CAS CS 365
InClass Note 3

Probability

1. Union Bound
    a. Pr(A U B U C) <= Pr(A) + Pr(B) + Pr(C)
    b. $$\boxed{\Pr(A_1 \cup \ldots \cup A_n) \leq \sum_{i=1}^{n} \Pr(A_i)}$$
    c. Example: True or False
        - Suppose X and Y are real random values.
        - Pr(X +Y >=0) <= Pr(X >=0) + Pr(Y >=0)
        - True
        - If X + Y >=0, at least one of X or Y should be greater than 0
        - {X,Y: X+Y>=0} is a subset of {X, Y: X>=0 U Y>=0} <= Pr(X >=0) + Pr(Y>=0)
2. Variance
    a. Is Var(X + Y) = Var(X) + Var(Y)? → False
    b. Var(x) = E((x-μ)^2)
    c. Var(X + Y) = E((X+Y-μ_x-μ_y)^2)
       = E((X-μ_x + Y-μ_y)^2)
    d. Covariance
       E((X-E(X)*(Y-E(Y))
       = E(XY-Xμ_y-Yμ_x+$\mu_x\mu_y$)
    e. Var (X+Y) = Var(X) + Var(Y) + 2 *coVar(X,Y)

3. Hats off (cont. from last time)
    a. Men with hats enter a room, and take off their hats. On their way out, hats are mixed up according to a random permutation, so each person receives a random hat. The probability each man gets his own hat is 1/n

        We define for each many i
        X_i = 1 if that person receives his hat
        = 0 otherwise
        Pr(X_1 = 1) = 1/n
        So, E(X_1) = 1 * 1/n + 0 * (1-1/n) = 1/n

        - Since the permutation is chosen uar, the probability each mean gets his own hat is 1/n

- Question: how many men in expectation will get their own hat?
- Answer:

  H_n = # of men who get their own hat

  $E(H_n) = E(x_1 + ... + x_n)$ = (lin exp) =($\sum$ i=1 to n) $E(x_1) = n/n = 1$

- Question: variance
- Answer:

  $Var(H_n) = Var(x_1 + \dots + x_n)$

  $= E((x_1 + ... + x_n - (\mu_1 + \dots \mu_n))^2)$

  $= E(((x_1 - \mu_1) + \dots (x_n - \mu_n))^2)$

  = ($\sum$ i=1 to n) $E((x_i - \mu_i)^2)$ + 2 * ($\sum$ i < j) $E((x_i - \mu_i)*(x_j - \mu_j))$

$$= \mathbb{V}\mathrm{ar}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{V}\mathrm{ar}[X_i] + 2\sum_{i<j} \mathbb{C}\mathrm{ov}[X_i, X_j]$$

$$\text{Reminder: } \mathbb{C}\mathrm{ov}[X,Y] = \left[(X - EX)(Y - EY)\right]$$
$$= E[XY] - (EX)(EY).$$

a. $Var(X_i) = E((x_i - p)^2)$

   $= E[X_i^2]$

   $= 1/n$

b. $Cov(X_i, X_j) = E(X_i * X_j) - E(X_i) * E(X_j)$

   $= E(X_i * X_j) - 1 * 1$

   Let $X_i * X_j = Z$

   $Z = 1$ if $X_i == X_j == 1$

   $= 0$ otherwise

   $E(Z) = 1/(n) * 1/(n-1)$

= n * 1/n * + n * (n-1) * 1/n * 1/(n-1) - 1

= 1 + 1 - 1

= 1

4. Linearity of Expectation
    a. Expected value of the sum of random variables is equal to the sum of their individual expectations
        - Important point: this holds regardless of whether they are independent
    b. Let $X_1, \ldots, X_k$ be random variables, $c_1,\ldots,c_k$ constants

    $$\boxed{\mathbb{E}\left[\sum_{i=1}^{k} c_i X_i\right] = \sum_{i=1}^{k} c_i \mathbb{E}[X_i]}$$

    c.
5. Digression
    a. Probability inequalities
        - Markov's inequality
        - Chebyshev's inequality
    b. Notions of convergence of random variables
    c. Limit theorems
        - Weak and strong laws of large numbers (LLN)
        - Central Limit Theorem (CLT)
6. Markov's inequality
    a. Let X be a non-negative random variable and suppose that E[X] exists
    b. For any $t > 0$, $\boxed{\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}}$
    c. Example: X ~ Bin(1000, 0.1). Then E[X] = 100.
    d. The tail probability Pr(x >= 400) <= ¼
    e. Proof:

    $$\mathbb{E}[X] = \int_0^\infty x f(x) dx = \int_0^t x f(x) dx + \int_t^\infty x f(x) dx \geq \int_t^\infty x f(x) dx \geq t \int_t^\infty f(x) dx = t \Pr(X > t)$$

7.