Linear Algebra

vectors and matrices

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}$$ → reals

is equivalent to a d-dimensional point ($\mathbb{R}^d$)

↳ column vector

$v^T = [v_1 \dots v_d]$

↳ row vector

} both will be stored as an array

$$A^{n \times d} = \begin{bmatrix} -a_1^T- \\ \vdots \\ -a_n^T- \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1d} \\ A_{i1} & A_{i2} & \cdots & A_{id} \\ A_{n1} & A_{n2} & \cdots & A_{nd} \end{bmatrix}$$

a_1^T (first row), b_1 (first column)

$A_{i,j}$ → col index (j)

↳ row index (i)

$$A = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_d \\ | & & | \end{bmatrix}$$

$a_1 + a_2 = (2+4, 4+1) = (6,5)$

when $a_1 = (2,4)$

$a_2 = (4,1)$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$ → dot product perspective of matrix vector multiplication

$A \quad x$

$\underline{2\times2} \times \underline{2\times1} = 2\times1$ matrix

"Linear combination of columns" perspective

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 4x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

ex)

$2x_1 + 5x_2 - 7x_3 = 10$

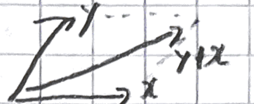
$-x_1 + 8x_2 - 2x_3 = -2$

$$A^{2\times3} \cdot x^{3\times1} = b^{2\times1}$$

$$\begin{bmatrix} 2 & 5 & -7 \\ -1 & 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

we are trying to find a linear combination of the three 2 dim columns of A such that this linear combination is equal to b

$x, y \in \mathbb{R}^d$ $\quad x+y = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_d \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_d+y_d \end{bmatrix}$



$A^{n\times d} + B^{n\times d} = C^{n\times d}$ where $C_{iy} = A_{iy} + B_{iy}$ for all $1 \le i \le n$

$C^{n\times m} = A^{n\times d} B^{d\times m}$ where $C_{iy} = \sum_{k=1}^{d} A_{ik} \cdot B_{ky} = a_i^T \cdot b_y$ $\quad 1 \le y \le d$

$$\begin{bmatrix} - a_1^T - \\ - a_i^T - \\ - a_n^T - \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ b_1 & b_y & b_m \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$A \cdot B \neq B \cdot A$ (matrix multiplication is not commutative)

$(AB)C = A(BC)$ associative

$A \cdot (B+C) = AB + AC$ distributive

Vector-vector product

$x^T y = (x_1 \cdots x_d)^{1\times d} \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}^{d\times 1} = x_1 y_1 + \dots + x_d y_d = \sum_{i=1}^{d} x_i y_i = \langle x, y \rangle \Rightarrow \in \mathbb{R}$, inner product

↳ Euclidean product / ℓ_2

$x^T y = (\text{length}(x))\ \text{length}(y) \cdot \cos(\theta)$

↳ $\|x\|_2$ $\quad = \|x\| = \sqrt{x_1^2 + \dots + x_d^2} = \sqrt{\langle x, x \rangle} = \sqrt{x^T x}$

↳ if omitted $\quad$ ↳ generalizes pythagoras theorem

- orthogonal of two vectors if $A \cdot B = 0$

$\underset{d\times 1 \ \ 1\times k}{(x\, y^T)} = \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_j \\ \vdots & & \vdots \\ x_d y_1 & \cdots & x_d y_j \end{pmatrix} \to d\times k \Rightarrow$ outer product, $\in \mathbb{R}^{d\times k}$

$\|x\|_2^2 = x^T \cdot x = \sum_{i=1}^{d} x_i^2$

L_p norms (well-defined for our purposes for $p \in [1, \infty)$)

$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}$ $\quad [\ \|x\|_p^p$

↑

denotes

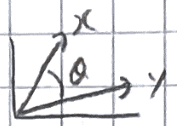
$$\|x\|_\infty = \max_{1\le i\le d} |x_i|$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Q $x^T y = x_1 y_1 + x_2 y_2$

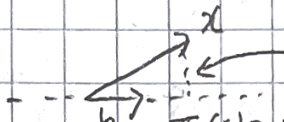
$\|x\| = \sqrt{x_1^2 + x_2^2}$

$\|y\| = \sqrt{y_1^2 + y_2^2}$



$$\cos\theta = \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2}\cdot\sqrt{y_1^2 + y_2^2}}$$ (projections)

Q How do we project? (why does trigonometry play a role?)



