-1}$ for agent $(n-1)$ depend on the action a_{n-2} according to

$$\frac{1 + \hat{\theta}_{n-1}}{1 - \hat{\theta}_{n-1}} = \frac{\sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-2}|A_{n-3}, l) \Pr(A_{n-3}|l) \Pr(l)}{\sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-2}|A_{n-3}, h) \Pr(A_{n-3}|h) \Pr(h)}$$

for $\hat{\theta}_{n-1} = \bar{\theta}_{n-1}$ or $\hat{\theta}_{n-1} = \underline{\theta}_{n-1}$ corresponding to $a_{n-2} = 1$ or $a_{n-2} = 0$, respectively. Since history A_{n-3} is unknown in both period n and $(n-1)$, this relationship enables player n to condition threshold $\hat{\theta}_n$ on only the two possible outcomes of a_{n-2}, greatly simplifying the problem and giving the following result.

Proposition 4. *For the social learning game with limited observable histories $\Gamma(a_{n-1})$ and the canonical signal structure, $\hat{\theta}_1 = 0$ and decision thresholds for $n \geq 2$ can be expressed recursively as*

$$\hat{\theta}_n = \begin{cases} \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) & \text{if } a_{n-1} = 1 \\ \underline{\theta}_n = \frac{1}{2}(1 + \underline{\theta}_{n-1}^2) & \text{if } a_{n-1} = 0 \end{cases}$$

As alluded to above and as thresholds $\hat{\theta}_n$ clearly show, the symmetric signal structure implies that the thresholds are also symmetric about zero for every n.

Corollary 1. *For the social learning game with limited observable histories $\Gamma(a_{n-1})$ and the canonical signal structure, $\bar{\theta}_n + \underline{\theta}_n = 0$.*
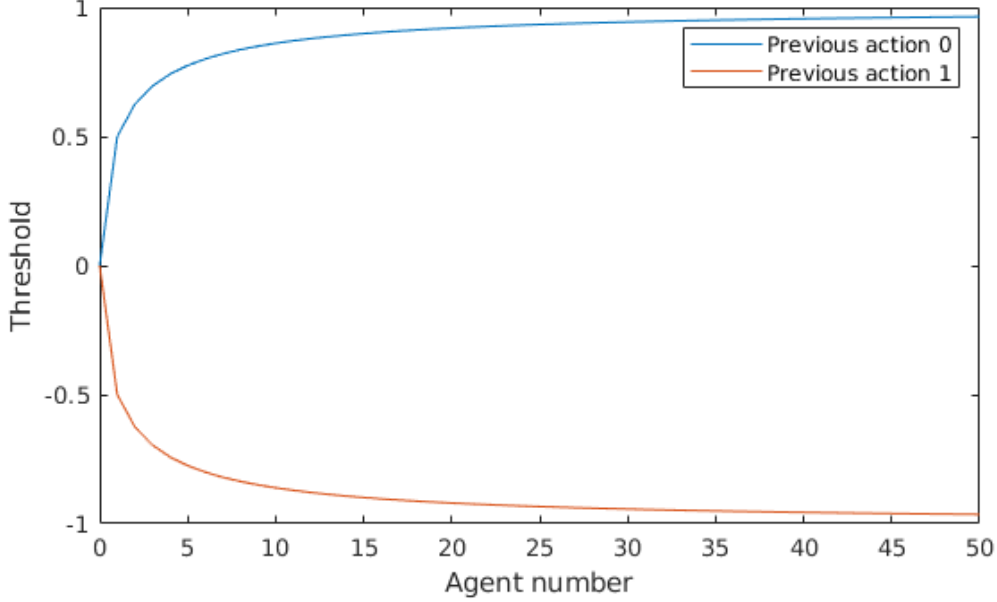
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Figure 2: Decision thresholds under limited histories of observation

The evolution of thresholds $\bar{\theta}_n$ and $\underline{\theta}_n$ is pictured below. As the figure shows, the bounds $\bar{\theta}_n$ and $\underline{\theta}_n$ diverge very quickly. This represents a higher standard of proof from signal θ_n in order deviate from previous action a_{n-1}.

The thresholds partition the signal space into three regions. When $\theta_n > \underline{\theta}_n$, the agent will follow their signal and play $a_n = 1$ independent of previous action a_{n-1}, believing the state $\omega = h$ to be more likely. When $\theta_n < \bar{\theta}_n$ the agent will believe $\omega = l$ is more likely and play $a_n = 0$. When $\theta_n \in (\bar{\theta}_n, \underline{\theta}_n)$, the threshold for following the private signal is not surpassed and the agent will always follow the previous action a_{n-1}.

The figure depicting thresholds in the case of limited observable history suggests convergence to the limits of the distribution, so that as the periods advance the signal strength required to deviate from imitation of the predecessor increases. This would imply complete learning even in the case of limited histories, which the following result confirms.

Proposition 5. *For the social learning game with limited observable histories $\Gamma(a_{n-1})$ and the canonical signal structure, thresholds $\bar{\theta}_n$ if $a_{n-1} = 1$ and $\underline{\theta}_n$ if $a_{n-1} = 0$ converge with $\lim_{n\to\infty} \bar{\theta}_n = -1$, $\lim_{n\to\infty} \underline{\theta}_n = 1$, and complete learning occurs.*

With fully observable histories, the threshold $\hat{\theta}_n$ was a recursive function of the previous agent's threshold, the function depending on the previous action a_{n-1}. Now, however, the threshold is solely determined by a_{n-1}, and as such sequences $\bar{\theta}_n$ and $\underline{\theta}_n$ take a predictable pattern. In fact as a result of this predictability of $\bar{\theta}_n$ and $\underline{\theta}_n$, it is possible that the martingale

$\hat{\theta}_n$ derived from fully history game $\Gamma(A_{n-1})$ does not converge as quickly as the thresholds in limited history game $\Gamma(a_{n-1})$. In fact, as the following figure shows, on average this is the case.
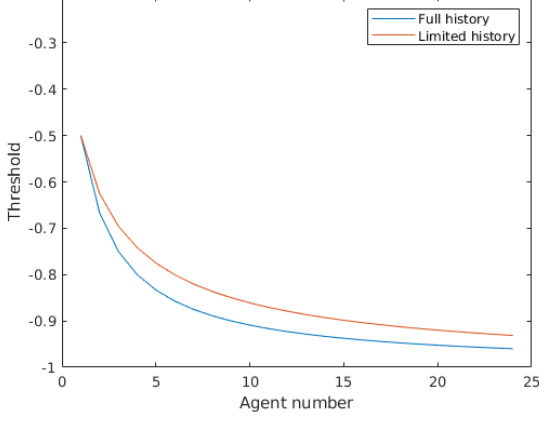

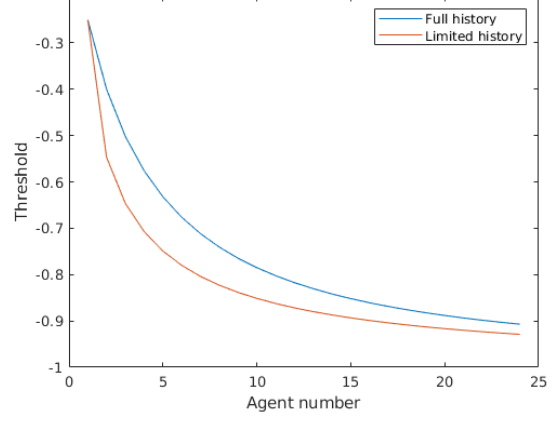Figure 3: Maximum threshold in state h

Figure 4: Expected threshold in state h

The panel on the left shows the maximal threshold values in the cases of full and limited history. In other words, these show the progression of the thresholds θ_n if $a_i = 1$ for all $i \leq n$. It is clear that with a history of only action $a_i = 1$ the threshold in the full information case converges more quickly than in the case of limited history. The right panel, however, shows that on average the threshold with limited history converges more quickly. In a sense, this reflects that with a limited history of observation, thresholds depend only on the previous action and are allowed to grow without respect to the full history. This fast growth is then reinforced by observing $a_{n-1} = 1$, given the strict threshold.

This interplay between history independence and a growing threshold suggests an increased possibility of error with limited observable histories. Comparing expected utilities highlights the welfare consequences of this error.

Figure 5 shows $\mathbb{E}[u(\theta)|H_n = A_{n-1}]$ and $\mathbb{E}[u(\theta)|H_n = a_{n-1}]$, the expected utilities with full and limited histories. It shows, as we would expect, that on average utility is higher with full information than under a limited history of observation. Even though the thresholds converge faster on average with a limited history, suggesting faster learning, in fact this reflects the loss of information as a result of limited observations.

This is again a depiction of the progression of expected thresholds θ_n, but the shaded region of figure 6 shows where $\mathbb{E}[\theta_n|H_n = A_{n-1}] > \theta > \mathbb{E}[\theta_n|H_n = a_{n-1}]$. This is where the realization of signal θ falls between the thresholds in the full history case and the limited history case. For signals in this region, agent n would follow their signal with a full history,
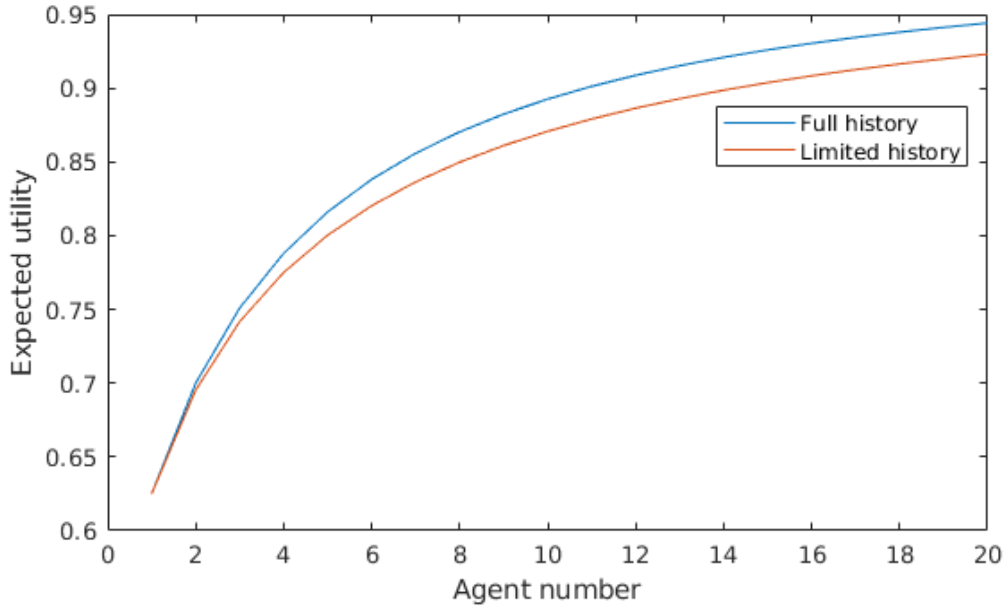
Figure 5: Expected utility in state h for full and limited histories of observation
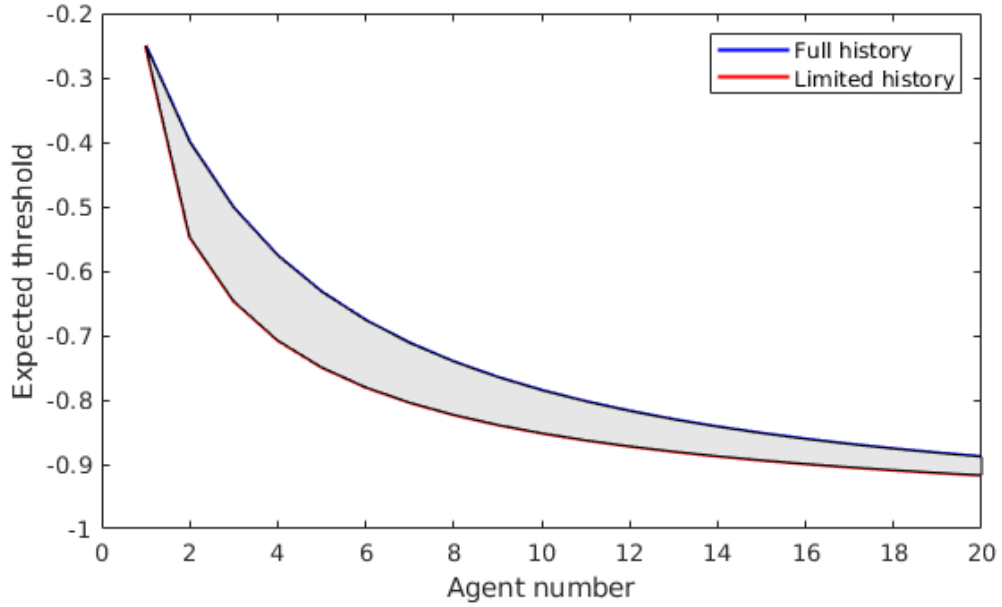

Figure 6: Expected information loss from limited observation

choosing $a_n = 1$ irrespective of the previous action, but would ignore the signal with a limited history, choosing $a_n = a_{n-1}$ even if $a_{n-1} = 0$. This increased possibility of error and propensity to discard information drives down expected utility under limited observational

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history.

3.3 Positional uncertainty

Now suppose that in addition to observing only the action of the preceding agent, each agent does not know their position. Instead agents hold beliefs μ^n over their positions, where μ_i^n is the probability agent n places on moving in position i. Since the first mover easily deduces being first by the absence of any preceding action, we introduce an agent in position 0 that chooses as the first agent in the case of no positional uncertainty: $a_0 = 1$ if and only if $\theta_0 \geq 0$.

Suppose beliefs μ take the form $\mu_n^n = \gamma$ and $\mu_i^n = \frac{1-\gamma}{N-1}$ for $i \neq n$, so that the agent in position n has a belief $\gamma \in [0,1]$ of their true position and spreads the additional probability $1 - \gamma$ uniformly across all other $N - 1$ positions. Then if agent n observes a_{n-1} the decision threshold is determined as before

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{\Pr(a_{n-1}|l)\Pr(l)}{\Pr(a_{n-1}|h)\Pr(h)}$$

$$= \frac{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|l)\Pr(l)}{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|h)\Pr(h)}$$

with the linear form of our signals and given that $\Pr(h) = \Pr(l)$

$$\frac{1+\hat{\theta}_n}{1-\hat{\theta}_n} = \frac{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|l)\Pr(l)}{\sum_{i=1}^N \mu_i^n \Pr(a_{i-1}|h)\Pr(h)}$$

$$\hat{\theta}_n = \frac{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1}|l) - \Pr(a_{i-1}|h))}{\sum_{i=1}^N \mu_i^n (\Pr(a_{i-1}|l) + \Pr(a_{i-1}|h))}$$

The assumed form of our probabilities μ^n yield the following result.

