

Positivity in matroid theory: MaTroCom Minicourse

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In algebraic geometry, “positivity” broadly refers to numerical properties enjoyed by certain classes of vector bundles on projective varieties. In this minicourse, we survey how a similar “positivity” arises in the combinatorics of matroids. We assume some familiarity with polyhedra and an acquaintance with matroids. Statements involving algebraic geometry (toric varieties) are in [a different color](#), which may be skipped.

Notation. Let $E = \{1, \dots, n\}$ be a finite set of cardinality n . For a subset $S \subseteq E$, denote by $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ the sum of standard basis vectors in \mathbb{R}^E . Let $\langle \cdot, \cdot \rangle$ denote the standard inner product. [Let \$T = \(\mathbb{C}^*\)^E\$ be the torus whose character lattice is \$\mathbb{Z}^E\$. A variety is reduced and irreducible.](#)

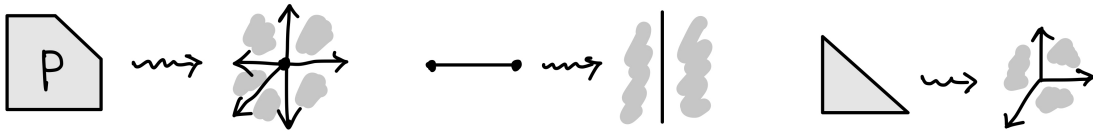
1 Overview

Let Σ be a unimodular projective fan in \mathbb{R}^E . *Projective* means that Σ is the (inner) normal fan Σ_P of a polytope P in \mathbb{R}^E . We allow $\dim P < n$, so the lineality space

$$\text{lin}(\Sigma) = (\text{the minimal cone of } \Sigma) = \{u \in \mathbb{R}^E : \langle u, x \rangle = 0 \ \forall x \in P\}$$

may be nontrivial of dimension ℓ . *Unimodular* means that for any cone $\sigma \in \Sigma$, the *primitive ray vectors* of $\sigma / \text{lin}(\Sigma)$ extends to a \mathbb{Z} -basis of $\mathbb{Z}^E / (\text{lin}(\Sigma) \cap \mathbb{Z}^E)$. Let $\Sigma(d)$ be the set of d -dimensional cones of Σ .

Example 1.1.



Recall the dimension-reversing bijection between the faces of a polytope P and the cones of its normal fan Σ_P given by:

$$\Sigma_P \ni \sigma \leftrightarrow \text{face}_\sigma(P) = \{p \in P : \langle p, v \rangle = \min_{q \in P} \langle q, v \rangle\} \text{ for any } v \text{ in the relative interior of } \sigma.$$

A lattice polytope Q is a *deformation* of Σ , denoted $Q \in \text{Def}(\Sigma)$, if its normal fan Σ_Q coarsens Σ . For instance, deformations of Σ in Example 1.1 include rectangles, standard simplices, etc.

[Let \$X_\Sigma\$ be the smooth projective toric variety associated to \$\Sigma / \text{lin}\(\Sigma\)\$, considered as a \$T\$ -variety. Recall similarly the bijection between the cones of \$\Sigma\$ and the torus-orbits of \$X_\Sigma\$. In particular, the](#)

maximal cones correspond to the torus-fixed points of X_Σ . A deformation Q of Σ corresponds to the base-point-free T -line bundle \mathcal{L}_Q on X_Σ whose complete linear system gives a map $X_\Sigma \rightarrow \mathbb{P}^{|Q \cap \mathbb{Z}^E| - 1}$ induced by the map of tori $t \mapsto (t^{\mathbf{m}})_{\mathbf{m} \in Q \cap \mathbb{Z}^E}$ [CLS11, Chapter 6].

We will learn about two well-studied rings $K(\Sigma)$ and $A^\bullet(\Sigma)$ attached to such Σ . In geometric terms, these are the Grothendieck K -ring of vector bundles and the Chow cohomology ring of the toric variety X_Σ . “GKM-varieties” is a good keyword for those wanting more geometric details. By associating to each matroid certain elements in these rings, one gains an insight into combinatorial properties of matroids via geometric methods.

