Spatial Statistics
Point Process Data Unit 2

PM569 Spatial Statistics

Lecture 10: November 17, 2017

Point Pattern Data

Review of point processes

- Simple stochastic models for point patterns do not have tractable distributions.
- To test models against data we use Monte Carlo tests (simulation-based).
- Monte Carlo steps:
  - Let u_1 be the observed value of a statistic U
  - Let u_i be the values of the statistic U generated by independent random sampling from the distribution of U under a simple hypothesis H_0 (the null hypothesis)
  - Let $u_{(j)}$ denote the jth largest among the u_i, $i = 1, ..., s$
  - Then, under H_0, $P\{u_1 = u_{(j)}\} = s^{-1}, j = 1, ..., s$ and rejection of H_0 on the basis that u_1 ranks kth largest or higher gives an exact one sided test of size k/s

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Review of point processes

- Monte Carlo methods are not precisely replicable since they rely on simulated data.
- An independent set of simulated realizations will result in a different estimated p-value than the first set of realizations.
- The larger number of simulations the more stable the resulting estimates.
- We use Monte Carlo methods to test whether our observations are a CSR with homogenous or inhomogeneous Poisson process, cluster process, regular process.

Point Pattern Data

Review of point processes

- Testing for CSR:
  - Often want to adjust for edge effects.
  - We test for CSR with Ripley's K, which involves a search window with bandwidth h.
  - We test for CSR based on inter-event distances being less than a threshold δ with H(h).
  - We test for for CSR based on nearest-neighbour distances with G(h).
- The types of spatial processes where the Poisson processes is the building block are:
  - Homogeneous Poisson process (constant intensity), used for testing CSR.
  - Inhomogeneous Poisson process (intensity varies across domain), used for testing CSR.
  - Poisson Cluster process (intensity varies for parents and/or children forming clusters), used for testing clustered patterns.
  - Simple inhibition processes, Markovian processes (Strauss and pairwise interaction), used for testing regular patterns.

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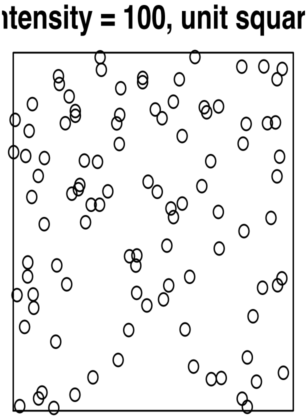
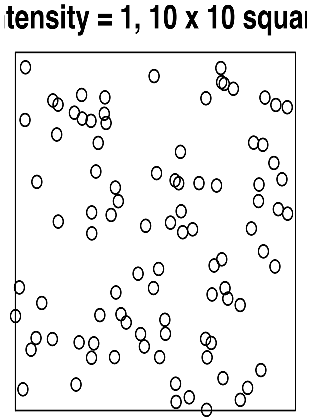
Homogeneous Poisson Process (CSR)





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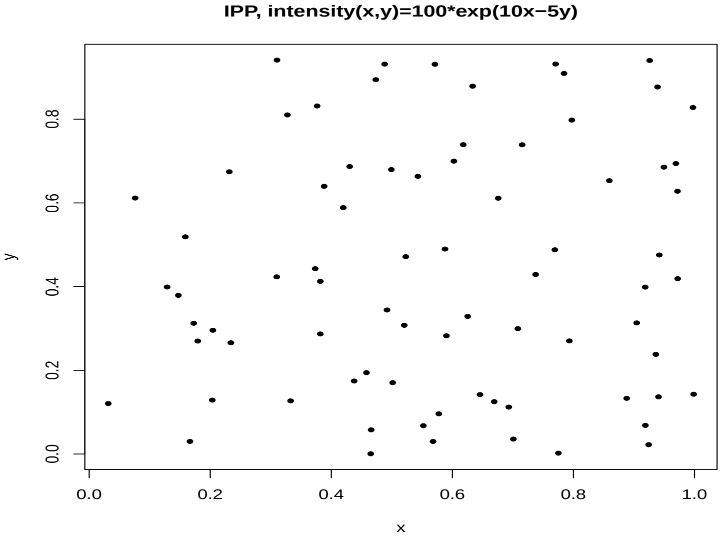
Inhomogeneous Poisson Process



