An Introduction to Self Similar Structures

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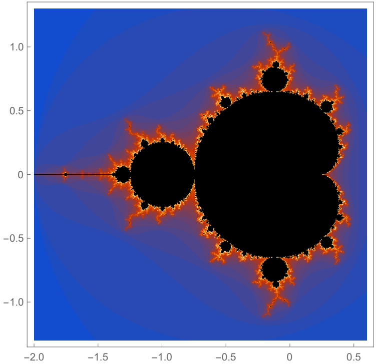
April 6th, 2018

What's a Fractal?

Fractal is not a precise term. There are many examples of fractals. Two commonly discussed types are *Julia Sets* such as the Mandelbrot Set and *Self Similar Sets* such as the Sierpinski Gasket.



The Mandelbrot set, which is *not* the topic of today's talk.

Self-Similarity

Self similar intuitively means that if you look at a small 'piece' of the set (or fractal) then it will look like the larger set, no matter how close you look.

Self-Similar Sets

Self similar sets are constructed using a finite number of special functions from real space to itself called *contractions*. Often, the contractions contract in a fixed way; these are called *similitudes*.

Contractions & Similitudes

Definition

Let (X, d) be a metric space. An injective map $f : X \to X$ is called a *contraction* if $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$.

Contractions are Lipschitz continuous with the Lipschitz constant usually represented in this context by r, sometimes by L.

Definition

Let (X, d) be a metric space. A contraction f is called a *similitude* if there is some $0 < r < 1$ such that $d(f(x), f(y)) = rd(x, y)$ for all x, y.

For similitudes, r is called the *contraction ratio*.

Sierpinski Gasket

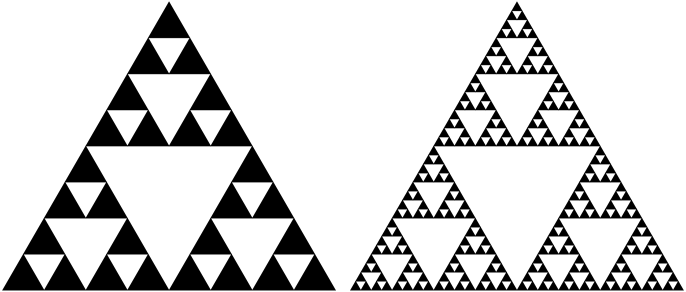
A Sierpinski Gasket is constructed with the three distinct points $q_1, q_2, q_3 \in \mathbb{R}^2$ of a triangle and three associated similitudes:

$$f_i(x) = \frac{x + q_i}{2}$$

where elements are treated as vectors. This has contraction ratio $1/2$ and 'pulls' space towards q_i.

Sierpinski Gasket

Define $F(A) = f_1(A) \cup f_2(A) \cup f_3(A)$ where A is a nonempty compact set. Define $F^n(A) = F \circ F \circ \cdots F(A)$ to be the n-fold composition of F. The Sierpinski Gasket is the limit $\lim_{n\to\infty} F^n(A)$ and visually:



where the left is $n = 3$ and the right is $n = 5$.

'Standard' Similitude

There are several kinds of similitudes, but the 'standard' one used in the Sierpinski Gasket, Carpet, and n-gasket uses a point q and a contraction ratio r and is defined:

$$f(x) = r(x - q) + q$$

where everything is treated as a vector in $\mathbb{R}^n$, usually $n = 2$.

Theorem

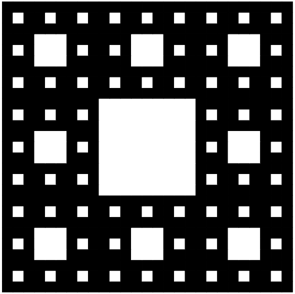
Similitudes f on $\mathbb{R}^n$ always have the form:

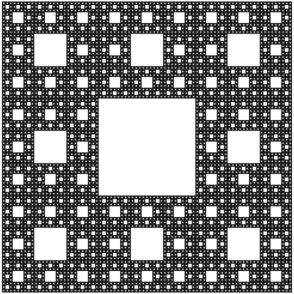
$$f(x) = rUx + a$$

where r is the contraction ratio, a is any point, and U is an element of the orthogonal group of $\mathbb{R}^n$, usually denoted $O(n)$.

