Exercises on graphs, networks, and incidence matrices

January 9, 2017

12.1

Incidence matrix $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. Vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in the nullspace of A. (1,0,0,0) is not in the row space of A because a row of incidence matrix represents the nodes through which the current flows and there are two nonzero elements instead of one in a row.

12.2

To find A^TCA:

$$A^TCA = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & -2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ -1 & 0 & -2 & 3 \end{bmatrix}$$

Since $A^TCAx = f$, we can find x by doing elimination:

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & -2 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -2 & 4 & -2 & -1 \\ 0 & -\frac{1}{2} & -2 & \frac{5}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{12}{5} & -\frac{12}{5} & -\frac{3}{5} \\ 0 & 0 & -\frac{12}{5} & \frac{12}{5} & \frac{3}{5} \end{array}\right] \to$$
$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & -1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

One solutions is $x = \begin{bmatrix} \frac{3}{2} \\ 1 \\ \frac{3}{4} \\ 1 \end{bmatrix}$.

Since current $y = -CAx$, $y = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ 1 \\ \frac{3}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ 1 \\ \frac{3}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$