Math 279R Exercises/Problems/Projects

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Exercises are meant to be relatively short "sanity checks." Problems are slightly more involved. Potential final project topic are listed throughout; it can be an exposition of progresses on a long-standing conjecture or an important topic in matroid theory not covered in lecture, or a short research project.

For full mark on the problem set portion (40%) of the final grade, you must complete at least 2/3 of the exercises and 1/2 of the problems. You may collaborate with others, or consult references, as long as you write-up your own solutions with clear indications of your sources.

Lecture 1

Exercise 1. From the basis exchange axiom, show that every bases of a matroid has the same cardinality.

Project 1. (Max distance separable conjecture). For a fixed finite field $\Bbbk = \mathbb{F}_q$, and a fixed integer $r \geq 0$, what is the largest N such that the uniform matroid $U_{r,N}$ of rank r on N elements is realizable? (This is a long-standing open problem).

Lecture 2

Exercise 2. Let M be a loopless matroid on a ground set E. Define a relation $//$ on E by declaring that $i//j$ if $\{ij\}$ not in any basis of M. Show that $//$ is an equivalence relation. Moreover, show that if $i \in E$ is a basis B of M, and $i//j$, then $B - i \cup j$ is also a basis of M.

Problem 1. Show that a graphical matroid is realizable over any field.

Problem 2. For rank 3 matroids (not necessarily simple), show that the f-vector of the matroid independence complex is log-concave.

Lecture 3

Exercise 3. If C is a circuit and H is a hyperplane of a matroid M, show that $|C \setminus H| \neq 1$.

Exercise 4. Let $M = (E, \mathcal{B})$ be a matroid. Prove the following:

(a) If $e \notin B$ for a basis B of M, then $B \cup e$ contains a unique circuit. (This circuit is called the **fundamental circuit of** (B, e)).

(b) Recall that the closure $\overline{S}$ of a subset S in M is the smallest flat of M containing it. If $S \subseteq T \subseteq E$, then $\overline{S} \subseteq \overline{T}$. Moreover, if S, T are two subsets of E, then $\overline{S \cup T} = \overline{\overline{S} \cup \overline{T}}$.

Problem 3. Prove the strong exchange axiom for a matroid $M = (E, \mathcal{B})$: If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that both $B_1 - x \cup y$ and $B_2 - y \cup x$ are in $\mathcal{B}$. The previous Exercise 4 may help.

Exercise 5. Let M be a matroid on $E = \{1, 2, 3, 4, 5\}$ whose bases are $\binom{E}{3} \setminus \{\{1, 2, 3\}, \{3, 4, 5\}\}$. This matroid is graphical; which graph is it? Compute the poset of flats of M. Write down a set of five concrete vectors in $L^\vee = \mathbb{C}^3$ that realize this matroid. Draw a pictorial model of the associated projective hyperplane arrangement.

Lecture 4

Exercise 6. Show that $\chi_G(q) = q^{\#\text{components of } G} \cdot \chi_{M(G)}(q)$.

Exercise 7. Formulate and prove the correct and precise version of the following statement: "The flats of a graphical matroid are partitions of the vertices."

Project 2. What is the Orlik-Solomon algebra of a matroid?

Project 3. Survey some recent developments on the topological/Igusa/motivic zeta functions of hyperplane arrangements and matroids.

Lecture 5

Problem 4. Let $L \subseteq \mathbb{C}^E$ be a linear subspace realizing a loopless matroid M. Show that

$$\overline{\chi}_M(1) = \chi_{top}(\mathbb{P}\mathring{L}),$$

where $\chi_{top}(\mathbb{P}\mathring{L})$ is the topological Euler characteristic of the associated projective hyperplane arrangement complement, by following the steps below.

(a) Verify that $\overline{\chi}_M(1) = \frac{d}{dq}\chi_M(q)\Big|_{q=1}$.

(b) For a flat F of M, consider the hyperplane arrangement in $\mathbb{P}L_F$ consisting of the hyperplanes $\mathbb{P}L_F \cap L_i$ for $i \notin F$. Show that the intersection poset of this hyperplane arrangement is isomorphic to the interval $[F, E]$ in the poset of flats of M.

(c) Recall (but not prove) the following two facts:

(i) $\chi_{top}(\mathbb{P}^m_{\mathbb{C}}) = m + 1$, and

(ii) if Y is a closed subvariety of a $\mathbb{C}$-variety X, then $\chi_{top}(X) = \chi_{top}(X \setminus Y) + \chi_{top}(Y)$.

