# Positivity in matroid theory: MaTroCom Minicourse

Christopher Eur

January 11, 2023

In algebraic geometry, "positivity" broadly refers to numerical properties enjoyed by certain classes of vector bundles on projective varieties. In this minicourse, we survey how a similar "positivity" arises in the combinatorics of matroids. We assume some familiarity with polyhedra and an acquaintance with matroids. Statements involving algebraic geometry (toric varieties) are in a different color, which may be skipped.

*Notation.* Let $E = \{1, \dots, n\}$ be a finite set of cardinality $n$. For a subset $S \subseteq E$, denote by $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ the sum of standard basis vectors in $\mathbb{R}^E$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product. Let $T = (\mathbb{C}^*)^E$ be the torus whose character lattice is $\mathbb{Z}^E$. A variety is reduced and irreducible.

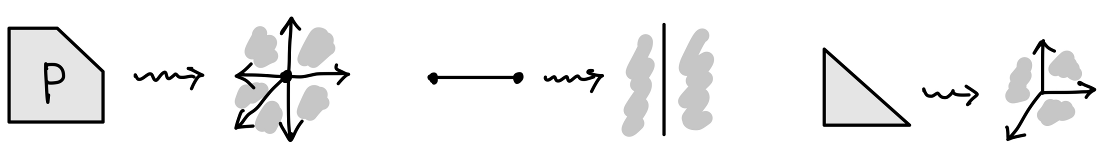
## 1 Overview

Let $\Sigma$ be a unimodular projective fan in $\mathbb{R}^E$. *Projective* means that $\Sigma$ is the (inner) normal fan $\Sigma_P$ of a polytope $P$ in $\mathbb{R}^E$. We allow $\dim P < n$, so the lineality space

$$\mathrm{lin}(\Sigma) = (\text{the minimal cone of } \Sigma) = \{u \in \mathbb{R}^E : \langle u, x \rangle = 0 \ \forall x \in P\}$$

may be nontrivial of dimension $\ell$. *Unimodular* means that for any cone $\sigma \in \Sigma$, the *primitive ray vectors* of $\sigma / \mathrm{lin}(\Sigma)$ extends to a $\mathbb{Z}$-basis of $\mathbb{Z}^E / (\mathrm{lin}(\Sigma) \cap \mathbb{Z}^E)$. Let $\Sigma(d)$ be the set of $d$-dimensional cones of $\Sigma$.

**Example 1.1.**




Recall the dimension-reversing bijection between the faces of a polytope $P$ and the cones of its normal fan $\Sigma_P$ given by:

$$\Sigma_P \ni \sigma \leftrightarrow \mathrm{face}_\sigma(P) = \{p \in P : \langle p, v \rangle = \min_{q \in P} \langle q, v \rangle\} \text{ for any } v \text{ in the relative interior of } \sigma.$$

A lattice polytope $Q$ is a *deformation* of $\Sigma$, denoted $Q \in \mathrm{Def}(\Sigma)$, if its normal fan $\Sigma_Q$ coarsens $\Sigma$. For instance, deformations of $\Sigma$ in Example 1.1 include rectangles, standard simplices, etc.

Let $X_\Sigma$ be the smooth projective toric variety associated to $\Sigma / \mathrm{lin}(\Sigma)$, considered as a $T$-variety. Recall similarly the bijection between the cones of $\Sigma$ and the torus-orbits of $X_\Sigma$. In particular, the maximal cones correspond to the torus-fixed points of $X_\Sigma$. A deformation $Q$ of $\Sigma$ corresponds to the base-point-free $T$-line bundle $\mathcal{L}_Q$ on $X_\Sigma$ whose complete linear system gives a map $X_\Sigma \to \mathbb{P}^{|Q\cap\mathbb{Z}^E|-1}$ induced by the map of tori $t \mapsto (t^{\mathbf{m}})_{\in Q\cap\mathbb{Z}^E}$ [CLS11, Chapter 6].

We will learn about two well-studied rings $K(\Sigma)$ and $A^\bullet(\Sigma)$ attached to such $\Sigma$. In geometric terms, these are the Grothendieck $K$-ring of vector bundles and the Chow cohomology ring of the toric variety $X_\Sigma$. "GKM-varieties" is a good keyword for those wanting more geometric details. By associating to each matroid certain elements in these rings, one gains an insight into combinatorial properties of matroids via geometric methods.

# 2 K-rings and matroid polytopes

## 2.1 K-rings, polytope algebras, and "piecewise" Laurent polynomials

We first describe the ring denoted $K_T(\Sigma)$ and then describe $K(\Sigma)$ as its quotient. It has three different descriptions, whose equivalence is a consequence of some major theorems.

