Positivity in matroid theory: MaTroCom Minicourse

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In algebraic geometry, "positivity" broadly refers to numerical properties enjoyed by certain classes of vector bundles on projective varieties. In this minicourse, we survey how a similar "positivity" arises in the combinatorics of matroids. We assume some familiarity with polyhedra and an acquaintance with matroids. Statements involving algebraic geometry (toric varieties) are in a different color, which may be skipped.

Notation. Let $E = \{1, \dots, n\}$ be a finite set of cardinality n. For a subset $S \subseteq E$, denote by $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ the sum of standard basis vectors in $\mathbb{R}^E$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product. Let $T = (\mathbb{C}^*)^E$ be the torus whose character lattice is $\mathbb{Z}^E$. A variety is reduced and irreducible.

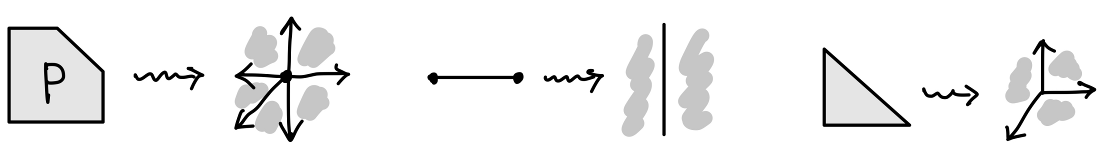
1 Overview

Let Σ be a unimodular projective fan in $\mathbb{R}^E$. *Projective* means that Σ is the (inner) normal fan Σ_P of a polytope P in $\mathbb{R}^E$. We allow $\dim P < n$, so the lineality space

$$\operatorname{lin}(\Sigma) = (\text{the minimal cone of } \Sigma) = \{u \in \mathbb{R}^E : \langle u, x \rangle = 0 \; \forall x \in P\}$$

may be nontrivial of dimension ℓ. *Unimodular* means that for any cone $\sigma \in \Sigma$, the *primitive ray vectors* of $\sigma / \operatorname{lin}(\Sigma)$ extends to a $\mathbb{Z}$-basis of $\mathbb{Z}^E / (\operatorname{lin}(\Sigma) \cap \mathbb{Z}^E)$. Let $\Sigma(d)$ be the set of d-dimensional cones of Σ.

Example 1.1.



Recall the dimension-reversing bijection between the faces of a polytope P and the cones of its normal fan Σ_P given by:

$$\Sigma_P \ni \sigma \leftrightarrow \operatorname{face}_\sigma(P) = \{p \in P : \langle p, v \rangle = \min_{q \in P} \langle q, v \rangle\} \text{ for any } v \text{ in the relative interior of } \sigma.$$

A lattice polytope Q is a *deformation* of Σ, denoted $Q \in \operatorname{Def}(\Sigma)$, if its normal fan Σ_Q coarsens Σ. For instance, deformations of Σ in Example 1.1 include rectangles, standard simplices, etc.

Let X_Σ be the smooth projective toric variety associated to $\Sigma / \operatorname{lin}(\Sigma)$, considered as a T-variety. Recall similarly the bijection between the cones of Σ and the torus-orbits of X_Σ. In particular, the maximal cones correspond to the torus-fixed points of X_Σ. A deformation Q of Σ corresponds to the base-point-free T-line bundle $\mathcal{L}_Q$ on X_Σ whose complete linear system gives a map $X_\Sigma \to \mathbb{P}^{|Q\cap\mathbb{Z}^E|-1}$ induced by the map of tori $t \mapsto (t^{\mathbf{m}})_{\in Q\cap \mathbb{Z}^E}$ [CLS11, Chapter 6].

We will learn about two well-studied rings $K(\Sigma)$ and $A^\bullet(\Sigma)$ attached to such Σ. In geometric terms, these are the Grothendieck K-ring of vector bundles and the Chow cohomology ring of the toric variety X_Σ. "GKM-varieties" is a good keyword for those wanting more geometric details. By associating to each matroid certain elements in these rings, one gains an insight into combinatorial properties of matroids via geometric methods.

2 K-rings and matroid polytopes

2.1 K-rings, polytope algebras, and "piecewise" Laurent polynomials

We first describe the ring denoted $K_T(\Sigma)$ and then describe $K(\Sigma)$ as its quotient. It has three different descriptions, whose equivalence is a consequence of some major theorems.

