Model description for "Origin of spatial variation in United States East Coast sea level trends during 1900–2017"

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1 Model description

Piecuch et al. (2018) describe a Bayesian hierarchical algorithm for analyzing tide gauge records, GPS data, sea level index points, and GIA model predictions to infer large-scale centennial trends in relative sea level, vertical land motion, and sea surface height along the United States East Coast during 1900–2017. See their online-only Methods for the basic algorithm description and general notation. Here we provide more details regarding the priors and full conditions used in that algorithm.

Prior distributions on scalar parameters and initial conditions Priors are given to scalar parameters and initial values to close the Bayesian framework. We use the convention that hyperparameters are given tildes to distinguish them from other parameters. Note that, for economy, we drop subscripts indicating vector or matrix dimension when it is clear from the context.

- $\boldsymbol{y}_0 \sim \mathcal{N}\left(\tilde{\eta}_{\boldsymbol{y}_0}\mathbf{1}, \tilde{\zeta}^2_{\boldsymbol{y}_0}\mathsf{I}\right)$. Initial conditions $\boldsymbol{y}_0$ have multivariate normal prior distribution with mean vector $\tilde{\eta}_{\boldsymbol{y}_0}\mathbf{1}$ and covariance matrix $\tilde{\zeta}^2_{\boldsymbol{y}_0}\mathsf{I}$. Values for $\tilde{\eta}_{\boldsymbol{y}_0}$ and $\tilde{\zeta}^2_{\boldsymbol{y}_0}$ are chosen as follows First, we fit a first-order, least-squares linear trend line to the available data at each tide gauge location. Second, we evaluate these fitted trend lines for all centered time steps $t_{k=1,\dots,K}$, storing the slopes (trends), intercepts, and initial conditions. Third, we define $\tilde{\eta}_{\boldsymbol{y}_0}$ as the average of the difference between pairs of stored initial conditions and intercepts, and we define $\tilde{\zeta}^2_{\boldsymbol{y}_0}$ as the variance across all initial condition-intercept pairs.

- $\alpha \sim \mathcal{N}\left(\tilde{\eta}_\alpha, \tilde{\zeta}^2_\alpha\right)$. Mean regional vertical land motion process trend unrelated to GIA α has normal prior distribution with mean $\tilde{\eta}_\alpha$ and variance $\tilde{\zeta}^2_\alpha$. We define $\tilde{\eta}_\alpha$ as the average of the vertical land motion trend values from the raw GPS data, and we set $\tilde{\zeta}^2_\alpha$ to 25 times their variance.

- $\beta \sim \mathcal{N}\left(\tilde{\eta}_\beta, \tilde{\zeta}^2_\beta\right)$. Spatial mean of site-specific intercepts $\boldsymbol{\iota}$ on the pre-industrial relative sea level process $\boldsymbol{Y}$ has normal prior distribution with mean $\tilde{\eta}_\beta$ and variance $\tilde{\zeta}^2_\beta$. To define the hyperparameters, we fit an ordinary least squares line to the index points at each saltmarsh, storing the slopes and intercepts of the fits at each location. We define $\tilde{\eta}_\beta$ as the average over all saltmarsh locations of the stored intercepts, and we define $\tilde{\zeta}^2_\beta$ as the variance in the stored intercepts.

- $\mu \sim \mathcal{N}\left(\tilde{\eta}_\mu, \tilde{\zeta}_\mu^2\right)$. Mean regional absolute sea level process trend unrelated to GIA μ has normal prior distribution with mean $\tilde{\eta}_\mu$ and variance $\tilde{\zeta}_\mu^2$. We define $\tilde{\eta}_\mu$ as the sum of the average of the tide gauge least-squares trend values mentioned above (in the description of the $\boldsymbol{y}_0$ hyperparameters) plus the average vertical land motion trend value from the raw GPS data, and we define $\tilde{\zeta}_\mu^2$ to 100 times the sum of their respective variances.

- $\nu \sim \mathcal{N}\left(\tilde{\eta}_\nu, \tilde{\zeta}_\nu^2\right)$. Spatial mean of tide gauge data biases ν has normal prior distribution with mean $\tilde{\eta}_\nu$ and variance $\tilde{\zeta}_\nu^2$. We define $\tilde{\eta}_\nu$ as the average of the tide gauge least-squares intercepts mentioned above (in describing the $\boldsymbol{y}_0$ hyperparameters) and we set $\tilde{\zeta}_\nu^2$ equal to 25 times their variance.

- $r \sim \mathcal{U}\left(\tilde{u}_r, \tilde{v}_r\right)$. Unitless AR(1) coefficient of the innovations r has uniform prior distribution with lower bound $\tilde{u}_r$ and upper bound $\tilde{v}_r$. We use $\tilde{u}_r = 0.1$ and $\tilde{v}_r = 0.9$.

- $\phi \sim \mathcal{N}_{\log}\left(\tilde{\eta}_\phi, \tilde{\zeta}_\phi^2\right)$. Inverse range of process innovations ϕ has log-normal prior distribution with "mean" $\tilde{\eta}_\phi$ and "variance" $\tilde{\zeta}_\phi^2$, and we use values of $\tilde{\eta}_\phi = -6.9$ and $\tilde{\zeta}_\phi^2 = 0.1225$. (Units are $\ln \mathrm{km}^{-1}$. These values roughly correspond to a 95% prior credible interval on the range $1/\lambda$ of 500 to 2,000 km.)

- $\lambda \sim \mathcal{N}_{\log}\left(\tilde{\eta}_\lambda, \tilde{\zeta}_\lambda^2\right)$. Inverse range of regional absolute sea level trends λ has log-normal prior distribution with "mean" $\tilde{\eta}_\lambda$ and "variance" $\tilde{\zeta}_\lambda^2$, and we use values of $\tilde{\eta}_\lambda = -6.9$ and $\tilde{\zeta}_\lambda^2 = 0.1225$. (Units are $\ln \mathrm{km}^{-1}$. These values roughly correspond to a 95% prior credible interval on the range $1/\lambda$ of 500 to 2,000 km.)

