Numerical Methods for Solving PDE's Arising in Option Pricing

An Overview

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Overview

- Review of Black-Scholes PDE for European Options
- Conversion to the Standard Diffusion Equation
- Finite Difference and Finite Elements Methods
- Pricing American options
- Methods for Exotic Options

The Black-Scholes PDE

Recall from Lecture 5 that we modeled the stock price S as an Itô process:

$$dS = \mu S\, dt + \sigma S\, dz \tag{1}$$

If $f(S,t)$ is a function of the stock price, then Itô's Lemma tells us

$$df = \sigma S \frac{\partial f}{\partial S} dz + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt. \tag{2}$$

(See Lecture 6.)

The Black-Scholes PDE

The random component dz in these equations can be eliminated using a simple hedging argument. We obtain the famed Black-Scholes partial differential equation:

$$\boxed{\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0.} \tag{3}$$

For now, we will consider a European call option to keep things simple.

The Black-Scholes PDE

To investigate the existence of solution to (3), we rewrite the equation in operator notation:

$$\left(\frac{\partial}{\partial t} - r + rS\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}\right) C = 0 \tag{4}$$

An operator of this form is called parabolic.

PDE theory asserts that in order for this problem to have a unique solution it is sufficient to know the value of the solution at one point in time and at $S = 0$ and $S = \infty$. These are called boundary conditions.

The Black-Scholes PDE

- Usually know value of solution at $t = 0$—an *initial* condition. In the Black-Scholes case, however, we know the *final* value of the solution—for a European option the payoff function at time $t = T$. (Black-Scholes PDE is "backwards" wrt time.)
- We know that if $S = 0$, the option is worthless, so $C(0, t) = 0$ for $0 \leq t \leq T$.
- We also know that if $S \to \infty$, exercise is almost certain. Thus $C(S, t) \sim S$ as $S \to \infty$.

The Heat Equation

- The prototypical example of a parabolic PDE is the heat (or diffusion) equation.
- The (one-dimensional) heat equation models the heat flow in a uniform medium (i.e., the thermal conductivity does not change with location) over time.
- The heat equation has been extensively studied. If we can transform the Black-Scholes equation into the heat equation, we can appeal to standard results and methods to study the solution to the Black-Scholes equation.

The Heat Equation

The heat equation on the domain $(x, t) \in (-\infty, \infty) \times [0, \infty)$ is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{5}$$

$$u(x, 0) = u_0(x), \quad -\infty < x < \infty \tag{6}$$

$$u(x, t) \sim F(x, t), \quad x \to \infty \tag{7}$$

$$u(x, t) \sim G(x, t), \quad x \to -\infty \tag{8}$$

The Heat Equation

The general solution to the heat equation

$$u(x,t) = \frac{1}{2\sqrt{t\pi}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4t}}\, dt \tag{9}$$

$$= u_0(x) * K(x,t), \tag{10}$$

where $K(x,t)$ is the heat kernel or fundamental solution:

$$K(x,t) = \begin{cases} (4\pi t)^{-1/2} e^{-x^2/4t}, & t > 0 \\ 0 & t \leq 0 \end{cases}. \tag{11}$$

and "$*$" represents convolution.

Conversion to the Heat Equation

To solve the Black-Scholes equation analytically, we convert it to a forward diffusion equation using the change of variables

$$S = Ke^x, \qquad t = T - 2\tau/\sigma^2, \qquad C = Kv(x, \tau) \quad (12)$$

Notice we are effectively now working with $\log S$ instead of S. Also notice that x and v will be dimensionless.

Conversion to the Heat Equation

This reduces the Black-Scholes equation to the equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{dv}{dx} - kv, \tag{13}$$

where $k = r/\frac{1}{2}\sigma^2$ is a dimensionless parameter. The boundary condition becomes

$$v(x,0) = \max(e^x - 1, 0) \tag{14}$$

Conversion to the Heat Equation

We can eliminate the $\frac{dv}{dx}$ term in (13) using an "integrating factor"

$$v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau), \tag{15}$$

where $\alpha = -0.5(k-1)$ and $\beta = -0.25(k+1)^2$.

