A Module on the Maths behind Q6-7

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Note: Practice questions are denoted by a blue box. There are three practice questions in this module; you are only required to do one. Animations are denoted by a red box. Sections that are labeled with "enrichment" are completely optional. Additionally, please answer the survey question immediately below.

Survey Question

After reading the module and doing the associated activities, summarize, in one to two sentences, the most important concept you learned through the module. If you have already seen everything in this module before, "nothing" is also a valid answer. Additionally, if anything was particularly confusing, please make a note of it here too.

A Whiff of Linear Algebra

In Q7, you encountered the beginnings of Dirac's bra-ket notation for quantum mechanics. To briefly review, recall the notion of a **q-vector** for **bras** and **kets**:

$$\text{A ket is } |\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{bmatrix}, \quad \text{while a bra is } \langle\psi| = \begin{bmatrix} \psi_1^* & \psi_2^* & \cdots \end{bmatrix}.$$

The funny stars at the top right of the bra vector denote the complex conjugate of the entries in the ket vector. However, for the purposes of this module, we will focus on when these entries are purely real numbers, or when $a^* = a$.

We will see that, with this restriction, the **inner product** of these two vectors is equivalent to the dot product between the two vectors. Recall that, for two vectors $\vec{v}$ and $\vec{w}$, the dot product is given by:

$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z.$$

This is the same as the inner product of two q-vectors, $\langle\psi|$ and $|\phi\rangle$, which is given by:

$$\langle\psi|\phi\rangle = \begin{bmatrix} \psi_1 & \psi_2 & \cdots \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{bmatrix} = \psi_1\phi_1 + \psi_2\phi_2 + \psi_3\phi_3 + \ldots = \sum_i \psi_i\phi_i.$$

The difference here is that the q-vectors can have any number of entries depending on the physical system being studied, while the spatial vectors $\vec{v}$ and $\vec{w}$ are strictly three-dimensional (since we live in three-dimensional space). For instance, if we were looking at q-vectors corresponding to the electron's spin in the z-direction, we would have a two-dimensional q-vector, since the electron can either be spin up or spin down.

Essentially, for two q-vectors, each with k components, their inner product is equivalent to the dot product in k-dimensional space!

Now that we have established their equivalence, let's see how the dot product or inner product can be interpreted as a measure of "similarity" between two vectors. It turns out that the dot product can be written in another way:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta),$$

where θ is the angle between the two vectors and $|\cdot|$ represents the magnitude of a vector, or its length. If the two vectors are parallel, then $\theta = 0$, and the dot product is maximized since $\cos\theta = 1$. If the two vectors are perpendicular, then $\theta = \pi/2$, and the dot product is zero. In this case, the vectors are known to be **orthogonal**. If $\theta = \pi$, the dot product is *minimized*, implying the vectors have maximum "anti-similarity". This is the same for the inner product of two q-vectors: if the two vectors are parallel or "maximally similar", the inner product is maximized, while if the two vectors are orthogonal, the inner product is zero.

What happens if you take the inner product of a q-vector with itself? Well, let's appeal to its equivalence to the dot product. If we take the dot product of a vector with itself, we get:

$$\vec{v} \cdot \vec{v} = v_x^2 + v_y^2 + v_z^2 = |\vec{v}|^2.$$

You may recognize this as the three-dimensional analogue of the Pythagorean theorem. Indeed, the square length of a vector is the sum of the squares of its components. This is the same for the inner product of a q-vector with itself, where we may think of the equation below as a k-dimensional analogue of the Pythagorean Theorem:

$$\langle \psi | \psi \rangle = \sum_i \psi_i^2 = |\psi|^2.$$

To enforce the equivalence between the dot product and the q-inner product, let's look at a few exercises.

Practice 1

Consider a mysterious physical system that can take on three physical states: A, B, C. State $|\alpha\rangle$ has two particles in state A and one particle in state B, so

$$|\alpha\rangle = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Meanwhile, state $|\beta\rangle$ has one particle in state B and two particles in state C, so

$$|\beta\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Practice 1

a) Calculate $\langle\alpha|\alpha\rangle$. Remember that $\langle\alpha|$ is the row vector version of $|\alpha\rangle$.

b) Calculate $\langle\beta|\beta\rangle$.

c) How similar are the two states $|\alpha\rangle$ and $|\beta\rangle$? Essentially, what is

$$\frac{\langle\alpha|\beta\rangle}{\sqrt{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}}? \tag{1}$$

d) What is the maximum attainable value of the expression in Eq. 1? Consider arbitrary q-vectors $|\alpha\rangle$ and $|\beta\rangle$. *Hint: you can make qualitative or quantitative arguments here.*