**Definition 2.1.** Let $K_T(\Sigma)$ be the Grothendieck $K$-ring of $T$-equivariant vector bundles on $X_\Sigma$, and let $K(\Sigma)$ be the non-equivariant $K$-ring. That is,

$$K_T(\Sigma) = \frac{\Big\{ \sum_i a_i[\mathcal{E}_i]^T : a_i \in \mathbb{Z},\ \mathcal{E}_i \text{ a } T\text{-equivariant vector bundle on } X_\Sigma \Big\}}{\Big\langle [\mathcal{E}]^T = [\mathcal{E}']^T + [\mathcal{E}'']^T : \exists \text{ a } T\text{-equivariant SES } 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \Big\rangle}, \quad \text{and}$$
$$K(\Sigma) = \frac{\Big\{ \sum_i a_i[\mathcal{E}_i] : a_i \in \mathbb{Z},\ \mathcal{E}_i \text{ a vector bundle on } X_\Sigma \Big\}}{\Big\langle [\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}''] : \exists \text{ a SES } 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \Big\rangle},$$

with the multiplication is given by tensor products.

The natural "forgetting $T$-equivariance" map $K_T(\Sigma) \to K(\Sigma)$ is a surjection [Mor93, Proposition 3]. This geometrically defined ring has the following combinatorial descriptions. For $\mathbf{m} \in \mathbb{Z}^E$, denote by $\mathbf{T}^{\mathbf{m}}$ the Laurent monomial $T_1^{m_1}\cdots T_n^{m_n} \in \mathbb{Z}[\mathbf{T}^\pm] = \mathbb{Z}[T_1^\pm, \dots, T_n^\pm]$.

**Theorem 2.2.** The rings $K_T(\Sigma)$ and $K(\Sigma)$ have the following equivalent descriptions:

1. For a polytope $Q \subset \mathbb{R}^E$, let $1_Q : \mathbb{R}^E \to \mathbb{Z}$ be its indicator function given by $1_Q(x) = 1$ if $x \in Q$ and $1_Q(x) = 0$ otherwise. Then, we have by [EHL, Theorem A.10]

$$K_T(\Sigma) \simeq \mathbb{I}(\Sigma) = \text{the subgroup of } \mathbb{Z}^{(\mathbb{R}^E)} \text{ generated by } \{1_Q \mid Q \in \mathrm{Def}(\Sigma)\}, \quad \text{and}$$
$$K(\Sigma) \simeq \overline{\mathbb{I}}(\Sigma) = \mathbb{I}(\Sigma)/\operatorname{transl}(\Sigma)$$

where $\operatorname{transl}(\Sigma)$ is the subgroup of $\mathbb{I}(\Sigma)$ generated by $\{1_Q - 1_{Q+u} \mid u \in \mathbb{Z}^m\}$. Multiplication in these rings are given by Minkowski sums of polytopes. Denote by $[Q]$ the class of $1_Q$ in $\overline{\mathbb{I}}(\Sigma)$. The ring $\overline{\mathbb{I}}(\Sigma)$ is also known as the *polytope algebra* [McM89].

2. For two maximal cones $\sigma$ and $\sigma'$ of $\Sigma$ sharing a wall (i.e. a codimension 1 face), let $\mathbf{m}(\sigma, \sigma')$ be the primitive vector normal to $\sigma \cap \sigma'$. Then, we have by [Nie74, VV03]

$$K_T(\Sigma) \simeq LP(\Sigma) = \left\{ (f_\sigma)_{\sigma \in \Sigma_{\max}} \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}] \;\middle|\; \begin{array}{l} f_\sigma - f_{\sigma'} \equiv 0 \bmod (1 - \mathbf{T}^{\mathbf{m}(\sigma,\sigma')}) \\ \text{for any } \sigma \text{ and } \sigma' \text{ sharing a wall} \end{array} \right\}, \quad \text{and}$$
$$K(\Sigma) \simeq \overline{LP}(\Sigma) = LP(\Sigma)/I_K$$

where $I_K$ is the ideal generated by $\{T_i - 1 : i \in E\}$ where $T_i$ here is considered as an element $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}]$ by $f_\sigma = T_i$ for all $\sigma$.