Conversion to the Heat Equation

This reduces (13) to the diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{16}$$

with boundary conditions

$$u(x,0) = u_0(x) \tag{17}$$

$$u(0,\tau) \to 0 \quad \text{as } x \to -\infty \tag{18}$$

$$u(x,\tau) \to g(x,\tau) \quad \text{as } x \to \infty, \tag{19}$$

Conversion to the Heat Equation

where the boundary functions are

$$u_0(x) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) \tag{20}$$

$$g(x,\tau) = e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau} \tag{21}$$

We can now find the solution $u(x,t)$ in terms of the fundamental solution (9). Some messy integration there, but technique is the same as that used in Lecture 6.

Conversion to the Heat Equation

To find the solution to the Black-Scholes equation, we simply transform our solution $u(x,t)$ back to the original variables to yield the solution to the Black-Scholes equation:

$$\boxed{C(S,t) = S\phi(d_1) - Ke^{-r(T-t)}\phi(d_2)} \tag{22}$$

where $\phi(x)$ is the CDF for the normal distribution and d_1 and d_2 are the constants introducted in Lecture 6.

Numerical Methods for PDE's

Two major methods:

- Finite Differences—approximate *derivatives*
- Finite Elements—approximate *integrals*

Both methods approximate a PDE by a matrix problem. For most of the problems we will consider here, both give the same matrix problem.

Finite Difference Method (FDM)

Approximate derivatives by difference quotients. More than one approximation.

In one dimension:

- Forward

$$f'(t) \approx \frac{f(t+\delta t) - f(t)}{\delta t} \tag{23}$$

- Centered

$$f'(t) \approx \frac{f(t+\delta t) - f(t-\delta t)}{2\delta t} \tag{24}$$

- Backward

$$f'(t) \approx \frac{f(t) - f(t-\delta t)}{\delta t} \tag{25}$$

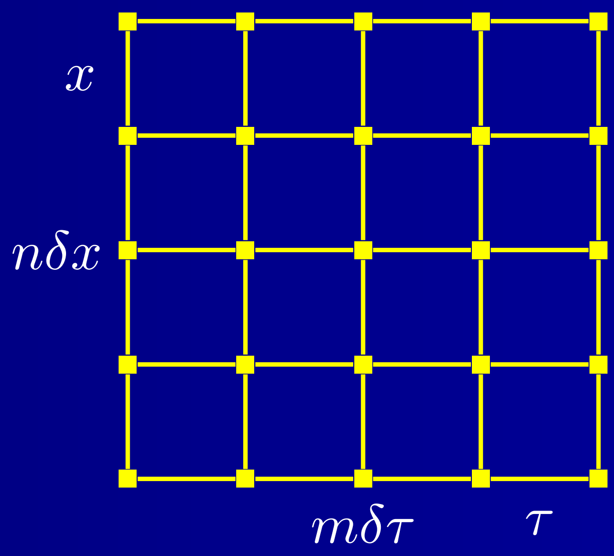
Finite Differences Method (FDM)

By repeated use of first-order formulas can get formulas for higher order derivatives. We will use a centered difference for $\frac{\partial^2 u}{\partial x^2}$:

$$\frac{\partial^2 u}{\partial x^2}(x,t) \approx \frac{u(x+\delta x,t) - 2u(x,t) + u(x-\delta x,t)}{(\delta x)^2} \tag{26}$$

Creating a Mesh

- Notation: $u_n^m = u(n\delta x, m\delta t)$
- Issue: discretize S in untransformed Black-Scholes, or discretize $\log S$ in transformed equation?
- Stock price has lognormal increments, so $\log S$ models the underlying Brownian motion better.
- Approx. infinite mesh by finite one. Needs some care.



