Econ 272a Winter 2022
Problem Set 1

January 14, 2022
(Due February 1, 2022)

In each period t, each firm i produces some quantity

$$q_{it}$$

with inputs k_{it} (capital), ℓ_{it} (labor), and m_{it} (materials). Throughout, lowercase (uppercase) letters refer to the natural logarithms (levels) of each variable.

Suppose the following:

A1. Technology

(a) The production function f is Cobb-Douglas in capital and labor, i.e.,

$$\begin{aligned} q_{it} &= f(k_{it}, \ell_{it}; \beta) + \omega_{it} + \varepsilon_{it} \\ &= \beta_0 + \beta_k k_{it} + \beta_\ell \ell_{it} + \omega_{it} + \varepsilon_{it}. \end{aligned} \tag{1}$$

(b) Each unit of production requires materials m_{it} in some fixed proportion to capital and labor.

(c) The level of capital evolves as a function of investment $I_{it} \geq 0$ and depreciation $\delta \in (0, 1)$, according to $K_{it} = (1 - \delta)K_{i,t-1} + I_{i,t-1}$ for all t.

(d) Productivity ω_{it} evolves as an AR(1) process, so that

$$\omega_{it} = \rho\omega_{i,t-1} + \xi_{it}. \tag{2}$$

(e) Productivity is one-dimensional, i.e., $\omega_{it} \in \mathbb{R}$.

A2. Conduct

(a) All firms face common output and materials prices that we normalize to 1.

(b) In each period t, each firm chooses ℓ_{it}, m_{it}, and investment i_{it}, taking as given their ω_{it} and factor prices. Firms maximize their sum of expected profits discounted with factor $1 - r$.

(c) Wages evolve according to an AR(1) process given by

$$P^\ell_{i,t+1} = \theta P^\ell_{it} + \nu_{it}$$

for every i and t.

(d) Convex investment costs, $C(I; \gamma_i) = \frac{1}{2}\gamma_i I^2$, differ by firm through $\gamma_i > 0$ but not over time.

1.1 Conceptual questions

1. For each assumption A1(a)–(e), give an example of an economically interesting question that the assumption rules out.

2. For each assumption A2(a)–(d), give an example of an economically interesting question that the assumption rules out.

3. Describe how each of the following pairs of assumptions interact:

- A1(b),(e)
- A1(d),(e)
- A2(a),(b)
- A1(c), A2(d)

4. State assumptions on (2) that justify the use of OLS to estimate (1). State assumptions on (2) that justify the use of a firm fixed effects estimator for (1). Deduce that this nests these empirical approaches as special cases of our model.

1.2 Simulation

1. Simulate exogenous variables:

(a) Fix the true parameters to $\beta = (\beta_0, \beta_k, \beta_\ell) = (0, 0.4, 0.6)$, $\rho = 0.7$, $\theta = 0.3$, $\delta = 0.2$, and $1 - r = 0.95$.

For $N = 1000$ firms, draw initial conditions for productivity and wages, assuming that $\omega_{i0} \sim \mathcal{N}(0, 0.3)$ and log wages $p^\ell_{i0} \sim \mathcal{N}(0, 0.1)$ for each i.

(b) Using the initial conditions obtained in (a), simulate paths for productivity and wages over $T = 50$ periods for each firm, assuming that the innovations $\xi_{it} \sim \mathcal{N}(0, 0.214)$ and $\ln \nu_{it} \sim \mathcal{N}(0, 0.1)$ are independently and identically distributed across i and t.

(c) Draw an investment cost γ_i for each firm, assuming that $\ln(1/\gamma_i) \sim \mathcal{N}(0, 0.6)$.

2. Solve for endogenous objects:

(a) Derive the optimal labor input conditional on capital, wages, and productivity using (1).

(b) Calculate the derivative of tomorrow's profits with respect to today's investment. Show that when the production function exhibits constant returns to scale ($\beta_k + \beta_\ell = 1$), your answer is independent of the level of capital.

(c) Suppose that in addition to $\beta_k + \beta_\ell = 1$, firms have perfect foresight (no uncertainty over future wages or productivity) and $\mathbb{E}[\exp(\varepsilon_{it})] = 1$. Argue that in this case, the investment policy equation,

κ, can be written

$$\kappa(\omega, p^\ell) = \frac{1}{\gamma_i} \cdot \beta_k \sum_{t=1}^{\infty} (1-\delta)^t (1-r)^t e^{\omega_{it}} \left(\frac{\beta_\ell e^{\omega_{it}}}{P_{it}^\ell} \right)^{\frac{\beta_\ell}{1-\beta_\ell}}$$

at $t = 0$.

(d) Use κ to approximate the paths of capital for each firm from $t = 0$ to $t = T$ under the assumptions of part (c).