Given $Q \in \mathrm{Def}(\Sigma)$, the claimed isomorphisms are given by $1_Q \mapsto (\mathbf{T}^{-\operatorname{face}_\sigma(Q)})_\sigma \in LP(\Sigma)$ and $1_Q \mapsto [\mathcal{L}_Q] \in K_T(\Sigma)$. The isomorphism $K_T(\Sigma) \simeq LP(\Sigma)$ is also described by $[\mathcal{E}]^T \mapsto \mathrm{Hilb}(\mathcal{E}|_{p_\sigma})_\sigma$, the restriction to the torus-fixed points.

These "$K$-rings" have distinguished maps to $\mathbb{Z}[\mathbf{T}^{\pm}]$ and $\mathbb{Z}$. Let $\chi^T : K_T(\Sigma) \to \mathbb{Z}[\mathbf{T}^{\pm}]$ and $\chi : K(\Sigma) \to \mathbb{Z}$ be the sheaf Euler characteristic maps. Combinatorial descriptions of these are:

**Theorem 2.3.** [CLS11, Ch. 9] For $1_Q \in \mathbb{I}(\Sigma)$, under the isomorphism $K_T(\Sigma) \simeq \mathbb{I}(\Sigma)$ we have

$$\chi^T(1_Q) = \sum_{\mathbf{m} \in Q \cap \mathbb{Z}^E} \mathbf{T}^{-\mathbf{m}} \quad \text{and} \quad \chi([Q])) = |Q \cap \mathbb{Z}^E|.$$

[Bri88, Ish90] For an element in $f \in K_T(\Sigma)$ given by $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}]$, we have

$$\chi^T(f) = \sum_{\sigma \in \Sigma_{\max}} \frac{f_\sigma}{\prod_{\substack{\mathbf{m} \text{ a primitive ray} \\ \text{generator of } \sigma^\vee}} (1 - \mathbf{T}^{-\mathbf{m}})} \quad \text{and} \quad \chi(f) = \chi^T(f)|_{T_1 = \cdots = T_n = 1}.$$

**Exercise 2.4.** Let $\Sigma$ be the normal fan of the polytope $P$ in the following. Verify the formula for $\chi^T$ in the case when $P = \mathrm{Conv}\{\mathbf{e}_1, \mathbf{e}_2\} \subset \mathbb{R}^2$ and $Q = \mathrm{Conv}\{2\mathbf{e}_1, 2\mathbf{e}_2\}$. If you'd like to get a feel of how quickly nontrivial it gets, try $P = Q = \mathrm{Conv}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \mathbb{R}^3$.

## 2.2 Permutohedral fan and base polytopes of matroids

Let $\mathfrak{S}_E$ be the permutation group of $E$. The *permutohedron* on $E$ is the polytope

$$\Pi_E = \text{convex hull of } \{w \cdot (0, \ldots, n-1) : w \in \mathfrak{S}_E\}.$$

Let the *permutohedral fan* $\Sigma_E$ be its normal fan in $\mathbb{R}^E$ with lineality space $\mathbb{R}\mathbf{e}_E$. It consists of the cones

$$\sigma_{\mathscr{C}} : \mathbb{R}_{\geq 0}\{\mathbf{e}_{S_1}, \ldots, \mathbf{e}_{S_k}\} + \mathbb{R}\mathbf{e}_E$$

for $\mathscr{C} = (\varnothing \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E)$ a nonempty proper chain of subsets of $E$.

**Exercise 2.5.** Show that the span of a codimension 1 cone $\sigma \in \Sigma_E(n-1)$ is the hyperplane $\{x_i = x_j\} \subset \mathbb{R}^E$ for some $i \neq j \in E$.

**Proposition 2.6.** [Pos09, ACEP20] A lattice polytope $Q \subset \mathbb{R}^E$ is a deformation of $\Sigma_E$ if and only if each edge of $Q$ is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in E$. Deformations of $\Sigma_E$ are also known as (integral) *generalized permutohedra.*

Matroids finally enter into our picture as follows.

**Theorem 2.7.** [GGMS87] For a collection $\mathcal{B} \subseteq 2^E$ of subsets of $E$, the polytope

$$\text{convex hull of } \{\mathbf{e}_B : B \in \mathcal{B}\} \subset \mathbb{R}^E.$$

is a generalized permutohedron if and only if $\mathcal{B}$ is the set of basis of a matroid $\mathrm{M}$ on $E$.

For a matroid $\mathrm{M}$ on $E$, we call the polytope in the theorem the *base polytope* of $\mathrm{M}$, denoted $P(\mathrm{M})$. The theorem implies that the base polytopes of matroids are exactly the generalized permutohedra contained in the unit cube $[0,1]^E$.