(d) Combine the previous three parts with the Möbius inversion formula to conclude $\overline{\chi}_M(1) = \chi_{top}(\mathbb{P}\mathring{L})$.

Exercise 8. Is the number of circuits of a matroid a Tutte-Grothendieck invariant?

Project 4 (Taken now)**.** Relate the new formula for the Tutte polynomial of a matroid given in [Kochol '21] to the internal-external activities formula.

Lecture 6

Exercise 9. Show that the sum of a base-point-free divisor and and an ample divisor is ample.

Lecture 7

Problem 5. For each X below, explicitly verify (HR$^{\leq 1}$) for the triple $(A^\bullet(X)_\mathbb{R}, \int_X, \mathscr{K}(X))$.

(a) $X = \mathbb{P}^2 \times \mathbb{P}^2$.

(b) $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

A boundary element in the nef cone $\overline{\mathscr{K}(X)}$ often fails to satisfy (HR$^{\leq 1}$) in several ways. Let $X = \mathrm{Bl}_p\,\mathbb{P}^3$, the blow-up of $\mathbb{P}^3$ at a point, which is the closure in $\mathbb{P}^3 \times \mathbb{P}^2$ of the rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given as the projection from the point p. Its ample cone is a 2-dimensional cone with two boundary rays, corresponding to the two distinguished maps $X \to \mathbb{P}^3$ and $X \to \mathbb{P}^2$.

(c) Show that one boundary ray gives a base-point-free divisor for which (HR^0) holds but (HR^1) fails, and that the other gives a base-point-free divisor for which (HR^0) fails but (HR^1) holds.

For those not familiar with cohomology rings of complex varieties and their ample cones, the restatement of this problem in a purely algebraic language is as follows:

(a) The ring is $A^\bullet = \mathbb{R}[x, y]/\langle x^3, y^3\rangle$ with $\int : x^2y^2 \mapsto 1$ and $\mathscr{K} = \{ax + by \mid a, b > 0\}$.
(b) The ring is $A^\bullet = \mathbb{R}[x, y, z]/\langle x^2, y^2, z^2\rangle$ with $\int : xyz \mapsto 1$ and $\mathscr{K} = \{ax + by + cz \mid a, b, c > 0\}$.
(c) The ring is $A^\bullet = \mathbb{R}[h, e]/\langle he, h^3 - e^3\rangle$ with $\int : h^3 \mapsto 1$ and $\mathscr{K} = \{ah + b(h - e) \mid a, b > 0\}$.

Lecture 8

Problem 6. The *normalization* $N(f)$ of a polynomial f is obtained by replacing each monomial $w_1^{m_1} \cdots w_n^{m_n}$ appearing in f with $\frac{w_1^{m_1} \cdots w_n^{m_n}}{m_1! \cdots m_n!}$.

(a) Let f be a bivariate homogeneous polynomial $f = \sum_{i=0}^d c_i x^i y^{d-i} \in \mathbb{R}_{\geq 0}[x, y]$ of degree d with nonnegative coefficients. Show that

$$N(f) \text{ is Lorentzian} \iff (c_0, \ldots, c_d) \text{ is log-concave with no internal zeros.}$$

(b) Let $f = x^3y + xy^3$. Conclude from (a) that f is not Lorentzian. Verify that: f is log-concave on $\mathbb{R}^2_{>0}$, but the triple $(A_f^\bullet, \deg, \mathscr{K}_f)$ fails (HR$^{\leq 1}$).

(c) Let $f = x^3y + x^2y^2 + xy^3$. Conclude from (a) that f is not Lorentzian. Verify that: The triple $(A_f^\bullet, \deg, \mathscr{K}_f)$ satisfies (HR$^{\leq 1}$), so f is log-concave on $\mathbb{R}^2_{>0}$, but the triple fails mixed (HR$^{\leq 1}$).

Problem 7. To go beyond bivariate polynomials as in the previous problem, let us define the following notion. For a homogeneous polynomial $f \in \mathbb{R}[w_1, \ldots, w_n]$ of degree d with nonnegative coefficients, we say that its coefficients form a *log-concave simplex* if, for any $1 \leq i < j \leq n$ and a monomial $\boldsymbol{w}^{\boldsymbol{m}}$ of degree $d' \leq d$, the coefficients of $\{w_i^k w_j^{d-d'-k} \boldsymbol{w}^{\boldsymbol{m}}\}_{0 \leq k \leq d-d'}$ in f form a log-concave sequence.