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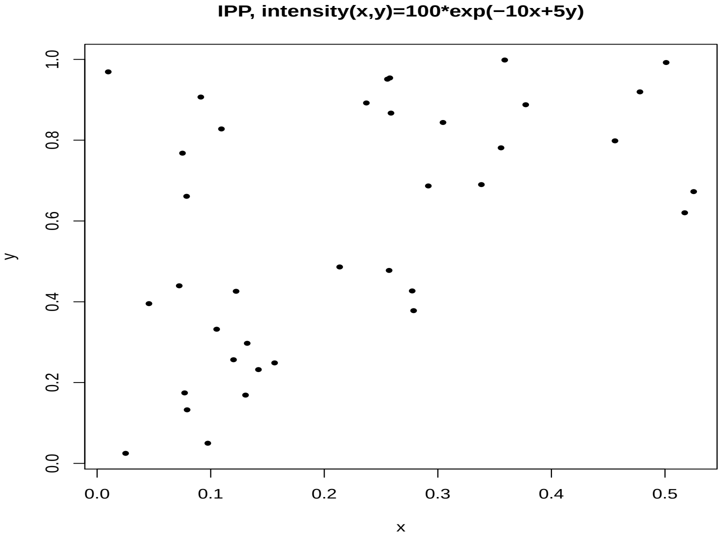
Inhomogeneous Poisson Process



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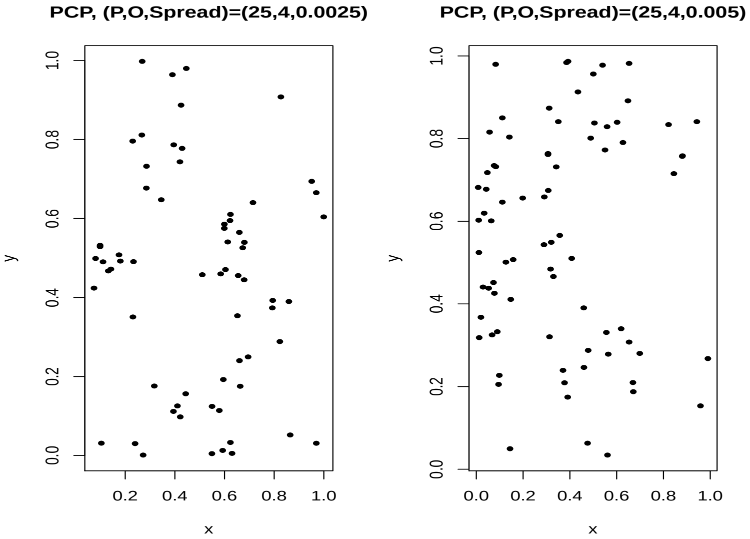
Poisson Clustered Process



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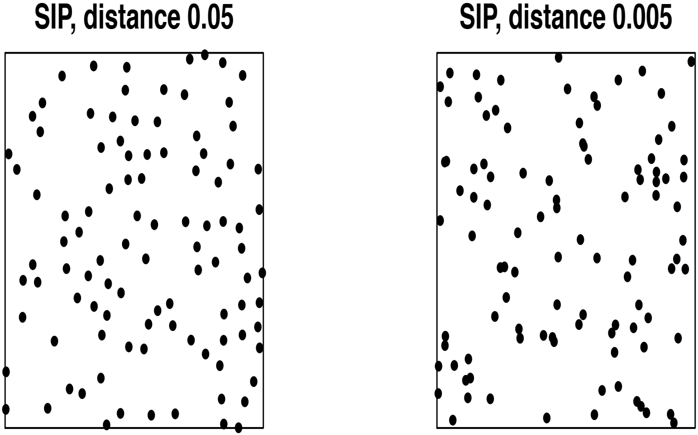
Simple Inhibition Process



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- Inhomogeneous Point Processes, where the intensity, λ, is not constant.
- Properties of a spatial point process in terms of the intensity function.
  - First order properties are described by the intensity function.

