

# Math 279R Exercises/Problems/Projects

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Exercises are meant to be relatively short “sanity checks.” Problems are slightly more involved. Potential final project topics are listed throughout; it can be an exposition of progress on a long-standing conjecture or an important topic in matroid theory not covered in lecture, or a short research project.

For full mark on the problem set portion (40%) of the final grade, you must complete at least  $2/3$  of the exercises and  $1/2$  of the problems. You may collaborate with others, or consult references, as long as you write-up your own solutions with clear indications of your sources.

## Lecture 1

**Exercise 1.** From the basis exchange axiom, show that every bases of a matroid has the same cardinality.

**Project 1.** (Max distance separable conjecture). For a fixed finite field  $\mathbb{k} = \mathbb{F}_q$ , and a fixed integer  $r \geq 0$ , what is the largest  $N$  such that the uniform matroid  $U_{r,N}$  of rank  $r$  on  $N$  elements is realizable? (This is a long-standing open problem).

## Lecture 2

**Exercise 2.** Let  $M$  be a loopless matroid on a ground set  $E$ . Define a relation  $//$  on  $E$  by declaring that  $i//j$  if  $\{ij\}$  not in any basis of  $M$ . Show that  $//$  is an equivalence relation. Moreover, show that if  $i \in E$  is a basis  $B$  of  $M$ , and  $i//j$ , then  $B - i \cup j$  is also a basis of  $M$ .

**Problem 1.** Show that a graphical matroid is realizable over any field.

**Problem 2.** For rank 3 matroids (not necessarily simple), show that the  $f$ -vector of the matroid independence complex is log-concave.

## Lecture 3

**Exercise 3.** If  $C$  is a circuit and  $H$  is a hyperplane of a matroid  $M$ , show that  $|C \setminus H| \neq 1$ .

**Exercise 4.** Let  $M = (E, \mathcal{B})$  be a matroid. Prove the following:

- (a) If  $e \notin B$  for a basis  $B$  of  $M$ , then  $B \cup e$  contains a unique circuit. (This circuit is called the **fundamental circuit of  $(B, e)$** ).

- (b) Recall that the closure  $\overline{S}$  of a subset  $S$  in  $M$  is the smallest flat of  $M$  containing it. If  $S \subseteq T \subseteq E$ , then  $\overline{S} \subseteq \overline{T}$ . Moreover, if  $S, T$  are two subsets of  $E$ , then  $\overline{S \cup T} = \overline{\overline{S} \cup \overline{T}}$ .

**Problem 3.** Prove the strong exchange axiom for a matroid  $M = (E, \mathcal{B})$ : If  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \setminus B_2$ , then there exists  $y \in B_2 \setminus B_1$  such that both  $B_1 - x \cup y$  and  $B_2 - y \cup x$  are in  $\mathcal{B}$ . The previous Exercise 4 may help.

**Exercise 5.** Let  $M$  be a matroid on  $E = \{1, 2, 3, 4, 5\}$  whose bases are  $\binom{E}{3} \setminus \{\{1, 2, 3\}, \{3, 4, 5\}\}$ . This matroid is graphical; which graph is it? Compute the poset of flats of  $M$ . Write down a set of five concrete vectors in  $L^\vee = \mathbb{C}^3$  that realize this matroid. Draw a pictorial model of the associated projective hyperplane arrangement.

## Lecture 4

**Exercise 6.** Show that  $\chi_G(q) = q^{\#\text{components of } G} \cdot \chi_{M(G)}(q)$ .

**Exercise 7.** Formulate and prove the correct and precise version of the following statement: “The flats of a graphical matroid are partitions of the vertices.”

**Project 2.** What is the Orlik-Solomon algebra of a matroid?

**Project 3.** Survey some recent developments on the topological/Igusa/motivic zeta functions of hyperplane arrangements and matroids.

## Lecture 5

**Problem 4.** Let  $L \subseteq \mathbb{C}^E$  be a linear subspace realizing a loopless matroid  $M$ . Show that

$$\overline{\chi}_M(1) = \chi_{top}(\mathbb{P}\mathring{L}),$$

where  $\chi_{top}(\mathbb{P}\mathring{L})$  is the topological Euler characteristic of the associated projective hyperplane arrangement complement, by following the steps below.

