ECON272A - Problem Set #1

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1.1 Conceptual Questions

Question 1

Assumption	Example Question	Ruled out because...
A1(a)	Does the elasticity of output with respect to inputs (capital and labor) change as output increases?	The Cobb-Douglas production function assumes that output elasticities (β_k, β_l) are constant for all levels of output.
A1(b)	Can more capital-efficient or labor-efficient materials be used in the production process?	Materials are required according to some fixed proportion to capital and labor.
A1(c)	Should the firm invest in training its workers (to increase human capital) in addition to physical capital?	There is a single type of capital.
A1(d)	Are firm investments able to improve the firm's productivity in the current period?	Productivity is modeled as an AR(1) process; ω_{it} depends only on $\omega_{i,t-1}$.
A1(e)	What is the labor productivity of the firm?	Productivity is one-dimensional and only reflects total factor productivity.

Question 2

Assumption	Example Question	Ruled out because...
A2(a)	How does the unobserved quality of output vary across firms?	Common output and materials prices imply no unobserved differences across firms and a homogeneous output market.
A2(b)	How do external technology shocks affect the firm's investments?	There are no idiosyncratic shocks to i_{it} and i_{it} depends only on state variables (k_{it}, ω_{it}).
A2(c)	Does the firm have monopsony power in the labor market?	Wages are determined by an AR(1) process.
A2(d)	How do changes in investment cost over time affect firms' use of capital in production?	The cost of investment varies only across firms but not over time.

Question 3

- Assumptions A1(b),(e) imply that productivity shocks are Hicks-neutral and do not affect the capital-labor ratio of the firm.
- Assumptions A1(d),(e) imply that the correlation coefficient ρ in (2) is a scalar value.
- Assumptions A2(a),(b) imply that there are no strategic interactions between firms operating in the same market, as firms are price-takers and make their input choices based only on their state variables (k_{it}, ω_{it}).
- Assumption A2(d) implies that the firm chooses the same level of optimal investment ($I_{it} = I^*$) in every period. This, with Assumption A1(c), implies one of the following 3 cases for the capital stock of the firm: (i) K grows at a constant rate (if $I^* > \delta K_{it}$), (ii) K has a steady-state (if $I^* = \delta K_{it}$) or (iii) K eventually falls to 0 (if $I^* < \delta K_{it}$).

Question 4

Equation (2) imposes the structural assumption that $\omega_{it} = \rho \omega_{i,t-1} + \xi_{it}$ where ξ_{it} is a mean zero i.i.d. random shock.

- To justify the use of OLS to estimate (1), we have to assume that $\omega_{it} = 0$ for all i and for all t. This is equivalent to assuming $\rho = 0$ in (2).
- To justify the use of FE to estimate (1), we have to assume that $\omega_{it} = \omega_i$ for all t. This is equivalent to assuming $\rho = 1$ in (2).

Therefore, the assumptions required for OLS and FE can be interpreted as special cases of the model that specify the value of ρ in equation (2).

1.2 Simulation

Questions 1, 2(d) to 2(g), 3

See below for code.

Question 2a

Let $A_{it} = \exp(\beta_0)\exp(\omega_{it})\exp(\varepsilon_{it})$. The production function is Cobb-Douglas in capital and labor, and can be written as:

$$Q_{it} = F(K_{it}, L_{it}) = A_{it} K_{it}^{\beta_k} L_{it}^{\beta_l}$$

Which gives the first-order condition (FOC) with respect to labor:

$$P_{it}^l = A_{it} K_{it}^{\beta_k} \beta_l L_{it}^{\beta_l - 1}$$

Re-arranging, the optimal labor input is:

$$L_{it} = \Big(\frac{A_{it} K_{it}^{\beta_k} \beta_l}{P_{it}^l}\Big)^{\frac{1}{1-\beta_l}}$$

