Spatial Statistics
Special Topics: Multivariate Spatial and Spatio-Temporal Statistics

PM569 Spatial Statistics

Lecture 11: November 13, 2015

Outline

- Multivariate spatial process models
  - Co-Kriging
  - Cross-Covariance
  - Co-Regionalization
- Spatio-Temporal processes

Multivariate Spatial Processes

We often want to analyze multivariate measurements obtained jointly at a set of spatial locations. For instance, several spatially dependent measures may be made at the same locations, such as monitoring stations measuring multiple pollutants (e.g. $PM_{2.5}$, O_3, and NO_2) or multiple property details in real estate (e.g. house price, number of bedrooms, and property size). For such data we would anticipate:

- association between measurements across locations.
- dependence between measurements within a particular location.

Multivariate Spatial Processes

To address multivariate data we need to develop cross-variograms (or equivalently, cross-covariance functions). If we consider $\mathbf{Z}(\mathbf{s})$, a px1 vector where $\mathbf{s} \in D$ and we seek to capture both the association within components of $\mathbf{Z}(\mathbf{s})$ (e.g. within multiple pollutants) and across locations $\mathbf{s}$, we define the cross-variogram:

$$\gamma_{ij}(h) = \frac{1}{2}E(Z_i(s+h) - Z_i(s))(Z_j(s+h) - Z_j(s))$$

where Z_i and Z_j are two variables measured at the same locations $\mathbf{s}$. The cross-covariance function is defined as:

$$C_{ij}(h) = E(Z_i(s+h) - \mu_i)(Z_j(s) - \mu_j)$$

The connection between the cross-covariance and cross-variogram is:

$$\gamma_{ij}(h) = C_{ij}(0) - \frac{1}{2}(C_{ij}(h) + C_{ij}(-h))$$

Spatio-Temporal Processes

- Many data that we deal with in spatial statistics are actually both spatial and temporal in nature.
  - $PM_{2.5}$ concentrations over the north east, daily for 1 year.
  - Trends in weather, measured at weather stations, over the last 50 years.
  - Housing prices over the last decade.
- We extend the definition of a stochastic spatial process to a stochastic spatio-temporal process:

$$\{Z(s,t) : s \in D(t) \subset \Re^2, t \in T\}$$

- The observation of attribute Z at location s is expanded to it's added temporal "location" at time t.
- The dependence of the domain D on time symbolizes the condition where the spatial domain changes over time.
- For simplicity, we may assume no spatial-temporal interaction, ie that $D(t) = D$.

Space-time semivariograms

- We can extend our semivariogram to include a temporal component. For a stationary process:

$$\begin{aligned}\gamma(h,k) &= \frac{1}{2}Var(Z(s,t) - Z(s+h,t+k)) \\ &= \frac{1}{2}E((Z(s,t) - Z(s+h,t+k))^2)\end{aligned}$$

- We can bin this:

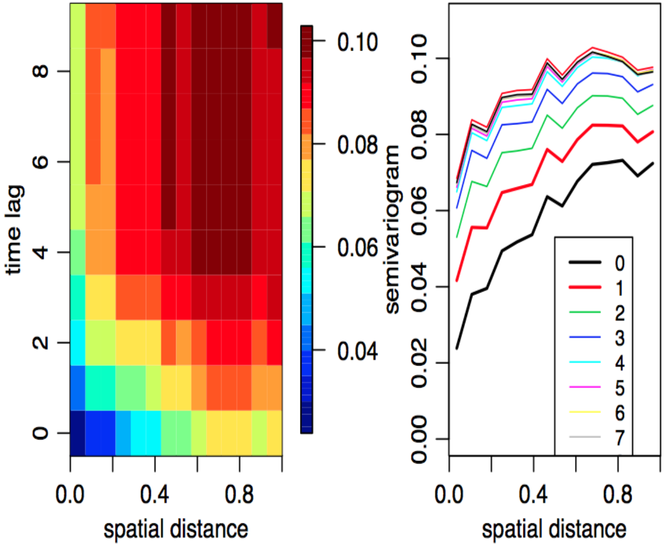
$$\hat{\gamma}(h,k) = \frac{1}{2N(h,k)}\sum_{N(h,k)}(Z(s_i,t) - Z(s_j,u))^2$$

