MATH 105LA FINAL PROJECT

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1 Introduction

In this project, we develop a method of calculating the error of the estimated solution $\mathbf{u}^{(k)}$ to the linear system

$$\begin{bmatrix} -2 & 1 & 0 & \cdot & \cdot & 0 \\ 1 & -2 & 1 & 0\cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \\ \cdot & & 0 & 1 & -2 & 1 \\ 0 & \cdot & \cdot & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix} = h^2 \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}$$

corresponding to the finite difference approximation for the differential equation

$$\frac{d^2u(x)}{dx^2} = f(x), \qquad x \in [0, \pi], \qquad f(x) = \sin(x), \qquad u(0) = u(\pi) = 0.$$

We implement the symmetric power method and compute the largest eigenvalue of T_j (as well as T_g and T_s) to show that the norm of the error will decay to zero as k approaches infinity and therefore the Jacobi method (and the Gauss-Seidel and SOR methods) converges to the exact solution $\mathbf{u}_{exact}$ of the linear system. Finally, we compare the speed of convergence of each method and the dependence of the convergence rate on the number of iterations.

2 Algorithm

Links to the MATLAB files:
symmjacobi.m
symmgauss.m
symmSOR.m

The symmjacobi function takes the inputs n (the dimension of A), tol (tolerance), and N (the maximum number of iterations) and outputs mu (the approximate dominant eigenvalue) and v (the approximate eigenvector). First, it determines the matrix A based on n and sets the initial eigenvector guess to be the vector of dimension n where each component is 1. Then it obtains D, L, U, and T_j from A and normalizes the initial vector guess. On each iteration, it finds y=T_j*v, mu=transpose(v)*y, the error, and v=y/norm(y), the kth eigenvector.

If norm(y) is 0, it states T_j has the eigenvalue 0 and ends the algorithm. When the error is less than the given tolerance, it ends the algorithm. Otherwise, it sets k=k+1 for the next iteration and stops when it reaches the maximum number of iterations.

The symmgauss and symmSOR functions work similarly, but symmgauss obtains T_g instead of T_j and symmSOR obtains T_s from symmjacobi by setting w=2/(1+sqrt(1-(symmjacobi(n,1e-6,300))^2)).

3 Answers

a) Let $\mathbf{u}_{exact} = \mathbf{u}$

$$D = \begin{bmatrix} -2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -2 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} -\frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{2} \end{bmatrix}$$

$$\text{(L+U)}= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{bmatrix} \Rightarrow (L+U)\mathbf{u}_{exact} = \begin{bmatrix} u_2 \\ u_1+u_3 \\ \vdots \\ u_{n-2}+u_n \\ u_{n-1} \end{bmatrix}$$

$$\Rightarrow -D^{-1}(L+U)\mathbf{u}_{exact} = \frac{1}{2}\begin{bmatrix} u_2 \\ u_1+u_3 \\ \vdots \\ u_{n-2}+u_n \\ u_{n-1} \end{bmatrix}$$

$$D^{-1}\mathbf{f} = -\frac{1}{2}h^2 \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} u_1 - \frac{1}{2}u_2 \\ -\frac{1}{2}u_1 + u_2 - \frac{1}{2}u_3 \\ \vdots \\ -\frac{1}{2}u_{n-2} + u_{n-1} - \frac{1}{2}u_n \\ -\frac{1}{2}u_{n-1} + u_n \end{bmatrix}$$

$$\Rightarrow -D^{-1}(L+U)\mathbf{u}_{exact} + D^{-1}\mathbf{f} = \begin{bmatrix} \frac{1}{2}u_2 + u_1 - \frac{1}{2}u_2 \\ \frac{1}{2}u_1 + \frac{1}{2}u_3 - \frac{1}{2}u_1 + u_2 - \frac{1}{2}u_3 \\ \vdots \\ \frac{1}{2}u_{n-2} + \frac{1}{2}u_n - \frac{1}{2}u_{n-2} + u_{n-1} - \frac{1}{2}u_n \\ \frac{1}{2}u_{n-1} - \frac{1}{2}u_{n-1} + u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \mathbf{u}_{exact}$$

$\Rightarrow \mathbf{u}_{exact}$ is a fixed point for the iteration

b)
$$\begin{aligned} \mathbf{e}^{(k+1)} &= \mathbf{u}^{(k+1)} - \mathbf{u}_{exact} \\ &= -D^{-1}(L+U)\mathbf{u}^{(k)} + D^{-1}\mathbf{f} + D^{-1}(L+U)\mathbf{u}_{exact} - D^{-1}\mathbf{f} \\ &= -D^{-1}(L+U)\mathbf{u}^{(k)} + D^{-1}(L+U)\mathbf{u}_{exact} \\ &= -D^{-1}(L+U)(\mathbf{u}^{(k)} - \mathbf{u}_{exact}) \\ &= -D^{-1}(L+U)\mathbf{e}^{(k)} \end{aligned}$$

c) $\mathbf{e}^{(k+1)} = T_j\mathbf{e}^k = T_j^{k+1}\mathbf{e}^0 \quad (*)$

$T_j\mathbf{v} = \lambda\mathbf{v} \Rightarrow T_j^{k+1}\mathbf{v} = \lambda^{k+1}\mathbf{v}$