Sierpinski Carpet

A Sierpinski Carpet is built with eight similitudes centered on the corners and midpoints of a square. The contraction ratio is $1/3$. Visually, the limit goes:

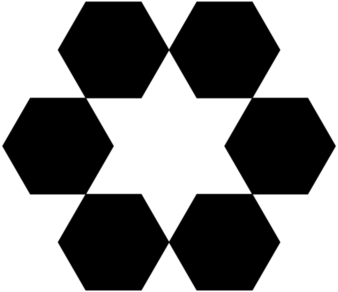


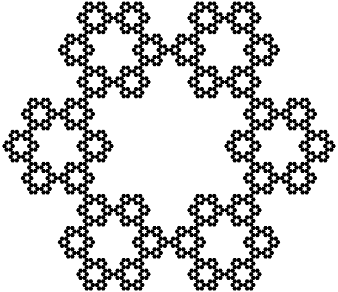


where the left is after 3 iterations.

Hexagasket

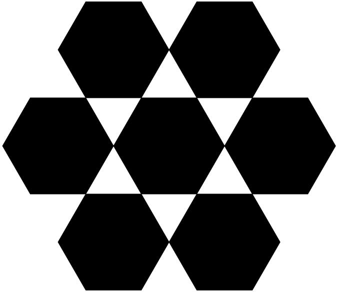
A Hexagasket is built with a hexagon and six similitudes at the vertices and a contraction ratio of $1/3$. Visually:

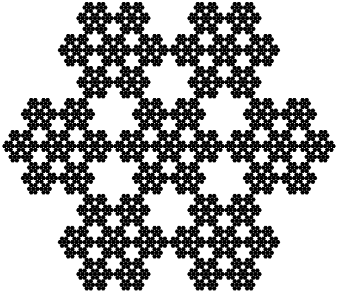




Lindstrøm Snowflake

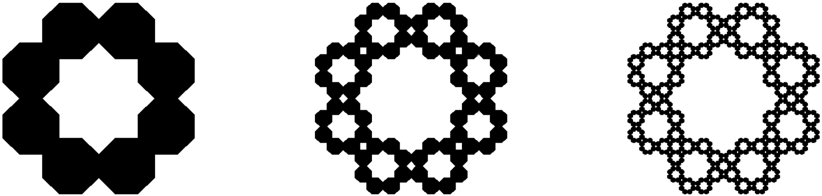
A hexagasket that adds the center:





Octagasket

An Octagasket is built with an octagon and eight similitudes in the usual place, and a contraction ratio of $1 - 1/(1+\sqrt{2})$. Visually:



Images taken from the Projective Octagasket program website (Cornell)

More Basic Examples

Interval

The interval is a self-similar set, we can construct it using *dyads* of decreasing width $1/2^m$. The similitudes are at 0 and 1, with ratio $1/2$.

Cantor Set

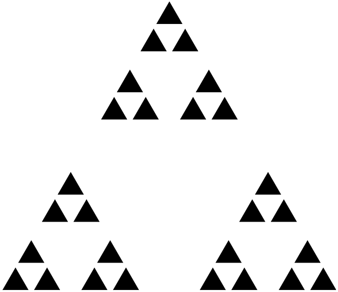
The Cantor Set uses the same similitudes as the interval but with contraction ratios $r < 1/2$. The usual third Cantor set uses $r = 1/3$.

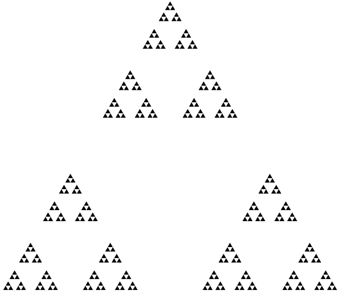
Square

Similar to the interval, we can create a dyadic construction of the square with $r = 1/2$ at the corners of the square.