Space-time semivariograms

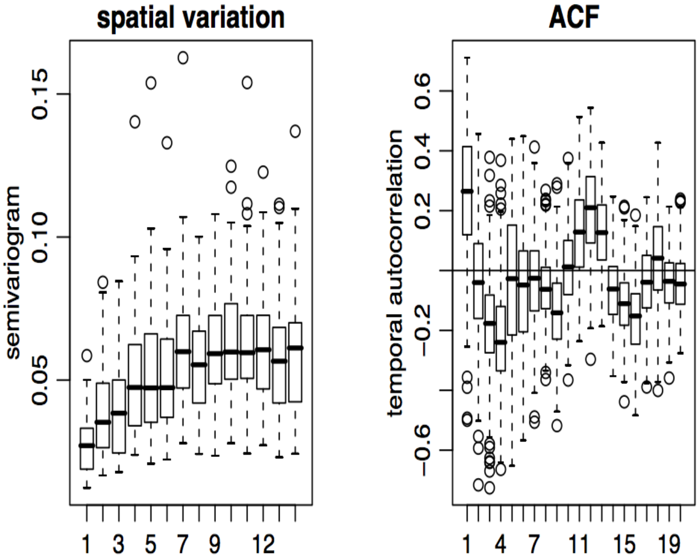
- We have a similar relationship as before between the space-time semivariogram and the space-time covariance
- For a stationary process:

$$\gamma(h, k) = C(0,0) - C(h, k)$$

Space-time semivariograms, $PM_{2.5}$ Example



Space-time semivariograms, $PM_{2.5}$ Example



Spatio-Temporal Processes

- As we saw with spatial data, we can model the spatio-temporal structure in either the covariance or in the mean
- Modeling spatio-temporal structure in the covariance leads to kriging-type formulations
- Most work is in defining valid spatio-temporal covariance models: the classes of these types of models is much more limited than spatial covariance
- The goal of space-time kriging is to make predictions at unobserved space-time points
- Statistical strength is borrowed over space and time to give efficient predictions and standard errors

Space-time Covariance Functions

- The general space-time covariance is:

$$Cov(Z_{it}, Z_{ju}) = Cov(s_i, s_j, t, u)$$

- For a stationary and isotropic process:

$$Cov(Z_{it}, Z_{ju}) = C(h, k)$$

- The challenge is to ensure the covariance functions are positive-definite
- Separable covariance functions are positive definite if their components are positive definite
- Separable product covariance:

$$Cov(Z_{it}, Z_{ju}) = Cov(s_i, s_j)Cov(t, u)$$

- Separable sum covariance:

$$Cov(Z_{it}, Z_{ju}) = Cov(s_i, s_j) + Cov(t, u)$$

Space-time Covariance Functions

- When separable, we ignore any space-time interaction
- This means we cannot capture correlations induced by transport or dynamics which is often seen in environmental processes
- Non-separable covariance is an area of active research, particularly by Stein (UChicago), Fuentes (NC State) and Gneiting (Washington)

Anisotropy and Space-Time

There is a parallel between modeling anisotropy in space through the covariance model and modeling space-time variation through the covariance model.

- Anisotropy: spatial covariance depends on direction, i.e. the covariance in one direction is different from the covariance in another direction
- In terms of space-time covariance we can think of the covariance in space as being different than the covariance in time
- Separable models: use isotropic models as building blocks to create an anisotropic model

Anisotropy and Space-Time

- For modeling anisotropy take the semivariogram, ie

$$\gamma(h) = \frac{1}{2}E((Z(s) - Z(s+h))^2)$$

- And define

$$\gamma(h) = \gamma_0(\sqrt{h'Qh}) = \gamma_0(||h||Q) = \gamma_0(||Ah||)$$

- Where $Q = A'A$ is positive definite and γ_0 is isotropic

Anisotropy and Space-Time

- For example, zonal anisotropy depends only on one component (Chiles and Delfiner, 1999)

$$\gamma(\mathbf{h}) = \gamma_0(h_z)$$

- Where $\mathbf{h} = (h_x, h_y, h_z)$ and we could assume h_z to be the time dimension
- The general model is a variogram defined as the sum of its components with at least one component being zonal. For example:

$$\gamma(\mathbf{h}) = \gamma_1(||h||) + \gamma_2(\sqrt{h_x^2 + h_y^2}) + \gamma_3(|h_z|)$$