- (a) Verify that  $\overline{\chi}_M(1) = \left. \frac{d}{dq} \chi_M(q) \right|_{q=1}$ .
- (b) For a flat  $F$  of  $M$ , consider the hyperplane arrangement in  $\mathbb{P}L_F$  consisting of the hyperplanes  $\mathbb{P}L_F \cap L_i$  for  $i \notin F$ . Show that the intersection poset of this hyperplane arrangement is isomorphic to the interval  $[F, E]$  in the poset of flats of  $M$ .
- (c) Recall (but not prove) the following two facts:
  - (i)  $\chi_{top}(\mathbb{P}\mathbb{C}^m) = m + 1$ , and
  - (ii) if  $Y$  is a closed subvariety of a  $\mathbb{C}$ -variety  $X$ , then  $\chi_{top}(X) = \chi_{top}(X \setminus Y) + \chi_{top}(Y)$ .
- (d) Combine the previous three parts with the Möbius inversion formula to conclude  $\overline{\chi}_M(1) = \chi_{top}(\mathbb{P}\mathring{L})$ .

**Exercise 8.** Is the number of circuits of a matroid a Tutte-Grothendieck invariant?

**Project 4** (Taken now). Relate the new formula for the Tutte polynomial of a matroid given in [Kochol '21] to the internal-external activities formula.

## Lecture 6

**Exercise 9.** Show that the sum of a base-point-free divisor and an ample divisor is ample.

## Lecture 7

**Problem 5.** For each  $X$  below, explicitly verify  $(\text{HR}^{\leq 1})$  for the triple  $(A^\bullet(X)_\mathbb{R}, \int_X, \mathcal{K}(X))$ .

- (a)  $X = \mathbb{P}^2 \times \mathbb{P}^2$ .
- (b)  $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .

A boundary element in the nef cone  $\overline{\mathcal{K}(X)}$  often fails to satisfy  $(\text{HR}^{\leq 1})$  in several ways. Let  $X = \text{Bl}_p \mathbb{P}^3$ , the blow-up of  $\mathbb{P}^3$  at a point, which is the closure in  $\mathbb{P}^3 \times \mathbb{P}^2$  of the rational map  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$  given as the projection from the point  $p$ . Its ample cone is a 2-dimensional cone with two boundary rays, corresponding to the two distinguished maps  $X \rightarrow \mathbb{P}^3$  and  $X \rightarrow \mathbb{P}^2$ .

- (c) Show that one boundary ray gives a base-point-free divisor for which  $(\text{HR}^0)$  holds but  $(\text{HR}^1)$  fails, and that the other gives a base-point-free divisor for which  $(\text{HR}^0)$  fails but  $(\text{HR}^1)$  holds.

For those not familiar with cohomology rings of complex varieties and their ample cones, the restatement of this problem in a purely algebraic language is as follows:

- (a) The ring is  $A^\bullet = \mathbb{R}[x, y]/\langle x^3, y^3 \rangle$  with  $\int : x^2 y^2 \mapsto 1$  and  $\mathcal{K} = \{ax + by \mid a, b > 0\}$ .
- (b) The ring is  $A^\bullet = \mathbb{R}[x, y, z]/\langle x^2, y^2, z^2 \rangle$  with  $\int : xyz \mapsto 1$  and  $\mathcal{K} = \{ax + by + cz \mid a, b, c > 0\}$ .
- (c) The ring is  $A^\bullet = \mathbb{R}[h, e]/\langle he, h^3 - e^3 \rangle$  with  $\int : h^3 \mapsto 1$  and  $\mathcal{K} = \{ah + b(h - e) \mid a, b > 0\}$ .

## Lecture 8

**Problem 6.** The *normalization*  $N(f)$  of a polynomial  $f$  is obtained by replacing each monomial  $w_1^{m_1} \cdots w_n^{m_n}$  appearing in  $f$  with  $\frac{w_1^{m_1} \cdots w_n^{m_n}}{m_1! \cdots m_n!}$ .

- (a) Let  $f$  be a bivariate homogeneous polynomial  $f = \sum_{i=0}^d c_i x^i y^{d-i} \in \mathbb{R}_{\geq 0}[x, y]$  of degree  $d$  with nonnegative coefficients. Show that

$$N(f) \text{ is Lorentzian} \iff (c_0, \dots, c_d) \text{ is log-concave with no internal zeros.}$$

- (b) Let  $f = x^3 y + x y^3$ . Conclude from (a) that  $f$  is not Lorentzian. Verify that:  $f$  is log-concave on  $\mathbb{R}_{>0}^2$ , but the triple  $(A_f^\bullet, \deg, \mathcal{K}_f)$  fails  $(\text{HR}^{\leq 1})$ .
- (c) Let  $f = x^3 y + x^2 y^2 + x y^3$ . Conclude from (a) that  $f$  is not Lorentzian. Verify that: The triple  $(A_f^\bullet, \deg, \mathcal{K}_f)$  satisfies  $(\text{HR}^{\leq 1})$ , so  $f$  is log-concave on  $\mathbb{R}_{>0}^2$ , but the triple fails mixed  $(\text{HR}^{\leq 1})$ .

