symmetric matrices and positive definiteness

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25.1

Writing out the first step of the proof:

$$Ax = \lambda x$$

left-hand side:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

right-hand side:

$$\begin{bmatrix} i^2 \\ i \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} \checkmark$$

Writing out the second step:

$$x^T Ax = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ & 1 \\ & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} -1 \\ i \end{bmatrix} = -i + i = 0$$

$$\lambda x^T x = i \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = i(i^2 + 1) = 0 \checkmark$$

The third step, $\lambda = \frac{x^T Ax}{x^T x}$, is under the assumption that $x^T x$ does not equal zero, and in the example above the product is zero. That's why the proof is false.

25.2

(a) If A and B are a positive definite symmetric matrices, for example,
$A = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. Then $AB = \begin{bmatrix} 7 & 23 \\ 15 & 76 \end{bmatrix}$ which is not symmetric. So, A is not a group.

(b) If Q is orthogonal, $(Q^{-1})^T Q^{-1} = QQ^{-1} = I$. If P and Q are orthogonal matrices, then $(PQ)^T PQ = Q^T P^T PQ = I$. The results are both still orthogonal; Q is a group.

(c)

$$(e^{tA})^{-1} = e^{-tA}$$

$$e^{mA}e^{nA} = e^{(m+n)A}$$

The exponential is a group.

(d)

$$D^{-1} = \frac{1}{det(D)}D = D$$

$$det(AB) = det(A)det(B) = 1$$

Both matrices still have determinant 1. D is a group.