① 1.

We are looking for $\lim_{n\to\infty} (1-\frac{1}{n})^{n\ln n}$

We can rewrite $(1-\frac{1}{n})^{n\ln(n)}$ as $e^{\ln(1-\frac{1}{n})^{n\ln(n)}}$, which equals

$e^{n\ln(n)\cdot\ln(1-\frac{1}{n})}$ by using properties of log.

$\ln(1-\frac{1}{n}) = \ln(1+(-\frac{1}{n}))$

The taylor series of $\ln(1+x)$ is $x-\frac{x^2}{2}+\frac{x^3}{3}\dots$

Substituting $-\frac{1}{n}$ to x, we get

$-\frac{1}{n}+\frac{1}{2n^2}-\frac{1}{3n^3}+\dots$

If we multiply $n\ln(n)$, we get $-\ln(n)-\frac{\ln(n)}{2n}-\frac{\ln(n)}{3n^2}+\dots$

This question is interested in the behavior as $n\to\infty$. Since $n, n^2, n^3, \dots$ grows faster than $\ln(n)$, we can rewrite the equation above as $-\ln(n)+O(1)$

Going back to the initial problem, we get

$$\lim_{n\to\infty} e^{n\ln(n)\cdot\ln(1-\frac{1}{n})}$$

$$= \lim_{n\to\infty} e^{-\ln(n)+O(1)}$$

$$= \lim_{n\to\infty} e^{-\ln(n)}\cdot e^{O(1)}$$

$$= \lim_{n\to\infty} \frac{1}{n}\cdot \text{constant } C$$

Since the constant is negligible and $\frac{1}{n}$ goes to 0 as $n\to\infty$,

$\lim_{n\to\infty} (1-\frac{1}{n})^{n\ln(n)} = 0.$

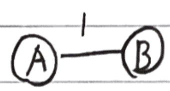
① 2.

If there are n nodes in a graph, the maximum number of edges it can have is $_nC_2$.

Assuming that a graph is undirected, N nodes can have maximum of $(N-1)$ edges.

↳ $N \cdot (N-1)$ edges.

However, since the graph is undirected, going from one node to another has the same edge from that another node to the original node.

Ex) 

A can go to B with path 1

B can go to A with path 1

Therefore, we need to divide the total number of edges by 2.

↳ $\frac{N \cdot (N-1)}{2}$, which equals $_nC_2$.

Out of $_nC_2$ edges, we want to find number of graphs with just m edges, which is $\binom{\binom{n}{2}}{m}$

or

(n choose 2) choose m

③

The question asks to prove that given $G(n,p)$ has m edges, all of the graph that has m edges are equally likely.

For example, if the graph has $n=3$ nodes and $m=2$ edges, the question is asking to prove that pr( | 2edges) = Pr( | 2 edges) = Pr( | 2edges) = $1/3$

Going back to the question, we need to prove pr(unique graph with m edges | m edges) are all equal to 1/ all graphs with m edges.

By bayes theorem, Pr(unique graph with m edges | m edges)

$$= \frac{Pr(\text{unique graph with } m \text{ edges})\, Pr(m \text{ edges} \mid \text{unique graph with } m \text{ edges})}{\sum Pr(m \text{ edges} \mid \text{unique graph with } m \text{ edges})\, Pr(\text{unique graph with } m \text{ edges})}$$

$Pr(\text{unique graph with } m \text{ edges}) = p^m (1-p)^{nC_2 - m}$

↳ there are m edges and $nC_2 - m$ no-edges

$Pr(m \text{ edges} \mid \text{unique graph with } m \text{ edges}) = 1$ since given that the unique graph has m edges, the probability of having m edges is 1

$\therefore$

$$= \frac{1 \cdot p^m (1-p)^{nC_2 - m}}{\sum p^m (1-p)^{nC_2 - m}}$$

↳ since the denominator is summing through all graphs with m edges, we are summing $p^m \cdot (1-p)^{nC_2 - m}$ for $\binom{\binom{n}{2}}{m}$ times.