Explicit Finite Differences

Discretize $\frac{\partial u}{\partial t}$ by forward difference, $\frac{\partial^2 u}{\partial x^2}$ by (26):

$$\frac{u(x,\tau+\delta\tau)-u(x,\tau)}{\delta\tau} \approx \frac{u(x+\delta x,\tau)-2*u(x,\tau)+u(x-\delta x,\tau)}{(\delta x)^2} \tag{27}$$

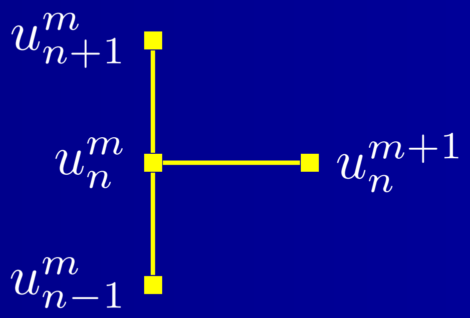
In terms of our mesh (see the previous slide) this equation says that

$$u_n^{m+1} \approx \alpha u_{n+1}^m + (1-2\alpha)u_n^m + \alpha u_{n-1}^m \tag{28}$$

where $\alpha = \delta\tau/(\delta x)^2$.

Explicit Finite Differences

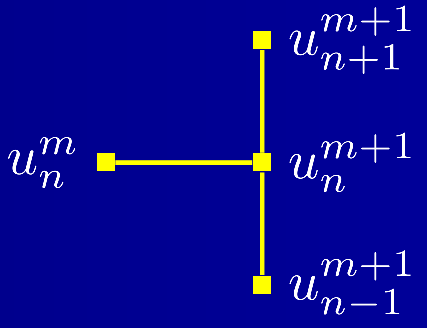
- Notice that u_n^{m+1} depends on the known quantities u_{n-1}^m, u_n^m, and u_{n+1}^m. Hence the name “explicit”.



- Notice structure is similar to a trinomial tree (going backwards in time). Up prob. is α, down prob. is α, stay put prob. is $(1-2\alpha)$.
- α is a stability parameter: need $0 < \alpha \leq \frac{1}{2}$ for to keep rounding errors from growing each step.
- Severely limits the spatial resolution of explicit finite differences.

Implicit Finite Differences

- Use backward approximation for $\frac{\partial u}{\partial t}$.
- Now u_{n-1}^{m+1}, u_n^{m+1}, and u_{n+1}^{m+1} all depend on u_n^m in an implicit manner.



- Requires us to solve a tridiagonal linear system to find values at the next time step.
- Always stable, for any value of $\alpha > 0$.

Crank-Nicolson Method

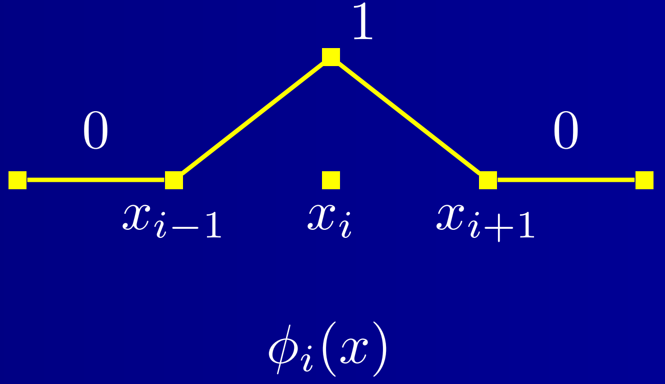
- Average explicit and implicit methods
- Stable for all $\alpha > 0$
- Converges faster than explicit or implicit methods ($O((\delta\tau)^2)$ vs. $O(\delta\tau)$)
- Again need to solve a linear system.

Finite Element Method (FEM)

- Suppose the solution u to the heat equation lies in some space (e.g. $L^2([0,1])$).
- Approximate u by $\tilde{u}$ in some finite-dimensional subspace of the original space
- Connected to the idea of a weak solution to a PDE: if L is a diff. operator and f is a “right-hand side”, find u s.t. $\langle Lu, v\rangle = \langle f, v\rangle$ for all test functions v. Here $\langle \circ, \star\rangle$ denotes integration wrt to some measure.

Finite Element Method (FEM)

- Discretize spatial domain $\{x_0, \ldots, x_n\}$.
- A typical test function (a.k.a shape function) is shown at right.