- $\rho \sim \mathcal{N}_{\log}\left(\tilde{\eta}_\rho, \tilde{\zeta}_\rho^2\right)$. Inverse range of regional vertical land motion trends ρ has log-normal prior distribution with "mean" $\tilde{\eta}_\rho$ and "variance" $\tilde{\zeta}_\rho^2$, and we use values of $\tilde{\eta}_\rho = -6.9$ and $\tilde{\zeta}_\rho^2 = 0.1225$. (Units are $\ln \mathrm{km}^{-1}$. These values roughly correspond to a 95% prior credible interval on the range $1/\rho$ of 500 to 2,000 km.)

- $\pi^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\pi^2}, \tilde{\chi}_{\pi^2}\right)$. Partial sill of regional absolute sea level trend π^2 has inverse gamma prior with shape $\tilde{\xi}_{\pi^2}$ and inverse scale $\tilde{\chi}_{\pi^2}$. This distribution can be interpreted as $2 \times \tilde{\xi}_{\pi^2}$ prior observations with a mean squared deviation of $\tilde{\chi}_{\pi^2} \big/ \tilde{\xi}_{\pi^2}$. We use $\tilde{\xi}_{\pi^2} = 0.5$ and define $\tilde{\chi}_{\pi^2}$ equal to one half the variance of the least-squares tide-gauge slopes mentioned above.

- $\sigma^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\sigma^2}, \tilde{\chi}_{\sigma^2}\right)$. Partial sill of the relative sea level innovations σ^2 has inverse gamma prior with shape parameter $\tilde{\xi}_{\sigma^2}$ and inverse scale $\tilde{\chi}_{\sigma^2}$. We use $\tilde{\xi}_{\sigma^2} = 0.5$. $\tilde{\chi}_{\sigma^2}$ is selected as follows: first, we linearly de-trend each tide-gauge relative sea level record; second, we assume the residuals behave as AR(1) red noise, and estimate the white-noise variance parameter; we define $\tilde{\chi}_{\sigma^2}$ as the average white-noise variance of those estimated white-noise variances.

- $\delta^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\delta^2}, \tilde{\chi}_{\delta^2}\right)$. Instrumental tide gauge error variance δ^2 has inverse gamma prior with shape parameter $\tilde{\xi}_{\delta^2}$ and inverse scale $\tilde{\chi}_{\delta^2}$. We use $\tilde{\xi}_{\delta^2} = 0.5$ and $\tilde{\chi}_{\delta^2} = 0.5 \times (0.01)^2$ [one observation with a mean squared deviation of (1 cm yr^{-1})^2].

- $\kappa^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\kappa^2}, \tilde{\chi}_{\kappa^2}\right)$. Spatial variance of site-specific intercepts $\boldsymbol{\iota}$ on the pre-industrial relative sea level process $\boldsymbol{Y}$ has has inverse gamma prior with shape parameter $\tilde{\xi}_{\kappa^2}$ and inverse scale $\tilde{\chi}_{\kappa^2}$. To define the hyperparameters, we fit an ordinary least squares line to the index points at each saltmarsh, storing the slopes and intercepts of the fits at each location. We use $\tilde{\xi}_{\kappa^2} = 0.5$ and set $\tilde{\chi}_{\kappa^2}$ equal to one-twentieth of the variance across all saltmarshes in the stored intercepts.

- $\epsilon^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\epsilon^2}, \tilde{\chi}_{\epsilon^2}\right)$. Spatial variance of spatiotemporally random residuals $\boldsymbol{f}$ on the pre-industrial relative sea level process $\boldsymbol{Y}$ has has inverse gamma prior with shape parameter $\tilde{\xi}_{\epsilon^2}$ and inverse scale $\tilde{\chi}_{\epsilon^2}$. We use $\tilde{\xi}_{\epsilon^2} = 0.5$ and $\tilde{\chi}_{\epsilon^2} = 0.5 \times (1 \text{ m})^2$ [one observation with a mean squared deviation of (1 m)^2].

- $\tau^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\tau^2}, \tilde{\chi}_{\tau^2}\right)$. Spatial variance in tide gauge data biases τ^2 has inverse gamma distribution with shape parameter $\tilde{\xi}_{\tau^2}$ and inverse scale $\tilde{\chi}_{\tau^2}$. We use $\tilde{\xi}_{\tau^2} = 0.5$ and define $\tilde{\chi}_{\tau^2}$ as one half the variance of the least-squares tide-gauge-record intercepts mentioned above (in description of the $\boldsymbol{y}_0$ prior and hyperparameters).

- $\varepsilon^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\varepsilon^2}, \tilde{\chi}_{\varepsilon^2}\right)$. Nugget effect of vertical land motion trend ε^2 has inverse gamma prior with shape parameter $\tilde{\xi}_{\varepsilon^2}$ and inverse scale $\tilde{\chi}_{\varepsilon^2}$. We use $\tilde{\xi}_{\varepsilon^2} = 0.5$ and $\tilde{\chi}_{\varepsilon^2} = 0.5 \times (0.001)^2$ [one observation with a mean squared deviation of (1 mm yr^{-1})^2].

- $\omega^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\omega^2}, \tilde{\chi}_{\omega^2}\right)$. Partial sill of regional vertical land motion trend ω^2 has inverse gamma prior with shape parameter $\tilde{\xi}_{\omega^2}$ and inverse scale $\tilde{\chi}_{\omega^2}$. We use $\tilde{\xi}_{\omega^2} = 0.5$ and set $\tilde{\chi}_{\omega^2}$ equal to one half the variance of the GPS vertical land motion trends.

- $\gamma^2 \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\gamma^2}, \tilde{\chi}_{\gamma^2}\right)$. Variance of tide gauge error trends γ^2 has inverse gamma prior with shape parameter $\tilde{\xi}_{\gamma^2}$ and inverse scale $\tilde{\chi}_{\gamma^2}$. We use $\tilde{\xi}_{\gamma^2} = 0.5$ and $\tilde{\chi}_{\gamma^2} = 0.5 \times (0.001)^2$ [one observation with a mean squared deviation of (1 mm yr^{-1})^2].