$$= \frac{p^m (1-p)^{nC_2 - m}}{(nC_2)C_m \cdot (p^m (1-p)^{nC_2 - m})} = \frac{1}{(nC_2)C_m} \text{ or } \frac{1}{\binom{\binom{n}{2}}{m}}$$

Since there are a total of $\binom{\binom{n}{2}}{m}$ graphs with m edges and each unique graph with m edges equals $\frac{1}{\binom{\binom{n}{2}}{m}}$, it is equally likely among all graphs that have m edges conditioned on G having m edges.

① 4.

In this question, we are proving that the number of nodes that have the degree of k is asymptotically equal to $\frac{c^k e^{-k}}{k!} n$ for any k.

There are n nodes in the graph and the probability that there exists an edge between two nodes is $p = \frac{c}{n}$

Let X_i be the degree of node i where $1 \leq i \leq n$.

Since this question is asking to find the asymptotic value, we are looking for the case where all nodes $1 \ldots n$ have degree of k

↳ $Pr(X_1 = k \wedge X_2 = k \wedge \ldots \wedge X_n = k)$

Each X_is follow the same binomial distribution

↳ There are total of n-1 nodes that it can be connected to.

There needs to be k edges for each node

$$Pr(X_i = k) = {}_{n-1}C_k \cdot p^k \cdot (1-p)^{n-1-k} \quad \text{as } n \to \infty$$

$$= {}_{n-1}C_k \cdot \left(\frac{c}{n}\right)^k \left(1-\frac{c}{n}\right)^{n-1-k} \quad \text{as } n \to \infty$$

$$= \frac{(n-1)!}{(n-1-k)!\,k!} \left(\frac{c}{n}\right)^k \left(1-\frac{c}{n}\right)^n \cdot \left(1-\frac{c}{n}\right)^{-1-k} \quad \text{as } n \to \infty$$

$$= \frac{c^k}{k!} \cdot \frac{(n-1)!}{(n-1-k)!\,n^k} \left(1-\frac{c}{n}\right)^n \left(1-\frac{c}{n}\right)^{-1-k} \quad \text{as } n \to \infty$$

$$\Rightarrow \frac{(n-1)!}{(n-1-k)!\,n^k} = \frac{(n-1)(n-2)\cdots(n-k)(n-1-k)!}{(n-1-k)!\,n^k}$$

- The $(n-1)(n-2)\cdots(n-k)$ can be written as $(n-1)^k$ asymptotically since you multiply (n − an integer) k times.
- $(n-1-k)!$ cancels out

$$= \frac{(n-1)^k}{(n)^k} = \left(\frac{n-1}{n}\right)^k = \left(1-\frac{1}{n}\right)^k$$

As $n \to \infty$, we get $(1-0)^k = 1$.

$$\Rightarrow \left(1-\frac{c}{n}\right)^n = e^{\ln\left(1-\frac{c}{n}\right)^n}$$
$$= e^{n\ln\left(1-\frac{c}{n}\right)}$$

The taylor series of $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

Plugging $x = -\frac{c}{n}$, we get $\ln(1-\frac{c}{n}) = \frac{-c}{n} - \frac{c^2}{2n^2} - \frac{c^3}{3n^3} - \ldots$

multiplying all terms by n, we get $n\ln(1-\frac{c}{n}) = -c - \frac{c^2}{2n} - \frac{c^3}{3n^2} - \ldots$

As $n \to \infty$, we get $-c + o(1)$

$$\lim_{n\to\infty} \left(1-\frac{c}{n}\right)^n = e^{n\ln\left(1-\frac{c}{n}\right)}$$
$$= e^{-(c+o(1))}$$
$$= e^{-c} \cdot e^{o(1)}$$
$$= e^{-c}$$

$\Rightarrow \left(1-\frac{c}{n}\right)^{-1-k}$ as $n \to \infty$ is simply $(1-0)^{-1-k} = 1$

$$\therefore \Pr(X_i = k) = \frac{c^k}{k!}(1)(e^{-c})(1) \text{ as } n \to \infty$$
$$= \frac{c^k e^{-c}}{k!} \text{ as } n \to \infty$$

However, we are concerned about
$Pr(X_1 = k \wedge X_2 = k \wedge \dots \wedge X_n = k)$.