“Sierpinski Dust”

If you contract by more than $1/2$ using the Sierpinski Gasket’s similitudes, there is no overlap and you get a Cantor Dust-like set:





Complex Operations as Similitudes

Multiplication

Recall that multiplication in $\mathbb{C}$ is essentially rotating and scaling simultaneously, that is, $z = re^{i\theta}$ acts on w by rotating w by θ radians and scaling it by r. If $0 < |z| = r < 1$ then multiplication is a similitude.

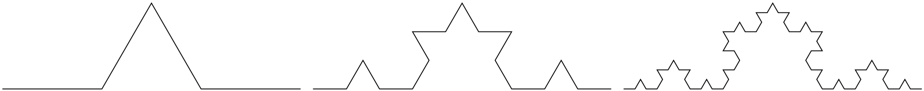
Conjugation

Conjugation (reflection across the real axis), that is, $z \mapsto \bar{z}$, is another important similitude.

These complex operations are used in the construction of the Koch Curve and the Hata Tree.

Koch Curve

There is an elegant two-similitude construction of the famous Koch Curve:



Koch Curve

Let $\alpha \in \mathbb{C}$ satisfy $|\alpha|^2 + |1-\alpha|^2 < 1$. Set:

$$f_1(z) = \alpha \bar{z}$$

$$f_2(z) = (1-\alpha)(\bar{z}-1) + 1$$

Hata Tree

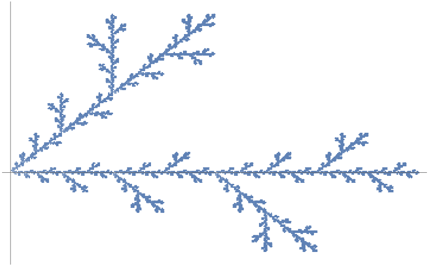
An interesting use of conjugation to create a tree-like fractal.

Hata's Tree-like Set

Let $\alpha \in \mathbb{C}$ such that $0 < |\alpha| < 1$ and $0 < |1 - \alpha| < 1$. Set:

$$f_1(z) = \alpha \bar{z}$$

$$f_2(z) = (1 - |\alpha|^2)\bar{z} + |\alpha|^2$$



Contraction Principle

Theorem

Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction. Then there exists a unique fixed point x^* of f. Moreover, for any $a \in X$, we have:

$$\lim_{n\to\infty} f^n(a) = x^*$$

where f^n is the n-fold composition of f with itself.

The Contraction Principle seems to be about points in space, but there are metric spaces where 'points' are more interesting.

Hausdorff Metric

Definition

Let A and B be nonempty compact sets in (X, d). Define for $\varepsilon > 0$:

$$A_\varepsilon = \{x \mid d(y,x) \leq \varepsilon, y \in A\}$$

Definition

The *Hausdorff Metric* on the space of nonempty compact subsets of a metric space (X, d) is:

$$\delta(A, B) = \inf\{\varepsilon \mid A_\varepsilon \supset B \wedge B_\varepsilon \supset A;\ \varepsilon > 0\}$$

Contraction Principle for Self Similar Sets

Definition

Let $f_1, f_2, \ldots, f_k$ be contractions in (X, d). Define $F : \mathcal{C}(X) \to \mathcal{C}(X)$, where $\mathcal{C}(X)$ is the space of nonempty compact sets in X, by:

$$F(A) = \bigcup_{i=1}^{k} f_i(A)$$

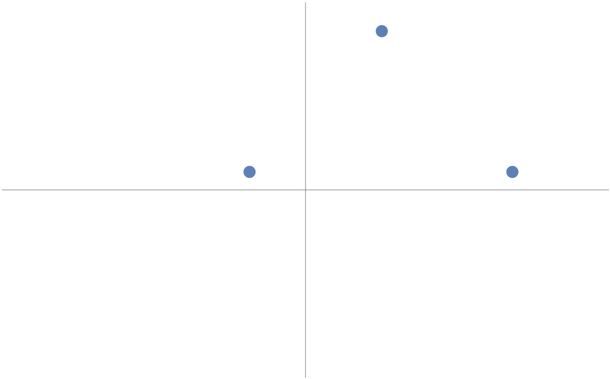
It is a fact that F is a contraction in $(\mathcal{C}(X), \delta)$.

