# Spatial Statistics
# Areal Data Unit 2

PM569 Spatial Statistics

Lecture 7: October 26, 2017

# Areal Data

- Spatial similarity
- Global indexes of spatial autocorrelation
- Local indexes of spatial autocorrelation (LISA)
- Spatial autoregressive models (CAR, SAR)

# Areal Data: Global Indexes of Spatial Autocorrelation

- The goal of global indexes of spatial autocorrelation is to summarize the degree to which similar observations tend to occur near each other
- Global indexes are summaries over the entire study area, akin to testing clustering rather than a test to detect individual clusters
- Indexes share a common structure: calculate the similarity of values at locations i and j then weight the similarity by the proximity of locations i and j
- High similarities with high weight indicate similar values that are close together; low similarities with high weight indicate dissimilar values that are close together

# Areal Data: Global Indexes of Spatial Autocorrelation

## Indexes of spatial autocorrelation

- We want to summarize similarity between nearby areal units
- Spatial autocorrelation is the the correlation of the same measurement taken at different areal units
- The similarity of values at locations $B_i$ and $B_j$ are weighted by the proximity of i and j
- The weight $w_{ij}$ defines proximity
- In general the extent of similarity is represented by the weighted average of similarity between areal units: indexes of spatial autocorrelation are built on this basic form:

$$\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} sim_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}}$$

# Areal Data: Global Indexes of Spatial Autocorrelation

## Moran's I

- Moran's I (1950) follows the basic form for global indexes of spatial autocorrelation with similarity between areal units i and j defined as the product of the respective difference between y$_i$ and y$_j$ with the overall mean
- Similarity $sim_{ij} = (y_i - \bar{y})(y_j - \bar{y})$
- Where $\bar{y} = \sum_{i=1}^{n} y_i / n$
- Divide the basic form by the sample variance to get the Moran's I statistic:
- $I = \frac{1}{s^2} \frac{\sum_i \sum_j (y_i - \bar{y})(y_j - \bar{y})}{\sum_i \sum_j w_{ij}}$
- Where $s^2 = \frac{\sum_i (y_i - \bar{y})^2}{n}$

# Areal Data: Global Indexes of Spatial Autocorrelation

## Moran's I

- I is a random variable having a distribution defined by the distributions of and interactions between the $y_i$
- When neighbouring regions have similar values (pattern is clustered), I will be positive
- When neighbouring regions have different values (pattern is regular), I will be negative
- When there is no correlation between neighbouring values: $E(I) = -\frac{1}{n-1}$
- When n$\rightarrow \infty$, E(I) $\rightarrow 0$
- $I$ is asymptotically normally distributed where $\frac{I+\frac{1}{n-1}}{\sqrt{Var(I)}} \sim N(0,1)$

# Areal Data: Global Indexes of Spatial Autocorrelation

### Moran's I

- Moran's I is similar to Pearson's correlation but it is not bounded on [-1,1] because of the spatial weights
- Null hypothesis: NO spatial association, i.e. $y_i$ iid
- Compare the z-score to a standard normal distribution
- The z-score that we compare to the standard normal is $z = \frac{I - E(I)}{\sqrt{Var(I)}}$ where $E(I) = -\frac{1}{n-1}$ and V(I) is a little complicated (shown later)

# Areal Data: Global Indexes of Spatial Autocorrelation

## Moran’s I in R using North Carolina SIDS data

```
# Define neighbours.  Choose k=2 NN
IDs<-row.names(as(nc, "data.frame"))
sids.kn2<-knn2nb(knearneigh(coordinates(nc), k=2,
RANN=FALSE), row.names=IDs)
# Convert to weight matrix (row standardized)
sids.kn2.w<-nb2listw(sids.kn2, style="W")
# Use moran.test for Moran’s I
moranSIDS<-moran.test(sids79.rate,sids.kn2.w)
```

# Areal Data: Global Indexes of Spatial Autocorrelation

**Moran's I in R using North Carolina SIDS data**

Result:

```
data:  sids79.rate
weights:  sids.kn2.w

Moran I statistic standard deviate = 2.4465, p-value
= 0.007213
alternative hypothesis:  greater
sample estimates:
Moran I statistic Expectation Variance
0.214682382 -0.010101010 0.008442014
```

The null hypothesis of no spatial correlation is rejected.