Since all $X_i = k$ has the same distribution, by applying union bound, we get

$Pr(X_1 = k \wedge X_2 = k \wedge \dots \wedge X_n = k) \leq n \, Pr(X_1 = k)$

$Pr(X_1 = k)$ is asymptotic to $\frac{c^k e^{-c}}{k!}$ as $n \to \infty$

Therefore, $n \cdot Pr(X_1 = k)$ is asympotic to $\frac{n \cdot c^k \cdot e^{-c}}{k!}$

$\therefore$ $Pr(X_1 = k \wedge X_2 = k \wedge \dots \wedge X_n = k) \leq n \cdot Pr(X_1 = k)$, which is asymptotically $\frac{c^k \cdot e^{-c}}{k!} n$

⑤

The question states that there exists an edge between two nodes if at least one of the two coins gives heads and no edge when both of the coins give tails.

↳ This is equivalent to graphing $G(n,p)$ where p equals the probability of giving at least one head from two coins.

Note that the two coins are independent to each other. The result (head or tail) of one coin doesn't affect the result of the second coin.

$$
\begin{aligned}
p(\text{coin } 1 = \text{head} \cup \text{coin } 2 = \text{head}) &= 1 - Pr(\text{coin } 1 \text{ tail} \cap \text{coin } 2 \text{ tail}) \\
&= 1 - Pr(\text{coin } 1 \text{ tail}) \cdot Pr(\text{coin } 2 \text{ tail}) \\
&= 1 - (1-p_1)(1-p_2) \\
&= 1 - (1 - p_1 - p_2 + p_1 p_2) \\
&= 1 - 1 + p_1 + p_2 - p_1 p_2 \\
&= p_1 + p_2 - p_1 p_2
\end{aligned}
$$

$\therefore$ This is equivalent to graphing $G(n, p_1 + p_2 - p_1 p_2)$

(6)

The question states that G is sampled from $G(n, 0.1)$.
This means that a graph has n nodes that have $p = 0.1$ on whether there exists an edge between two nodes.

The central limit theorem states that the standardized sample mean tends to the standard normal distribution regardless of the fact that the original variables are not normally distributed.

We are concerned about the degree of a node.
There are total of $n-1$ nodes that each node can be connected to by an edge.
Let X_i denote the degree to node i where $1 \leq i \leq n$
The distribution here is binomial, with $n = (n-1)$ and $p = 0.1$, and every X_i follows the same distribution

$E(X_i) = np = (n-1) \cdot (0.1)$

$Var(X_i) = np(1-p) = (n-1)(0.1)(0.9)$

The question states to give the range of the degree of any node within 99% confidence interval.
99% confidence interval, approximately, is 2.576 standard deviations away from the mean, according to the class slides

$Std(X) = \sqrt{Var(X)} = \sqrt{(n-1)(0.1)(0.9)} = \sqrt{(n-1)(0.09)} = 0.3\sqrt{(n-1)}$

$\therefore$ The range that the degree of a node lies within probability of at least 99% is $(n-1)(0.1) \pm 2.576(0.3\sqrt{n-1}) = 0.1n - 0.1 \pm 0.7728\sqrt{n-1}$

or

$$(n-1)(0.1) - 2.576(0.3\sqrt{n-1}) \leq X_i \leq (n-1)(0.1) + 2.576(0.3\sqrt{n-1})$$

$$= 0.1n - 0.1 - 0.7728\sqrt{n-1} \leq X_i \leq 0.1n - 0.1 + 0.7728\sqrt{n-1}$$