STA9792 - Homework 2

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Question 2.1

Please see the code script named `STA9792 - Homework 2 code.R`, under section **Question 2.1** for references to answers

The program is written in R, and uses data from the Excel spreadsheet `Excel HW02.xlsm` from worksheet `Prob 2.1 CAPM Regression` and `Stock Data`

Using the data, we utilized R's builtin linear regression modeling capabilities, and iterated over all 500 stocks from the `Stock Data` worksheet

Each `lm()` call automatically performs the hypothesis t-test on the regressor (i.e. market index) but only on $\beta = 0$ null hypothesis. Therefore we performed our own. For each regression model we found the β t-value using this formulation:

$$t_{\hat{\beta}} = \frac{\hat{\beta} - 1}{se_{\hat{\beta}}}$$

With a t-critical value to be 1.645

Of the 500 stocks we performed the CAPM model on, 377 had significant β values

Question 2.3

Please see the code script named `STA9792 - Homework 2 code.R`, under section **Question 2.3** for references to answers

The program is written in R, and uses data from the Excel spreadsheet `Excel HW02.xlsm` from worksheet `Prob 2.3 Probability Model`

We first convert the provided probability into a linear regression form by doing the following:

- $new_prob = \frac{prob}{1-prob}$
- Then $new_prob = ln(new_prob)$

This linearizes the form into a standard linear regression

A linear regression on the regressors $X1$, $X2$, and $X3$ is performed

The resulting probabilities is reversed back:

- $\frac{ln(pred_prob)}{1+ln(pred_prob)}$

The predicted probabilities is then compared to the original probabilities. We got a MSE of 0.0304

Question 2.4

Let $\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$

Our error function (residual sum of squares) is:

$$L = \sum_{i=1}^{n} (y - \hat{y})^2$$

subject to the constraint of $\beta_1 + \beta_2 = 0.5$. Let:

$$g(\beta_j) = \beta_1 + \beta_2 - 0.5 = 0$$

Therefore our Lagrange objective function is:

$$E = \sum_{i=1}^{n} (y - \hat{y})^2 + \lambda \cdot g(\beta_j)$$

$$E = \sum_{i=1}^{n} (y - \hat{y})^2 + \lambda(\beta_1 + \beta_2 - 0.5)$$

Taking the partial derivative of E with respect to β and setting to zero, we get the following:

$$\frac{\partial}{\partial \beta_j} E = \frac{\partial}{\partial \beta_j} [\sum_{i=1}^{n} (y - \hat{y})^2 + \lambda(\beta_1 + \beta_2 - 0.5)] = 0$$

$$\frac{\partial}{\partial \beta_j} E = 2 \sum_{i=1}^{n} (y - \hat{y})(-\frac{\partial}{\partial \beta_j} \hat{y}) + \lambda \frac{\partial}{\partial \beta_j} (\beta_1 + \beta_2 - 0.5) = 0$$

Hence, the first order conditions will be the following:

$$\frac{\partial}{\partial \beta_0} E = 2 \sum_{i=1}^{n} (y - (\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3))(-1) = 0$$

$$\frac{\partial}{\partial \beta_1} E = 2 \sum_{i=1}^{n} (y - (\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3))(-x_1) + \lambda = 0$$

$$\frac{\partial}{\partial \beta_2} E = 2 \sum_{i=1}^{n} (y - (\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3))(-x_2) + \lambda = 0$$

$$\frac{\partial}{\partial \beta_3} E = 2 \sum_{i=1}^{n} (y - (\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3))(-x_3) = 0$$

Rewritting:

$$\sum \beta_0 + \sum \beta_1 x_1 + \sum \beta_2 x_2 + \sum \beta_3 x_3 = \sum y_i$$

$$\sum \beta_0 x_1 + \sum \beta_1 x_1 x_1 + \sum \beta_2 x_2 x_1 + \sum \beta_3 x_3 x_1 = -\frac{\lambda}{2} + \sum y_i x_{i,1}$$

$$\sum \beta_0 x_2 + \sum \beta_1 x_1 x_2 + \sum \beta_2 x_2 x_2 + \sum \beta_3 x_3 x_2 = -\frac{\lambda}{2} + \sum y_i x_{i,2}$$

$$\sum \beta_0 x_3 + \sum \beta_1 x_1 x_3 + \sum \beta_2 x_2 x_3 + \sum \beta_3 x_3 x_3 = -\frac{\lambda}{2} + \sum y_i x_{i,3}$$

$$
\begin{bmatrix}
n & \sum x_{i,1} & \sum x_{i,2} & \sum x_{i,3} \\
\sum x_{i,1} & \sum x_{i,1}^2 & \sum x_{i,2}x_{i,1} & \sum x_{i,3}x_{i,1} \\
\sum x_{i,2} & \sum x_{i,1}x_{i,2} & \sum x_{i,2}^2 & \sum x_{i,3}x_{i,2} \\
\sum x_{i,3} & \sum x_{i,1}x_{i,3} & \sum x_{i,3}x_{i,2} & \sum x_{i,3}^2
\end{bmatrix}
\cdot
\begin{bmatrix}
\beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3
\end{bmatrix}
=
\begin{bmatrix}
\sum y_i \\ \sum y_i x_{i,1} \\ \sum y_i x_{i,2} \\ \sum y_i x_{i,3}
\end{bmatrix}
-
\begin{bmatrix}
\frac{\lambda}{2} \\ \frac{\lambda}{2} \\ \frac{\lambda}{2} \\ \frac{\lambda}{2}
\end{bmatrix}
$$