Anisotropy and Space-Time

From Chiles and Delfiner, 1999

- The problem with linear models is that the semi-variogram is not always negative definite, for example:

$$\gamma(h) = \gamma_1(h_x) + \gamma_2(h_y)$$

- Which is obtained from a random process such that

$$Z(x, y) = Z_1(x) + Z_2(y)$$

- And the combination becomes

$$Z(x, y) - Z(x, y + v) - Z(x + u, u) + Z(x + u, y + v)$$

- Which is zero

Anisotropy and Space-Time

From Chiles and Delfiner, 1999

- Better to use mixtures which are a linear combination of isotropic semi-variograms

$$\gamma(\mathbf{h}) = \mu_1\gamma_0(|h_x|) + \mu_2\gamma_0(\frac{|h_x + h_y|}{\sqrt{2}}) + \mu_3\gamma_0(|h_y|) + \mu_4\gamma_0(\frac{|h_x - h_y|}{\sqrt{2}}$$

- or product models of the covariance:

$$Cov(h_1, h_2, ..., h_n) = \prod_{i=1}^{n} C_i(h_i)$$

- Which is obtained from a random process (with Z_i's being independent random processes):

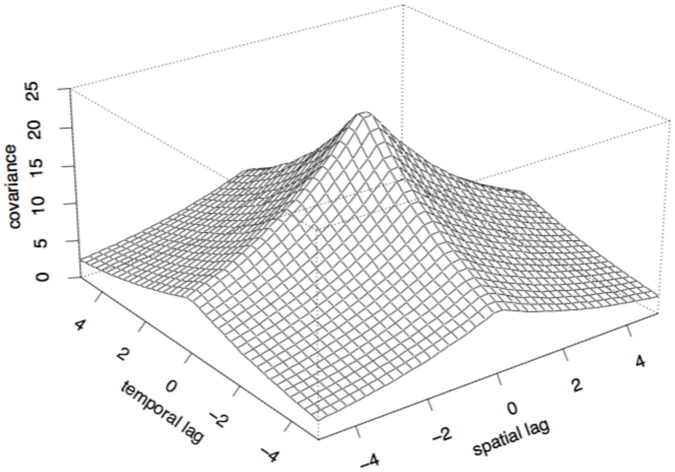
$$Z(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} Z_i(x_i)$$

Separable Space-Time Covariance A separable covariance model assumes separability between the spatial and temporal component, meaning the covariance function can be given by:

$$C(h, k) = C_s(h)C_t(k)$$

The problem is that many physical phenomena do not satisfy separability, so this model needs to be used carefully.

Separable Space-Time Covariance



Non-separable Space-Time

Work by Stein, Cressie-Huang and Fuentes approaches this with spectral methods

- The general models are of the form

$$Cov(\mathbf{x}, t) = \int \exp^{i\mathbf{x}^T\omega} k(\omega) f(\omega, t) d\omega$$

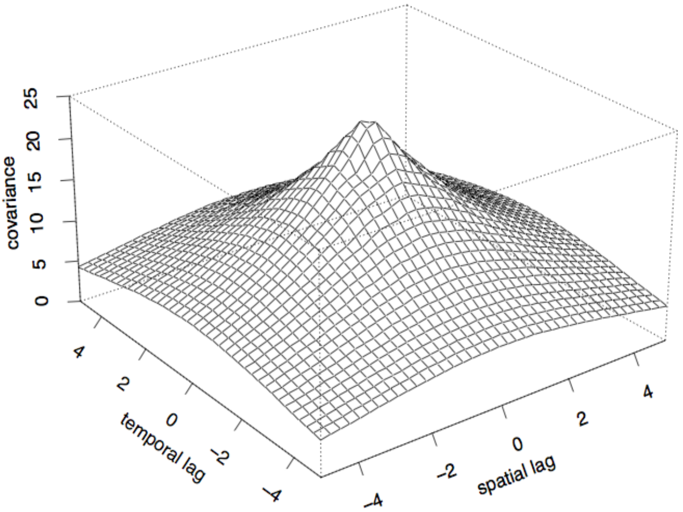
- Where ω is the frequency and for each frequency, $f(\omega, t)$ is a continuous temporal autocorrelation function and $k(\omega) > 0$
- The spectral density, f is the Fourier transform of the spatial-temporal covariance function

$$f(\omega, \tau) = \frac{1}{(2\pi)^{d+1}} \int_{\Re^d} \int_t \exp(-i\omega^T x - i\tau t) Cov(x, t) dx dt$$