(a) Show that if $N(f)$ is Lorentzian, then the coefficients of f form a log-concave simplex.

(b) In contrast to the bivariate case, give an example of a polynomial f whose support is M-convex and whose coefficients form a log-concave simplex, but $N(f)$ is not Lorentzian.

Lecture 9

Exercise 10. Let M be a loopless matroid. Verify that the poset of flats of the truncation matroid $Tr(M)$ is obtained by removing the corank 1 layer of the poset of flats of the matroid M. Conclude that

$$\overline{\chi}_{Tr(M)}(q) = \frac{\overline{\chi}_M(q) - \overline{\chi}_M(0)}{q}.$$

Problem 8. Let M be a matroid on a ground set E with a total order $<$. A *broken circuit* is a subset of E of the form $C \setminus i$, where C is a circuit of M and i is the smallest element in C. The *BC-complex* of M is the simplicial complex on vertices E whose minimal non-faces are the broken circuits of M.

(a) Pick a loopless matroid M of rank 3. Compute its BC-complex and compare its f-vector to the characteristic polynomial of M.

(b) Show that BC-complex of a matroid is pure (i.e. every facet is of the same cardinality).

(c) Show that the number of facets of the BC-complex of M equals the absolute value of the constant coefficient of the characteristic polynomial χ_M.

See [Brylawski '77 "The Broken-circuit complex"] for more on this complex.

Exercise 11. Let $L \subseteq \Bbbk^E$ be a linear subspace over a field $\Bbbk$ whose associated matroid M is loopless. In this problem we show that the degree of the reciprocal linear space

$$\mathbb{P}L^{-1} = \text{the closure of } \left\{ [\tfrac{1}{x_0} : \tfrac{1}{x_2} : \cdots : \tfrac{1}{x_n}] \in \mathbb{P}^n \;\middle|\; [x_0 : x_2 : \cdots : x_n] \in \mathbb{P}\mathring{L} \right\}.$$

is equal to $|\overline{\chi}_M(0)|$ as follows.

(a) Show that for each circuit C of M, there exists a linear form $\ell_C = \sum_{i \in C} a_i x_i$, unique up to scaling, such that $\ell_C \in I_{\mathbb{P}L}$, where $I_{\mathbb{P}L}$ is the defining ideal of $\mathbb{P}L$ as a subvariety of $\mathbb{P}^n$.

(b) For each circuit C of M, denote by ℓ'_C the polynomial obtained by inverting the variables in ℓ_C and clearing out denominators. That is,

$$\ell'_C = \sum_{i \in C} a_i \prod_{j \in C \setminus \{i\}} x_j.$$

[Proudfoot-Speyer '06, Theorem 4] states the following: The set $\{\ell'_C \mid C \text{ a circuit of } M\}$ is a universal Gröbner basis for $I_{\mathbb{P}L^{-1}}$, where $I_{\mathbb{P}L^{-1}}$ is the defining ideal of $\mathbb{P}L^{-1} \subseteq \mathbb{P}^{n-1}$. Using this theorem and Problem 8, conclude that $\deg \mathbb{P}L^{-1} = |\overline{\chi}_M(0)|$. You may need some standard results about Stanley-Reisner ideals.

Lecture 10

Exercise 12. Let M be the uniform matroid of rank 3 on 4 elements, and let $L \subset \mathbb{C}^4$ be a realization of M.

(a) Compute the reduced characteristic polynomial of M.

(b) Verify that the reciprocal linear space $\mathbb{P}L^{-1}$ is a cubic surface, in agreement with the computation in part (a).

(c) The surface $\mathbb{P}L^{-1}$ has four singular points and six lines, one for each pair of points (google "Cayley nodal cubic" for an image!); can you see where they come from? The surface $\mathbb{P}L^{-1}$ has three additional lines (for a total of nine lines); where do they come from?

(d) Let $H \subset \mathbb{P}^3$ be a general hyperplane. Verify that the image of $\mathbb{P}L \cap H$ under the Cremona transformation $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is a twisted cubic (a rational normal curve), in agreement with part (a).

In the next two problems, we will prove for any rank 3 matroid M the log-concavity of the coefficients of $\overline{\chi}_M(q)$ in two ways. Assume rank 3 and loopless for the matroids in the next two exercises.

Problem 9. First, an elementary approach that does not generalize to higher ranks:

(a) Verify that we may assume that M is *simple*, i.e. that every rank 1 flat of M is a singleton. Under this assumption, show that $\overline{\chi}_M(q) = q^2 - (|E| - 1)q + (1 - |E| + \sum_{\mathrm{rk}_M(F)=2}(|F| - 1))$.