# Areal Data: Global Indexes of Spatial Autocorrelation

**Geary's** $c$

- Geary (1954) devides the contiguity ratio or Geary's c
- Similarity $sim_{ij} = (y_i - y_j)^2$
- If regions i and j have similar values, sim$_{ij}$ wil be small
- $c = \frac{n-1}{2\sum_i (y_i - \bar{y})^2} \frac{\sum_i \sum_j w_{ij}(y_i - y_j)^2}{\sum_i \sum_j w_{ij}}$
- Like Moran's it is a weighted average, but here it is scaled by a measure of the overall variation around the mean, $\bar{y}$

# Areal Data: Global Indexes of Spatial Autocorrelation

**Geary's** $c$

- $c$ ranges from 0 to 2 with 0 indicating perfect positive spatial correlation and 2 indicating perfect negative spatial correlation
- $c$ is not a Pearson correlation (related to the Durbin-Watson statistic)
- Low values of Geary's c denote positive autocorrelation and high values indicate negative correlation
- Expected value, $E(c) = 1$ under spatial independence

# Areal Data: Global Indexes of Spatial Autocorrelation

## Geary's $c$ in R using North Carolina SIDS data

```
# Define neighbours.  Choose k=2 NN
IDs<-row.names(as(nc, "data.frame"))
sids.kn2<-knn2nb(knearneigh(coordinates(nc), k=2,
RANN=FALSE), row.names=IDs)
# Convert to weight matrix (row standardized)
sids.kn2.w<-nb2listw(sids.kn2, style="W")
# Use geary.test for Geary's c
gearySIDS<-geary.test(sids79.rate,sids.kn2.w)
```

# Areal Data: Global Indexes of Spatial Autocorrelation

### Geary's $c$ in R using North Carolina SIDS data

Result:

```
data:  sids79.rate
weights:  sids.kn2.w

Geary C statistic standard deviate = 1.6166, p-value
= 0.05299
alternative hypothesis:  Expectation greater than
statistic
sample estimates:
Geary C statistic Expectation Variance
0.83688116 1.00000000 0.01018165
```

We have a marginal p-value for negative spatial autocorrelation.

# Areal Data: Global Indexes of Spatial Autocorrelation

## Moran's $I$

- Inference is performed on $I$ under the randomization assumption or by Monte Carlo tests
- Null hypothesis is always that there is no spatial association
- We have a normality assumption of spatial independence such that all observations follow iid gaussian distribution
- Random permutations of the null distribution are computed

# Areal Data: Global Indexes of Spatial Autocorrelation

## Moran's $I$

- Randomization means the observations are assigned at random in the $B_i$ areal units
- Test statistic is z= (observed-expected)/s.d expected
- $E(I) = \frac{-1}{(n-1)}$ under null hypothesis of no autocorrelation
- $V(I)$ is more complicated and is dependent upon the weight matrix. The normal approximation of the variance under randomization is (Cliff and Ord, 1981):

$$V(I) = \frac{ns_1 - s_2 s_3}{(n-1)(n-2)(n-3)(\sum_i \sum_j w_{ij})^2}$$

$$s_1 = (n^2 - 3n + 3)(0.5 \sum_i \sum_j (w_{ij} + w_{ji})^2) - n(\sum_i (\sum_j w_{ij} + \sum_j w_{ji})^2) + 3(\sum_i \sum_j w_{ij})^2$$

$$s_2 = \frac{n^{-1} \sum_i (y_i - \bar{y})^4}{(n^{-1} \sum_i (y_i - \bar{y})^2)^2}$$

$$s_3 = 0.5 \sum_i \sum_j (w_{ij} + w_{ji})^2 - 2n(0.5 \sum_i \sum_j (w_{ij} + w_{ji})^2) + 6(\sum_i \sum_j w_{ij})^2$$