Definition 2.1. Let $K_T(\Sigma)$ be the Grothendieck K-ring of T-equivariant vector bundles on X_Σ, and let $K(\Sigma)$ be the non-equivariant K-ring. That is,

$$K_T(\Sigma) = \frac{\left\{ \sum_i a_i[\mathcal{E}_i]^T : a_i \in \mathbb{Z},\ \mathcal{E}_i \text{ a } T\text{-equivariant vector bundle on } X_\Sigma \right\}}{\left\langle [\mathcal{E}]^T = [\mathcal{E}']^T + [\mathcal{E}'']^T : \exists \text{ a } T\text{-equivariant SES } 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \right\rangle}, \quad \text{and}$$
$$K(\Sigma) = \frac{\left\{ \sum_i a_i[\mathcal{E}_i] : a_i \in \mathbb{Z},\ \mathcal{E}_i \text{ a vector bundle on } X_\Sigma \right\}}{\left\langle [\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}''] : \exists \text{ a SES } 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \right\rangle},$$

with the multiplication is given by tensor products.

The natural "forgetting T-equivariance" map $K_T(\Sigma) \to K(\Sigma)$ is a surjection [Mor93, Proposition 3]. This geometrically defined ring has the following combinatorial descriptions. For $\mathbf{m} \in \mathbb{Z}^E$, denote by $\mathbf{T}^{\mathbf{m}}$ the Laurent monomial $T_1^{m_1}\cdots T_n^{m_n} \in \mathbb{Z}[\mathbf{T}^{\pm}] = \mathbb{Z}[T_1^{\pm},\dots,T_n^{\pm}]$.

Theorem 2.2. The rings $K_T(\Sigma)$ and $K(\Sigma)$ have the following equivalent descriptions:

1. For a polytope $Q \subset \mathbb{R}^E$, let $1_Q : \mathbb{R}^E \to \mathbb{Z}$ be its indicator function given by $1_Q(x) = 1$ if $x \in Q$ and $1_Q(x) = 0$ otherwise. Then, we have by [EHL, Theorem A.10]

$$K_T(\Sigma) \simeq \mathbb{I}(\Sigma) = \text{the subgroup of } \mathbb{Z}^{(\mathbb{R}^E)} \text{ generated by } \{1_Q \mid Q \in \mathrm{Def}(\Sigma)\}, \quad \text{and}$$
$$K(\Sigma) \simeq \overline{\mathbb{I}}(\Sigma) = \mathbb{I}(\Sigma)/\operatorname{transl}(\Sigma)$$

where $\operatorname{transl}(\Sigma)$ is the subgroup of $\mathbb{I}(\Sigma)$ generated by $\{1_Q - 1_{Q+u} \mid u \in \mathbb{Z}^m\}$. Multiplication in these rings are given by Minkowski sums of polytopes. Denote by $[Q]$ the class of 1_Q in $\overline{\mathbb{I}}(\Sigma)$. The ring $\overline{\mathbb{I}}(\Sigma)$ is also known as the *polytope algebra* [McM89].

2. For two maximal cones σ and σ' of Σ sharing a wall (i.e. a codimension 1 face), let $\mathbf{m}(\sigma, \sigma')$ be the primitive vector normal to $\sigma \cap \sigma'$. Then, we have by [Nie74, VV03]

$$K_T(\Sigma) \simeq LP(\Sigma) = \left\{ (f_\sigma)_{\sigma \in \Sigma_{\max}} \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}] \;\middle|\; \begin{array}{l} f_\sigma - f_{\sigma'} \equiv 0 \bmod (1 - \mathbf{T}^{\mathbf{m}(\sigma,\sigma')}) \\ \text{for any } \sigma \text{ and } \sigma' \text{ sharing a wall} \end{array} \right\}, \quad \text{and}$$
$$K(\Sigma) \simeq \overline{LP}(\Sigma) = LP(\Sigma)/I_K$$

where I_K is the ideal generated by $\{T_i - 1 : i \in E\}$ where T_i here is considered as an element $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}]$ by $f_\sigma = T_i$ for all σ.

Given $Q \in \mathrm{Def}(\Sigma)$, the claimed isomorphisms are given by $1_Q \mapsto (\mathbf{T}^{-\operatorname{face}_\sigma(Q)})_\sigma \in LP(\Sigma)$ and $1_Q \mapsto [\mathcal{L}_Q] \in K_T(\Sigma)$. The isomorphism $K_T(\Sigma) \simeq LP(\Sigma)$ is also described by $[\mathcal{E}]^T \mapsto \mathrm{Hilb}(\mathcal{E}|_{p_\sigma})_\sigma$, the restriction to the torus-fixed points.