Proposition 6. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n, and the canonical signal structure, a threshold equilibrium can be defined recursively as*

$$\bar{\theta}_1 = \frac{N(\gamma - 2) + 1}{2(N-1)} + 2\left(\frac{1-\gamma}{N-1}\right)\sum_{i=0}^{N-1}\Pr(a_i = 1|l)$$

$$\underline{\theta}_1 = \frac{N(3\gamma - 2) - 1}{2(N-1)} + 2\left(\frac{1-\gamma}{N-1}\right)\sum_{i=0}^{N-1}\Pr(a_i = 0|l)$$

and for $n \geq 2$

$$\bar{\theta}_n = 2 \left(\frac{N\gamma - 1}{N - 1} \right) \Pr(a_{n-1} = 1|l) + \bar{\theta}_1 - \frac{N\gamma - 1}{2(N - 1)}$$

$$\underline{\theta}_n = 2 \left(\frac{N\gamma - 1}{N - 1} \right) \Pr(a_{n-1} = 0|l) + \underline{\theta}_1 - \frac{3(N\gamma - 1)}{2(N - 1)}$$

As the thresholds make clear, the action dependent signals $\bar{\theta}_n$ and $\underline{\theta}_n$ exhibit the same symmetry about zero as in the case without positional uncertainty.

Corollary 2. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n, and the canonical signal structure, $\bar{\theta}_n + \underline{\theta}_n = 0$.*

The figure below shows the evolution of thresholds which display the downward trend that we have come to expect, suggestive of convergence to a limit.
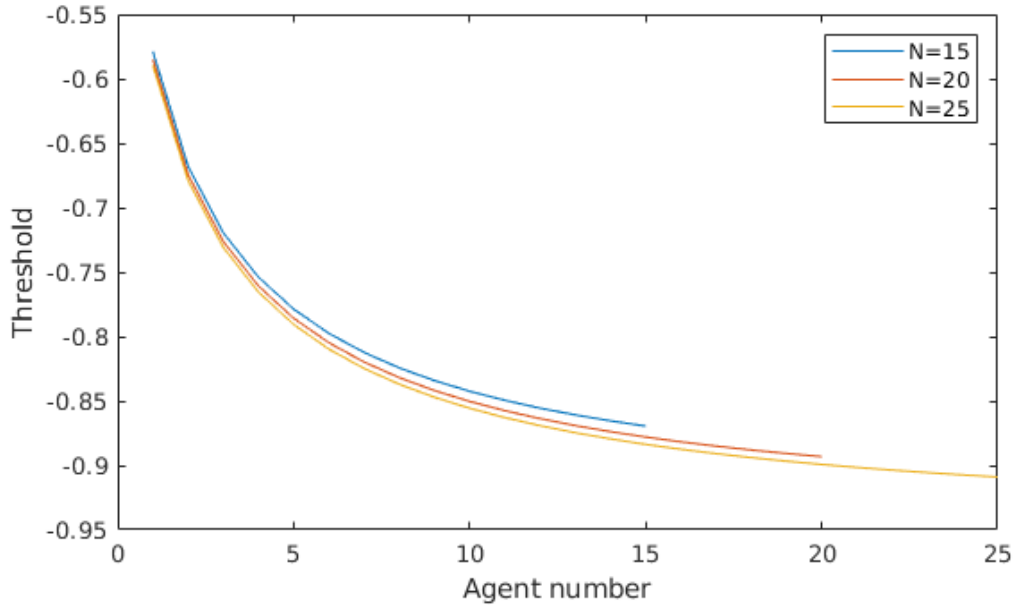

Figure 7: Thresholds under posional uncertainty for various N

Now, however, each agent n holds the belief $\gamma < 1$ that they act in the position n that they indeed do. This lack of certainty over position could translate into a lack of certainty over the true state of the world, leading to a limit $\bar{\theta} > -1$ or $\bar{\theta} < 1$. Fortunately, it turns out that if such a limit exists, this limit must be $\bar{\theta} = -1$ or $\bar{\theta} = 1$. While the speed of this convergence will depend on belief parameter γ, complete learning occurs in the limit for all beliefs.

Proposition 7. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n, and the canonical signal structure, $\lim_{n\to\infty} \bar{\theta}_n = -1$ and $\lim_{n\to\infty} \underline{\theta}_n = 1$ if such limits exist. Moreover, if these limits exist then complete learning occurs.*

Notice also that the positional probability beliefs depend on N, and thus so do the thresholds. As the figure above shows, the larger is the number of agents N, the faster is the convergence of the threshold to its limit. As the number of agents increases, the belief of moving in any position other than n becomes diluted. This applies particularly to early positions where the predecessor faced a relatively low threshold, thereby inducing the successor to require a higher standard of proof. As this probability decreases, each agent relies more strongly on the true prior action a_{n-1}, thus leading to faster convergence of the threshold.

Since the object of interest will be the evolution of learning as the number of agents observing histories increases, we will focus on thresholds for large N, which take a convenient form.

Proposition 8. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n, and the canonical signal structure*

$$\lim_{N\to\infty} \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) - \left(\frac{1-\gamma}{2}\right)(1 + \bar{\theta}_{n-1})^2$$

$$\lim_{N\to\infty} \underline{\theta}_n = \frac{1}{2}(1 + \bar{\theta}_{n-1}^2) + \left(\frac{1-\gamma}{2}\right)(1 + \bar{\theta}_{n-1})^2$$

An obvious consequence of the form the thresholds take is that the introduction of term $\left(\frac{1-\gamma}{2}\right)(1 + \bar{\theta}_{n-1})^2$ leads to a lower (higher) value of threshold $\bar{\theta}_n(\underline{\theta}_n)$ for every n. This leads to a faster convergence to the limit, a rate which increases as belief γ decreases.

The above graph shows this relationship between γ and the rate of convergence, and in fact at the extreme of $\gamma \to 0$ the threshold converges immediately to $\bar{\theta}_n = -1$ for all n. Immediate convergence is the result of total positional uncertainty, whereby it makes more sense to each agent to follow the action of the previous agent because they have no concept of their own signal's informational value.

As in the case of limited history we can compare expected utility to get a more complete story of the welfare implications of this convergence. We would expect that the increased rate of convergence to lead to a further loss of welfare, increasing the region where agents ignore their information.

As expected, the figure above shows this exact result. The panel on the left shows a lower expected utility under positional uncertainty characterized by $\gamma = 0.75$, while the right panel shows an even further loss of utility for $\gamma = 0.5$. In fact expected utility in the case of limited history can be shown to take an explicit form.
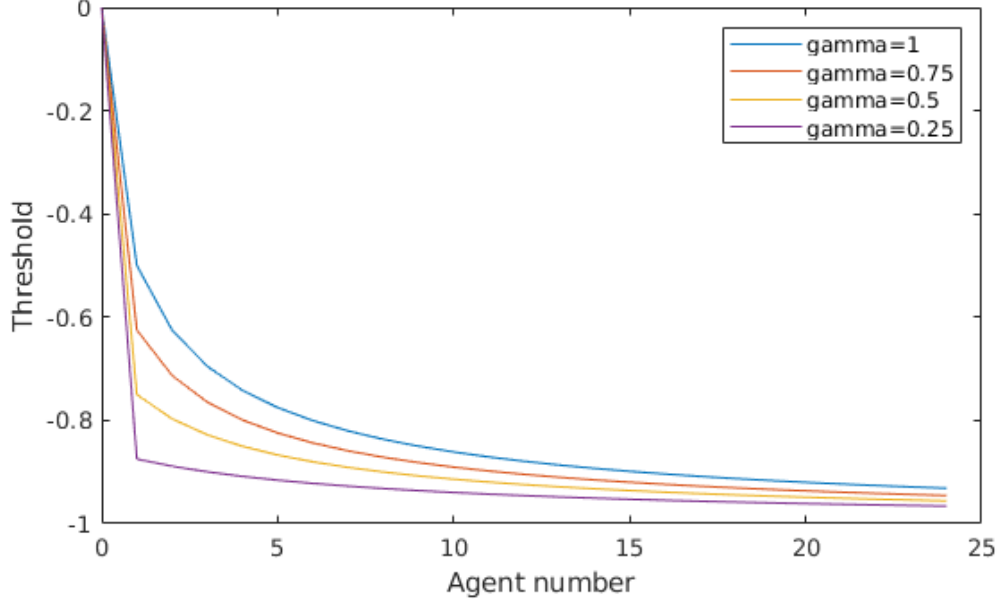
Figure 8: Thresholds under posional uncertainty with large N for various γ

Proposition 9. *Under positional uncertainty*

$$\mathbb{E}[u(\theta)|H_n = a_{n-1}, \mu^n] = \frac{1}{2\gamma}(1 + \bar{\theta}_n^2) + \frac{1-\gamma}{2\gamma}(\bar{\theta}_n^2 + 2\bar{\theta}_n - 1)$$

Comparative analysis on the parameter γ confirms the result that expected utility under positional uncertainty $\mathbb{E}[u(\theta)|H_n = a_{n-1}, \mu^n]$ indeed decreases as uncertainty γ increases.

4 The general case

While much of the above used the canonical signal structure $F_h(\theta) = \frac{1}{4}(1 + \theta^2)$ and $F_l(\theta) = 1 - \frac{1}{4}(1 - \theta^2)$, many of the results hold for more general signal structures that satisfy the assumptions **(A1)**-**(A4)**. Of course, the result of complete learning should be no surprise, as it has been the focus of much theoretical work in the area of social learning.

Proposition 10. *For the social learning game with fully observable histories $\Gamma(A_{t-1})$ and a signal structure F satisfying **(A1)**-**(A4)**, state dependent thresholds $\hat{\theta}_n(\omega)$ converge with $\lim_{n\to\infty} \hat{\theta}_n(l) = 1$, $\lim_{n\to\infty} \hat{\theta}_n(h) = -1$, and complete learning occurs.*

The focus of this work, social learning in an environment with a limited history of observation, also features complete learning in the more general setting.
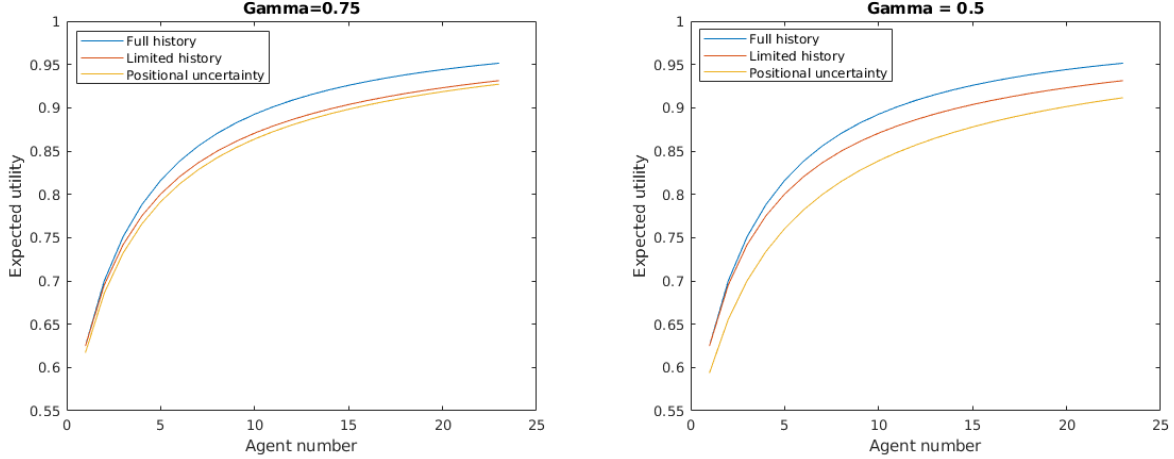
Figure 9: Relationship between expected utility and gamma in state h

Proposition 11. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$ and a signal structure F satisfying **(A1)-(A4)**, thresholds $\bar{\theta}_n$ if $a_{n-1} = 1$ and $\underline{\theta}_n$ if $a_{n-1} = 0$ converge with $\lim_{n\to\infty} \bar{\theta}_n = -1$, $\lim_{n\to\infty} \underline{\theta}_n = 1$, and complete learning occurs.*

Moreover, complete learning in an environment of positional uncertainty also holds for a general signal structure F.

Proposition 12. *For the social learning game with limited observable histories $\Gamma(a_{t-1})$, positional uncertainty μ^n, and a signal structure F satisfying **(A1)-(A4)**, $\lim_{n\to\infty} \bar{\theta}_n = -1$ and $\lim_{n\to\infty} \underline{\theta}_n = 1$ if such limits exist. Moreover, if these limits exist then complete learning occurs.*

5   Concluding remarks

The traditional model of social learning offers powerfully intuitive results on how the courtship of private information and observation leads to informed economic decision making. But this marriage is only as strong as the assumptions it stands upon. In particular, if the assumptions of fully observable histories and certainty about position in the sequence of actors come into question, there are behavioral and welfare consequences that alter the learning dynamic. By addressing this we gain a richer depiction of an environment in which agents learn from an appreciably less learned starting point.

The introduction of limited observation of preceding actions to the standard social learning model changes the integration of information, but the limit result of complete learning remains. With agents only conditioning on the previous action, the threshold equilibria take

a predictable form, on average converging more quickly to the limit ensuring the correct action. Despite this, the increased possibility of discarding information increases the possibility for error, leading to a lower expected utility for each agent in finite time.

Complete learning in the limit continues to hold even when agents are uncertain of their position in the sequence. In fact, the threshold equilibria converge more quickly the higher is the uncertainty over position, exacerbating the reduction in expected utility of the limited history case. While the pace of learning in terms of welfare decreases with positional uncertainty, complete learning in the limit does not depend on its existence or magnitude.

