Doing Bayesian analysis for glacial isostatic adjustment

Chris Piecuch, Woods Hole Oceanographic Institution

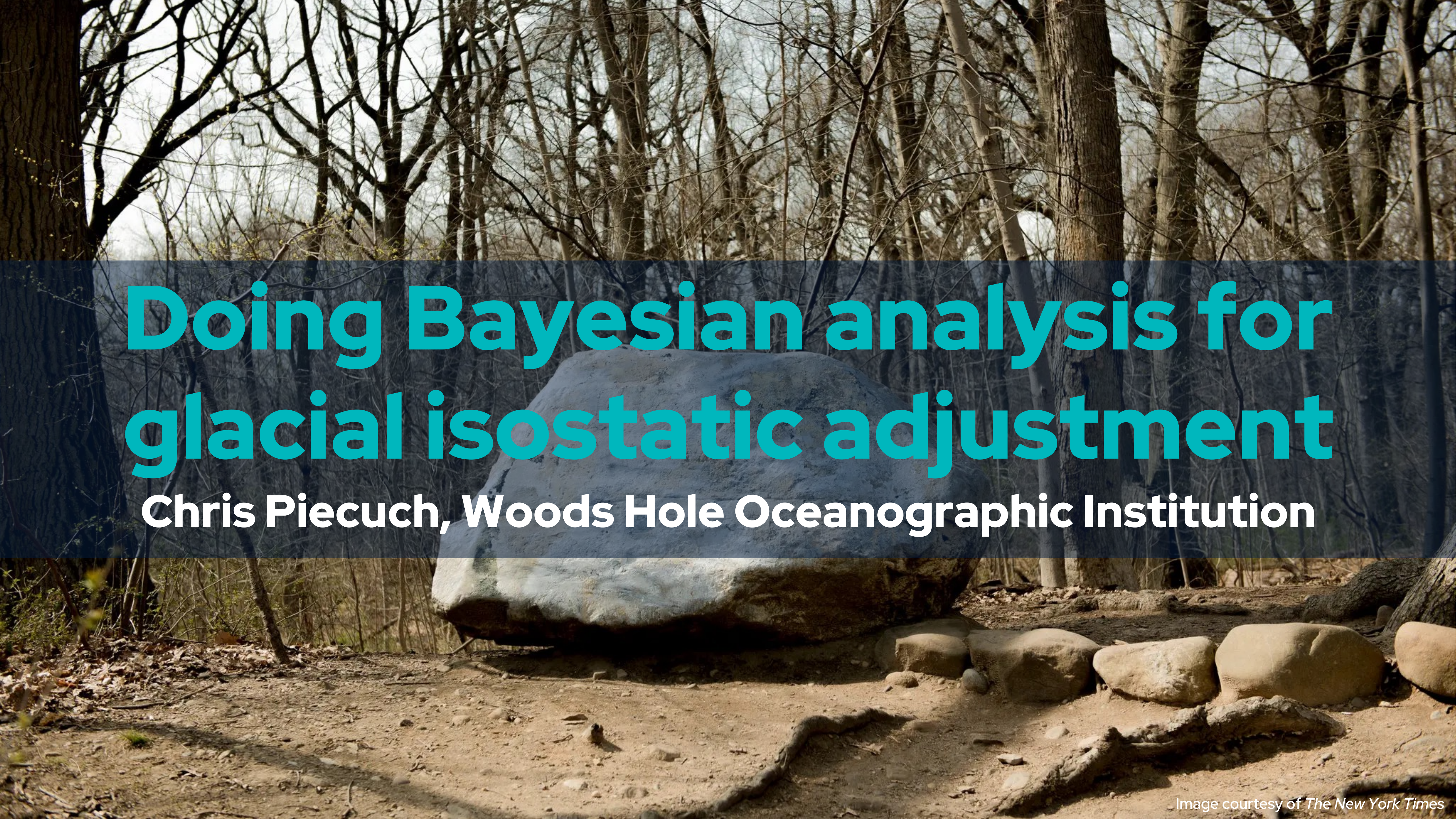
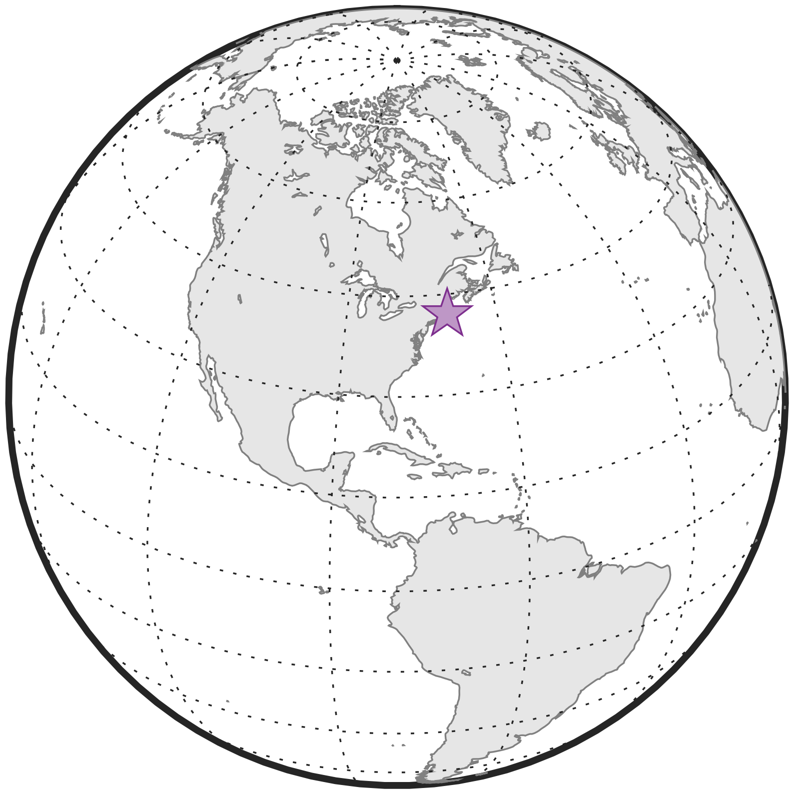


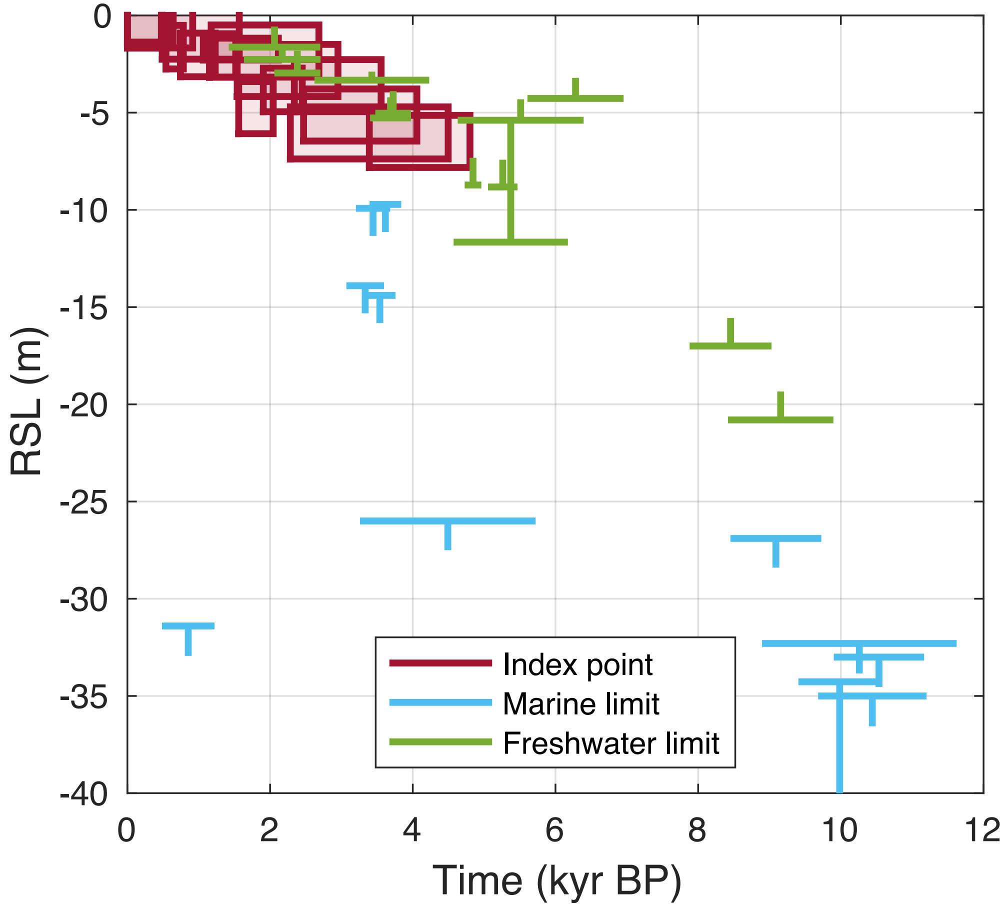
Image courtesy of *The New York Times*

What do we have?

What do we want?

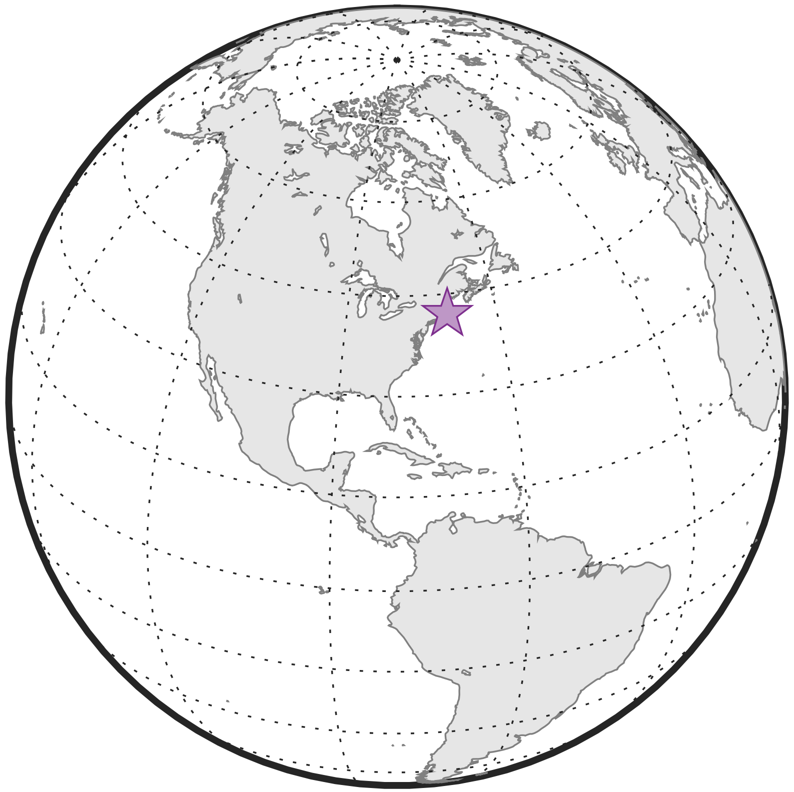
We have uncertain proxies; we want past rates of change

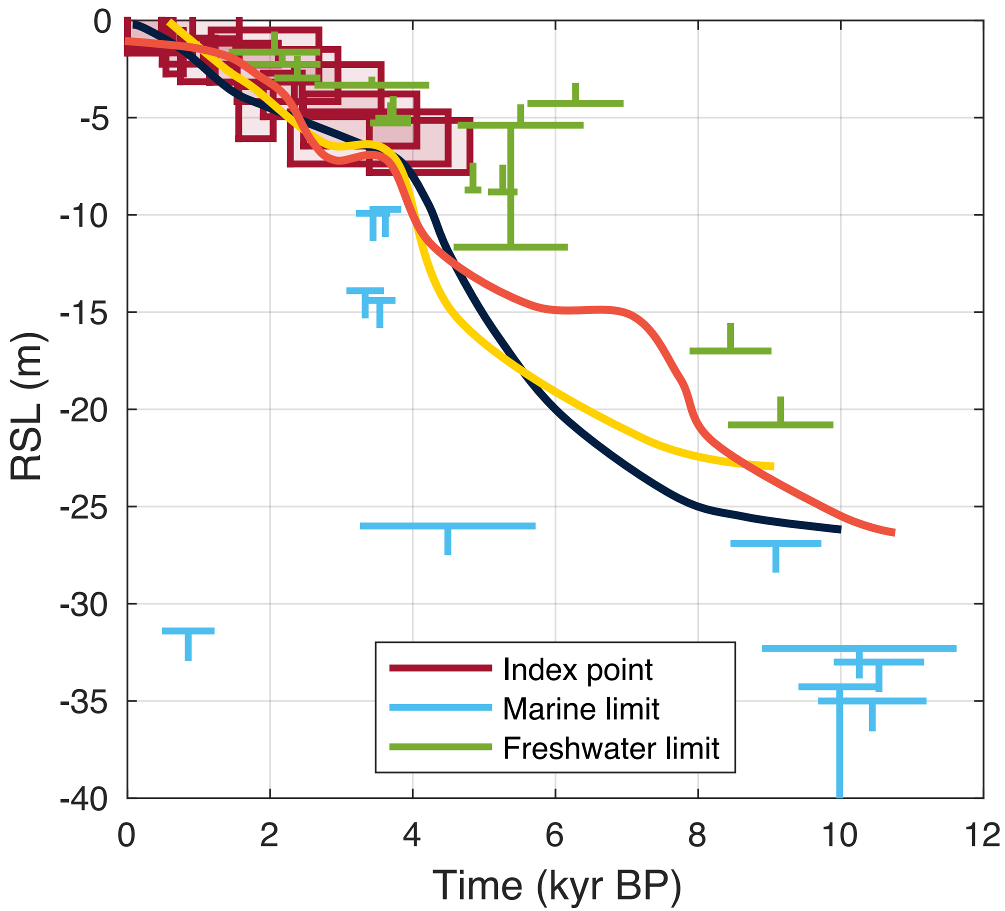




Data compiled by Engelhart and Horton (2012)