**Remark 2.8.** When a matroid $\mathrm{M}$ of rank $r$ has a realization by $L \subseteq \mathbb{C}^E$, that is, a point in the Grassmannian $Gr(r;E)$ with the usual $T$-action, we have that $\overline{T \cdot L}$ is isomorphic to the toric variety of the base polytope. The line bundle $\mathcal{L}_{P(M)}$ is the pullback of $\mathcal{O}(1)$ on $Gr(r;E)$ along the composition $X_E \to X_{P(\mathrm{M})} \to Gr(r;E)$.

**Exercise 2.9.** Deduce the greedy algorithm property of matroids from the fact that base polytopes of matroids are generalized permutohedra.

The following notion of *valuativity* is a powerful tool in the study of matroid invariants [AFR10, DF10, AS22, BEST, FS].

**Definition 2.10.** For $0 \le r \le n$, define the (rank $r$) *valuative group* by

$$\mathrm{Val}_r(E) = \text{the subgroup of } \mathbb{I}(\Sigma_E) \text{ generated by } \{1_{P(\mathrm{M})} : \mathrm{M} \text{ a matroid on } E \text{ of rank } r\}.$$

A function $f$ on the set of matroids on $E$ with values in an abelian is *valuative* if it factors through $\bigoplus_{r=0}^n \mathrm{Val}_r(E)$.

An element $i \in E$ is a *loop* in a matroid $\mathrm{M}$ if $i$ is in no basis of $\mathrm{M}$, or equivalently $P(\mathrm{M}) \subset \{x_i = 0\}$. Dually, an element $i \in E$ is a *coloop* if $i$ is in every basis of $\mathrm{M}$, or equivalently $P(\mathrm{M}) \subset \{x_i = 1\}$. Let $\mathrm{Val}_r^\circ(E)$ be the subgroup of $\mathrm{Val}_r(E)$ generated by the loopless matroids.

**Exercise 2.11.** Let $E = \{1,2,3,4\}$. Compute that the ranks of the groups $\mathrm{Val}_r^\circ(E)$ for $r = 0, \dots, 4$ are $0, 1, 11, 11, 1$. Compare this to the $h$-vector of the simple polytope $\Pi_E$ (i.e. the $h$-vector of the simplicial complex which is the full barycentric subdivision of the boundary of the 3-simplex—that is, take stellar subdivisions of all facets, and then of all original codimension 1 faces, and so forth).

**Theorem 2.12.** We have an isomorphism

$$\bigoplus_{r=1}^n \mathrm{Val}_r^\circ(E) \xrightarrow{\sim} \overline{\mathbb{I}}(\Sigma_E) \quad \text{given by} \quad 1_{P(\mathrm{M})} \mapsto [P(\mathrm{M})].$$

This is a special case of [EL, Theorem 6.1]. Similar statements can be found in [EHL, Proposition 7.4] and [ELS, Corollary 2.16].

**Corollary 2.13.** For any valuative function $f$ on the set of loopless matroids with values in $\mathbb{Z}$, there exists loopless matroids $\mathrm{M}_1, \dots, \mathrm{M}_k$ and integers $a_1, \dots, a_k$ such that

$$f(\mathrm{M}) = \chi\big([P(\mathrm{M})] \cdot (a_1[P(\mathrm{M}_1)] + \cdots + a_k[P(\mathrm{M}_k)])\big) \quad \text{for all loopless matroid } \mathrm{M} \text{ on } E.$$

*Proof.* Combine the theorem with the fact [AP15, Theorem 6.1] that the bilinear map $K(\Sigma) \times K(\Sigma) \to \mathbb{Z}$ defined by $(a, b) \mapsto \chi(a \cdot b)$, is non-degenerate. □

In fact, any valuative function on the loopless matroids extends to all matroids by sending matroids with loops to zero. That is, the conclusion of the corollary holds for all valuative function on matroids that is zero on matroids with loops.

*Sketch of the proof of Theorem 2.12.* The key fact is that the intersection of an (integral) generalized permutohedra with a coordinate half-space is an (integral) generalized permutohedra. Then, one considers tiling $\mathbb{R}^E$ by integral translates of unit cubes, and observes that base polytopes of matroids are translates of each other if and only if the matroids differ by converting some loops to coloops. □

**Exercise 2.14.** Finish the proof of the theorem, given the key fact.

# 3 Chow cohomology and Bergman fans

By *primitive ray generators* $\mathcal{R}(\Sigma)$ of a fan $\Sigma$ with possibly nontrivial lineality space, we mean a set of vectors in $\mathbb{Z}^E$ whose images in $\mathbb{R}^E/\operatorname{lin}(\Sigma)$ are the primitive ray generators of $\Sigma/\operatorname{lin}(\Sigma)$. Fix such a choice, and denote $u_\rho \in \mathcal{R}(\Sigma)$ for each $\rho \in \Sigma(\ell+1)$. Let us also fix a $\mathbb{Z}$-basis $(u_1, \ldots, u_\ell)$ of $\operatorname{lin}(\Sigma) \cap \mathbb{Z}^E$.