$$\lambda(x) = \lim_{|dx| \to 0} \frac{E[N(dx)]}{|dx|}$$

  - The first order properties are the mean properties of the random process, describing the expcted density of events in any location of the region
  - Clusters appear in areas of high intensity
  - Under IPP and CPC, clusters occur due to heterogeneities in the intensity function and individual event locations remain independent of one another

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- First order properties are described by the intensity function
- Example: consider the constant risk hypothesis
  - Each person has the same risk of disease, but we expect more cases in areas with more people at risk
  - Clusters of cases in high population areas will violate CSR but not the constant risk hypothesis
  - We are interested in clustering of disease events after accounting for known variations in population density
  - This requires a generalization of the intensity where we define it as a spatially varying function over the study area
  - As population size increases, so should the expected number of cases

Inhomogeneous Poisson Process intensity function

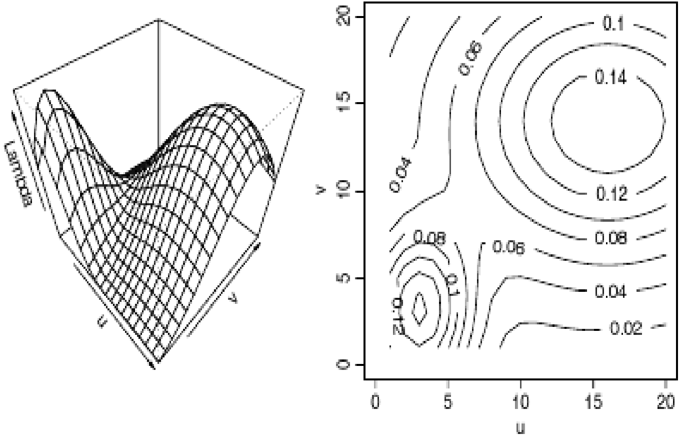

FIG. 5.5 Example intensity function, $\lambda(s)$, for a heterogeneous Poisson point process defined for $s = (u, v)$ and $u, v \in (0, 20)$.

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- The inhomogeneous Poisson process shows lack of events between the modes
- More events around the mode (16,14) and a narrower peaked area around (3,3)
- Collections of events suggest areas of higher intensity
- Single realizations make it hard to identify the specific areas of these modes
- Useful to simulate multiple realizations of the process

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- Second order properties are described by the inter-relationships between events

$$\lambda(x,y) = \lim_{|dx|,|dy|\to 0} \frac{E[N(dx)N(dy)]}{|dx||dy|}$$

- This allows us to describe how often events occur within a given distance of other events
- The second order properties are similar to variance/covariance of the process
- Allows us to summarize the spatial dependence between events over a wide range of possible spatial scales
- The Ripley's K function is a second-order statistic

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- Recall the K function for distance h:

$$K(h) = \frac{E[\# \text{ events within h of randomly chosen event}]}{\lambda}$$

- The second order properties gives us insight into the global aspects of the point pattern
- Are there general patterns of clustering or regularity with respect to CSR or another pattern?

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Cox processes

- Spatial clustering with a spatially varying intensity function of the inhomogeneous Poisson process
- Varying $\lambda(x)$ and $\lambda(x)$ is a realization of a stochastic process
- Property 1) it is a non-negative valued stochastic process

$$\{\Lambda(x); x \in \Re^2\}$$

- Property 2) the events for an inhomogenous poisson process with intensity function $\lambda(x)$

$$\{\Lambda(x) = \lambda(x); x \in \Re^2\}$$

Cox processes

- The Cox process is homogeneous iff $\Lambda(x)$is homogeneous:

$$E[\Lambda(x)] = \lambda \forall x$$

$$E[\Lambda(x)\Lambda(x+h)]\text{depends only on}||h||$$

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Cox processes

- The Cox process is linked to the clustered Poisson process
- Aggregation into clusters may be a result of environmental heterogeneity
- Clusters of events in regions of high intensity
- Cox processes are considered doubly stochastic, intensity is heterogeneous but also may be a random quantity
- $\lambda(x)$ can be drawn from some probability distribution of possible intensity functions over the study area