(e) Solve for the optimal labor input in each period t given the trajectory of capital, wages, and productivity.

(f) Calculate output using your results from (d) and (e) and draws of measurement error $\varepsilon_{it} \sim \mathcal{N}(0, 0.1)$ (note that this contradicts our assumption above that $\mathbb{E}[\exp(\varepsilon_{it})] = 1$).

(g) Calculate "optimal" materials as output prior to measurement error.

3. Assemble q_{it}, k_{it}, ℓ_{it}, and m_{it} into a dataframe.

1.3 Estimation

0. Download the R dataframe `acf_sim.RData` from our website. Load the dataframe into your preferred computational environment (e.g., R). Note the variable names: `firm_id`, `year`, `k` (capital), `k_1` (lagged capital), `q` (output), `q_1` (lagged output), `labor` (labor), and `labor_1` (lagged labor).

1. Estimate β using a linear regression of q_{it} on k_{it} and ℓ_{it}.

2. Estimate β using a linear regression with fixed effects for i.

3. Control function

(a) Approximate the nonparametric relationship $q_{it} = \Phi(k_{it}, \ell_{it}, m_{it})$ using a third-degree polynomial in k_{it}, ℓ_{it}, and m_{it}. Estimate this polynomial and tabulate the fitted values $\hat{\Phi}_{it} \equiv \hat{\Phi}(k_{it}, \ell_{it}, m_{it})$.

(b) Code a function for the production function in (1); i.e., code a function that takes a parameter guess β and data on a firm's inputs and returns $f(k_{it}, \ell_{it}; \beta)$.

(c) Use your work in (a) and (b) to construct $\hat{\omega}_{it}(\beta) = \hat{\Phi}_{it} - f(k_{it}, \ell_{it}; \beta)$. Then code a function that takes a parameter guess β and returns the implied productivity innovation $\hat{\xi}_{it}(\beta)$ as the residual of a linear regression of $\hat{\omega}_{it}(\beta)$ on $\hat{\omega}_{i,t-1}(\beta)$.

(d) Assemble the ACF instruments, $Z_{it} = (1, k_{it}, \ell_{i,t-1})$. Explain why these are valid instruments; i.e., explain why (i) they satisfy the exclusion restriction $\mathbb{E}[Z'_{it}\xi_{it} | \hat{\omega}_{i,t-1}] = 0$ and (ii) they might be relevant shifters of time-t inputs.

(e) Construct a GMM objective function using the moments

$$\mathbb{E}[Z'_{it}\xi_{it} | \hat{\omega}_{i,t-1}] = 0$$

evaluated with the $\hat{\xi}_{it}(\beta)$ obtained in (c) and the instruments in (d). Use either the identity matrix or $[Z'Z]^{-1}$ as a weight matrix.

(f) Solve for the value $\hat{\beta}$ that minimizes the objective in (e) using a grid search or a solver like `optim` in R.

(g) Construct K bootstrap draws of the dataset, each $k \in \{1, 2, \ldots, K\}$ of size $N^{(k)} = 1000$, by sampling the original dataset (with replacement). Let $K = 40$.

(h) Code a loop that iterates (a), (e), and (f) over K datasets to recover K estimates of $\hat{\beta}$. Report 90% confidence intervals for β_k and β_ℓ using the distribution of bootstrap estimates.

4. (*Extra credit*). Estimate "dynamic panel" coefficients using a similar line of attack as in the previous question. Drop step (a), modify step (c) to directly use the AR(1) equation in (2), and include ρ in the vector β of parameters over which you search in step (f).

5. Tabulate the production function coefficients β_k and β_ℓ from OLS, fixed effects, ACF, and dynamic panel (*if applicable*) estimators from questions 1–4.

1.4 Counterfactuals

1. Plot the empirical distribution of productivity across all firms at time T using the estimates from the ACF procedure in the previous question.

2. Capital "misallocation"

(a) The ex-ante marginal product of capital is given by

$$e^{\hat{\omega}_{it}} \frac{\partial F(K_{it}, L_{it}; \beta)}{\partial K}.$$

Plot its empirical distribution across all firms at time $t = T$.

(b) What is the average (unweighted) marginal product of capital across firms?

(c) What are the first and ninth deciles of this distribution?

3. Capital reallocation

(a) Sum capital across all firms to obtain the time-T economy's total capital stock $K_T = \sum_{i=1}^{N} K_{iT}$.

(b) Suppose that capital can be freely reallocated across firms. Write an algorithm to assign capital to firms with the highest marginal product of capital, and iterate to find the allocation of capital across firms that sums to K_T and maximizes total output.

(c) What is the marginal product of capital at the allocation in (b)?

(d) What is the total output across firms under the capital allocation in (b) and how does it compare with total output under the allocation of capital in the data?