Case study on foreign exchange risk management

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1 Introduction

In this case study, we assume the risk management of a private equity fund with investments in three different currencies (USD, Euro and GBP). The fund's domestic currency is USD, and this is the currency in which its earnings are measured. Thus, the fund is exposed to FX risk, and we are asked to construct a strategy to mitigate that risk using knock-in options.

More concretely, we are asked to construct a portfolio for the fund which includes a set of cash flows in the 3 given currencies. We assume these cash flows are deterministic. Then, we are asked to calculate the portfolio's internal rate of return (IRR) at risk. Finally, we are asked to construct appropriate options such that the IRR at the lower tail of the IRR distribution is minimized.

The most crucial part of our task is the choice of model for the financial quantities that determine the value and risk of our portfolio. Over the next section, we will conduct an overview of the considered models; then, we will proceed to calculate the IRR at risk of our portfolio under these models, and construct options that will improve it.

To facilitate the calibration of our models, we are provided with financial data ranging from 2000 to 2019, containing spot exchange rates, forward exchange rates, and implied volatilities of those rates. Furthermore, the data contains yield curves for bonds in all 3 currencies, along with interest rate implied volatilities extrapolated from interest rate swaptions. In this report, we explain the models we use; the details of the implementation are seen in the accompanied files.

We exhibit 2 models, one model which assumes the evolution of the FX rates depends on constant interest rates and constant volatilities, and one which assumes the volatility to be time-dependent.

2 Constant interest rates - Constant volatility

Our first concern is choosing an appropriate model for the evolution of the involved exchange rates. Our choice is the Geometric Brownian Motion (GBM) model of Garman and Kohlhagen, where the evolution of a spot exchange rate $R(t)$ as a function of time t is described by the following SDE (time is measured in years, and $R(t)$ is quoted in terms of units of domestic currency per unit of foreign currency):

$$dR(t) = \mu R(t) + \sigma R(t) dW(t) \tag{1}$$

where $W(t)$ is a standard Brownian motion. It is easy to derive

$$R(t) = R(0) e^{(\mu - \sigma^2/2)t + \sigma W(t)}, \tag{2}$$

and

$$\mathrm{E}\left[R(t)\right] = R(0) e^{\mu t}. \tag{3}$$

Under this model, we need to evaluate μ and σ. A standard assumption to evaluate these parameters is to assume the existence of a domestic risk-free interest rate $r_d(t)$, and a foreign risk-free interest rate $r_f(t)$; for our starting model, we assume these to be constant. Since these interest rates are a theoretical construct (after all, not all entities can invest in the same products, or borrow at the same rate), we need to choose an appropriate value for r_d and r_f; a common proxy for these interest rates is the short-term rate of return of the smallest available maturity government bond, and this is the choice we make here.

Another alternative for calculating the interest rates would be to use the provided forward exchange rates. Under our model, a forward exchange rate is a deterministic function of the spot exchange rate and the 2 associated interest rates, which we remind are assumed to be constant. Therefore, we can fit the interest rates so that the mean squared error between the prices predicted by our model and the observed prices is minimized. This would actually be the only meaningful use of the provided forward rates, since, unlike FX options, under our current model the forward rates are not associated at all with the volatility of the underlying spot rates.

Investing 1 unit of currency using the risk-free interest rate r will return e^{rt} units in time t. Under the risk-neutral measure associated with choosing $e^{r_d t}$ as numeraire, we have

$$\mu = r_d - r_f. \tag{4}$$

Observe that under this measure, the values of both the domestic and the foreign money market accounts are martingales. In order to evaluate σ, one typically calibrates the model to fit given prices of simple FX options; in our case, we are not given explicit such prices, but rather we are given the volatilities implied by such options. Of course, in reality these volatilities vary over time, in contrast to our model. Furthermore, even at a single given point in time, the volatilities implied by options of different maturities differ to a degree that indicates this behavior is systemic rather than a result of noise. Nonetheless, for the moment we have a model with a single volatility parameter per exchange rate; in order to extract a single volatility for our model, we will use the implied volatilites to calculate the prices of the options they were implied from, and then we will fit σ to minimize the mean squared error between the option prices and the ones predicted from our constant volatility model.