# Areal Data: Global Indexes of Spatial Autocorrelation

**Moran's $I$**

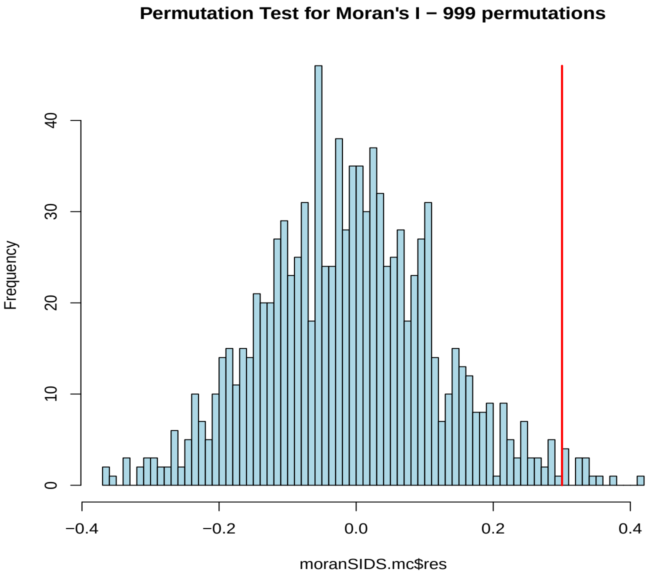
- Monte Carlo approach repeats randomization of the observations into the areal units a large number of times (e.g. $N_{sim} \sim 999$)
- For each randomization the Moran's I statistic is calculated
- Compare the observed Moran's $I$ to the random set
- If the actual $I$ falls at the 5th/95th percentile (or smaller/greater) then it is significant at $\alpha = 0.05$

# Areal Data: Global Indexes of Spatial Autocorrelation

- R output for monte carlo estimate of global Moran's I
- `moran.mc(sids79.rate,sids.kn1.w,nsim=999)`

  ```
  Monte-Carlo simulation of Moran's I
  data:  sids79.rate
  weights:  sids.kn1.w
  number of simulations + 1:  1000
  statistic = 0.3003, observed rank = 987, p-value
  = 0.013
  alternative hypothesis:  greater
  ```
- The null hypothesis of no spatial correlation is rejected.

# Areal Data: Global Indexes of Spatial Autocorrelation



# Areal Data: Global Indexes of Spatial Autocorrelation

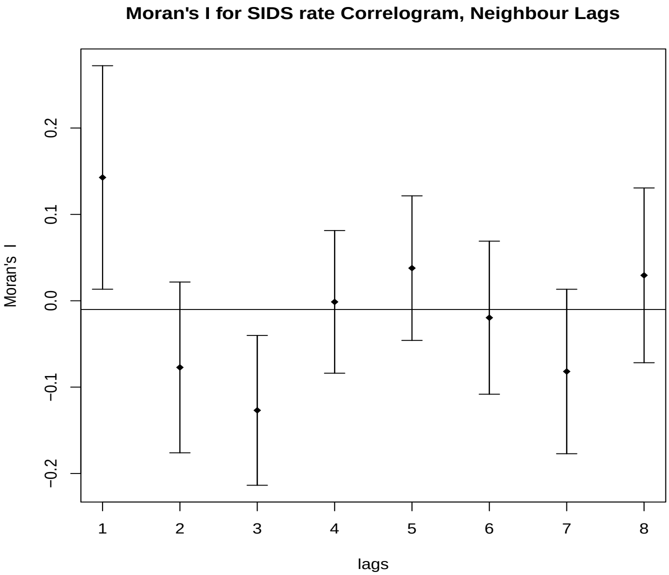
Issues with spatial autocorrelation tests:

- they assume that the mean trend has been removed, for example elevation effect when examining spatial pattern in precipitation
- this assumption is in centering the mean $y_i - \bar{y}$ as it's equivalent to saying the correct model has constant mean and the spatial pattern is represented in the spatial weights
- removing trend may be impossible if you don't have covariate data
- spatial weights may be misspecified for testing autocorrelation, for instance too few neighbour weights when spatial pattern is based on larger distances or vice versa
- they require $min(N) \sim 20$ to provide asymptotically accurate results

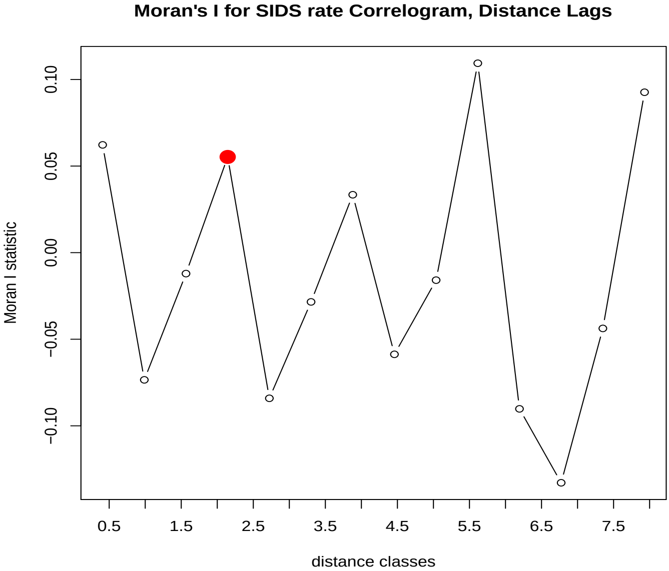
# Areal Data: Global Indexes of Spatial Autocorrelation

- We can look at correlograms of the Moran's **I** statistic to determine appropriate number of neighbours or distance
- Calculate $I$ based on knn for a range of k (e.g. 1,...,8) or number of borders shared (e.g. queen, rook)
- Calculate $I$ based on $d_{ij}$ for a range of distances

# Areal Data: Global Indexes of Spatial Autocorrelation



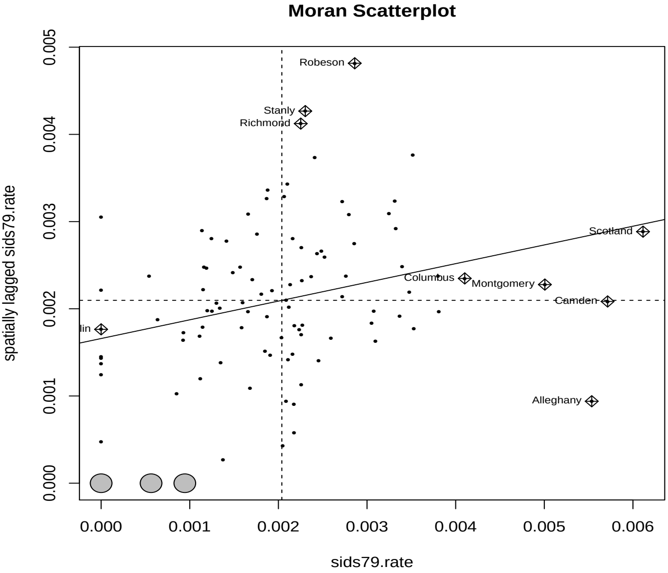
# Areal Data: Global Indexes of Spatial Autocorrelation



# Areal Data: Quasi Local Indexes of Spatial Autocorrelation

- Global tests of spatial autocorrelation can be broken down into components
- We can construct local tests that identify clusters
- One preliminary step is to construct a Moran's I scatterplot, where variable of interest (e.g. SIDS rate) is on the x-axis and their spatially lagged values on the y-axis

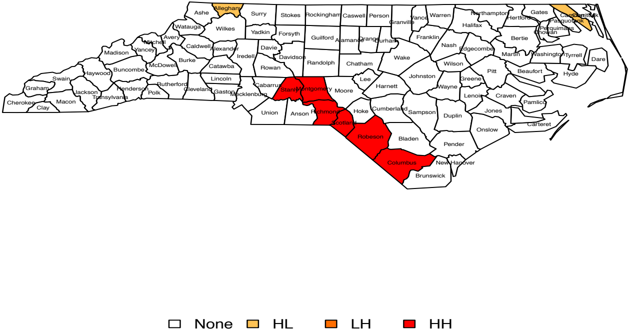
# Areal Data: Quasi Local Indexes of Spatial Autocorrelation



# Areal Data: Quasi Local Indexes of Spatial Autocorrelation

- This plot can be divided into quadrants of low-low, low-high, high-low and high-high
- The global Moran's $I$ is the slope of this plot lm(wx x) where wx is the spatially lagged value of the values on the x-axis (e.g. SIDS rates)
- Can also detect outliers by testing for influence measures

# Areal Data: Quasi Local Indexes of Spatial Autocorrelation



# Areal Data: Local Indexes of Spatial Autocorrelation

- Global measures (Moran's I or Geary's c) are a single value that apply to the entire study area
- The same pattern or process occurs over the entire geographic area
- Global statistic suggests that there is clustering but does not identify areas of particular clusters
- Global test is often used first to determine if there is evidence of spatial association
- Want to detect local areas of similar values, need a local statistic
- LISAs are decompositions of global indicators into the contribution of each individual observation (i.e. $B_i \in D$)
- As a result the sum of LISAs is proportional to the equivalent global indicator
- Local Moran's I, Getis-Ord G*

# Areal Data: Local Indexes of Spatial Autocorrelation

- With LISAs, each observation gives an indication of the extent of significant spatial clustering of similar values located around that observation
- The locations "around" one particular observation is defined as a neighbourhood and is formalized with the spatial adjacency weights matrix, W
- Recall, W can be based on sharing a border (full or partial) or distance
- Row standardization of W helps with interpretation of the statistic

# Areal Data: Local Indexes of Spatial Autocorrelation

LISAs can be used to detect

- Clusters (areal units with similar neighbours): Local Moran's I
- Hotspots (areal units with dissimilar neighbours): Getis-Ord G*

# Areal Data: Local Indexes of Spatial Autocorrelation

**Moran's $I$ vs Local Moran's $I$**

$$I = \frac{1}{s^2} \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} (y_i - \bar{y})(y_j - \bar{y})}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}}$$

$$s^2 = \frac{\sum_{i}^{n} (y_i - \bar{y})^2}{n}$$

$$I_i = \frac{y_i - \bar{y}}{s} \sum_{j}^{n} w_{ij} \frac{(y_j - \bar{y})}{s}$$

- $I_i$ is calculated for each areal unit $B_i$

# Areal Data: Local Indexes of Spatial Autocorrelation

**Local Moran's *I***

$$I_i = \frac{y_i - \bar{y}}{s} \sum_{j}^{n} w_{ij} \frac{(y_j - \bar{y})}{s}$$

- Have a value of $I$ for each $B_i$
- Sometimes the $I_i$ are mapped to indicate units with high values indicating stronger local autocorrelation
- More often, z-score and significance of z-score is plotted
- As before, test statistics are generated under randomization
- Since in a local setting, there are multiple comparisons being made (neighbours sharing observations when calculating $I_I$) we need a Bonferroni adjustment

# Areal Data: Local Indexes of Spatial Autocorrelation

**Local Moran's $I$**

$$I_i = \frac{y_i - \bar{y}}{s} \sum_{j}^{n} w_{ij} \frac{(y_j - \bar{y})}{s}$$

- Have a value of $I$ for each $B_i$
- Sometimes the $I_i$ are mapped to indicate units with high values indicating stronger local autocorrelation
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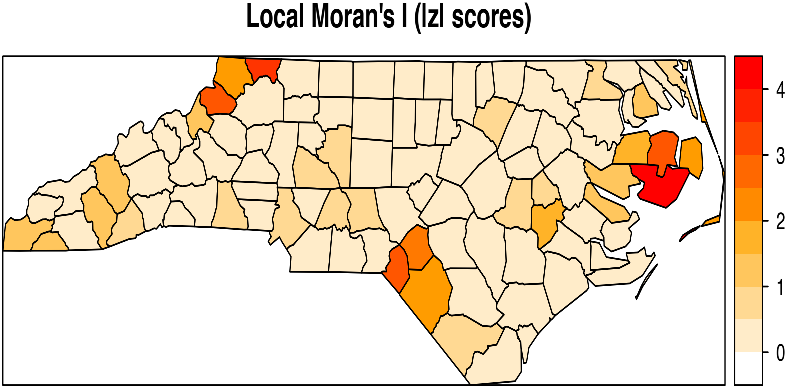
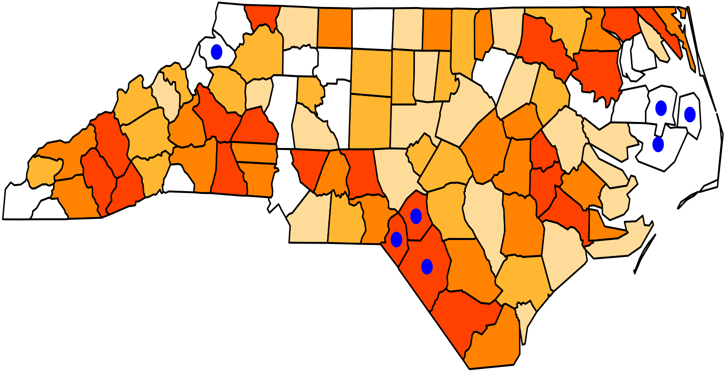
# Areal Data: Local Indexes of Spatial Autocorrelation