**Example 3.1.** For the permutohedral fan $\Sigma_E$, our choice of $\mathcal{R}(\Sigma_E)$ will always be $\{\mathbf{e}_S : \varnothing \subsetneq S \subsetneq E\}$, and we fix $\{\mathbf{e}_E\}$ as the basis of $\operatorname{lin}(\Sigma_E)$.

## 3.1 Cohomology rings, Minkowski weights, and piecewise polynomials

Like the $K$-ring case, we describe (torus-equivariant) Chow cohomology rings in three different ways. We will need the following notion of Minkowski weights (a.k.a. tropical cycles).

**Definition 3.2.** A $d$-dimensional *Minkowski weight* on $\Sigma$ is a function $w : \Sigma(d) \to \mathbb{Z}$ such that for any $\tau \in \Sigma(d-1)$, it satisfies

$$\sum_{\sigma \supset \tau} w(\sigma) u_{\sigma \setminus \tau} \in \operatorname{span}_{\mathbb{R}}(\tau).$$

where $u_{\sigma\setminus\tau}$ denotes the unique primitive ray generator in $\sigma$ not in $\tau$. The group of $d$-dimensional Minkowski weights is denoted $\operatorname{MW}_d(\Sigma)$.

**Example 3.3.**


**Example 3.4.** As $\Sigma$ is a complete (i.e. its support is all of $\mathbb{R}^E$), the constant function $\Sigma(n) \to \mathbb{Z}$ sending $\sigma \mapsto 1$ for all $\sigma \in \Sigma(n)$ is the Minkowski weight of weight $n$ up to scaling.

**Exercise 3.5.** For a deformation $Q$ of $\Sigma$, the function $w_Q : \Sigma(n-1) \to \mathbb{Z}$ given by

$$w_Q(\sigma) = \text{(lattice) length of } \operatorname{face}_\sigma(Q)$$

is a Minkowski weight.

**Remark 3.6.** Given a *very affine variety*, i.e. a subvariety $X$ of $(\mathbb{C}^*)^E$, its *tropicalization* $\operatorname{trop}(X)$ is a pure-dimensional polyhedral complex in $\mathbb{R}^E$ with appropriate weights on the maximal cells. It defines a Minkowski weight on any $\Sigma$ containing a subfan whose support equals the support of $\operatorname{trop}(X)$. Moreover, under the isomorphism in Theorem 3.7, this Minkowski weight equals the Chow homology class $[\overline{X}]$ of closure of $X$ in $X_\Sigma$. See [MS15, Ch. 6].

Let $A^\bullet_T(\Sigma)$ and $A^\bullet(\Sigma)$ be the *T-equivariant Chow cohomology ring* and *Chow cohomology ring* of $X_\Sigma$, respectively. These geometric rings have the following combinatorial descriptions. For $\mathbf{m} \in \mathbb{Z}^E$, denote by $\mathbf{m} \cdot \mathbf{t}$ the polynomial $m_1 t_1 + \cdots + m_n t_n \in \mathbb{Z}[\mathbf{t}] = \mathbb{Z}[t_1, \ldots, t_n]$.

**Theorem 3.7.** The rings $A^\bullet_T(\Sigma)$ and $A^\bullet(\Sigma)$ have the following equivalent descriptions:

1. [CLS11, Theorems 12.4.4 & 12.4.14] Let $[\ell] = \{1, \ldots, \ell\}$, indexing the fixed basis $\{u_1, \ldots, u_\ell\}$ of $\operatorname{lin}(\Sigma) \cap \mathbb{Z}^E$. As standard graded polynomial rings we have

$$A^\bullet_T(\Sigma) = \frac{\mathbb{Z}[x_\rho : \rho \in \Sigma(\ell+1) \cup [\ell]]}{\left\langle \prod_{\rho \in S} x_\rho : S \subseteq \Sigma(\ell+1) \cup [\ell] \text{ such that } \{u_\rho\}_{\rho \in S} \text{ not in a common cone of } \Sigma \right\rangle}, \quad \text{and}$$
$$A^\bullet(\Sigma) = \frac{A^\bullet_T(\Sigma)}{\left\langle \sum_{\rho \in \Sigma(\ell+1) \cup [\ell]} \langle u_\rho, m \rangle x_\rho : m \in \mathbb{Z}^E \right\rangle}.$$

2. [FS97] For $0 \leq i \leq n - \ell$, denote by $\mathrm{MW}^i(\Sigma) = \mathrm{MW}_{n-\ell-i}(\Sigma)$. We have

$$A^\bullet(\Sigma) \simeq \mathrm{MW}^\bullet(\Sigma)$$

where the multiplication in $\mathrm{MW}$ is given by *stable intersections* [FS97].

