SKELET #17 DOES NOT HALT

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1. Description

It is known that Skelet #17 is equivalent to the following process: begin with the state $S = (0, 2, 4, 0)$. Then the transition rules are as follows:

- *Overflow:* If $S = (2a_1 + 1, 2a_2, \dots, 2a_\ell)$, then transition to $(0, 2a_1 + 2, 2a_2, \dots, 2a_\ell)$.
- *Halt:* If $S = (0, 0, 2a_1, \dots, 2a_\ell)$, then HALT.
- *Empty:* If $S = (2a_1, 2a_2, \dots, 2a_\ell)$ and $(a_1, a_2) \neq (0, 0)$, then transition to $(0, 0, 2a_1 + 1, 2a_2, \dots, 2a_\ell - 1)$.
- *Halve:* If $S = (a_1, \dots, a_\ell, -1)$, then transition to $(a_1, \dots, a_\ell)$.
- *Increment:* Otherwise, S is necessarily of the form $(a_1, \dots, a_{i-1}, 2a_i + 1, 2a_{i+1}, \dots, 2a_\ell)$. In this case, transition to $(a_1, \dots, a_{i-1} + 1, 2a_i + 1, 2a_{i+1}, \dots, 2a_\ell - 1)$.

Remark 1.1. Note that the conventions of savask's document are slightly different from ours: what corresponds to "Overflow" there is "Overflow + Empty" here, and what corresponds to "Halve" there is "Increment + Halve" here.

The goal of this article is to show that the above process does not halt. Namely, we will establish the following theorem:

Theorem 1.2. *We have* $(0, 2, 4, \dots, 2^{2k}, 0) \to (0, 2, 4, \dots, 2^{2k+2}, 0)$ *for all* $k \in \mathbb{N}$*. Moreover, during this, we pass* Overflow *precisely once.*

As a result, we are able to show:

Corollary 1.3. *Skelet #17 never halts.*

2. Basics

2.1. **State variables.** For a state $S = (a_1, \dots, a_{\ell+1})$, introduce the following state variables:

- Let $n := n(S)$ denote the number associated to the Gray code $(\bar{a_1}, \dots, \bar{a_\ell})$. Here we let $\bar{a_i} := a_i \bmod 2$.
- Let $\ell := \ell(S)$ denote the size of S, i.e. $|S| - 1$.
- Let $\sigma := \sigma(S) \in \{-1, +1\}$ denote the direction of S: namely, $\sigma = +1$ if $\sum_{i=1}^{\ell+1} a_i$ is odd, and vice versa.

Under the transition rules, the state variables change as follows:

Rule	n	ℓ	σ
Overflow	$2^\ell - 1 \to 0$	$\ell \to \ell + 1$	$+1 \to -1$
Empty	$0 \to 2^\ell - 1$	$\ell \to \ell + 2$	$-1 \to -1$
Halve	$n \to \lfloor n/2 \rfloor$	$\ell \to \ell - 1$	switch
Increment	$n \to n + \sigma$	stay	stay

2.2. **Increments.** For $r \in \mathbb{R}$, let $\langle r \rangle$ denote the nearest integer to r, where we round up if r is a half-integer. Define the function

$$d_j(a, b) := \left| \left\langle \frac{a}{2^j} \right\rangle - \left\langle \frac{b}{2^j} \right\rangle \right|.$$

The next proposition states how a state's coordinates change under a series of increments:

Proposition 2.1. *For* $S := (a_\ell, \dots, a_1, a_0)$ *and* $S' := (b_\ell, \dots, b_1, b_0)$*, suppose we have* $S \to S'$ *via a series of increments. In addition, let* $n := n(S)$ *and* $n' := n(S')$*. Then for all* $i \in [1, \ell]$ *we have* $b_i = a_i + d_i(n, n')$*.*

Proof. Indeed, $d_i(n, n')$ is precisely the number of times the i^{th} to last digit changes when incrementing the Gray code of n to the Gray code of n'. □

3. Regular empty states

3.1. **Conventions.** From this section onwards, fix $k \in \mathbb{N}$ and let $S_k := (0, 2, 2^2, \ldots, 2^{2k}, 0)$. Then it suffices to show that $S_k \to S_{k+1}$. From now on, every state S we will consider will implicitly satisfy $S_k \to S$, but not $S_{k+1} \to S$.

Call a state E *empty* if $n(E) = 0$. If E is empty, then let $N(E)$ denote the next empty state after E, and let $T(E)$ denote the transition $E \to N(E)$. If $E = (a_\ell, \ldots, a_1, a_0)$, then for $i \in [0, \ell]$, denote $E[i] := (a_\ell, \ldots, a_{i+1}, a_i + 2, a_{i-1}, \ldots, a_0)$.

