Statistical Signal Processing: Homework 2

This is the second out of three homework sheets for the 2019 Statistical Signal Processing problem class. Each homework sheet includes three different examples resulting in a total number of nine homework examples over progress of the course. The presented homework examples are mostly MATLAB examples with a strong real world focus.

The grading of the problem class is based on student presentations (Power Point or Latex presentation) and exercise interviews for five out of the nine examples. For each of the three homework sheets, students have to present one example, which will be selected by the lecturers (*must show*). Students will be informed three days prior to the deadline of the problem sheet, about the number of the example, which has to be presented. The remaining two examples can be selected by the students itself (*will show*). Hereby only one of the remaining two examples per problem sheet can be selected. Note, that the grading is an individual grading.

All presentations and necessary MATLAB-files have to be provided by the deadline to ssp@emt.tugraz.at. This holds for all presentations (*must show* and selected *will show*). Please follow the guidelines for the upload, which are provided by an additional sheet on the server.

The presentation slides have to consider all necessary derivations, algorithmic considerations, diagrams, conclusions, etc.. A pure presentation of the final results will lead to reduction of your points. Also take care to the representation of necessary diagrams. Slides can be in German of English. Note, that your slides have to include all graphics. A reference to a generating MATLAB-script is not accepted.

Deadline: June 28, 2019

Example 1

ML-estimators are among the most used estimators for many technical problems. The general form of an ML estimators is given by

$$\hat{\boldsymbol{x}}_{\mathrm{ML}} = \arg\max_{\boldsymbol{x}} L(\boldsymbol{x}, \tilde{\boldsymbol{d}}) \tag{1}$$

where $L(\boldsymbol{x}, \tilde{\boldsymbol{d}})$ is the likelihood function. The likelihood is a probability measure, which answers the question: "What is the probability for the data (measurements) $\tilde{\boldsymbol{d}} \in \mathbb{R}^M$ being caused by $\boldsymbol{x} \in \mathbb{R}^N$." An often less notices aspect in estimator design is the noise model. However, the noise model is essential for the structure of the likelihood function. For an additive noise model the likelihood function is given by

$$L(\boldsymbol{x}, \tilde{\boldsymbol{d}}) = \pi(\tilde{\boldsymbol{d}}|\boldsymbol{x}) = \pi_V(F(\boldsymbol{x}) - \tilde{\boldsymbol{d}}), \tag{2}$$

where π_V is the probability density function of the measurement noise. However, for different noise models the likelihood function is of different structure.

In this example we want to perform ML-estimation for a (scalar) constant x, which is disturbed by multiplicative noise. Hence, the measurement process is given by

$$\tilde{d}[n] = a[n]x. \tag{3}$$

The gain $a[n]$ is the realization of the random variable A, which is i.i.d. with $A \propto \mathcal{N}(\mu_A, \sigma_A^2)$. The likelihood function for this example is given by

$$L(x, \tilde{\boldsymbol{d}}) = \frac{1}{x^M} \pi_A\left(\frac{\tilde{\boldsymbol{d}}}{x}\right). \tag{4}$$

Please provide answers to the following points:

1. Derive an analytic form of the ML estimator. Note that μ_A and σ_A are known parameter for this example.
2. Use the expectation operator and show, that also the modified mean estimator

$$\hat{x}_{\mathrm{mean}} = \frac{1}{\mu_A M} \sum_{i=0}^{M-1} \tilde{d}[i] \tag{5}$$

   is a meaningful estimator for this estimation problem.
3. Use Monte Carlo analysis and validate the performance of your ML estimator. Compare this result with respect to the performance of the mean estimator. For your analysis vary the following parameters:
   - Vary the data length M
   - Vary the variance of A

   Provide meaningful results for the estimator bias, the estimator variance and the MSE.

Example 2

Model based signal processing gains its power by combining measurements and model knowledge (data model, noise model, measurement model and prior models). In Data Reconciliation (DR), the task is to process raw measurements from industrial processes and provide corrected results. Figure 1 illustrates one classical application of DR. The measured mass flows Q_1 to Q_4 do not fulfill the mass balance conditions. DR is used to process the raw measurements and provide estimates, which fulfill the mass balance conditions.

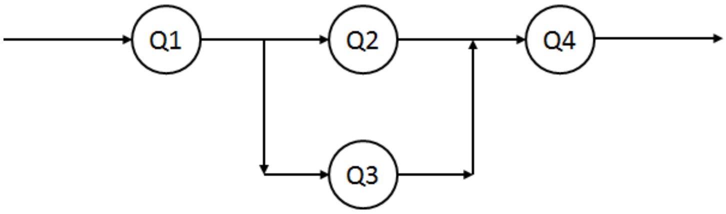

Figure 1: Example for Data Reconciliation. The measured mass flows Q_1 to Q_4 do not fulfil the mass balance condition.