These "K-rings" have distinguished maps to $\mathbb{Z}[\mathbf{T}^{\pm}]$ and $\mathbb{Z}$. Let $\chi^T : K_T(\Sigma) \to \mathbb{Z}[\mathbf{T}^{\pm}]$ and $\chi : K(\Sigma) \to \mathbb{Z}$ be the sheaf Euler characteristic maps. Combinatorial descriptions of these are:

Theorem 2.3. [CLS11, Ch. 9] For an element $1_Q \in \mathbb{I}(\Sigma)$, under the isomorphism $K_T(\Sigma) \simeq \mathbb{I}(\Sigma)$ we have

$$\chi^T(1_Q) = \sum_{\mathbf{m} \in Q \cap \mathbb{Z}^E} \mathbf{T}^{-\mathbf{m}} \quad \text{and} \quad \chi([Q)]) = |Q \cap \mathbb{Z}^E|.$$

[Bri88, Ish90]For an element in $f \in K_T(\Sigma)$ given by $(f_\sigma)_\sigma \in \prod_{\sigma \in \Sigma_{\max}} \mathbb{Z}[\mathbf{T}^{\pm}]$, we have

$$\chi^T(f) = \sum_{\sigma \in \Sigma_{\max}} \frac{f_\sigma}{\prod_{\substack{\mathbf{m} \text{ a primitive ray} \\ \text{generator of } \sigma^\vee}} (1 - \mathbf{T}^{-\mathbf{m}})} \quad \text{and} \quad \chi(f) = \chi^T(f)|_{T_1 = \cdots = T_n = 1}.$$

2.2 Permutohedral fan and base polytopes of matroids

Let $\mathfrak{S}_E$ be the permutation group of E. The *permutohedron* on E is the polytope

$$\Pi_E = \text{convex hull of } \{w \cdot (0, \ldots, n-1) : w \in \mathfrak{S}_E\}.$$

Let the *permutohedral fan* Σ_E be its normal fan in $\mathbb{R}^E$ with lineality space $\mathbb{R}\mathbf{e}_E$. It consists of the cones

$$\mathbb{R}_{\geq 0}\{\mathbf{e}_{S_1}, \ldots, \mathbf{e}_{S_k}\} + \mathbb{R}\mathbf{e}_E$$

for $\varnothing \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ a nonempty proper chain of subsets of E.

Proposition 2.4. [Pos09, ACEP20] A lattice polytope $Q \subset \mathbb{R}^E$ is a deformation of Σ_E if and only if each edge of Q is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i, j \in E$. Deformations of Σ_E are also known as (integral) *generalized permutohedra.*

Matroids finally enter into our picture as follows.

Theorem 2.5. [GGMS87] For a collection $\mathcal{B} \subseteq 2^E$ of subsets of E, the polytope

$$\text{convex hull of } \{\mathbf{e}_B : B \in \mathcal{B}\} \subset \mathbb{R}^E.$$

is a generalized permutohedron if and only if $\mathcal{B}$ is the set of basis of a matroid M on E.

For a matroid M on E, we call the polytope in the theorem the *base polytope* of M, denoted $P(\mathrm{M})$. The theorem implies that the base polytopes of matroids are exactly the generalized permutohedra contained in the unit cube $[0,1]^E$.

Remark 2.6. When a matroid M of rank r has a realization by $L \subseteq \mathbb{C}^E$, that is, a point in the Grassmannian $Gr(r;E)$ with the usual T-action, we have that $\overline{T \cdot L}$ is isomorphic to the toric variety of the base polytope. The line bundle $\mathcal{L}_{P(M)}$ is the pullback of $\mathcal{O}(1)$ on $Gr(r;E)$ along the composition $X_E \to X_{P(\mathrm{M})} \to Gr(r;E)$.

Exercise 2.7. Deduce the greedy algorithm property of matroids from the fact that base polytopes of matroids are generalized permutohedra.

The following notion of *valuativity* is a powerful tool in the study of matroid invariants [AFR10, DF10, AS22, BEST, FS].

Definition 2.8. For $0 \leq r \leq n$, define the (rank r) *valuative group* by

$$\mathrm{Val}_r(E) = \text{the subgroup of } \mathbb{I}(\Sigma_E) \text{ generated by } \{1_{P(\mathrm{M})} : \mathrm{M} \text{ a matroid on } E \text{ of rank } r\}.$$

A function f on the set of matroids on E with values in an abelian is *valuative* if it factors through $\bigoplus_{r=0}^n \mathrm{Val}_r(E)$.

An element $i \in E$ is a *loop* in a matroid M if i is in no basis of M, or equivalently $P(\mathrm{M}) \subset \{x_i = 0\}$. Dually, an element $i \in E$ is a *coloop* if i is in every basis of M, or equivalently $P(\mathrm{M}) \subset \{x_i = 1\}$. Let $\mathrm{Val}_r^\circ(E)$ be the subgroup of $\mathrm{Val}_r(E)$ generated by the loopless matroids.

Exercise 2.9. Let $E = \{1,2,3,4\}$. Compute that the ranks of the groups $\mathrm{Val}_r^\circ(E)$ for $r = 0, \ldots, 4$ are $0, 1, 11, 11, 1$. Compare this to the h-vector of the simple polytope Π_E.

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