To sum up, so far we have calculated the 3 relevant interest rates $r_1(t)$ (USD), $r_2(t)$ (Euro) and $r_3(t)$ (GBP), and we have a model for the foreign exchange rates whose evolution is described by the following system of SDEs:

$$
\begin{aligned}
dR_{12}(t) &= \mu_{12}R_{12}(t) + \sigma_{12}R_{12}(t)dW_{12}(t)\\
dR_{23}(t) &= \mu_{23}R_{23}(t) + \sigma_{23}R_{23}(t)dW_{23}(t)\\
dR_{31}(t) &= \mu_{31}R_{31}(t) + \sigma_{31}R_{31}(t)dW_{31}(t),
\end{aligned}
$$

where $R_{ij}(t)$ is how many units of currency i we can buy with 1 unit of currency j at time t. Of course, the 3 Brownian motions are not independent (another way to write the above system is to express the noise of each FX rate as the weighted sum of 3 independent Brownian motions; these weights correspond to the correlation between each rate and motion). Furthermore, due to no-arbitrage, the third motion is redundant, since it is completely determined by the other two. Finally, although technically the exchange rates are not assets, the underlying assets of our model are the domestic and foreign money market accounts, which depend on the corresponding interest and exchange rates

In order to calculate the IRR of our portfolio, we will need to be able to sample from the joint distribution of the 3 Brownian motions (in fact, just the 2 exchange rates associated to USD will be enough in our case, since we never exchange between Euro and GBP). In order to sample from the joint distribution, we need the correlation between the involved Brownian motions; from no-arbitrage we have:

$$
\begin{aligned}
R_{23}(t) &= R_{13}(t)\cdot R_{21}(t)\\
R_{23}(0)e^{(\mu_{23}-\sigma_{23}^2/2)t+\sigma_{23}W_{23}(t)} &= R_{13}(0)e^{(\mu_{13}-\sigma_{13}^2/2)t+\sigma_{13}W_{13}(t)}\cdot R_{21}(0)e^{(\mu_{21}-\sigma_{21}^2/2)t+\sigma_{21}W_{21}(t)}\\
e^{(-\sigma_{23}^2/2)t+\sigma_{23}W_{23}(t)} &= e^{(-\sigma_{13}^2/2)t+\sigma_{13}W_{13}(t)}\cdot e^{(-\sigma_{21}^2/2)t+\sigma_{21}W_{21}(t)}\\
(-\sigma_{23}^2/2)t+\sigma_{23}W_{23}(t) &= (-\sigma_{13}^2/2)t+\sigma_{13}W_{13}(t)+(-\sigma_{21}^2/2)t+\sigma_{21}W_{21}(t)\\
\mathrm{Var}\left[(-\sigma_{23}^2/2)t+\sigma_{23}W_{23}(t)\right] &= \mathrm{Var}\left[(-\sigma_{13}^2/2)t+\sigma_{13}W_{13}(t)+(-\sigma_{21}^2/2)t+\sigma_{21}W_{21}(t)\right]\\
\mathrm{Var}\left[\sigma_{23}W_{23}(t)\right] &= \mathrm{Var}\left[\sigma_{13}W_{13}(t)+\sigma_{21}W_{21}(t)\right]
\end{aligned}
$$

Now, we know that $W_{ij}(t)$ is a normally distributed random variable with 0 mean and variance t, for all i, j. We will assume the correlation between any two involved Brownian motions is constant over time; using the sum of variances rule we get:

$$
\begin{aligned}
\sigma_{23}^2t^2 &= \sigma_{13}^2t^2+\sigma_{21}^2t^2+2\rho_{13,21}\sigma_{13}\sigma_{21}t^2\\
\sigma_{23}^2 &= \sigma_{13}^2+\sigma_{21}^2+2\rho_{13,21}\sigma_{13}\sigma_{21}\\
\rho_{13,21} &= \frac{\sigma_{23}^2-\sigma_{13}^2-\sigma_{21}^2}{2\sigma_{13}\sigma_{21}}.
\end{aligned}
$$

Since we have already calculated the volatilities, we now have all the necessary information to calculate the IRR of our portfolio. To do so, we simulate the involved stohastic processes using Monte Carlo simulation.