Theorem

Let (X, d) be a complete metric space. Then $(\mathcal{C}(X), \delta)$ is complete. By the contraction principle on $(\mathcal{C}(X), \delta)$, F has a unique fixed 'point' K. We call K the self-similar set associated with $f_1, f_2, \ldots, f_k$.

Initial Compact Set is Irrelevant

One result of the Contraction Principle is that your 'initial set' is irrelevant, the fractal is determined only by the functions. You can see this visually with the Sierpinski Gasket:



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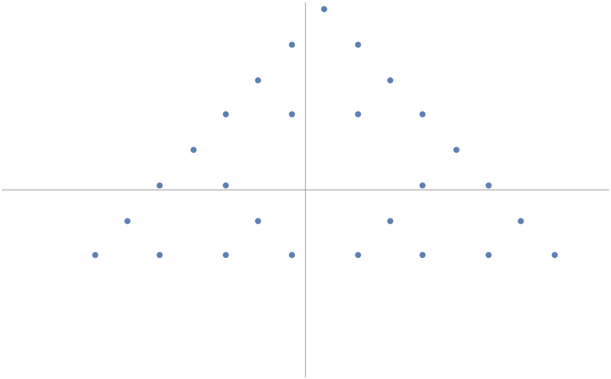

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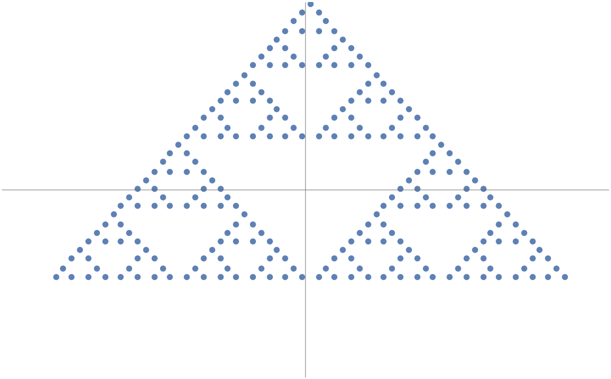

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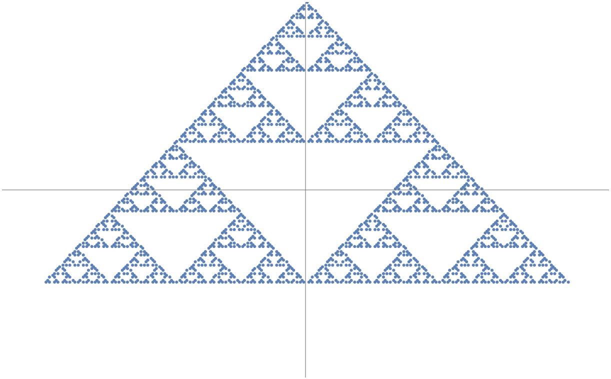

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and on the Hata Tree:

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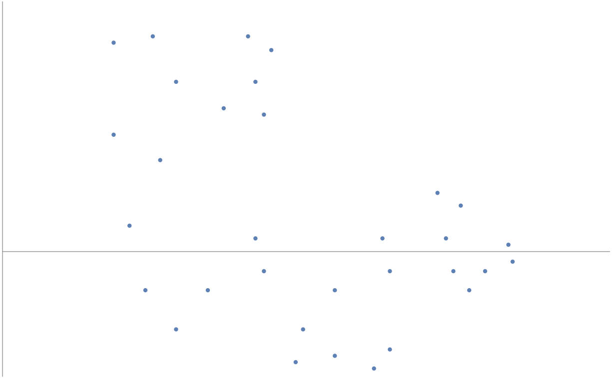

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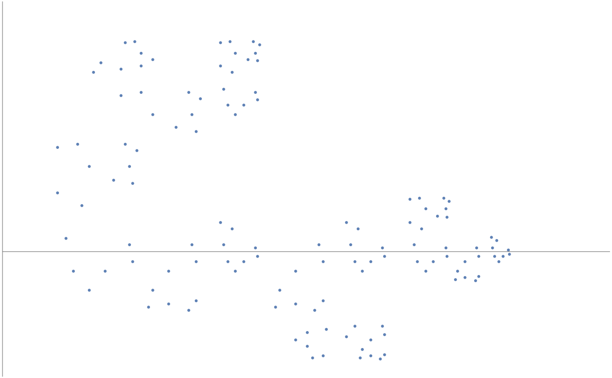