3.2. **Regularity.**

Definition 3.1. Let E be an empty state.

(1) We say that E is *regular* if $T(E)$ consists of only two Halve rules such that the rest are Increment rules (aside from the Empty rule at the start). If in addition, all previous empty states between S_k and E are regular, then say E is *strongly regular*.
(2) Suppose E is regular. Define $h(E) := (h_1, h_2)$ to be the two values of n immediately after each Halve rule is applied in $T(E)$.
(3) Let E be regular. Say that E is *nice* if $N(E) = E[i]$ for some $i \in [0, \ell(E)]$.
(4) Suppose E is nice. Let $i(E) \in [0, \ell(E)]$ denote the index i for which $N(E) = E[i]$.

Note that $h(E)$ determines $T(E)$ completely. In addition, we must necessarily have $h_2 < 2^{2k+1}$, or else we would run into the Overflow rule before the second Halve rule.

Example 3.2. Let $E := (2, 2, 6, 8, 18, 0)$. Then E is regular with $h(E) = (15, 17)$. In particular, n will start at 31 after the Empty rule, immediately Halve to $n = 15$, where after a series of Increments are applied to make $n = 34$. Then the Halve rule is applied to yield $n = 17$, and after a series of Increments we have $n = 0$, which corresponds to the state $E' = (2, 2, 6, 8, 20, 0)$. In particular we also see that E is nice with $i(E) = 1$ - as we shall soon see, this is not a coincidence.

Lemma 3.3. *Let $E = (a_\ell, \ldots, a_1, a_0)$ be a regular empty state with $h(E) = (h_1, h_2)$, and suppose we have $i \in [0, \ell]$ such that $E[i]$ is also regular. Then:*

$$h(E[i]) = \begin{cases} (h_1 - 1, h_2) & i = 0 \\ (h_1, h_2 + 1) & i = 1 \\ (h_1, h_2) & i \geq 2. \end{cases}$$

Proof. In all cases, we have $n(E) = n(E[i])$. In $T(E)$, if n is fixed, then only the last two digits affect the outcome of $h(E)$. Thus $h(E[i]) = h(E)$ if $i \geq 2$. If $i = 1$, then h_1 stays the same, but we must Increment two extra times before the second Halve rule; this increases h_2 by 1. Similarly, if $i = 0$, then we must Increment two extra times before the first Halve rule, thereby decreasing h_1 by 1. □

Proposition 3.4. *Suppose E is strongly regular. Then E is nice.*

Proof. Induct on E. For the base case, note that $N(S_k) = (2, 2, 4, \ldots, 2^{2k}, 0) = S_k[2k + 1]$. For the inductive step, assume that the hypothesis holds for E, and that $N(E) = E[i]$ is regular. By Lemma 3.3, it follows that $T(E^i)$ has at least as many Increments as $T(E)$ does, so in particular all indices of $N(N(E)) = N(E[i])$ are at least as large as those in $N(E) = E[i]$. But $T(E[i])$ increases the sum of the indices by precisely 2 (note that this comes from the two Halve rules), so $T(E[i])$ must increase exactly one of the indices by 2. Thus $N(E[i]) = E[i][j]$ for some $0 \leq j \leq \ell(E)$, completing the inductive step. □

Proposition 3.5. *Suppose both E and $N(E)$ are nice, and let $h(N(E)) = (h_1, h_2)$. Then:*

$$i(N(E)) = \begin{cases} \nu_2(h_1 + 1) & i(E) = 0 \\ \nu_2(h_2) + 1 & i(E) = 1 \\ i(E) - 2 & i(E) \geq 2. \end{cases}$$

Proof. Let $i := i(E)$. If $i \geq 2$, then by Lemma 3.3, $h(N(E)) = h(E)$. But $N(E)$ is just E with the i^{th} index raised by 2, and the i^{th} index will become the $(i-2)^{th}$ index upon another application of N. Thus $i(N(E)) = i - 2$.

If $i = 1$, then Lemma 3.3 says $h(E) = (h_1, h_2 - 1)$. Since $T(N(E))$ increases exactly one index by 2, by Proposition 2.1, it follows that such an index is the unique $j \in [0, \ell(E)]$ such that $d_j(h_2, h_2 - 1) > 0$. This occurs precisely when $h_2/2^j$ is a half-integer, i.e. $\nu_2(h_2) = j - 1$.

