Spatial Statistics
Point Process Data Unit 2

PM569 Spatial Statistics

Lecture 8: October 23, 2015

Review of point processes

- Simple stochastic models for point patterns do not have tractable distributions
- To test models against data we use Monte Carlo tests (simulation-based)
- Monte Carlo steps:
  - Let u_1 be the observed value of a statistic U
  - Let u_i be the values of the statistic U generated by independent random sampling from the distribution of U under a simple hypothesis H_0 (the null hypothesis)
  - Let $u_{(j)}$ denote the jth largest among the u_i, $i = 1, ..., s$
  - Then, under H_0, $P\{u_1 = u_{(j)}\} = s^{-1}, j = 1, ..., s$ and rejection of H_0 on the basis that u_1 ranks kth largest or higher gives an exact one sided test of size k/s

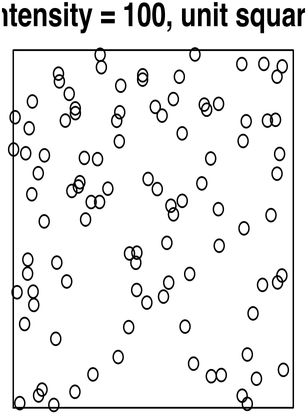
Review of point processes

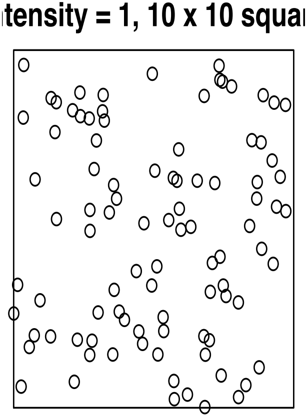
- Monte Carlo methods are not precisely replicable since they rely on simulated data
- An independent set of s simulated realizations will result in a different estimated p-value than the first set of realizations
- The larger number of simulations the more stable the resulting estimates
- We use Monte Carlo to test whether our observations are CSR, Inhomogeneous Poisson process, cluster process, regular process

Review of point processes

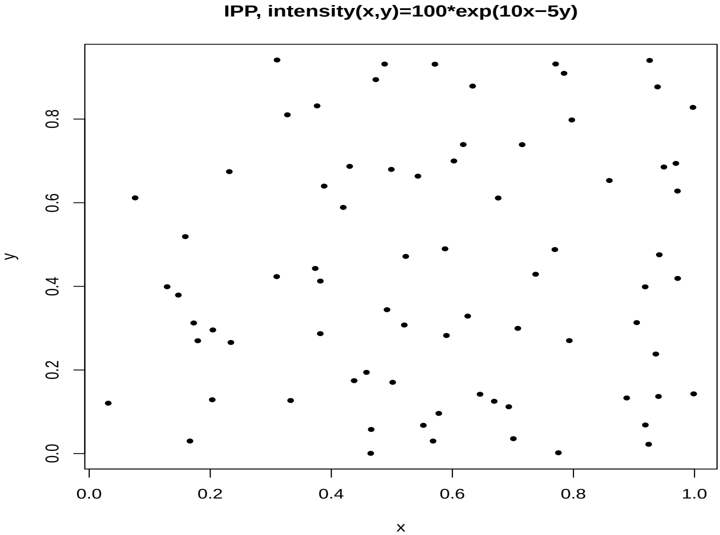
- Testing for CSR
  - Adjusting for edge effect
  - Testing for CSR with Ripley's K
  - Testing for CSR based on inter-event distances, H(h)
  - Testing for CSR based on nearest-neighbour distances, G(h)
- Spatial processes, Poisson processes are the building block
  - Homogeneous Poisson process (constant intensity)
  - Inhomogeneous Poisson process (intensity varies across domain)
  - Poisson Cluster process (intensity varies for parents and/or children forming clusters)
  - Simple inhibition processes, Markovian processes (Strauss and pairwise interaction) for regular patterns