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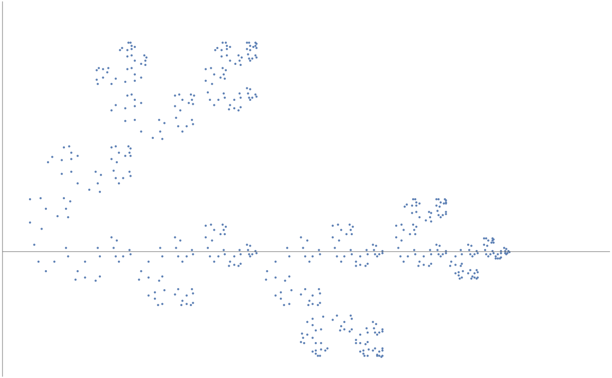

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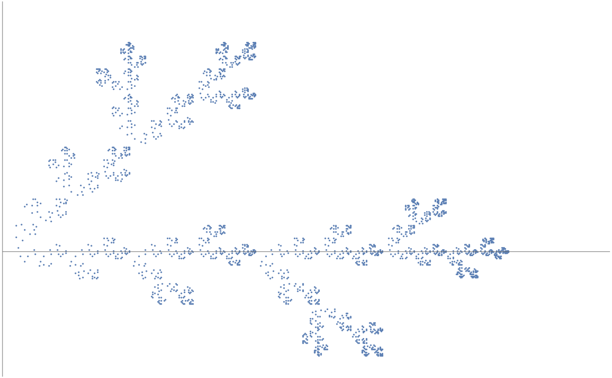

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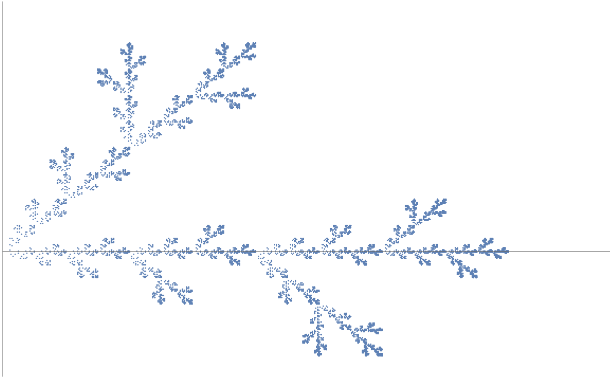


Can We Build Fractals as Infinite Words?

Singletons are compact sets, so let $a \in X$. A self-similar set K with similitudes $f_1, \cdots, f_k$ can be found as the 'limit' of the sequence of sets:

$$F^0(a) = \{a\}$$

$$F^1(a) = \{f_1(a), f_2(a), \ldots, f_k(a)\}$$

$$F^2(a) = \{f_1 \circ f_1(a), f_2 \circ f_1(a), \ldots, f_k \circ f_1(a), f_1 \circ f_2(a), \ldots, \ldots, f_k \circ f_k(a)\}$$

and so on. We can select distinct 'paths' through the sequence of sets by looking at infinite-length 'words' built from an alphabet $\{1, 2, \ldots, k\}$. We can also interpret this as a k-ary tree. But be careful - two different infinite words or paths may lead to the same point!

Word Spaces

Definition

- Let S be a finite set, the *alphabet*. Usually $S = \{1, \ldots, k\}$ where there are k similitudes.
- Let W_m be the space of finite words of length m built with S, for example $4235 \in W_4$ when $S = \{1, 2, \ldots, 5\}$.
- Let $W_* = \cup_{m \geq 0} W_m$ be the *finite word space*. Elements are usually denoted by $w = w_1 w_2 w_3 \ldots w_m$ where $w_i \in S$.
- Let Σ be the *infinite word space* or *shift space*. Elements are usually denoted by $\omega = \omega_1 \omega_2 \cdots$, using lowercase omega instead of w.

