ECON272A - Problem Set #1

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1.1 Conceptual Questions

Question 1

Assumption	Example Question	Ruled out because...
A1(a)	Does the elasticity of output with respect to inputs (capital and labor) change as output increases?	Output elasticities (β_k, β_l) are constant for all levels of output.
A1(b)	Are there fixed material costs incurred from production?	Materials are in fixed proportion to capital and labor.
A1(c)	How does investment risk affect the capital accumulation of the firm?	Capital accumulation is deterministic.
A1(d)	Can firm investments lead to endogenous productivity improvements in the same period?	Productivity is modeled as an AR(1) process; ω_{it} depends only on $\omega_{i,t-1}$.
A1(e)	Is productivity growth factor-augmenting for the firm?	$\omega_{it} \in \mathbb{R}$ and only captures factor-neutral productivity.

Question 2

Assumption	Example Question	Ruled out because...
A2(a)	Do firms produce differentiated varieties?	All firms produce the same output and belong to the same market.
A2(b)	Do different firms have different values for future profits?	Firms discount the future with a common factor $(1 - r)$.
A2(c)	Does the firm have monopsony power in the labor market?	Wages are determined by an AR(1) process.
A2(d)	How do changes in investment cost over time affect firms?	The cost of investment varies only across firms but not over time.

Question 3

- Assumptions A1(b),(e) imply that productivity shocks are Hicks-neutral.
- Assumptions A1(d),(e) imply that the correlation coefficient ρ in (2) is a scalar value.
- Assumptions A2(a),(b) imply that any two or more firms with the same state variables (k_{it}, ω_{it}) will face the same profit maximization problem and behave identically.
- Assumption A2(d) implies that the firm chooses the same level of optimal investment ($I_{it} = I^*$) in every period. This, with Assumption A1(c), implies one of the following 3 cases for the capital stock of the firm: (i) K grows at a constant rate (if $I^* > \delta K_{it}$), (ii) K has a steady-state (if $I^* = \delta K_{it}$) or (iii) K eventually falls to 0 (if $I^* < \delta K_{it}$).

Question 4

Equation (2) imposes the structural assumption that $\omega_{it} = \rho\omega_{i,t-1} + \xi_{it}$ where ξ_{it} is a mean zero i.i.d. random shock.

- To justify the use of OLS to estimate (1), we have to assume that $\omega_{it} = 0$ for all i and for all t. This is equivalent to assuming $\rho = 0$ in (2).
- To justify the use of FE to estimate (1), we have to assume that $\omega_{it} = \omega_i$ for all t. This is equivalent to assuming $\rho = 1$ in (2).

Therefore, the assumptions required for OLS and FE can be interpreted as special cases of the model that specify the value of ρ in equation (2).

1.2 Simulation

Questions 1, 2(d) to 2(g), 3

See below for code.

Question 2a

Let $A_{it} = \exp(\beta_0)\exp(\omega_{it})\exp(\varepsilon_{it})$. The production function is Cobb-Douglas in capital and labor, and can be written as:

$$Q_{it} = F(K_{it}, L_{it}) = A_{it}K_{it}^{\beta_k}L_{it}^{\beta_l}$$

Which gives the first-order condition (FOC) with respect to labor:

$$P_{it}^l = A_{it}K_{it}^{\beta_k}\beta_l L_{it}^{\beta_l - 1}$$

Re-arranging, the optimal labor input is:

$$L_{it} = \Big(\frac{A_{it}K_{it}^{\beta_k}\beta_l}{P_{it}^l}\Big)^{\frac{1}{1-\beta_l}}$$

Question 2b

Capital evolves according to:

$$K_{i,t+1} = (1-\delta)K_{it} + I_{it}$$

with convex investment costs:

$$C(I_{it};\gamma_i) = \frac{1}{2}\gamma_i I_{it}^2$$

Next-period production can thus be expressed as:

$$Q_{i,t+1} = A_{i,t+1}K_{i,t+1}^{\beta_k}L_{i,t+1}^{\beta_l} = A_{i,t+1}[(1-\delta)K_{it} + I_{it}]^{\beta_k}L_{i,t+1}^{\beta_l}$$