$\Rightarrow \mathbf{e}^{(k+1)} = \lambda^{k+1}\mathbf{v}$ is a nontrivial solution to (*) if and only if λ is an eigenvalue of T_j and $\mathbf{v}$ is the associated eigenvector.

Suppose that $|\lambda| \geq 1$. Then $\rho(T_j) \geq |\lambda| \geq 1 \Rightarrow \|\mathbf{e}^{k+1}\|_2 = \|\lambda^{k+1}\mathbf{v}\|_2 = |\lambda|^{k+1}\|\mathbf{v}\|_2 \not\to 0$ as $k \to \infty$

Suppose that $\rho(T_j) < 1$. Then $|\lambda| \leq \rho(T_j) < 1 \Rightarrow \|\mathbf{e}^{(k+1)}\|_2 = |\lambda|^{k+1}\|\mathbf{v}\|_2 \to 0$ **as** $k \to \infty$

d) Jacobi

Using tol=1e-6, and N=300:

- n=3: For N=10, 100, and 1000, mu=0.6667 $\Rightarrow$ $\rho(T_j)$=0.6667<1 $\Rightarrow$ $\|\mathbf{e}^{(k+1)}\|_2 \to 0$ as $k \to \infty$

```
>> [mu,v]=symmjacobi(3,1e-6,300)
The maximum number of iterations exceeded
mu =

    0.6667


v =

    0.5774
    0.5774
    0.5774
```

- n=5: For N=10, 100, and 1000, mu=0.8571 $\Rightarrow$ $\rho(T_j)$=0.8571<1 $\Rightarrow$ $\|\mathbf{e}^{(k+1)}\|_2 \to 0$ as $k \to \infty$

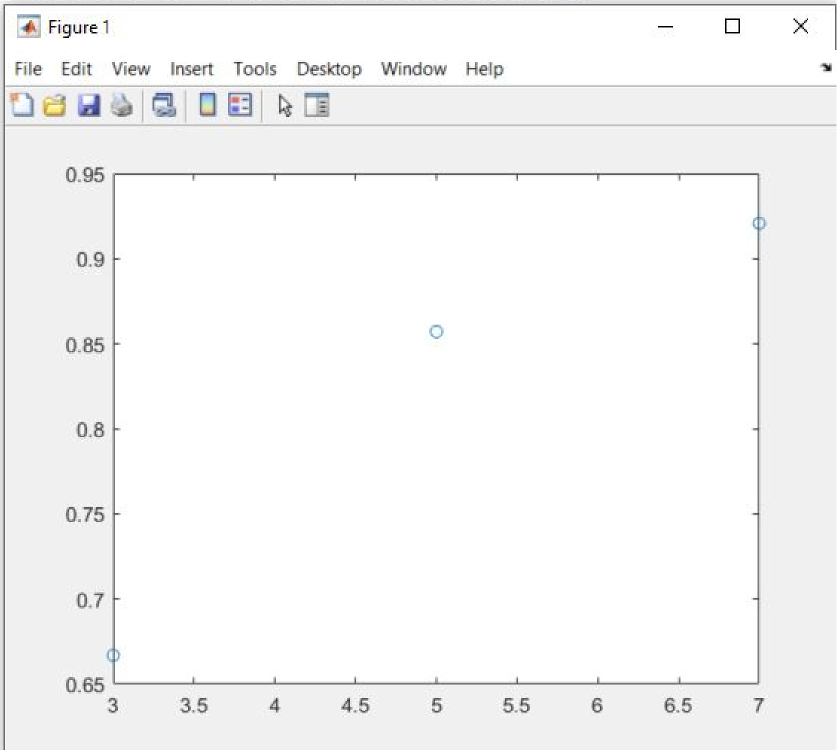
```
>> [mu,v]=symmjacobi(5,1e-6,300)
The maximum number of iterations exceeded
mu =

    0.8571


v =

    0.3086
    0.4629
    0.6172
    0.4629
    0.3086
```

- n=7: For N=10, 100, and 1000, mu=0.9210 $\Rightarrow$ $\rho(T_j)$=0.9210<1 $\Rightarrow$ $\|\mathbf{e}^{(k+1)}\|_2 \rightarrow 0$ as $k \rightarrow \infty$

```
>> [mu,v]=symmjacobi(7,1e-6,300)
The maximum number of iterations exceeded
mu =

    0.9210


v =

    0.1988
    0.3393
    0.4798
    0.4798
    0.4798
    0.3393
    0.1988
```

