MATB24 Final Notes

▼ Extra problem for practice with no solution attached:

FIND A LEAST-SQUARES SOLUTION OF $Ax=b$ FOR

$$A=\begin{bmatrix}1&1&0&0\\1&1&0&0\\1&0&1&0\\1&0&1&0\\1&0&0&1\\1&0&0&1\end{bmatrix} \quad b=\begin{bmatrix}-3\\-1\\0\\2\\5\\1\end{bmatrix}$$

Least Squares Problems

FIND THE LEAST SQUARES SOLUTION TO THE INCONSISTENT SYSTEM $Ax=b$ FOR $A=\begin{bmatrix}1&1\\0&1\\1&2\end{bmatrix}$, $b=\begin{bmatrix}2\\1\\2\end{bmatrix}$

▼ Solve the problem above:

Step 1: Ensure the system is inconsistent.

Solve the system (ie. gaussian elimination) to make sure that we cannot obtain a solution.

$$\left[\begin{array}{cc|c}1&1&2\\0&1&1\\1&2&2\end{array}\right] \sim \left[\begin{array}{cc|c}1&1&2\\0&1&1\\0&-1&0\end{array}\right] \sim \left[\begin{array}{cc|c}1&1&2\\0&1&1\\0&0&1\end{array}\right]$$

$$0x_1+0x_2=1$$

Step 2: Calculate A^TA and A^Tb.

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 1+0+2 \\ 1+0+2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+0+2 \\ 2+1+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Step 3: Solve for $\bar{x}$ where .

$$\underline{\underline{A^T A}} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \qquad \underline{\underline{A^T b}} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 3 & 6 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 3 & 6 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 0 & 3/2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 3/2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2/3 \end{array}\right] \qquad \hat{x} = \begin{matrix} 1 \\ 2/3 \end{matrix}$$

Step 4: Solve $A\bar{x} = \bar{b}$ where $\bar{x} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 2/3 \\ 7/3 \end{bmatrix}$$

Step 5: Calculate error $||b - \bar{b}||$.

$$\|b-\hat{b}\| = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 2/3 \\ 7/3 \end{bmatrix} = \left\| \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \end{bmatrix} \right\|$$

$$= \sqrt{1/9 + 1/9 + 1/9} = \sqrt{1/3} = \boxed{\frac{1}{3}\sqrt{3}}$$

TODO Left-over exercise questions from lectures:

- Prove that the set M generated by $d(x) \in F[x]$ is in fact an ideal.

To prove ideal, need to satisfy non-zero vector in M, closure under $(+;\cdot)$ & closure under mult. by elements of the ring (outside elements)

i) $0 \in \mathbb{F}[x] \Rightarrow 0 \in M$ since $d(x) \in \mathbb{F}[x]$ generates M.

ii) sps. $f(x), g(x) \in M$, some scalar c;

$$f(x) = a(x)d(x) \qquad g(x) = b(x)d(x) \Rightarrow f(x) + cg(x) = a(x)d(x) + cb(x)d(x) = [a(x) + cb(x)]d(x) \in M.$$

iii) sps. $f(x) \in M$, $g(x) \in \mathbb{F}[x]$

$\Rightarrow f(x) = a(x)d(x)$

$\Rightarrow g(x)f(x) = g(x)a(x)d(x) = [g(x)a(x)]d(x) \in M$ $\therefore$ M is an ideal.