2 K-rings and matroid polytopes

2.1 K-rings, polytope algebras, and “piecewise” Laurent polynomials

We first describe the ring denoted $K_T(\Sigma)$ and then describe $K(\Sigma)$ as its quotient. It has three different descriptions, whose equivalence is a consequence of some major theorems.

Definition 2.1. Let $K_T(\Sigma)$ be the Grothendieck K -ring of T -equivariant vector bundles on X_Σ , and let $K(\Sigma)$ be the non-equivariant K -ring. That is,

$$K_T(\Sigma) = \frac{\left\{ \sum_i a_i [\mathcal{E}_i]^T : a_i \in \mathbb{Z}, \mathcal{E}_i \text{ a } T\text{-equivariant vector bundle on } X_\Sigma \right\}}{\left\langle [\mathcal{E}]^T = [\mathcal{E}']^T + [\mathcal{E}'']^T : \exists \text{ a } T\text{-equivariant SES } 0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0 \right\rangle}, \quad \text{and}$$

$$K(\Sigma) = \frac{\left\{ \sum_i a_i [\mathcal{E}_i] : a_i \in \mathbb{Z}, \mathcal{E}_i \text{ a vector bundle on } X_\Sigma \right\}}{\left\langle [\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}''] : \exists \text{ a SES } 0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0 \right\rangle},$$

with the multiplication is given by tensor products.

The natural “forgetting T -equivariance” map $K_T(\Sigma) \rightarrow K(\Sigma)$ is a surjection [Mor93, Proposition 3]. This geometrically defined ring has the following combinatorial descriptions. For $\mathbf{m} \in \mathbb{Z}^E$, denote by $\mathbf{T}^{\mathbf{m}}$ the Laurent monomial $T_1^{m_1} \cdots T_n^{m_n} \in \mathbb{Z}[\mathbf{T}^\pm] = \mathbb{Z}[T_1^\pm, \dots, T_n^\pm]$.

Theorem 2.2. The rings $K_T(\Sigma)$ and $K(\Sigma)$ have the following equivalent descriptions:

1. For a polytope $Q \subset \mathbb{R}^E$, let $1_Q : \mathbb{R}^E \rightarrow \mathbb{Z}$ be its indicator function given by $1_Q(x) = 1$ if $x \in Q$ and $1_Q(x) = 0$ otherwise. Then, we have by [EHL, Theorem A.10]

$$K_T(\Sigma) \simeq \mathbb{I}(\Sigma) = \text{the subgroup of } \mathbb{Z}^{(\mathbb{R}^E)} \text{ generated by } \{1_Q \mid Q \in \text{Def}(\Sigma)\}, \quad \text{and}$$

$$K(\Sigma) \simeq \bar{\mathbb{I}}(\Sigma) = \mathbb{I}(\Sigma) / \text{transl}(\Sigma)$$

where $\text{transl}(\Sigma)$ is the subgroup of $\mathbb{I}(\Sigma)$ generated by $\{1_Q - 1_{Q+u} \mid u \in \mathbb{Z}^m\}$. Multiplication in these rings are given by Minkowski sums of polytopes. Denote by $[Q]$ the class of 1_Q in $\bar{\mathbb{I}}(\Sigma)$. The ring $\bar{\mathbb{I}}(\Sigma)$ is also known as the *polytope algebra* [McM89].