We have uncertain proxies; we want past rates of change





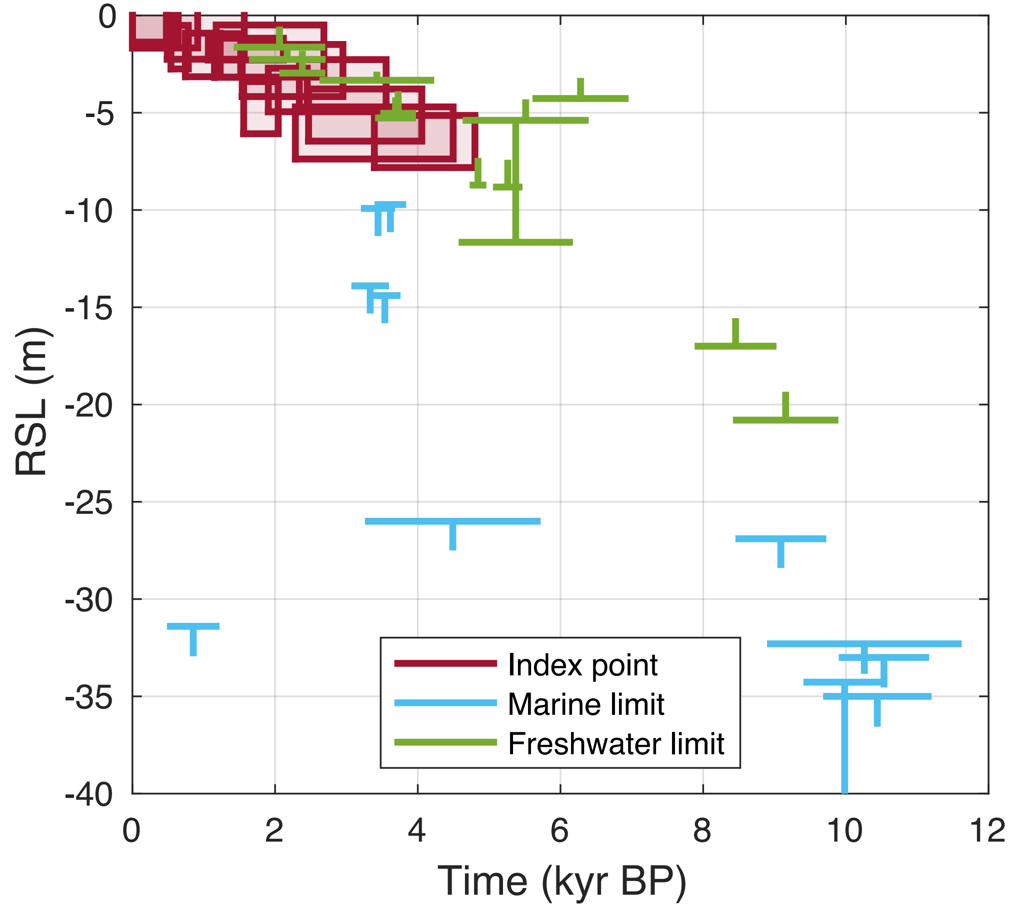
Data compiled by Engelhart and Horton (2012)

We have uncertain proxies; we want past rates of change

So how do we do this?

How do we get what we want given what we have?

Simplest thing is to fit a line through the best-estimate ages and relative sea levels from the index points using ordinary least squares (OLS)



Data compiled by Engelhart and Horton (2012)

We have uncertain proxies; we want past rates of change

We specify the model

$$y_n = \alpha x_n + \beta + \varepsilon_n$$

where

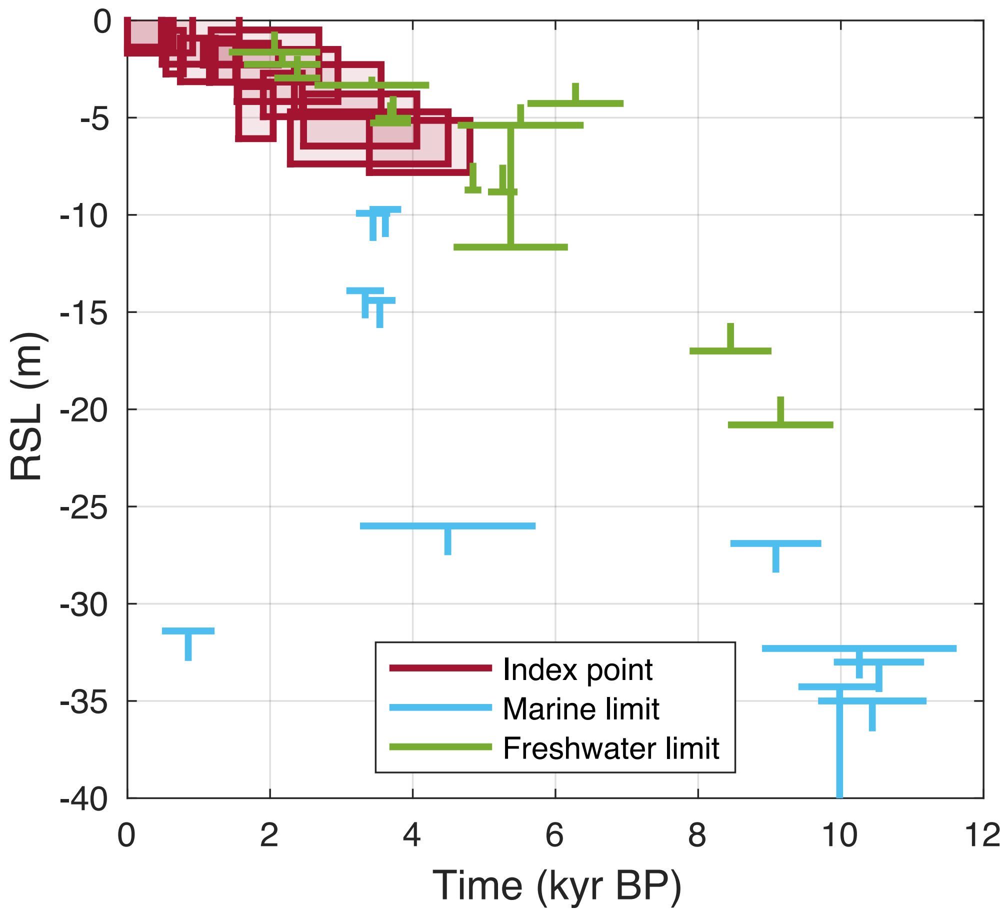
y_n is the nth value of sea level

x_n is the nth value of time

α is the slope

β is the intercept

ε_n is the nth residual (random unstructured noise)



Data compiled by Engelhart and Horton (2012)

We have uncertain proxies; we want past rates of change

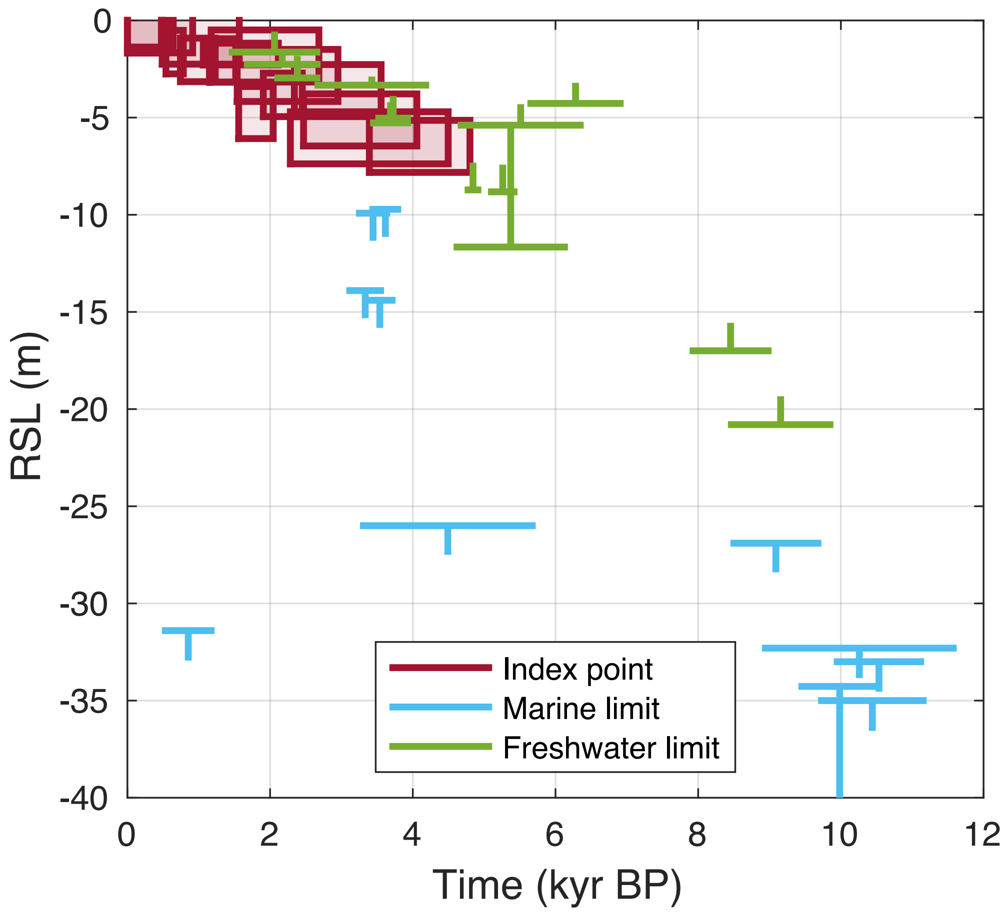
We solve it by minimizing the sum of the squared residuals

$$\sum_n (y_n - \alpha x_n - \beta)^2$$

OLS gives solutions $\hat{\alpha}$ for $\hat{\beta}$

$$\hat{\alpha} = \frac{\sum_n (x_n - \bar{x})(y_n - \bar{y})}{\sum_n (x_n - \bar{x})^2}$$

$$\hat{\beta} = \bar{y} - \hat{\alpha}\bar{x}$$



Data compiled by Engelhart and Horton (2012)