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Cox processes

$$\Lambda(x) = \mu \sum_{i=1}^{\infty} h(x - X_i)$$

- $\mu > 0$, $h(\cdot)$ is a bivariate pdf, and X_i are points from a Poisson process
- The Cox process can also be thought of as a specific case of a Poisson cluster process with number of offspring having intensity μ and dispersion around parents with pdf $h(\cdot)$

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Cox processes

- The log-Gaussian Cox process is another form of the Cox process

$$\Lambda(x) = \exp(Z(x))$$

- $Z(x)$ is a Gaussian process.
- If $Z(x)$ is stationary with mean μ, variance σ^2 and correlation $\rho(h)$:
  - $\lambda = \exp(\mu + 0.5\sigma^2)$
  - $\gamma(h) = \exp(\sigma\rho(h))$
- The log-Gaussian Cox process can be fit in R spatstat with the rLGCP() function

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Simple Inhibition Process

- This process is used to describe regular patterns
- Often related to interactions or contagions where the occurrence of an event raises or lowers the probability of subsequent events nearby
- Useful for modeling the spread of infectious disease (contagion) or an application where an event precludes the occurrence of other events in a nearby area such as animal territories (inhibition)
- **Contagion** typically refers to the increased likelihood of events occurring near other events
- **Inhibition** may be absolute, where there is a specified distance around which *no* other events may occur, or it may be probabilistic where there is small but positive probability of an event occurring near other events

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Simple Inhibition Process

- Models for inhibition or contagion processes are Markov point processes or Gibbs processes
- The general idea is to take a CSR and ”delete” points within a distance less than a threshold δ
- Under a Markov process, the existence of an event in a region depends on the locations of events in a neighbourhood (where neighbourhoods are within regions)
- There are two ways to do this: 1) to simulate CSR then delete all within a distance δ, and 2) to simulate CSR, record when event was simulated, then delete an event if it is within distance δ of an older event

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Simple Inhibition Process

- We use the packing intensity to describe simple inhibition processes:

  $$\tau = \lambda \pi (\delta/2)^2$$

  Where λ is the intensity, giving τ to be the proportion of the region A covered by non-overlapping discs of diameter δ

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Simple Inhibition Process

- For simple inhibition process 1) we take a a Poisson process with intensity ρ and thin it by the deletion of pairs of events that are less than δ apart
- In this case, the probability that an event "survives" is $\exp(-\pi\rho\delta^2)$ giving the intensity of a simple inhibition process as:

$$\lambda = \rho \exp(-\pi\rho\delta^2)$$

- The second order properties can be expressed as:

$$\lambda(h) = \rho^2 \exp(-\rho U_\delta(h)) \qquad\qquad h \geq \delta$$

- $\lambda(h) = 0$ when $0 < h < \delta$, and $U_\delta(h)$ is the area of the union of two discs with equal radius δ and centers distance h apart

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Simple Inhibition Process

- For simple inhibition process 2) we take a a Poisson process with intensity ρ and thin it by the deletion of pairs of "older" events that are less than δ apart
- The expressions are the same as for process 1) but with the addition of the sequential piece (this process is referred to as the simple sequential inhibition process)
- Let X_i be a sequence of n events in A, and $d(x, y)$ be the distance between two points x and y. Then:
  - X_1 is simulated from a uniform distribution in A
  - Given (past) $\{X_j = x_j, j = 1, ...(i-1)\}$, then X_i (present) is uniformly distributed on the intersection of A with $\{y : d(y, x_j) \geq \delta j = 1, ...(i-1)\}$
- So the simple sequential inhibition process has packing intensity:

$$\tau = \frac{n\pi(\delta/2^2)}{|A|}$$

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- In R (spatstat), the functions for thinning processes 1) and 2) described above are called rMaternI and rMaternII
- The simple sequential inhibition process, called rSSI is similar but slightly different:
    - Each new point is generated uniformly in the window and independently of preceding points
    - If a new point lies within distance δ from an existing point then it is rejected and another random point is generated
    - The SSI process ends when no points can be added

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Markov point processes

- The general idea of a Markov point process lies in conditioning, whereby the existence of an event in a finite region A depends on the locations of events in a neighbourhood
- Inhibition processes are a special form of Markov process: the conditional intensity of an event at a point x given the realization of the process in the remainder of the region A depends on the existence (or otherwise) of an event within distance δ of x
- General Markov processes were introduced by Ripley and Kelly (1977)
- Markov point processes are characterized by the likelihood ratio with respect to a Poisson process of unit intensity