- Show that $F[x]$ is a principal ideal. What is the generator?
- Let $f(x) = x^3 - 1$. Find the prime factors of $f(x)$ over R and C.

prime factors of $f(x)$ over $\mathbb{R}$ and $\mathbb{C}$:

$$f(x) = x^3 - 1 = (x-1)(x^2+x+1) = \begin{cases} (x-1)(x^2+x+1), & f(x) \in \mathbb{R} \\ (x-1)\left(x - \frac{i\sqrt{3}-1}{2}\right)\left(x + \frac{i\sqrt{3}-1}{2}\right), & f(x) \in \mathbb{C} \end{cases}$$

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x-1 & x^3 + 0x^2 + 0x - 1 \\ & x^3 - x^2 \\ \hline & \quad x^2 + 0x \\ & \quad x^2 + x \\ \hline & \qquad x - 1 \\ & \qquad x - 1 \\ \hline & \qquad\quad 0 \end{array}$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \frac{\pm i\sqrt{3} - 1}{2}$$

- Let $D : F[x] \to F[x]$ be the differentiation operator. Let W be the subspace of polynomials of degree less than or equal to n. $W = \{f(x) = c_o + c_1 x + ... + c_k x^k | c_j \in F, k \leq n\}$. Is W invariant under D.

  Let $f \in W \Rightarrow (Df)(x) = c_1 + 2c_2 x + ... + kc_k x^{k-1}$
  $\Rightarrow Df \in W$

  Therefore showing that W is invariant under D.

- Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$. Is A similar to a triangular matrix over R.

  $det(xI - A) = x^3$

  Therefore the minimal polynomial is either x, x^2 or x^3. But $A \neq 0$ and $A^2 \neq 0$ so the minimal polynomial must be x^3.

  Note, since the minimal polynomial x^3 does not have distinct linear factors ($deg(x) = 3$), then A is triangularizable, but not diagonalizable. Thus A is similar to a triangular matrix over R.

- Theorem 6.8.2 proof. Complete the proof. Show that when $r \geq max(r_j)$, then $N^r = 0$. Additionally, show that if a matrix N is strictly upper or lower triangular, then N is nilpotent.

- If A is the companion matrix of a minimal polynomial p_α, then p_α is both the minimal and the characteristic polynomial for A. Prove this (there are two ways this can be proven)

$$A = \begin{bmatrix} 0 & 0 & 0 & & 0 & -c_0 \\ 1 & 0 & 0 & & 0 & -c_1 \\ \vdots & 1 & 0 & \vdots & 0 & \vdots \\ \vdots & \vdots & 1 & \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots & & 0 & -c_{k-2} \\ 0 & 0 & \vdots & & 1 & -c_{k-1} \end{bmatrix} \qquad p(A) \text{ is obv. minimal.}$$

Let's see if $p(A) = f(A)$.

$$\begin{aligned} f(A) &= \det(xI - A) \\ &= \det\begin{pmatrix} x & & & c_0 \\ -1 & x & & c_1 \\ & -1 & \ddots & \vdots \\ & & -1 & x + c_{k-1} \end{pmatrix} \\ &= (x - c_0)(x - c_1)(x - c_2)\ldots \\ &= p(A) \end{aligned}$$

▼ Prove *Theorem 7.2.1. (refer to important theorems and definitions)*

Proof. Consider when $p = p_i^k$ when p_i is a prime factor. As $p_i(T)^n = 0$ and $p_i(T)^{k-1} \neq 0$, a vector $\alpha \in V$ exists such that $p_i(T)^{n-1}\alpha \neq 0$.

Now consider any minimal polynomial $p = p_1^{r_1} \cdots p_k^{r_1}$, applying the primary decomposition theorem, we can obtain $V = W_1 \oplus \cdots \oplus W_k$, where each W_i is the restriction of T to W_i, where the minimal polynomial of W_i equals $p_i^{r_i}$. For each W_i, a vector α_i exists such that $p_i^{r_i}$ is the T-annihilator of α_i. Let $\alpha = \sum_{i=1}^{k} \alpha_k$.

Let g be the T-annihilator of α, by definition g divides p. Let $f(T)$ be any non-zero multiple of g, then $\sum \alpha_{i=1}^{k} f(T)\alpha = 0 \implies F(T)\alpha_i = 0$. Therefore all $p_i^{r_i}$ divide f. As f is any non-zero multiple of g, p divided g.

Now we will prove that $Z(\alpha; T)$ is T-admissible.