Full conditional distributions Because we regard model parameters as uncertain, the normalizing constant in Bayes' rule is unknown. Hence, the full posterior distribution [equation (8) in Methods] cannot be evaluated analytically, and numerical methods must be employed. The MCMC algorithm outlined in the methods section uses full conditional distributions for all process and parameters values. For a given process or parameter X, the full conditional $p\left(X|\cdot\right)$ follows from dropping terms in the full posterior distribution that do not treat X as a variable. For economy, we omit subscripts relating to matrix or vector dimension.

The full conditional distribution of $\boldsymbol{y}_0$ is multivariate normal:

$$p\left(\boldsymbol{y}_0|\cdot\right) \propto p\left(\boldsymbol{y}_0\right) p\left(\boldsymbol{y}_1|\boldsymbol{y}_0, \boldsymbol{b}, r, \sigma^2, \phi\right), \text{ thus }, \tag{1}$$

$$\boldsymbol{y}_0|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{y}_0}\boldsymbol{V}_{\boldsymbol{y}_0}, \Psi_{\boldsymbol{y}_0}\right), \tag{2}$$

$$\boldsymbol{V}_{\boldsymbol{y}_0} \doteq \frac{\tilde{\eta}_{\boldsymbol{y}_0}}{\tilde{\zeta}^2_{\boldsymbol{y}_0}}\mathbf{1} + \Sigma^{-1}\left[r\boldsymbol{y}_1 - r\left(t_1 - rt_0\right)\boldsymbol{b}\right], \tag{3}$$

$$\Psi_{\boldsymbol{y}_0} \doteq \left[\frac{1}{\tilde{\zeta}^2_{\boldsymbol{y}_0}}\mathsf{I} + r^2\Sigma^{-1}\right]^{-1}. \tag{4}$$

The full conditional distribution of $\boldsymbol{y}_k$ $(0 < k < K)$ is multivariate normal:

$$p\left(\boldsymbol{y}_k|\cdot\right) \propto p\left(\boldsymbol{z}_k|\boldsymbol{y}_k, \delta^2, \boldsymbol{\ell}, \boldsymbol{a}\right) p\left(\boldsymbol{y}_k|\boldsymbol{y}_{k-1}, \boldsymbol{b}, r, \sigma^2, \phi\right) p\left(\boldsymbol{y}_{k+1}|\boldsymbol{y}_k, \boldsymbol{b}, r, \sigma^2, \phi\right), \text{ thus }, \tag{5}$$

$$\boldsymbol{y}_k|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{y}_k} \boldsymbol{V}_{\boldsymbol{y}_k}, \Psi_{\boldsymbol{y}_k}\right), \tag{6}$$

$$\boldsymbol{V}_{\boldsymbol{y}_k} \doteq \frac{\mathsf{H}_k^\mathsf{T}\left[\boldsymbol{z}_k - \mathsf{F}_k\left(\boldsymbol{\ell} + \boldsymbol{a}t_k\right)\right]}{\delta^2} + \Sigma^{-1}\Big[r\left(\boldsymbol{y}_{k-1} + \boldsymbol{y}_{k+1}\right) + \left(1 + r^2\right) t_k \boldsymbol{b} - r\left(t_{k-1} + t_{k+1}\right)\boldsymbol{b}\Big], \tag{7}$$

$$\Psi_{\boldsymbol{y}_k} \doteq \left[\frac{\mathsf{H}_k^\mathsf{T}\mathsf{H}_k}{\delta^2} + \left(1 + r^2\right)\Sigma^{-1}\right]^{-1}. \tag{8}$$

The full conditional distribution of $\boldsymbol{y}_K$ is multivariate normal:

$$p\left(\boldsymbol{y}_K|\cdot\right) \propto p\left(\boldsymbol{z}_K|\boldsymbol{y}_K, \delta^2, \boldsymbol{\ell}, \boldsymbol{a}\right) p\left(\boldsymbol{y}_K|\boldsymbol{y}_{K-1}, \boldsymbol{b}, r, \sigma^2, \phi\right), \text{ thus }, \tag{9}$$

$$\boldsymbol{y}_K|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{y}_K} \boldsymbol{V}_{\boldsymbol{y}_K}, \Psi_{\boldsymbol{y}_K}\right), \tag{10}$$

$$\boldsymbol{V}_{\boldsymbol{y}_K} \doteq \frac{\mathsf{H}_K^\mathsf{T}\left[\boldsymbol{z}_K - \mathsf{F}_K\left(\boldsymbol{\ell} + \boldsymbol{a}t_K\right)\right]}{\delta^2} + \Sigma^{-1}\Big[r\boldsymbol{y}_{K-1} + \left(t_K - rt_{K-1}\right)\boldsymbol{b}\Big], \tag{11}$$

$$\Psi_{\boldsymbol{y}_K} \doteq \left[\frac{\mathsf{H}_K^\mathsf{T}\mathsf{H}_K}{\delta^2} + \Sigma^{-1}\right]^{-1}. \tag{12}$$

The full conditional distribution of $\boldsymbol{Y}$ is multivariate normal:

$$p\left(\boldsymbol{Y}|\cdot\right) \propto p\left(\boldsymbol{Z}|\boldsymbol{Y}\right) p\left(\boldsymbol{Y}|\boldsymbol{w}_g, \boldsymbol{u}_g, \boldsymbol{T}, \boldsymbol{\iota}, \epsilon^2\right), \text{ thus }, \tag{13}$$

$$\boldsymbol{Y}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{Y}} \boldsymbol{V}_{\boldsymbol{Y}}, \Psi_{\boldsymbol{Y}}\right), \tag{14}$$

$$\boldsymbol{V}_{\boldsymbol{Y}} \doteq \mathsf{\Gamma}^{-1}\boldsymbol{Z} + \frac{1}{\epsilon^2}\left[\mathrm{diag}\left(\boldsymbol{T}\right)\mathsf{G}\left(\boldsymbol{w}_g - \boldsymbol{u}_g\right) + \mathsf{D}\boldsymbol{\iota}\right], \tag{15}$$

$$\Psi_{\boldsymbol{Y}} \doteq \left[\mathsf{\Gamma}^{-1} + \frac{1}{\epsilon^2}\mathsf{I}\right]^{-1}. \tag{16}$$