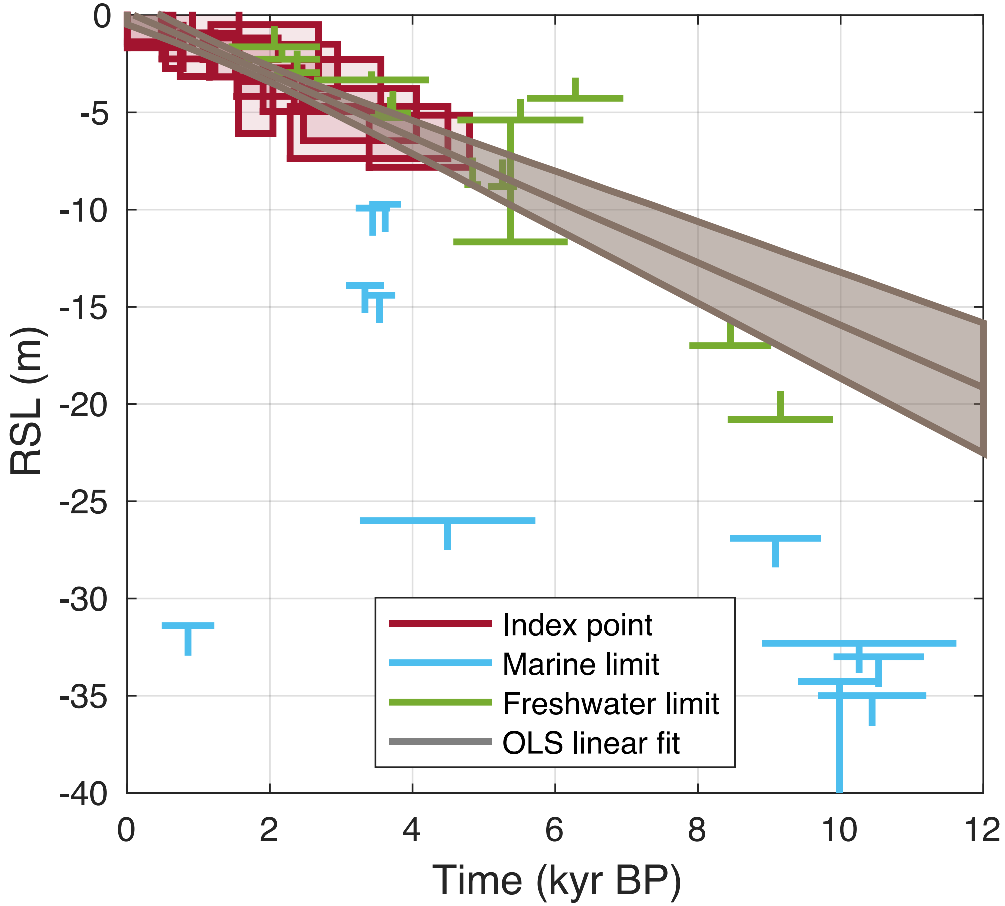
We have uncertain proxies; we want past rates of change

When we do this, we get the value $\hat{\alpha} = 1.6 \pm 0.3$ mm yr^{-1}

But what about the error bars on the observations?

Do we have to prescribe a particular functional form?

And how do we incorporate limiting or bounding data?



Data compiled by Engelhart and Horton (2012)

Think probabilistically

Think hierarchically

Think conditionally

Observations are conditioned on processes

Processes are conditioned on parameters

Everything is uncertain!

Probability postulates

Say that S is the set ("sample space") of all possible events and A is an event ("a subset of S")

Postulate #1

- The probability of A is a non-negative real number

$$P(A) \geq 0$$

Postulate #2

- Something must happen

$$P(S) = 1$$

Postulate #3

- If $A, B, C, \ldots$ are mutually exclusive events, then

$$P(A \cup B \cup C \cup \cdots) = P(A) + P(B) + P(C) + \ldots$$

Taken from Miller and Miller (2004)

Some standard distributions

Normal

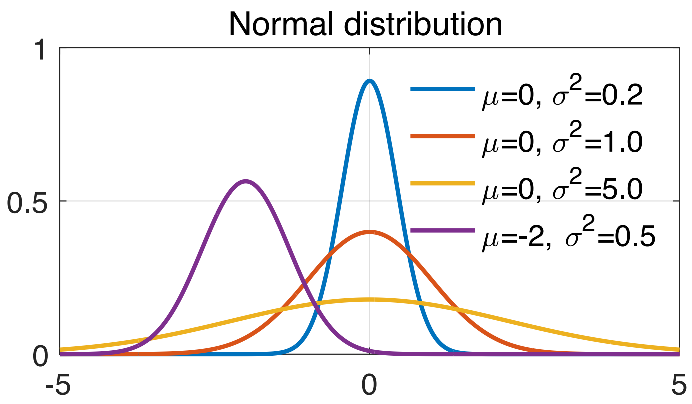
- $P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

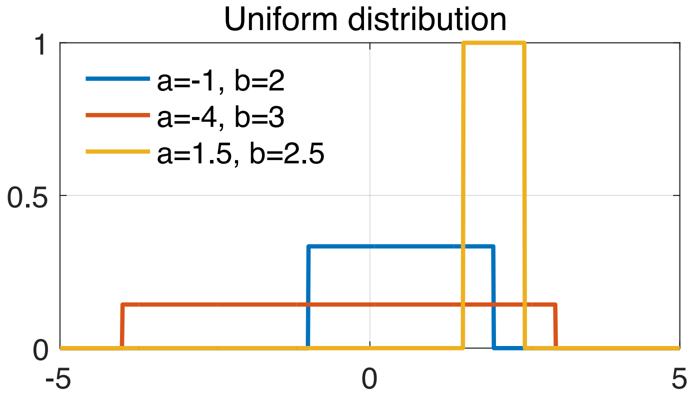
Uniform

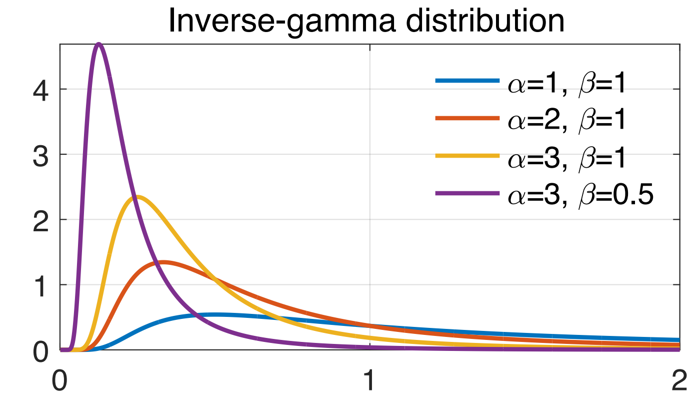
- $P(x) = \begin{cases} \frac{1}{b-a}, x \in [a, b] \\ 0, x \notin [a, b] \end{cases}$

Inverse-gamma

- $P(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}$







Probability distributions

Marginal distribution

- $\boldsymbol{P}(\boldsymbol{A})$ is the probability that A occurs

Joint distribution

- $\boldsymbol{P}(\boldsymbol{A}, \boldsymbol{B})$ is the probability that A and B both occur

Conditional distribution

- $\boldsymbol{P}(\boldsymbol{A}|\boldsymbol{B})$ is the probability that A occurs given that B is certain to occur (or already occurred)

These three distributions are related according to

$$\boldsymbol{P}(\boldsymbol{A}|\boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}, \boldsymbol{B}) / \boldsymbol{P}(\boldsymbol{B})$$

Bayes theorem

Bayes theorem

$$P(A|B) = P(B|A)P(A)/P(B) \propto P(B|A)P(A)$$

Posterior Likelihood Prior Normalizing constant

Example: what is the temperature θ outside?

- Prior (climatology) $p(\theta) = N(\mu, \sigma^2); \mu = 17℃, \sigma = 2℃$
- Likelihood (data) $p(T|\theta) = N(\theta, \delta^2); T = 20℃, \delta = 1℃$
- Posterior (temperature estimate θ given the data T)

$$p(\theta|T) = N\left(\frac{\frac{\mu}{\sigma^2} + \frac{T}{\delta^2}}{\frac{1}{\sigma^2} + \frac{1}{\delta^2}}, \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\delta^2}}\right) = N(19.4℃, 0.64℃^2)$$

Back to the data

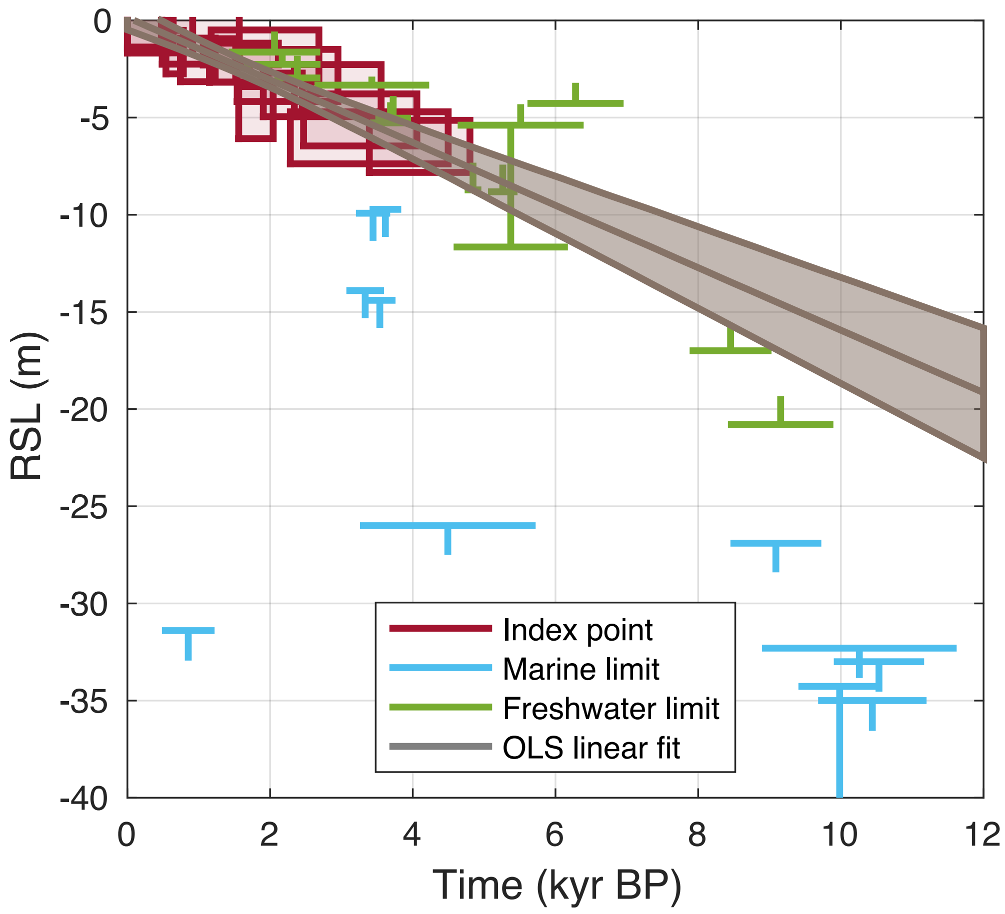
Let’s think about what we did earlier

When we got the OLS slope, we did two things without realizing

1. We made a modeling choice
2. We made an analysis choice

What if we make different modeling and analysis choices?

Does that allow us to address the questions we had before?



Data compiled by Engelhart and Horton (2012)

A series of four Bayesian models

Start—
Ordinary least squares;
parametric;
perfect data;
without limiting dates

End—
Bayesian;
nonparametric;
uncertain data;
with limiting dates

First model: linear/parametric process, perfect data, without limiting data

Model 1: linear/parametric process, perfect observations, without limiting data

Let's use the same model but change the analysis approach

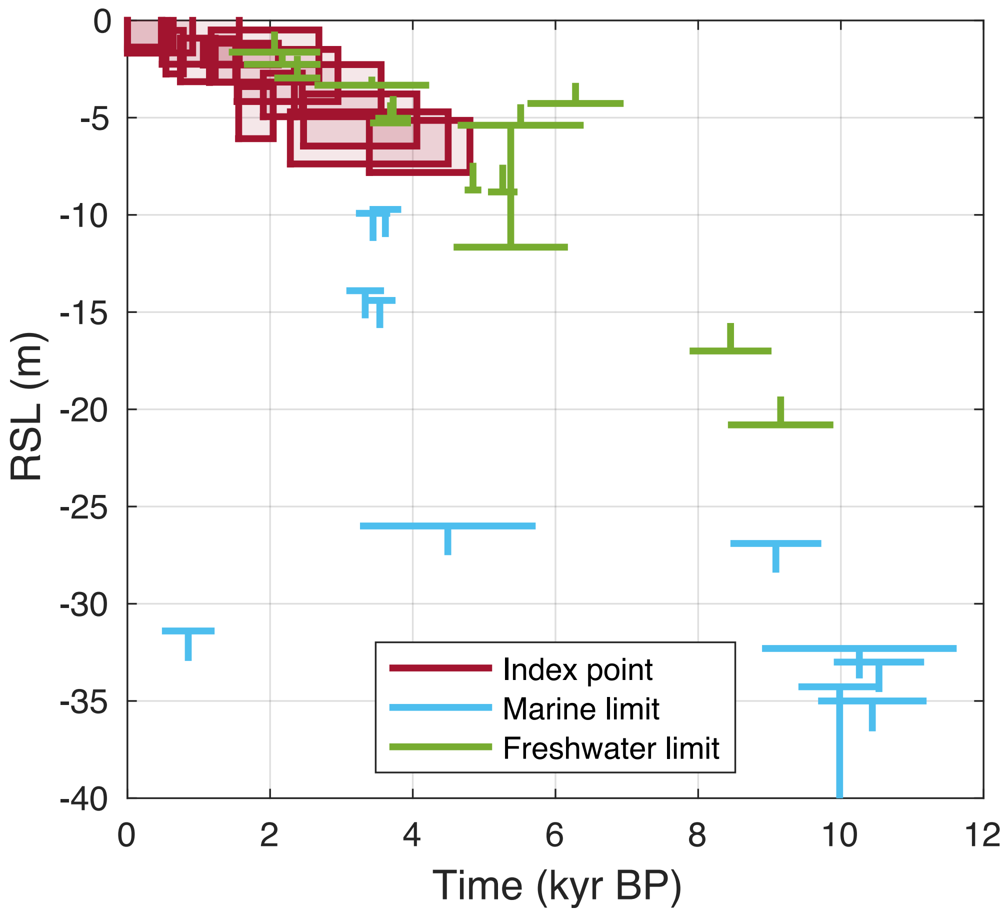
We want the posterior distribution

$$p(\alpha, \beta, \gamma^2 | x_1, \ldots, y_1, \ldots)$$

To get that, Bayes rule says that we need likelihood & prior distributions

The likelihood is our linear model, now written in a probabilistic form

$$p(y_n | x_n, \alpha, \beta, \gamma^2) = N(\alpha x_n + \beta, \gamma^2)$$



Data compiled by Engelhart and Horton (2012)