The minimal polynomial $p = x^k f$, where $0 \leqslant k \leqslant n$ and f does not have a factor $(x - 0)$. By the primary decomposition theorem we can divide by V into two T-invariant spaces, W_1 and W_2, were the minimal polynomial of W_1 is x^k and the minimal polynomial of W_2 is f.

Assume $\beta \notin Z(\alpha; T)$, $\beta = \gamma + \gamma'$, where $\gamma \in W_1$ and $\gamma' \in W_2$. Since $Z(\alpha; T)$ is T-invariant it can also be split into two T-invariant subspaces W_1' and W_2'. With the minimal polynomial x^k and f respectively. If $T\beta \in Z(\alpha; T)$, then $T\gamma \in W_1'$ and $T\gamma' \in W_2'$.

We know that there exists a vector $\alpha_1 \in W_1'$ such that $\alpha_1, T\alpha_1, \cdots, T^{k-1}\alpha_1$ forms a basis for W_1'. Since the minimal polynomial for $W_1 = x^k$, $T^{k-1}(T\gamma_1) = 0$. Therefore $T\gamma = \sum\limits_{i=1}^{n} a_i T^i \alpha = T \sum\limits_{i=1}^{n} a_i T^{i-1} \alpha = T\hat{\gamma}$, where $\hat{\gamma} \in W_1'$.

We also know that there exists a vector $\alpha_2 \in W_2'$ such that $\alpha_2, T\alpha_2, \cdots, T^{k-1}\alpha_2$ forms a basis for W_2'. Since $T\beta \in Z(\alpha; T)$, $T\gamma'$ is in W_2'. Since x is not a factor of f, $dim(TW_2') = dim(W_2') = k_2$. This means that $T\alpha_2, \cdots, T^k\alpha_2$ also forms a basis for W_2'. Therefore $T\gamma' = \sum\limits_{i=1}^{k_2} a_i T^i \alpha_2 = T \sum\limits_{i=1}^{k_2} a_i T^{i-1} \alpha_2 = T\hat{\gamma'}$, where $\hat{\gamma'} \in W_2'$.

Since $\hat{\gamma} + gam\hat{m}a' = v \in W_1' + W_2' = Z(\alpha; T)$ and $T\beta = Tv$, we can concluded that $Z(\alpha; T)$ is T-invariant. $\square$

▼ Using $x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$, show that $< \alpha|\beta >= y^* G x$.

recall; $\langle \alpha | \beta \rangle = \langle \alpha | \cdot | \beta \rangle = \overline{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}^T \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1^* & a_2^* & \cdots & a_n^* \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} =$

$= a_1^* b_1 + a_2^* b_2 + a_3^* b_3 + \dots + a_n^* b_n$

Consider $Gx = g_1 x_1 + g_2 x_2 + \dots + g_n x_n$.

$\Rightarrow y^* G x = y_1^*(g_1 x_1) + y_2^*(g_2 x_2) + \dots + y_n^*(g_n x_n)$

Notice the similarities b/w $\langle \alpha | \beta \rangle$ and $y^* G x$, as it shows that $y^* G x$ is of form $\langle \alpha | \beta \rangle$, hence $\langle \alpha | \beta \rangle = y^* G x$. $\square$

▼ Prove the 4 identities of an inner product space.

1. $\|c\alpha\| = c\|\alpha\|$

2. $\|\alpha\| > 0, \ \forall \alpha \neq 0$

3. $|\langle\alpha|\beta\rangle| \leqslant \|\alpha\| * \|\beta\|$ ***(Cauchy-Schwartz inequality)***

4. $\|\alpha + \beta\| \leqslant \|\alpha\| + \|\beta\|$ ***(triangle inequality)***

▼ Answers to Proofs:

Let V be an inner product space. The $\forall \alpha, \beta \in V$ and $\forall c \in \mathbb{F}$,