- Using $n-1$ shape functions, our approximation to u is

$$u_n(x) = \sum_{i=1}^{n-1} \alpha_i \phi_i(x) \tag{29}$$

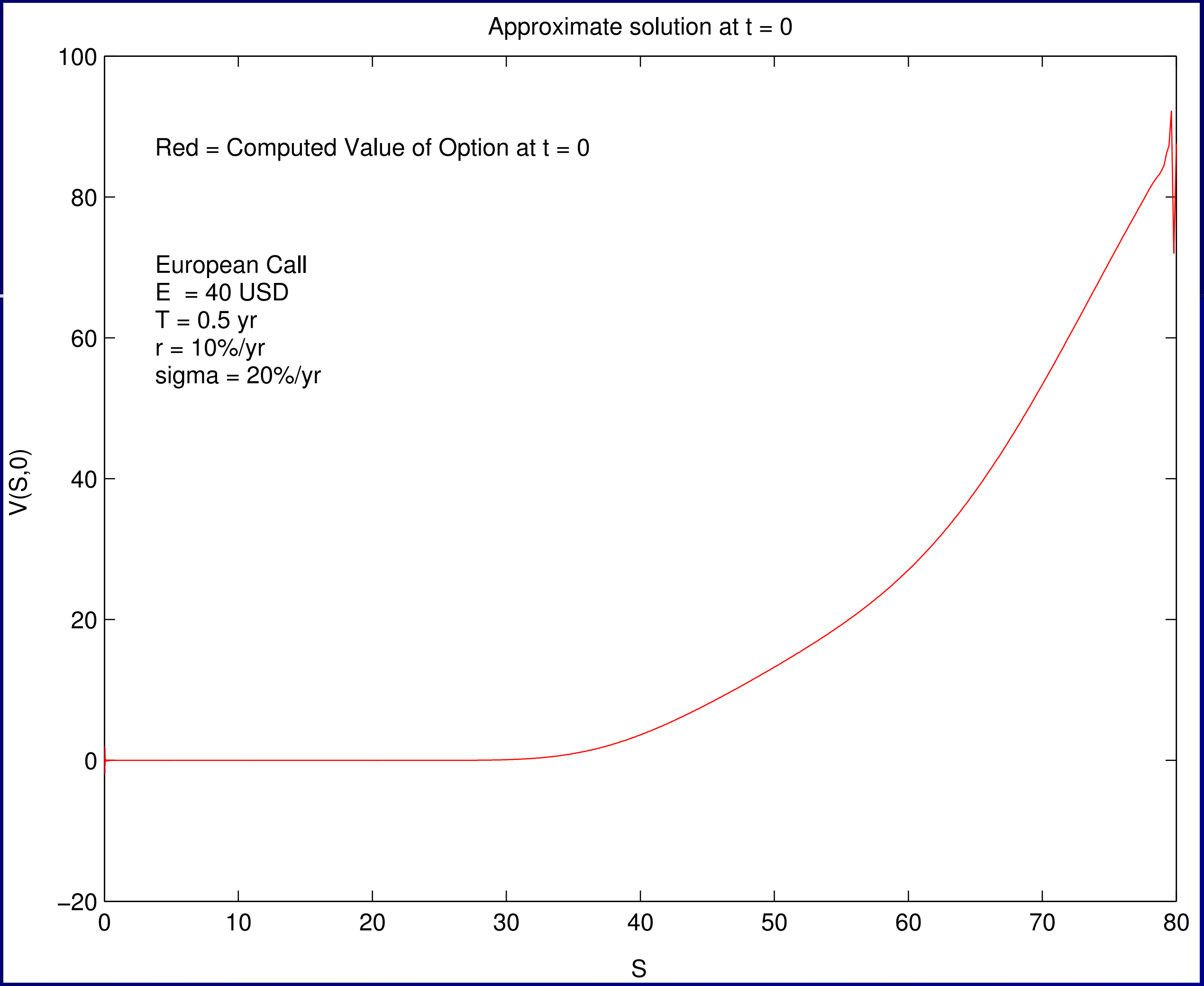
where the α_i's are coefficients to be determined by plugging into weak form of PDE.