The full conditional distribution of $\boldsymbol{a}$ is multivariate normal:

$$p\left(\boldsymbol{a}|\cdot\right) \propto p\left(\boldsymbol{a}|\gamma^2\right) \prod_{k=1}^{K} p\left(\boldsymbol{z}_k|\boldsymbol{y}_k, \delta^2, \boldsymbol{\ell}, \boldsymbol{a}\right), \text{ thus }, \tag{17}$$

$$\boldsymbol{a}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{a}} \boldsymbol{V}_{\boldsymbol{a}}, \Psi_{\boldsymbol{a}}\right), \tag{18}$$

$$\boldsymbol{V}_{\boldsymbol{a}} \doteq \frac{1}{\delta^2}\sum_{k=1}^{K} t_k \mathsf{F}_k^{\mathsf{T}}\left(\boldsymbol{z}_k - \mathsf{H}_k\boldsymbol{y}_k - \mathsf{F}_k\boldsymbol{\ell}\right), \tag{19}$$

$$\Psi_{\boldsymbol{a}} \doteq \left[\frac{1}{\gamma^2}\mathsf{I} + \frac{1}{\delta^2}\sum_{k=1}^{K} t_k^2 \mathsf{F}_k^{\mathsf{T}}\mathsf{F}_k\right]^{-1}. \tag{20}$$

The full conditional distribution of $\boldsymbol{b}$ is multivariate normal:

$$p\left(\boldsymbol{b}|\cdot\right) \propto p\left(\boldsymbol{b}|\boldsymbol{u}, \boldsymbol{w}_g, \mu, \pi^2, \lambda\right) \prod_{k=1}^{K} p\left(\boldsymbol{y}_k|\boldsymbol{y}_{k-1}, \boldsymbol{b}, r, \sigma^2, \phi\right), \text{ thus }, \tag{21}$$

$$\boldsymbol{b}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{b}}\boldsymbol{V}_{\boldsymbol{b}}, \Psi_{\boldsymbol{b}}\right), \tag{22}$$

$$\boldsymbol{V}_{\boldsymbol{b}} \doteq \mathsf{\Pi}^{-1}\left(\mu\mathbf{1} + \boldsymbol{w}_g - \boldsymbol{u}\right) + \mathsf{\Sigma}^{-1}\sum_{k=1}^{K}\left(t_k - rt_{k-1}\right)\left(\boldsymbol{y}_k - r\boldsymbol{y}_{k-1}\right), \tag{23}$$

$$\Psi_{\boldsymbol{b}} \doteq \left[\mathsf{\Pi}^{-1} + \mathsf{\Sigma}^{-1}\sum_{k=1}^{K}\left(t_k - rt_{k-1}\right)^2\right]^{-1}. \tag{24}$$

The full conditional distribution of $\boldsymbol{\iota}$ is multivariate normal:

$$p\left(\boldsymbol{\iota}|\cdot\right) \propto p\left(\boldsymbol{\iota}|\beta, \kappa^2\right) p\left(\boldsymbol{Y}|\boldsymbol{w}_g, \boldsymbol{u}_g, \boldsymbol{T}, \boldsymbol{\iota}, \epsilon^2\right), \text{ thus }, \tag{25}$$

$$\boldsymbol{\iota}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{\iota}}\boldsymbol{V}_{\boldsymbol{\iota}}, \Psi_{\boldsymbol{\iota}}\right), \tag{26}$$

$$\boldsymbol{V}_{\boldsymbol{\iota}} \doteq \frac{\beta}{\kappa^2}\mathbf{1} + \frac{1}{\epsilon^2}\mathsf{D}^{\mathsf{T}}\left[\boldsymbol{Y} - \operatorname{diag}\left(\boldsymbol{T}\right)\mathsf{G}\left(\boldsymbol{w}_g - \boldsymbol{u}_g\right)\right], \tag{27}$$

$$\Psi_{\boldsymbol{\iota}} \doteq \left[\frac{1}{\kappa^2}\mathsf{I} + \frac{1}{\epsilon^2}\mathsf{D}^{\mathsf{T}}\mathsf{D}\right]^{-1}. \tag{28}$$

The full conditional distribution of $\boldsymbol{u}$ is multivariate normal:

$$p\left(\boldsymbol{u}|\cdot\right) \propto p\left(\boldsymbol{b}|\boldsymbol{u}, \boldsymbol{w}_g, \mu, \pi^2, \lambda\right) p\left(\boldsymbol{u}|\boldsymbol{u}_g, \alpha, \omega^2, \rho\right) p\left(\boldsymbol{v}|\boldsymbol{u}, \varepsilon^2\right), \text{ thus }, \tag{29}$$

$$\boldsymbol{u}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{u}}\boldsymbol{V}_{\boldsymbol{u}}, \Psi_{\boldsymbol{u}}\right), \tag{30}$$

$$\boldsymbol{V}_{\boldsymbol{u}} \doteq \mathsf{\Pi}^{-1}\left(\mu\mathbf{1} + \boldsymbol{w}_g - \boldsymbol{b}\right) + \mathsf{\Omega}^{-1}\left(\boldsymbol{u}_g + \alpha\mathbf{1}\right) + \frac{1}{\varepsilon^2}\boldsymbol{v}, \tag{31}$$

$$\Psi_{\boldsymbol{u}} \doteq \left[\mathsf{\Pi}^{-1} + \mathsf{\Omega}^{-1} + \frac{1}{\varepsilon^2}\mathsf{I}\right]^{-1}. \tag{32}$$

The full conditional distribution of $\boldsymbol{v}$ is multivariate normal:

$$p\left(\boldsymbol{v}|\cdot\right) \propto p\left(\boldsymbol{v}|\boldsymbol{u}, \varepsilon^2\right) p\left(\boldsymbol{x}|\boldsymbol{v}\right), \text{ thus }, \tag{33}$$

$$\boldsymbol{v}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{v}} \boldsymbol{V}_{\boldsymbol{v}}, \Psi_{\boldsymbol{v}}\right), \tag{34}$$

$$\boldsymbol{V}_{\boldsymbol{v}} \doteq \frac{1}{\varepsilon^2}\boldsymbol{u} + \mathsf{E}^{\mathsf{T}}\Delta^{-1}\boldsymbol{x}, \tag{35}$$

$$\Psi_{\boldsymbol{v}} \doteq \left[\frac{1}{\varepsilon^2}\mathsf{I} + \mathsf{E}^{\mathsf{T}}\Delta^{-1}\mathsf{E}\right]^{-1}. \tag{36}$$