Prove: $\|c\alpha\| = c\|\alpha\|$

$$\begin{aligned} \|c\alpha\| &= \sqrt{\langle c\alpha | c\alpha\rangle} \\ &= \sqrt{c\langle\alpha | c\alpha\rangle} \\ &= \sqrt{c\bar{c}\langle\alpha|\alpha\rangle} \\ &= \sqrt{c^2\langle\alpha|\alpha\rangle} \\ &= c\sqrt{\langle\alpha|\alpha\rangle} \\ &= c\|\alpha\| \ \blacksquare \end{aligned}$$

Prove: $\|\alpha\| > 0, \forall\alpha \neq 0$

Note: $\langle\alpha|\alpha\rangle = 0$ if $\alpha = 0$

Since, $\alpha \neq 0 \Rightarrow \langle\alpha|\alpha\rangle \neq 0$

Note that a property of inner products is that $\langle\alpha|\alpha\rangle > 0$, $\alpha \neq 0$.

So $\langle\alpha|\alpha\rangle > 0$

$\Rightarrow \|\alpha\|^2 > 0$

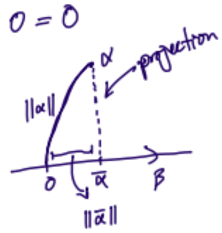
$\Rightarrow \|\alpha\| > 0 \ \blacksquare$

Prove: $|\langle\alpha|\beta\rangle| \leq \|\alpha\|\cdot\|\beta\|$

$\Rightarrow |\alpha\cdot\beta| \leq \|\alpha\|\cdot\|\beta\|$

Consider if $\beta = 0$, $0 \leq \|\alpha\|\cdot 0 = 0$

Suppose $\beta \neq 0$,



$\therefore \ \bar{\alpha} = \frac{\langle\alpha|\beta\rangle}{\|\beta\|^2}\beta$

Note $\|\bar{\alpha}\| \leq \|\alpha\|$

$\left\|\frac{\langle\alpha|\beta\rangle}{\|\alpha\|^2}\alpha\right\| \leq \|\alpha\|$

$\left|\frac{\langle\alpha|\beta\rangle}{\|\beta\|^2}\right|\cdot\|\beta\| \leq \|\alpha\|$

$\frac{|\langle\alpha|\beta\rangle|}{\|\beta\|^{\cancel{2}}}\cancel{\|\beta\|} \leq \|\alpha\|$

$|\langle\alpha|\beta\rangle| \leq \|\alpha\|\cdot\|\beta\| \ \blacksquare$

▼ Proofs from Examples 8.6/8.7.

Example 8.6.

The standard basis $e_1, \cdots, e_n$ are orthonormal in both $\mathbb{R}^2$ and $\mathbb{C}^2$, with respect the dot product, but not with to respect to $\langle\alpha|\beta\rangle = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2$

prove it to yourself.

Example 8.7.

Let $V = \mathbb{C}^{n\times n}$ with the trace inner product $(\langle A|B\rangle = tr(B * A)$. The set $\{E^{p,q} : 1 \leqslant p, q \leqslant n\}$, where $E^{p,q}$ has zero entries everywhere except the p, qth entry, which is equal to 1.

Prove it to yourself.

$$\text{Let } \alpha_1 = \tfrac{1}{\sqrt{2}}(1,1), \alpha_1 = \tfrac{1}{\sqrt{2}}(1,-1) \in \mathbb{R}^2.$$

▼ Given the above information. Prove Normal.

$\alpha_1 = \frac{1}{\sqrt{2}}(1,1)$, $\alpha_2 = \frac{1}{\sqrt{2}}(1,-1)$

Prove Normal $\Rightarrow$ Prove $\|\alpha_1\| = 1$ and $\|\alpha_2\| = 1$

$\|\alpha_1\| = \sqrt{\langle\alpha_1|\alpha_1\rangle} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1$

$\|\alpha_2\| = \sqrt{\langle\alpha_2|\alpha_2\rangle} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1$ $\square$

▼ Continuing from last question. Prove orthogonal.

