Exercises on Factorization into A = LU

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4.1

First, we go through elimination on matrix A. To eliminate the first column:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix}$$

To eliminate the second column:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To summarize the elimination process, we have $E_2E_1A = U$ where $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$.

We can rewrite the equation into $A = E_1^{-1}E_2^{-1}U$ and $A = LU$. Then it leads to

$$L = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

4.2

To eliminate the first column:

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

To eliminate the second column:

$$\begin{bmatrix} a\ a\ a\ a \\ 0\ b-a\ b-a\ b-a \\ 0\ b-a\ c-a\ c-a \\ 0\ b-a\ c-a\ d-a \end{bmatrix} \begin{bmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ \ -1\ 1\ 0 \\ 0\ \ -1\ 0\ 1 \end{bmatrix} = \begin{bmatrix} a\ a\ a\ a \\ 0\ b-a\ b-a\ bia \\ 0\ 0\ c-b\ c-b \\ 0\ 0\ c-b\ d-b \end{bmatrix}$$

To eliminate the third column:

$$\begin{bmatrix} a\ a\ a\ a \\ 0\ b-a\ b-a\ bia \\ 0\ 0\ c-b\ c-b \\ 0\ 0\ c-b\ d-b \end{bmatrix} \begin{bmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ \ -1\ 1 \end{bmatrix} = \begin{bmatrix} a\ a\ a\ a \\ 0\ b-a\ b-a\ bia \\ 0\ 0\ c-b\ c-b \\ 0\ 0\ 0\ d-c \end{bmatrix}$$

To summarize, we have $E_3E_2E_1A = U$, so $A = E_1^{-1}E_2^{-1}E_3^{-1}U$. Therefore,

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1\ 0\ 0\ 0 \\ 1\ 1\ 0\ 0 \\ 1\ 1\ 1\ 0 \\ 1\ 1\ 1\ 1 \end{bmatrix}$$

The four conditions to get $A = LU$ with four pivots are: $a \neq 0$, $a \neq b$, $b \neq c$ and $c \neq d$.