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A1. Appendix

Proof of Proposition 1. Suppose agent n observes history H_n and receives signal θ_n. Then n will choose $a_n = 1$ if

$$\Pr(h|\theta_n, H_t) \geq \Pr(l|\theta_n, H_t) \iff \frac{\Pr(\theta_n, H_t|h)\Pr(h)}{\Pr(\theta_n, H_t)} \geq \frac{\Pr(\theta_n, H_t|l)\Pr(l)}{\Pr(\theta_n, H_t)}$$

$$\iff f_h(\theta_n)\Pr(H_t|h) \geq f_l(\theta_n)\Pr(H_t|l) \iff \frac{f_h(\theta_n)}{f_l(\theta_n)} \geq \frac{\Pr(H_t|l)}{\Pr(H_t|h)}$$

Given that $\frac{f_h(\theta_n)}{f_l(\theta_n)}$ is strictly increasing in θ_n there must be some $\hat{\theta}_n$ for which $\Pr(h|\theta_n, H_t) \geq \Pr(l|\theta_n, H_t)$ if $\theta_n \geq \hat{\theta}_n$ and $\Pr(h|\theta_n, H_t) < \Pr(l|\theta_n, H_t)$ otherwise. Thus n follows a threshold strategy. □

Proof of Proposition 2. Applying the result of lemma 1 to our canonical signal structure defined by $F_h(\theta) = \frac{1}{4}(1 + \theta)^2$ and $F_l(\theta) = 1 - \frac{1}{4}(1 - \theta)^2$, if $a_{n-1} = 1$,

$$\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} = \frac{\frac{1}{4}(1 - \hat{\theta}_{n-1})^2 \frac{1}{2}(1 + \hat{\theta}_{n-1})}{[1 - \frac{1}{4}(1 + \hat{\theta}_{n-1})^2]\frac{1}{2}(1 - \hat{\theta}_{n-1})} = \frac{(1 - \hat{\theta}_{n-1})(1 + \hat{\theta}_{n-1})}{3 - 2\hat{\theta}_{n-1} - \hat{\theta}_{n-1}^2}$$

$$= \frac{(1 - \hat{\theta}_{n-1})(1 + \hat{\theta}_{n-1})}{(3 + \hat{\theta}_{n-1})(1 - \hat{\theta}_{n-1})} = \frac{1 + \hat{\theta}_{n-1}}{3 + \hat{\theta}_{n-1}}$$

$$\implies \hat{\theta}_n = \frac{-1}{2 + \hat{\theta}_{n-1}}$$

and if $a_{n-1} = 0$

$$\frac{1 + \hat{\theta}_n}{1 - \hat{\theta}_n} = \frac{[1 - \frac{1}{4}(1 - \hat{\theta}_{n-1})^2]\frac{1}{2}(1 + \hat{\theta}_{n-1})}{\frac{1}{4}(1 + \hat{\theta}_{n-1})^2 \frac{1}{2}(1 - \hat{\theta}_{n-1})} = \frac{3 + 2\hat{\theta}_{n-1} - \hat{\theta}_{n-1}^2}{(1 + \hat{\theta}_{n-1})(1 - \hat{\theta}_{n-1})}$$

$$= \frac{(3 - \hat{\theta}_{n-1})(1 + \hat{\theta}_{n-1})}{(1 + \hat{\theta}_{n-1})(1 - \hat{\theta}_{n-1})} = \frac{3 - \hat{\theta}_{n-1}}{1 - \hat{\theta}_{n-1}}$$

$$\implies \hat{\theta}_n = \frac{1}{2 - \hat{\theta}_{n-1}}$$

Thus we have the recursive result that

$$\hat{\theta}_n = \begin{cases} \frac{-1}{2 + \hat{\theta}_{n-1}} & \text{if } a_{n-1} = 1 \\ \frac{1}{2 - \hat{\theta}_{n-1}} & \text{if } a_{n-1} = 0 \end{cases}$$

□

Proof of Proposition 3. Given the recursive form of the threshold strategy determined for the canonical case,

$$
\begin{aligned}
\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] &= \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 1|h) + \left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 0|h) \\
&= \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) (1 - \Pr(a_{n-1} = 0|h)) + \left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) \Pr(a_{n-1} = 0|h) \\
&= F_h(\hat{\theta}_{n-1}) \left(\left(\frac{1}{2 - \hat{\theta}_{n-1}} \right) - \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) \right) + \left(\frac{-1}{2 + \hat{\theta}_{n-1}} \right) \\
&= \frac{1}{4}(1 + \hat{\theta}_{n-1})^2 \left(\frac{4}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \right) - \frac{1}{2 + \hat{\theta}_{n-1}} \\
&= \frac{1 + 2\hat{\theta}_{n-1} + \hat{\theta}_{n-1}^2}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \frac{2 - \hat{\theta}_{n-1}}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] - \hat{\theta}_{n-1} &= \frac{\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \hat{\theta}_{n-1} \\
&= \frac{\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \frac{4\hat{\theta}_{n-1} - \hat{\theta}_{n-1}^3}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{\hat{\theta}_{n-1}^3 + \hat{\theta}_{n-1}^2 - \hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} \\
&= \frac{(\hat{\theta}_{n-1} + 1)^2(\hat{\theta}_{n-1} - 1)}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}
\end{aligned}
$$

Since $-1 \leq \hat{\theta}_{n-1} \leq 1$, all terms of $\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] - \hat{\theta}_{n-1}$ are positive except $(\hat{\theta}_{n-1} - 1) \leq 0$ so that $\mathbb{E}[\hat{\theta}_n | \hat{\theta}_{n-1}, h] \leq \hat{\theta}_{n-1}$. Then conditional on $\omega = h$, $\hat{\theta}_n$ is a supermartingale bounded below by -1 and thus must converge to a limit almost everywhere.

Similarly,

$$\mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, l] = \left(\frac{-1}{2 + \hat{\theta}_{n-1}}\right)\Pr(a_{n-1} = 1|l) + \left(\frac{1}{2 - \hat{\theta}_{n-1}}\right)\Pr(a_{n-1} = 0|l)$$

$$= \left(\frac{-1}{2 + \hat{\theta}_{n-1}}\right)\Pr(a_{n-1} = 1|l) + \left(\frac{1}{2 - \hat{\theta}_{n-1}}\right)(1 - \Pr(a_{n-1} = 1|l))$$

$$= (1 - F_l(\hat{\theta}_{n-1}))\left(\left(\frac{-1}{2 + \hat{\theta}_{n-1}}\right) - \left(\frac{1}{2 - \hat{\theta}_{n-1}}\right)\right) + \left(\frac{1}{2 - \hat{\theta}_{n-1}}\right)$$

$$= \frac{1}{4}(1 - \hat{\theta}_{n-1})^2\left(\frac{-4}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}\right) + \frac{1}{2 - \hat{\theta}_{n-1}}$$

$$= \frac{2 + \hat{\theta}_{n-1}}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \frac{1 - 2\hat{\theta}_{n-1} + \hat{\theta}_{n-1}^2}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

$$= \frac{-\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

so that

$$\mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, l] - \hat{\theta}_{n-1} = \frac{-\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \hat{\theta}_{n-1}$$

$$= \frac{-\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})} - \frac{4\hat{\theta}_{n-1} - \hat{\theta}_{n-1}^3}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

$$= \frac{\hat{\theta}_{n-1}^3 - \hat{\theta}_{n-1}^2 - \hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

$$= \frac{(\hat{\theta}_{n-1} + 1)(\hat{\theta}_{n-1} - 1)^2}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

Since $-1 \leq \hat{\theta}_{n-1} \leq 1$ all terms are nonnegative so that $\mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, l] \geq \hat{\theta}_{n-1}$. Then conditional on $\omega = l$, $\hat{\theta}_n$ is a submartingale bounded above by 1 and thus must converge to a limit almost everywhere.

Let $\bar{\theta} = \lim_{n \to \infty} \hat{\theta}_n(h)$ be the limit of the supermartingale $\hat{\theta}_n$ conditional on the state $\omega = h$. Then

$$\bar{\theta} = \lim_{n \to \infty} \mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, h] = \lim_{n \to \infty} \frac{\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} - 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

$$\implies \bar{\theta} = \frac{\bar{\theta}^2 + 3\bar{\theta} - 1}{(2 + \bar{\theta})(2 - \bar{\theta})}$$

which reduces to $(\bar{\theta} + 1)^2(\bar{\theta} - 1) = 0$. Then either $\bar{\theta} = 1$ or $\bar{\theta} = -1$. But as we saw above, $\hat{\theta}_{n-1} = 0$ if $n = 1$ so that $\mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, h] \leq 0$ for all $n \geq 1$. This only leaves $\bar{\theta} = -1$.

Let $\underline{\theta} = \lim_{n \to \infty} \hat{\theta}_n(l)$ be the limit of the submartingale $\hat{\theta}_n$ conditional on the state $\omega = l$. Then

$$\underline{\theta} = \mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, l] = \lim_{n \to \infty} \frac{-\hat{\theta}_{n-1}^2 + 3\hat{\theta}_{n-1} + 1}{(2 + \hat{\theta}_{n-1})(2 - \hat{\theta}_{n-1})}$$

$$\implies \underline{\theta} = \frac{-\underline{\theta}^2 + 3\underline{\theta} + 1}{(2 + \hat{\theta})(2 - \underline{\theta})}$$

which reduces to $(\underline{\theta} + 1)(\underline{\theta} - 1)^2 = 0$. Then either $\underline{\theta} = 1$ or $\underline{\theta} = -1$. But as we saw above, $\hat{\theta}_{n-1} = 0$ if $n = 1$ so that $\mathbb{E}[\hat{\theta}_n|\hat{\theta}_{n-1}, l] \geq 0$ for all $n \geq 1$. This only leaves $\underline{\theta} = 1$.

Finally, $\Pr(a_n = 1|h) = \Pr(\theta_n > \hat{\theta}_n|h) = 1 - F_h(\hat{\theta}_n) = 1 - \frac{1}{4}(1 + \hat{\theta}_n)^2$. Similarly $\Pr(a_n = 0|l) = \Pr(\theta_n \leq \hat{\theta}_n|l) = F_l(\hat{\theta}_n) = 1 - \frac{1}{4}(1 - \hat{\theta}_n)^2$. So then

$$\lim_{n \to \infty} \Pr(a_n = 1|h) = \lim_{n \to \infty} 1 - \frac{1}{4}\left(1 + \hat{\theta}_n(h)\right)^2 = 1 - \frac{1}{4}\left(1 + \lim_{n \to \infty} \hat{\theta}_n(h)\right)^2 = 1$$

$$\lim_{n \to \infty} \Pr(a_n = 0|l) = \lim_{n \to \infty} 1 - \frac{1}{4}\left(1 - \hat{\theta}_n(l)\right)^2 = 1 - \frac{1}{4}\left(1 - \lim_{n \to \infty} \hat{\theta}_n(l)\right)^2 = 1$$

so that $\lim_{n \to \infty} \Pr(a_n = 1|h) = \lim_{n \to \infty} \Pr(a_n = 0|l) = 1$ and complete learning occurs. $\quad\square$

Proof of Proposition 4. As shown above, the threshold for n is given by

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{\Pr(a_{n-1}|l)\Pr(l)}{\Pr(a_{n-1}|h)\Pr(h)}$$

$$= \frac{\sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1}|A_{n-2}, l)\Pr(A_{n-2}|l)\Pr(l)}{\sum_{A_{n-2} \in \mathcal{A}^{n-2}} \Pr(a_{n-1}|A_{n-2}, h)\Pr(A_{n-2}|h)\Pr(h)}$$

$$= \frac{\sum_{a_{n-2} \in \{0,1\}} \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-1}|a_{n-2}, A_{n-3}, l)\Pr(a_{n-2}|A_{n-3}, l)\Pr(A_{n-3}|l)}{\sum_{a_{n-2} \in \{0,1\}} \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-1}|a_{n-2}, A_{n-3}, h)\Pr(a_{n-2}|A_{n-3}, h)\Pr(A_{n-3}|h)}$$

Notice that since $H_{n-1} = a_{n-2}$ the action of $(n-1)$ is independent of history A_{n-3} so this reduces to

$$= \frac{\sum_{a_{n-2} \in \{0,1\}} \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-1}|a_{n-2}, l)\Pr(a_{n-2}|A_{n-3}, l)\Pr(A_{n-3}|l)}{\sum_{a_{n-2} \in \{0,1\}} \sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-1}|a_{n-2}, h)\Pr(a_{n-2}|A_{n-3}, h)\Pr(A_{n-3}|h)}$$

$$= \frac{\sum_{a_{n-2} \in \{0,1\}} \Pr(a_{n-1}|a_{n-2}, l)\sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-2}|A_{n-3}, l)\Pr(A_{n-3}|l)}{\sum_{a_{n-2} \in \{0,1\}} \Pr(a_{n-1}|a_{n-2}, h)\sum_{A_{n-3} \in \mathcal{A}^{n-3}} \Pr(a_{n-2}|A_{n-3}, h)\Pr(A_{n-3}|h)}$$

$$= \frac{\sum_{a_{n-2} \in \{0,1\}} \Pr(a_{n-1}|a_{n-2}, l)\Pr(a_{n-2}|l)}{\sum_{a_{n-2} \in \{0,1\}} \Pr(a_{n-1}|a_{n-2}, h)\Pr(a_{n-2}|h)}$$

The threshold for agent $(n-1)$ is given by the solution

$$\frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} = \frac{\Pr(a_{n-2}|l)\Pr(l)}{\Pr(a_{n-2}|h)\Pr(h)}$$

If thresholds $\bar{\theta}_{n-1}$ and $\underline{\theta}_{n-1}$ correspond to $a_{n-2} = 1$ and $a_{n-2} = 0$ respectively then

$$\frac{f_h(\bar{\theta}_{n-1})}{f_l(\bar{\theta}_{n-1})} = \frac{\Pr(a_{n-2}=1|l)\Pr(l)}{\Pr(a_{n-2}=1|h)\Pr(h)} \quad \text{and} \quad \frac{f_h(\underline{\theta}_{n-1})}{f_l(\underline{\theta}_{n-1})} = \frac{\Pr(a_{n-2}=0|l)\Pr(l)}{\Pr(a_{n-2}=0|h)\Pr(h)}$$

Then $\Pr(a_{n-2}=1|h)\Pr(h) = \frac{f_l(\bar{\theta}_{n-1})}{f_h(\bar{\theta}_{n-1})}Pr(a_{n-2}=1|l)\Pr(l)$ and $\Pr(a_{n-2}=0|l)\Pr(l) = \frac{f_h(\underline{\theta}_{n-1})}{f_l(\underline{\theta}_{n-1})}\Pr(a_{n-2}=0|h)\Pr(h)$. Then the thresholds become