Using the labor FOC, the FOC with respect to investment-today:

$$\begin{aligned}
\gamma_i I_{it} &= A_{i,t+1}L_{i,t+1}^{\beta_l}\beta_k[(1-\delta)K_{it} + I_{it}]^{\beta_k - 1} \\
&= A_{i,t+1}\Big(\frac{A_{i,t+1}K_{i,t+1}^{\beta_k}\beta_l}{P_{i,t+1}^l}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k[K_{i,t+1}]^{\beta_k-1} \\
&= A_{i,t+1}\Big(\frac{A_{i,t+1}\beta_l}{P_{i,t+1}^l}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k[K_{i,t+1}]^{\frac{\beta_k\beta_l}{1-\beta_l}+\beta_k-1} \\
&= A_{i,t+1}\Big(\frac{A_{i,t+1}\beta_l}{P_{i,t+1}^l}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k
\end{aligned}$$

which is independent of the level of capital because if $\beta_k + \beta_l = 1$, then

$$\frac{\beta_k\beta_l}{1-\beta_l} + \beta_k - 1 = \frac{\beta_k\beta_l + \beta_k(1-\beta_l) - 1 + \beta_l}{1-\beta_l} = \frac{(\beta_k+\beta_l)-1}{1-\beta_l} = 0$$

Question 2c

Given that $\beta_0 = 0$ and $\mathbb{E}[\exp(\varepsilon_{it})] = 1$,

$$\mathbb{E}[A_{it}] = \mathbb{E}[\exp(0)\exp(\omega_{it})\exp(\varepsilon_{it})] = \exp(\omega_{it})\mathbb{E}[\exp(\varepsilon_{it})] = \exp(\omega_{it})$$

Where $\exp(\omega_{it})$ is outside the expectation operator because of Assumption A2(b) (each firm takes as given their ω_{it} in each period t). Now we can write the optimal investment rule as:

$$\begin{aligned}
I_{it} &= \frac{1}{\gamma_i}\mathbb{E}[A_{i,t+1}]\Big(\frac{\mathbb{E}[A_{i,t+1}]\beta_l}{P_{i,t+1}^l}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k \\
&= \frac{1}{\gamma_i}\exp(\omega_{i,t+1})\Big(\frac{\exp(\omega_{i,t+1})\beta_l}{P_{i,t+1}^l}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k
\end{aligned}$$

The investment policy equation is the present discounted value of future investment flows (net of depreciation) at $t = 0$:

$$\begin{aligned}
\kappa(\{\omega_{it}, P_{it}^l\}_{t=1}^{\infty}) &= \sum_{t=1}^{\infty}(1-\delta)^t(1-r)^t I_{it} \\
&= \sum_{t=1}^{\infty}(1-\delta)^t(1-r)^t\frac{1}{\gamma_i}\exp(\omega_{it})\Big(\frac{\exp(\omega_{it})\beta_l}{P_{it}^l}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k \\
&= \frac{1}{\gamma_i}\cdot\beta_k\sum_{t=1}^{\infty}(1-\delta)^t(1-r)^t\exp(\omega_{it})\Big(\frac{\exp(\omega_{it})\beta_l}{P_{it}^l}\Big)^{\frac{\beta_l}{1-\beta_l}}
\end{aligned}$$