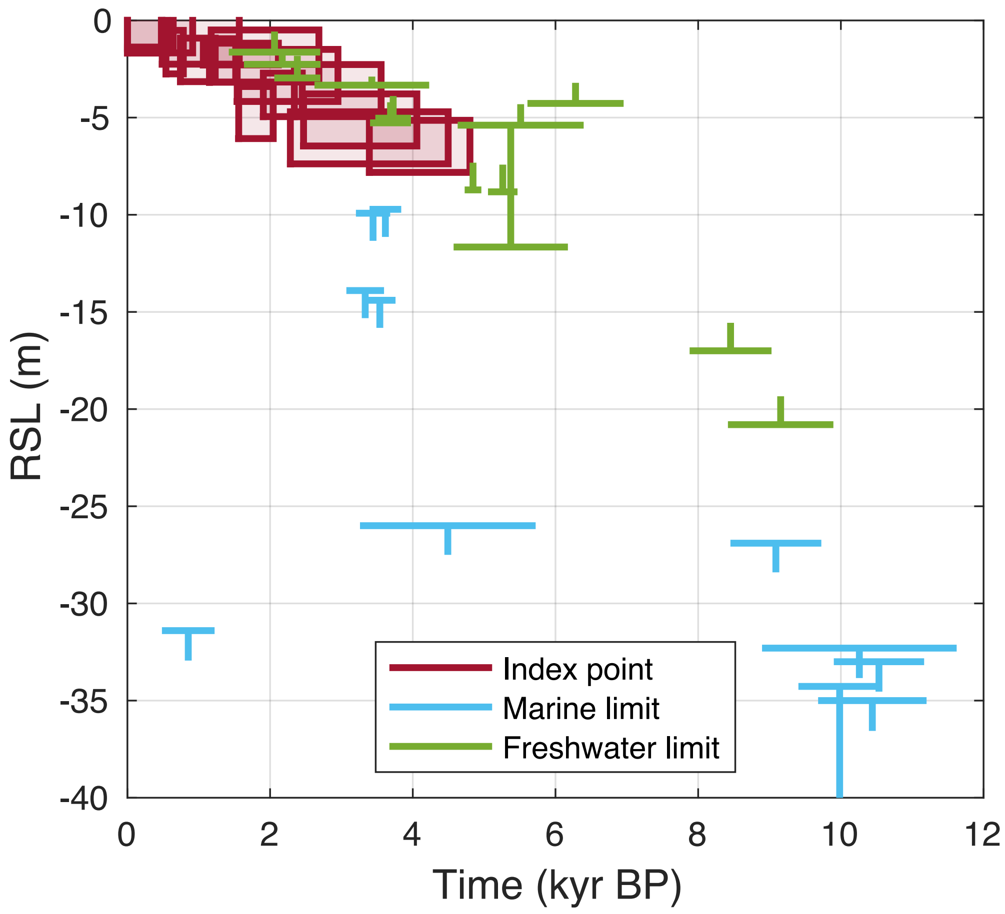
Model 1: linear/parametric process, perfect observations, without limiting data

The prior distributions express our belief about the parameters before we consider the data; let's choose

$$p(\alpha) = N(\tilde{\eta}_\alpha, \tilde{\zeta}_\alpha^2)$$
$$p(\beta) = N(\tilde{\eta}_\beta, \tilde{\zeta}_\beta^2)$$
$$p(\gamma^2) = G^{-1}(\tilde{\xi}_\gamma, \tilde{\chi}_\gamma)$$

Bringing the pieces together, we find

$$p(\alpha, \beta, \gamma^2 | x_1, \dots, y_1, \dots)$$
$$\propto p(\alpha) p(\beta) p(\gamma^2) \prod_n p(y_n | x_n, \alpha, \beta, \gamma^2)$$



Data compiled by Engelhart and Horton (2012)

Model 1: linear/parametric process, perfect observations, without limiting data

We cannot sample this joint posterior, but we can sample the full conditional posteriors of each of the parameters

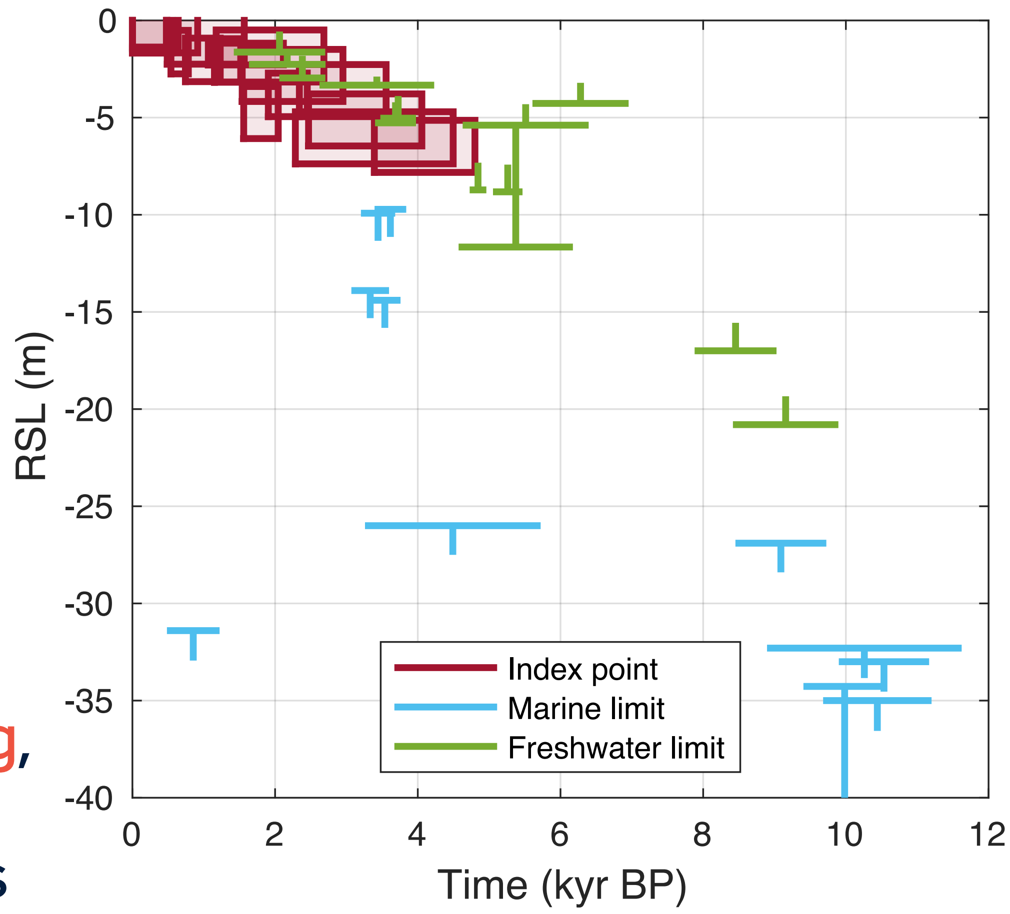
For example, for $\boldsymbol{\alpha}$, we have

$$p(\boldsymbol{\alpha}|\cdot) = N(\boldsymbol{\psi_\alpha V_\alpha}, \boldsymbol{\psi_\alpha})$$

$$\boldsymbol{V_\alpha} = \boldsymbol{\tilde{\zeta}_\alpha^{-2}\tilde{\eta}_\alpha + \gamma^{-2}\sum_{k=1}^{n} x_k(y_k - \beta)}$$

$$\boldsymbol{\psi_\alpha} = \boldsymbol{1/(\tilde{\zeta}_\alpha^{-2} + \gamma^{-2}\sum_{k=1}^{n} x_k^2)}$$

This forms the basis of Gibbs sampling, which is a powerful way to numerically sample from known distribution forms



Data compiled by Engelhart and Horton (2012)

Model 1: linear/parametric process, perfect observations, without limiting data

Gibbs sampler

First, make initial guesses for the parameters you seek $\alpha_{(0)}, \beta_{(0)}, \gamma^2_{(0)}$

Second, iteratively sample from the full conditional posterior distributions

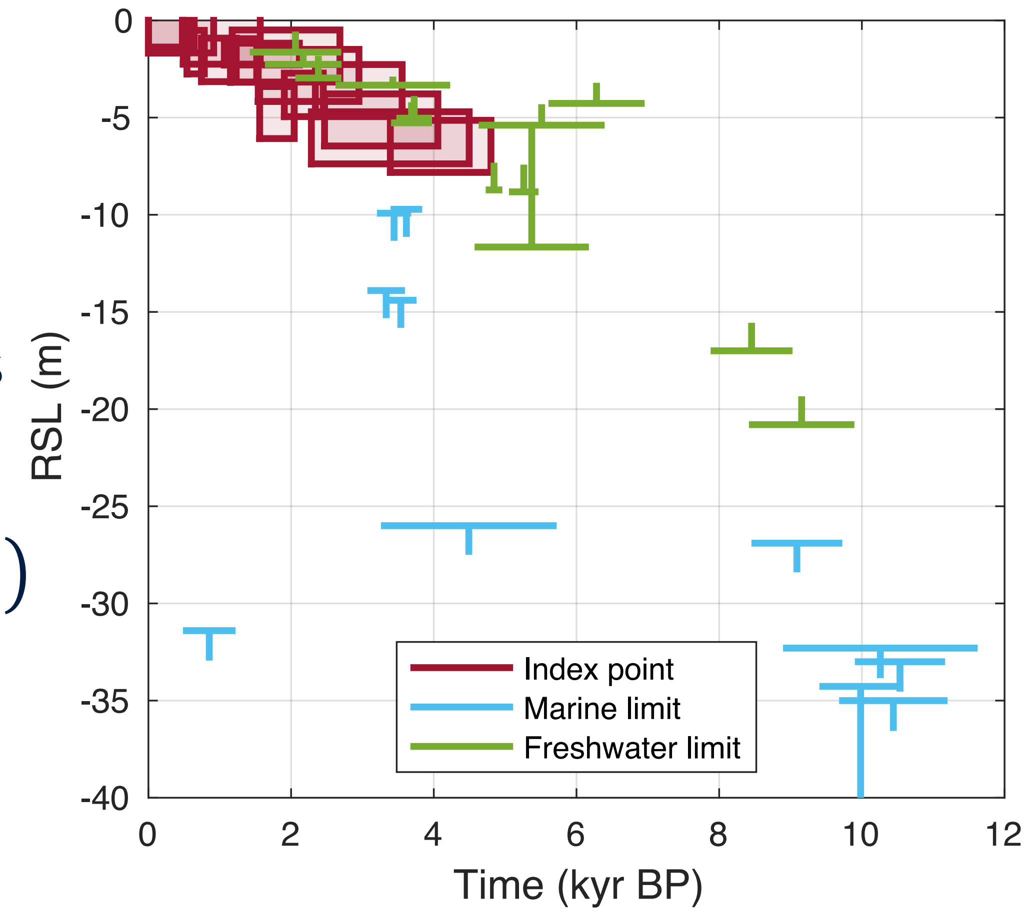
for $m = 1, \dots, M$

  draw $\alpha_{(m)}$ from $p\left(\alpha \middle| \beta_{(m-1)}, \gamma^2_{(m-1)}, \dots\right)$

  draw $\beta_{(m)}$ from $p(\beta | \alpha_{(m)}, \gamma^2_{(m-1)}, \dots)$

  draw $\gamma^2_{(m)}$ from $p(\gamma^2 | \alpha_{(m)}, \beta_{(m)}, \dots)$

end



Data compiled by Engelhart and Horton (2012)