Shifts and Shift Spaces

Definition

- Define $\sigma_i(\omega) = i\omega$ via concatenation, where $i \in S$.
- If $\omega = \omega_1\omega_2\omega_3\cdots$, define $\sigma(\omega) = \omega_2\omega_3\cdots$.

Essentially, σ_i 'tacks on' i and σ deletes the first letter. Note in particular that for any $i \in S$ we have $\sigma \circ \sigma_i(\omega) = \omega$.

Definition

- If $w = w_1 w_2 \cdots w_k$ then define $\sigma_w = \sigma_{w_1} \circ \sigma_{w_2} \circ \cdots \circ \sigma_{w_k}$.
- Define $\Sigma_w = \{\sigma_w(\omega) \mid \omega \in \Sigma\}$.
- Define $f_w = f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_k}$.
- Define $K_w = f_w(K)$.

Shift Space Metric

We can construct a metric on Σ as follows.

Definition

Let $0 < r < 1$, and for $\omega, \nu \in \Sigma$:

$$R(\omega, \nu) = \min\{n \mid \sigma^n(\omega) \neq \sigma^n(\nu),\ n \geq 0\}$$

and define the *shift space metric* by:

$$d_\Sigma(\omega, \nu) = r^{R(\omega,\nu)}$$

Theorem

The shift space metric is a metric, and (Σ, d_Σ) is a compact metric space. Each σ_i is a similitude.

Theorem

Define: $\pi : (\Sigma, d_\Sigma) \mapsto (K, d)$ *by:*

$$\pi(\omega) = \cap_{m \geq 1} K_{\omega_1 \omega_2 \ldots \omega_m}$$

This is well defined ($\cap_{m \geq 1} K_{\omega_1 \omega_2 \ldots \omega_m}$ *is a single point) and* π *is a continuous surjection. In addition, we have an important property:*

$$\pi \circ \sigma_i = f_i \circ \pi$$

This creates a connection between infinite words and points in a self-similar set. However, for 'interesting' self-similar sets, π is not injective.

Fractals from n-ary Trees

One interpretation of an infinite word space with $\#S = n$ is as an n-ary infinite tree. So we can view the Sierpinski gasket as a limit of an operation on a compact set, as the quotient space of an infinite word space, or as a 'quotient' space on an infinite ternary tree.

Question: What kind of results from the theory of trees can we find (probability theory or computability theory)?

The Critical Set

A *self similar structure* is formed with the same connection between self similar sets and the shift space via π, except the set is merely a compact metrizable topological space (no metric is used).

Definition

Let $f_1, \ldots, f_n$ be the similitudes of a self similar structure $L = (K, S, f_{i \in S})$.

- Define:

$$\mathcal{C}_{L,K} = \bigcup_{i \neq j} (f_i(K) \cap f_j(K))$$

- Define:

$$\mathcal{C}_L = \pi^{-1}(\mathcal{C}_{L,K})$$

We call $\mathcal{C}_L$ the *critical set* of the self similar structure.

The Post Critical Set

Definition

- Define:

$$\mathcal{P}_L = \{\sigma^n(\omega) \mid \omega \in \mathcal{C}_L,\ n \geq 1\}$$

  we call $\mathcal{P}_L$ the *post-critical set* and say that a self similar structure is *post-critically finite* when $\mathcal{P}_L$ is finite. Often we simply refer to "p.c.f. fractals" and that's what this is.

- Define:

$$V_0 = \pi(P_L)$$

- Define:

$$V_{m+1} = F(V_m)$$

  where F is the associated union of similitudes.

Open Set Condition

Definition

Let $(K, \{f_i\}_{i\in S}, S)$ be a self-similar structure where $K \subset \mathbb{R}^n$ and f_i are all similitudes. We say the structure satisfies the *open set condition* if there is an open set U such that $F(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for all $i \neq j$.

Moran's Theorem

If $(K, S, \{f_i\}_{i\in S})$ is a self-similar structure with common contraction ratio r that satisfies the open set condition, then the Hausdorff dimension of K is:

$$\dim_H(K) = \frac{\log(1/\#S)}{\log r}$$

There is a more general case of Moran's Theorem for when the similitudes have different ratios.