Python code for '1.2 Simulation'

```
import pandas as pd
import numpy as np

# (1a) Set seed and parameters
np.random.seed(123456789)
N=1000
T=50
beta_zero=0
beta_k=0.4
beta_l=0.6
rho=0.7
theta=0.3
delta=0.2
r=0.05

# (1b) Initial conditions for productivity and wages
sigma_omega=np.sqrt(0.3)
sigma_logwage=np.sqrt(0.1)
omega_zero=np.random.normal(0,sigma_omega,(N,1)) # omega_0 ~ N(0,0.3)
wage_zero=np.exp(np.random.normal(0,sigma_logwage,(N,1))) # log wage_0 ~ N(0,0.1)

# (1b) Shocks to productivity and wages (i.i.d. across firms and time)
sigma_xi=np.sqrt(0.214)
sigma_nu=np.sqrt(0.1)
xi=np.random.normal(0,sigma_xi,(N,T)) # xi ~ N(0,0.214)
nu=np.exp(np.random.normal(0,sigma_nu,(N,T))) # log nu ~ N(0,0.1)

# (1c) Paths for productivity and wages
M=np.zeros((N,T))
omega=np.concatenate((omega_zero,M),axis=1)
wage=np.concatenate((wage_zero,M),axis=1)

for t in range(T):
        omega[:,t+1]=rho*omega[:,t]+xi[:,t] # Productivity path from t=0 to t=50
        wage[:,t+1]=theta*wage[:,t]+nu[:,t] # Wage path from t=0 to t=50

# (1c) Generate time-invariant investment costs for firms
sigma_g=np.sqrt(0.6)
g=np.random.normal(0,sigma_g,(N,1)) # g = log(1/gamma)~N(0,0.6)
gamma=1/np.exp(g) # gamma = 1/exp(g)

# (2d) Path for investments from t=0 to t=49 (use FOC for investments)
inv=np.zeros((N,T))
for i in range(N):
        for t in range(T):
                inv[i,t]=1/gamma[i]*beta_k*np.exp(omega[i,t+1])*\
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                np.power((np.exp(omega[i,t+1])*beta_l/wage[i,t+1]),(beta_l/(1-beta_l)))

# (2d) Path for capital from t=0 to t=50 (use equation for capital growth)
# Assume that every firm begins with 1 unit of capital at t=0
capital=np.ones((N,T+1))
for t in range(T):
        capital[:,t+1]=(1-delta)*capital[:,t]+inv[:,t]

log_capital=np.log(capital)

# (2e) Path for labor from t=0 to t=50 (use FOC for labor)
labor=np.power((np.exp(omega)*np.power(capital,beta_k)*beta_l)/wage,1/(1-beta_l))
log_labor=np.log(labor)

# (2f) Simulate measurement error
sigma_err=np.sqrt(0.1)
err=np.random.normal(0,sigma_err,(N,T+1))

# (2f) Path for output (use equation 1)
log_output=beta_zero+beta_k*log_capital+beta_l*log_labor+omega+err
output=np.exp(log_output)

# (2g) Path for materials (output prior to measurement error)
log_materials=log_output-omega-err
materials=np.exp(log_materials)

# (3) Dataframe for log_output (q_it), log_capital (k_it),
# log_labor (l_it), log_materials (m_it)

# (3) Index for N firms
firms=pd.DataFrame(np.linspace(1,N,num=N), columns=["firm_id"])
firm_id=pd.concat([firms]*51,ignore_index=True)

# (3) Index for T+1 periods
time=pd.Series(np.linspace(0,T,num=T+1)).repeat(1000)
time=pd.DataFrame(time, columns=["time"]).reset_index()

# (3) Reshape 'N x (T+1)' matrices into columns of length N*(T+1)
q_it=pd.DataFrame(log_output.flatten('F'), columns=["log_output"])
k_it=pd.DataFrame(log_capital.flatten('F'), columns=["log_capital"])
l_it=pd.DataFrame(log_labor.flatten('F'), columns=["log_labor"])
m_it=pd.DataFrame(log_materials.flatten('F'), columns=["log_materials"])

# (3) Assemble dataframe
dataframe=firm_id.join(time['time']).join(q_it).join(k_it).join(l_it).join(m_it)
```