Math 279R Exercises/Problems/Projects

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Exercises are meant to be relatively short "sanity checks." Problems are slightly more involved. Potential final project topic are listed throughout; it can be an exposition of progresses on a long-standing conjecture or an important topic in matroid theory not covered in lecture, or a short research project.

For full mark on the problem set portion (40%) of the final grade, you must complete at least 2/3 of the exercises and 1/2 of the problems. You may collaborate with others, or consult references, as long as you write-up your own solutions with clear indications of your sources.

Lecture 1

Exercise 1. From the basis exchange axiom, show that every bases of a matroid has the same cardinality.

Project 1. (Max distance separable conjecture). For a fixed finite field $\Bbbk = \mathbb{F}_q$, and a fixed integer $r \geq 0$, what is the largest N such that the uniform matroid $U_{r,N}$ of rank r on N elements is realizable? (This is a long-standing open problem).

Lecture 2

Exercise 2. Let M be a loopless matroid on a ground set E. Define a relation $//$ on E by declaring that $i//j$ if $\{ij\}$ not in any basis of M. Show that $//$ is an equivalence relation. Moreover, show that if $i \in E$ is a basis B of M, and $i//j$, then $B - i \cup j$ is also a basis of M.

Problem 1. Show that a graphical matroid is realizable over any field.

Problem 2. For rank 3 matroids (not necessarily simple), show that the f-vector of the matroid independence complex is log-concave.

Lecture 3

Exercise 3. If C is a circuit and H is a hyperplane of a matroid M, show that $|C \setminus H| \neq 1$.

Exercise 4. Let $M = (E, \mathcal{B})$ be a matroid. Prove the following:

(a) If $e \notin B$ for a basis B of M, then $B \cup e$ contains a unique circuit. (This circuit is called the **fundamental circuit of** (B, e)).

(b) Recall that the closure $\overline{S}$ of a subset S in M is the smallest flat of M containing it. If $S \subseteq T \subseteq E$, then $\overline{S} \subseteq \overline{T}$. Moreover, if S, T are two subsets of E, then $\overline{S \cup T} = \overline{\overline{S} \cup \overline{T}}$.

Problem 3. Prove the strong exchange axiom for a matroid $M = (E, \mathcal{B})$: If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that both $B_1 - x \cup y$ and $B_2 - y \cup x$ are in $\mathcal{B}$. The previous Exercise 4 may help.

Exercise 5. Let M be a matroid on $E = \{1, 2, 3, 4, 5\}$ whose bases are $\binom{E}{3} \setminus \{\{1, 2, 3\}, \{3, 4, 5\}\}$. This matroid is graphical; which graph is it? Compute the poset of flats of M. Write down a set of five concrete vectors in $L^\vee = \mathbb{C}^3$ that realize this matroid. Draw a pictorial model of the associated projective hyperplane arrangement.

Lecture 4

Exercise 6. Show that $\chi_G(q) = q^{\#\text{components of } G} \cdot \chi_{M(G)}(q)$.

Exercise 7. Formulate and prove the correct and precise version of the following statement: "The flats of a graphical matroid are partitions of the vertices."

Project 2. What is the Orlik-Solomon algebra of a matroid?

Project 3. Survey some recent developments on the topological/Igusa/motivic zeta functions of hyperplane arrangements and matroids.

Lecture 5

Problem 4. Let $L \subseteq \mathbb{C}^E$ be a linear subspace realizing a loopless matroid M. Show that

$$\overline{\chi}_M(1) = \chi_{top}(\mathbb{P}\mathring{L}),$$

where $\chi_{top}(\mathbb{P}\mathring{L})$ is the topological Euler characteristic of the associated projective hyperplane arrangement complement, by following the steps below.

(a) Verify that $\overline{\chi}_M(1) = \frac{d}{dq}\chi_M(q)\Big|_{q=1}$.

(b) For a flat F of M, consider the hyperplane arrangement in $\mathbb{P}L_F$ consisting of the hyperplanes $\mathbb{P}L_F \cap L_i$ for $i \notin F$. Show that the intersection poset of this hyperplane arrangement is isomorphic to the interval $[F, E]$ in the poset of flats of M.

(c) Recall (but not prove) the following two facts:

(i) $\chi_{top}(\mathbb{P}^m_{\mathbb{C}}) = m + 1$, and

(ii) if Y is a closed subvariety of a $\mathbb{C}$-variety X, then $\chi_{top}(X) = \chi_{top}(X \setminus Y) + \chi_{top}(Y)$.

(d) Combine the previous three parts with the Möbius inversion formula to conclude $\overline{\chi}_M(1) = \chi_{top}(\mathbb{P}\mathring{L})$.

Exercise 8. Is the number of circuits of a matroid a Tutte-Grothendieck invariant?

Project 4 (Taken now)**.** Relate the new formula for the Tutte polynomial of a matroid given in [Kochol '21] to the internal-external activities formula.

Lecture 6

Exercise 9. Show that the sum of a base-point-free divisor and and an ample divisor is ample.

Lecture 7

Problem 5. For each X below, explicitly verify (HR$^{\leq 1}$) for the triple $(A^\bullet(X)_{\mathbb{R}}, \int_X, \mathscr{K}(X))$.

(a) $X = \mathbb{P}^2 \times \mathbb{P}^2$.

(b) $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

A boundary element in the nef cone $\overline{\mathscr{K}(X)}$ often fails to satisfy (HR$^{\leq 1}$) in several ways. Let $X = \mathrm{Bl}_p\,\mathbb{P}^3$, the blow-up of $\mathbb{P}^3$ at a point, which is the closure in $\mathbb{P}^3 \times \mathbb{P}^2$ of the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given as the projection from the point p. Its ample cone is a 2-dimensional cone with two boundary rays, corresponding to the two distinguished maps $X \to \mathbb{P}^3$ and $X \to \mathbb{P}^2$.

(c) Show that one boundary ray gives a base-point-free divisor for which (HR^0) holds but (HR^1) fails, and that the other gives a base-point-free divisor for which (HR^0) fails but (HR^1) holds.

For those not familiar with cohomology rings of complex varieties and their ample cones, the restatement of this problem in a purely algebraic language is as follows:

(a) The ring is $A^\bullet = \mathbb{R}[x,y]/\langle x^3, y^3\rangle$ with $\int : x^2y^2 \mapsto 1$ and $\mathscr{K} = \{ax + by \mid a, b > 0\}$.

(b) The ring is $A^\bullet = \mathbb{R}[x,y,z]/\langle x^2, y^2, z^2\rangle$ with $\int : xyz \mapsto 1$ and $\mathscr{K} = \{ax + by + cz \mid a, b, c > 0\}$.

(c) The ring is $A^\bullet = \mathbb{R}[h,e]/\langle he, h^3 - e^3\rangle$ with $\int : h^3 \mapsto 1$ and $\mathscr{K} = \{ah + b(h-e) \mid a, b > 0\}$.