Exercises on projection matrices and least squares

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16.1

We can rewrite the problem as $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. Multiply each side by A^T, we want to find $\hat{x}$ that satisfies $A^T A\hat{x} = A^T b$.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$$

We can solve the above and get $\hat{x} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$.

16.2

$$p = A\hat{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$$

$$Pe = P(b - p) = Pb - Pp = p - p = 0$$

16.3

When $b = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$, b is perpendicular to the column space of A.

16.4

$hatx$ remains the same as before. The error $e = 0$ because b is in the column space of A.

16.5

The error vector e is in the left nullspace of A. p is in the column space of A. $\hat{x}$ is in the row space of A. The nullspace of A is the zero vector.

16.6

In an attempt to solve $\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, we multiply both sides by A^T.

$$A^T A = \begin{bmatrix} -5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

Solving the above, we get $\hat{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore, the best line is $y = 1 - t$.