Solutions for HW10

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Practice 1

Consider a mysterious physical system that can take on three physical states: A, B, C. State $|\alpha\rangle$ has two particles in state A and one particle in state B, so

$$|\alpha\rangle = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Meanwhile, state $|\beta\rangle$ has one particle in state B and two particles in state C, so

$$|\beta\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

a) Calculate $\langle\alpha|\alpha\rangle$. Remember that $\langle\alpha|$ is the row vector version of $|\alpha\rangle$.

b) Calculate $\langle\beta|\beta\rangle$.

c) How similar are the two states $|\alpha\rangle$ and $|\beta\rangle$? Essentially, what is

$$\frac{\langle\alpha|\beta\rangle}{\sqrt{\langle\alpha|\alpha\rangle\,\langle\beta|\beta\rangle}}? \tag{1}$$

d) What is the maximum attainable value of the expression in Eq. 1? Consider arbitrary q-vectors $|\alpha\rangle$ and $|\beta\rangle$. *Hint: you can make qualitative or quantitative arguments here.*

Solution:

a)

$$\langle \alpha | \alpha \rangle = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2^2 + 1^2 + 0^2 = |\alpha|^2 = 5.$$

b)

$$\langle \beta | \beta \rangle = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0^2 + 1^2 + 2^2 = |\beta|^2 = 5.$$

c) The reason we have the denominator is to noramlize the inner product. With this in mind, we have that

$$\frac{\langle \alpha | \beta \rangle}{\sqrt{\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle}} = \frac{\langle \alpha | \beta \rangle}{|\alpha| \, |\beta|}$$

Now, we can calculate the numerator:

$$\langle \alpha | \beta \rangle = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 2(0) + 1(1) + 0(2) = 1.$$

Therefore,

$$\frac{\langle \alpha | \beta \rangle}{\sqrt{\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle}} = \frac{1}{\sqrt{5}\sqrt{5}} = \frac{1}{5}.$$

d) The maximum value of the expression in Eq. 1 can be obtained when we just set the q-vectors to be the same (this does indeed reflect maximal similarity!). In this case, we have that

$$\frac{\langle \alpha | \alpha \rangle}{\sqrt{\langle \alpha | \alpha \rangle \langle \alpha | \alpha \rangle}} = \frac{|\alpha|^2}{\sqrt{|\alpha|^2 \, |\alpha|^2}} = \frac{|\alpha|^2}{|\alpha|^2} = 1.$$

Practice 2

A beam of electrons are sent through a Stern-Gerlach apparatus, like the SG machines in Q7. The beam is initially in some state $|+\theta\rangle$ as in Table Q7.1. As it passes through an SG_z detector, the following data is recorded:

State	Counts
Spin up	75
Spin down	25

a) What is θ? *Hint: the probabilities are* ***normalized****, meaning* $|\langle+\theta|+z\rangle|^2 + |\langle+\theta|-z\rangle|^2 = 1$.

b) What would the data look like if the initial state was instead $|-x\rangle$?

Solution:

a) Using the hint, we first calculate

$$\langle+\theta|+z\rangle = \cos(\theta/2), \quad \langle+\theta|-z\rangle = \sin(\theta/2).$$

One can see that the normalization condition is a good sanity check, as

$$|\langle+\theta|+z\rangle|^2 + |\langle+\theta|-z\rangle|^2 = \cos^2(\theta) + \sin^2(\theta) = 1.$$

However, we also know that the probabilities are

$$P(+z) = |\langle+\theta|+z\rangle|^2 = \cos^2(\theta/2) = \frac{75}{100} = \frac{3}{4}, \quad P(-z) = |\langle+\theta|-z\rangle|^2 = \sin^2(\theta/2) = \frac{25}{100} = \frac{1}{4}.$$

With this in mind, we can solve for θ, which technically yields two solutions. We will focus on one for now, where

$$\theta = 2\cos^{-1}\left(\sqrt{\frac{3}{4}}\right) = 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 2 \times \frac{\pi}{6} = \frac{\pi}{3}.$$

Another valid solution is $\theta = -\pi/3$.

b) If the initial state was $|-x\rangle$, then the probabilities would be

$$P(+z) = |\langle-x|+z\rangle|^2 = \frac{1}{2}, \quad P(-z) = |\langle-x|-z\rangle|^2 = \frac{1}{2}.$$

Therefore, the data would be

State	Counts
Spin up	50
Spin down	50

Practice 3

Consider the two q-vectors $|\psi\rangle = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$ and $|\phi\rangle = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$. Are they orthogonal?

Solution:

Let's take their (Hermitian) inner product:

$$\langle\psi|\phi\rangle = \begin{bmatrix} 1-i & -1 \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} = (1-i)(1+i) + (-1)(2) = 1 + i - i + 1 - 2 = 0.$$

Since the inner product is zero, the two q-vectors are orthogonal.