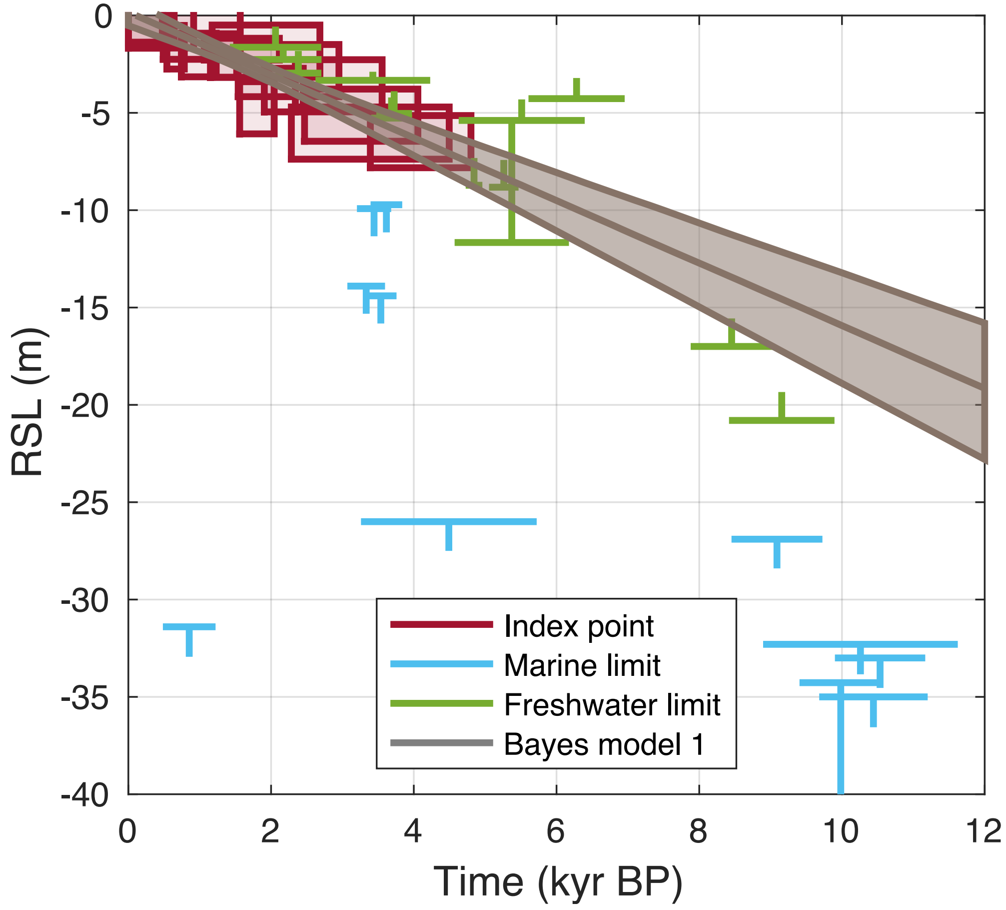
Model 1: linear/parametric process, perfect observations, without limiting data

We choose wide uninformative priors

When we run the Gibbs sampler, we get $\alpha = 1.6 \pm 0.3$ mm yr^{-1} (95% CI)

Did we just do a lot of work to get the same thing we had earlier with OLS?

Yes & no. You'll see the advantages of the Bayesian approach shortly.



Data compiled by Engelhart and Horton (2012)

Second model: linear/parametric process, imperfect data, without limiting data

Model 2: linear/parametric process, imperfect observations, without limiting data

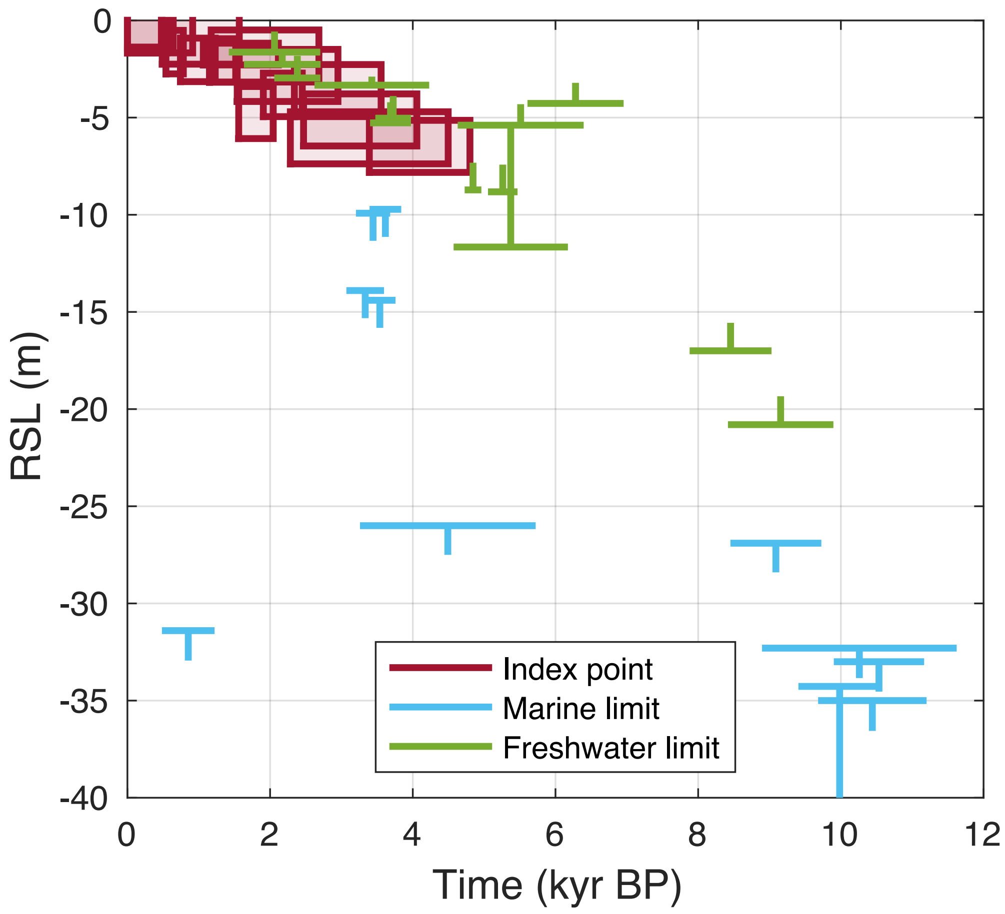
How do we factor in the errors on the proxy reconstructions?

Earlier, we had a process level and a prior level; now let's add more equations in a data level

Say that index points are noisy versions of the true processes

$$p(z_n) = N(y_n, \delta_n^2)$$
$$p(w_n) = N(x_n, \epsilon_n^2)$$

where z_n, w_n are sea level, age data, and δ_n^2, ϵ_n^2 are the errors



Data compiled by Engelhart and Horton (2012)

Model 2: linear/parametric process, imperfect observations, without limiting data

All other equations are the same

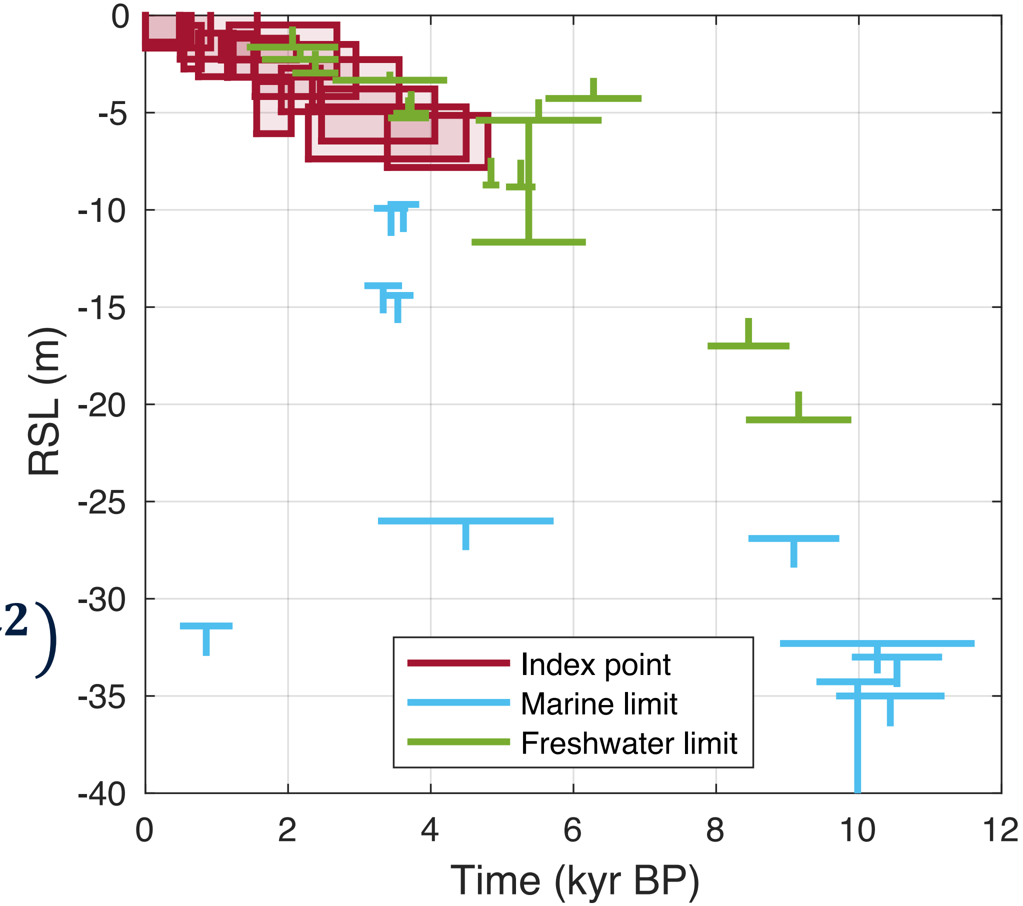
Now x_n and y_n are uncertain(!), so our posterior distribution is

$$p(\alpha, \beta, \gamma^2, x_1, \ldots, y_1, \ldots \mid w_1, \ldots, z_1, \ldots)$$

Using Bayes, & our prior, data, & process equations, this becomes

$$\propto p(\alpha)p(\beta)p(\gamma^2) \prod_n [p(y_n \mid x_n, \alpha, \beta, \gamma^2) \times p(z_n \mid y_n)\, p(w_n \mid x_n)]$$

Draw solutions for $\alpha, \beta, \gamma^2, x, y$ again based on Gibbs sampling



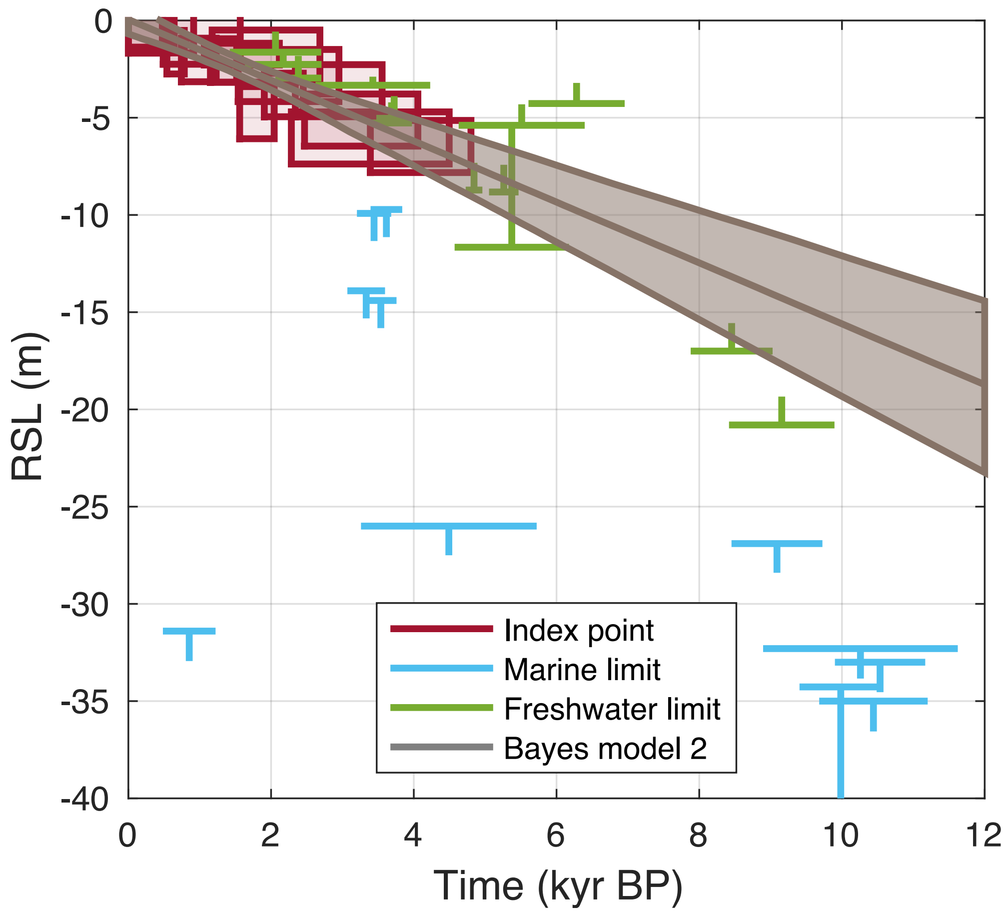
Data compiled by Engelhart and Horton (2012)

Model 2: linear/parametric process, imperfect observations, without limiting data

Now we determine a sea level rate of $\alpha = 1.6 \pm 0.4$ mm yr⁻¹ (95% CI)

The error bars are now wider because we include observational uncertainty in the design of our Bayesian model

We arguably obtain a more realistic linear trend in sea level, but is a linear trend the most appropriate model?