- Plotting largest eigenvalue against n:

```
>> n=[3,5,7];mu=[0.6667,0.8571,0.9210];plot(n,mu,'o')
>>
```



e) Gauss-Seidel

Using tol=1e-6, and N=300:

- n=3: For N=10, 100, and 1000, mu=0.5000 ⇒ $\rho(T_g)$=0.5000<1 ⇒ $\|\mathbf{e}^{(k+1)}\|_2 \to 0$ as $k \to \infty$

```
>> [mu,v]=symmgauss(3,1e-6,300)
The maximum number of iterations exceeded
mu =

    0.5000


v =

    0.6667
    0.6667
    0.3333
```

- n=5: For N=10, 100, and 1000, mu=0.7500 $\Rightarrow$ $\rho(T_g)$=0.7500<1 $\Rightarrow$ $||\mathbf{e}^{(k+1)}||_2 \rightarrow 0$ as k $\rightarrow \infty$

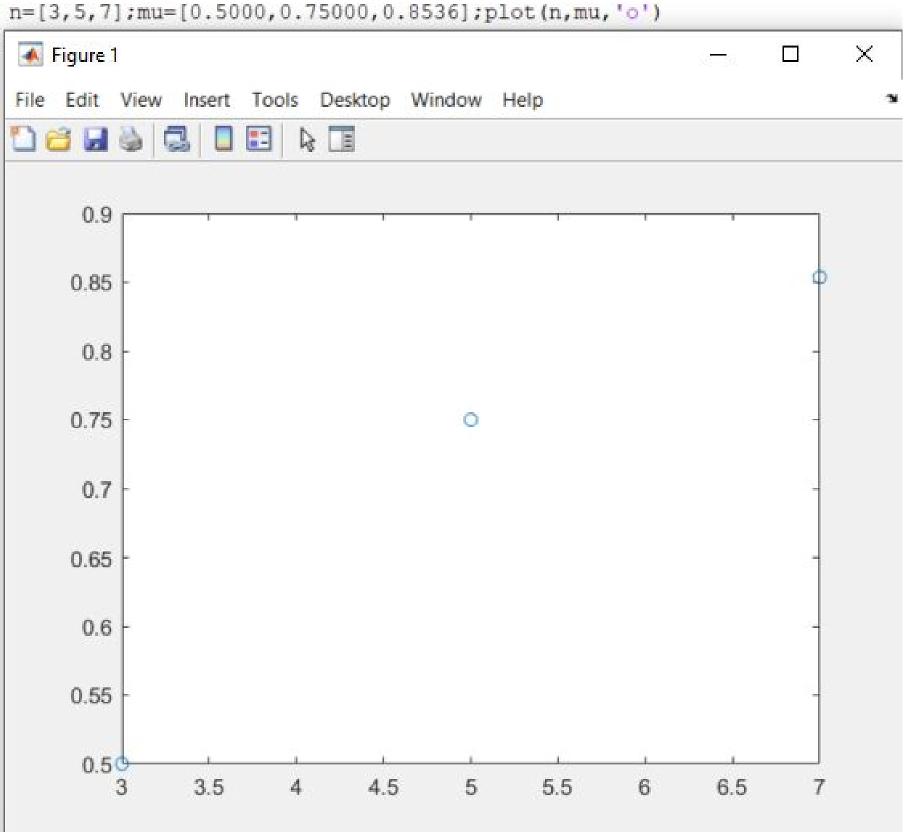
```
>> [mu,v]=symmgauss(5,1e-6,300)
The maximum number of iterations exceeded
mu =

    0.7500


v =

    0.3758
    0.5637
    0.5637
    0.4227
    0.2114
```

- n=7: For N=10, 100, and 1000, mu=0.8536 $\Rightarrow$ $\rho(T_g)$=0.8536<1 $\Rightarrow$ $||\mathbf{e}^{(k+1)}||_2 \rightarrow 0$ as k $\rightarrow \infty$

```
>> [mu,v]=symmgauss(7,1e-6,300)
The maximum number of iterations exceeded
mu =

    0.8536


v =

    0.2395
    0.4088
    0.4935
    0.4935
    0.4212
    0.2979
    0.1489
```

- Plotting largest eigenvalue against n:

```
>> n=[3,5,7];mu=[0.5000,0.75000,0.8536];plot(n,mu,'o')
>>
```



SOR

Using tol=1e-6, and N=300:

- n=3: For N=10, mu=0.2921 and for N=100, N=1000, mu=0.2918 $\Rightarrow$ $\rho(T_s)=0.2918<1$

