Exercises on singular value decomposition

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29.1

$$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$det(A^T A) = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{5}}{2}$$

We know that $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$, so we only need to verify that the two entries in the given Σ square up to λ. And indeed, $(\frac{1+\sqrt{5}}{2})^2 = \frac{3+\sqrt{5}}{2}$ and $(\frac{\sqrt{5}-1}{2})^2 = \frac{3-\sqrt{5}}{2}$.

29.2

If A has orthogonal columns, then $A^T A = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}$ is a diagonal matrix with σ^2 on the diagonal. Since we also know that $A^T A = V\Sigma^2 V^T$, it leads to $V = I$. Lastly, since $A = U\Sigma V = U\Sigma$, we have:

$$U = A\Sigma^{-1} = \begin{bmatrix} \frac{w_1}{\sigma_1} & \frac{w_2}{\sigma_2} & \dots & \frac{w_n}{\sigma_n} \end{bmatrix}$$