- This employs Lebesque measures where the spectral density f satisfies

$$K(x) = \int_{\Re^d} \exp(-i\omega^T x) f(\omega) d\omega$$

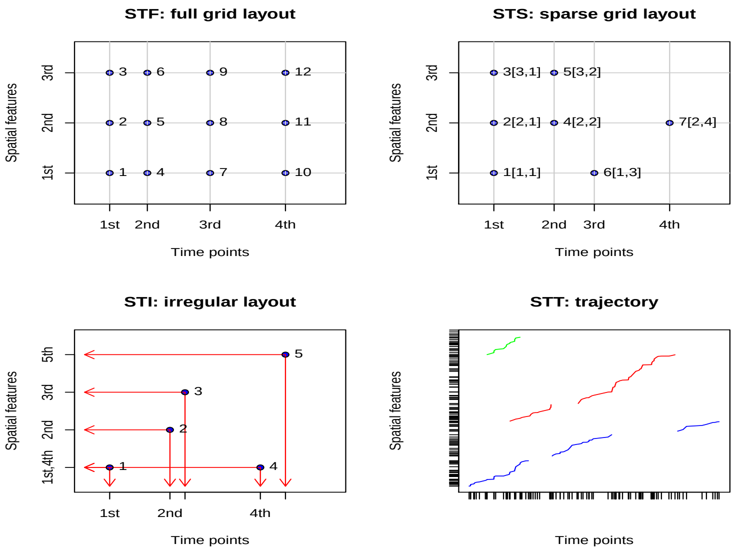
Non-separable Space-Time Covariance



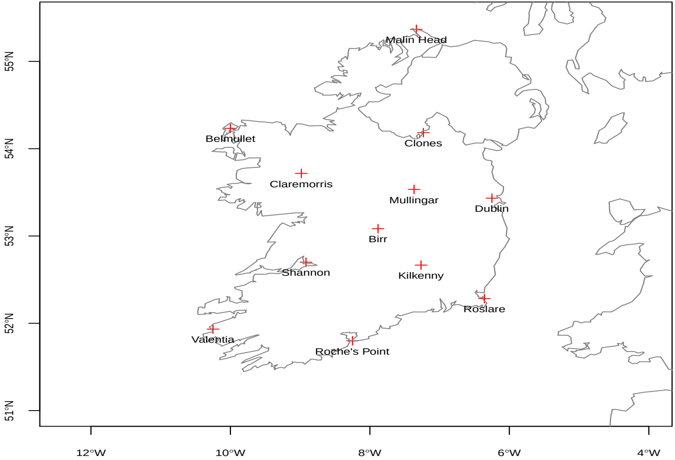
Spatio-Temporal Analysis in R

The R package `gstat` is able to perform spatio-temporal kriging exploiting the functionalities of the package `spacetime`, which was developed by the same team as `gstat`. In `spacetime` we have two ways to represent spatio-temporal data: STFDF and STIDF formats. STFDF objects are a complete space time grid. In other words a spatio-temporal object is created using the n spatial locations at m time points. The spatio-temporal grid is of size $n \times m$.
STIDF objects are irregular spatio-temporal objects, where both space and time change dynamically. For example, data collected on top of trams moving around the city of Zurich. This means that the location of the sensors is not consistent throughout the sampling window.

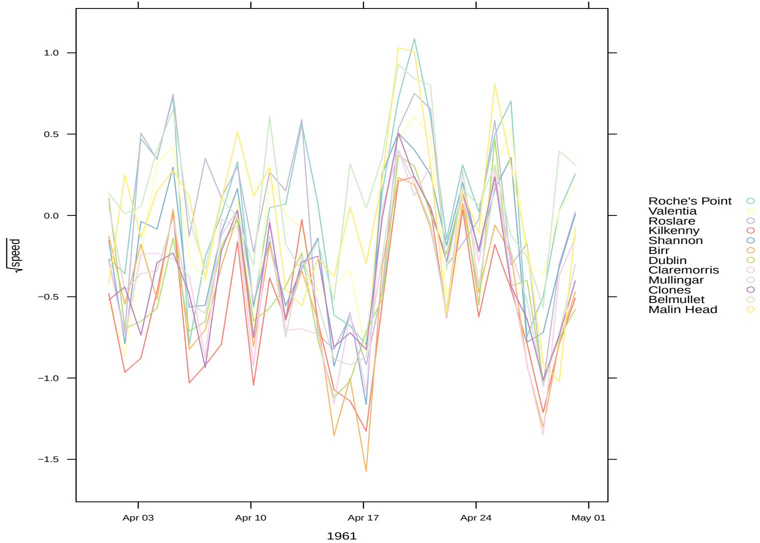
Spatio-Temporal Analysis in R



An example of regular space-time data (STFDF) is daily wind (m/s) measurements at 12 weather stations in Ireland.