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Markov point processes

- Let's call the likelihood ratio $f(\cdot)$
- If $\mathbf{X} = \{x_1, ..., x_n\}$ denotes a finite set of points in A then $f(\mathbf{X})$ indicates how much more likely is the configuration of events $\mathbf{X}$ than a homogeneous point process (with unit intensity)
- We can factorize the likelihood ratio to:

$$f(\mathbf{X}) = \alpha \prod_{i=1}^{n} g_i(x_i) \prod_{j>i} g_{ij}(x_i, x_j) ... g_{12...n}(x_1, x_2, ..., x_n)$$

- Where α is a normalizing constant
- We also define two points x and y in A to be neighbours if $d(x, y) < \delta$ for some $\delta > 0$ where $d(x, y)$ is the distance between x and y
- We also define a clique (recall areal data) as a set of mutual neighbours, and the neighbourhood of x to be the set of points $\{y \in A : 0 < d(x, y), \delta\}$

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Markov point processes

- The point process with these definitions is Markov with range δ if the conditional intensity at the point x given the configuration of the other events in A depends only on the configuration in the neighbourhood of x
- The g-functions from the above equation are unity *unless* the x form a clique

Examples of Markov point processes: the Strauss process

$$f(\mathbf{X}) = \alpha\beta^n\gamma^p$$

- Where α is the normalizing constant, β is the intensity of the process, γ is the interaction between neighbours, and p is the number of distinct pairs of neighbours in $\mathbf{X}$
- If $\gamma = 1$ then the Strauss process gives a Poisson process with intensity β
- if $\gamma = 0$ then the Strauss process gives a simple inhibition process because no two events may be neighbours
- In R spatstat, the Strauss process is simulated with `rStrauss`

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Examples of Markov point processes: the pairwise interaction process

$$f(\mathbf{X}) = \alpha\beta^n \prod_{i \neq j} h\{d(x_i, x_j)\}$$

- Where α is the normalizing constant, β is the intensity of the process, $h(d)$ is non-negative for all distances and the product is over all pairs of distinct points in $\mathbf{X}$
- The additional restriction is that $h(d)$ is bounded and that $h(d) = 0$ for all distances less than some $\delta > 0$
- This restriction limits the number of events in A by imposing a minimum allowable distance δ between any two events
- The pairwise interaction process may be fit in R `spatstat` using the `rmh` function

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Examples of Markov point processes: the pairwise interaction process

- The pairwise interaction process may be simulated using the following steps (MCMC):
    1. For the initial realization, consider n points $\{x_1, ...x_n\}$
    2. Delete one of the points in $\{x_1, ...x_n\}$
    3. Generate a point y from a uniform distribution in A, and accept y with probability $p(y)$
    4. Repeat 2-3 until the MCMC converges

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Review of point processes

- Testing for CSR:
  - Adjusting for edge effect
  - Testing for CSR with Ripley's K.
  - Testing for CSR based on inter-event distances, H(h).
  - Testing for CSR based on nearest-neighbour distances, G(h).
- Spatial processes, Poisson processes are the building block:
  - Homogeneous Poisson process (constant intensity).
  - Inhomogeneous Poisson process (intensity varies across domain).
  - Poisson Cluster process (intensity varies for parents and/or children forming clusters), Cox process when the cluster intensity is spatially varying.
  - Simple inhibition processes, Markovian processes (Strauss and pairwise interaction) for regular patterns.
- In this lecture we will focus on fitting point process models and on methods for detecting clusters.

Fitting point process models

- Given our set of observed point events $\{x_1, ...x_n\}$ in region A we wish to fit a model (which is stationary and isotropic)
- Model fitting is approached by estimating the parameters of the particular process
  - Example: fitting the parameters ρ, μ and σ^2 of a clustered process
  - Example: fitting a parametric form of the intensity of an inhomogeneous poisson process
- We use familiar fitting methods: Least squares, Maximum Likelihood and non-parametric methods.