Homogeneous Poisson Process (CSR)

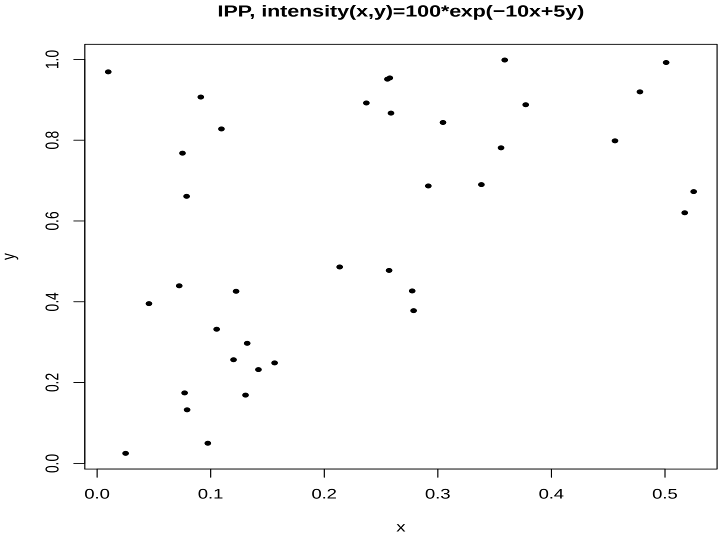




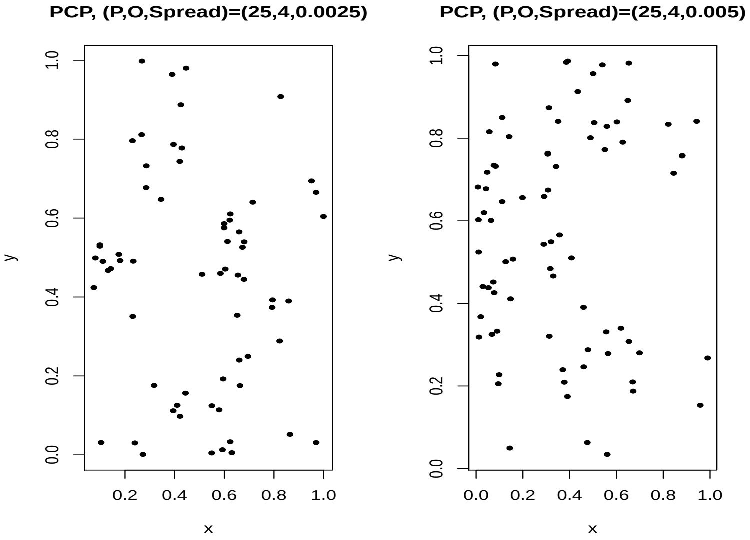
Inhomogeneous Poisson Process



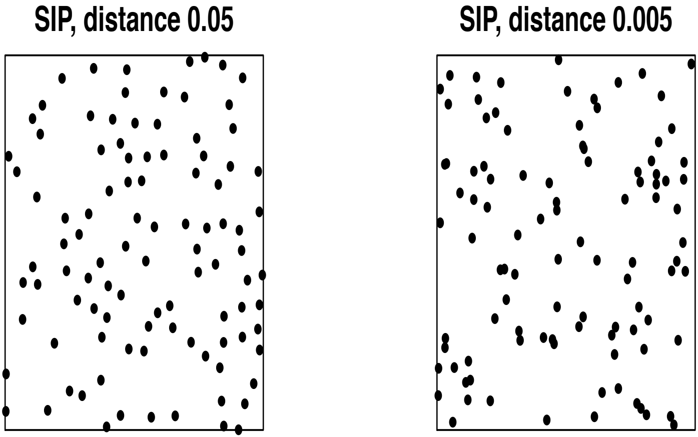
Inhomogeneous Poisson Process



Poisson Clustered Process



Simple Inhibition Process



Point Pattern Data

- Inhomogeneous Point Processes, where the intensity, λ, is not constant.
- Properties of a spatial point process in terms of the intensity function.
  - First order properties are described by the intensity function.