Data compiled by Engelhart and Horton (2012)

Third model: nonlinear/nonparametric process, imperfect data, without limiting data

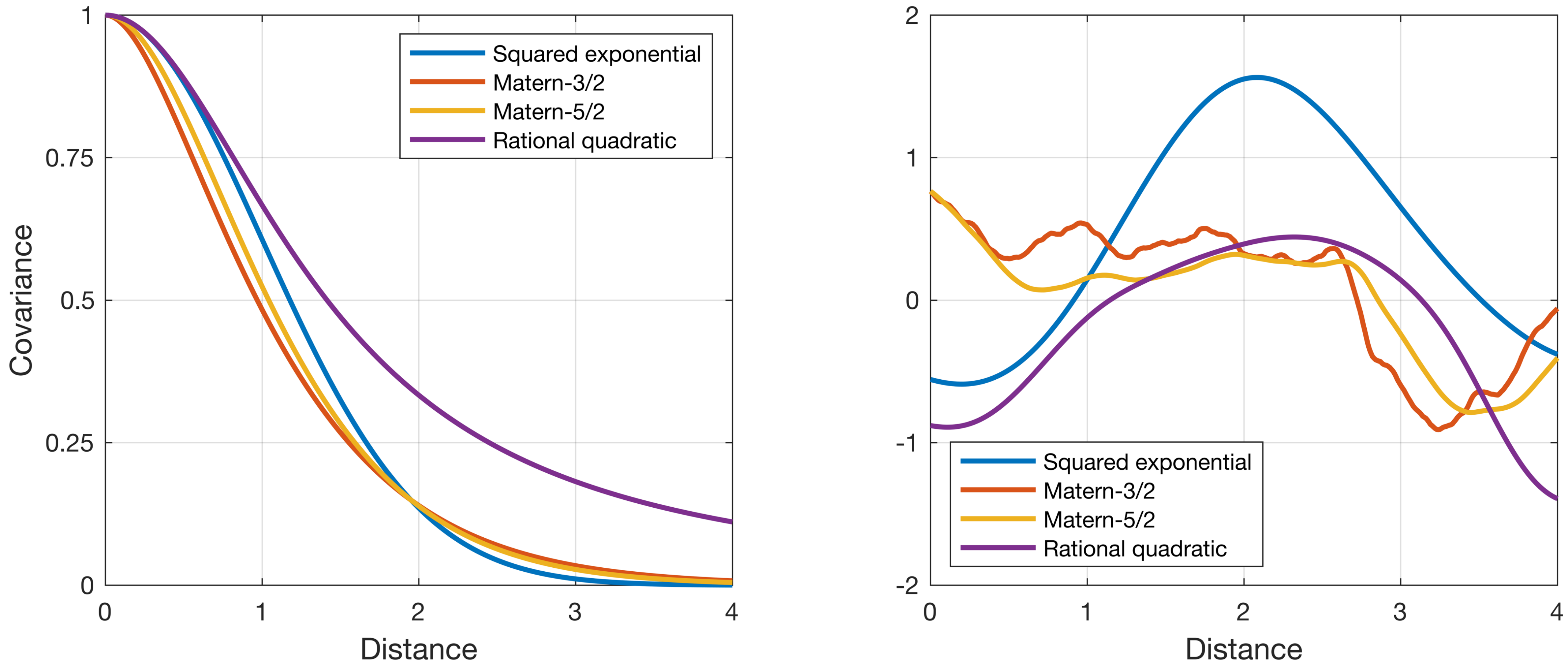
We must change how we view the process

We will still use a Gaussian distribution, but ...

We were putting all structure in the mean;
now we'll put the structure in the variance

We were prescribing a functional form;
now we'll specify a relation between points

Covariance functions (kernels)



Model 3: nonlinear/nonparametric process, imperfect observations, without limiting data

We'll use the same data model, but write a new process model

Let's represent sea level y as a random process with structure

$$y \sim N(0, L + G)$$

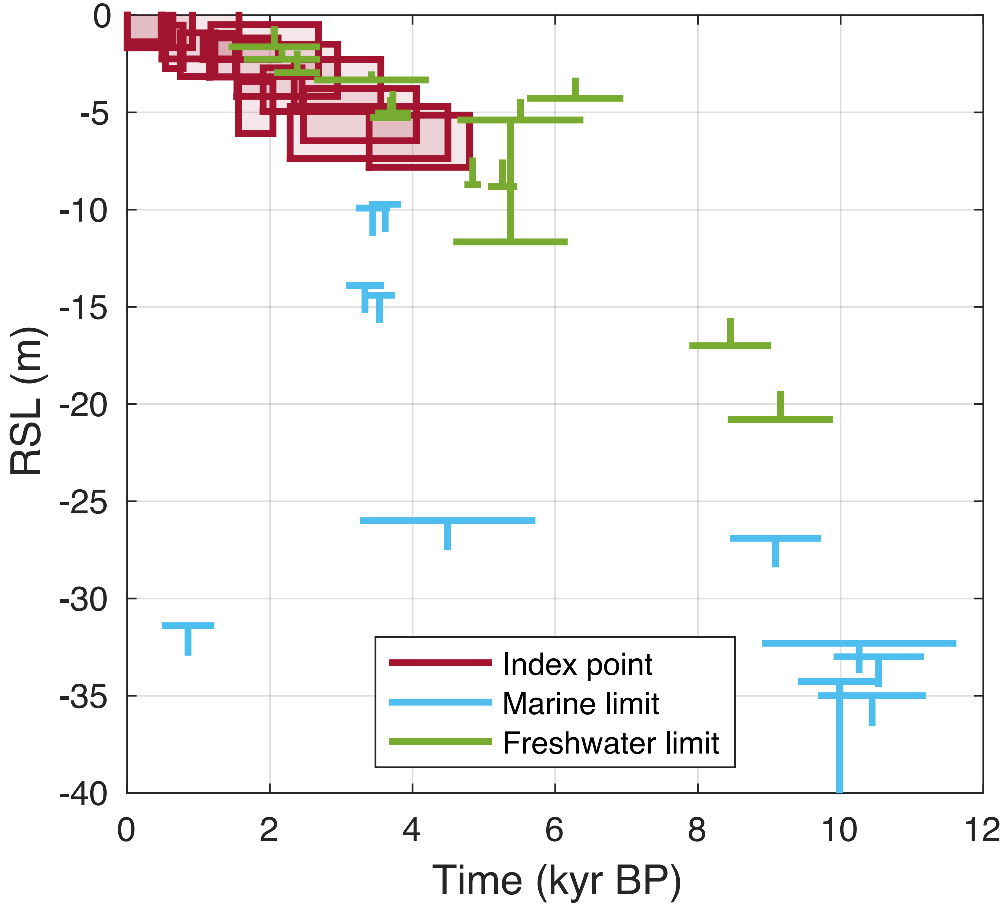
local (uncorrelated)

$$L_{jk} = \lambda^2 \delta(x_j, x_k)$$

regional (correlated)

$$G_{jk} = \rho^2 C(x_j, x_k, \varphi)$$

where δ is the Kronecker delta and C the covariance function (compare to Khan et al. 2015)



Data compiled by Engelhart and Horton (2012)

Model 3: nonlinear/nonparametric process, imperfect observations, without limiting data

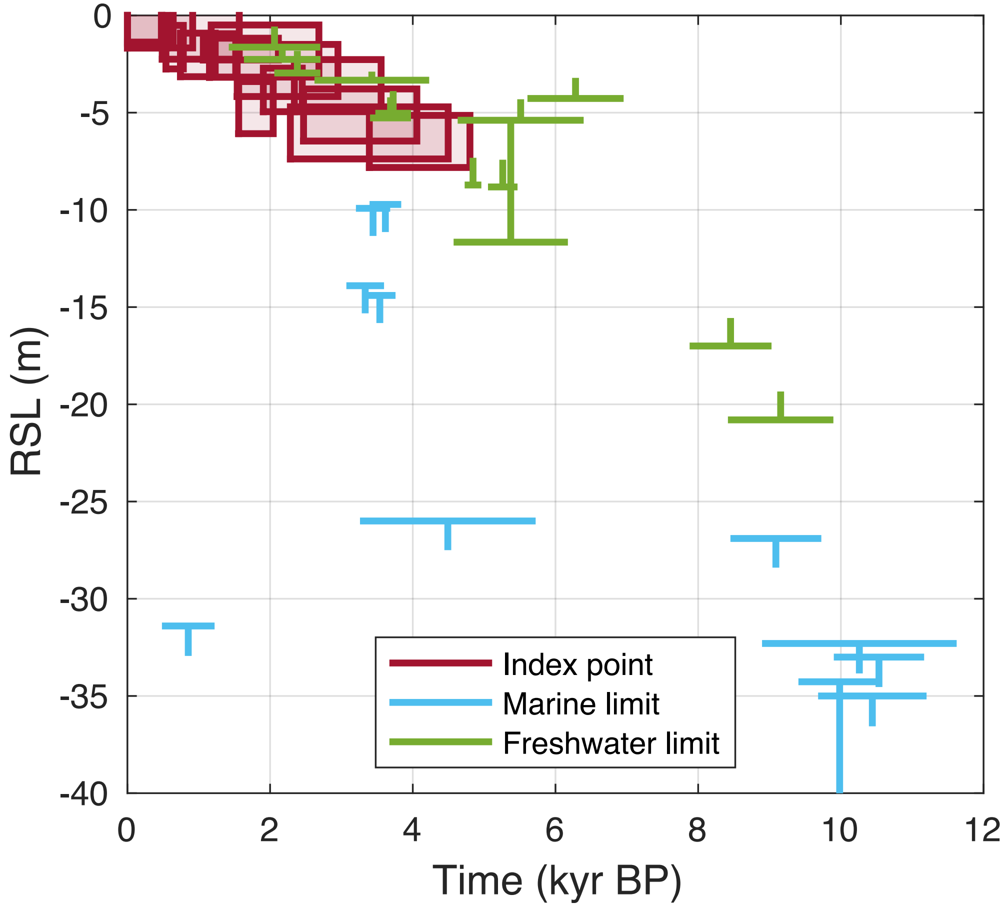
Place priors on $\lambda^2, \rho^2, \varphi$

The new posterior distribution is

$$p\left(\lambda^2, \rho^2, \varphi, x_1, \ldots, y_1, \ldots \middle| w_1, \ldots, z_1, \ldots\right)$$
$$\propto p(\lambda^2) p(\rho^2) p(\varphi) \prod_n \left[p(z_n | y_n)\right.$$
$$\left.\times\ p(w_n | x_n) p\left(y_n \middle| x, y_{m \neq n}, \lambda^2, \rho^2, \varphi\right)\right]$$

Posterior is complicated, and full conditional posteriors don't all have standard distributions

We can't use Gibbs sampler, but we can use Metropolis sampler



Data compiled by Engelhart and Horton (2012)

Model 3: nonlinear/nonparametric process, imperfect observations, without limiting data