Following the Bayesian philosophy, the state vector $\boldsymbol{x}$ to be estimated should be non redundant. However, for DR it is common to create the state vector $\boldsymbol{x}$ in accordance to the measurements. The mass balance equations are incorporated by means of the prior $\pi(\boldsymbol{x})$. Then the MAP estimator turns into a constrained optimization problem given by

$$\hat{\boldsymbol{x}}_{\text{MAP}} = \arg\min_{\boldsymbol{x}} (\boldsymbol{H}\boldsymbol{x} - \tilde{\boldsymbol{d}})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{H}\boldsymbol{x} - \tilde{\boldsymbol{d}}) \tag{6}$$

$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \tag{7}$$

Please provide answers to the following points:

- Provide the matrices $\boldsymbol{H}$ and $\boldsymbol{A}$ for the network depicted in figure 1 to formulate the DR problem as given the equations (6) and (7).
- Again formulate a DR approach, but in this case use a non redundant state vector $\boldsymbol{x}$. E.g. $\boldsymbol{x}^T = [\ \hat{Q}_1 \quad \hat{Q}_2\ ]$. For this you have to incorporate the balance equations into the matrix $\boldsymbol{H}$.
- Solve the DR problem for the following measurements

Nr.	Q_i	$u(Q_i)$
	kt/h	kt/h
1	100	1.5
2	30	2
3	60	2
4	110	4

Hereby $u(\cdot)$ denotes the standard uncertainty according to the GUM (Guide for uncertainty in Measurements). The standard uncertainty has the meaning of a standard deviation. To solve the problem you can use the MATLAB function `fmincon`. We also derived an analytic solution in the optimization class.

- We now assume, that the mass flow meter Q_3 is under maintenance. However, we still need information about Q_3. Figure 2 illustrates this scenario. Recast the DR problem (find the matrices $\boldsymbol{H}$ and $\boldsymbol{A}$) and estimate all flows using the measurements Q_1, Q_2 and Q_4 only.

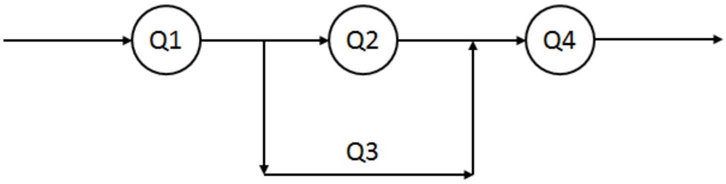

Figure 2: Example for Data Reconciliation and estimation of not measured quantities.

Example 3

This example is the same sound localization problem as used in the first homework sheet, but now a Bayesian inference approach shall be used.

The example is about the localization of a sound source in a domain Ω. To locate the position

$$\boldsymbol{x}_\mathrm{s} = \begin{bmatrix} x_\mathrm{s} & y_\mathrm{s} \end{bmatrix}^T,$$

of a sound source, where x_s and y_s are cartesian coordinates, a set of N_mic microphones are placed on encircling positions around the source.

Let $\boldsymbol{x}_{\mathrm{m},i}$ denote the cartesian position of the i^th microphone. Then the time of flight (ToF) of the sound signal to the i^th microphone is given by

$$t_i = \frac{1}{c}||\boldsymbol{x}_{\mathrm{m},i} - \boldsymbol{x}_\mathrm{s}||_2$$

where $c = 343\,\mathrm{m/s}$ is the speed of sound.

In this example you are provided by the MATLAB function

```
[dt] = func_func_soundlocalization_HW3(XYmic,ID),
```

which already provides you the (noisy) time differences between the microphone signals. Hereby the elements of the vector `dt` provide the information (MATLAB notation)

$$\begin{aligned} \mathrm{dt}(1) &= t_\mathrm{Mic1} - t_\mathrm{Mic2} \\ \mathrm{dt}(2) &= t_\mathrm{Mic1} - t_\mathrm{Mic3} \\ &\vdots \\ \mathrm{dt}(\mathrm{M}) &= t_\mathrm{MicN-1} - t_\mathrm{MicN} \end{aligned}$$

The parameter `ID` is your student ID (personalized problem). The source position is known to fulfill the condition $||\boldsymbol{x}_\mathrm{s}||_2 \leq 100\,\mathrm{m}$ (source is within a centered circle with a radius of 100 m). You can place an arbitrary number of microphones by means of the $N_\mathrm{mic} \times 2$ matrix `XYmic`.

Run the `m`-file `Demo_EX2.m` for a demo implementation of the measurement process.

Your task in this example is to apply MCMC to estimate the position $\boldsymbol{x}_\mathrm{s}$ from the time difference measurements of the N_mic microphones.

Please provide answers to the following points:

1. Provide a noise model for the measurement process.

2. Provide the posterior distribution.
3. Use the Metropolis Hastings (MH) algorithm and sample from the posterior distribution.
4. Compute the mean and the covariance of the position estimate and illustrate them in a proper way.