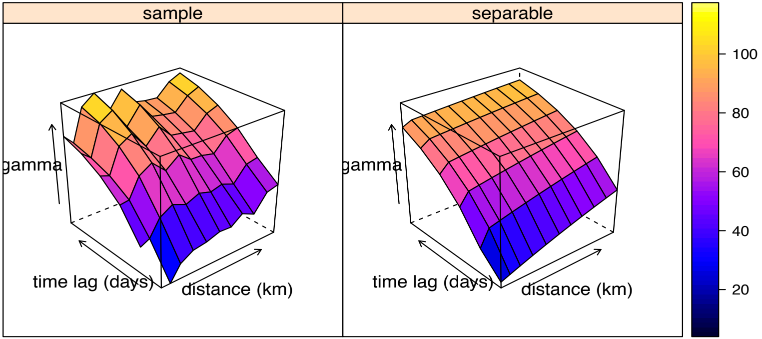


Time series at each location.



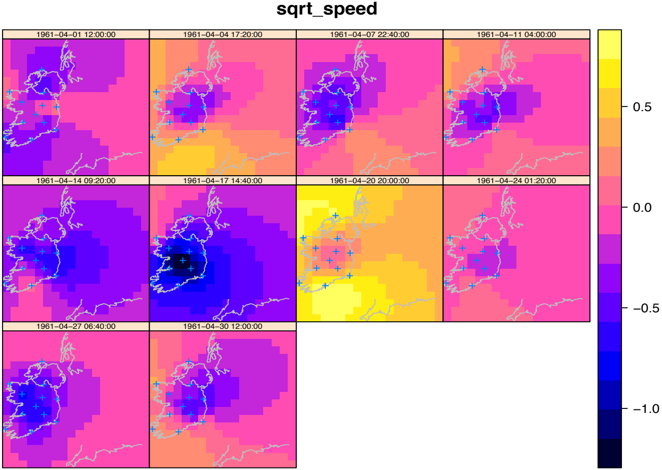
Spatio-Temporal Analysis in R

Applying a space-time variogram using `vgmST()`, we specify a spatial variogram and a temporal variogram:
Spatial: give initial parameters for partial sill, model (exponential, spherical, gaussian, matern), range, nugget. These will be in terms of distance units (e.g. km) Temporal: give initial parameters for partial sill, model (same as above), range. These will be in terms of temporal units (e.g. month, or day, or year)



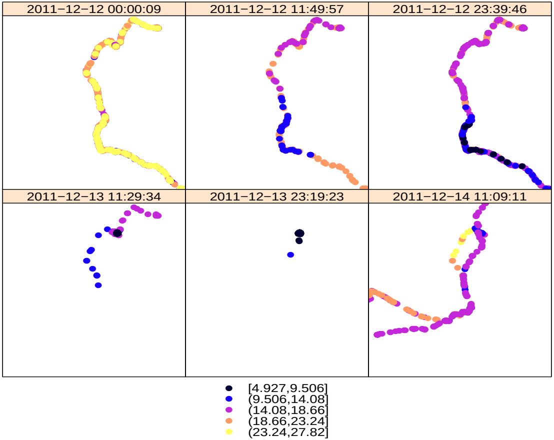
Spatio-Temporal Analysis in R

Using that S-T variogram to perform kriging, we get the following predicted surfaces (for 10 days)

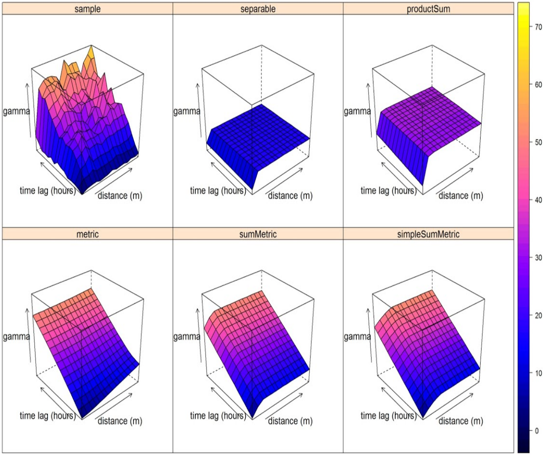


Spatio-Temporal Analysis in R

An example of irregular space-time data (STIDF) is ozone data collected along a path. A monitor was mounted on a train in Zurich, and measured ozone along a route, but the samples were gathered at different dates and times.