The case $i = 0$ is similar: by Lemma 3.3, we have $h(E) = (h_1 + 1, h_2)$, and then the unique j for which $T(N(E))$ increases index j by 2, is precisely the $j \in [0, \ell(E)]$ satisfying $d_{j+1}(h_1 + 1, h_1) > 0$. From this we immediately obtain $\nu_2(h_1 + 1) = (j + 1) - 1 = j$. ☐

In light of Proposition 3.5, define $N'(E) := N^{\lceil (i(E)+1)/2 \rceil}(E)$; this function applies N repeatedly until $i(E) \in \{0, 1\}$, so in particular, if $i(E) = i$, then we will have

$$N'(E) = \begin{cases} E[i][i-2]\cdots[3][1] & i \text{ odd} \\ E[i][i-2]\cdots[2][0] & i \text{ even.} \end{cases}$$

By Proposition 3.5, we immediately obtain:

Corollary 3.6. *Suppose both E and $N'(E)$ are nice, and let $h(N'(E)) = (h_1, h_2)$. Then:*

$$i(N'(E)) = \begin{cases} \nu_2(h_1 + 1) & i(E) \text{ even} \\ \nu_2(h_2) + 1 & i(E) \text{ odd.} \end{cases}$$

4. Computing states after S_k

4.1. Counting the occurrences of $i(E)$.

We aim to prove the following result:

Theorem 4.1. *We have*

$$\kappa(i) := \#\{E \text{ strongly regular} : i(E) = i\} = \begin{cases} 2^{2k} - 2 & i = 0 \\ 2^{2k+1-i} & i \in [1, 2k+1] \text{ odd} \\ 2^{2k+1-i} - 1 & i \in [1, 2k+1] \text{ even.} \end{cases}$$

In order to prove Theorem 4.1, we must state a few things.

Definition 4.2. A *run* is a sequence $h(E), h(N'(E)), \ldots, h((N')^d(E))$ such that it equals $(h_1, h_2), (h_1, h_2+1), \ldots, (h_1, h_2+d)$ or $(h_1, h_2), (h_1 - 1, h_2), \ldots, (h_1 - d, h_2)$. Call the former sequence a *run on the second index* and the latter sequence a *run on the first index*. The number d is called the *length* of the run. A run is *maximal* if it cannot be extended on either side.

Lemma 4.3. *Let $h(E), h(N'(E)), \ldots, h((N')^d(E))$ be a maximal run of length d on the second index such that all $(N')^i(E)$ are strongly regular. Then such a run must be immediately followed by a maximal run of length d on the first index, i.e. $h((N')^d(E)), h((N')^{d+1}(E)), \ldots, h((N')^{2d}(E))$ is a maximal run on the first index. In addition, if we let $h(E) = (h_1, h_2)$, then we must have $h_1 + h_2 = 2^{2k+1} - 1$.*

Proof. Induct on E. In the base case, $E = S_k$, with $h(S_k) = (2^{2k} - 1, 2^{2k})$, so $h_1 + h_2 = 2^{2k+1} - 1$. Now let E be strongly regular such that the conditions in the lemma are satisfied. Then for all $i \in [1, d]$, we must have that the quantity $1 + \nu_2(h_2)$ for $h((N')^i(E))$ agrees with the quantity $\nu_2(1 + h_1)$ for $h((N')^{i+d}(E))$; this is proven by inducting on i. In particular, we obtain that $h((N')^d(E)), h((N')^{d+1}(E)), \ldots, h((N')^{2d}(E))$ is a maximal run on the second index. ☐

Corollary 4.4. *The last strongly regular E (i.e. the E such that $N'(E)$ is no longer regular) must satisfy $h(E) = (1, 2^{2k+1} - 1)$.*

Proof. Start with $E = S_k$, and repeatedly apply $N'(\cdot)$, noting Lemma 4.3. Eventually, the second index of $h(E)$ will increase to $2^{2k+1} - 2$, at which point Corollary 3.6 says that $i(E) = \nu_2(2^{2k+1} - 2) + 1 = 2$. By Lemma 3.3, this means that the first index of $h(E)$ will start to decrease when we next apply $N'(\cdot)$. By Lemma 4.3, it will decrease to $2^{2k+1} - 1 - (2^{2k+1} - 2) = 1$ before the second index of $h(E)$ starts to increase again. But it will only increase to $2^{2k+1} - 1$, since afterwards, E will not be regular anymore. This completes the proof. ☐