Hedging After calculating the distribution of the IRR, our next task is to insure our returns in case they are observed to be on the lower tail of the IRR curve. We focus on the lower 5th percentile. As already mentioned, we will use barrier options in order to hedge against this risk. The main reasoning behind our hedging strategy will be the following: since we have a series of independent cash flows, the two strategies that seem most reasonable are either to try and cover the risk of each cash flow separately by constructing one option for each cash flow, or to construct a singular option that insures our whole investment. We choose the first idea, mainly because it makes it conceptually easier to choose the barrier and the strike of each corresponding option.

Let us expand a bit more, by performing a thought experiment where we only have positive cash flows returning at a single point in time t. The eventual IRR is a function of the two involved exchange rates (call them A_t and B_t). Since the IRR is a function of both A_t and B_t, the more principled approach would be to look into the joint distribution of the two exchange rates and identify the part of the probability space that corresponds to the lower 5th percentile of the IRR. However, in order to make full use of the fact that we can express the IRR as a function of both exchange rates, we would have to issue barrier options that become activated when both A_t and B_t reach a certain threshold. Although such options can be evaluated in theory, they tend to be too exotic. Observe however, that if the joint distribution as given to us, using such options we could achieve a perfect hedge.

Hence, we choose to look at each exchange rate individually (this does not mean that we assume they are independent; we will come back to this point). Then, our task becomes conceptually straightforward: since we look at the IRR as a function of e.g. A_t, and since the IRR is a monotone function of the rate, then insuring against the 5th percentile of the IRR is equivalent to insuring against the 5th percentile of A_t. Furthermore, since under our model A_t is essentially a time-dependent deterministic function times the exponential of a normal random variable, we conclude that we have to insure ourselves against the event of the aforementioned normal variable being more than 2 standard deviations away from the mean.

More concretely, let's assume $A_t = e^X$ with $X \sim N(0, \sigma^2 t)$ (the time-dependent part of A_t is important for option pricing, but not for the intuition we want to give here). Then, in order to manage the risk of the IRR (again, assume A_t is how much domestic currency 1 unit of foreign currency buys, i.e., large A_t means large IRR), we issue an option with barrier equal to $e^{-1.6\sqrt{t}\sigma}$[1]. Next, we have to choose the strike: we choose strike equal to $e^{1.6\sqrt{t}\sigma}$, and since $1 + x$ is a good approximation for e^x when x is around 0, choosing nominal value equal to half the cash flow at time t means that the combined cash flow is exchanged at a rate of approximately 1[2] (this approximation deteriorates as t gets larger but not too much since σ is relatively very small; we could get exactly 1 in a straightforward manner but we find this level of precision beyond

[1]Observe that $\Pr[N(0, \sigma^2) \geq 1.6\sigma] \approx 0.05$.
[2]Observe that 1 is the starting rate in this example.

the scope of this study, and we find the symmetry to be quite intuitive and, perhaps more importantly, pleasing).

Notice how barrier options are the right instrument for our purposes; buying and selling appropriate amounts of at-the-money options could give a perfect hedge, but that is not what we are looking for. Instead, we minimize our losses in the lower tail of the IRR distribution, at the cost of our returns at the upper tail of the IRR distribution (we have to sell a symmetric option in order to finance buying the original one). Moreover, observe that, while the choice of the barrier is clearly determined by the percentile of the exchange rate distrbution against which we want to cover our returns, there is no single way to determine the strike. The logic behind the choice we made is that we want to improve the lower 5th percentile of the IRR distribution at the expense of the upper 5th percentile distribution (since after activation, we are left with an option that pays out a positive amount of cash in any case except for when we fall into the upper 5th percentile). However, we are always free to tinker with the strike in order to shape the final IRR distribution in a different way.

Indeed, while the above choice of barriers and strikes serves to explain our train of thought, this is hardly the best choice. Indeed, our strategy addresses only IRR values at the 5% level before hedging; even if we set the probability of every such value to 0, the previous 10% level will become the new 5% level. Thinking along these lines, one can see that the best choice of barriers would be such that we address the lower 25% tail. Indeed, we are able to verify this empirically.