**Problem 7.** To go beyond bivariate polynomials as in the previous problem, let us define the following notion. For a homogeneous polynomial  $f \in \mathbb{R}[w_1, \dots, w_n]$  of degree  $d$  with nonnegative coefficients, we say that its coefficients form a *log-concave simplex* if, for any  $1 \leq i < j \leq n$  and a monomial  $w^m$  of degree  $d' \leq d$ , the coefficients of  $\{w_i^k w_j^{d-d'-k} w^m\}_{0 \leq k \leq d-d'}$  in  $f$  form a log-concave sequence.

- (a) Show that if  $N(f)$  is Lorentzian, then the coefficients of  $f$  form a log-concave simplex.
- (b) In contrast to the bivariate case, give an example of a polynomial  $f$  whose support is M-convex and whose coefficients form a log-concave simplex, but  $N(f)$  is not Lorentzian.

## Lecture 9

**Exercise 10.** Let  $M$  be a loopless matroid. Verify that the poset of flats of the truncation matroid  $Tr(M)$  is obtained by removing the corank 1 layer of the poset of flats of the matroid  $M$ . Conclude that

$$\bar{\chi}_{Tr(M)}(q) = \frac{\bar{\chi}_M(q) - \bar{\chi}_M(0)}{q}.$$

**Problem 8.** Let  $M$  be a matroid on a ground set  $E$  with a total order  $<$ . A *broken circuit* is a subset of  $E$  of the form  $C \setminus i$ , where  $C$  is a circuit of  $M$  and  $i$  is the smallest element in  $C$ . The *BC-complex* of  $M$  is the simplicial complex on vertices  $E$  whose minimal non-faces are the broken circuits of  $M$ .

- (a) Pick a loopless matroid  $M$  of rank 3. Compute its BC-complex and compare its  $f$ -vector to the characteristic polynomial of  $M$ .
- (b) Show that BC-complex of a matroid is pure (i.e. every facet is of the same cardinality).
- (c) Show that the number of facets of the BC-complex of  $M$  equals the absolute value of the constant coefficient of the characteristic polynomial  $\chi_M$ .

See [Brylawski '77 "The Broken-circuit complex"] for more on this complex.

**Exercise 11.** Let  $L \subseteq \mathbb{k}^E$  be a linear subspace over a field  $\mathbb{k}$  whose associated matroid  $M$  is loopless. In this problem we show that the degree of the reciprocal linear space

$$\mathbb{P}L^{-1} = \text{the closure of } \left\{ \left[ \frac{1}{x_0} : \frac{1}{x_2} : \dots : \frac{1}{x_n} \right] \in \mathbb{P}^n \mid [x_0 : x_2 : \dots : x_n] \in \mathring{\mathbb{P}}L \right\}.$$

is equal to  $|\bar{\chi}_M(0)|$  as follows.

- (a) Show that for each circuit  $C$  of  $M$ , there exists a linear form  $\ell_C = \sum_{i \in C} a_i x_i$ , unique up to scaling, such that  $\ell_C \in I_{\mathbb{P}L}$ , where  $I_{\mathbb{P}L}$  is the defining ideal of  $\mathbb{P}L$  as a subvariety of  $\mathbb{P}^n$ .
- (b) For each circuit  $C$  of  $M$ , denote by  $\ell'_C$  the polynomial obtained by inverting the variables in  $\ell_C$  and clearing out denominators. That is,

$$\ell'_C = \sum_{i \in C} a_i \prod_{j \in C \setminus \{i\}} x_j.$$

[Proudfoot-Speyer '06, Theorem 4] states the following: The set  $\{\ell'_C \mid C \text{ a circuit of } M\}$  is a universal Gröbner basis for  $I_{\mathbb{P}L^{-1}}$ , where  $I_{\mathbb{P}L^{-1}}$  is the defining ideal of  $\mathbb{P}L^{-1} \subseteq \mathbb{P}^{n-1}$ . Using this theorem and Problem 8, conclude that  $\deg \mathbb{P}L^{-1} = |\bar{\chi}_M(0)|$ . You may need some standard results about Stanley-Reisner ideals.

## Lecture 10

**Exercise 12.** Let  $M$  be the uniform matroid of rank 3 on 4 elements, and let  $L \subset \mathbb{C}^4$  be a realization of  $M$ .