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Fitting point process models: Least Squares

- We start with $K(h)$ and the estimator $\hat{K}(h)$ (or L, or G, or H) for parameter fitting
- This is useful when the mathematical form of $K(h)$ is known either explicitly or as an integral (which is true for some point processes)
- If $K(h)$ is not known we use the simulated realizations
- Example, to fit a homogeneous poisson cluster process we have parameters $\theta = (\rho, \sigma)$ and the Ripley's K function is:

$$K(h, \theta) = \pi h^2 + \frac{1}{\rho}(1 - \exp(-h^2/(4\sigma^2)))$$

- And we estimate $\hat{K}(h)$ from the data

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Fitting point process models: Least Squares

- Given the theoretical K-function and the estimator $\hat{K}(h)$ we minimize the deviance:

$$D(\theta) = \int_0^{h_0} [(\hat{K}(h))^c - (K(h,\theta))^c]^2 dh$$

- Where h_0 is the maximum distance which is typically chosen as $1/3$ to $1/2$ of the width of a rectangular region, and c is the power transformation
- The power transformation controls the sampling fluctuations in $\hat{K}(h)$ which can increase with h and have influence on $\hat{\theta}$ (i.e. it is a variance stabilizer)
- Examples of c are c=0.5 for a pattern that is not too different from CSR, c=0.25 for cluster patters. However, choose a variety of c values in practice in order to see how sensitive the results are

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Fitting point process models: Least Squares Estimation Steps

1. Compute the edge corrected $\hat{K}(h)$

$$\hat{K}(h) = \frac{|A|}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} I(h_{i,j} \leq h)$$

2. Choose a theoretical model for $K(h, \theta)$ where θ are the parameters of the model
3. Find $\hat{\theta}$ that minimizes the deviance for a given c

$$D(\theta) = \int_0^{h_0} [(\hat{K}(h))^c - (K(h,\theta))^c]^2 dh$$

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Fitting point process models: Least Squares Estimation

- When $K(h, \theta)$ is unknown because there is no closed form, use the simulated method (for s simulations):

$$\bar{K}_s(h, \theta) = \frac{1}{n} \sum_{i=1}^{s} \hat{K}_i(h, \theta)$$

- Finding $\bar{K}_s(h, \theta)$ for each value of θ can be prohibitive computationally
  1. Start with a small number of simulations, s
  2. Find a first approximation of $\hat{\theta}$
  3. Repeat with a larger value for s

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Fitting point process models: Least Squares Estimation Steps

- A weighted version of the deviance, shown to have asymptotic properties (consistency and asymptotic normality), is often used:

$$D(\theta) = \int_0^{h_0} w(h)[(\hat{K}(h))^c - (K(h,\theta))^c]^2 dh$$

- The weight $w(h)$ is a weight on the distance also controls the variance
- When $c = 0.5$ and $w(h) = 1$ we have the Poisson cluster process
- See Guam and Sherman, J R Stat Soc (2007) for asymptotic properties

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Fitting point process models: Least Squares Estimation

- In R `spatstat`, cluster or Cox point process models are fit with least squares estimation through the `kppm` function with the `method="mincontrast"` option
- To fit a log-Gaussian Cox point process, use the function `lgcp.estK`
- To fit the Matern cluster process (type I or II), use the function `matclust.estK`

Fitting point process models: Maximum Likelihood

- To fit inhomogeneous Poisson, Strauss and pairwise interaction processes we need to rely on likelihood methods
- Recall for the inhomogeneous Poisson process:
    - $N(A)$ is Poisson with mean $\int_A \lambda(x)dx$
    - Conditional on $N(A) = n$, the n events in A form an independent random sample from A with a probability distribution function proportional to $\lambda(x)$
- We can define the process based on its conditional intensity
- Namely, the conditional probability of finding a point of the process inside an infinitesimal neighbourhood du of the location u given the complete point pattern $\mathbf{x}$ is $\lambda(u, \mathbf{x})du$