Pricing European Calls and Puts

Convert to heat equation, solve, convert back.

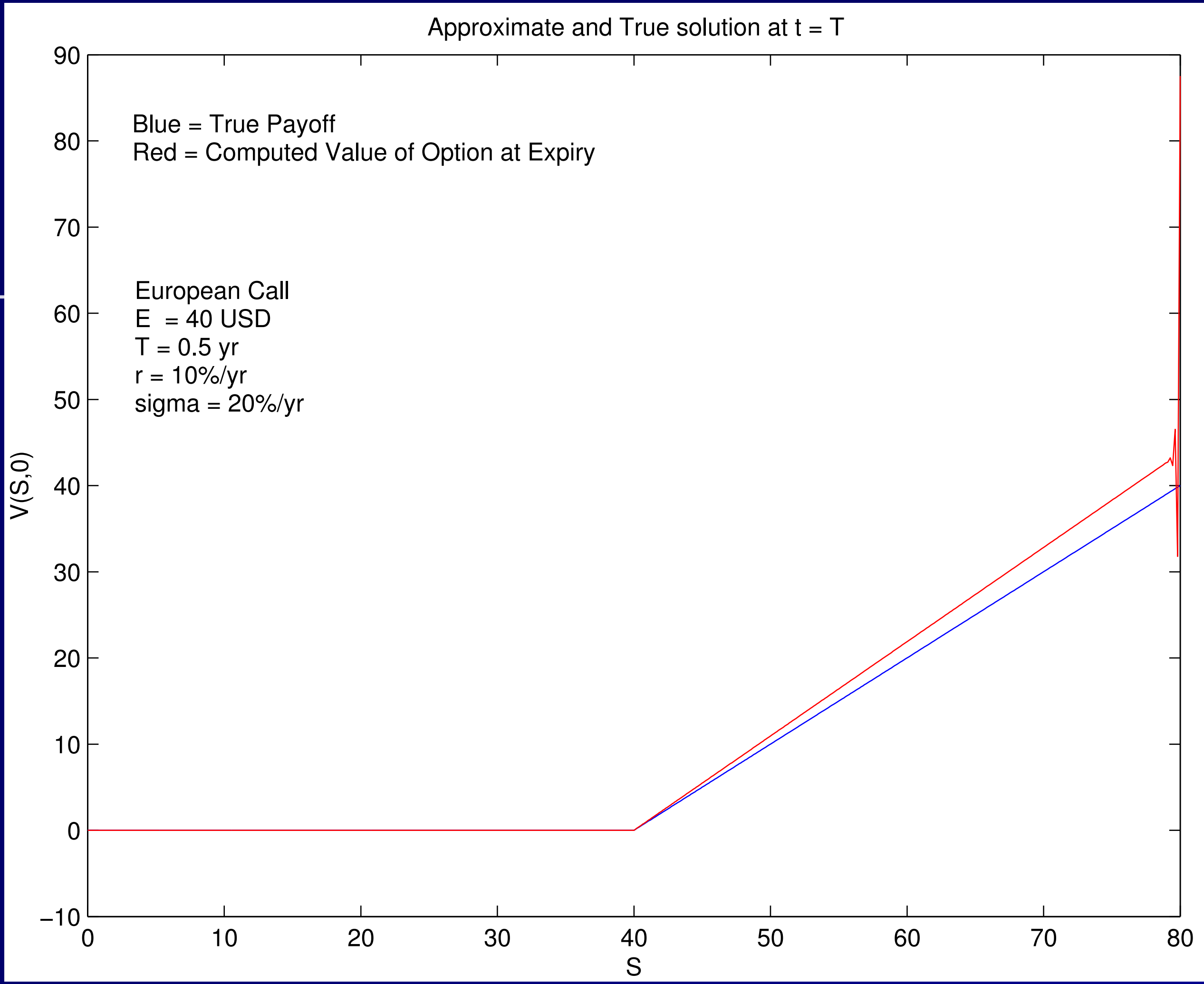
- Could use FDM or FEM in space
- Crank-Nicholson in time
- Large sparse symmetric linear system—use an iterative method (such as conjugate gradients with a good preconditioner) for best results.