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{\Pr(a_{n-1}|l)\Pr(l)}{\Pr(a_{n-1}|h)\Pr(h)}$$

$$= \frac{\Pr(a_{n-1}|a_{n-2}=1,l)\Pr(a_{n-2}=1|l)\Pr(l) + \Pr(a_{n-1}|a_{n-2}=0,l)\Pr(a_{n-2}=0|l)\Pr(l)}{\Pr(a_{n-1}|a_{n-2}=1,h)\Pr(a_{n-2}=1|h)\Pr(h) + \Pr(a_{n-1}|a_{n-2}=0,h)\Pr(a_{n-2}=0|h)\Pr(h)}$$

$$= \frac{\Pr(a_{n-1}|a_{n-2}=1,l)\Pr(a_{n-2}=1|l)\Pr(l) + \Pr(a_{n-1}|a_{n-2}=0,l)\frac{f_h(\underline{\theta}_{n-1})}{f_l(\underline{\theta}_{n-1})}\Pr(a_{n-2}=0|h)\Pr(h)}{\Pr(a_{n-1}|a_{n-2}=1,h)\frac{f_l(\bar{\theta}_{n-1})}{f_h(\bar{\theta}_{n-1})}Pr(a_{n-2}=1|l)\Pr(l) + \Pr(a_{n-1}|a_{n-2}=0,h)\Pr(a_{n-2}=0|h)\Pr(h)}$$

As noted, $\bar{\theta}_2 = -\frac{1}{2}$ and $\underline{\theta}_2 = \frac{1}{2}$ so that $\bar{\theta}_2 + \underline{\theta}_2 = 0$. Also, $\Pr(a_1 = 1|l) = (1 - F_l(0)) = \frac{1}{4}$ and $\Pr(a_1 = 0|h) = F_h(0) = \frac{1}{4}$ so that $\Pr(a_1 = 1|l) = \Pr(a_1 = 0|h)$. Conjecture that $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ and $\Pr(a_{n-2}=1|l) = \Pr(a_{n-2}=0|h)$ for $n \geq 4$. Then since $\Pr(h) = \Pr(l)$, this reduces to

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{f_h(\bar{\theta}_{n-1})[\Pr(a_{n-1}|a_{n-2}=1,l)f_l(\underline{\theta}_{n-1}) + \Pr(a_{n-1}|a_{n-2}=0,l)f_h(\underline{\theta}_{n-1})]}{f_l(\underline{\theta}_{n-1})[\Pr(a_{n-1}|a_{n-2}=1,h)f_l(\bar{\theta}_{n-1}) + \Pr(a_{n-1}|a_{n-2}=0,h)f_h(\bar{\theta}_{n-1})]}$$

If $a_{n-1} = 1$

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{f_h(\bar{\theta}_{n-1})[\Pr(\theta > \bar{\theta}_{n-1}|l)f_l(\underline{\theta}_{n-1}) + \Pr(\theta > \underline{\theta}_{n-1}|l)f_h(\underline{\theta}_{n-1})]}{f_l(\underline{\theta}_{n-1})[\Pr(\theta > \bar{\theta}_{n-1}|h)f_l(\bar{\theta}_{n-1}) + \Pr(\theta > \underline{\theta}_{n-1}|h)f_h(\bar{\theta}_{n-1})]}$$

$$= \frac{f_h(\bar{\theta}_{n-1})[(1 - F_l(\bar{\theta}_{n-1}))f_l(\underline{\theta}_{n-1}) + (1 - F_l(\underline{\theta}_{n-1}))f_h(\underline{\theta}_{n-1})]}{f_l(\underline{\theta}_{n-1})[(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + (1 - F_h(\underline{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})]}$$

$$= \frac{f_h(\bar{\theta}_{n-1})[(1 - F_l(\bar{\theta}_{n-1}))f_l(-\bar{\theta}_{n-1}) + (1 - F_l(-\bar{\theta}_{n-1}))f_h(-\bar{\theta}_{n-1})]}{f_l(-\bar{\theta}_{n-1})[(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + (1 - F_h(-\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})]}$$

and by symmetry of the signal functions

$$= \frac{(1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})}{(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})}$$

and similarly if $a_{n-1} = 0$

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{f_h(\bar{\theta}_{n-1})[F_l(\bar{\theta}_{n-1})f_l(\underline{\theta}_{n-1}) + F_l(\underline{\theta}_{n-1})f_h(\underline{\theta}_{n-1})]}{f_l(\underline{\theta}_{n-1})[F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) + F_h(\underline{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]}$$

$$= \frac{f_h(\bar{\theta}_{n-1})[F_l(\bar{\theta}_{n-1})f_l(-\bar{\theta}_{n-1}) + F_l(-\bar{\theta}_{n-1})f_h(-\bar{\theta}_{n-1})]}{f_l(-\bar{\theta}_{n-1})[F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) + F_h(-\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]}$$

$$= \frac{F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + (1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1})}{F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) + (1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})}$$

For our canonical signal structure defined by $F_h(\theta) = \frac{1}{4}(1+\theta)^2$ and $F_l(\theta) = 1 - \frac{1}{4}(1-\theta)^2$, if $a_{n-1} = 1$

$$
\begin{aligned}
\frac{1+\hat{\theta}_n}{1-\hat{\theta}_n} &= \frac{\frac{1}{4}(1-\bar{\theta}_{n-1})^2 \frac{1}{2}(1+\bar{\theta}_{n-1}) + \frac{1}{4}(1+\bar{\theta}_{n-1})^2 \frac{1}{2}(1-\bar{\theta}_{n-1})}{[1-\frac{1}{4}(1+\bar{\theta}_{n-1})^2]\frac{1}{2}(1-\bar{\theta}_{n-1}) + [1-\frac{1}{4}(1-\bar{\theta}_{n-1})^2]\frac{1}{2}(1+\bar{\theta}_{n-1})} \\
&= \frac{(1-\bar{\theta}_{n-1})^2(1+\bar{\theta}_{n-1}) + (1+\bar{\theta}_{n-1})^2(1-\bar{\theta}_{n-1})}{(3-2\bar{\theta}_{n-1}-\bar{\theta}_{n-1}^2)(1-\bar{\theta}_{n-1}) + (3+2\bar{\theta}_{n-1}-\bar{\theta}_{n-1}^2)(1+\bar{\theta}_{n-1})} \\
&= \frac{(1-\bar{\theta}_{n-1})(1+\bar{\theta}_{n-1})[(1-\bar{\theta}_{n-1}) + (1+\bar{\theta}_{n-1})]}{6 - 2\bar{\theta}_{n-1}^2 + \bar{\theta}_{n-1}(4\bar{\theta}_{n-1})} \\
&= \frac{2(1-\bar{\theta}_{n-1}^2)}{6 + 2\bar{\theta}_{n-1}^2} \\
\implies \hat{\theta}_n &= -\frac{1}{2}(1+\bar{\theta}_{n-1}^2)
\end{aligned}
$$

and if $a_{n-1} = 0$

$$
\begin{aligned}
\frac{1+\hat{\theta}_n}{1-\hat{\theta}_n} &= \frac{[1-\frac{1}{4}(1-\bar{\theta}_{n-1})^2]\frac{1}{2}(1+\bar{\theta}_{n-1}) + [1-\frac{1}{4}(1+\bar{\theta}_{n-1})^2]\frac{1}{2}(1-\bar{\theta}_{n-1})}{\frac{1}{4}(1+\bar{\theta}_{n-1})^2\frac{1}{2}(1-\bar{\theta}_{n-1}) + \frac{1}{4}(1-\bar{\theta}_{n-1})^2\frac{1}{2}(1+\bar{\theta}_{n-1})} \\
&= \frac{(3+2\bar{\theta}_{n-1}-\bar{\theta}_{n-1}^2)(1+\bar{\theta}_{n-1}) + (3-2\bar{\theta}_{n-1}-\bar{\theta}_{n-1}^2)(1-\bar{\theta}_{n-1})}{(1+\bar{\theta}_{n-1})^2(1-\bar{\theta}_{n-1}) + (1-\bar{\theta}_{n-1})^2(1+\bar{\theta}_{n-1})} \\
&= \frac{6 - 2\bar{\theta}_{n-1}^2 + \bar{\theta}_{n-1}(4\bar{\theta}_{n-1})}{(1+\bar{\theta}_{n-1})(1-\bar{\theta}_{n-1})[(1+\bar{\theta}_{n-1}) + (1-\bar{\theta}_{n-1})]} \\
&= \frac{6 + 2\bar{\theta}_{n-1}^2}{2(1-\bar{\theta}_{n-1}^2)} \\
\implies \hat{\theta}_n &= \frac{1}{2}(1+\bar{\theta}_{n-1}^2)
\end{aligned}
$$

Thus we have the recursive result that

$$
\hat{\theta}_n = \begin{cases} -\frac{1}{2}(1+\bar{\theta}_{n-1}^2) & \text{if } a_{n-1} = 1 \\ \frac{1}{2}(1+\bar{\theta}_{n-1}^2) & \text{if } a_{n-1} = 0 \end{cases}
$$

Finally, recall that we conjectured that $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ and that $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$ and the resulting threshold $\hat{\theta}_n$ also satisfied $\bar{\theta}_n + \underline{\theta}_n = 0$. Moreover,

$$
\begin{aligned}
\Pr(a_{n-1} = 1|l) &= \Pr(a_{n-1} = 1|a_{n-2} = 1, l)\Pr(a_{n-2} = 1|l) + \Pr(a_{n-1} = 1|a_{n-2} = 0, l)\Pr(a_{n-2} = 0|l) \\
&= (1 - F_l(\bar{\theta}_{n-1}))\Pr(a_{n-2} = 1|l) + (1 - F_l(\underline{\theta}_{n-1}))(1 - \Pr(a_{n-2} = 1|l))
\end{aligned}
$$

and by symmetry of F_l and F_h and $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ this becomes

$$
\begin{aligned}
&= F_h(\underline{\theta}_{n-1}))\Pr(a_{n-2} = 1|l) + F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{n-2} = 1|l)) \\
&= F_h(\bar{\theta}_{n-1}) + \Pr(a_{n-2} = 1|l))(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))
\end{aligned}
$$

$$\Pr(a_{n-1}=0|h) = \Pr(a_{n-1}=0|a_{n-2}=1,h)\Pr(a_{n-2}=1|h) + \Pr(a_{n-1}=0|a_{n-2}=0,h)\Pr(a_{n-2}=0|h)$$
$$= F_h(\bar{\theta}_{n-1})(1-\Pr(a_{n-2}=0|h)) + F_h(\underline{\theta}_{n-1})\Pr(a_{n-2}=0|h)$$
$$= F_h(\bar{\theta}_{n-1}) + \Pr(a_{n-2}=0|h)(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))$$

So that

$$\Pr(a_{n-1}=1|l) - \Pr(a_{n-1}=0|h) = (\Pr(a_{n-2}=1|l) - \Pr(a_{n-2}=0|h))(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))$$

and $\Pr(a_{n-1}=1|l) = \Pr(a_{n-1}=0|h)$ since $\Pr(a_{n-2}=1|l) = \Pr(a_{n-2}=0|h)$. So by induction, $\bar{\theta}_n + \underline{\theta}_n = 0$ and $\Pr(a_n=1|l) = \Pr(a_n=0|h)$ for all n. $\qquad\square$

Proof of Proposition 5. We found above that in equilibrium $\bar{\theta}_1 = 0$ and $\bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2)$ for $n \geq 2$ so that

$$\bar{\theta}_{n+1} - \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_n^2) - \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_n)^2 < 0$$

Then $\bar{\theta}_n$ is a decreasing sequence bounded below by -1 and as such must converge. Moreover

$$\lim_{n\to\infty} |\bar{\theta}_{n+1} - \bar{\theta}_n| = \lim_{n\to\infty} \frac{1}{2}(1 + \bar{\theta}_n)^2 = 0 \iff \lim_{n\to\infty} \bar{\theta}_n = -1$$

And thus $\bar{\theta}_n \to -1$. Since we proved above that $\underline{\theta}_n = -\bar{\theta}_n$, $\underline{\theta}_n \to 1$.

We showed in the proof of proposition 4 that $Pr(a_{n-1}=1|l) = \Pr(a_{n-1}=0|h)$. But since $Pr(a_{n-1}=1|l) = 1 - Pr(a_{n-1}=0|l)$ and $\Pr(a_{n-1}=0|h) = 1 - Pr(a_{n-1}=1|h)$ it must also be that $Pr(a_{n-1}=0|l) = Pr(a_{n-1}=1|h)$.