Metropolis sampler for parameter ϑ

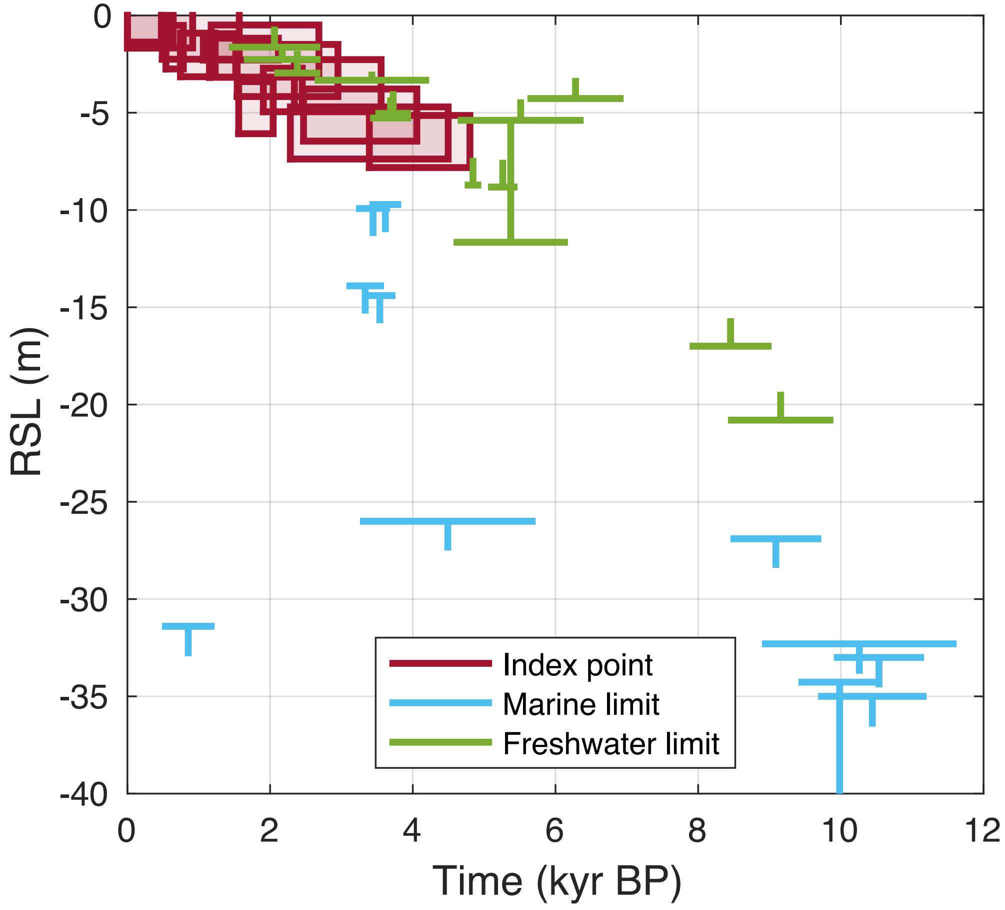
Goal is to sample a new estimate ϑ^{new} based on the current estimate ϑ^{now}

1. Propose a new value $\vartheta^{*} = \vartheta^{now} + \xi$ with ξ from a symmetric distribution

2. Then compute the ratio

$$R_M = \frac{P(\vartheta^{*} | \cdot)}{P(\vartheta^{now} | \cdot)}$$

3. Accept the proposition and define $\vartheta^{new} = \vartheta^{*}$ with probability $min(1, R_M)$



Data compiled by Engelhart and Horton (2012)

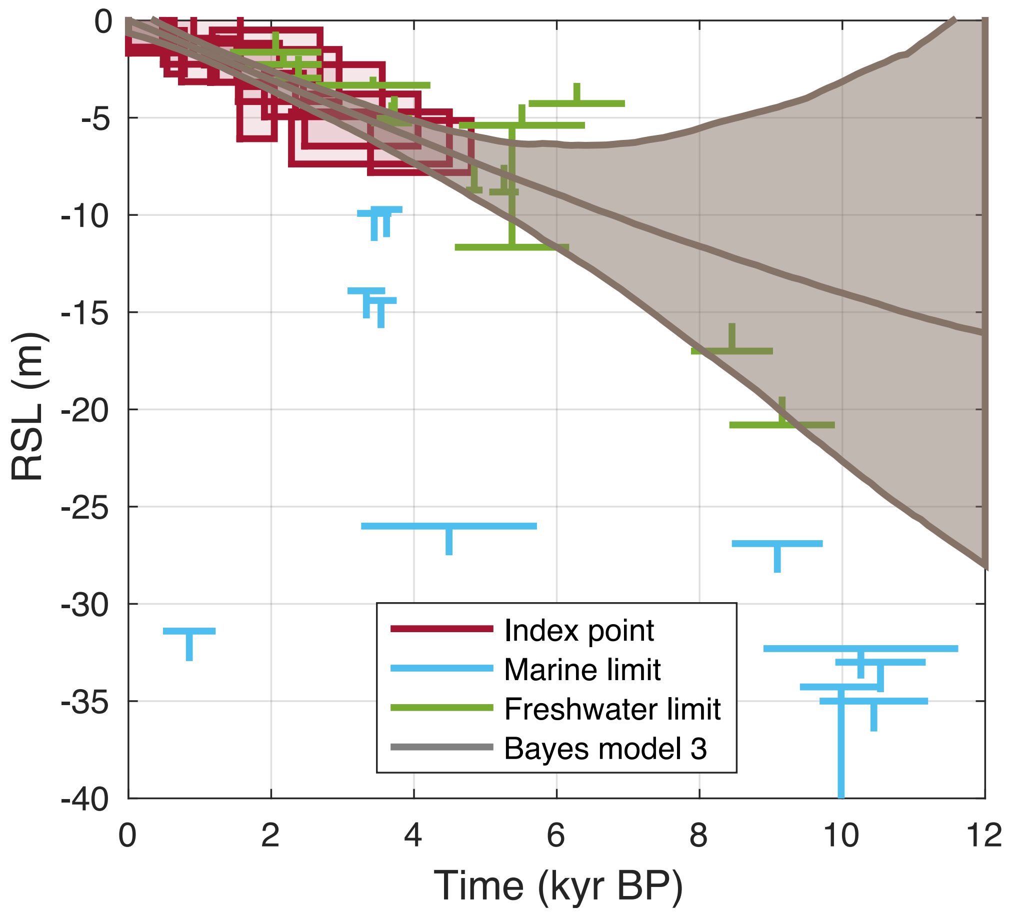
Model 3: nonlinear/nonparametric process, imperfect observations, without limiting data

Now we determine a variable rate of sea-level change over the Holocene

Rates are 1.5 ± 0.6, 1.5 ± 0.7, and 1.2 ± 1.9 mm yr^{-1} at 0, 4, and 8 kyr

Model more completely represents the structure of the index points, and the uncertainties characterizing them

But we're still omitting limiting dates!



Data compiled by Engelhart and Horton (2012)

Model 3: nonlinear/nonparametric process, imperfect observations, without limiting data

Hold up. We didn't specify a function for sea level, so how did I just estimate sea-level values at regular time points?

To do this, I used posterior prediction: knowing sea levels y and times x, I can estimate sea levels y^* at other times x^*

$$p(y^*|x, y, x^*) = N\left(C(x^*, x)C(x, x)^{-1}\, y,\ \rho^2 C(x^*, x^*) - \rho^4 C(x^*, x)\left[\rho^2 C(x, x) + \lambda^2 I\right]^{-1} C(x, x^*)\right)$$

This follows from Gaussian identities and is at the core of Gaussian Process regression

Fourth model: nonlinear/nonparametric process, imperfect data, with limiting data

Model 4: nonlinear/nonparametric process, imperfect observations, with limiting data

Last model! Almost there!

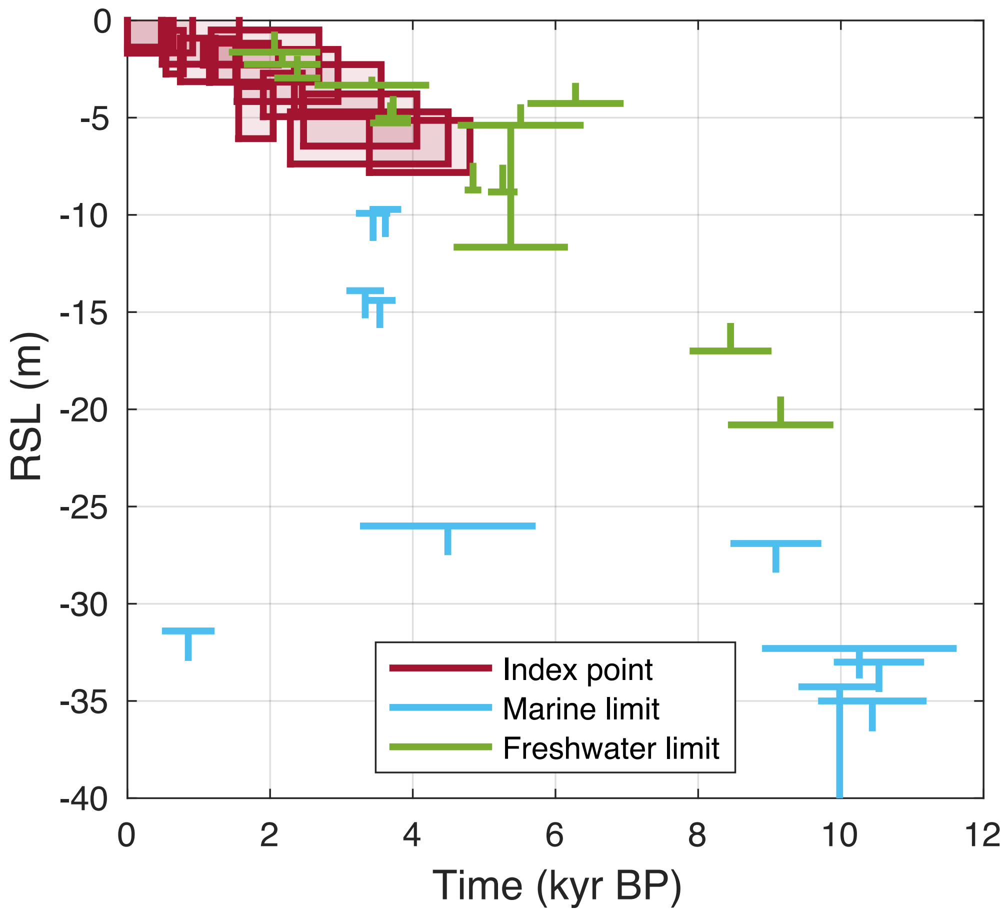
Adjust data equation and add process equations for marine limit m and freshwater limit f

Data–

$$p(z_n) = \begin{cases} N(m_n, \delta_n^2) \\ N(y_n, \delta_n^2) \\ N(f_n, \delta_n^2) \end{cases}$$

Process–

$$p(m_n) = U(-\infty, y_n)$$
$$p(f_n) = U(y_n, +\infty)$$



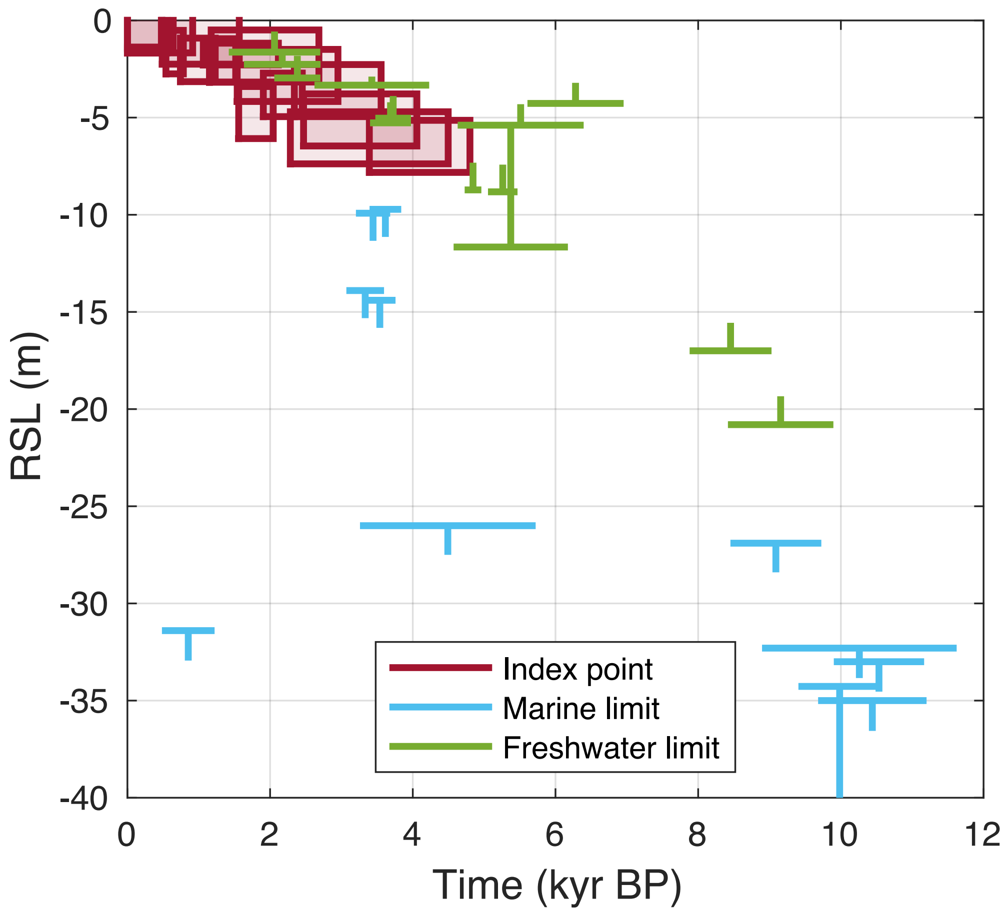
Data compiled by Engelhart and Horton (2012)