- (a) Compute the reduced characteristic polynomial of  $M$ .
- (b) Verify that the reciprocal linear space  $\mathbb{P}L^{-1}$  is a cubic surface, in agreement with the computation in part (a).
- (c) The surface  $\mathbb{P}L^{-1}$  has four singular points and six lines, one for each pair of points (google “Cayley nodal cubic” for an image!); can you see where they come from? The surface  $\mathbb{P}L^{-1}$  has three additional lines (for a total of nine lines); where do they come from?
- (d) Let  $H \subset \mathbb{P}^3$  be a general hyperplane. Verify that the image of  $\mathbb{P}L \cap H$  under the Cremona transformation  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$  is a twisted cubic (a rational normal curve), in agreement with part (a).

In the next two problems, we will prove for any rank 3 matroid  $M$  the log-concavity of the coefficients of  $\bar{\chi}_M(q)$  in two ways. Assume rank 3 and loopless for the matroids in the next two exercises.

**Problem 9.** First, an elementary approach that does not generalize to higher ranks:

- (a) Verify that we may assume that  $M$  is *simple*, i.e. that every rank 1 flat of  $M$  is a singleton. Under this assumption, show that  $\bar{\chi}_M(q) = q^2 - (|E| - 1)q + (1 - |E| + \sum_{\text{rk}_M(F)=2} (|F| - 1))$ .
- (b) Using the cover-partition axiom for flats of matroids, conclude the log-concavity of the coefficients of  $\bar{\chi}_M(q)$  for a rank 3 matroid  $M$ .

**Problem 10.** Second, an approach that generalizes to higher ranks, but is not as conceptually straightforward initially.

- (a) Let  $F$  be a rank 2 flat of  $M$ . Verify that  $\int_M \alpha x_F = 0$  and  $\int_M x_F^2 = -1$ .
- (b) Show that the symmetric bilinear form  $A^1(M)_{\mathbb{R}} \times A^1(M)_{\mathbb{R}} \rightarrow \mathbb{R}$  defined by  $(x, y) \mapsto \int_M xy$  has the signature  $(+, -, -, \dots, -)$ .
- (c) Conclude the desired log-concavity.

## Lecture 11

**Exercise 13.** If a cone  $\sigma$  in  $N_{\mathbb{R}}$  has dimension  $\dim N_{\mathbb{R}}$  and is strongly convex, do  $\sigma$  and  $\sigma^{\vee}$  have the same number of rays?

**Exercise 14.** Recall that the torus  $T$  with its character lattice  $M$  acts on the character ring  $\mathbb{k}[M]$  by  $t \cdot \chi^m = \chi^{-m}(t)\chi^m$ . What happens when the action is changed to be  $t \cdot \chi^m = \chi^m(t)\chi^m$ ? Verify these changes explicitly on the toric variety of  $\sigma = \text{Cone}(\mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_2) \subset \mathbb{R}^2$ .

## Lecture 12

**Exercise 15.** Realize the blow-up of a point in an affine plane as a toric variety of a fan. Verify the orbit-cone correspondence.

**Problem 11.** Show that the blow-up of a linear subspace in an affine space can be realized as a toric variety of a fan. How does “stellar subdivision” relate?

**Exercise 16.** Let  $\tau$  be a face of a cone  $\sigma$  in  $N_{\mathbb{R}}$ . Show that the orbit-closure  $V(\sigma)$ , as a closed subvariety in  $U_{\sigma}$ , is given by a ring map  $\mathbb{k}[S_{\sigma}] \rightarrow \mathbb{k}[\sigma^{\vee} \cap \tau^{\perp} \cap M]$ .

## Lecture 13

**Exercise 17.** Suppose  $P \subset M_{\mathbb{R}}$  is a zonotope, i.e. the Minkowski sum of line segments  $\ell_1, \dots, \ell_k$  in  $M_{\mathbb{R}}$ . Show that a polytope  $Q$  is a deformation of  $P$  if and only if every edge  $Q$  is parallel to  $\ell_i$  for some  $i = 1, \dots, k$ .

**Problem 12.** Show that if  $P \subseteq M_{\mathbb{R}}$  is a  $(M)$ -lattice polytope that has IDP (i.e. is normal), then the embedded projective variety  $X_{P \cap M} \subseteq \mathbb{P}(\mathbb{C}^{P \cap M})$  is a projectively normal embedding of the toric variety  $X_{\Sigma_P}$ .

## Lecture 14

**Exercise 18.** Compute extremal rays of the nef cone of the permutohedral variety of dimension 2.

**Problem 13.** Show that the dimension of the base polytope  $P(M)$  of a matroid  $M$  on a ground set  $E$  is equal to  $|E| - \# \text{comp}(M)$  where  $\# \text{comp}(M)$  is the number of connected components of  $M$ .