Question 2b

Capital evolves according to:

$$K_{i,t+1} = (1-\delta)K_{it} + I_{it}$$

with convex investment costs:

$$C(I_{it}; \gamma_i) = \frac{1}{2}\gamma_i I_{it}^2$$

Next-period production can thus be expressed as:

$$Q_{i,t+1} = A_{i,t+1} K_{i,t+1}^{\beta_k} L_{i,t+1}^{\beta_l} = A_{i,t+1}[(1-\delta)K_{it} + I_{it}]^{\beta_k} L_{i,t+1}^{\beta_l}$$

Using the labor FOC, the FOC with respect to investment-today:

$$\begin{aligned}
\gamma_i I_{it} &= A_{i,t+1} L_{i,t+1}^{\beta_l} \beta_k [(1-\delta)K_{it} + I_{it}]^{\beta_k - 1} \\
&= A_{i,t+1}\Big(\frac{A_{i,t+1} K_{i,t+1}^{\beta_k} \beta_l}{P_{i,t+1}^l}\Big)^{\frac{\beta_l}{1-\beta_l}} \beta_k [K_{i,t+1}]^{\beta_k - 1} \\
&= A_{i,t+1}\Big(\frac{A_{i,t+1}\beta_l}{P_{i,t+1}^l}\Big)^{\frac{\beta_l}{1-\beta_l}} \beta_k [K_{i,t+1}]^{\frac{\beta_k\beta_l}{1-\beta_l} + \beta_k - 1} \\
&= A_{i,t+1}\Big(\frac{A_{i,t+1}\beta_l}{P_{i,t+1}^l}\Big)^{\frac{\beta_l}{1-\beta_l}} \beta_k
\end{aligned}$$

which is independent of the level of capital because if $\beta_k + \beta_l = 1$, then

$$\frac{\beta_k\beta_l}{1-\beta_l} + \beta_k - 1 = \frac{\beta_k\beta_l + \beta_k(1-\beta_l) - 1 + \beta_l}{1-\beta_l} = \frac{(\beta_k + \beta_l) - 1}{1-\beta_l} = 0$$

Question 2c

Given that $\beta_0 = 0$ and $\mathbb{E}[\exp(\varepsilon_{it})] = 1$,

$$\mathbb{E}[A_{it}] = \mathbb{E}[\exp(0)\exp(\omega_{it})\exp(\varepsilon_{it})] = \exp(\omega_{it})\mathbb{E}[\exp(\varepsilon_{it})] = \exp(\omega_{it})$$

Where $\exp(\omega_{it})$ is outside the expectation operator because of Assumption A2(b) (each firm takes as given their ω_{it} in each period t). Now we can write the optimal investment rule as:

$$\begin{aligned} I_{it} &= \frac{1}{\gamma_i}\mathbb{E}[A_{i,t+1}]\Big(\frac{\mathbb{E}[A_{i,t+1}]\beta_l}{P^l_{i,t+1}}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k \\ &= \frac{1}{\gamma_i}\exp(\omega_{i,t+1})\Big(\frac{\exp(\omega_{i,t+1})\beta_l}{P^l_{i,t+1}}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k \end{aligned}$$

The investment policy equation is the present discounted value of future investment flows (net of depreciation) at $t = 0$:

$$\begin{aligned} \kappa(\{\omega_{it}, P^l_{it}\}^{\infty}_{t=1}) &= \sum_{t=1}^{\infty}(1-\delta)^t(1-r)^t I_{it} \\ &= \sum_{t=1}^{\infty}(1-\delta)^t(1-r)^t\frac{1}{\gamma_i}\exp(\omega_{it})\Big(\frac{\exp(\omega_{it})\beta_l}{P^l_{it}}\Big)^{\frac{\beta_l}{1-\beta_l}}\beta_k \\ &= \frac{1}{\gamma_i}\cdot\beta_k\sum_{t=1}^{\infty}(1-\delta)^t(1-r)^t\exp(\omega_{it})\Big(\frac{\exp(\omega_{it})\beta_l}{P^l_{it}}\Big)^{\frac{\beta_l}{1-\beta_l}} \end{aligned}$$