The full conditional distribution of $\boldsymbol{\ell}$ is multivariate normal:

$$p\left(\boldsymbol{\ell}|\cdot\right) \propto p\left(\boldsymbol{\ell}|\nu, \tau^2\right) \prod_{k=1}^{K} p\left(\boldsymbol{z}_k|\boldsymbol{y}_k, \delta^2, \boldsymbol{\ell}, \boldsymbol{a}\right), \text{ thus }, \tag{37}$$

$$\boldsymbol{\ell}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{\ell}} \boldsymbol{V}_{\boldsymbol{\ell}}, \Psi_{\boldsymbol{\ell}}\right), \tag{38}$$

$$\boldsymbol{V}_{\boldsymbol{\ell}} \doteq \frac{\nu}{\tau^2}\mathbf{1} + \frac{1}{\delta^2}\sum_{k=1}^{K} \mathsf{F}_k^{\mathsf{T}}\left(\boldsymbol{z}_k - \mathsf{H}_k\boldsymbol{y}_k - \mathsf{F}_k\boldsymbol{a}t_k\right), \tag{39}$$

$$\Psi_{\boldsymbol{\ell}} \doteq \left[\frac{1}{\tau^2}\mathsf{I} + \frac{1}{\delta^2}\sum_{k=1}^{K}\mathsf{F}_k^{\mathsf{T}}\mathsf{F}_k\right]^{-1}. \tag{40}$$

The full conditional distribution of $\boldsymbol{w}_g$ is multivariate normal:

$$p\left(\boldsymbol{w}_g|\cdot\right) \propto p\left(\boldsymbol{w}_g\right) p\left(\boldsymbol{b}|\boldsymbol{u}, \boldsymbol{w}_g, \mu, \pi^2, \lambda\right) p\left(\boldsymbol{Y}|\boldsymbol{w}_g, \boldsymbol{u}_g, \boldsymbol{T}, \boldsymbol{\iota}, \epsilon^2\right), \text{ thus }, \tag{41}$$

$$\boldsymbol{w_g}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{w}_g} \boldsymbol{V}_{\boldsymbol{w}_g}, \Psi_{\boldsymbol{w}_g}\right), \tag{42}$$

$$\boldsymbol{V}_{\boldsymbol{w}_g} \doteq \tilde{\mathsf{Z}}_{\boldsymbol{w}_g}^{-1} \tilde{\boldsymbol{\eta}}_{\boldsymbol{w}_g} + \Pi^{-1}\left(\boldsymbol{b} + \boldsymbol{u} - \mu \mathbf{1}\right) + \frac{1}{\epsilon^2} \mathsf{G}^{\mathsf{T}} \mathrm{diag}\left(\boldsymbol{T}\right)^{\mathsf{T}} \left[\boldsymbol{Y} + \mathrm{diag}\left(\boldsymbol{T}\right) \mathsf{G} \boldsymbol{u}_g - \mathsf{D}\boldsymbol{\iota}\right], \tag{43}$$

$$\Psi_{\boldsymbol{w}_g} \doteq \left[\tilde{\mathsf{Z}}_{\boldsymbol{w}_g}^{-1} + \Pi^{-1} + \frac{1}{\epsilon^2} \mathsf{G}^{\mathsf{T}} \mathrm{diag}\left(\boldsymbol{T}\right)^{\mathsf{T}} \mathrm{diag}\left(\boldsymbol{T}\right) \mathsf{G}\right]^{-1}. \tag{44}$$

The full conditional distribution of $\boldsymbol{u}_g$ is multivariate normal:

$$p\left(\boldsymbol{u}_g|\cdot\right) \propto p\left(\boldsymbol{u}_g\right) p\left(\boldsymbol{u}|\boldsymbol{u}_g, \alpha, \omega^2, \rho\right) p\left(\boldsymbol{Y}|\boldsymbol{w}_g, \boldsymbol{u}_g, \boldsymbol{T}, \boldsymbol{\iota}, \epsilon^2\right), \text{ thus }, \tag{45}$$

$$\boldsymbol{u_g}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{u}_g} \boldsymbol{V}_{\boldsymbol{u}_g}, \Psi_{\boldsymbol{u}_g}\right), \tag{46}$$

$$\boldsymbol{V}_{\boldsymbol{u}_g} \doteq \tilde{\mathsf{Z}}_{\boldsymbol{u}_g}^{-1} \tilde{\boldsymbol{\eta}}_{\boldsymbol{u}_g} + \Omega^{-1}\left(\boldsymbol{u} - \alpha \mathbf{1}\right) + \frac{1}{\epsilon^2} \mathsf{G}^{\mathsf{T}} \mathrm{diag}\left(\boldsymbol{T}\right)^{\mathsf{T}} \left[\mathrm{diag}\left(\boldsymbol{T}\right) \mathsf{G} \boldsymbol{w}_g + \mathsf{D}\boldsymbol{\iota} - \boldsymbol{Y}\right], \tag{47}$$

$$\Psi_{\boldsymbol{u}_g} \doteq \left[\tilde{\mathsf{Z}}_{\boldsymbol{u}_g}^{-1} + \Omega^{-1} + \frac{1}{\epsilon^2} \mathsf{G}^{\mathsf{T}} \mathrm{diag}\left(\boldsymbol{T}\right)^{\mathsf{T}} \mathrm{diag}\left(\boldsymbol{T}\right) \mathsf{G}\right]^{-1}. \tag{48}$$