By lemma 2,

$$\Pr(a_{n-1}=1|h) = \frac{f_l(\bar{\theta}_n)}{f_l(\bar{\theta}_n) + f_h(\bar{\theta}_n)} = \frac{\frac{1}{2}(1-\bar{\theta}_n)}{\frac{1}{2}(1-\bar{\theta}_n) + \frac{1}{2}(1+\bar{\theta}_n)} = \frac{1}{2}(1-\bar{\theta}_n)$$

Then $\lim_{n\to\infty}\Pr(a_{n-1}=1|h) = \lim_{n\to\infty}\frac{1}{2}(1-\bar{\theta}_n) = 1$. Since $\Pr(a_{n-1}=0|l) = \Pr(a_{n-1}=1|h)$, $\lim_{n\to\infty}\Pr(a_{n-1}=0|1) = 1$. Thus complete learning occurs. $\qquad\square$

Proof of Proposition 6. Assume $\Pr(a_{i-2}) = \frac{1}{2}$ given no a priori information possible about this action. As noted above,

$$\hat{\theta}_n = \frac{\sum_{i=1}^{N} \mu_i^n (\Pr(a_{i-1}|l) - \Pr(a_{i-1}|h))}{\sum_{i=1}^{N} \mu_i^n (\Pr(a_{i-1}|l) + \Pr(a_{i-1}|h))}$$

$$\Pr(a_{i-1}|l) + \Pr(a_{i-1}|h) = \sum_{a_{i-2} \in \mathcal{A}} \Pr(a_{i-1}|a_{i-2}, l)\Pr(a_{i-2}|l) + \sum_{a_{i-2} \in \mathcal{A}} \Pr(a_{i-1}|a_{i-2}, h)\Pr(a_{i-2}|h)$$

$$= F_l(\bar{\theta}_{n-1})\Pr(a_{i-2}=1|l) + F_l(\underline{\theta}_{n-1})\Pr(a_{i-2}=0|l)$$
$$+ F_h(\bar{\theta}_{n-1})\Pr(a_{i-2}=1|h) + F_h(\underline{\theta}_{n-1})\Pr(a_{i-2}=0|h)$$
$$= F_l(\bar{\theta}_{n-1})\Pr(a_{i-2}=1|l) + F_l(\underline{\theta}_{n-1})(1 - \Pr(a_{i-2}=1|l))$$
$$+ F_h(\bar{\theta}_{n-1})\Pr(a_{n-2}=1|h) + F_h(\underline{\theta}_{n-1})(1 - \Pr(a_{i-2}=1|h))$$

If $\Pr(a_{i-2}|l)\Pr(l) + \Pr(a_{i-2}|h)\Pr(h) = \frac{1}{2}$, $\Pr(a_{i-2}|l) + \Pr(a_{i-2}|h) = 1$ and

$$= F_l(\bar{\theta}_{n-1})\Pr(a_{i-2}=1|l) + F_l(\underline{\theta}_{n-1})(1 - \Pr(a_{i-2}=1|l))$$
$$+ F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{i-2}=1|l)) + F_h(\underline{\theta}_{n-1})\Pr(a_{i-2}=1|l)$$
$$= \Pr(a_{i-2}=1|l)[F_l(\bar{\theta}_{n-1}) - F_l(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}) + F_h(\underline{\theta}_{n-1})] + F_l(\underline{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})$$
$$= \Pr(a_{i-2}=1|l)\left(\frac{1}{2}(\underline{\theta}_{n-1}^2 - \bar{\theta}_{n-1}^2)\right) + 1 + \frac{1}{4}\left(2(\underline{\theta}_{n-1} + \bar{\theta}_{n-1}) + \bar{\theta}_{n-1}^2 - \underline{\theta}_{n-1}^2\right)$$
$$= 1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1}) + (\underline{\theta}_{n-1}^2 - \bar{\theta}_{n-1}^2)\left(\frac{1}{4} - \frac{1}{2}\Pr(a_{i-2}=1|l)\right)$$
$$= 1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1})\left(1 + (\underline{\theta}_{n-1} - \bar{\theta}_{n-1})\left(\frac{1}{2} - \Pr(a_{i-2}=1|l)\right)\right)$$

And the thresholds become

$$\bar{\theta}_n = \frac{\sum_{i=1}^N \mu_i^n(\Pr(a_{i-1}=1|l) - \Pr(a_{i-1}=1|h))}{\sum_{i=1}^N \mu_i^n(1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1})\left(1 + (\underline{\theta}_{n-1} - \bar{\theta}_{n-1})\left(\frac{1}{2} - \Pr(a_{i-2}=1|l)\right)\right)}$$

$$\underline{\theta}_n = \frac{\sum_{i=1}^N \mu_i^n(\Pr(a_{i-1}=0|l) - \Pr(a_{i-1}=0|h))}{\sum_{i=1}^N \mu_i^n(1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1})\left(1 + (\underline{\theta}_{n-1} - \bar{\theta}_{n-1})\left(\frac{1}{2} - \Pr(a_{i-2}=1|l)\right)\right)}$$

It is easy to see then that

$$\bar{\theta}_n + \underline{\theta}_n = \frac{\sum_{i=1}^N \mu_i^n(\Pr(a_{i-1}=1|l) - \Pr(a_{i-1}=1|h)) + \sum_{i=1}^N \mu_i^n(\Pr(a_{i-1}=0|l) - \Pr(a_{i-1}=0|h))}{\sum_{i=1}^N \mu_i^n(1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1})\left(1 + (\underline{\theta}_{n-1} - \bar{\theta}_{n-1})\left(\frac{1}{2} - \Pr(a_{i-2}=1|l)\right)\right)}$$
$$= \frac{\sum_{i=1}^N \mu_i^n\left([\Pr(a_{i-1}=1|l) + \Pr(a_{i-1}=0|l)] - [\Pr(a_{i-1}=1|h) + \Pr(a_{i-1}=0|h)]\right)}{\sum_{i=1}^N \mu_i^n(1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1})\left(1 + (\underline{\theta}_{n-1} - \bar{\theta}_{n-1})\left(\frac{1}{2} - \Pr(a_{i-2}=1|l)\right)\right)} = 0$$

Since $\bar{\theta}_n + \underline{\theta}_n = 0$, for each i

$$\Pr(a_{i-1}|l) + \Pr(a_{i-1}|h) = 1 + \frac{1}{2}(\underline{\theta}_{n-1} + \bar{\theta}_{n-1})\left(1 + (\underline{\theta}_{n-1} - \bar{\theta}_{n-1})\left(\frac{1}{2} - \Pr(a_{i-2}=1|l)\right)\right)$$
$$= 1$$

and the threshold takes the form

$$\hat{\theta}_n = \sum_{i=1}^{N} \mu_i^n (\Pr(a_{i-1}|l) - \Pr(a_{i-1}|h))$$

Moreover, given that $\Pr(a_{i-1}|l) + \Pr(a_{i-1}|h) = 1$,

$$\hat{\theta}_n = 2 \sum_{i=1}^{N} \mu_i^n \Pr(a_{i-1}|l) - 1$$

If $\mu_n^n = \gamma$ and $\mu_i^n = \frac{1-\gamma}{N-1}$ for $i \neq n$ then

$$\hat{\theta}_n = 2\gamma \Pr(a_{n-1}|l) + 2 \sum_{i \neq n}^{N} \left(\frac{1-\gamma}{N-1}\right) \Pr(a_{i-1}|l) - 1$$

$$= 2\gamma \Pr(a_{n-1}|l) - 2 \left(\frac{1-\gamma}{N-1}\right) \Pr(a_{n-1}|l) + 2 \sum_{i=1}^{N} \left(\frac{1-\gamma}{N-1}\right) \Pr(a_{i-1}|l) - 1$$

$$= 2 \left(\frac{N\gamma - 1}{N - 1}\right) \Pr(a_{n-1}|l) - 1 + 2 \left(\frac{1-\gamma}{N-1}\right) \sum_{i=0}^{N-1} \Pr(a_i|l)$$

Consider $a_{n-1} = 1$. Since $\Pr(a_0 = 1|l) = \frac{1}{4}$ threshold $\bar{\theta}_1$ takes the form

$$\bar{\theta}_1 = 2 \left(\frac{N\gamma - 1}{N - 1}\right) \frac{1}{4} - 1 + 2 \left(\frac{1-\gamma}{N-1}\right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l)$$

$$= \frac{N(\gamma - 2) + 1}{2(N - 1)} + 2 \left(\frac{1-\gamma}{N-1}\right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l)$$

and for $n \geq 2$

$$\bar{\theta}_n = 2 \left(\frac{N\gamma - 1}{N - 1}\right) \Pr(a_{n-1} = 1|l) + \bar{\theta}_1 - \frac{N\gamma - 1}{2(N - 1)}$$

If $a_{n-1} = 0$, since $\Pr(a_0 = 0|l) = \frac{3}{4}$ threshold $\underline{\theta}_1$ takes the form

$$\underline{\theta}_1 = 2 \left(\frac{N\gamma - 1}{N - 1}\right) \frac{3}{4} - 1 + 2 \left(\frac{1-\gamma}{N-1}\right) \sum_{i=0}^{N-1} \Pr(a_i = 0|l)$$

$$= \frac{N(3\gamma - 2) - 1}{2(N - 1)} + 2 \left(\frac{1-\gamma}{N-1}\right) \sum_{i=0}^{N-1} \Pr(a_i = 0|l)$$

and for $n \geq 2$

$$\underline{\theta}_n = 2 \left(\frac{N\gamma - 1}{N - 1}\right) \Pr(a_{n-1} = 0|l) + \underline{\theta}_1 - \frac{3(N\gamma - 1)}{2(N - 1)}$$

$\square$

Proof of Proposition 7. By definition $\bar{\theta}_n = 2\left(\frac{N\gamma-1}{N-1}\right)\Pr(a_{n-1}=1|l) + \bar{\theta}_1 - \frac{N\gamma-1}{2(N-1)}$ so that

$$\bar{\theta}_{n+1} - \bar{\theta}_n = 2\left(\frac{N\gamma-1}{N-1}\right)(\Pr(a_n=1|l) - \Pr(a_{n-1}=1|l))$$

From lemma 3, $\Pr(a_n=1|l) = \frac{1}{4}(1+\bar{\theta}_n)^2 - \bar{\theta}_n\Pr(a_{n-1}=1|l)$ and this becomes

$$\bar{\theta}_{n+1} - \bar{\theta}_n = 2\left(\frac{N\gamma-1}{N-1}\right)\left[\frac{1}{4}(1+\bar{\theta}_n)^2 - \bar{\theta}_n\Pr(a_{n-1}=1|l) - \Pr(a_{n-1}=1|l)\right]$$

$$= \left(\frac{N\gamma-1}{2(N-1)}\right)\left[(1+\bar{\theta}_n)^2 - 4\Pr(a_{n-1}=1|l)(1+\bar{\theta}_n)\right]$$

$$= \left(\frac{N\gamma-1}{2(N-1)}\right)(1+\bar{\theta}_n)\left[1+\bar{\theta}_n - 4\Pr(a_{n-1}=1|l)\right]$$

Suppose $\bar{\theta}_n$ converges to $\bar{\theta}$. Then

$$\left(\frac{N\gamma-1}{2(N-1)}\right)(1+\bar{\theta})\left[1+\bar{\theta} - 4\lim_{n\to\infty}\Pr(a_n=1|l)\right] = 0$$

This is satisfied if $\bar{\theta} = -1$. If $\bar{\theta} > -1$ then

$$4\lim_{n\to\infty}\Pr(a_n=1|l) = 1+\bar{\theta}$$

By definition $\underline{\theta}_n = 2\left(\frac{N\gamma-1}{N-1}\right)\Pr(a_{n-1}=0|l) + \underline{\theta}_1 - \frac{3(N\gamma-1)}{2(N-1)}$ so that

$$\underline{\theta}_{n+1} - \underline{\theta}_n = 2\left(\frac{N\gamma-1}{N-1}\right)(\Pr(a_n=0|l) - \Pr(a_{n-1}=0|l))$$

From lemma 3, $\Pr(a_n=0|l) = \frac{1}{4}(3+\underline{\theta}_n)(1-\underline{\theta}_n) + \underline{\theta}_n\Pr(a_{n-1}=0|l)$ and this becomes

$$\underline{\theta}_{n+1} - \underline{\theta}_n = 2\left(\frac{N\gamma-1}{N-1}\right)\left(\frac{1}{4}(3+\underline{\theta}_n)(1-\underline{\theta}_n) + \underline{\theta}_n\Pr(a_{n-1}=0|l) - \Pr(a_{n-1}=0|l)\right)$$

$$= 2\left(\frac{N\gamma-1}{N-1}\right)\frac{1}{4}(1-\underline{\theta}_n)((3+\underline{\theta}_n) - 4\Pr(a_{n-1}=0|l))$$

Suppose $\underline{\theta}_n$ converges to $\underline{\theta}$. Then

$$2\left(\frac{N\gamma-1}{N-1}\right)\frac{1}{4}(1-\underline{\theta})\left((3+\underline{\theta}) - 4\lim_{n\to\infty}\Pr(a_n=0|l)\right)$$

This is satisfied if $\underline{\theta} = 1$. If $\underline{\theta} < 1$ then

$$4\lim_{n\to\infty}\Pr(a_n=0|l) = 3+\underline{\theta}$$

Then we have two cases

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(i)

$$\lim_{n\to\infty} \bar{\theta}_n = -1 \text{ and } \lim_{n\to\infty} \underline{\theta}_n = 1$$

(ii)

$$\lim_{n\to\infty} \Pr(a_n = 0|l) = \frac{1}{4}(3+\underline{\theta}) \text{ and } \lim_{n\to\infty} \Pr(a_n = 1|l) = \frac{1}{4}(1+\bar{\theta})$$

As shown above,

$$\bar{\theta}_n = 2\left(\frac{N\gamma-1}{N-1}\right)\Pr(a_{n-1}=1|l) - 1 + 2\left(\frac{1-\gamma}{N-1}\right)\sum_{i=0}^{N-1}\Pr(a_i=1|l)$$

If $\bar{\theta}_n \to \bar{\theta}$, then $\Pr(a_{n-1}=1|l) \to \lim_{n\to\infty}\Pr(a_n=1|l)$, $\sum_{i=0}^{N-1}\Pr(a_i=1|l) \to N\lim_{n\to\infty}\Pr(a_n=1|l)$, and

$$\bar{\theta} = 2\lim_{n\to\infty}\Pr(a_n=1|l) - 1$$

Similarly, if $\underline{\theta}_n \to \underline{\theta}$

$$\underline{\theta} = 2\lim_{n\to\infty}\Pr(a_n=0|l) - 1$$

In case (i) $\bar{\theta} = -1$ and $\underline{\theta} = 1$ so $\lim_{n\to\infty}\Pr(a_n=1|l) = 0$ and $\lim_{n\to\infty}\Pr(a_n=0|l) = 1$. In case (ii)

$$\bar{\theta} = 2\lim_{n\to\infty}\Pr(a_n=1|l) - 1 = \frac{1}{2}(1+\bar{\theta}) - 1 = \frac{1}{2}\bar{\theta} - \frac{1}{2}$$

so that $\bar{\theta} = -1$. Similarly, if $\underline{\theta}_n \to \underline{\theta}$ then

$$\underline{\theta} = 2\lim_{n\to\infty}\Pr(a_n=0|l) - 1 = \frac{1}{2}(3+\underline{\theta}) - 1 = \frac{1}{2}\underline{\theta} + \frac{1}{2}$$

so that $\underline{\theta} = 1$. From above $\Pr(a_n = 1|h) = 1 - \Pr(a_n = 1|l)$, and by the definition of the limit, $\lim_{n\to\infty}\Pr(a_n = 1|h) = 1 - \lim_{n\to\infty}\Pr(a_n = 1|l) = 1 - \frac{1}{4}(1+\bar{\theta}) = 1$ and $\lim_{n\to\infty}\Pr(a_n = 0|l) = \frac{1}{4}(3+\underline{\theta}) = 1$

Thus in either case $\lim_{n\to\infty}\Pr(a_n = 1|h) = 1$ and $\lim_{n\to\infty}\Pr(a_n = 0|l) = 1$ so complete learning occurs. □

Proof of Proposition 8. From proposition 6,

$$\bar{\theta}_1 = \frac{N(\gamma-2)+1}{2(N-1)} + 2\left(\frac{1-\gamma}{N-1}\right)\sum_{i=0}^{N-1}\Pr(a_i=1|l)$$

$$\bar{\theta}_n = 2\left(\frac{N\gamma-1}{N-1}\right)\Pr(a_{n-1}=1|l) + \bar{\theta}_1 - \frac{N\gamma-1}{2(N-1)}$$