Bernoulli Measure

Definition

Let (Σ, d_Σ) be the metric shift space with alphabet S. For each $i \in S$ select scalars p_i so that $0 < p_i < 1$ and $\sum_{i \in S} p_i = 1$ and set $p = (p_i)_{i \in S}$. Let $\tilde{\mu}^p$ be defined on Σ_w for any w by $\tilde{\mu}^p(\Sigma_w) = \prod_{i=1}^{k} p_{w_i}$.

Theorem

There is a unique regular Borel measure on (Σ, d_Σ) that is the same as $\tilde{\mu}^p$ on Σ_w. Denote it by μ^p, it is called the Bernoulli measure with weight p.

Self Similar Measure

Definition

The *self similar measure* with weight p on (K, S, f_i) with associated π is defined on sets A for which $\pi^{-1}(A)$ is Bernoulli measurable. We define:

$$\nu^p(A) = \mu^p\left(\pi^{-1}(A)\right)$$

It is Borel regular and satisfies: $\nu^p(A) = \sum_{i \in S} \nu\left(f_i^{-1}(A)\right)$

Theorem

For any $w \in W_k$,

$$\nu^p(K_w) = p_{w_1} p_{w_2} \cdots p_{w_k}$$

if and only if the set of points in K with infinite preimages in Σ has Bernoulli measure 0. Otherwise $\nu^p(K_w) \geq p_{w_1} p_{w_2} \cdots p_{w_k}$.

‘Trinity’ and Motivation

What *is* analysis? What do people mean by ‘an’ analysis?

Dirichlet Forms

$\mathcal{E}$

Markov Semigroups

P_t

$\Delta = \lim_{t \downarrow 0} \frac{P_t - I}{t}$

$\langle -\Delta f, g \rangle = \mathcal{E}(f, g)$
connection

Laplacians

Δ

Hille-Yosida
Theorem

There are fundamental differential equations that use a Laplacian, which is sometimes indirectly constructed via the diagram above. Examples include Poisson's equation $\Delta u = f$ and the heat equation $\Delta u = u_t$. Brownian motion can be used to create Markov semigroups which is why probability theory is often used in analysis on fractals.

The Dirichlet Form on the Sierpinski Gasket

Dirichlet forms on post-critically finite fractals can be constructed as a limit on graph approximations. (Another approach uses probability theory.)

Definition

Consider the Sierpinski Gasket in $\mathbb{R}^2$. Define $V_m = F^m(V_0)$. Let $\sim$ denote an edge connection by letting $x \sim y$ if $d(x,y) = 2^{-m}$. Define the (graph) Dirichlet form on the Sierpinski Gasket on $(V_m, \sim)$ by:

$$E_m(f,g) = \sum_{x \sim y} (f(x) - f(y))(g(x) - g(y))$$

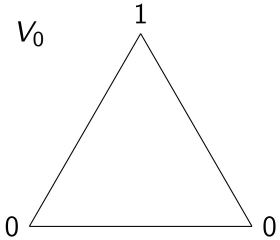
and define the normalized (graph) Dirichlet form by:

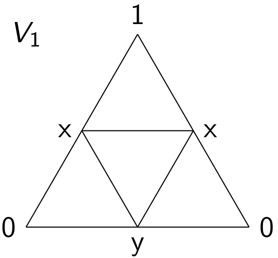
$$\mathcal{E}_m(f,g) = \left(\frac{5}{3}\right)^m E_m(f,g)$$

This begs the question: where does the (5/3) come from?

The Dirichlet Form on the Sierpinski Gasket

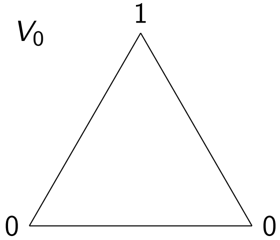
For a general Dirichlet form you can interpret $\mathcal{E}(f, f)$ to be the 'energy' of the function on the space. A function that minimizes energy is *harmonic*. If we define boundary conditions on V_0, we have an energy E_0 we ask what extension of those conditions to V_1 will keep the energy E_1 minimized.