Example on next slide.



Price at time $t = 0$ of a 6-month European call with strike price 40 USD. (See Hull, p249).

The mesh consisted of 5000 points in the x-direction and 60 points in the τ direction. Total MATLAB time was approx. 10 minutes on a (loaded) P4 1.5 GHz with 512 MB RAM. Note the endpoint instability due to the truncation of the infinite domain.



Price at expiry of a the 6-month European call mentioned in the previous slide. (See Hull, p249).

Pricing American Options

- American options are potentially worth more than their European cousins. Cannot blindly apply the above methods to American options!
- When exercise early is not optimal, the American option should behave just like the European option, i.e., it should satisfy the Black-Scholes equation.
- When early exercise is optimal, arbitrage argument for Black-Scholes only tells us that the return for the portfolio cannot exceed the return on a risk-free deposit.

Pricing American Options

Thus an American option satisfies a Black-Scholes inequality:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf \leq 0. \qquad (30)$$

Pricing American Options

- By “no arbitrage”, value of option cannot fall below intrinsic value (payoff)
- Thus must have $P(S, t) \geq \max(K - S, 0)$ for a put.
- When early exercise is optimal, option value is the payoff value.
- Problem: *a priori* we don’t know where the exercise boundary is!
- This is an example of a free boundary problem—we have a boundary condition, but the location of the boundary must be determined as part of the solution.

Free Boundary Problems

Example: The Stefan Problem

- Apply heat to a block of ice at one end. The ice melts, but not uniformly.
- Under some simplifying assumptions, the diffusion equation still holds for the liquid part.
- Boundary between water and ice is unknown *a priori*, changes over time.
- Modeled like the heat equation, but with conditions on the free boundary.

Pricing American Options

- We make the same change of variables as in the European case to convert the Black-Scholes equation into a diffusion equation.
- Boundary functions are now

$$u_0(x) = \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0) \tag{31}$$

$$g(x,\tau) \geq e^{\frac{1}{4}((k-1)^2+4k)\tau} u(x,0) \tag{32}$$

- Could probably apply numerical methods here, but there is a better way!

Linear Complementarity Form

- Essentially a “weak” formulation of the problem—recall the construction of the finite element method
- LC form of Black-Scholes for American put:

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \right) (u(x,\tau) - g(x,\tau)) = 0 \tag{33}$$

$$\left(\frac{\partial(}{\partial u})\tau - \frac{\partial^2 u}{\partial x^2} \right) \geq 0 \qquad (u(x,\tau) - g(x,\tau)) \geq 0 \tag{34}$$

