Example 9.

Sensitivity of hydraulic head at a point and at a discrete time to spatially uniform initial conditions under transient flow conditions

0. Forward model

Governing equation:

$$K\,b\,\frac{\partial^2 h}{\partial x^2} + R = S_s\,b\,\frac{\partial h}{\partial t} \tag{1}$$

(2)

Boundary conditions:

$$-K\,b\,\frac{dh(x)}{dx} = 0\,, \qquad x = 0 = \Gamma_2 \tag{3}$$

$$h(x,t) = h_{\Gamma_{1_L}}\,, \qquad x = L = \Gamma_{1_L} \tag{4}$$

(5)

Initial conditions:

$$h(x,t) = h_0\,, \qquad t = 0 \tag{6}$$

(7)

Closed-form solution:

Not available (8)

(9)

Spatial derivatives from differentiation:

Not available (10)

(11)

In [133…
```
from IPython.display import HTML, display
def set_background(color):
    script = (
        "var cell = this.closest('.code_cell');"
        "var editor = cell.querySelector('.input_area');"
        "editor.style.background='{}';"
        "this.parentNode.removeChild(this)").format(color)
    display(HTML('<img src onerror="{}">'.format(script)))
```

In [134…
```
from warnings import filterwarnings
filterwarnings("ignore", category=DeprecationWarning)

import numpy as np

K, Ss, R, b, L, BC1h, ICh, ocol, nper = 10., 1e-2, 1e-1/1000., 10., 10000., 0., 50., 5000, 1000
X = np.arange(L)
```

1. Direct sensitivity

Not available (12)

2. Perturbation sensitivity

(13)

$$\frac{\partial h(x')}{\partial h_0} \approx \frac{h(x, h_0 + \Delta h_0) - h(x, h_0)}{\Delta h_0} \tag{14}$$

(15)

2a. Analytical

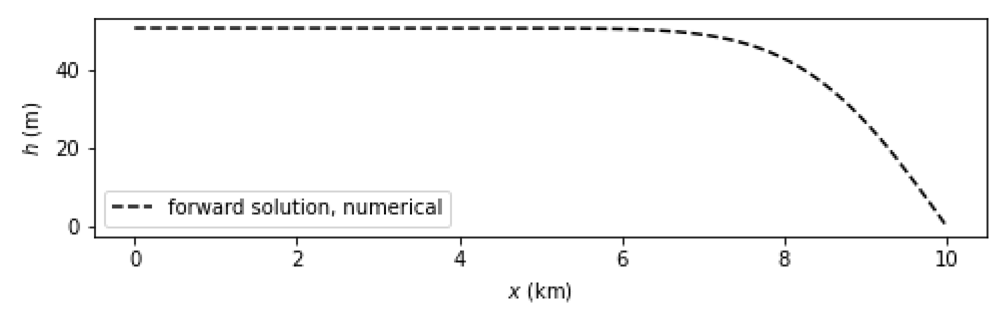
Not available (16)

2b. Semi-analytical

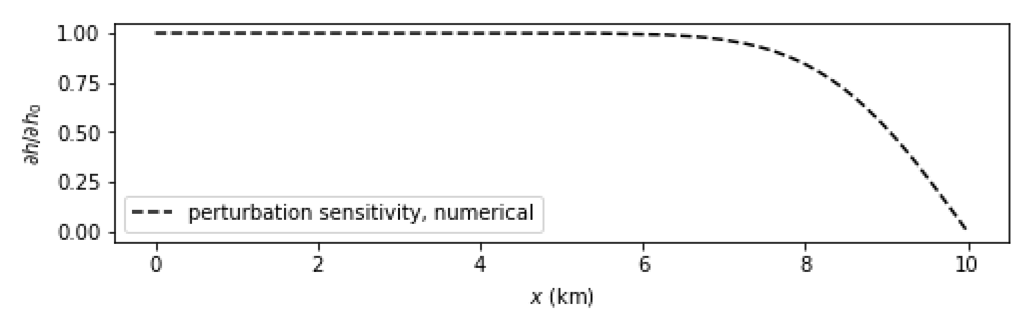
Not available (17)