From proposition 7, $\lim_{i \to \infty} \Pr(a_i = 1|l) = 0$ so that $\lim_{N \to \infty} 2\left(\frac{1-\gamma}{N-1}\right) \sum_{i=0}^{N-1} \Pr(a_i = 1|l) = 0$. Thus $\lim_{N \to \infty} \bar{\theta}_1 = -\frac{2-\gamma}{2}$ and $\lim_{N \to \infty} \bar{\theta}_n = 2\gamma \Pr(a_{n-1} = 1|l) - \frac{2-\gamma}{2} - \frac{\gamma}{2} = 2\gamma \Pr(a_{n-1} = 1|l) - 1$. From lemma 3, $\Pr(a_{n-1} = 1|l) = \frac{1}{4}(1 + \bar{\theta}_{n-1})^2 - \bar{\theta}_{n-1} \Pr(a_{n-2} = 1|l)$. Moreover, $\lim_{N \to \infty} \bar{\theta}_{n-1} = 2\gamma \Pr(a_{n-2} = 1|l) - 1$ which implies $\Pr(a_{n-2} = 1|l) = \frac{\bar{\theta}_{n-1}+1}{2\gamma}$ and

$$
\begin{aligned}
\Pr(a_{n-1} = 1|l) &= \frac{1}{4}(1 + \bar{\theta}_{n-1})^2 - \bar{\theta}_{n-1}\frac{\bar{\theta}_{n-1} + 1}{2\gamma} \\
&= \frac{1}{4}(1 + 2\bar{\theta}_{n-1} + \bar{\theta}_{n-1}^2) - \frac{\bar{\theta}_{n-1}^2 + \bar{\theta}_{n-1}}{2\gamma} \\
&= \frac{1}{4\gamma}(\gamma + 2\bar{\theta}_{n-1}(\gamma - 1) + \bar{\theta}_{n-1}^2(\gamma - 2))
\end{aligned}
$$

Then as $N \to \infty$,

$$
\begin{aligned}
\bar{\theta}_n = 2\gamma \Pr(a_{n-1} = 1|l) - 1 &= 2\gamma\left[\frac{1}{4\gamma}(\gamma + 2\bar{\theta}_{n-1}(\gamma - 1) + \bar{\theta}_{n-1}^2(\gamma - 2))\right] - 1 \\
&= \frac{1}{2}\left(\gamma + \bar{\theta}_{n-1}^2(\gamma - 2)\right) + \bar{\theta}_{n-1}(\gamma - 1) - 1 \\
&= -\frac{2-\gamma}{2}\left(1 + \bar{\theta}_{n-1}^2\right) - (1 - \gamma)\bar{\theta}_{n-1} \\
&= -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) - \frac{1-\gamma}{2}(1 + \bar{\theta}_{n-1})^2
\end{aligned}
$$

Thus $\lim_{N \to \infty} \bar{\theta}_n = -\frac{1}{2}(1 + \bar{\theta}_{n-1}^2) - \frac{1-\gamma}{2}(1 + \bar{\theta}_{n-1})^2$ and since $\underline{\theta}_n = -\bar{\theta}_n$ for all n, $\lim_{N \to \infty} \underline{\theta}_n = \frac{1}{2}(1 + \bar{\theta}_{n-1}^2) + \frac{1-\gamma}{2}(1 + \bar{\theta}_{n-1})^2$. $\qquad \square$

Proof of Proposition 9. By definition

$$
\mathbb{E}[u(\theta)|H_n, h] = \Pr(a_n = 1|h) - \Pr(a_n = 1|l) = 1 - 2\Pr(a_n = 1|l)
$$

From proposition 8, $\Pr(a_n = 1|l) = \frac{1}{4\gamma}(\gamma + 2\bar{\theta}_n(\gamma - 1) + \bar{\theta}_n^2(\gamma - 2))$, so that

$$
\begin{aligned}
\mathbb{E}[u(\theta)|H_n, h] &= 1 - 2\left(\frac{1}{4\gamma}\right)(\gamma + 2\bar{\theta}_n(\gamma - 1) + \bar{\theta}_n^2(\gamma - 2)) \\
&= \left(\frac{1}{2\gamma}\right)(\gamma + 2\bar{\theta}_n(1 - \gamma) + \bar{\theta}_n^2(2 - \gamma)) \\
&= \frac{1}{2\gamma}(1 + \bar{\theta}_n^2) + \frac{1-\gamma}{2\gamma}(\bar{\theta}_n^2 + 2\bar{\theta}_n - 1)
\end{aligned}
$$

$\qquad \square$

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Proof of Proposition 10. Recall from lemma 1 that the threshold strategy is defined recursively by

$$\frac{f_h(\hat\theta_n)}{f_l(\hat\theta_n)} = \begin{cases} \frac{(1-F_l(\hat\theta_{n-1}))f_h(\hat\theta_{n-1})}{(1-F_h(\hat\theta_{n-1}))f_l(\hat\theta_{n-1})} & \text{if } a_{n-1} = 1 \\ \frac{F_l(\hat\theta_{n-1})f_h(\hat\theta_{n-1})}{F_h(\hat\theta_{n-1})f_l(\hat\theta_{n-1})} & \text{if } a_{n-1} = 0 \end{cases}$$

Then

$$\mathbb{E}\left[\frac{f_h(\hat\theta_n)}{f_l(\hat\theta_n)}\Big|\theta_{n-1},l\right] = \Pr(a_{n-1}=0|l)\frac{F_l(\hat\theta_{n-1})f_h(\hat\theta_{n-1})}{F_h(\hat\theta_{n-1})f_l(\hat\theta_{n-1})} + \Pr(a_{n-1}=1|l)\frac{(1-F_l(\hat\theta_{n-1}))f_h(\hat\theta_{n-1})}{(1-F_h(\hat\theta_{n-1}))f_l(\hat\theta_{n-1})}$$

$$= \frac{f_h(\hat\theta_{n-1})}{f_l(\hat\theta_{n-1})}\left[\Pr(a_{n-1}=0|l)\frac{F_l(\hat\theta_{n-1})}{F_h(\hat\theta_{n-1})} + (1-\Pr(a_{n-1}=0|l))\frac{(1-F_l(\hat\theta_{n-1}))}{(1-F_h(\hat\theta_{n-1}))}\right]$$

$$= \frac{f_h(\hat\theta_{n-1})}{f_l(\hat\theta_{n-1})}\left[F_l\left(\frac{F_l}{F_h} - \frac{1-F_l}{1-F_h}\right) + \frac{1-F_l}{1-F_h}\right]$$

$$= \frac{f_h(\hat\theta_{n-1})}{f_l(\hat\theta_{n-1})}\left[F_l\left(\frac{F_l(1-F_h)-F_h(1-F_l)}{F_h(1-F_h)}\right) + \frac{F_h(1-F_l)}{F_h(1-F_h)}\right]$$

$$= \frac{f_h(\hat\theta_{n-1})}{f_l(\hat\theta_{n-1})}\frac{F_l^2-2F_lF_h+F_h}{F_h(1-F_h)}$$

$$= \frac{f_h(\hat\theta_{n-1})}{f_l(\hat\theta_{n-1})}\frac{(F_l-F_h)^2}{F_h(1-F_h)} + \frac{f_h(\hat\theta_{n-1})}{f_l(\hat\theta_{n-1})}$$

Thus $\mathbb{E}\left[\frac{f_h(\hat\theta_n)}{f_l(\hat\theta_n)}|\theta_{n-1},l\right] - \frac{f_h(\hat\theta_{n-1})}{f_l(\hat\theta_{n-1})} = \frac{f_h(\hat\theta_{n-1})}{f_l(\hat\theta_{n-1})}\frac{(F_l(\hat\theta_{n-1})-F_h(\hat\theta_{n-1}))^2}{F_h(\hat\theta_{n-1})(1-F_h(\hat\theta_{n-1}))}$, and by strict First Order Stochastic Dominance this is strictly positive for interior signals θ. Then since it is a submartingale the likelihood ratio $\frac{f_h(\hat\theta_n)}{f_l(\hat\theta_n)}$ either converges to a limit or diverges, but since $F_l(\theta) - F_h(\theta) > 0$ for interior signals it cannot converge to a limit and hence must diverge. By the monotone

likelihood ratio property, since $\lim_{n\to\infty} \frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \infty$ it must be that $\lim_{n\to\infty} \hat{\theta}_n = 1$. Similarly,

$$
\begin{aligned}
\mathbb{E}\left[\frac{f_l(\hat{\theta}_n)}{f_h(\hat{\theta}_n)}|\theta_{n-1}, h\right] &= \Pr(a_{n-1}=0|h)\frac{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})}{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})} + \Pr(a_{n-1}=1|h)\frac{(1-F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})}{(1-F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})} \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})}\left[\Pr(a_{n-1}=0|h)\frac{F_h(\hat{\theta}_{n-1})}{F_l(\hat{\theta}_{n-1})} + (1-\Pr(a_{n-1}=0|h))\frac{(1-F_h(\hat{\theta}_{n-1}))}{(1-F_l(\hat{\theta}_{n-1}))}\right] \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})}\left[F_h\left(\frac{F_h}{F_l}-\frac{1-F_h}{1-F_l}\right)+\frac{1-F_h}{1-F_l}\right] \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})}\left[F_h\left(\frac{F_h(1-F_l)-F_l(1-F_h)}{F_l(1-F_l)}\right)+\frac{F_l(1-F_h)}{F_l(1-F_l)}\right] \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})}\frac{F_h^2-2F_hF_l+F_l}{F_l(1-F_l)} \\
&= \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})}\frac{(F_h-F_l)^2}{F_l(1-F_l)}+\frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})}
\end{aligned}
$$

Then $\mathbb{E}\left[\frac{f_l(\hat{\theta}_n)}{f_h(\hat{\theta}_n)}|\theta_{n-1}, h\right] - \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})} = \frac{f_l(\hat{\theta}_{n-1})}{f_h(\hat{\theta}_{n-1})}\frac{(F_h(\hat{\theta}_{n-1})-F_l(\hat{\theta}_{n-1}))^2}{F_l(\hat{\theta}_{n-1})(1-F_l(\hat{\theta}_{n-1}))}$ so the likelihood ratio $\frac{f_l(\hat{\theta}_n)}{f_h(\hat{\theta}_n)}$ is a submartingale conditional on $\omega = h$. Strict FOSD again implies that the likelihood ratio diverges so that $\hat{\theta}_n = -1$.

Together these results imply

$$
\begin{aligned}
\lim_{n\to\infty}\Pr(a_n=0|l) &= \lim_{n\to\infty}F_l(\hat{\theta}_n) = F(1) = 1 \\
\lim_{n\to\infty}\Pr(a_n=1|h) &= \lim_{n\to\infty}(1-F_h(\hat{\theta}_n)) = 1 - F(-1) = 1
\end{aligned}
$$

so that complete learning occurs. $\qquad\square$

Proof of Proposition 11.

$$
\begin{aligned}
\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} - \frac{f_h(\bar{\theta}_{n-1})}{f_l(\bar{\theta}_{n-1})} &= \frac{(1-F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})+F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})}{(1-F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1})+F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})} - \frac{f_h(\bar{\theta}_{n-1})}{f_l(\bar{\theta}_{n-1})} \\
&= \frac{f_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})-F_l(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})+F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})^2}{f_l(\bar{\theta}_{n-1})[(1-F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1})+F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]} \\
&\quad - \frac{f_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})-F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})+F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})^2}{f_l(\bar{\theta}_{n-1})[(1-F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1})+F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]} \\
&= \frac{(f_h(\bar{\theta}_{n-1})-f_l(\bar{\theta}_{n-1}))(F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})+F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}))}{f_l(\bar{\theta}_{n-1})[(1-F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1})+F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})]}
\end{aligned}
$$

Then $\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} - \frac{f_h(\bar{\theta}_{n-1})}{f_l(\bar{\theta}_{n-1})} \leq 0 \iff f_h(\bar{\theta}_{n-1}) \leq f_l(\bar{\theta}_{n-1})$. Symmetry about 0 implies $\frac{f_h(0)}{f_l(0)} = 1$ and with the monotone likelihood ratio assumption $f_h(\bar{\theta}_{n-1}) \leq f_l(\bar{\theta}_{n-1})$ for $\bar{\theta}_{n-1} \leq 0$. Thus

the likelihood ratio $f_h(\bar{\theta}_{n-1}) \leq f_l(\bar{\theta}_{n-1})$ is a decreasing sequence bounded below so it must converge. Moreover, strict FOSD implies that $f_h(\bar{\theta}_{n-1}) < f_l(\bar{\theta}_{n-1})$ if $\bar{\theta}_{n-1} < 0$. As we showed above $\bar{\theta}_1 < 0$ and by the monotonicity of the likelihood ratio $\theta_n < \bar{\theta}_{n-1}$ for all $n \geq 1$. Then the likelihood ratio strictly decreases in n until $F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) = 0$, or when $F_l(\bar{\theta}_{n-1})$ and $F_h(\bar{\theta}_{n-1})$ converge to 0. Thus $\lim_{n\to\infty} \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = 0$.