Fitting point process models: Conditional Intensity

- For example, CSR has conditional intensity $\lambda(u, \mathbf{x}) = \lambda$
- The IPP has conditional intensity $\lambda(u, \mathbf{x}) = \lambda(u)$
- Sometimes the IPP trend is denoted as $\beta(u)$ and indicates "spatial trend"
- The Strauss process has conditional intensity $\lambda(u, \mathbf{x}) = \beta^n \gamma^p$ where β is the intensity, γ is the interaction parameter, and p is the number of points of $\mathbf{x}$ that lie within a distance δ of u (i.e. pairs of neighbours)
- For example, the Strauss process with $\gamma < 1$ dependence between points is reflected in the fact that the conditional probability of finding a point of the process at the location u is reduced if other points of the process are present within a distance δ. And when $\gamma = 0$, the conditional probability of finding a point at u is zero if there are any other points of the process within a distance δ of this location.

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Fitting point process models: Pseudolikelihood

- Because maximum likelihood is difficult for point process models, the log of the pseudolikelihood is maximized, using the conditional intensity
- For a point process governed by parameter θ the pseudolikelihood is:

$$PL(\theta; x) = \prod_{i=1}^{n} \lambda_\theta(x_i; x) \exp(\int_A \lambda_\theta(u; x) du)$$

- The maximum pseudolikelihood estimate of θ minimizes the above equation

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Fitting point process models: Pseudolikelihood

- We need to take the log of the pseudolikelihood equation and approximate the integral using a "quadrature" scheme (see Berman and Turner, 1992)

$$\int_A \lambda_\theta(u; x) du \approx \sum_{j=1}^{m} \lambda_\theta(u_j, x) w_j$$

- Where u_j are "quadrature points" in A and $w_j \geq 0$ are the "quadrature weights"

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Fitting point process models: Pseudolikelihood

- The quadrature points can be chosen as all data points, x_i and the addition of some dummy points u_j, i.e. $\{x_1, ..., x_n\} \subset \{u_1, ..., u_m\}$
- Then the log pseudolikelihood can be written:

$$\log PL(\theta; x) = \sum_{j=1}^{m} z_j \log \lambda_\theta(u_j; x) - w_j \lambda_\theta(u_j; x)$$

- Where $z_j = 1$ if u_j is a data point, and $z_j = 0$ if u_j is a dummy point

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Fitting point process models in R

- In R `spatstat` the function `ppm` fits models by pseudolikelihood based on the conditional intensity $\lambda_\theta(u, x)$
- The model must be loglinear in the parameters θ
- For example, the Strauss process can be written:

$$\log \lambda(u, x) = \log \beta + \log \gamma p$$

- So $\theta = (\log \beta, \log \gamma)$ are the "regular parameters" and the parameter driving the interaction, p is the "irregular parameter"

Fitting point process models in R

- Thus in `spatstat` `ppm` the conditional intensity is split into first and higher order terms:

$$\log \lambda_\theta(u, x) = \eta S(u) + \phi V(u, x)$$

- The first order term $S(u)$ describes the spatial inhomogeneity of the intensity (including covariate effects) and the higher order term $V(u, x)$ describes the interactions between points

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Fitting point process models in R

- The general form of `ppm` is `ppm(X, trend, interaction,...)`
- The trend argument specifies any spatial trend or covariate effects and is written as an R formula
- The default trend formula is ~ 1, which indicates $\lambda(u) = 1$, corresponding to a process without spatial trend or covariate effects. The formula $\sim x$ indicates the vector statistic $\lambda(x, y) = (1, x)$ corresponding to a spatial trend of the form $\exp(\alpha + \beta x)$, where α, β are coefficient parameters to be estimated, while $\sim x + y$ indicates $\lambda(x, y) = (1, x, y)$ corresponding to $\exp(\alpha + \beta x + \gamma y)$

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Fitting point process models in R

- The general form of `ppm` is `ppm(X, trend, interaction,...)`
- The interaction term represents the interaction function $V(u, x)$
- For example, the Strauss function with interaction distance $\delta = 0.1$ is fit with `ppm(X,  1, Strauss(r=0.1))`
- Note that the `ppm` with a specified higher order term calls the first order term for the intensity β rather than λ
- `spatstat` automatically creates a quadrature scheme but it can be controlled by the user through the function `quadscheme`

Point Pattern Data: Detecting Clusters

- So far we have simulated point processes, found statistics that indicate a global measure of what pattern there may be, and fit models to specific types of point patterns.
- If we want to find where clusters of observations are located, we need a different set of tools called scan statistics.
- Goals of scan statistics:
  - To determine areas where the number of events is inconsistent with the number observed over the rest of the study area.
  - Compare local rates of events (or case/control ratios) to detect clusters.