Is it orthogonal?

Check: $\langle \alpha_1 | \alpha_2 \rangle = 0$

$\langle \alpha_1 | \alpha_2 \rangle = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{2} = 0$

$\therefore$ orthogonal as well $\square$

Note since α_1, α_2 normal & orthogonal $\Rightarrow$ orthonormal.

▼ Continuing from last question. Verify the corollary for this example.

Verify corollary

$$\begin{aligned}
\langle \beta | \alpha_1 \rangle \alpha_1 + \langle \beta | \alpha_2 \rangle \alpha_2 &= \frac{a+b}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + \frac{a-b}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \\
&= \frac{1}{2}\left[(a+b, a+b) + (a-b, b-a)\right] \\
&= \frac{1}{2}(a+b+a-b, a+b+b-a) \\
&= \frac{1}{2}(2a, 2b) \\
&= (a, b) = \beta \, \square
\end{aligned}$$

▼ Prove that a linear operator $E(\beta)$ is linear.

Proof:

$E(\beta)(cx+y) = E(\beta)(cx) + E(\beta)(y) = cE(\beta)(x) + E(\beta)(y) \, \square$

▼ Prove that a linear operator $E(\beta)$ is a projection.

Proof: WTS: $E^2(\beta) = E(\beta)$

$E^2(\beta)x = E(\beta)\big(E(\beta)(x)\big) = E(\beta^2)(x)$ b/c $E(\beta)$ is lin. operator.

Now show $E(\beta^2) = E(\beta)$.

Note that projecting twice = projecting once. $\therefore E(\beta^2) = E(\beta) \Rightarrow$
$E^2(\beta) = E(\beta)$ $\square$

▼ Prove that $range(E) = W$.

▼ Let W be a finite-dimensional subspace of an inner product space V and let E be the orthogonal projections of V on W. Then E is a projection of V onto W, $W^\perp$ is the nullspace of E, and $V = W \bigoplus W^\perp$.

Step 1: Show E is a projection

Image $\in W$
orth. proj. $E(v)$ of W is by def'n closest point to $v \in W \Rightarrow E(v) \in W$

Projection
$E(v) \in W \Rightarrow E(E(v))$ projects to itself $\Rightarrow E(E(v)) = E(v)$

Step 2: Show E is lin. operator

$x, y \in V$, c is some arbitrary scalar

$E(cx+y) = E(cx) + E(y) = cE(x) + E(y)$

Step 3: Show $W^\perp$ is nullspace of E.

If $E(v) = 0 \Rightarrow$ orthogonal proj. of v onto W is 0.

$\therefore v$ is orthogonal to every vector in $W \Rightarrow v \in W^\perp \Rightarrow W^\perp =$ nullspace(E)

Step 4: Show it is a direct sum

orthogonal proj. $E(v)$ is sum of $x \in W$ and $y \in W^{\perp}$

$\therefore v = x + y$, decomposition unique $\therefore V = W \oplus W^{\perp}$ $\square$

▼ Prove Theorem 8.3.3.

Let $T \in \mathcal{L}V$ *and suppose* $\beta = \{\alpha_1, \cdots, \alpha_n\}$ *is an* ***ordered orthonormal basis*** *for* V. *Let* $A = [T]_\beta = (A_{kj})$, *where* $1 \leqslant k, j \leqslant n$.

$$A_{kj} = \langle T\alpha_j | \alpha_k \rangle, \quad \forall 1 \leqslant j, k \leqslant n$$

▼ Determine if $A = \begin{bmatrix} i/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$ are unitary operators using the standard inner product.

Determine if the following matrices are unitary operators.