Finally, let us come back to the discussion about examining the two rates independently. Surely, we lose some efficiency by not issuing options conditioned on the joint distribution, but the fact that we buy and sell symmetric *pairs* of options helps mitigate this inaccuracy. In informal terms, if the two rates are extremely positively correlated then we lose on both pairs of options or win on both pairs of options (since the normal distribution is symmetric, win and loss are equiprobable). If the two rates are extremely anti-correlated, whenever we win on one pair of options we lose on the other; and if the two rates are more or less independent then our framework of looking at them individually was right to begin with. This point can be extended to argue about why considering each cash flow independently is also not a bad idea: surely we lose some value by not considering the joint distribution of the different $\{A_t\}_{t\geq 0}$, but since there is considerable correlation between the IRR dropping below the 5th percentile and many of our barriers being activated that loss is not significant.

To conclude, let us formally describe the options we construct (the choice of barrier and strike values, as well as the nominal value, will vary in practice). For a given cash flow F_T (in foreign currency) and a given exchange rate $R(t)$ with

$$R(t) = R(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)},$$

we construct and buy a knock-in barrier option (checked daily) with barrier $B_1 = R(0)e^{(\mu-\sigma^2/2)T}e^{-1.6\sigma\sqrt{T}}$, which after activation becomes a European put option (right to sell foreign currency at a fixed rate) with strike $K_1 =$

$R(0)e^{(\mu-\sigma^2/2)T}e^{1.6\sigma\sqrt{T}}$, expiry at T and nominal value ideally set to $F_T/2$; this will not be always possible since we want the nominal values of both options to be at most F_T. The payoff at maturity of this put option (in domestic currency) is

$$\max\{0, K_1 - R(T)\} \cdot F_T/2.$$

Symmetrically, we construct and sell a knock-in barrier option (checked daily) with barrier $B_2 = R(0)e^{(\mu-\sigma^2/2)T}e^{1.6\sigma\sqrt{T}}$, which after activation becomes a European call option (right to buy foreign currency at a fixed rate) with strike $K_1 = R(0)e^{(\mu-\sigma^2/2)T}e^{-1.6\sigma\sqrt{T}}$, expiry at T and nominal value set so that the values of the two options sum up to 0; if that would lead the two nominal values to exceed F_T, then we modify them accordingly. The payoff at maturity of this call option (in domestic currency) is

$$\max\{0, R(T) - K_2\} \cdot N_T/2,$$

where N_T is a nominal value we choose so that the total value (at time 0) of buying and selling the two options adds up to 0.

Since the values of the two options are highly path-dependent, we evaluate them using Monte Carlo simulation. An interesting note is that we do not actually need to evaluate the two options, but rather the ratio of their values; this could be useful in more complicated models with complicated numeraires. An honorable mention should be made in favor of using time-dependent barriers such as $B(t) = R(0)e^{(\mu-\sigma^2/2)t}e^{-1.6\sigma\sqrt{t}}$, since such an approach, if carefully implemented, could lead to pairs of options whose values trivially add up to 0, thus completely removing the need to actually evaluate them.

3 Constant interest rates - Time-dependent volatility

Next, we introduce our second model. In the first one, we assumed the volatility of the underlying rates to be constant. However, this assumption is far from realistic, as is observed in reality. To remedy this, we will construct a local volatility model for the exchange rates, i.e., a model where the volatility is a deterministic function of time:

$$dR(t) = \mu R(t) + \sigma(t)R(t)dW(t).$$

The main purpose of this model is to capture the phenomenon of the volatility surface not being flat. Another alternative would be to use a fully stohastic volatility model; however, the complications arising when trying to compute option prices in closed form and the resulting complications in calibrating are beyond the scope of this study.