By lemma 2,

$$\Pr(a_n = 1 | h) = \frac{f_l(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1}) + f_h(\bar{\theta}_{n+1})} = \frac{1}{1 + \frac{f_h(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1})}}$$

so that $\lim_{n\to\infty} \Pr(a_n = 1 | h) = \frac{1}{1 + \lim_{n\to\infty} \frac{f_h(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1})}} = 1$. By lemma 3 $\Pr(a_n = 1 | l) = \Pr(a_n = 0 | h)$ so $\lim_{n\to\infty} \Pr(a_n = 0 | l) = 1$ and complete learning occurs. $\qquad\square$

Proof of Proposition 12. From above the likelihood ratio takes the form

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{\sum_{i=1}^{N} \mu_i^n \Pr(a_{i-1}|l) \Pr(l)}{\sum_{i=1}^{N} \mu_i^n \Pr(a_{i-1}|h) \Pr(h)}$$

With $\mu_n^n = \gamma$ and $\mu = \mu_i^n = \frac{1-\gamma}{N-1}$ for $i \neq n$ this becomes

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{(\gamma - \mu)\Pr(a_{n-1}|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)}{(\gamma - \mu)\Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)}$$

If $\bar{\theta}_n$ is the threshold in response to $a_{n-1} = 1$ then

$$\frac{f_h(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1})} - \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{(\gamma - \mu)\Pr(a_n|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)}{(\gamma - \mu)\Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)} - \frac{(\gamma - \mu)\Pr(a_{n-1}|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)}{(\gamma - \mu)\Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)}$$

$$\frac{[(\gamma - \mu)\Pr(a_n|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)][(\gamma - \mu)\Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}{[(\gamma - \mu)\Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu)\Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}$$

$$- \frac{[(\gamma - \mu)\Pr(a_{n-1}|l) + \mu \sum_{i=0}^{N-1} \Pr(a_i|l)][(\gamma - \mu)\Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}{[(\gamma - \mu)\Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu)\Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}$$

$$= \frac{(\gamma - \mu)^2[\Pr(a_n|l)\Pr(a_{n-1}|h) - \Pr(a_{n-1}|l)\Pr(a_n|h)]}{[(\gamma - \mu)\Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu)\Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}$$

$$+ \frac{(\gamma - \mu)\mu \left[\sum_{i=0}^{N-1} \Pr(a_i|l)(\Pr(a_{n-1}|h) - \Pr(a_n|h)) + \sum_{i=0}^{N-1} \Pr(a_i|h)(\Pr(a_n|l) - \Pr(a_{n-1}|l)) \right]}{[(\gamma - \mu)\Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu)\Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}$$

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and with the results in lemma 8 this becomes

$$
= \frac{(\gamma - \mu)^2 [F_h(\bar{\theta}_n) \Pr(a_{n-1} = 1|h) - F_l(\bar{\theta}_n) \Pr(a_{n-1} = 1|l)]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}
$$

$$
+ \frac{(\gamma - \mu)\mu \left[\sum_{i=0}^{N-1} \Pr(a_i|h)(F_h(\bar{\theta}_n) - \Pr(a_{n-1} = 1|l)[F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]) \right]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}
$$

$$
- \frac{(\gamma - \mu)\mu \left[\sum_{i=0}^{N-1} \Pr(a_i|l)(F_l(\bar{\theta}_n) - \Pr(a_{n-1} = 1|h)[F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]) \right]}{[(\gamma - \mu) \Pr(a_{n-1}|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)][(\gamma - \mu) \Pr(a_n|h) + \mu \sum_{i=0}^{N-1} \Pr(a_i|h)]}
$$

which converges to

$$
\rightarrow \frac{\gamma^2 [F_h(\bar{\theta}) \Pr(a = 1|h) - F_l(\bar{\theta}) \Pr(a = 1|l)]}{[\gamma \Pr(a|h) + \Pr(a|h)][\gamma \Pr(a|h) + \Pr(a|h)]}
$$

$$
+ \frac{\gamma \left[\Pr(a|h)(F_h(\bar{\theta}) - \Pr(a = 1|l)[F_l(\bar{\theta}) + F_h(\bar{\theta})]) \right]}{[\gamma \Pr(a|h) + \Pr(a|h)][\gamma \Pr(a|h) + \Pr(a|h)]}
$$

$$
- \frac{\gamma \left[\Pr(a|l)(F_l(\bar{\theta}) - \Pr(a = 1|h)[F_l(\bar{\theta}) + F_h(\bar{\theta})]) \right]}{[\gamma \Pr(a|h) + \Pr(a|h)][\gamma \Pr(a|h) + \Pr(a|h)]}
$$

$$
= \frac{\gamma(1 + \gamma)[F_h(\bar{\theta}) \Pr(a = 1|h) - F_l(\bar{\theta}) \Pr(a = 1|l)]}{[\gamma \Pr(a|h) + \Pr(a|h)][\gamma \Pr(a|h) + \Pr(a|h)]}
$$

Then $\lim_{n \to \infty} \frac{f_h(\bar{\theta}_{n+1})}{f_l(\bar{\theta}_{n+1})} - \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = 0$ if $F_h(\bar{\theta}) \Pr(a = 1|h) = F_l(\bar{\theta}) \Pr(a = 1|l)$. And given that $\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{\Pr(a_{n-1} = 1|l)}{\Pr(a_{n-1} = 1|h)}$,

$$
\frac{f_h(\bar{\theta})}{f_l(\bar{\theta})} = \lim_{n \to \infty} \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \lim_{n \to \infty} \frac{\Pr(a_{n-1} = 1|l)}{\Pr(a_{n-1} = 1|h)} = \frac{\Pr(a = 1|l)}{\Pr(a = 1|h)} = \frac{F_h(\bar{\theta})}{F_l(\bar{\theta})}
$$

so that $\frac{f_h(\bar{\theta})}{f_l(\bar{\theta})} = \frac{F_h(\bar{\theta})}{F_l(\bar{\theta})}$. But $\frac{F_h(\theta)}{F_l(\theta)} < 1$ for all interior θ and by symmetry and the MLRP $\frac{f_h(\theta)}{f_l(\theta)} > 1$ for $\theta > 0$, so $\bar{\theta} \in [-1, 0)$. By lemma 3 $\frac{f_h(\theta)}{f_l(\theta)} > \frac{F_h(\theta)}{F_l(\theta)}$ for all interior θ so the only remaining candidate is $\bar{\theta} = -1$. Indeed, by L'Hopital's rule, $\lim_{\theta \to -1} \frac{F_h(\theta)}{F_l(\theta)} = \lim_{\theta \to -1} \frac{f_h(\theta)}{f_l(\theta)} = \frac{f_h(-1)}{f_l(-1)}$ so that $\bar{\theta} = -1$.

From lemma 5, $\Pr(a_n = 1|h) = \Pr(a_n = 0|l) = \frac{F_l(\bar{\theta}_{n+1})}{F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})}$

$$
\lim_{n \to \infty} \frac{F_l(\bar{\theta}_{n+1})}{F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})} = \lim_{\theta \to \bar{\theta}} \frac{1}{1 + \frac{F_h(\theta)}{F_l(\theta)}} = \lim_{\theta \to -1} \frac{1}{1 + \frac{f_h(\theta)}{f_l(\theta)}} = 1
$$

so that $\lim_{n \to \infty} \Pr(a_n = 1|h) = \lim_{n \to \infty} \Pr(a_n = 0|l) = 1$ and complete learning occurs. $\square$

A2. Useful Lemmas

Lemma 1. *For a general signal structure* $\Pr(\theta_n \leq \theta|h) = F_h(\theta)$ *and* $\Pr(\theta_n \leq \theta|l) = F_l(\theta)$ *that admit distribution functions* f_h, f_l *characterized by the monotone likelihood ratio property,*

cutoff strategies $\hat{\theta}_n$ *are determined recursively by*

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \begin{cases} \frac{(1-F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})}{(1-F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 1 \\ \frac{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 0 \end{cases}$$

Proof. We determined that the threshold for each signal is implicitly defined by $\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \frac{\Pr(A_{n-1}|l)\Pr(l)}{\Pr(A_{n-1}|h)\Pr(h)}$. Since this is true for all n, $\frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})} = \frac{\Pr(A_{n-2}|l)\Pr(l)}{\Pr(A_{n-2}|h)\Pr(h)}$ so that $\Pr(A_{n-2}|l)\Pr(l) = \frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})}\Pr(A_{n-2}|h)\Pr(h)$.

In the case of $a_{n-1} = 1$, $\theta_{n-1} \geq \hat{\theta}_{n-1}$ and

$$\begin{aligned} \frac{\Pr(A_{n-1}|l)\Pr(l)}{\Pr(A_{n-1}|h)\Pr(h)} &= \frac{\Pr(\theta_{n-1} \geq \hat{\theta}_{n-1}|l)\Pr(A_{n-2}|l)\Pr(l)}{\Pr(\theta_{n-1} \geq \hat{\theta}_{n-1}|h)\Pr(A_{n-2}|h)\Pr(h)} \\ &= \frac{(1 - F_l(\hat{\theta}_{n-1}))\Pr(A_{n-2}|l)\Pr(l)}{(1 - F_h(\hat{\theta}_{n-1}))\Pr(A_{n-2}|h)\Pr(h)} \\ &= \frac{(1 - F_l(\hat{\theta}_{n-1}))\frac{f_h(\hat{\theta}_{n-1})}{f_l(\hat{\theta}_{n-1})}\Pr(A_{n-2}|h)\Pr(h)}{(1 - F_h(\hat{\theta}_{n-1}))\Pr(A_{n-2}|h)\Pr(h)} \\ &= \frac{(1 - F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})}{(1 - F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})} \end{aligned}$$

while if $a_{n-1} = 0$ then $\theta < \hat{\theta}_{n-1}$ and

$$\frac{\Pr(A_{n-1}|h)\Pr(h)}{\Pr(A_{n-1}|l)\Pr(l)} = \frac{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})}$$

Thus the cutoff strategy is defined recursively by

$$\frac{f_h(\hat{\theta}_n)}{f_l(\hat{\theta}_n)} = \begin{cases} \frac{(1-F_l(\hat{\theta}_{n-1}))f_h(\hat{\theta}_{n-1})}{(1-F_h(\hat{\theta}_{n-1}))f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 1 \\ \frac{F_l(\hat{\theta}_{n-1})f_h(\hat{\theta}_{n-1})}{F_h(\hat{\theta}_{n-1})f_l(\hat{\theta}_{n-1})} & \text{if } a_{n-1} = 0 \end{cases}$$

$\square$

Lemma 2. *In the case of limited histories in the sense that $A_n = a_n$, if $Pr(a_{n-1} = 0|l) = Pr(a_{n-1} = 1|h)$ then*

$$\Pr(a_{n-1} = 1|h) = \frac{f_l(\bar{\theta}_n)}{f_l(\bar{\theta}_n) + f_h(\bar{\theta}_n)}$$

Proof. As noted above, in the case of limited histories beliefs are recursively related according to

$$\Pr(a_{n-1} = 1|h) = \frac{f_l(\bar{\theta}_n)}{f_h(\bar{\theta}_n)}Pr(a_{n-1} = 1|l) = \frac{f_l(\bar{\theta}_n)}{f_h(\bar{\theta}_n)}(1 - Pr(a_{n-1} = 0|l))$$

and with $Pr(a_{n-1} = 0|l) = Pr(a_{n-1} = 1|h)$,

$$\Pr(a_{n-1} = 1|h) = \frac{f_l(\bar{\theta}_n)}{f_l(\bar{\theta}_n) + f_h(\bar{\theta}_n)}$$

$\square$

Lemma 3. *For the canonical signal structure, if* $\bar{\theta}_n + \underline{\theta}_n = 0$*, then*

$$\Pr(a_n = 1|l) = \frac{1}{4}(1 + \bar{\theta}_n)^2 - \bar{\theta}_n \Pr(a_{n-1} = 1|l) \qquad \text{and}$$

$$\Pr(a_n = 0|l) = \frac{1}{4}(3 + \underline{\theta}_n)(1 - \underline{\theta}_n) + \underline{\theta}_n \Pr(a_{n-1} = 0|l)$$

Proof.

$$\Pr(a_n = 1|l) = \Pr(a_n = 1|a_{n-1} = 1, l)\Pr(a_{n-1} = 1|l) + \Pr(a_n = 1|a_{n-1} = 0, l)\Pr(a_{n-1} = 0|l)$$

$$= (1 - F_l(\bar{\theta}_n))\Pr(a_{n-1} = 1|l) + (1 - F_l(\underline{\theta}_n))\Pr(a_{n-1} = 0|l)$$

$$= (1 - F_l(\bar{\theta}_n))\Pr(a_{n-1} = 1|l) + F_h(\bar{\theta}_n)(1 - \Pr(a_{n-1} = 1|l))$$

$$= \Pr(a_{n-1} = 1|l)\left[\frac{1}{4}(1 - \bar{\theta}_n)^2 - \frac{1}{4}(1 + \bar{\theta}_n)^2\right] + \frac{1}{4}(1 + \bar{\theta}_n)^2$$

$$= \frac{1}{4}(1 + \bar{\theta}_n)^2 - \bar{\theta}_n \Pr(a_{n-1} = 1|l)$$

$$\Pr(a_n = 0|l) = 1 - \Pr(a_n = 1|l) = \bar{\theta}_n \Pr(a_{n-1} = 1|l) - \frac{1}{4}(1 + \bar{\theta}_n)^2$$

$$= 1 - \frac{1}{4}(1 - \bar{\theta}_n)^2 - \bar{\theta}_n \Pr(a_{n-1} = 0|l) = \frac{1}{4}(3 + 2\bar{\theta}_n - \bar{\theta}_n^2) - \bar{\theta}_n \Pr(a_{n-1} = 0|l)$$

$$= \frac{1}{4}(3 - \bar{\theta}_n)(1 + \bar{\theta}_n) - \bar{\theta}_n \Pr(a_{n-1} = 0|l) = \frac{1}{4}(3 + \underline{\theta}_n)(1 - \underline{\theta}_n) + \underline{\theta}_n \Pr(a_{n-1} = 0|l)$$

$\square$

Lemma 4. *If* $\frac{f_h(\theta)}{f_l(\theta)}$ *exhibits the strict Monotone Likelihood Ratio Property in the sense that* $\theta_1 > \theta_0$ *implies* $\frac{f_h(\theta_1)}{f_l(\theta_1)} > \frac{f_h(\theta_0)}{f_l(\theta_0)}$ *then*

(i) $\frac{f_h(\theta)}{f_l(\theta)} > \frac{F_h(\theta)}{F_l(\theta)}$ *for all* $\theta \in supp(F)^\circ$

(ii) F_l *strictly First Order Stochastically Dominates* F_h *in that* $F_l(\theta) > F_h(\theta)$ *for all* $\theta \in supp(F)^\circ$

Proof. Let $\theta_0, \theta_1 \in supp(F)^\circ$ with $\theta_1 > \theta_0$. Then by the MLRP $f_h(\theta_1)f_l(\theta_0) > f_h(\theta_0)f_l(\theta_1)$ and integrating with respect to θ_0,

(i)

$$\int_{-\infty}^{\theta_1} f_h(\theta_1)f_l(\theta_0)d\theta_0 > \int_{-\infty}^{\theta_1} f_h(\theta_0)f_l(\theta_1)d\theta_0$$

$$\implies f_h(\theta_1)F_l(\theta_1) > F_h(\theta_1)f_l(\theta_1)$$

$$\implies \frac{f_h(\theta_1)}{f_l(\theta_1)} > \frac{F_h(\theta_1)}{F_l(\theta_1)}$$

(ii) Integrating instead with respect to θ_1,

$$\int_{\theta_0}^{-\infty} f_h(\theta_1)f_l(\theta_0)d\theta_1 > \int_{\theta_0}^{-\infty} f_h(\theta_0)f_l(\theta_1)d\theta_1$$

$$\implies (1 - F_h(\theta_0))f_l(\theta_0) > f_h(\theta_0)(1 - F_l(\theta_0))$$

$$\implies \frac{(1 - F_h(\theta_0))}{(1 - F_l(\theta_0))} > \frac{f_h(\theta_0)}{f_l(\theta_0)}$$

Combining the above gives $\frac{(1-F_h(\theta_0))}{(1-F_l(\theta_0))} > \frac{F_h(\theta_1)}{F_l(\theta_1)}$, or $F_l(\theta_1) > F_h(\theta_1)$ for any interior θ_1.