3. Recall the notation that $\mathbf{m}(\sigma, \sigma')$ denotes the primitive vector normal to $\sigma \cap \sigma'$ for two maximal cones $\sigma$ and $\sigma'$ of $\Sigma$ sharing a wall. Then, we have [Bri96, Bri97]

$$A^\bullet_T(\Sigma) \simeq PP(\Sigma) = \left\{ (f_\sigma)_{\sigma \in \Sigma_{\max}} \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{t}] \;\middle|\; \begin{array}{l} f_\sigma - f_{\sigma'} \equiv 0 \bmod \mathbf{m}(\sigma, \sigma') \cdot \mathbf{T} \\ \text{for any } \sigma \text{ and } \sigma' \text{ sharing a wall} \end{array} \right\}, \quad \text{and}$$
$$A^\bullet(\Sigma) \simeq \overline{PP}(\Sigma) = PP(\Sigma)/I_A$$

where $I_A$ is the ideal generated by $\{t_i\}$ where $t_i$ here is considered as an element $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{t}]$ by $f_\sigma = t_i$ for all $\sigma$.

Given a deformation $Q$, one obtains

$$D_Q = \sum_{\rho \in \Sigma(\ell+1) \cup [\ell]} - \min_{m \in Q} \langle u_\rho, m \rangle x_\rho \in A^\bullet_T(\Sigma) \quad \text{and} \quad (-\operatorname{face}_\sigma(Q) \cdot \mathbf{t})_\sigma \in PP(\Sigma).$$

Under the claimed isomorphism $A^\bullet_T(\Sigma) \simeq PP(\Sigma)$, these two elements agree, and their image in $A^\bullet(\Sigma)$ is the Minkowski weight $w_Q$ in Exercise 3.5 under the isomorphism $A^\bullet(\Sigma) \simeq \mathrm{MW}^\bullet(\Sigma)$.

**Example 3.8.** For $\Sigma = \Sigma_E$, for reach "ray" $\rho_S = \mathbb{R}_{\geq 0}\mathbf{e}_S + \mathbb{R}\mathbf{e}_E$ of $\Sigma_E$, let us denote $x_{\rho_S}$ by $x_S$. Then, we compute that

$$A_T^\bullet(\Sigma_E) = \frac{\mathbb{Z}[x_S : \varnothing \subsetneq S \subseteq E]}{\langle x_S x_{S'} : S, S' \text{ incomparable}\rangle} \quad \text{and} \quad A^\bullet(\Sigma_E) = \frac{A_T^\bullet(\Sigma_E)}{\langle \sum_{S \ni i} x_S : i \in E\rangle}.$$

**Theorem 3.9.** There is an isomorphism, called the *degree map* and denoted $\deg_\Sigma : A^{n-\ell}(\Sigma) \xrightarrow{\sim} \mathbb{Z}$, such that $\deg_\Sigma(\prod_{\rho \in \sigma(\ell+1)} x_\rho) = 1$ for all $\sigma \in \Sigma_{\max}$. The degree map also has description as $\deg_\Sigma(D_Q^{n-\ell}) = \mathrm{Volume}(Q)$ for any $Q \in \mathrm{Def}(\Sigma)$.

Here, Volume is defined by declaring that the volume of a full-dimensional unit simplex in $\mathrm{lin}(\Sigma)^\perp = \{x \in \mathbb{R}^E : \langle x, y\rangle = 0 \text{ for all } y \in \mathrm{lin}(\Sigma)\}$ is equal to 1.

## 3.2 Bergman fans of matroids

**Definition 3.10.** A *flat* of a matroid $\mathrm{M}$ on $E$ is a subset $F \subseteq E$ such that

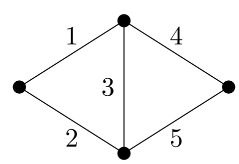
$$\max\{|(F \cup i) \cap B| : B \text{ a basis of } \mathrm{M}\} > \max\{|F \cap B| : B \text{ a basis of } \mathrm{M}\} \quad \text{for all } i \in E \setminus F.$$

Or equivalently, a flat is a subset of $E$ that is maximal for its rank.