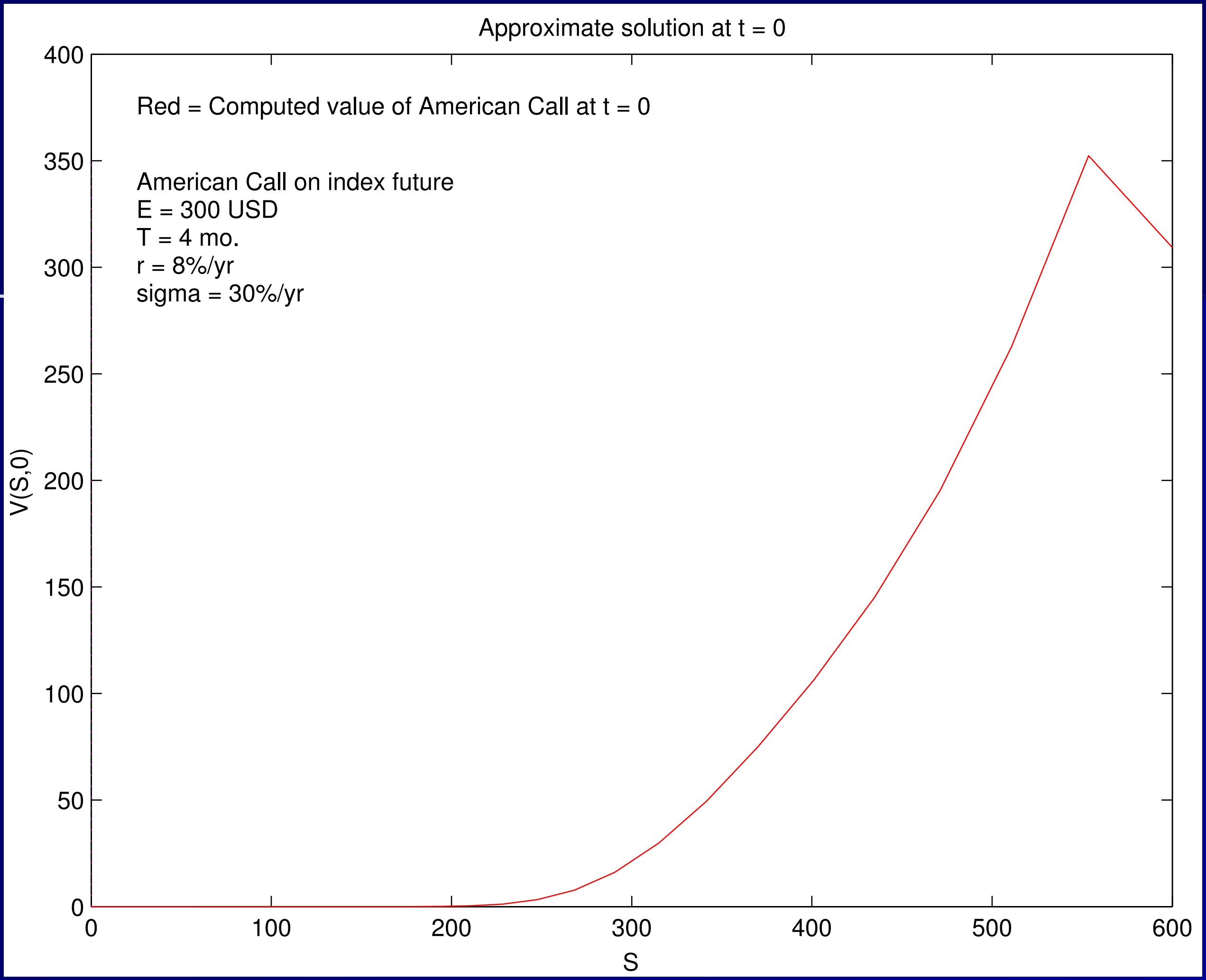
- Advantage: no explict mention of free boundary!