$$\lambda(x) = \lim_{|dx| \to 0} \frac{E[N(dx)]}{|dx|}$$

  - The first order properties are the mean properties of the random process, describing the expcted density of events in any location of the region
  - Clusters appear in areas of high intensity
  - Under IPP and CPC, clusters occur due to heterogeneities in the intensity function and individual event locations remain independent of one another

Point Pattern Data

- First order properties are described by the intensity function
- Example: consider the constant risk hypothesis
  - Each person has the same risk of disease, but we expect more cases in areas with more people at risk
  - Clusters of cases in high population areas will violate CSR but not the constant risk hypothesis
  - We are interested in clustering of disease events after accounting for known variations in population density
  - This requires a generalization of the intensity where we define it as a spatially varying function over the study area
  - As population size increases, so should the expected number of cases

Inhomogeneous Poisson Process intensity function

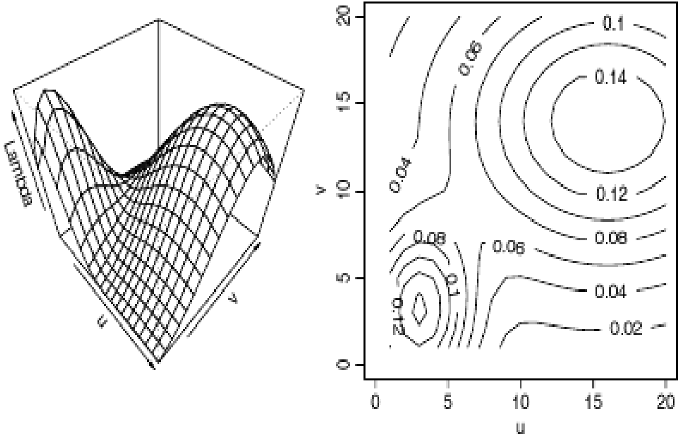


FIG. 5.5 Example intensity function, $\lambda(s)$, for a heterogeneous Poisson point process defined for $s = (u, v)$ and $u, v \in (0, 20)$.

Point Pattern Data

- The inhomogeneous Poisson process shows lack of events between the modes
- More events around the mode (16,14) and a narrower peaked area around (3,3)
- Collections of events suggest areas of higher intensity
- Single realizations make it hard to identify the specific areas of these modes
- Useful to simulate multiple realizations of the process

Point Pattern Data

- Second order properties are described by the inter-relationships between events

$$\lambda(x, y) = \lim_{|dx|,|dy| \to 0} \frac{E[N(dx)N(dy)]}{|dx||dy|}$$

- This allows us to describe how often events occur within a given distance of other events
- The second order properties are similar to variance/covariance of the process
- Allows us to summarize the spatial dependence between events over a wide range of possible spatial scales
- The Ripley's K function is a second-order statistic

Point Pattern Data

- Recall the K function for distance h:

$$K(h) = \frac{E[\# \text{ events within h of randomly chosen event}]}{\lambda}$$

- The second order properties gives us insight into the global aspects of the point pattern
- Are there general patterns of clustering or regularity with respect to CSR or another pattern?

Cox processes

- Spatial clustering with a spatially varying intensity function of the inhomogeneous Poisson process
- Varying $\lambda(x)$ and $\lambda(x)$ is a realization of a stochastic process
- Property 1) it is a non-negative valued stochastic process

$$\{\Lambda(x); x \in \Re^2\}$$

- Property 2) the events for an inhomogenous poisson process with intensity function $\lambda(x)$

$$\{\Lambda(x) = \lambda(x); x \in \Re^2\}$$

Cox processes

- The Cox process is homogeneous iff $\Lambda(x)$is homogeneous:

$$E[\Lambda(x)] = \lambda \forall x$$

$$E[\Lambda(x)\Lambda(x+h)] \text{depends only on} ||h||$$

Point Pattern Data

Cox processes

- The Cox process is linked to the clustered Poisson process
- Aggregation into clusters may be a result of environmental heterogeneity
- Clusters of events in regions of high intensity
- Cox processes are considered doubly stochastic, intensity is heterogeneous but also may be a random quantity
- $\lambda(x)$ can be drawn from some probability distribution of possible intensity functions over the study area