2. For two maximal cones σ and σ' of Σ sharing a wall (i.e. a codimension 1 face), let $\mathbf{m}(\sigma, \sigma')$ be the primitive vector normal to $\sigma \cap \sigma'$. Then, we have by [Nie74, VV03]

$$K_T(\Sigma) \simeq LP(\Sigma) = \left\{ (f_\sigma)_{\sigma \in \Sigma_{\max}} \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^\pm] \mid \begin{array}{l} f_\sigma - f_{\sigma'} \equiv 0 \pmod{(1 - \mathbf{T}^{\mathbf{m}(\sigma, \sigma')})} \\ \text{for any } \sigma \text{ and } \sigma' \text{ sharing a wall} \end{array} \right\}, \quad \text{and}$$

$$K(\Sigma) \simeq \overline{LP}(\Sigma) = LP(\Sigma)/I_K$$

where I_K is the ideal generated by $\{T_i - 1 : i \in E\}$ where T_i here is considered as an element $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^\pm]$ by $f_\sigma = T_i$ for all σ .

Given $Q \in \text{Def}(\Sigma)$, the claimed isomorphisms are given by $1_Q \mapsto (\mathbf{T}^{-\text{face}_\sigma(Q)})_\sigma \in LP(\Sigma)$ and $1_Q \mapsto [\mathcal{L}_Q] \in K_T(\Sigma)$. The isomorphism $K_T(\Sigma) \simeq LP(\Sigma)$ is also described by $[\mathcal{E}]^T \mapsto \text{Hilb}(\mathcal{E}|_{p_\sigma})_\sigma$, the restriction to the torus-fixed points.

These “ K -rings” have distinguished maps to $\mathbb{Z}[\mathbf{T}^\pm]$ and \mathbb{Z} . Let $\chi^T : K_T(\Sigma) \rightarrow \mathbb{Z}[\mathbf{T}^\pm]$ and $\chi : K(\Sigma) \rightarrow \mathbb{Z}$ be the sheaf Euler characteristic maps. Combinatorial descriptions of these are:

Theorem 2.3. [CLS11, Ch. 9] For $1_Q \in \mathbb{I}(\Sigma)$, under the isomorphism $K_T(\Sigma) \simeq \mathbb{I}(\Sigma)$ we have

$$\chi^T(1_Q) = \sum_{\mathbf{m} \in Q \cap \mathbb{Z}^E} \mathbf{T}^{-\mathbf{m}} \quad \text{and} \quad \chi([Q]) = |Q \cap \mathbb{Z}^E|.$$

[Bri88, Ish90] For an element in $f \in K_T(\Sigma)$ given by $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^\pm]$, we have

$$\chi^T(f) = \sum_{\sigma \in \Sigma_{\max}} \frac{f_\sigma}{\prod_{\substack{\mathbf{m} \text{ a primitive ray} \\ \text{generator of } \sigma^\vee}} (1 - \mathbf{T}^{-\mathbf{m}})} \quad \text{and} \quad \chi(f) = \chi^T(f)|_{T_1=\dots=T_n=1}.$$

Exercise 2.4. Verify the theorem in the case when $P = Q = \text{Conv}\{\mathbf{e}_1, \mathbf{e}_2\} \subset \mathbb{R}^2$. (If you’d like to get a feel of how quickly nontrivial it gets, try $P = Q = \text{Conv}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \mathbb{R}^e$).

2.2 Permutohedral fan and base polytopes of matroids

Let \mathfrak{S}_E be the permutation group of E . The *permutohedron* on E is the polytope

$$\Pi_E = \text{convex hull of } \{w \cdot (0, \dots, n-1) : w \in \mathfrak{S}_E\}.$$

Let the *permutohedral fan* Σ_E be its normal fan in \mathbb{R}^E with lineality space $\mathbb{R}\mathbf{e}_E$. It consists of the cones

$$\sigma_{\mathcal{C}} : \mathbb{R}_{\geq 0}\{\mathbf{e}_{S_1}, \dots, \mathbf{e}_{S_k}\} + \mathbb{R}\mathbf{e}_E$$

for $\mathcal{C} = (\emptyset \subsetneq S_1 \subsetneq \dots \subsetneq S_k \subsetneq E)$ a nonempty proper chain of subsets of E .