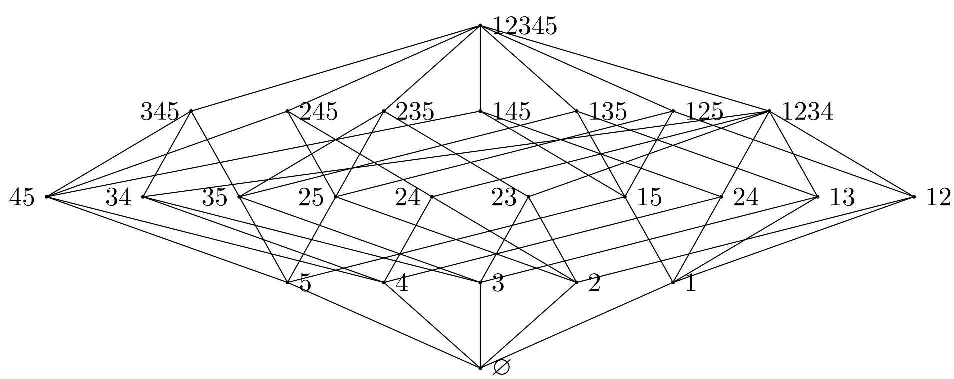
Note that $\varnothing$ is a flat of $\mathrm{M}$ if and only if $\mathrm{M}$ is loopless. The set of flats of a matroid $\mathrm{M}$ forms a poset under inclusion. This poset is a *lattice* with meet and join defined by

$$F \wedge F' = F \cap F' \quad \text{and} \quad F \vee F' = \text{the smallest flat containing } F \cup F'.$$

**Example 3.11.** Let $\mathrm{M}$ be the graphical matroid of the graph




The lattice of flats of $\mathrm{M}$ is as follows.



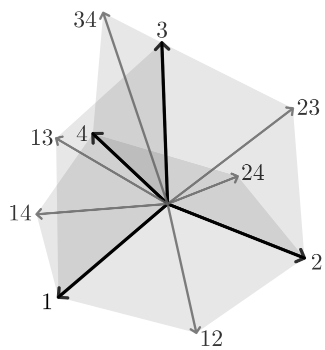
**Definition 3.12.** The Bergman fan of a loopless matroid $\mathrm{M}$ is a subfan $\Sigma_{\mathrm{M}}$ of $\Sigma_E$ consisting of cones

$$\sigma_{\mathscr{F}} = \mathrm{cone}\{\mathbf{e}_{F_1}, \dots, \mathbf{e}_{F_k}\} + \mathbb{R}\mathbf{e}_E$$

for every chain $\mathscr{F} = \{\varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E\}$ of nonempty proper flats of $\mathrm{M}$.

**Example 3.13.** When $\mathrm{M} = U_{n,n}$ (sometimes called the *Boolean matroid* on $E$), we have that $\Sigma_{\mathrm{M}} = \Sigma_E$.

**Example 3.14.** The following is the Bergman of $U_{3,4}$ with the lineality space $\mathbb{R}\mathbf{e}_E$ quotiented out.




**Proposition 3.15.** Let $\mathrm{M}$ be a loopless matroid of rank $r$ on $E$.

(a) [AHK18, Proposition 5.2] The function

$$w_{\mathrm{M}} : \Sigma_E(r) \to \mathbb{Z} \quad \text{defined by} \quad w_{\mathrm{M}}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in \Sigma_{\mathrm{M}} \\ 0 & \text{otherwise.} \end{cases}$$

(b) [BEST, Theorem 7.6] For $\sigma \in \Sigma_E(n)$, denote by $B_\sigma(\mathrm{M}^\perp)$ the basis of the dual matroid $\mathrm{M}^\perp$ such that $\mathbf{e}_{B_\sigma(\mathrm{M}^\perp)} = \mathrm{face}_\sigma(\mathrm{M}^\perp)$. Define an element $w_{\mathrm{M}}^T$ in $PP(\Sigma_E)$ by

$$(w_{\mathrm{M}})^T_\sigma = \prod_{i \in B_\sigma(\mathrm{M}^\perp)} -t_i = (-1)^{n-r} t^{\mathbf{e}_{B_\sigma(\mathrm{M}^\perp)}} \quad \text{for every } \sigma \in \Sigma(n).$$

Then, its image in $\overline{PP}(\Sigma_E)$, under the isomorphism $\overline{PP}(\Sigma_E) \simeq \mathrm{MW}^\bullet(\Sigma)$, equals $w_{\mathrm{M}}$.