The full conditional distribution of $\boldsymbol{T}$ is multivariate normal:

$$p\left(\boldsymbol{T}|\cdot\right) \propto p\left(\boldsymbol{Y}|\boldsymbol{w}_g, \boldsymbol{u}_g, \boldsymbol{T}, \boldsymbol{\iota}, \epsilon^2\right) p\left(\boldsymbol{S}|\boldsymbol{T}\right), \text{ thus }, \tag{49}$$

$$\boldsymbol{T}|\cdot \sim \mathcal{N}\left(\Psi_{\boldsymbol{T}} \boldsymbol{V}_{\boldsymbol{T}}, \Psi_{\boldsymbol{T}}\right), \tag{50}$$

$$\boldsymbol{V}_{\boldsymbol{T}} \doteq \frac{1}{\epsilon^2} \mathrm{diag}\left[\mathsf{G}\left(\boldsymbol{w}_g - \boldsymbol{u}_g\right)\right]^{\mathsf{T}} \left(\boldsymbol{Y} - \mathsf{D}\boldsymbol{\iota}\right) + \Xi^{-1} \boldsymbol{S}, \tag{51}$$

$$\Psi_{\boldsymbol{T}} \doteq \left[\frac{1}{\epsilon^2} \mathrm{diag}\left\{\mathsf{G}\left(\boldsymbol{w}_g - \boldsymbol{u}_g\right)\right\}^{\mathsf{T}} \left\{\mathsf{G}\left(\boldsymbol{w}_g - \boldsymbol{u}_g\right)\right\} + \Xi^{-1}\right]^{-1}. \tag{52}$$

The full conditional distribution of α is normal:

$$p\left(\alpha|\cdot\right) \propto p\left(\alpha\right) p\left(\boldsymbol{u}|\boldsymbol{u}_g, \alpha, \omega^2, \rho\right), \text{ thus }, \tag{53}$$

$$\alpha|\cdot \sim \mathcal{N}\left(\psi_\alpha V_\alpha, \psi_\alpha\right), \tag{54}$$

$$V_\alpha \doteq \frac{\tilde{\eta}_\alpha}{\tilde{\zeta}_\alpha^2} + \mathbf{1}^\mathsf{T}\Omega^{-1}\left(\boldsymbol{u} - \boldsymbol{u}_g\right), \tag{55}$$

$$\psi_\alpha \doteq \left(\frac{1}{\tilde{\zeta}_\alpha^2} + \mathbf{1}^\mathsf{T}\Omega^{-1}\mathbf{1}\right)^{-1}. \tag{56}$$

The full conditional distribution of β is normal:

$$p\left(\beta|\cdot\right) \propto p\left(\beta\right) p\left(\boldsymbol{\iota}|\beta, \kappa^2\right), \text{ thus }, \tag{57}$$

$$\beta|\cdot \sim \mathcal{N}\left(\psi_\beta V_\beta, \psi_\beta\right), \tag{58}$$

$$V_\beta \doteq \frac{\tilde{\eta}_\beta}{\tilde{\zeta}_\beta^2} + \frac{1}{\kappa^2}\mathbf{1}^\mathsf{T}\boldsymbol{\iota}, \tag{59}$$

$$\psi_\beta \doteq \left(\frac{1}{\tilde{\zeta}_\beta^2} + \frac{N_s}{\kappa^2}\right)^{-1}. \tag{60}$$

The full conditional distribution of μ is normal:

$$p\left(\mu|\cdot\right) \propto p\left(\mu\right) p\left(\boldsymbol{b}|\boldsymbol{u}, \boldsymbol{w}_g, \mu, \pi^2, \lambda\right), \text{ thus }, \tag{61}$$

$$\mu|\cdot \sim \mathcal{N}\left(\psi_\mu V_\mu, \psi_\mu\right), \tag{62}$$

$$V_\mu \doteq \frac{\tilde{\eta}_\mu}{\tilde{\zeta}_\mu^2} + \mathbf{1}^\mathsf{T}\Pi^{-1}\left(\boldsymbol{b} + \boldsymbol{u} - \boldsymbol{w}_g\right), \tag{63}$$

$$\psi_\mu \doteq \left(\frac{1}{\tilde{\zeta}_\mu^2} + \mathbf{1}^\mathsf{T}\Pi^{-1}\mathbf{1}\right)^{-1}. \tag{64}$$

The full conditional distribution of ν is normal:

$$p\left(\nu|\cdot\right) \propto p\left(\nu\right) p\left(\boldsymbol{\ell}|\nu, \tau^2\right), \text{ thus }, \tag{65}$$

$$\nu|\cdot \sim \mathcal{N}\left(\psi_\nu V_\nu, \psi_\nu\right), \tag{66}$$

$$V_\nu \doteq \frac{\tilde{\eta}_\nu}{\tilde{\zeta}_\nu^2} + \frac{1}{\tau^2}\mathbf{1}^\mathsf{T}\boldsymbol{\ell}, \tag{67}$$

$$\psi_\nu \doteq \left(\frac{1}{\tilde{\zeta}_\nu^2} + \frac{M}{\tau^2}\right)^{-1}. \tag{68}$$

The full conditional distribution of r is truncated normal:

$$p\left(r|\cdot\right) \propto p\left(r\right) \prod_{k=1}^{K} p\left(\boldsymbol{y}_k|\boldsymbol{y}_{k-1}, \boldsymbol{b}, r, \sigma^2, \phi\right), \text{ thus }, \tag{69}$$

$$r|\cdot \sim \mathcal{N}_{[\tilde{u}_r, \tilde{v}_r]}\left(\psi_r V_r, \psi_r\right), \tag{70}$$

$$V_r \doteq \sum_{k=1}^{K} \left(\boldsymbol{y}_{k-1} - \boldsymbol{b}t_{k-1}\right)^\mathsf{T} \Sigma^{-1} \left(\boldsymbol{y}_k - \boldsymbol{b}t_k\right), \tag{71}$$

$$\psi_r \doteq \left[\sum_{k=1}^{K} \left(\boldsymbol{y}_{k-1} - \boldsymbol{b}t_{k-1}\right)^\mathsf{T} \Sigma^{-1} \left(\boldsymbol{y}_{k-1} - \boldsymbol{b}t_{k-1}\right)\right]^{-1}. \tag{72}$$