1. $A = \begin{bmatrix} i/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$

$\langle (i/\sqrt{2}, -i/\sqrt{2}) | (i/\sqrt{2}, i/\sqrt{2}) \rangle = \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) + \left(\frac{i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{2} = 0$

$\langle (i/\sqrt{2}, -i/\sqrt{2}) | (i/\sqrt{2}, -i/\sqrt{2}) \rangle = \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) + \left(\frac{i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$

$\langle (i/\sqrt{2}, i/\sqrt{2}) | (i/\sqrt{2}, i/\sqrt{2}) \rangle = \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) + \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$

$\therefore$ unitary operator

Solution could be wrong, plz check.

▼ Determine if $A = \begin{bmatrix} i/\sqrt{2} & 1 \\ i/\sqrt{2} & -1 \end{bmatrix}$ are unitary operators using the standard inner product.

2. $A = \begin{bmatrix} \boxed{\begin{matrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{matrix}} & \boxed{\begin{matrix} 1 \\ -1 \end{matrix}} \end{bmatrix}$ $\langle (1/\sqrt{2}, 1/\sqrt{2}) \mid (1,-1) \rangle = \left(\frac{1}{\sqrt{2}}\right)(1) + \left(\frac{1}{\sqrt{2}}\right)(-1) = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$

$\langle (1/\sqrt{2}, 1/\sqrt{2}) \mid (1/\sqrt{2}, 1/\sqrt{2}) \rangle = \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} - \frac{1}{2} = -1$

$\therefore$ not unitary operator

Solution could be wrong, plz check.

▼ What is a real $n \times n$ matrix such that $\sigma(A)$ has no entries, but $\sigma(A^2) = \{-1\}$?

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

▼ Given $2x_1y_1 + 3x_1y_1 - 5x_2y_1 + x_2y_2$. Is this an inner product?

$2x_1y_1 + 3x_1y_2 - 5x_2y_1 + x_2y_2$. Is this an inner product?

$x A y^T = (x_1, x_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= (x_1, x_2) \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix}$

$= ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2$

$\Rightarrow a = 2,\ b = 3,\ c = -5,\ d = 1$

$\Rightarrow A = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} \Rightarrow A \neq A^T$ $\therefore$ not inner product. $\square$

▼ Let W be the subspace of R^2 spanned by the vector (3,4). Using the standard inner product, let E be the orthogonal projection of R^2 onto W. Find:
a) a formula for $E(x_1, x_2)$;
b) the matrix of E in the standard ordered basis;
c) $W^{\perp}$
d) an orthonormal basis in which E is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

a) projection formula: projection of α onto β where $\alpha = (3,4)$ projecting onto $(x_1, x_2) \in W$

$$E(x_1, x_2) = \frac{\langle (x_1,x_2) | (3,4) \rangle}{\|(3,4)\|}(3,4) = \frac{3x_1 + 4x_2}{\sqrt{9+16}}(3,4) = \left(\frac{9x_1+12x_2}{5}, \frac{12x_1+16x_2}{5}\right)$$

b) $[E]_{\{e_1,e_2\}} = \begin{bmatrix} 9/5 & 12/5 \\ 12/5 & 16/5 \end{bmatrix} \Leftarrow \frac{9}{5}x_1y_1 + \frac{12}{5}x_2y_1 + \frac{12}{5}x_1y_2 + \frac{16}{5}x_2y_2$

c) $W^\perp$:

$\langle \alpha | (3,4) \rangle = 0$

$\Rightarrow \langle (x_1, x_2) | (3,4) \rangle = 0$

$\Rightarrow 3x_1 + 4x_2 = 0$

$\Rightarrow 3x_1 = -4x_2$

$\Rightarrow W^\perp = \text{span}\{(-4, 3)\}$

d) orthogonal basis: $\{W, W^\perp\} = \{(3,4), (-4,3)\}$

then, normalize each vector: $\|(3,4)\| = \sqrt{9+16} = \sqrt{25} = 5$

$\|(4,-3)\| = \sqrt{16+9} = \sqrt{25} = 5$

$\Rightarrow$ orthonormal basis: $\left\{\frac{(3,4)}{5}, \frac{(-4,3)}{5}\right\}$.

▼ Let V be the subspace of $R[x]$ of polynomials of degree at most 3. Equip V with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$.

a) Find the orthogonal complement of the subspace of scalar polynomials.

b) Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$.

a) $\langle 1 | (a+bx+cx^2+dx^3) \rangle = 0 \Rightarrow \int_0^1 (a+bx+cx^2+dx^3)\,dx = 0$

$\Rightarrow a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} = 0$

$\Rightarrow a = \frac{-b}{2} - \frac{c}{3} - \frac{d}{4}$

letting $b, c, d \in \mathbb{R}$ gives basis, $\{2-x, 3-x^2, 4-x^3\}$

b) $v_1 = 1$

$\beta_1 = \frac{1}{\|1\|} = \frac{1}{1} = 1$

$v_2 = x - \langle x | 1 \rangle 1 = x - \int_0^1 x\,dx = x - \frac{1}{2}$

$\beta_2 = \frac{x - 1/2}{\|x - 1/2\|} = \frac{x - 1/2}{\sqrt{\langle x - \frac{1}{2} | x - \frac{1}{2} \rangle}} = \frac{x - 1/2}{\sqrt{\int_0^1 (x-\frac{1}{2})(x-\frac{1}{2})dx}} = \frac{x - 1/2}{\sqrt{\int_0^1 (x^2 - x + \frac{1}{4})dx}} = \sqrt{12}(x-2)$

$v_3 = \ldots$

$\beta_3 = \ldots$

$v_4 = \ldots$

$\beta_4 = \ldots$

Ends at β_4 b/c $\deg(\{1, x, x^2, x^3\}) = 4$.

▼ Not a real problem solving question, just a reminder of change of basis.

$[T]_\alpha^\beta$: write α in terms of β. So each vector of α will be written in terms of basis β, which becomes column vectors of the new matrix.

Possible Short answer questions:

- Give an example of a matrix that is not diagonalizable.
- Give matrix that is unitary but not hermitian.

▼ Give matrix that is normal but not hermitian.

$$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

- Give matrix that is normal but not self-adjoint.
- Give matrix that is normal but not unitary.
- Give matrix that is unitary and normal.
- ▼ Give matrix that is unitary but not orthogonal.

  $$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

- ▼ Give matrix that is orthogonal but not unitary.

  $$\begin{bmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{bmatrix}$$

- ▼ Is a complex symmetric matrix self-adjoint?

  No, check with $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

- ▼ Is a complex symmetric matrix normal?

  No, check with $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

- ▼ Give an example of a 2x2 matrix A such that A^2 is normal, but A is not normal.

  Use nilpotent matrix with minimal polynomial x^2 that is not normal. Example $\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$.

- ▼ Give a matrix that is orthonormal.

  1/√2 -1/√2 0
  1/√6 1/√6 -2/√6
  1/√3 1/√3 1/√3

Some important theorems and definitions:

Orthogonal Projection

$I - E$ is the orthogonal projection of V onto $W^\perp$, and $range(I - E) = W^\perp$.

Cayley-Hamilton Theorem

Let $T \in L(V), dim(V) < \infty$. Let $f(x) = det(xI - T)$ be the characteristic polynomial for T. Then $f(T) = 0$. Thus the minimal polynomial p for T divides f. In fact it has the same roots as f (LCM). Similarly, if A is an $n \times n$ matrix with characteristic polynomial $f(x)$, then $f(A) = 0$.

Triangularizability

$T \in L(V)$ is triangularizable if an ordered basis β exists for V such that $[T]_\beta$ is (upper) triangular.

Primary Decomposition Theorem

Let $T \in L(V), dim(V) < \infty$. Let $p(x)$ be the minimal polynomial for T, as suppose

$$p(X) = p_1(x)^{r_1} ... p_k(x)^{r_k}$$

where the p_j are distinct irreducible monic polynomials and $r_j \geq 1$. Let $W_j = nullspace(p_j(T))^{r_j}$ for $j = 1, ..., k$. Then

1. $V = W \bigoplus ... \bigoplus W_k$
2. each W_i is invariant under T
3. $T_i = (p_i(x))^{r_i}$

Theorem 6.8.2

Let $T \in L(V)$, where V is a complex vector space with $dim(V) < \infty$. Then there is a diagonalizable $D \in L(V)$ and a nilpotent $N \in L(V)$ such that

$$T = D + N$$

Proof:

$$T = TI = T(E_1 + E_2 + ... + E_k) = c_1 E_1 + ... + c_k E_k = D$$
$$\Rightarrow N = T - D = (T - c_1 I)E_1 + ... + (T - c_k I)E_k$$
$$\Rightarrow N^r = (T - c_1)^r E_1 + ... + (T - c_k I)^2 E_k$$

As an exercise, complete the proof (refer to exercise questions above).

Theorem 7.2.1

Let $T \in L(V)$ and p be the minimal polynomial. Then there exists a vector α such that the T-annihilator of α is the minimal polynomial. The invariant subspace $Z(\alpha; T)$ is T-admissible.

Hermitian Inner Product

The inner product for complex numbers.

Polarization Identity (Inner products)

Real domain:

$$\langle \alpha | \beta \rangle = \frac{1}{4}\|\alpha + \beta\|^2 - \frac{1}{4}\|\alpha - \beta\|^2 \tag{8.1}$$

Complex case domain:

$$\langle \alpha | \beta \rangle = \frac{1}{4}\|\alpha + \beta\|^2 - \frac{1}{4}\|\alpha - \beta\|^2 + i\frac{1}{4}\|\alpha + i\beta\|^2 - i\frac{1}{4}\|\alpha - i\beta\|^2 \tag{8.2}$$

Application of Cauchy-Schwartz inequality to the four inner products discussed

1. (Dot product) $|\sum\limits_{k=1}^{n} x_k\overline{y}_k| \leqslant \left(\sum\limits_{k=1}^{n} |x_n|^2\right)^{1/2} \left(\sum\limits_{k=1}^{n} |y_n|^2\right)^{1/2}$.

2. $|x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2| \leqslant |(x_1 - x_2)^{+}3x_2^2)^{1/2}((y_1 - y_2)^2 + 3y^2)^{1/2}$

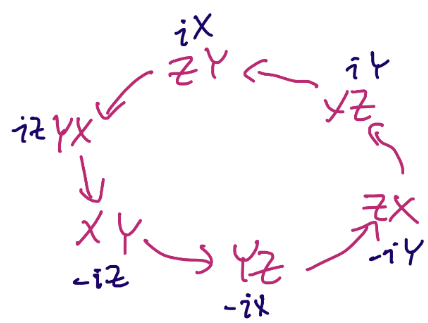
3. $|tr(AB^*)| \leqslant (tr(A^*A))^{1/2}(tr(B^*B))^{1/2}$

4. $|\int\limits_0^1 f(t)\overline{g(t)}dt| \leqslant \left(\int\limits_0^1 |f(t)|^2 dt\right)^{1/2} \left(\int\limits_0^1 |f(t)|^2 dt\right)^{1/2}$

Pauli Matrices Properties

$XY = -iZ,\ YZ = -iX,\ ZX = -iY,\ YX = iZ,\ ZY = iX,\ XZ = iY$

Also, $XX = YY = ZZ = I$



- Pauli matrices are all unitary and self-adjoint; their spectrum is $\{\pm 1\}$.

Adjoint Existence and Uniqueness (key equality for some Inner Product Proofs)

Suppose that $T \in L(V)$ for some inner product space V. There exists a *unique* linear operator $T^* \in L(V)$ such that for all $\alpha, \beta \in V$,

$$\langle T(\alpha) \mid \beta \rangle = \langle \alpha \mid T^*(\beta) \rangle$$