**Exercise 3.16.** If $\mathrm{M}$ is a loopless matroid of rank $n-1$ on $E$, then $w_{\mathrm{M}} = D_{P(\mathrm{M}^\perp)}$.

The following is a key application of the Hodge theory of matroids developed in [AHK18].

**Theorem 3.17.** Let $Q_1$ and $Q_2$ be deformations of $\Sigma_E$, and $\mathrm{M}$ a loopless matroid of rank $r$. Then,

$$\big(\deg_{\Sigma_E}(w_{\mathrm{M}} \cdot D_{Q_1}^{r-1}), \deg_{\Sigma_E}(w_{\mathrm{M}} \cdot D_{Q_1}^{r-2} D_{Q_2}), \dots, \deg_{\Sigma_E}(w_{\mathrm{M}} \cdot D_{Q_1}^{r-1-i} D_{Q_2}^{i}), \dots, \deg_{\Sigma_E}(w_{\mathrm{M}} \cdot D_{Q_2}^{r-1})\big)$$

is a log-concave nonnegative sequence with no internal zeros.

Setting $Q_1$ and $Q_2$ to be the standard and the opposite simplex, the authors of [AHK18] resolved the long-standing conjecture on a log-concavity of the coefficients of $T_{\mathrm{M}}(1+x, 0)$.

# References


[ACEP20] Federico Ardila, Federico Castillo, Christopher Eur, and Alexander Postnikov. Coxeter submodular functions and deformations of Coxeter permutahedra. *Adv. Math.*, 365:107039, 2020. 3

[AFR10] Federico Ardila, Alex Fink, and Felipe Rincón. Valuations for matroid polytope subdivisions. *Canad. J. Math.*, 62(6):1228–1245, 2010. 4

[AHK18] Karim Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. *Ann. of Math. (2)*, 188(2):381–452, 2018. 8

[AP15] Dave Anderson and Sam Payne. Operational $K$-theory. *Doc. Math.*, 20:357–399, 2015. 5

[AS22] Federico Ardila and Mario Sanchez. Valuations and the Hopf Monoid of Generalized Permutahedra. *International Mathematics Research Notices*, 01 2022. rnab355. 4

[BEST] Andrew Berget, Christopher Eur, Hunter Spink, and Dennis Tseng. Tautological classes of matroids. arXiv:2103.08021. 4, 8

[Bri88] Michel Brion. Points entiers dans les polyèdres convexes. *Ann. Sci. École Norm. Sup. (4)*, 21(4):653–663, 1988. 3

[Bri96] Michel Brion. Piecewise polynomial functions, convex polytopes and enumerative geometry. pages 25–44, 1996. 6

[Bri97] M. Brion. Equivariant Chow groups for torus actions. *Transform. Groups*, 2(3):225–267, 1997. 6

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011. 2, 3, 6

[DF10] Harm Derksen and Alex Fink. Valuative invariants for polymatroids. *Adv. Math.*, 225(4):1840–1892, 2010. 4

[EHL] Christopher Eur, June Huh, and Matt Larson. Intersection theory of polymatroids. arXiv:2207.10605. 2, 4

[EL] Christopher Eur and Matt Larson. Stellahedral geometry of matroids. arXiv:2301.00831. 4

[ELS] Alex Eur, Christopher Fink, Matt Larson, and Hunter Spink. Signed permutohedra, delta-matroids, and beyond. arXiv:2209.06752. 4

[FS] Luis Ferroni and Benjamin Schröter. Valuative invariants for large classes of matroids. arXiv:2208.04893. 4

[FS97] William Fulton and Bernd Sturmfels. Intersection theory on toric varieties. *Topology*, 36(2):335–353, 1997. 6

[GGMS87] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. in Math.*, 63(3):301–316, 1987. 4

[Ish90] Masa-Nori Ishida. Polyhedral Laurent series and Brion's equalities. *Internat. J. Math.*, 1(3):251–265, 1990. 3

[McM89] Peter McMullen. The polytope algebra. *Adv. Math.*, 78(1):76–130, 1989. 2

[Mor93] Robert Morelli. The $K$-theory of a toric variety. *Adv. Math.*, 100(2):154–182, 1993. 2

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015. 6

[Nie74] Andreas Nielsen. Diagonalizably linearized coherent sheaves. *Bull. Soc. Math. France*, 102:85–97, 1974. 3

[Pos09] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009. 3

[VV03] Gabriele Vezzosi and Angelo Vistoli. Higher algebraic $K$-theory for actions of diagonalizable groups. *Invent. Math.*, 153(1):1–44, 2003. 3