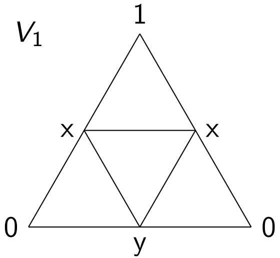




Let's set some simple 'one sided' boundary conditions and try to extend it to V_1 while minimizing E_1. Let one boundary point equal 1 and the rest 0.

The Dirichlet Form on the Sierpinski Gasket





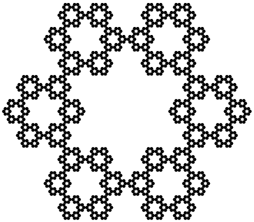
Let f be our initial function on the boundary and $\tilde{f}$ the extension. Note that $E_0(f) = 2$ and that:

$$E_1(\tilde{f}) = 2(x-1)^2 + 2x^2 + 2y^2 + 2(x-y)^2$$

Using standard calculus, the minimum is when $x = 2/5$ and $y = 1/5$, and $E_1(\tilde{f}) = 6/5$. We would like $E_1(\tilde{f}) = E_0(f) = 2$, so we multiply by $5/3$.

Dirichlet Form on Post-Critically Finite Sets

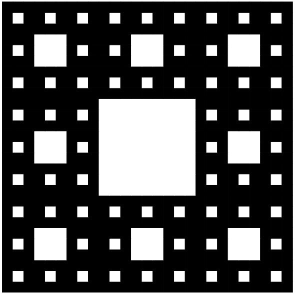
To construct a Dirichlet form on a general post-critically finite set, we use a similar approach to the Sierpinski Gasket by looking at graphs built with V_0 and V_1. However, there are more variables, and situations may get complicated, for example, look at the hexagasket:

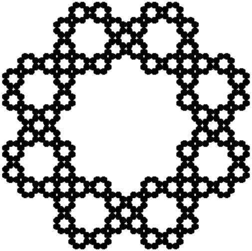


Notice that some cells have two adjacent cells, and other cells have three adjacent cells. This is an example of why it is harder to find the coefficients for the Dirichlet form on a general p.c.f. fractal.

Analysis on Non Post Critically Finite Sets

Currently an active field of research. Consider the Sierpinski Carpet and Octagasket (both non-p.c.f.):





There are intersections $f_i(K) \cap f_j(K)$ that are lines and/or Cantor sets, both of which create difficulties in analysis as they are not simple "junction points" like post-critically finite sets have.

Thank You!

References & Credits

- Content on self-similar sets was taken largely from *Analysis on Fractals* by Jun Kigami.
- Content on Dirichlet forms on self-similar sets was taken from *Differential Equations on Fractals: A Tutorial* by Robert S. Strichartz.
- The ‘Trinity’ is a simplified version of a diagram from *Introduction to the Theory of Non-Symmetric Dirichlet Forms* by Zhi-Ming Ma and Michael Röckner.
- All images were created in Mathematica or *tikz* with the exception of the Octagasket ones.

'Self similar structure' is a broader term than self similar set. We define a triple:

$$(K, S, f_{i\in S})$$

where $f_{i\in S}$ is shorthand for $\{f_i\}_{i\in S}$. K is a compact metrizable topological space, S is a finite set, and f_i are *continuous injections* from K to itself.

Definition

If $(K, S, f_{i\in S})$ has a $\pi : \Sigma \to K$ that is a continuous surjection and $f_i \circ \pi = \sigma_i \circ \pi$ for any i, then $(K, S, F_{i\in S})$ is a *self similar structure*.

Definition

Two self-similar structures with associated π_1, π_2 and a shared S are *isomorphic* if $\pi_2 \circ \pi_1^{-1}$ is a homeomorphism.

It may be possible to construct a self-similar structure with 'fewer' similitudes or with a subset of W^*.

Definition

Let $(K, S, f_{i\in S})$ be a self similar structure with post critical set V_0. If V_0 has empty interior, then the structure is *minimal*.

There is a long list of logically equivalent for minimality, see *Analysis on Fractals* by Kigami for details.

Connectedness

Theorem

Path (or arcwise) connectedness is equivalent to connectedness in self-similar structures.