Pricing American Options

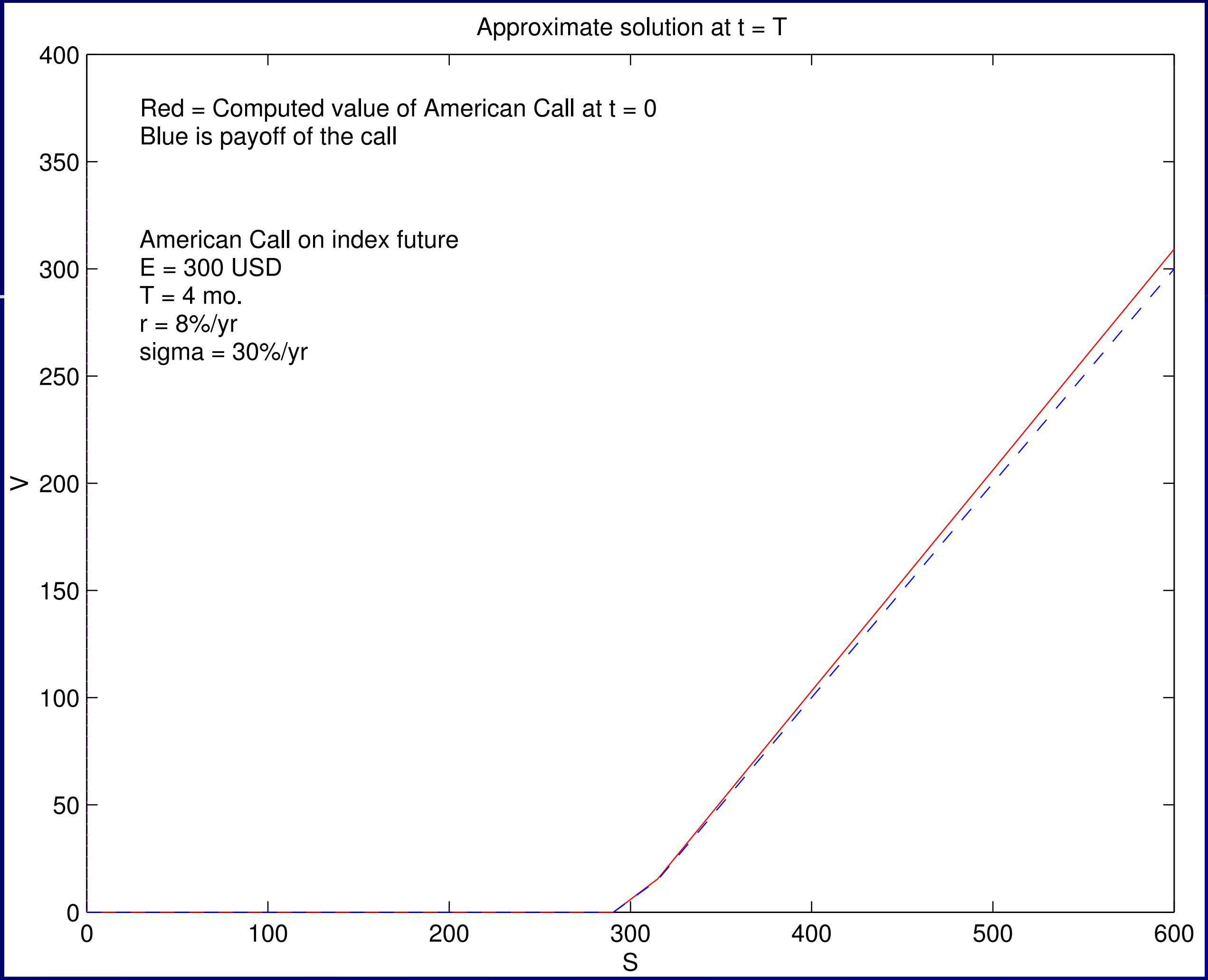
- Discretize LC form using FDM or FEM with Crank-Nicolson.
- Need to solve a constrained matrix problem of the form

$$Ax \geq b, \quad x \geq c, \quad (x-c)^T(Ax-b) = 0$$

- Projected Successive Over Relaxation algorithm is the standard tool for this.
- Solve for free boundary after the fact using its definition.



Price at time $t = 0$ of a 4-month American call on index futures with strike price 300 USD. (See Hull, p400). The mesh consisted of 250 points in the x-direction and 120 points in the τ direction. Total MATLAB time was approx. 10 minutes on a (loaded) P4 1.5 GHz with 512 MB RAM. Note the endpoint instability due to the truncation of the infinite domain.



Price at time $t = T$ of same option.

Methods for Other Types of Options

- European/American call/put with constant dividend yield. Slight chance to above method, very easy.
- European/American Cash-or-nothing call. Different payoff function, but an identical methodology can be used. Must be careful about the jump discontinuity in the payoff.

Methods for Other Types of Options

- Other vanilla barrier options. Same change of variables, extra boundary condition. Use the “method of images” with the above methodology.
- Asian options. Need a little bit of everything!
- Loopback options. For European, use the ”method of images”. For American, use LC form.

Methods for Other Types of Options

- Russian options (lookback with infinite time horizon). LC form seems to be a good strategy in most setups.
- Passport options (call on the balance of a trading account—see Wilmott’s web site). Hamilton-Jacobi-Bellman equations, multiple free boundaries...
- Parisian options (barrier, but barrier only active if price remains beyond the barrier for prespecified time). Wilmott has a solution using basic methods we described above.

Questions for Further Study

Some questions for further study.

Conservation Law for American Options?

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