(b) Using the cover-partition axiom for flats of matroids, conclude the log-concavity of the coefficients of $\overline{\chi}_M(q)$ for a rank 3 matroid M.

Problem 10. Second, an approach that generalizes to higher ranks, but is not as conceptually straightforward initially.

(a) Let F be a rank 2 flat of M. Verify that $\int_M \alpha x_F = 0$ and $\int_M x_F^2 = -1$.

(b) Show that the symmetric bilinear form $A^1(M)_{\mathbb{R}} \times A^1(M)_{\mathbb{R}} \to \mathbb{R}$ defined by $(x, y) \mapsto \int_M xy$ has the signature $(+, -, -, \cdots, -)$.

(c) Conclude the desired log-concavity.

Lecture 11

Exercise 13. If a cone σ in $N_{\mathbb{R}}$ has dimension $\dim N_{\mathbb{R}}$ and is strongly convex, do σ and $\sigma^\vee$ have the same number of rays?

Exercise 14. Recall that the torus T with its character lattice M acts on the character ring $\Bbbk[M]$ by $t \cdot \chi^m = \chi^{-m}(t)\chi^m$. What happens when the action is changed to be $t \cdot \chi^m = \chi^m(t)\chi^m$? Verify these changes explicitly on the toric variety of $\sigma = \mathrm{Cone}(\mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_2) \subset \mathbb{R}^2$.

Lecture 12

Exercise 15. Realize the blow-up of a point in an affine plane as a toric variety of a fan. Verify the orbit-cone correspondence.

Problem 11. Show that the blow-up of a linear subspace in an affine space can be realized as a toric variety of a fan. How does "stellar subdivision" relate?

Exercise 16. Let τ be a face of a cone σ in $N_{\mathbb{R}}$. Show that the orbit-closure $V(\sigma)$, as a closed subvariety in U_σ, is given by a ring map $\Bbbk[S_\sigma] \to \Bbbk[\sigma^\vee \cap \tau^\perp \cap M]$.

Lecture 13

Exercise 17. Suppose $P \subset M_{\mathbb{R}}$ is a zonotope, i.e. the Minkowski sum of line segments $\ell_1, \dots, \ell_k$ in $M_{\mathbb{R}}$. Show that a polytope Q is a deformation of P if and only if every edge Q is parallel to ℓ_i for some $i = 1, \dots, k$.

Problem 12. Show that if $P \subseteq M_{\mathbb{R}}$ is a (M-)lattice polytope that has IDP (i.e. is normal), then the embedded projective variety $X_{P\cap M} \subseteq \mathbb{P}(\mathbb{C}^{P\cap M})$ is a projectively normal embedding of the toric variety X_{Σ_P}.

Lecture 14

Exercise 18. Compute the extremal rays of the nef cone of the permutohedral variety of dimension 2.

Problem 13. Show that the dimension of the base polytope $P(M)$ of a matroid M on a ground set E is equal to $|E| - \#\operatorname{comp}(M)$ where $\#\operatorname{comp}(M)$ is the number of connected components of M.

Lecture 15

Exercise 19. Let Σ be a simpicial fan. Show that the ring $A^\bullet(\Sigma)$ presented as a quotient of the polynomial ring $\mathbb{Z}[x_\rho \mid \rho \in \Sigma(1)]$ is generated as a $\mathbb{Z}$-module by (the images of) square-free monomials.

Exercise 20. Let Σ be a complete fan of dimension n. Show that $\mathrm{MW}_n(\Sigma) \simeq \mathbb{Z}$.

Lecture 16

Problem 14. Let Σ be a smooth projective fan of dimension n. For a base-point-free toric divisor D on X_Σ, let φ_D and P_D be the associated piecewise-linear function and the deformation polytope. Let $[D] \in A^1(X_\Sigma)$ be the divisor class of D. Show that

$[D] \cap \Delta_\Sigma = V_{\mathrm{trop}}(\varphi_D)$, and that for any $\sigma \in \Sigma(n-1)$, $([D] \cap \Delta_\Sigma)(\sigma) =$ lattice length of the edge P_D^σ.

Exercise 21. Let M be a loopless matroid of rank r. Show that the Bergman fan Σ_M with all of its maximal cones given weight 1 is a well-defined Minkowski weight of dimension $r - 1$. (Hint: The cover-partition axiom may be useful).