Exercise 2.5. Show that the span of a codimension 1 cone $\sigma \in \Sigma_E(n-1)$ is the hyperplane $\{x_i = x_j\} \subset \mathbb{R}^E$ for some $i \neq j \in E$.

Proposition 2.6. [Pos09, ACEP20] A lattice polytope $Q \subset \mathbb{R}^E$ is a deformation of Σ_E if and only if each edge of Q is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in E$. Deformations of Σ_E are also known as (integral) *generalized permutohedra*.

Matroids finally enter into our picture as follows.

Theorem 2.7. [GMS87] For a collection $\mathcal{B} \subseteq 2^E$ of subsets of E , the polytope

$$\text{convex hull of } \{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^E.$$

is a generalized permutohedron if and only if \mathcal{B} is the set of basis of a matroid M on E .

For a matroid M on E , we call the polytope in the theorem the *base polytope* of M , denoted $P(M)$. The theorem implies that the base polytopes of matroids are exactly the generalized permutohedra contained in the unit cube $[0, 1]^E$.

Remark 2.8. When a matroid M of rank r has a realization by $L \subseteq \mathbb{C}^E$, that is, a point in the Grassmannian $Gr(r; E)$ with the usual T -action, we have that $\overline{T \cdot L}$ is isomorphic to the toric variety of the base polytope. The line bundle $\mathcal{L}_{P(M)}$ is the pullback of $\mathcal{O}(1)$ on $Gr(r; E)$ along the composition $X_E \rightarrow X_{P(M)} \rightarrow Gr(r; E)$.

Exercise 2.9. Deduce the greedy algorithm property of matroids from the fact that base polytopes of matroids are generalized permutohedra.

The following notion of *valuativity* is a powerful tool in the study of matroid invariants [AFR10, DF10, AS22, BEST, FS].

Definition 2.10. For $0 \leq r \leq n$, define the (rank r) *valuative group* by

$$\text{Val}_r(E) = \text{the subgroup of } \mathbb{I}(\Sigma_E) \text{ generated by } \{1_{P(M)} : M \text{ a matroid on } E \text{ of rank } r\}.$$

A function f on the set of matroids on E with values in an abelian is *valuative* if it factors through $\bigoplus_{r=0}^n \text{Val}_r(E)$.

An element $i \in E$ is a *loop* in a matroid M if i is in no basis of M , or equivalently $P(M) \subset \{x_i = 0\}$. Dually, an element $i \in E$ is a *coloop* if i is in every basis of M , or equivalently $P(M) \subset \{x_i = 1\}$. Let $\text{Val}_r^\circ(E)$ be the subgroup of $\text{Val}_r(E)$ generated by the loopless matroids.

Exercise 2.11. Let $E = \{1, 2, 3, 4\}$. Compute that the ranks of the groups $\text{Val}_r^\circ(E)$ for $r = 0, \dots, 4$ are 0, 1, 11, 11, 1. Compare this to the h -vector of the simple polytope Π_E (i.e. the h -vector of the simplicial complex which is the full barycentric subdivision of the boundary of the 3-simplex—that is, take stellar subdivisions of all facets, and then of all original codimension 1 faces, and so forth).

Theorem 2.12. We have an isomorphism

$$\bigoplus_{r=1}^n \text{Val}_r^\circ(E) \xrightarrow{\sim} \mathbb{I}(\Sigma_E) \quad \text{given by} \quad 1_{P(M)} \mapsto [P(M)].$$

This is a special case of [EL, Theorem 6.1]. Similar statements can be found in [EHL, Proposition 7.4] and [ELS, Corollary 2.16].