Similarly, ST variograms can be fit. Here we look at different types, not only the separable model.



In addition to the separable model, the other ST variograms in the spacetime library are:

- Product Sum: does not assume separability, the function is $C(h,k) = jC_s(h)C_t(k) + C_s(h) + C_t(k)$ where j is a parameter value greater than 0.
- Metric: assumes identical covariance for space and time but includes as spatio-temporal anisotropy $C(h,k) = C(\sqrt{h^2 + (j \cdot k)^2})$.
- Sum Metric: includes spatial and temporal covariance terms and anisotropy $C(h,k) = C_s(h) + C_t(k) + C(\sqrt{h^2 + (j \cdot k)^2})$.

Spatio-Temporal GAM models

- If we model the space-time structure in the mean we can use a spatio-temporal gam model.
- Using the same formulation as before, but adding a smooth for time will give us a separable space-time model but where it is specified in the mean.

$$
\begin{aligned}
Z_{it} = g(s_i) + f(t) + \epsilon_{it} \\
\epsilon \sim N(0, \sigma^2)
\end{aligned}
$$

alternatively time could be represented as a factor, but $f(t)$ gives the flexibility of specifying more complex temporal trends.
In R:

```
mod <- gam(z s(x,y,k=150)+s(t,k=30))
```

Spatio-Temporal GAM models

- We can consider non-separable space-time smoothing using space-time basis functions

$$Z_{it} = g(s_i) + f(t) + \eta(s_i, t) + \epsilon_{it}$$

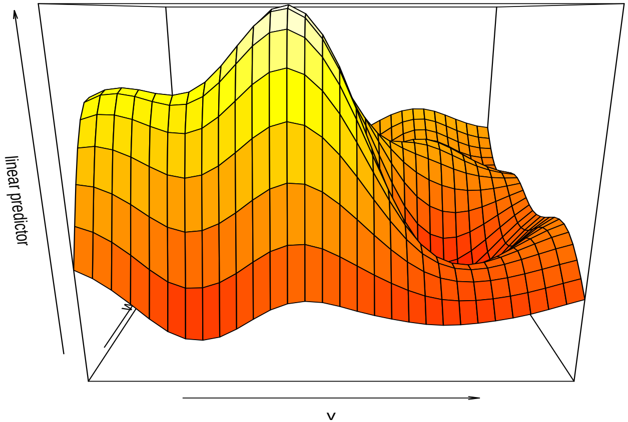
- The basis function η has both space and time in it, so presumably we can multiply a spatial and a temporal basis function together: $b_s(s) \otimes b_t(t)$.
- This cross product basis is known as a tensor-product basis function, and it allows for smoothing over space and time simultaneously.

Spatio-Temporal GAM models

The mgcv package has functions for tensor product bases. They are specified with three variables in the t2 smooth function. Here the spatial smooth is a thin plate spline (as before) and the temporal smooth is a cubic regression spline. The product of the two gives the tensor product.

Spatio-Temporal GAM models

```
b = gam(y
t2(v,w,u,k=c(25,5),d=c(2,1),bs=c("tp","cr"),full=TRUE),
method="ML")
```



Spatio-Temporal Areal Data

- Recall the SAR model

$$Y - X\beta = C(Y - X\beta) + \epsilon$$

  where $C = \rho W$ where W is the spatial adjacency matrix.
- The STAR (spatio-temporal autoregressive model) has a similar form,

$$Y - X\beta = C(Y - X\beta) + \epsilon$$

  where $D = \rho_s S + \rho_t T + \rho_{st} ST + \rho_{ts} TS$, T is the adjacency matrix in time and S is the adjacency matrix in space. D is lower triangular because time proximity is unidirectional.

Spatio-Temporal Areal Data

$D = \rho_s S + \rho_t T + \rho_{st} ST + \rho_{ts} TS$
Particular cases:

- Additive: $D = \rho_s S + \rho_t T$
- Filter for time and then for space: $D = \rho_s S + \rho_t T - \rho_s \rho_t ST$
- Filter for space then for time: $D = \rho_s S + \rho_t T - \rho_s \rho_t TS$
- Linear combination of filters:
  $D = \rho_s S + \rho_t T - w \rho_s \rho_t ST - (1 - w) \rho_s \rho_t ST$