$\square$

Lemma 5. *Under the assumptions of social learning with limited history and general signals as in proposition 8, decision thresholds $\bar{\theta}_n$ if $a_{n-1} = 1$ and $\underline{\theta}_n$ if $a_{n-1} = 0$ satisfy $\bar{\theta}_1 < 0$ and $\bar{\theta}_n + \underline{\theta}_n = 0$. Moreover $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$ for all n.*

Proof. Consider the decision of the first agent. By the assumption of symmetry on F_l, F_h, agent 0 will play $a_0 = 1$ if and only if $\theta \geq 0$. As above, with prior $\Pr(h) = \frac{1}{2}$, agent 1 will then set thresholds $\bar{\theta}_1$ if $a_0 = 1$ and $\underline{\theta}_1$ if $a_0 = 0$ such that

$$\frac{f_h(\bar{\theta}_1)}{f_l(\bar{\theta}_1)} = \frac{\frac{1}{2}(1 - F_l(0))}{\frac{1}{2}(1 - F_h(0))} \quad \text{and} \quad \frac{f_h(\underline{\theta}_1)}{f_l(\underline{\theta}_1)} = \frac{\frac{1}{2}(F_l(0))}{\frac{1}{2}(F_h(0))}$$

Then $\frac{f_h(\bar{\theta}_1)}{f_l(\bar{\theta}_1)} < 1$ since $F_l(0) > F_h(0)$ by strict FOSD. Thus $\bar{\theta}_1 < 0$. Also, since symmetry gives $F_l(0) = 1 - F_h(0)$,

$$\frac{f_l(-\bar{\theta}_1)}{f_h(-\bar{\theta}_1)} = \frac{f_h(\bar{\theta}_1)}{f_l(\bar{\theta}_1)} = \frac{1 - F_l(0)}{1 - F_h(0)} = \frac{F_h(0)}{F_l(0)}$$

so that $\frac{f_l(-\bar{\theta}_1)}{f_h(-\bar{\theta}_1)} = \frac{f_l(\underline{\theta}_1)}{f_h(\underline{\theta}_1)}$. Then by the strict monotonicity of the likelihood ratio, $\underline{\theta}_1 = -\bar{\theta}_1$.

In the base case of $n = 2$, $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ and $\Pr(a_{n-2} = 1|l) = 1 - F_l(0) = F_h(0) = \Pr(a_{n-2} = 0|h)$. Conjecture $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ and $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$ for general

$n \geq 2$. As we in the proof of proposition 3 this implies that for general n the agent will set threshold $\bar{\theta}_n$ if $a_{n-1} = 1$ and $\underline{\theta}_n$ if $a_{n-1} = 0$ such that

$$\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{(1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})}{(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})}$$

and similarly if $a_{n-1} = 0$

$$\frac{f_h(\underline{\theta}_n)}{f_l(\underline{\theta}_n)} = \frac{F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1}) + (1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1})}{F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1}) + (1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1})}$$

By symmetry

$$\frac{f_l(-\bar{\theta}_n)}{f_h(-\bar{\theta}_n)} = \frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{(1 - F_l(\bar{\theta}_{n-1}))f_h(\bar{\theta}_{n-1}) + F_h(\bar{\theta}_{n-1})f_l(\bar{\theta}_{n-1})}{(1 - F_h(\bar{\theta}_{n-1}))f_l(\bar{\theta}_{n-1}) + F_l(\bar{\theta}_{n-1})f_h(\bar{\theta}_{n-1})} = \frac{f_l(\underline{\theta}_n)}{f_h(\underline{\theta}_n)}$$

which implies $\frac{f_l(-\bar{\theta}_n)}{f_h(-\bar{\theta}_n)} = \frac{f_l(\underline{\theta}_n)}{f_h(\underline{\theta}_n)}$. By the strict monotonicity of the likelihood ratio $\underline{\theta}_n = -\bar{\theta}_n$. Moreover,

$$\Pr(a_{n-1} = 1|l) = \Pr(a_{n-1} = 1|a_{n-2} = 1, l)\Pr(a_{n-2} = 1|l) + \Pr(a_{n-1} = 1|a_{n-2} = 0, l)\Pr(a_{n-2} = 0|l)$$
$$= (1 - F_l(\bar{\theta}_{n-1}))\Pr(a_{n-2} = 1|l) + (1 - F_l(\underline{\theta}_{n-1}))(1 - \Pr(a_{n-2} = 1|l))$$

and by symmetry of F_l and F_h and $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$ this becomes

$$= F_h(\underline{\theta}_{n-1}))\Pr(a_{n-2} = 1|l) + F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{n-2} = 1|l))$$
$$= F_h(\bar{\theta}_{n-1}) + \Pr(a_{n-2} = 1|l))(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))$$

$$\Pr(a_{n-1} = 0|h) = \Pr(a_{n-1} = 0|a_{n-2} = 1, h)\Pr(a_{n-2} = 1|h) + \Pr(a_{n-1} = 0|a_{n-2} = 0, h)\Pr(a_{n-2} = 0|h)$$
$$= F_h(\bar{\theta}_{n-1})(1 - \Pr(a_{n-2} = 0|h)) + F_h(\underline{\theta}_{n-1})\Pr(a_{n-2} = 0|h)$$
$$= F_h(\bar{\theta}_{n-1}) + \Pr(a_{n-2} = 0|h)(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))$$

So that

$$\Pr(a_{n-1} = 1|l) - \Pr(a_{n-1} = 0|h) = (\Pr(a_{n-2} = 1|l) - \Pr(a_{n-2} = 0|h))(F_h(\underline{\theta}_{n-1}) - F_h(\bar{\theta}_{n-1}))$$

and $\Pr(a_{n-1} = 1|l) = \Pr(a_{n-1} = 0|h)$ since $\Pr(a_{n-2} = 1|l) = \Pr(a_{n-2} = 0|h)$ by our induction conjecture. So by induction, $\bar{\theta}_n + \underline{\theta}_n = 0$ and $\Pr(a_n = 1|l) = \Pr(a_n = 0|h)$ for all n. $\qquad\square$

Lemma 6. *Information structures F_l and F_h satisfying **(A1)** - **(A4)**, equilibrium thresholds satisfy $\bar{\theta}_n + \underline{\theta}_n = 0$.*

Proof. If the first agent is aware of their position, then $\underline{\theta}_1 + \bar{\theta}_1 = 0$ as shown in the proof of lemma 5.

By induction, suppose $\bar{\theta}_{n-1} + \underline{\theta}_{n-1} = 0$. The decision thresholds for agent n can be expressed as

$$\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{1 - F_l(\bar{\theta}_{n-1})}{1 - F_h(\bar{\theta}_{n-1})} \quad \text{and} \quad \frac{f_h(\underline{\theta}_n)}{f_l(\underline{\theta}_n)} = \frac{1 - F_l(\underline{\theta}_{n-1})}{1 - F_h(\underline{\theta}_{n-1})}$$

Then

$$\frac{f_h(\bar{\theta}_n)/f_l(\bar{\theta}_n)}{f_h(-\underline{\theta}_n)/f_l(-\underline{\theta}_n)} = \frac{[1 - F_l(\bar{\theta}_{n-1})][1 - F_h(-\underline{\theta}_{n-1})]}{[1 - F_h(\bar{\theta}_{n-1})][1 - F_l(-\underline{\theta}_{n-1})]} = \frac{[1 - F_l(\bar{\theta}_{n-1})][1 - F_h(\bar{\theta}_{n-1})]}{[1 - F_h(\bar{\theta}_{n-1})][1 - F_l(\bar{\theta}_{n-1})]} = 1$$

where the penultimate equality results from the induction assumption. Thus $\frac{f_h(\bar{\theta}_n)}{f_l(\bar{\theta}_n)} = \frac{f_h(-\underline{\theta}_n)}{f_l(-\underline{\theta}_n)}$ and since the strict monotone likelihood ratio holds, it must be that $\bar{\theta}_n = -\underline{\theta}_n$. □

Lemma 7. *Under mutual symmetry of F_l and F_h,*

$$\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l) = F_h(\bar{\theta}_n) - \Pr(a_{n-1} = 1|l)[F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]$$
$$\Pr(a_n = 1|h) - \Pr(a_{n-1} = 1|h) = F_l(\bar{\theta}_n) - \Pr(a_{n-1} = 1|h)[F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]$$

and

$$\Pr(a_n = 1|l)\Pr(a_{n-1} = 1|h) - \Pr(a_{n-1} = 1|l)\Pr(a_n = 1|h)$$
$$= F_h(\bar{\theta}_n)\Pr(a_{n-1} = 1|h) - F_l(\bar{\theta}_n)\Pr(a_{n-1} = 1|l)$$

Proof.

$$\Pr(a_n = 1|l) = \Pr(a_n = 1|a_{n-1} = 1, l)\Pr(a_{n-1} = 1|l) + \Pr(a_n = 1|a_{n-1} = 0, l)\Pr(a_{n-1} = 0|l)$$
$$= (1 - F_l(\bar{\theta}_n))\Pr(a_{n-1} = 1|l) + (1 - F_l(\underline{\theta}_n))\Pr(a_{n-1} = 0|l)$$
$$= (1 - F_l(\bar{\theta}_n))\Pr(a_{n-1} = 1|l) + F_h(\bar{\theta}_n)(1 - \Pr(a_{n-1} = 1|l))$$
$$= F_h(\bar{\theta}_n) + \Pr(a_{n-1} = 1|l)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))$$

and

$$\Pr(a_n = 1|h) = \Pr(a_n = 1|a_{n-1} = 1, h)\Pr(a_{n-1} = 1|h) + \Pr(a_n = 1|a_{n-1} = 0, h)\Pr(a_{n-1} = 0|h)$$
$$= (1 - F_h(\bar{\theta}_n))\Pr(a_{n-1} = 1|h) + (1 - F_h(\underline{\theta}_n))\Pr(a_{n-1} = 0|h)$$
$$= (1 - F_h(\bar{\theta}_n))\Pr(a_{n-1} = 1|h) + F_l(\bar{\theta}_n)(1 - \Pr(a_{n-1} = 1|h))$$
$$= F_l(\bar{\theta}_n) + \Pr(a_{n-1} = 1|h)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))$$

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The first two desired equalities are easily obtained by rearranging the above equations while the third is given by

$$\Pr(a_n = 1|l)\Pr(a_{n-1} = 1|h) - \Pr(a_{n-1} = 1|l)\Pr(a_n = 1|h)$$
$$= [F_h(\bar{\theta}_n) + \Pr(a_{n-1} = 1|l)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))]\Pr(a_{n-1} = 1|h)$$
$$- \Pr(a_{n-1} = 1|l)[F_l(\bar{\theta}_n) + \Pr(a_{n-1} = 1|h)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))]$$
$$= F_h(\bar{\theta}_n)\Pr(a_{n-1} = 1|h) - F_l(\bar{\theta}_n)\Pr(a_{n-1} = 1|l)$$

$$\square$$

Lemma 8. *If $\hat{\theta}_n$ converges to a limit then $\Pr(a_n|l)$ and $\Pr(a_n|h)$ also converge.*

Proof. By lemma 7, $\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l) = F_h(\bar{\theta}_n) - \Pr(a_{n-1} = 1|l)[F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)]$. If $\Pr(a_n|l)$ does not converge, let $\varepsilon > 0$ for which for any N there is always some $n \geq N$ with $|\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l)| > \varepsilon$. Suppose $\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l) > \varepsilon$. Then

$$\Pr(a_{n-1} = 1|l) < \frac{F_h(\bar{\theta}_n)}{(F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n))} - \varepsilon$$

Then

$$\Pr(a_{n+1} = 1|l) - \Pr(a_n = 1|l) = F_h(\bar{\theta}_{n+1}) - \Pr(a_n = 1|l)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))$$
$$= F_h(\bar{\theta}_{n+1}) - (F_h(\bar{\theta}_n) + \Pr(a_{n-1} = 1|l)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))$$
$$= F_h(\bar{\theta}_{n+1}) - F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) - \Pr(a_{n-1} = 1|l)(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))$$
$$> F_h(\bar{\theta}_{n+1}) - F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))$$
$$- (1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))\left(\frac{F_h(\bar{\theta}_n)}{F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)} - \varepsilon\right)$$
$$= F_h(\bar{\theta}_{n+1}) - F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) + F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))$$
$$+ \varepsilon(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1})) - \frac{F_h(\bar{\theta}_n)(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))}{F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)}$$
$$= \frac{F_h(\bar{\theta}_{n+1})F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n)F_l(\bar{\theta}_{n+1})}{F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n)} + \varepsilon(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))$$

Given the convergence of $\bar{\theta}_n$, n can be made large enough that

$$F_h(\bar{\theta}_{n+1})F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n)F_l(\bar{\theta}_{n+1}) > -\varepsilon(1 - F_l(\bar{\theta}_n) - F_h(\bar{\theta}_n))(F_l(\bar{\theta}_{n+1}) + F_h(\bar{\theta}_{n+1}))(F_l(\bar{\theta}_n) + F_h(\bar{\theta}_n))$$

so that $\Pr(a_{n+1}|l) > \Pr(a_n = 1|l)$ for $n \geq N$. Thus $\Pr(a_n = 1|l)$ is an increasing sequence, bounded above and must converge. If $\Pr(a_n = 1|l) - \Pr(a_{n-1} = 1|l) < -\varepsilon$ then $\Pr(a_n = 1|l)$ is a decreasing sequence bounded below and must converge. Since $\Pr(a_n = 0|l) = 1 - \Pr(a_n = 1|l)$ this must converge as well. Finally, an analogous proof shows the convergence of $\Pr(a_n|h)$. $\square$