  $\Rightarrow$ $\|\mathbf{e}^{(k+1)}\|_2 \to 0$ as $k \to \infty$

```
>> w=2/(1+sqrt(1-(symmjacobi(3,1e-6,300))^2));[mu,v]=symmSOR(w,3,1e-6,300)
The maximum number of iterations exceededThe maximum number of iterations exceeded
mu =

    0.2918


v =

    0.7741
    0.5914
    0.2259
```

- n=5: For N=10, mu=0.4522 and for N=100, N=1000, mu=0.4268 $\Rightarrow$ $\rho(T_s)=0.4268<1$

  $\Rightarrow$ $\|\mathbf{e}^{(k+1)}\|_2 \to 0$ as $k \to \infty$

```
>> w=2/(1+sqrt(1-(symmjacobi(5,1e-6,300))^2));[mu,v]=symmSOR(w,5,1e-6,300)
The maximum number of iterations exceededThe maximum number of iterations exceeded
mu =

    0.4268


v =

    0.5526
    0.6252
    0.4716
    0.2668
    0.1006
```

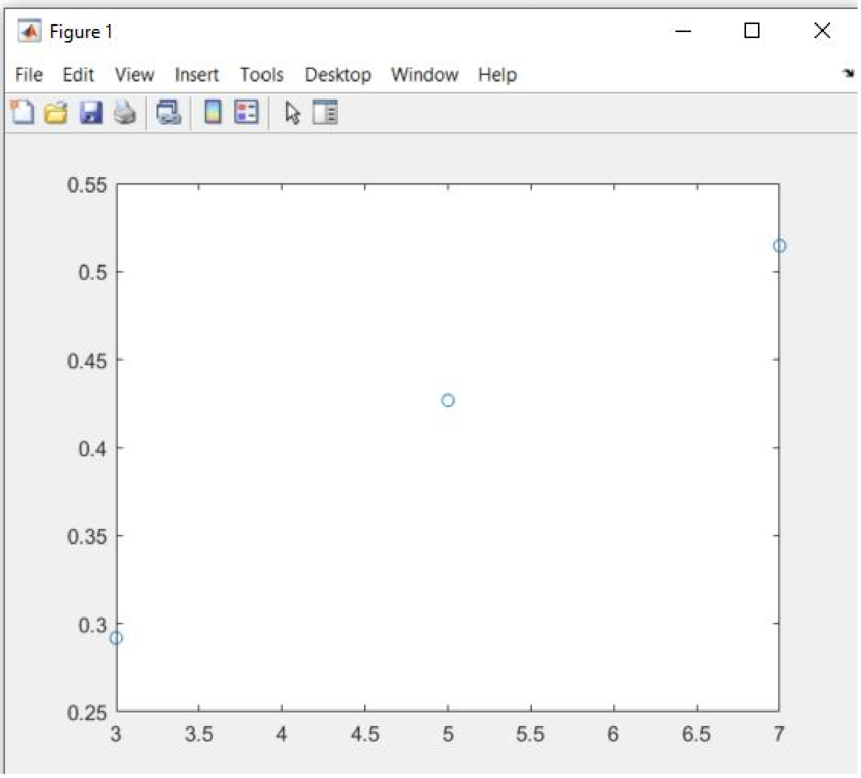
- n=7: For N=10, mu=0.5255 and for N=100, N=1000, mu=0.5146 $\Rightarrow$ $\rho(T_s)=0.5146<1$

  $\Rightarrow$ $\|\mathbf{e}^{(k+1)}\|_2 \to 0$ as $k \to \infty$

```
>> w=2/(1+sqrt(1-(symmjacobi(7,1e-6,300))^2));[mu,v]=symmSOR(w,7,1e-6,300)
The maximum number of iterations exceededThe maximum number of iterations exceeded
mu =

    0.5146


v =

    0.4159
    0.5512
    0.5167
    0.4012
    0.2659
    0.1460
    0.0567
```

- Plotting largest eigenvalue against n:

```
>> n=[3,5,7];mu=[0.2918,0.4268,0.5146];plot(n,mu,'o')
>>
```



f) The smaller $\rho(T)$ is, the faster $\rho(T)^k$ converges to 0 and thus the faster λ^k and $\|\mathbf{e}^{(k+1)}\|_2 = \|\lambda^k \mathbf{e}^{(0)}\|_2$ converge to 0. Since the estimated $\rho(T_j)$, $\rho(T_g)$, and $\rho(T_s)$ satisfy $\rho(T_s) \leq \rho(T_g) \leq \rho(T_j)$

for all n, the SOR method converges the fastest. For all methods, $\rho(T)$ increases as n increases. Therefore, as n increases, the rate of convergence decreases.