Since $\sigma(t)$ is not given to us explicitly, we will have to infer it from the given data. We are given implied volatilities on a range of maturities; due to our fund's cash flow structure, we focus on the first 5 years. The number of the maturities we will use is limited by the numerical stability of the process of minimizing the mean squared error of observed and predicted prices. If we are given n maturities, we will approximate the real $\sigma(t)$ with a polynomial of degree at most n, with the actual choice being made with numerical stability in mind; Taylor's theorem implies that this is a decent aproximation for the real $\sigma(t)$. Of course there are other basis on which $\sigma(t)$ can be analyzed (e.g. the Fourier basis). To sum up, given n implied volatilities, we approximate $\sigma(t)$ as follows:

$$\sigma(t) \approx p_0 + \sum_{1 \leq i \leq n} p_i x^i.$$

During calibration, we will fit the above coefficients such that predicted option prices are as close to the observed ones as possible. Solving the SDE underlying our model is basic:

$$R(T) = R(0)e^{mT - \int_0^T \sigma^2(t)/2dt + \int_0^T \sigma(t)dW(t)} = R(0)e^{mT - \int_0^T \sigma^2(t)/2dt} e^{N(0, \int_0^T \sigma^2(t)dt)}.$$

Observe that evaluating option prices with maturity T with respect to this model is equivalent to evaluating option prices in our original model with volatility

$$\sqrt{\int_0^T \sigma^2(t)dt}.$$

Furthermore, under the risk-neutral measure associated with the domestic numeraire, the drift is still $\mu = r_d - r_f$, and the values of domestic and foreign money market accounts are still martingales. Finally, the correlation coefficient between the two involved Brownian motions is time-dependent, but determining it at any point in time stays the same, since at any point in time we still have to evaluate the correlation of two normal random variables, given their variances and the variance of their sums; the only difference is that now we have different variances and correlations over time. What will change with our new approach (apart from the process of evaluating the volatility) will be the barriers and strikes, since now an option with expiry T will have its barrier and strike set using a formula similar to before, with $\sqrt{\int_0^T \sigma^2(t)dt}$ replacing $\sigma\sqrt{T}$.

Having completed the description of our models, we refer to the accompanying material for the implementation and empirical evaluation of our models.

4 Concluding remarks

We would like to close this report with a few remarks:

- comparing the two models: unfortunately, the sample size is far too small to draw conclusions with statistical significance when we test the two models on historic data, and even if both models are considered accurate representations of reality, their relative performance on the given data reduces to the question of which model's volatility curve is closer to the realized one. That being said, one has to note that the second model almost always achieves better average squared error with respect to historic option prices.

- other potenial models: the models we presented here only had uncertainty coming in the form of explaining the evolution of the interest rates as geometric Brownian motions. We also implemented a model where the involved interest rates are stohastic and exhibit mean-reversion. The numeraire for this model would be the domestic zero-coupon bond price as a function of time, and calibration was done with respect to the yield curve we were provided. However, we did not include this model as it was not clear whether the extra uncertainty introduced was not already present in the form of the uncertainty of the exchange rates. Ideally, we would prefer to have a model with stohastic volatility; however calibrating that model would not be accurate, since we are only provided with a slice of the empirical volatility surface.

- barrier option considerations: as we already mentioned, we believe that ideally we should construct options with time-dependent barriers. The advantage of such a strategy would be two-fold: on the one hand we would be able to translate our intuition exactly into the barrier values, and on the other hand we could have saved a lot of computational time. To see this, observe that with our present barriers we have to sample a certain number of realizations of the involved Brownian motions, and based on that we get the discounted payoffs of the barrier options. With time-dependent barriers, our task would reduce to sampling the maximum of these motions over time, which is straightforward.

- computational issues: as things stand, the bulk of the computational burden lies on evaluating the barrier options. The main problem is the number of the options that have to be evaluated, combined with the fact that we for each such option we have to sample a sufficiently large number of realizations of the involved motions for every day over 5 years. If we consider the number of options, then one idea would be to only use put options; in this case we would not have a self-financing strategy, but we would have to pay a premium at day one. Apart from this idea, it is hard to imagine how we could reduce the number of options and have the same effect, without making the options too exotic. For example, we could write options on the IRR itself, and computing their fair value would not be difficult; it would be difficult however to trade them. Putting the number of options aside, one way to reduce the computational burden would be to use options with continuous monitoring, and try to derive closed-form prices. Finally, one other idea would be to price the barrier options by reusing the sampled motions; one should be careful when playing with fire however, since this idea would increase the variance of our error drastically.