Cox processes

$$\Lambda(x) = \mu \sum_{i=1}^{\infty} h(x - X_i)$$

- $\mu > 0$, $h(\cdot)$ is a bivariate pdf, and X_i are points from a Poisson process
- The Cox process can also be thought of as a specific case of a Poisson cluster process with number of offspring having intensity μ and dispersion around parents with pdf $h(\cdot)$

Cox processes

- The log-Gaussian Cox process is another form of the Cox process

$$\Lambda(x) = \exp(Z(x))$$

- $Z(x)$ is a Gaussian process.
- If $Z(x)$ is stationary with mean μ, variance σ^2 and correlation $\rho(h)$:
    - $\lambda = \exp(\mu + 0.5\sigma^2)$
    - $\gamma(h) = \exp(\sigma\rho(h))$
- The log-Gaussian Cox process can be fit in R spatstat with the rLGCP() function

Simple Inhibition Process

- This process is used to describe regular patterns
- Often related to interactions or contagions where the occurrence of an event raises or lowers the probability of subsequent events nearby
- Useful for modeling the spread of infectious disease (contagion) or an application where an event precludes the occurrence of other events in a nearby area such as animal territories (inhibition)
- **Contagion** typically refers to the increased likelihood of events occurring near other events
- **Inhibition** may be absolute, where there is a specified distance around which *no* other events may occur, or it may be probabilistic where there is small but positive probability of an event occurring near other events

Simple Inhibition Process

- Models for inhibition or contagion processes are Markov point processes or Gibbs processes
- The general idea is to take a CSR and "delete" points within a distance less than a threshold δ
- Under a Markov process, the existence of an event in a region depends on the locations of events in a neighbourhood (where neighbourhoods are within regions)
- There are two ways to do this: 1) to simulate CSR then delete all within a distance δ, and 2) to simulate CSR, record when event was simulated, then delete an event if it is within distance δ of an older event

Simple Inhibition Process

- We use the packing intensity to describe simple inhibition processes:

$$\tau = \lambda\pi(\delta/2)^2$$

Where λ is the intensity, giving τ to be the proportion of the region A covered by non-overlapping discs of diameter δ

Simple Inhibition Process

- For simple inhibition process 1) we take a a Poisson process with intensity ρ and thin it by the deletion of pairs of events that are less than δ apart
- In this case, the probability that an event "survives" is $\exp(-\pi\rho\delta^2)$ giving the intensity of a simple inhibition process as:

$$\lambda = \rho \exp(-\pi\rho\delta^2)$$

- The second order properties can be expressed as:

$$\lambda(h) = \rho^2 \exp(-\rho U_\delta(h)) \qquad\qquad h \geq \delta$$

- $\lambda(h) = 0$ when $0 < h < \delta$, and $U_\delta(h)$ is the area of the union of two discs with equal radius δ and centers distance h apart