Practice 2

A beam of electrons are sent through a Stern-Gerlach apparatus, like the SG machines in Q7. The beam is initially in some state $|+\theta\rangle$ as in Table Q7.1. As it passes through an SG_z detector, the following data is recorded:

State	Counts
Spin up	75
Spin down	25

a) What is θ? *Hint: the probabilities are **normalized**, meaning* $|\langle+\theta|+z\rangle|^2 + |\langle+\theta|-z\rangle|^2 = 1$.

b) What would the data look like if the initial state was instead $|-x\rangle$?

Everything below this sentence is enrichment material.

An Introduction to Complex Numbers (Enrichment)

Complex numbers were first invented by an Italian mathematician named Gerolamo Cardano in the 16th century to help find a general solution to cubic equations. The key ingredient to complex numbers is the imaginary number i, which by definition satisfies

$$i^2 = -1.$$

Having this, we can now define a complex number z as

$$z = a + bi,$$

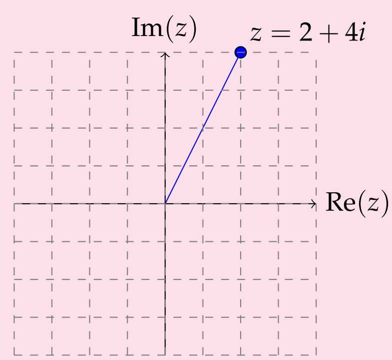
where a and b are real numbers. The real part of z is denoted by $\mathrm{Re}(z)$, while the imaginary part of z is denoted by $\mathrm{Im}(z)$. The complex conjugate of z, denoted by z^*, is given by

$$z^* = a - bi.$$

A special property of complex numbers and their conjugate is that the product of a complex number and its conjugate is always real:

$$zz^* = (a + bi)(a - bi) = a^2 + b^2 = |z|^2,$$

where $|z|$ is the magnitude of the complex number z. One can intuitively imagine the complex plane as a two-dimensional plane, where the x-axis is the real part of the complex number and the y-axis is the imaginary part. Thus, we can "plot" any complex number on this plane. Such a diagram is called an **Argand diagram**. To illustrate, we can plot the complex number $z = 2 + 4i$ on the Argand diagram, as shown on the right:



Now that we have established the nature of bras and kets, we can now introduce the **inner product** of two vectors. In regular Euclidean space, the inner product of two vectors is the dot product, as highlighted above. However, with complex numbers, the inner product is slightly different, and is given the name **Hermitian inner product**. The Hermitian inner product of two vectors $\langle\psi|$ and $|\phi\rangle$ is given by

$$\text{For } \langle\psi| = \begin{bmatrix} \psi_1^* & \psi_2^* & \cdots \end{bmatrix} \text{ and } |\phi\rangle = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{bmatrix}, \quad \langle\psi|\phi\rangle = \sum_i \psi_i^* \phi_i.$$

You may notice that if we reverse the order of the vectors, we have

$$\langle\phi|\psi\rangle = \sum_i \phi_i^* \psi_i = \left(\sum_i \psi_i^* \phi_i\right)^* = \langle\psi|\phi\rangle^*.$$

This is a property unique to the Hermitian inner product: the regular (Euclidean) inner product does not have this property, since all real numbers satisfy $a = a^*$.

An Introduction to Complex Numbers (Enrichment)

As outlined in Q7.3, there are a couple more properties and vocabulary to take note of. If we take the inner product of a q-vector with itself, we get

$$\langle\psi|\psi\rangle = \sum_i \psi_i^* \psi_i = \sum_i |\psi_i|^2.$$

This looks a lot like the *square* of the magnitude of the q-vector. Indeed! It turns out that the inner product of a q-vector with itself is equal to the square of the q-vector's magnitude.

Also recall that the angle between two vectors can be found using the dot product for the Euclidean case. Using this as a heuristic, we see that if $\theta = \pi/2$, the dot product must be zero, since $\cos(\pi/2) = 0$. This translates to the vectors being perpendicular, or more generally, **orthogonal** to each other. In the context of quantum mechanics, we say that two vectors are orthogonal if their (Hermitian) inner product is zero.

Practice 3

Consider the two q-vectors $|\psi\rangle = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$ and $|\phi\rangle = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$. Are they orthogonal?