The full conditional distribution of ϕ is nonstandard. To allow a Metropolis step with a symmetric proposal distribution, we define the log-transformed variable $\Phi \doteq \ln \phi$, which has the following full conditional form:

$$p\left(\phi|\cdot\right) \propto p\left(\phi\right) \prod_{k=1}^{K} p\left(\boldsymbol{y}_k, \boldsymbol{y}_{k-1}, \boldsymbol{b}, r, \sigma^2, \phi\right) \text{, thus ,} \tag{73}$$

$$p\left(\Phi|\cdot\right) \propto \left|\mathsf{S}\right|^{-K/2} \exp\left\{ -\frac{1}{2\tilde{\zeta}_\phi^2}\left[\Phi - \tilde{\eta}_\phi\right]^2 - \frac{1}{2\sigma^2}\sum_{k=1}^{K} \Delta\boldsymbol{y}_{k,k-1}^{\mathsf{T}} \mathsf{S}^{-1} \Delta\boldsymbol{y}_{k,k-1} \right\}, \tag{74}$$

$$\Delta\boldsymbol{y}_{k,k-1} \doteq \boldsymbol{y}_k - r\boldsymbol{y}_{k-1} - \left(t_k - rt_{k-1}\right)\boldsymbol{b}, \tag{75}$$

$$\mathsf{S}_{ij} \doteq \left(\mathsf{c}_{ij}\right) \exp\left(-e^{\Phi}\left|\mathbf{s}_i - \mathbf{s}_j\right|\right). \tag{76}$$

The Metropolis algorithm uses a normal jumping distribution with a variance of $(0.05)^2$. Full conditional samples of ϕ are found by exponentiating draws of the Φ posterior distribution.

The full conditional distribution of λ is nonstandard. To allow a Metropolis step with a symmetric proposal distribution, we define the log-transformed variable $\Lambda \doteq \ln\lambda$, which has the following full conditional form:

$$p\left(\lambda|\cdot\right) \propto p\left(\lambda\right) p\left(\boldsymbol{b}|\boldsymbol{u}, \boldsymbol{w}_g, \mu, \pi^2, \lambda\right) \text{, thus ,} \tag{77}$$

$$p\left(\Lambda|\cdot\right) \propto \left|\mathsf{L}\right|^{-1/2} \exp\left\{ -\frac{1}{2\tilde{\zeta}_\lambda^2}\left[\Lambda - \tilde{\eta}_\lambda\right]^2 - \frac{1}{2\pi^2}\left[\boldsymbol{b} + \boldsymbol{u} - \mu\mathbf{1} - \boldsymbol{w}_g\right]^{\mathsf{T}} \mathsf{L}^{-1}\left[\boldsymbol{b} + \boldsymbol{u} - \mu\mathbf{1} - \boldsymbol{w}_g\right] \right\}, \tag{78}$$

$$\mathsf{L}_{ij} \doteq \exp\left(-e^{\Lambda}\left|\mathbf{s}_i - \mathbf{s}_j\right|\right). \tag{79}$$

The Metropolis algorithm uses a normal jumping distribution with a variance of $(0.05)^2$. Full conditional samples of λ are found by exponentiating draws of the Λ posterior distribution.

The full conditional distribution of ρ is nonstandard. To allow a Metropolis step with a symmetric proposal distribution, we define the log-transformed variable $\varrho \doteq \ln \rho$, which has the following full conditional form:

$$p\left(\rho|\cdot\right) \propto p\left(\rho\right) p\left(\boldsymbol{u}|\boldsymbol{u}_g, \alpha, \omega^2, \rho\right), \text{ thus }, \tag{80}$$

$$p\left(\varrho|\cdot\right) \propto \left|\mathsf{T}\right|^{-1/2} \exp\left\{-\frac{1}{2\tilde{\zeta}_\rho^2}\left[\varrho - \tilde{\eta}_\rho\right]^2 - \frac{1}{2\omega^2}\left[\boldsymbol{u} - \alpha\mathbf{1} - \boldsymbol{u}_g\right]^\mathsf{T} \mathsf{T}^{-1}\left[\boldsymbol{u} - \alpha\mathbf{1} - \boldsymbol{u}_g\right]\right\}, \tag{81}$$

$$\mathsf{T}_{ij} \doteq \exp\left(-e^{\varrho}\left|\mathbf{s}_i - \mathbf{s}_j\right|\right). \tag{82}$$

The Metropolis algorithm uses a normal jumping distribution with a variance of $(0.05)^2$. Full conditional samples of ρ are found by exponentiating draws of the ϱ posterior distribution.

The full conditional distribution of π^2 is inverse gamma:

$$p\left(\pi^2|\cdot\right) \propto p\left(\pi^2\right) p\left(\boldsymbol{b}|\boldsymbol{u}, \boldsymbol{w}_g, \mu, \pi^2, \lambda\right), \text{ thus }, \tag{83}$$

$$\pi^2|\cdot \sim \mathcal{G}^{-1}\left[\tilde{\xi}_{\pi^2} + \frac{N}{2}, \tilde{\chi}_{\pi^2} + \frac{1}{2}\left(\boldsymbol{b} + \boldsymbol{u} - \mu\mathbf{1} - \boldsymbol{w}_g\right)^\mathsf{T} \mathsf{L}^{-1}\left(\boldsymbol{b} + \boldsymbol{u} - \mu\mathbf{1} - \boldsymbol{w}_g\right)\right], \tag{84}$$

$$\mathsf{L}_{ij} \doteq \exp\left(-\lambda\left|\mathbf{s}_i - \mathbf{s}_j\right|\right). \tag{85}$$

The full conditional distribution of σ^2 is inverse gamma:

$$p\left(\sigma^2|\cdot\right) \propto p\left(\sigma^2\right) \prod_{k=1}^{K} p\left(\boldsymbol{y}_k|\boldsymbol{y}_{k-1}, \boldsymbol{b}, r, \sigma^2, \phi\right), \text{ thus }, \tag{86}$$

$$\sigma^2|\cdot \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\sigma^2} + \frac{NK}{2}, \tilde{\chi}_{\sigma^2} + \frac{1}{2}\sum_{k=1}^{K} \Delta\boldsymbol{y}_{k,k-1}^{\mathsf{T}} \mathsf{S}^{-1} \Delta\boldsymbol{y}_{k,k-1}\right), \tag{87}$$

$$\Delta\boldsymbol{y}_{k,k-1} \doteq \boldsymbol{y}_k - r\boldsymbol{y}_{k-1} - \left(t_k - rt_{k-1}\right)\boldsymbol{b}, \tag{88}$$

$$\mathsf{S}_{ij} \doteq \left(\mathsf{c}_{ij}\right)\exp\left(-\phi\left|\mathbf{s}_i - \mathbf{s}_j\right|\right). \tag{89}$$