Point Pattern Data: Detecting Clusters

- First attempts at cluster detection methods were developed in the 1980s: geographical analysis machine and the cluster evaluation permutation procedure.
- Early cluster detection methods:
  - Graphical in nature
  - Divides the region up into a fine grid
  - Uses a search window, which is a circle of predefined radius, larger than the grid spacing in order for circles to overlap
  - Centers the circle over each grid cell, then moves across the region
  - The number of cases occurring within the search window are counted
  - The circle is drawn on the map if the count observed within the circle exceeds some tolerance level
  - The tolerance level may be defined as the observed count exceeding all of the counts associated with that circle under random selection (N=499)

Point Pattern Data: Detecting Clusters

- The circle in these cases is considered a circular uniform kernel
- In kernel estimation, the kernel is centered on the data locations, in scan statistics, the circle is centered on the grid

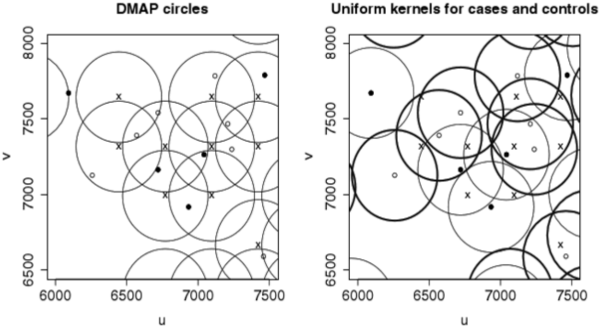

FIG. 6.8 Equivalence between the case/control ratio within circles surrounding grid points (left-hand plot) and the ratio of intensity (density) functions based on circular uniform kernels (right-hand plot). Case locations appear as filled small circles, control locations as open small circles, and grid points as "×" symbols. In the left-hand plot, large circles represent radii of 300 units around each grid point. In the right-hand plot, dark circles represent control kernel radii, and lighter circles represent control kernel radii (both 300 units).

Point Pattern Data: Spatial Scan Statistics

- A scan statistics involves the definition of a moving window and a statistical comparison measurement (count or rate) within the window to the same measurement outside the window.
- Kulldorff (1995, 1997) defines a spatial scan statistic that is similar to the geographical analysis machine, but with an inferential framework.
- Scan statistics tend to consider circular windows with variable radii ranging from the smallest distance between a pair of cases to the user-defined upper bound.
- The circles can be centered on either grid locations (like earlier methods)or the set of locations. Different results will be seen depending on what is chosen.

Point Pattern Data: Spatial Scan Statistics

Setup:
Let $N_{1,in}$ represent the number of case locations and $N_{in} = N_{0,in} + N_{1,in}$ be the total number of people at risk (or number of case and control locations) inside a particular window.
Let $N_{1,out}$ represent the number of case locations and $N_{out} = N_{0,out} + N_{1,out}$ be the total number of people at risk (or number of case and control locations) outside the window.
The test statistic is:

$$T_{scan} = \max(\frac{N_{1,in}}{N_{in}})^{N_{1,in}}(\frac{N_{1,out}}{N_{out}})^{N_{1,out}} I(\frac{N_{1,in}}{N_{in}} > \frac{N_{1,out}}{N_{out}})$$

I(.) is the indicator function (i.e. we only maximize over windows where the observed rate inside the window exceeds that outside the window)

Point Pattern Data: Spatial Scan Statistics

- The maximum observed likelihood ratio statistic provides a test of overall general clustering and an indication of the most likely clusters with significance determined by Monte Carlo testing of the constant risk hypothesis.
- SaTScan is a software package that enables this.
- There is also a new R package called scanstatistics that looks promising `https://cran.r-project.org/web/packages/scanstatistics/vignettes/introduction.html`