Exercises on differential equations and e^{At}

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23.1

$\frac{d||u(t)||^2}{dt} = \frac{d(u_1^2+u_2^2+u_3^2)}{dt} = 2u_1u_1'+2u_2u_2'+2u_3u_3' = 2u_1(cu_2-bu_3)+2u_2(au_3-cu_1)+2u_3(bu_1-au_2) = 0$
This means that

23.2

Diagonalize A:

$$det(A-\lambda I) = det\begin{bmatrix}1-\lambda 1\\0\ 3-\lambda\end{bmatrix} = (1-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, 3$$

When $\lambda = 1$, $(A-\lambda I)x = \begin{bmatrix}0 & 1\\0 & 2\end{bmatrix} x = 0 \rightarrow x = \begin{bmatrix}1\\0\end{bmatrix}$.

When $\lambda = 3$, $(A-\lambda I)x = \begin{bmatrix}-2 & 1\\0 & 0\end{bmatrix} x = 0 \rightarrow x = \begin{bmatrix}1\\2\end{bmatrix}$.

Therefore, $A = \begin{bmatrix}1 & 1\\0 & 3\end{bmatrix} = S\Lambda S^{-1}$ where $S = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$ and $\Lambda = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$.

$$e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}e^t & 0\\0 & e^{3t}\end{bmatrix}\cdot\frac{1}{2}\cdot\begin{bmatrix}2 & -1\\ & 0\ 1\end{bmatrix} = \begin{bmatrix}e^t & e^{3t}\\0 & 2e^{3t}\end{bmatrix}\cdot\frac{1}{2}\cdot\begin{bmatrix}2 & -1\\ & 0\ 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2e^t & -e^t+e^{3t}\\ & 0\ 2e^{3t}\end{bmatrix}$$

$$e^{At}|_{t=0} = \frac{1}{2}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

$$\frac{d(d^{At})}{dt} = \frac{1}{2}\begin{bmatrix}2e^t & -e^t+3e^{3t}\\ & 0\ 6e^{3t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 2\\0 & 6\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 3\end{bmatrix} = A$$

$$\frac{ud}{dt} = Au\checkmark$$