Proof of Theorem 4.1. We start with S_k: here, $h(S_k) = (2^{2k} - 1, 2^{2k})$. For each $d \in [1, 2^{2k} - 2]$, consider when h_2 increases from $2^{2k} + d - 1$ to $2^{2k} + d$ after one application of N'. If we let $v := \nu_2(2^{2k} + d) + 1$, then this increase will yield exactly one occurrence of $i(E) = i$ for each element i of $\{v, v - 2, \ldots, v\%2 + 2, v\%2\}$. Similarly, when h_1 decreases from $2^{2k} - d$ to $2^{2k} - d - 1$, then when we let $v' := \nu_2(2^{2k} - d)$, this decrease will yield exactly one occurrence of $i(E) = i$ for each element i of $\{v', v' - 2, \ldots, v'\%2 + 2, v'\%2\}$. But $\nu_2(2^{2k} - d) = \nu_2(2^{2k} + d) = \nu_2(d)$, so the union of the two sets mentioned must be exactly $\{0, 1, \ldots, \nu_2(d) + 1\}$.

Counting $\{0, 1, \ldots, \nu_2(d) + 1\}$ as d ranges in the interval $[1, 2^{2k} - 2]$ yields:

$i(E)$	0	1	2	3	$\cdots$	$2k$
# occurrences	$2^{2k}-2$	$2^{2k}-2$	$2^{2k-1}-1$	$2^{2k-2}-1$	$\cdots$	2^1-1

This counts everything except for the $i(E)$ arising from the two instances $E = S_k$ and $h(E) = (1, 2^{2k+1}-1)$; these are precisely the sets $\{2k+1, 2k-1, \ldots, 3, 1\}$ and $\{1\}$, respectively. Incorporating these into the above table yields the desired result:

$i(E)$	0	1	2	3	$\cdots$	$2k-1$	$2k$
# occurrences	$2^{2k}-2$	2^{2k}	$2^{2k-1}-1$	2^{2k-2}	$\cdots$	2^2	2^1-1

We are done. □

4.2. **End of proof.** At last, we are ready to prove Theorem 1.2. From Theorem 4.1, we get that S_k necessarily transitions to the state

$$(0+2\kappa(2k+1), 2+2\kappa(2k), 4+2\kappa(2k-1), \ldots, 2^{2k}+2\kappa(1), 0+2\kappa(0))$$

which evaluates to

$$(2, 3\cdot 2^1-2, 3\cdot 2^2, 3\cdot 2^3-2, \ldots, 3\cdot 2^{2k-1}-2, 3\cdot 2^{2k}, 2^{2k+1}-4).$$

Applying the Empty rule yields

$$(0,0,3,3\cdot 2^1-2, 3\cdot 2^2, 3\cdot 2^3-2, \ldots, 3\cdot 2^{2k-1}-2, 3\cdot 2^{2k}, 2^{2k+1}-5) \qquad (n,\sigma) = (2^{2k+1}-1, -1)$$

after which applying $2^{2k+1}-4$ Increment rules yields

$$(0,0,2^2,2^3-2,2^4,2^5-2,\ldots,2^{2k-1}-2,2^{2k},2^{2k+1}-3,2^{2k+2}-2,-1) \qquad (n,\sigma)=(3,-1)$$

and one Halve rule later we get

$$(0,0,2^2,2^3-2,2^4,2^5-2,\ldots,2^{2k-1}-2,2^{2k},2^{2k+1}-3,2^{2k+2}-2) \qquad (n,\sigma)=(1,+1).$$

Applying $2^{2k+2}-2$ Increment rules yields

$$(1,2,2^3,2^4-2,2^5,2^6-2,\ldots,2^{2k}-2,2^{2k+1},2^{2k+2}-4,0) \qquad (n,\sigma)=(2^{2k+2}-1,+1)$$

after which applying an Overflow rule yields

$$E_k := (0,2,2,2^3,2^4-2,\ldots,2^{2k}-2,2^{2k+1},2^{2k+2}-4,0) \qquad (n,\sigma)=(0,-1).$$

Note that E_k is regular with $h(E_k) = (2^{2k+2}-1, 2^{2k+2}-2)$ and $i(E_k) = 2k+1$, so by Proposition 3.5, applying $N'(\cdot)$ to E_k yields

$$E'_k := (0,2,2^2,2^3,2^4,\ldots,2^{2k},2^{2k+1},2^{2k+2}-2,0).$$

By Lemma 3.3 applied to $h(E_k)$, we have $h(E'_k) = (2^{2k+2}-1, 2^{2k+2}-1)$; therefore, Proposition 3.5 tells us that $i(E'_k) = \nu_2(2^{2k+2}-1)+1 = 1$. Hence, we obtain

$$N(E'_k) = (0,2,2^2,2^3,2^4,\ldots,2^{2k},2^{2k+1},2^{2k+2},0),$$

which is just S_{k+1}.