Unit 1 Exam

January 10, 2017

1

Soltuions to $Ax = b$ have the form $x = x_p + cx_s$. Since Ax has one solution to some right-hand side b, it means that there are no free variables and the rank r equals the number of columns n. Since Ax has no solution to some other right-hand side b, it means that the rank of row space is smaller than the number of rows. In other words, $r < m$. To summarize, $r = n < m$.

2

(a) Since $E_{23}E_{31}E_{21}A = I$, it leads to $A^{-1} = E_{21}^{-1}E_{31}^{-1}E_{23}^{-1}$.

$$A^{-1} = E_{23}E_{31}E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix}$$

(b) $A = E_{21}^{-1}E_{31}^{-1}E_{23}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

(c) Since $U = I$ in this $A = LU$ equation, $L = A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$.

3

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & c-3 & -4 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) When $c - 3 \neq 0$ or $c \neq 3$, there are three pivots in A. The basis for the column space of A is $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. When $c = 0$, there are only two pivots in A so the basis is $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

(b) When $c - 3 \neq 0$ or $c \neq 3$, there are three pivots and one free variable in A so there is one special solution. The basis for the nullspace of A is $\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. When $c = 3$, there are two free variables and thus two special solutions. The basis in this case would be $\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) Perform row reduction on $Ax = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$, we have

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 1 \\ 0 & c-3 & -4 & -4 & c-3 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

When $c - 3 \neq 0$ or $c \neq 3$, the complete solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. When $c = 3$, the complete solution is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + f \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

4

(a) The nullspace of A equals $n - R(A)$. Since $R(A) >= 0 and n = 5$, the dimension of $N(A)$ is between 2 and 5 (inclusive).

(b) Since there are three pivots and two free variables, the column space of A has dimension of 3 and the basis is column 1, 4, 5 of A.

(c) In vector space M of 3 by 3 matrices, subspace S spanned by rref contain matrices of form

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$