Simple Inhibition Process

- For simple inhibition process 2) we take a a Poisson process with intensity ρ and thin it by the deletion of pairs of "older" events that are less than δ apart
- The expressions are the same as for process 1) but with the addition of the sequential piece (this process is referred to as the simple sequential inhibition process)
- Let X_i be a sequence of n events in A, and $d(x, y)$ be the distance between two points x and y. Then:
  - X_1 is simulated from a uniform distribution in A
  - Given (past) $\{X_j = x_j, j = 1, ...(i-1)\}$, then X_i (present) is uniformly distributed on the intersection of A with $\{y : d(y, x_j) \geq \delta j = 1, ...(i-1)\}$
- So the simple sequential inhibition process has packing intensity:

$$\tau = \frac{n\pi(\delta/2^2)}{|A|}$$

Point Pattern Data

- In R (spatstat), the functions for thinning processes 1) and 2) described above are called rMaternI and rMaternII
- The simple sequential inhibition process, called rSSI is similar but slightly different:
  - Each new point is generated uniformly in the window and independently of preceding points
  - If a new point lies within distance δ from an existing point then it is rejected and another random point is generated
  - The SSI process ends when no points can be added

Markov point processes

- The general idea of a Markov point process lies in conditioning, whereby the existence of an event in a finite region A depends on the locations of events in a neighbourhood
- Inhibition processes are a special form of Markov process: the conditional intensity of an event at a point x given the realization of the process in the remainder of the region A depends on the existence (or otherwise) of an event within distance δ of x
- General Markov processes were introduced by Ripley and Kelly (1977)
- Markov point processes are characterized by the likelihood ratio with respect to a Poisson process of unit intensity

Markov point processes

- Let's call the likelihood ratio $f(\cdot)$
- If $\mathbf{X} = \{x_1, ..., x_n\}$ denotes a finite set of points in A then $f(\mathbf{X})$ indicates how much more likely is the configuration of events $\mathbf{X}$ than a homogeneous point process (with unit intensity)
- We can factorize the likelihood ratio to:

$$f(\mathbf{X}) = \alpha \prod_{i=1}^{n} g_i(x_i) \prod_{j>i} g_{ij}(x_i, x_j) ... g_{12...n}(x_1, x_2, ..., x_n)$$

- Where α is a normalizing constant
- We also define two points x and y in A to be neighbours if $d(x, y) < \delta$ for some $\delta > 0$ where $d(x, y)$ is the distance between x and y
- We also define a clique (recall areal data) as a set of mutual neighbours, and the neighbourhood of x to be the set of points $\{y \in A : 0 < d(x, y), \delta\}$

Markov point processes

- The point process with these definitions is Markov with range δ if the conditional intensity at the point x given the configuration of the other events in A depends only on the configuration in the neighbourhood of x
- The g-functions from the above equation are unity *unless* the x form a clique

Examples of Markov point processes: the Strauss process

$$f(\mathbf{X}) = \alpha\beta^n\gamma^p$$

- Where α is the normalizing constant, β is the intensity of the process, γ is the interaction between neighbours, and p is the number of distinct pairs of neighbours in $\mathbf{X}$
- If $\gamma = 1$ then the Strauss process gives a Poisson process with intensity β
- if $\gamma = 0$ then the Strauss process gives a simple inhibition process because no two events may be neighbours
- In R spatstat, the Strauss process is simulated with `rStrauss`

Examples of Markov point processes: the pairwise interaction process

$$f(\mathbf{X}) = \alpha\beta^n \prod_{i \neq j} h\{d(x_i, x_j)\}$$

- Where α is the normalizing constant, β is the intensity of the process, $h(d)$ is non-negative for all distances and the product is over all pairs of distinct points in $\mathbf{X}$
- The additional restriction is that $h(d)$ is bounded and that $h(d) = 0$ for all distances less than some $\delta > 0$
- This restriction limits the number of events in A by imposing a minimum allowable distance δ between any two events
- The pairwise interaction process may be fit in R `spatstat` using the `rmh` function

Examples of Markov point processes: the pairwise interaction process

- The pairwise interaction process may be simulated using the following steps (MCMC):
  1. For the initial realization, consider n points $\{x_1, ...x_n\}$
  2. Delete one of the points in $\{x_1, ...x_n\}$
  3. Generate a point y from a uniform distribution in A, and accept y with probability $p(y)$
  4. Repeat 2-3 until the MCMC converges