The full conditional distribution of δ^2 is inverse gamma:

$$p\left(\delta^2|\cdot\right) \propto p\left(\delta^2\right) \prod_{k=1}^{K} p\left(\boldsymbol{z}_k|\boldsymbol{y}_k, \delta^2, \boldsymbol{\ell}, \boldsymbol{a}\right), \text{ thus }, \tag{90}$$

$$\delta^2|\cdot \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\delta^2} + \frac{1}{2}\sum_{k=1}^{K} M_k, \tilde{\chi}_{\delta^2} + \frac{1}{2}\sum_{k=1}^{K} \boldsymbol{W}_k^{\mathsf{T}}\boldsymbol{W}_k\right), \tag{91}$$

$$\boldsymbol{W}_k \doteq \boldsymbol{z}_k - \mathsf{H}_k\boldsymbol{y}_k - \mathsf{F}_k\left(\boldsymbol{\ell} + \boldsymbol{a}t_k\right). \tag{92}$$

The full conditional distribution of κ^2 is inverse gamma:

$$p\left(\kappa^2|\cdot\right) \propto p\left(\kappa^2\right) p\left(\boldsymbol{\iota}|\beta, \kappa^2\right), \text{ thus }, \tag{93}$$

$$\kappa^2|\cdot \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\kappa^2} + \frac{N_s}{2}, \tilde{\chi}_{\kappa^2} + \frac{1}{2}\left[\iota - \beta\mathbf{1}\right]^{\mathsf{T}}\left[\iota - \beta\mathbf{1}\right]\right). \tag{94}$$

The full conditional distribution of ϵ^2 is inverse gamma:

$$p\left(\epsilon^2|\cdot\right) \propto p\left(\epsilon^2\right) p\left(\boldsymbol{Y}|\boldsymbol{w}_g, \boldsymbol{u}_g, \boldsymbol{T}, \boldsymbol{\iota}, \epsilon^2\right), \text{ thus }, \tag{95}$$

$$\epsilon^2|\cdot \sim \mathcal{G}^{-1}\left(\tilde{\xi}_{\epsilon^2} + \frac{N_d}{2}, \tilde{\chi}_{\epsilon^2} + \frac{1}{2}\left[\boldsymbol{Y} - \text{diag}\left(\boldsymbol{T}\right)\mathsf{G}\left(\boldsymbol{w}_g - \boldsymbol{u}_g\right)\right]^{\mathsf{T}}\left[\boldsymbol{Y} - \text{diag}\left(\boldsymbol{T}\right)\mathsf{G}\left(\boldsymbol{w}_g - \boldsymbol{u}_g\right)\right]\right). \tag{96}$$

The full conditional distribution of τ^2 is inverse gamma:

$$p\left(\tau^2|\cdot\right) \propto p\left(\tau^2\right) p\left(\boldsymbol{\ell}|\nu, \tau^2\right), \text{ thus }, \tag{97}$$

$$\tau^2|\cdot \sim \mathcal{G}^{-1}\left[\tilde{\xi}_{\tau^2} + \frac{M}{2}, \tilde{\chi}_{\tau^2} + \frac{1}{2}\left(\boldsymbol{\ell} - \nu\mathbf{1}\right)^{\mathsf{T}}\left(\boldsymbol{\ell} - \nu\mathbf{1}\right)\right]. \tag{98}$$

The full conditional distribution of ε^2 is inverse gamma:

$$p\left(\varepsilon^2|\cdot\right) \propto p\left(\varepsilon^2\right) p\left(\boldsymbol{v}|\boldsymbol{u}, \varepsilon^2\right), \text{ thus }, \tag{99}$$

$$\varepsilon^2|\cdot \sim \mathcal{G}^{-1}\left[\tilde{\xi}_{\varepsilon^2} + \frac{N}{2}, \tilde{\chi}_{\varepsilon^2} + \frac{1}{2}\left(\boldsymbol{v} - \boldsymbol{u}\right)^{\mathsf{T}}\left(\boldsymbol{v} - \boldsymbol{u}\right)\right]. \tag{100}$$

The full conditional distribution of ω^2 is inverse gamma:

$$p\left(\omega^2|\cdot\right) \propto p\left(\omega^2\right) p\left(\boldsymbol{u}|\boldsymbol{u}_g, \alpha, \omega^2, \rho\right), \text{ thus }, \tag{101}$$

$$\omega^2|\cdot \sim \mathcal{G}^{-1}\left[\tilde{\xi}_{\omega^2} + \frac{N}{2}, \tilde{\chi}_{\omega^2} + \frac{1}{2}\left(\boldsymbol{u} - \boldsymbol{u}_g - \alpha\mathbf{1}\right)^{\mathsf{T}}\mathsf{T}^{-1}\left(\boldsymbol{u} - \boldsymbol{u}_g - \alpha\mathbf{1}\right)\right], \tag{102}$$

$$\mathsf{T}_{ij} \doteq \exp\left(-\rho\left|\mathbf{s}_i - \mathbf{s}_j\right|\right). \tag{103}$$

The full conditional distribution of γ^2 is inverse gamma:

$$p\left(\gamma^2|\cdot\right) \propto p\left(\gamma^2\right) p\left(\boldsymbol{a}|\gamma^2\right), \text{ thus }, \tag{104}$$

$$\gamma^2|\cdot \sim \mathcal{G}^{-1}\left[\tilde{\xi}_{\gamma^2} + \frac{M}{2}, \tilde{\chi}_{\gamma^2} + \frac{1}{2}\boldsymbol{a}^\mathsf{T}\boldsymbol{a}\right]. \tag{105}$$