Python code for '1.2 Simulation'

```
import pandas as pd
import numpy as np

# (1a) Set seed and parameters
np.random.seed(123456789)
N=1000
T=50
beta_zero=0
beta_k=0.4
beta_l=0.6
rho=0.7
theta=0.3
delta=0.2
r=0.05

# (1b) Initial conditions for productivity and wages
sigma_omega=np.sqrt(0.3)
sigma_logwage=np.sqrt(0.1)
omega_zero=np.random.normal(0,sigma_omega,(N,1)) # omega_0 ~ N(0,0.3)
wage_zero=np.exp(np.random.normal(0,sigma_logwage,(N,1))) # log wage_0 ~ N(0,0.1)

# (1b) Shocks to productivity and wages (i.i.d. across firms and time)
sigma_xi=np.sqrt(0.214)
sigma_nu=np.sqrt(0.1)
xi=np.random.normal(0,sigma_xi,(N,T)) # xi ~ N(0,0.214)
nu=np.exp(np.random.normal(0,sigma_nu,(N,T))) # log nu ~ N(0,0.1)

# (1c) Paths for productivity and wages
M=np.zeros((N,T))
omega=np.concatenate((omega_zero,M),axis=1)
wage=np.concatenate((wage_zero,M),axis=1)

for t in range(T):
        omega[:,t+1]=rho*omega[:,t]+xi[:,t] # Productivity path from t=0 to t=50
        wage[:,t+1]=theta*wage[:,t]+nu[:,t] # Wage path from t=0 to t=50

# (1c) Generate time-invariant investment costs for firms
sigma_g=np.sqrt(0.6)
g=np.random.normal(0,sigma_g,(N,1)) # g = log(1/gamma)~N(0,0.6)
gamma=1/np.exp(g) # gamma = 1/exp(g)

# (2d) Path for investments from t=0 to t=49 (use FOC for investments)
inv=np.zeros((N,T))
for i in range(N):
        for t in range(T):
                inv[i,t]=1/gamma[i]*beta_k*np.exp(omega[i,t+1])*\
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                np.power((np.exp(omega[i,t+1])*beta_l/wage[i,t+1]),(beta_l/(1-beta_l)))

# (2d) Path for capital from t=0 to t=50 (use equation for capital growth)
# Assume that every firm begins with 1 unit of capital at t=0
capital=np.ones((N,T+1))
for t in range(T):
        capital[:,t+1]=(1-delta)*capital[:,t]+inv[:,t]

log_capital=np.log(capital)

# (2e) Path for labor from t=0 to t=50 (use FOC for labor)
labor=np.power((np.exp(omega)*np.power(capital,beta_k)*beta_l)/wage,1/(1-beta_l))
log_labor=np.log(labor)

# (2f) Simulate measurement error
sigma_err=np.sqrt(0.1)
err=np.random.normal(0,sigma_err,(N,T+1))

# (2f) Path for output (use equation 1)
log_output=beta_zero+beta_k*log_capital+beta_l*log_labor+omega+err
output=np.exp(log_output)

# (2g) Path for materials (output prior to measurement error)
log_materials=log_output-err
materials=np.exp(log_materials)

# (3) Dataframe for log_output (q_it), log_capital (k_it),
# log_labor (l_it), log_materials (m_it)

# (3) Index for N firms
firms=pd.DataFrame(np.linspace(1,N,num=N), columns=["firm_id"])
firm_id=pd.concat([firms]*51,ignore_index=True)

# (3) Index for T+1 periods
time=pd.Series(np.linspace(0,T,num=T+1)).repeat(1000)
time=pd.DataFrame(time, columns=["time"]).reset_index()

# (3) Reshape 'N x (T+1)' matrices into columns of length N*(T+1)
q_it=pd.DataFrame(log_output.flatten('F'), columns=["log_output"])
k_it=pd.DataFrame(log_capital.flatten('F'), columns=["log_capital"])
l_it=pd.DataFrame(log_labor.flatten('F'), columns=["log_labor"])
m_it=pd.DataFrame(log_materials.flatten('F'), columns=["log_materials"])

# (3) Assemble dataframe
dataframe=firm_id.join(time['time']).join(q_it).join(k_it).join(l_it).join(m_it)
```