1-a

$$y_1 = [1 \ x_1 \ x_1^2 \ x_1^3 \ldots x_1^d] \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix} \qquad y_2 = [1 \ x_2 \ x_2^2 \ x_2^3 \ldots x_2^d] \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix}$$

$$y_3 = [1 \ x_3 \ x_3^2 \ x_3^3 \ldots x_3^d] \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix} \qquad y_4 = [1 \ x_4 \ x_4^2 \ x_4^3 \ldots x_4^d] \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix}$$

$$y_5 = [1 \ x_5 \ x_5^2 \ x_5^3 \ldots x_5^d] \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix} \qquad y_n = [1 \ x_n \ x_n^2 \ x_n^3 \ldots x_n^d] \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix}$$

$$\Rightarrow \underset{Y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \\ y_n \end{bmatrix}} = \underset{w}{\begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^d \\ 1 & x_2 & x_2^2 & x_2^3 & \ldots & x_2^d \\ 1 & x_3 & x_3^2 & x_3^3 & \ldots & x_3^d \\ 1 & x_4 & x_4^2 & x_4^3 & \ldots & x_4^d \\ 1 & x_5 & x_5^2 & x_5^3 & \ldots & x_5^d \\ \vdots & \vdots & \vdots & \vdots & & \\ 1 & x_n & x_n^2 & x_n^3 & \ldots & x_n^d \end{bmatrix}}$$

w - $1 \times (d+1)$ matrix

~~vecto~~ vector of size $d+1$

y - $n \times 1$ matrix

vector of size n

1-b

A - $n \times (d+1)$ matrix

1-c We have: $n = d+1$
A matrix

$$\det(A) = \prod_{1 \le i \le j \le n} (x_j - x_i)$$

$n=2$

$$\Rightarrow A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \quad \det(A) = (x_2 - x_1)$$

Let $n=k \Rightarrow n = k+1$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^n \\ 1 & x_4 & x_4^2 & x_4^3 & \cdots & x_4^n \\ 1 & x_5 & x_5^2 & x_5^3 & \cdots & x_5^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{k+1} & x_{k+1}^2 & x_{k+1}^3 & & x_{k+1}^{k+1} \end{bmatrix}$$

$\Rightarrow C_{k+1} - x_1 C_k \Rightarrow$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & 0 \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{k+1} - x_1 x_2^k \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^{k+1} - x_1 x_3^k \\ 1 & x_4 & x_4^2 & x_4^3 & \cdots & x_4^{k+1} - x_1 x_4^k \\ 1 & x_5 & x_5^2 & x_5^3 & \cdots & x_5^{k+1} - x_1 x_5^k \\ \vdots & & & & & \\ 1 & x_{k+1} & x_{k+1}^2 & x_{k+1}^3 & \cdots & x_{k+1}^{k+1} - x_1 x_{k+1}^k \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & 0 \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & (x_2 - x_1) x_2^k \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & (x_3 - x_1) x_3^k \\ 1 & x_4 & x_4^2 & x_4^3 & \cdots & (x_4 - x_1) x_4^k \\ 1 & x_5 & x_5^2 & x_5^3 & \cdots & (x_5 - x_1) x_5^k \\ \vdots & \vdots & \vdots & \vdots & & \\ 1 & x_{k+1} & x_{k+1}^2 & x_{k+1}^3 & \cdots & (x_{k+1} - x_1) x_{k+1}^k \end{bmatrix}$$

$C_k - x_1 C_{k-1}$,

$C_{k-1} - x_1 C_{k-2}$,

$\vdots$

$C_4 - x_1 C_3$,

$C_3 - x_1 C_2$,

$C_2 - x_1 C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & (x_2 - x_1)x_2 & (x_2 - x_1)x_2^2 & \cdots & (x_2 - x_1)x_2^k \\ 1 & x_3 - x_1 & (x_3 - x_1)x_3 & (x_3 - x_1)x_3^2 & \cdots & (x_3 - x_1)x_3^k \\ 1 & x_4 - x_1 & (x_4 - x_1)x_4 & (x_4 - x_1)x_4^2 & \cdots & (x_4 - x_1)x_4^k \\ 1 & x_5 - x_1 & (x_5 - x_1)x_5 & (x_5 - x_1)x_5^2 & \cdots & (x_5 - x_1)x_5^k \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{k+1} - x_1 & (x_{k+1} - x_1)x_{k+1} & (x_{k+1} - x_1)x_{k+1}^2 & \cdots & (x_{k+1} - x_1)x_{k+1}^k \end{bmatrix}$$

$$\det(A) = \det \begin{bmatrix} x_2 - x_1 & (x_2 - x_1)x_2 & (x_2 - x_1)x_2^2 & \cdots & (x_2 - x_1)x_2^k \\ x_3 - x_1 & (x_3 - x_1)x_3 & (x_3 - x_1)x_3^2 & \cdots & (x_3 - x_1)x_3^k \\ x_4 - x_1 & (x_4 - x_1)x_4 & (x_4 - x_1)x_4^2 & \cdots & (x_4 - x_1)x_4^k \\ x_5 - x_1 & (x_5 - x_1)x_5 & (x_5 - x_1)x_5^2 & \cdots & (x_5 - x_1)x_5^k \\ \vdots & \vdots & \vdots & & \vdots \\ x_{k+1} - x_1 & (x_{k+1} - x_1)x_{k+1} & (x_{k+1} - x_1)x_{k+1}^2 & & (x_{k+1} - x_1)x_{k+1}^k \end{bmatrix}$$

$$= \prod_{2 \le j \le (k+1)} (x_j - x_1) \det \begin{bmatrix} 1 & x_2 & x_2 & \cdots & x_2^k \\ 1 & x_3 & x_3 & \cdots & x_3^k \\ 1 & x_4 & x_4 & \cdots & x_4^k \\ 1 & x_5 & x_5 & \cdots & x_5^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{k+1} & x_{k+1}^2 & \cdots & x_{k+1}^k \end{bmatrix} = \prod_{2 \le i \le j \le (k+1)} (x_j - x_i)$$

$$\Rightarrow \left(\prod_{2 \le j \le (k+1)} (x_j - x_1) \right) \left(\prod_{2 \le i < j \le (k+1)} (x_j - x_i) \right) = \prod_{1 \le i < j \le (k+1)} (x_j - x_i)$$

$$\Rightarrow \det(A) = \prod_{1 \le i < j \le n} (x_j - x_i)$$

1-d

We have $\det(A) = \prod_{1 \le i < j \le n} (x_j - x_i)$

$\det(A) \neq 0 \Leftrightarrow x_j \neq x_i$ for all $1 \le i < j \le n$

1-e

$\det(A) \neq 0 \Rightarrow A^{-1}$ exist

$Aw = y \Rightarrow A^{-1} y = w$

$w = A^{-1} y$

2. We have:

$n > d+1$ $\quad Aw = y$ $\quad$ A - $n \times (d+1)$ matrix
w - $1 \times (d+1)$ matrix or vector of size $(d+1)$
y - vector of size n

1. Write matrix A included with y as : y $\quad [A/y]$
2. Write matrix A into row echelon form and the same operations are applied on y which are applied on A.
3. Let ~~Row~~ reduced form of A ~~as~~ be B, y be Y

$\Rightarrow Bw = Y$

4. Equating both sides we can find the elements of w.