Corollary 2.13. For any valutive function f on the set of loopless matroids with values in \mathbb{Z} , there exists loopless matroids M_1, \dots, M_k and integers a_1, \dots, a_k such that

$$f(M) = \chi([P(M)] \cdot (a_1[P(M_1)] + \dots + a_k[P(M_k)])) \quad \text{for all loopless matroid } M \text{ on } E.$$

Proof. Combine the theorem with the fact [AP15, Theorem 6.1] that the bilinear map $K(\Sigma) \times K(\Sigma) \rightarrow \mathbb{Z}$ defined by $(a, b) \mapsto \chi(a \cdot b)$, is non-degenerate. \square

In fact, any valutive function on the loopless matroids extends to all matroids by sending matroids with loops to zero. That is, the conclusion of the corollary holds for all valutive function on matroids that is zero on matroids with loops.

Sketch of the proof of Theorem 2.12. The key fact is that the intersection of an (integral) generalized permutohedra with a coordinate half-space is an (integral) generalized permutohedra. Then, one considers tiling \mathbb{R}^E by integral translates of unit cubes, and observes that base polytopes of matroids are translates of each other if and only if the matroids differ by converting some loops to coloops. \square

Exercise 2.14. Finish the proof of the theorem, given the key fact.

3 Chow cohomology and Bergman fans

By *primitive ray generators* $\mathcal{R}(\Sigma)$ of a fan Σ with possibly nontrivial lineality space, we mean a set of vectors in \mathbb{Z}^E whose images in $\mathbb{R}^E / \text{lin}(\Sigma)$ are the primitive ray generators of $\Sigma / \text{lin}(\Sigma)$. Fix such a choice, and denote $u_\rho \in \mathcal{R}(\Sigma)$ for each $\rho \in \Sigma(\ell + 1)$. Let us also fix a \mathbb{Z} -basis (u_1, \dots, u_ℓ) of $\text{lin}(\Sigma) \cap \mathbb{Z}^E$.

Example 3.1. For the permutohedral fan Σ_E , our choice of $\mathcal{R}(\Sigma_E)$ will always be $\{e_S : \emptyset \subsetneq S \subsetneq E\}$, and we fix $\{e_E\}$ as the basis of $\text{lin}(\Sigma_E)$.

3.1 Cohomology rings, Minkowski weights, and piecewise polynomials

Like the K -ring case, we describe (torus-equivariant) Chow cohomology rings in three different ways. We will need the following notion of Minkowski weights (a.k.a. tropical cycles).

Definition 3.2. A d -dimensional *Minkowski weight* on Σ is a function $w : \Sigma(d) \rightarrow \mathbb{Z}$ such that for any $\tau \in \Sigma(d - 1)$, it satisfies

$$\sum_{\sigma \supset \tau} w(\sigma) u_{\sigma \setminus \tau} \in \text{span}_{\mathbb{R}}(\tau).$$

where $u_{\sigma \setminus \tau}$ denotes the unique primitive ray generator in σ not in τ . The group of d -dimensional Minkowski weights is denoted $\text{MW}_d(\Sigma)$.

Example 3.3.



Example 3.4. As Σ is a complete (i.e. its support is all of \mathbb{R}^E), the constant function $\Sigma(n) \rightarrow \mathbb{Z}$ sending $\sigma \mapsto 1$ for all $\sigma \in \Sigma(n)$ is the Minkowski weight of weight n up to scaling.

Exercise 3.5. For a deformation Q of Σ , the function $w_Q : \Sigma(n-1) \rightarrow \mathbb{Z}$ given by

$$w_Q(\sigma) = (\text{lattice}) \text{ length of } \text{face}_\sigma(Q)$$

is a Minkowski weight.

Remark 3.6. Given a *very affine variety*, i.e. a subvariety X of $(\mathbb{C}^*)^E$, its *tropicalization* $\text{trop}(X)$ is a pure-dimensional polyhedral complex in \mathbb{R}^E with appropriate weights on the maximal cells. It defines a Minkowski weight on any Σ containing a subfan whose support equals the support of $\text{trop}(X)$. Moreover, under the isomorphism in ??, this Minkowski weight equals the Chow homology class $[\overline{X}]$ of closure of X in X_Σ . See [MS15, Ch. 6].

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