2c. Numerical

In [135…
```
import flopy

dpar = 1e-2
nrow, ncol = 1, int(L)
M0 = flopy.modflow.Modflow(modelname='model', exe_name='../mf2005.exe')
flopy.modflow.ModflowDis(M0, nlay=1, nrow=nrow, ncol=ncol, nper=nper, delr=1., delc=1., top=0., botm=-b,
                         steady=False, perlen=np.ones(nper), nstp=1)
flopy.modflow.ModflowBas(M0, ibound=np.hstack([np.ones([nrow, ncol-1], dtype=int), -1*np.ones([1,1])]),
                         strt=np.hstack([ICh*np.ones([nrow, ncol-1], dtype=float), np.atleast_2d([0.])]))
flopy.modflow.ModflowLpf(M0, hk=K, vka=-999., ss=Ss, sy=-999., ipakcb=53)
flopy.modflow.ModflowRch(M0, nrchop=1, rech={0:R}, ipakcb=53)
flopy.modflow.ModflowPcg(M0, hclose=1e-6, rclose=1e-6)
flopy.modflow.ModflowOc(M0, stress_period_data={(i,0): ['save head'] for i in range(nper)})
M0.write_input()
success, buff = M0.run_model(silent=True)
H0 = flopy.utils.binaryfile.HeadFile('model.hds').get_data(kstpkper=[0,nper-1])[0,0,:]

%matplotlib inline
import matplotlib as mpl
import matplotlib.pyplot as plt
plt.subplots(figsize=[8,2])
plt.plot(X/1000., H0, 'k--', mfc='none', label='forward solution, numerical')
plt.xlabel('$x$ (km)')
plt.ylabel('$h$ (m)')
plt.legend(loc=3);
```



In [136…
```
M1 = M0
flopy.modflow.ModflowBas(M1, ibound=np.hstack([np.ones([nrow, ncol-1], dtype=int), -1*np.ones([1,1])]),
                         strt=np.hstack([(ICh+ICh*dpar)*np.ones([nrow, ncol-1], dtype=float),
                                         np.atleast_2d([0.])]))
M1.write_input()
success, buff = M1.run_model(silent=True)
H1 = flopy.utils.binaryfile.HeadFile('model.hds').get_data(kstpkper=[0,nper-1])[0,0,:]
dhdICh_p = (H1-H0)/(ICh*dpar)
benchmark = dhdICh_p[ocol]

%matplotlib inline
import matplotlib as mpl
import matplotlib.pyplot as plt
f,s = plt.subplots(figsize=[8,2])
plt.plot(X/1000., dhdICh_p, 'k--', mfc='none', label='perturbation sensitivity, numerical')
plt.xlabel('$x$ (km)')
plt.ylabel(r'$\partial h / \partial h_0$')
plt.legend(loc=3);
```



In [137…
```
set_background('rgba(0, 200, 0, 0.2)')
print('%.6f'% dhdICh_p[ocol])
```

```
0.999619
```

3. Adjoint sensitivity

$$\frac{\partial h(x')}{\partial h_0} = \int_X \psi_1^*(x,0)\;S_s\;b\;dx \tag{18}$$

Governing equation:

$$K\,b\,\frac{\partial^2 \psi_1^*}{\partial x^2} + \delta(x - x')\;\delta(\tau) = S_s\,b\,\frac{\partial \psi_1^*}{\partial \tau} \tag{19}$$

(20)

Boundary conditions:

$$-K\,b\,\frac{d\psi_1^*}{dx} = 0\,, \qquad x = 0 = \Gamma_2 \tag{21}$$

$$\psi_1^*(x,\tau) = 0\,, \qquad x = L = \Gamma_{1_L} \tag{22}$$

(23)

Terminal conditions:

$$\psi_1^*(x,\tau) = 0\,, \qquad t = t_{final} \tag{24}$$

(25)

Closed-form solution:

Not available (26)

3a. Analytical

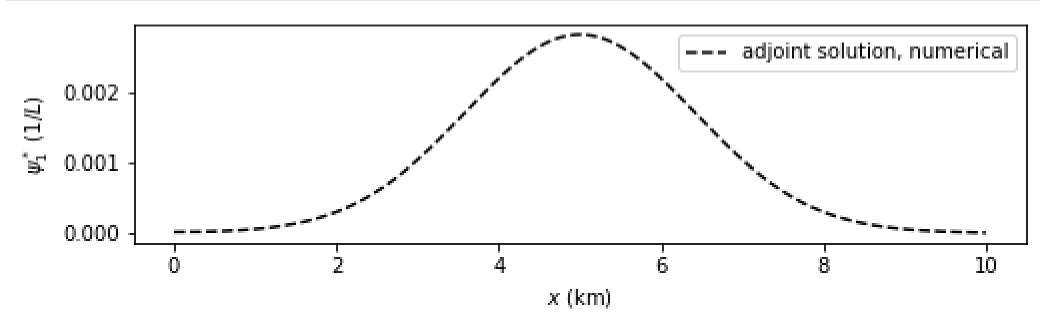
Not available (27)