# Areal Data: Local Indexes of Spatial Autocorrelation

Statistically significant Local Moran's I as blue dots



# Areal Data: Local Indexes of Spatial Autocorrelation

**Getis-Ord $G$ vs Local Getis-Ord $G*$**

$$G = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} y_i y_j}{\sum_{i=1}^{n}\sum_{j=1}^{n} y_i y_j}$$

$$G_i* = \frac{\sum_{j=1}^{n} w_{ij} y_j}{\sum_{j=1}^{n} y_j}$$

# Areal Data: Local Indexes of Spatial Autocorrelation

## Getis-Ord $G$

- Again, we compute the z-score for spatially randomized G to determine if it is significantly different than our observed
- z=observed-expected/s.d. observed
- $E(G) = \frac{\sum_i \sum_j w_{ij}}{n(n-1)}$
- V(G) is complicated
- the sign of the z-score is important; positive z means high values cluster together, negative values means low values cluster together
- the p-value must be computed to determine significance of G

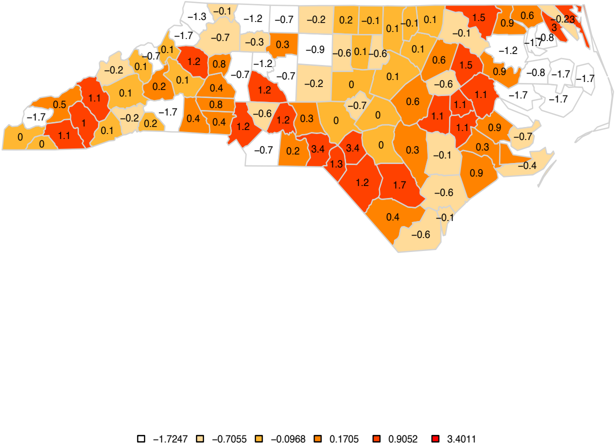
# Areal Data: Local Indexes of Spatial Autocorrelation

**Local Getis-Ord** $G^*$

- Similar to local Moran's $I_i$, $G_i^*$ is calculated for each areal unit
- A group of areal units with high $G_i^*$ indicates a "hotspot" where as low $G_i^*$ means a "coldspot"

# Areal Data: Local Indexes of Spatial Autocorrelation

Getis-Ord G* for SIDS79 rates



# Areal Data

Issues with spatial autocorrelation tests:

- They assume that the mean trend has been removed, for example household income effect when examining spatial pattern in SIDS rate
- One solution is to run a linear model and then test for spatial association on residuals
- Use autoregressive models

# Areal Data: Models

**Simultaneous Autoregressive models**

- Similar to universal kriging or regression kriging
- Use regression on values from neighbouring areal units to account for spatial dependence
- Autocorrelation reflects self regression where you use observations of the outcome at other locations as additional covariates in the model
- $Y(s) \sim MVN(X\beta, \Sigma)$
- In universal kriging we modeled $\Sigma$ as a parametric function of distance
- In areal modeling, we restrict distances to those between our areal units

# Areal Data: Models

## Simultaneous Autoregressive models

- We represent $\Sigma$ as our residual errors $\epsilon(s_i) = \sum_j b_{ij}\epsilon(s_j) + \nu(s_i)$ and apply spatial correlation to these residuals

$$Y(s_i) = x(s_i)\beta + \sum_j b_{ij}\epsilon(s_j) + \nu(s_i)$$

$$Y(s_i) = x(s_i)\beta + \sum_j b_{ij}[Y(s_j) - x(s_j)\beta] + \nu(s_i)$$

The degree of spatial dependence is through the term $\sum_j b_{ij}[Y(s_j) - x(s_j)\beta]$

# Areal Data: Models

**Simultaneous Autoregressive models**

- SAR models are often represented in matrix form
- From the equation on the previous slide,

$$Y = X^T\beta + B(Y - X^T\beta) + \nu$$

There are 3 types of SAR models, SAR error (spatial error models), SAR lag (spatial lag models), and SAR mixed (spatial mixed models). They all depend on the neighbourhoods chosen and the spatial weight matrices.
More on this next week.