Model 4: nonlinear/nonparametric process, imperfect observations, with limiting data

Final posterior distribution is

$$p\left(\lambda^2, \rho^2, \varphi, x_1, \ldots, y_1, \ldots, m_1, \ldots, f_1, \ldots \,\middle|\, w_1, \ldots, z_1, \ldots\right)$$
$$\propto p(\lambda^2)p(\rho^2)p(\varphi) \prod_n \left[p(w_n|x_n) \times p(z_n|y_n, m_n, f_n)p(m_n|y_n) \times p(f_n|y_n)p\left(y_n\middle|x, y_{m\neq n}, \lambda^2, \rho^2, \varphi\right)\right]$$

We use Metropolis algorithm and posterior prediction to generate solutions at regular points in time



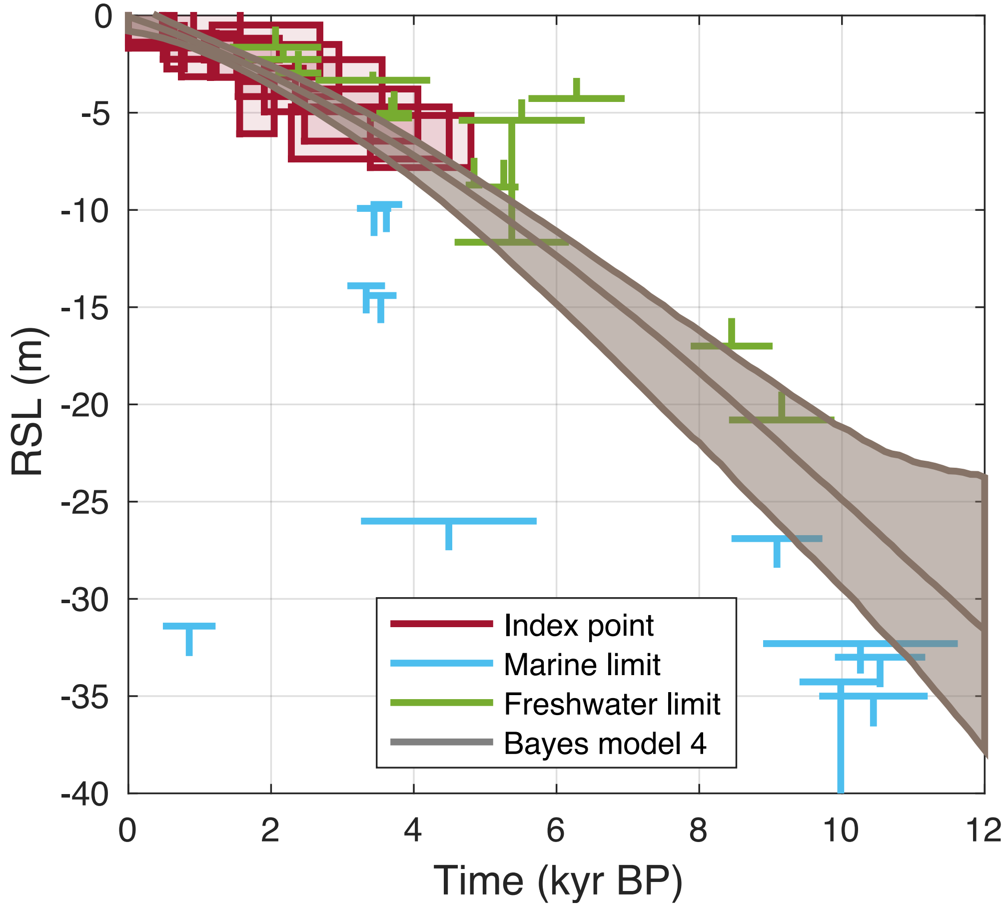
Data compiled by Engelhart and Horton (2012)

Model 4: nonlinear/nonparametric process, imperfect observations, with limiting data

Now we determine variable rates of sea-level change over the Holocene that are consistent with all data types

Model shows higher rates earlier in time, which is more consistent with our physical intuition

Rates are 1.3 ± 0.8, 2.4 ± 0.6, and 3.1 ± 0.9 mm yr^{-1} at 0, 4, and 8 kyr



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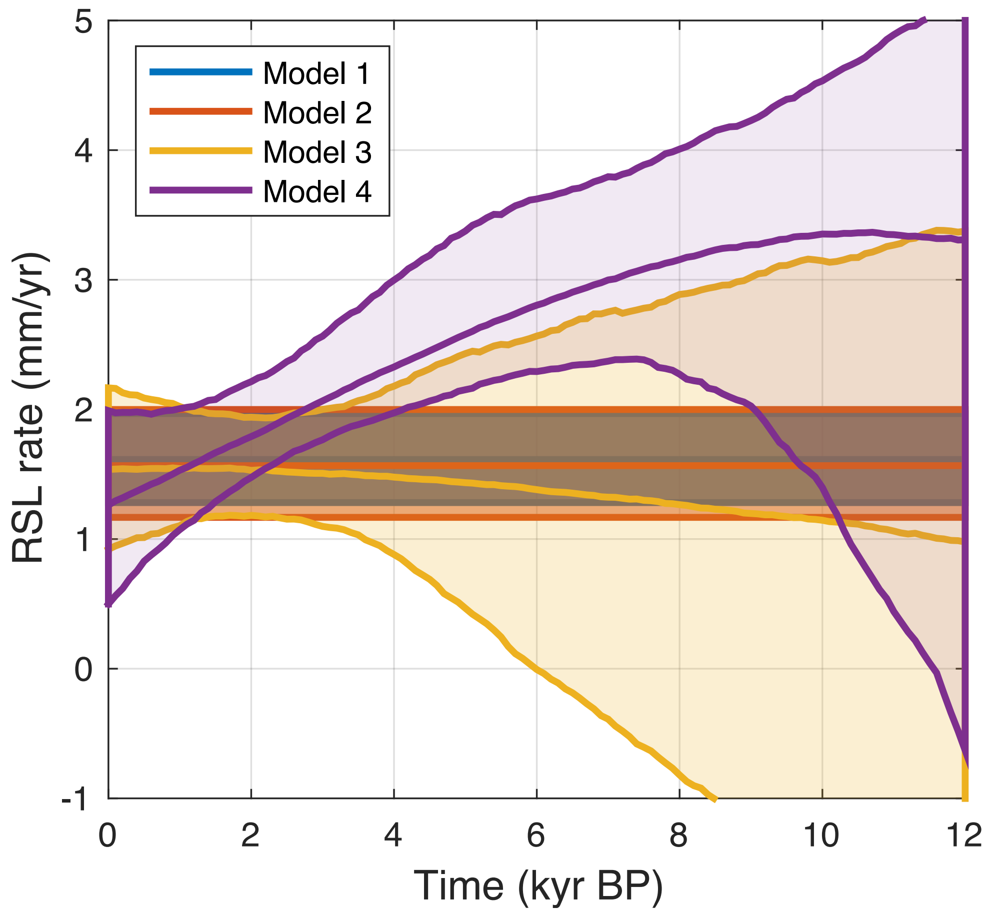

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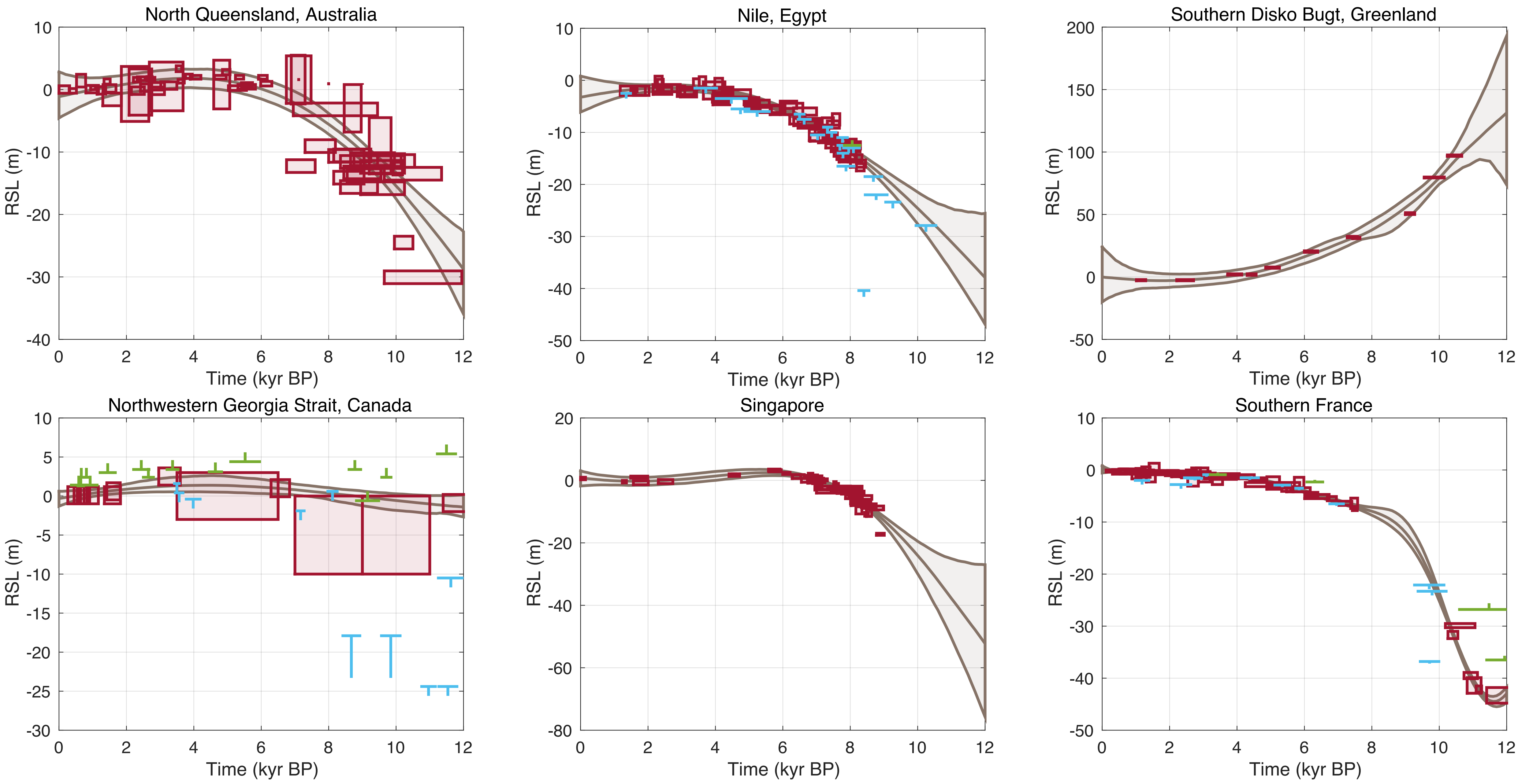

Data compiled by Khan et al. (2015)

Summary

Bayesian modeling involves thinking about problems probabilistically, hierarchically, and conditionally

This approach offers a flexible framework for analyzing sea-level reconstructions and quantifying uncertainty

You don’t need to be a statistician to understand these methods and use them informedly in your own research

Summary

codes/slides available at

github.com/christopherpiecuch/bayesGavle

get/stay in touch

cpiecuch@whoi.edu

twitter.com/chrispiecuch

Literature (freely available books)

Cressie and Wikle (2019)

https://spacetimewithr.org/

Gelman et al. (2013)

http://www.stat.columbia.edu/~gelman/book/BDA3.pdf

Rasmussen and Williams (2006)

https://gaussianprocess.org/gpml/chapters/RW.pdf

Literature (a very incomplete list)

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Cahill et al. (2015) https://www.jstor.org/stable/24522592

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Open-source code from others

Erica Ashe https://github.com/ericaashe

Niamh Cahill https://github.com/ncahill89

Bob Kopp https://github.com/bobkopp

Maeve Upton https://maeveupton.github.io/reslr/

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