3b. Semi-analytical

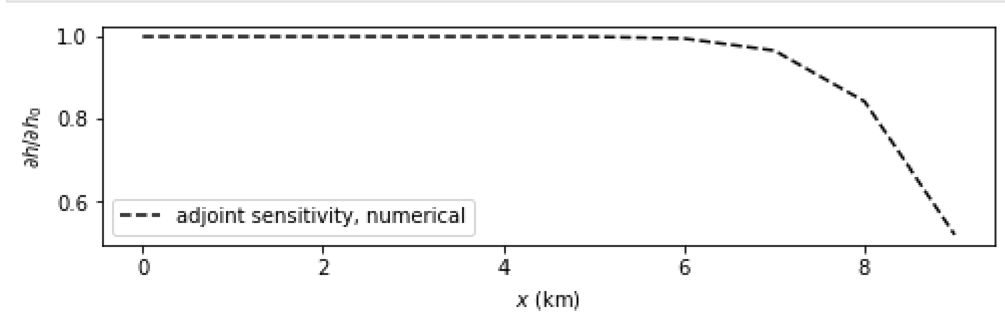
Not available (28)

3c. Numerical

In [138…
```
M0 = flopy.modflow.Modflow(modelname='model', exe_name='../mf2005.exe')
flopy.modflow.ModflowDis(M0, nlay=1, nrow=1, ncol=ncol, nper=nper, delr=1., delc=1., top=0., botm=-b,
                         steady=False, perlen=np.ones(nper), nstp=1)
strt = np.zeros([nrow, ncol], dtype=float)
strt[0,ocol] = 1./Ss/b
flopy.modflow.ModflowBas(M0, ibound=np.hstack([np.ones([nrow, ncol-1], dtype=int), -1*np.ones([1,1])]),
                         strt=strt)
flopy.modflow.ModflowLpf(M0, hk=K, vka=-999., ss=Ss, sy=-999., ipakcb=53)
flopy.modflow.ModflowPcg(M0, hclose=1e-6, rclose=1e-6)
flopy.modflow.ModflowOc(M0, stress_period_data={(i,0): ['save head'] for i in range(nper)})
M0.write_input()
success, buff = M0.run_model(silent=True)
A = np.ravel(flopy.utils.binaryfile.HeadFile('model.hds').get_data(kstpkper=[0,nper-1])[0,0,:])

%matplotlib inline
import matplotlib as mpl
import matplotlib.pyplot as plt
f,s = plt.subplots(figsize=[8,2])
plt.plot(X/1000., A, 'k--', mfc='none', label='adjoint solution, numerical')
plt.xlabel('$x$ (km)')
plt.ylabel(r'$\psi^*_1$ (1/$L$)')
plt.legend();
```



In [139…
```
dhdICh = np.empty(0)
for oc in [int(x) for x in X[::1000]]:
    M1 = M0
    strt = np.zeros([nrow, ncol], dtype=float)
    strt[0,oc] = 1./Ss/b
    flopy.modflow.ModflowBas(M1, ibound=np.hstack([np.ones([nrow, ncol-1], dtype=int), -1*np.ones([1,1])]),
                             strt=strt)
    M1.write_input()
    success, buff = M1.run_model(silent=True)
    A = flopy.utils.binaryfile.HeadFile('model.hds').get_data(kstpkper=[0,nper-1])[0,0,:]
    dhdICh = np.append(dhdICh, np.sum(A*Ss*b))

%matplotlib inline
import matplotlib as mpl
import matplotlib.pyplot as plt
f,s = plt.subplots(figsize=[8,2])
plt.plot(X[::1000]/1000., dhdICh, 'k--', mfc='none', label='adjoint sensitivity, numerical')
plt.xlabel('$x$ (km)')
plt.ylabel(r'$\partial h / \partial h_0$')
plt.legend(loc=3);
```



In [140…
```
set_background('rgba(0, 200, 0, 0.2)')
print('%.6f (%.6f)'% (dhdICh_a[int(np.where(X[::1000]==float(ocol))[0])], benchmark))
```

```
0.999999 (0.999619)
```

In [ ]: