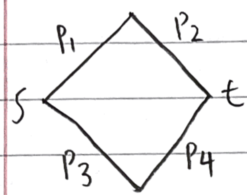
PS 1

① $P(\text{success from } s \text{ to } t) = 1 - P(\text{failure from } s \text{ to } t)$

- Divide into two partition for $P(\text{failure from } s \text{ to } t)$

ⓐ When P_5 fails, water does not flow under these circumstances:

- $P_1 \cap P_3$ fail, $P_1 \cap P_4$ fail, $P_2 \cap P_4$ fail, $P_2 \cap P_3$ fail



Since the possibility of each pipe breaking are independent, we can denote $P_1 \cap P_3$ fail as $P_1 \cdot P_3$, ..., and etc.

Let $A = P_1P_3$, $B = P_1P_4$, $C = P_2P_3$, $D = P_2P_4$.

Using Inclusion/Exclusion formula,

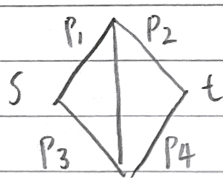
$$P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D) \quad ①$$
$$- (P(A \cap B) + P(A \cap C) + P(A \cap D) + P(B \cap C) + P(B \cap D) + P(C \cap D)) \quad ②$$
$$+ P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) \quad ③$$
$$- P(A \cap B \cap C \cap D) \quad ④$$

$\Downarrow$

$$P_5 (P_1P_3 + P_1P_4 + P_2P_3 + P_2P_4$$
$$- (P_1^2 P_3 P_4 + P_1 P_2 P_3^2 + P_1 P_2 P_4^2 + P_2^2 P_3 P_4 + 2 P_1 P_2 P_3 P_4)$$
$$+ (P_1^2 P_2 P_3^2 P_4 + P_1^2 P_2 P_3 P_4^2 + P_1 P_2^2 P_3 P_4^2 + P_1 P_2^2 P_3^2 P_4)$$
$$- (P_1^2 P_2^2 P_3^2 P_4^2)$$

ⓑ When P_5 does not fail, water does not flow under these conditions:

- $P_1 \cap P_3$ fail, $P_2 \cap P_4$ fail



Again, due to the independency, we denote

$A = P_1P_3$ and $B = P_2P_4$

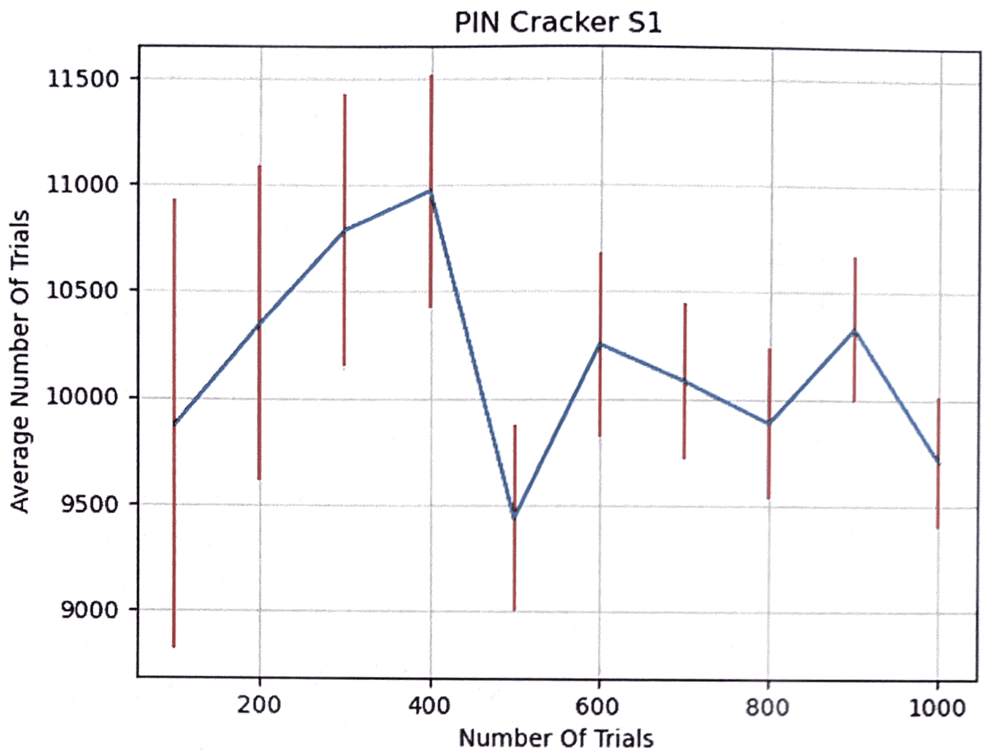
Using Inclusion/Exclusion formula,

$$(1-P_5)(P_1P_3 + P_2P_4 - P_1P_2P_3P_4)$$

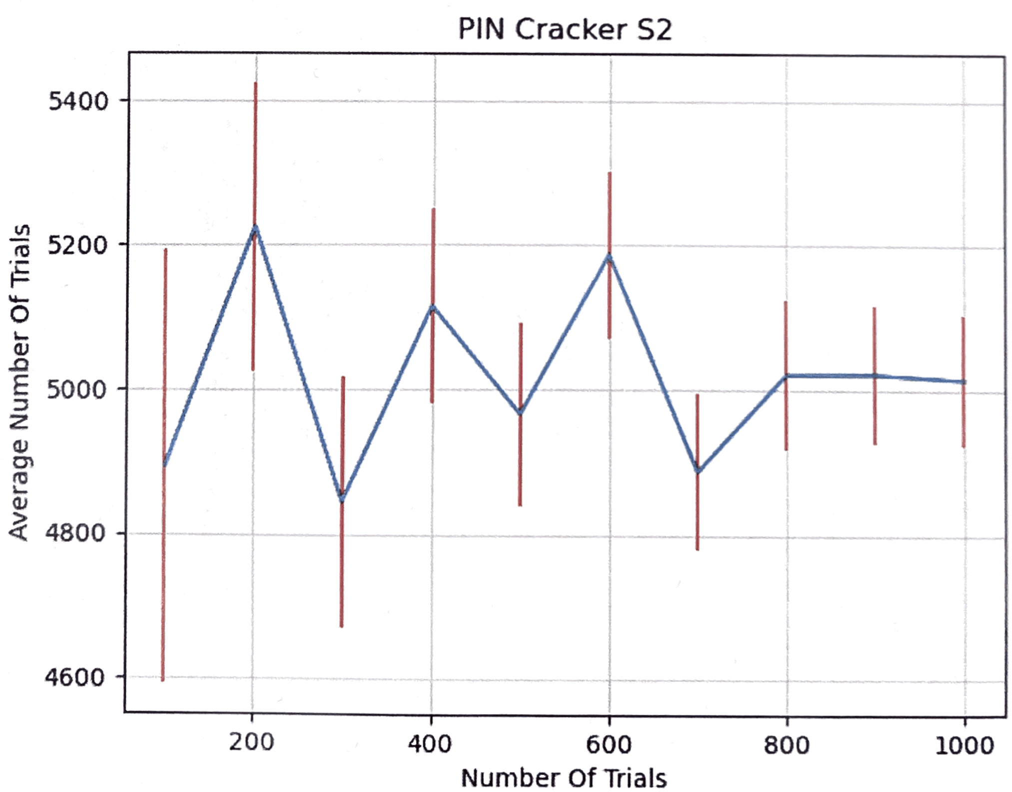
Final Answer:

$$P(\text{success}) = 1 - P(\text{failure})$$
$$= 1 - (P_5)((P_1P_3 + P_1P_4 + P_2P_3 + P_2P_4) - (P_1^2P_3P_4 + P_1P_2P_3^2 + P_1P_2P_4^2 + P_2^2P_3P_4 + 2P_1P_2P_3P_4)$$
$$+ (P_1^2P_2P_3^2P_4 + P_1^2P_2P_3P_4^2 + P_1P_2^2P_3P_4^2 + P_1P_2^2P_3^2P_4) - (P_1^2P_2^2P_3^2P_4^2))$$
$$- (1-P_5)(P_1P_3 + P_2P_4 - P_1P_2P_3P_4)$$

2.1



PIN Cracker S1
11500
11000
10500
10000
9500
9000
Average Number Of Trials
200
400
600
800
1000
Number Of Trials



PIN Cracker S2
5400
5200
5000
4800
4600
Average Number Of Trials
200
400
600
800
1000
Number Of Trials

2.2 i) John tries random PIN until he gets the correct PIN.
Each time John enters a random PIN, there are total of 10^4 combinations
$\hookrightarrow p = \frac{1}{10000}$
Since he continues to try until he gets the correct PIN, we can apply geometric distribution where $p = 0.0001$
Expected value of geometric distribution
$\hookrightarrow \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} p \Rightarrow \frac{1}{p}$

$$\boxed{E[X_1] = \frac{1}{p} = \frac{1}{0.0001} = 10000 \text{ trials}}$$

ii) This time, John does same method as S_1, but keeps track of the random PIN so that he tries a new PIN each time
Expected value
$\hookrightarrow E[X_2] = \sum_{n=1}^{10000} n \cdot P_r(X_n = PIN)$, but the value of Pr changes every trial

$\Downarrow$

Trial 1: $n=1,\ p = \frac{1}{10^4}$

Trial 2: $n=2,\ p = \left(\frac{10^4-1}{10^4}\right)\left(\frac{1}{10^4-1}\right)$

Trial 3: $n=3,\ p = \left(\frac{10^4-1}{10^4}\right)\left(\frac{10^4-2}{10^4-1}\right)\left(\frac{1}{10^4-2}\right)$

In trial 2, he guesses the PIN wrong with $p = \frac{10^4-1}{10^4}$ since there is only one correct PIN. Then, out of 10^4-1 remaining PINs, he has to guess it correctly. This pattern continues until trial 10000 in the worst case, where he is guaranteed to get the correct PIN, after trying all possible 4 digits.
If we add them all up,

$$\boxed{E[X_2] = \sum_{n=1}^{10000} n \cdot p = 1\left(\frac{1}{10^4}\right) + 2\left(\frac{\cancel{10^4-1}}{10^4}\right)\left(\frac{1}{\cancel{10^4-1}}\right) + \ldots\ 10000\left(\frac{\cancel{10^4-1}}{10^4}\right)\left(\frac{\cancel{10^4-2}}{\cancel{10^4-1}}\right)\cdots\left(\frac{1}{\cancel{2}}\right)\cdot 1 = \frac{\sum_{n=1}^{10000} n}{10^4} = \frac{50{,}005{,}000}{10^4} = 5000.5 \text{ trials}}$$

2.3 For $i=1$,

$$P(X_1 \geq 7000) \leq \frac{E[X_1]}{7000} \leq \frac{10000}{7000}$$

For $i=2$,

$$P(X_2 \geq 7000) \leq \frac{E[X_2]}{7000} \leq \frac{5000.5}{7000}$$

Markov's inequality does not always provide meaningful bounds. The equation of Markov's inequality is

$$P(X \geq t) \leq \frac{E[X]}{t}.$$

According to the law of probability, probability of an event is always $0 \leq Pr(E) \leq 1$.

If the value of $E[x]$ is greater than t, like S_1 in the question where $E[x] = 10000 > t = 7000$, the value of $E[x]/t > 1$.

When this occurs, Markov's inequality tells us that the upper bound of the Probability is higher than 1, which is the fact that we already know. In other words, if $E[x] > t$, Markov's inequality provides a result that we already know by the definition of probability, and therefore the information is not useful.

$\therefore$ Markov's inequality does not always provide a meaningful bound, and is only meaningful when $E[x] < t$.

3.1

In order to prove the independence of two fair dice summing up to 7 and score on the first die, we need to prove the independence for all possible value for first die

Let Event A = two dice sum up to 7,
B = first die value

$P(A \cap B=1) = P(A) \cdot P(B=1)$
$P(A \cap B=2) = P(A) \cdot P(B=2)$
$\vdots$
$P(A \cap B=6) = P(A) \cdot P(B=6)$

There are 6 out of 36 combinations that give the sum of two dice to 7 → (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)

Since all dice are fair, the numbers (1-6) on the dice are equally likely to occur
$P(B=1) = P(B=2) = \ldots = P(B=6) = 1/6$

When the first die is 1, the second die must be 6 for the sum to be 7
When the first die is 2, the second die must be 5 for the sum to be 7
$\vdots$
When the first die is 6, the second die must be 1 for the sum to be 7

If we let C = second die value,
$P(A \cap B=1) = P(B=1 \cap C=6) = 1/36$, which equals $P(A) \cdot P(B=1) = \frac{6}{36} \cdot \frac{1}{6} = 1/36$
$P(A \cap B=2) = P(B=2 \cap C=5) = 1/36$, which equals $P(A) \cdot P(B=2) = 6/36 \cdot 1/6 = 1/36$
$\vdots$
$P(A \cap B=6) = P(B=6 \cap C=1) = 1/36$, which equals $P(A) \cdot P(B=6) = 6/36 \cdot 1/6 = 1/36$

$\therefore$ The events of two dice summing up to 7 and score on the first dice are independent

3.2

In order to prove $E[xy]^2 \leq E(x^2)E(y^2)$, let $C=(X-aY)^2$ as random variable.

No matter what $X-aY$ is, the square of the value is always positive, giving C always a positive value.
There fore, $E(C) \geq 0$.

If we compute $E[C]$, we get

$$E[(X-aY)^2] = E[x^2 - 2axy + a^2y^2]$$
$$= E[x^2] - 2aE[xy] + a^2E[y^2]$$

Since $E[C] \geq 0$, $E[x^2] - 2aE[xy] + a^2E[y^2] \geq 0$.

Now, let $a = \frac{E[xy]}{E[y^2]}$, but $E[y^2] \neq 0$

Then, we get $E[x^2] - 2\frac{E[xy]E[xy]}{E[y^2]} + \frac{E[xy]^2}{E[y^2]^{\cancel{2}}} \cdot \cancel{E[y^2]} \geq 0$

$$E[x^2] - 2\frac{E[xy]^2}{E[y^2]} + \frac{E[xy]^2}{E[y^2]} \geq 0$$

$$E[x^2] - \frac{E[xy]^2}{E[y^2]} \geq 0$$

$$E[x^2] \geq \frac{E[xy]^2}{E[y^2]}$$

$$E[x^2]E[y^2] \geq E[xy]^2.$$

$\therefore E[xy]^2 \leq E[x^2]E[y^2]$.

The equality, in this case, holds when
$E[x^2] - 2aE[xy] + a^2E[y^2] = E[(x-ay)^2] = E[C] = 0$.
$(x-ay)^2 = 0$ when $x = ay$.
$\therefore$ The equality holds only when $x = ay$.

3.3

$$f(x,y) = \log_{10}\left(1+\frac{1}{10x+y}\right) = \log_{10}\left(\frac{10x+y+1}{10x+y}\right),\ 1 \le x \le 9,\ 0 \le y \le 9$$

- Mass function of first significant digit X:

$$P_X(x) \begin{cases} \log_{10}\left(\frac{11}{10}\right) + \log_{10}\left(\frac{12}{11}\right) + \dots + \log_{10}\left(\frac{20}{19}\right) = \log_{10}\left(\frac{2}{1}\right) \approx 0.3010 & \text{if } x=1 \\ \log_{10}\left(\frac{21}{20}\right) + \log_{10}\left(\frac{22}{21}\right) + \dots + \log_{10}\left(\frac{30}{29}\right) = \log_{10}\left(\frac{3}{2}\right) \approx 0.1760 & \text{if } x=2 \\ \log_{10}\left(\frac{31}{30}\right) + \log_{10}\left(\frac{32}{31}\right) + \dots + \log_{10}\left(\frac{40}{39}\right) = \log_{10}\left(\frac{4}{3}\right) \approx 0.1249 & \text{if } x=3 \\ \log_{10}\left(\frac{41}{40}\right) + \log_{10}\left(\frac{42}{41}\right) + \dots + \log_{10}\left(\frac{50}{49}\right) = \log_{10}\left(\frac{5}{4}\right) \approx 0.0969 & \text{if } x=4 \\ \log_{10}\left(\frac{51}{50}\right) + \log_{10}\left(\frac{52}{51}\right) + \dots + \log_{10}\left(\frac{60}{59}\right) = \log_{10}\left(\frac{6}{5}\right) \approx 0.0792 & \text{if } x=5 \\ \log_{10}\left(\frac{61}{60}\right) + \log_{10}\left(\frac{62}{61}\right) + \dots + \log_{10}\left(\frac{70}{69}\right) = \log_{10}\left(\frac{7}{6}\right) \approx 0.0669 & \text{if } x=6 \\ \log_{10}\left(\frac{71}{70}\right) + \log_{10}\left(\frac{72}{71}\right) + \dots + \log_{10}\left(\frac{80}{79}\right) = \log_{10}\left(\frac{8}{7}\right) \approx 0.0580 & \text{if } x=7 \\ \log_{10}\left(\frac{81}{80}\right) + \log_{10}\left(\frac{82}{81}\right) + \dots + \log_{10}\left(\frac{90}{89}\right) = \log_{10}\left(\frac{9}{8}\right) \approx 0.0512 & \text{if } x=8 \\ \log_{10}\left(\frac{91}{90}\right) + \log_{10}\left(\frac{92}{91}\right) + \dots + \log_{10}\left(\frac{100}{99}\right) = \log_{10}\left(\frac{10}{9}\right) \approx 0.0458 & \text{if } x=9 \end{cases}$$

- From the mass function above, we can observe that the probability is the highest when the first significant digit $X=1$. In addition, as x increases, the probability for the specific X decreases. Therefore, digits 1 and 9 are not equally likely to be the first digit.

$$E[X] = 1\cdot\log_{10}\left(\tfrac{2}{1}\right) + 2\cdot\log_{10}\left(\tfrac{3}{2}\right) + 3\cdot\log_{10}\left(\tfrac{4}{3}\right) + 4\cdot\log_{10}\left(\tfrac{5}{4}\right)$$
$$+ 5\cdot\log_{10}\left(\tfrac{6}{5}\right) + 6\cdot\log_{10}\left(\tfrac{7}{6}\right) + 7\cdot\log_{10}\left(\tfrac{8}{7}\right) + 8\cdot\log_{10}\left(\tfrac{9}{8}\right)$$
$$+ 9\cdot\log_{10}(10/9)$$
$$= 3.4402$$

3.4

$f(x,y) = \lambda^2 e^{-\lambda y}$ for $0 \leq x \leq y < \infty$

To find the conditional density, we need to compute

$$f_{Y|X=x}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \text{ where } f_X(x) \neq 0$$

$$f_X(x) = \int_x^\infty f_{xy}(x,y)\,dy$$

$$= \int_x^\infty \lambda^2 e^{-\lambda y}\,dy$$

$$= \lambda^2 \int_x^\infty e^{-\lambda y}\,dy$$

$$= \lambda^2 \left[\frac{e^{-\lambda y}}{-\lambda}\right]_x^\infty$$

$$= -\lambda \left[e^{-\lambda y}\right]_x^\infty$$

$$= -\lambda \left[0 - e^{-\lambda x}\right]$$

$$= \lambda e^{-\lambda x}$$

$$f_{Y|X=x}(y|x) = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda y + \lambda x} = \lambda e^{-\lambda(y-x)}$$

$$\boxed{f_{Y|X=x}(y|x) \begin{cases} \lambda e^{-\lambda(y-x)} & \text{for } 0 \leq x \leq y < \infty \\ 0 & \text{otherwise} \end{cases}}$$