Q: How do we project x on b?

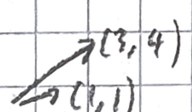
$\pi_b(x)$ = projection of x on b

ex) $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\pi_b(x) = \pi_{\binom{1}{0}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

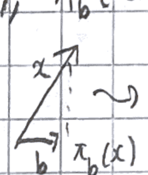
$\pi_{\binom{0}{1}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$

This example is easy, because $b = \binom{1}{0}, \binom{0}{1}$ → standard orthonormal basis for $\mathbb{R}^2$

($e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ is standard basis for $\mathbb{R}^n$)



Clearly, $\pi_b(x) = \lambda\cdot b$ for some $\lambda \in \mathbb{R}$



$\rightsquigarrow x - \pi_b(x) \rightarrow x - \pi_b(x) \perp b$

$$(x - \pi_b(x))^T b = 0$$

$$x^T b = \pi_b(x)^T \cdot b$$

$$\lambda b^T b = x^T b$$

$$\lambda = \frac{x^T b}{b^T b} = \frac{b^T x}{\|b\|_2^2}$$

projection matrix

$$\text{So } \pi_b(x) = \lambda b = \left(\frac{b^T x}{b^T b}\right) b = \left[\left(\frac{b\cdot b^T}{b^T\cdot b}\right)\right]^{d\times d} x^{d\times 1}$$

Let $P = \frac{bb^T}{b^Tb}$

$$P^2 = \left(\frac{bb^T}{b^Tb}\right)\left(\frac{bb^T}{b^Tb}\right) = \frac{b(b^Tb)b^T}{(b^Tb)(b^Tb)} = \frac{bb^T}{b^Tb} = P$$

$\because\ x^Tb = x_1b_1 + \dots + x_d b_d = b^T x$

$b = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$b^Tb = \frac{1}{2} + \frac{1}{2} = 1$

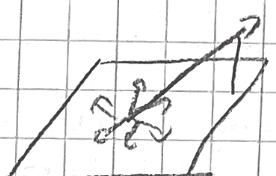
$bb^T = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$

$P = \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$ $P_x = \begin{pmatrix}1/2\,x_1 + 1/2\,x_2\\ 1/2\,x_1 + 1/2\,x_2\end{pmatrix}$

$P^2 = P \rightarrow \frac{bb^T}{b^Tb}$

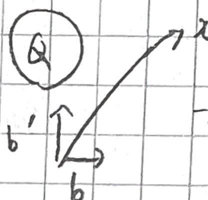
↳ $(P \cdot x) = \pi_b(x)$

$\pi_b(\pi_b(x)) = \pi_b(x)$



$\{x_1, \dots x_n\}$ $\mathbb{R}^3$

Q



→ How to find b'?

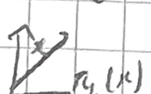
idea 1

find b'. $b = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$, then $b' = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$ and repeat

$P' = \frac{(b'b'^T)}{b'^Tb'}$ $\pi_{b'}(x) = P' \cdot x$

Idea 2

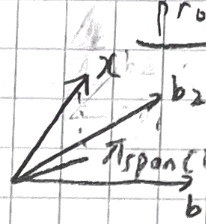
$(I - P)x = x - Px = x - \pi_b(x)$



$P = \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$ (on $b = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$)

Claim $P' = I - P$ $I - P = \begin{pmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{pmatrix} = \frac{(b')(b')^T}{(b'^T)(b')} = \frac{b'b'^T}{1}$

Projection on general space



$\pi_{span(b_1, b_2)}(x) = \pi_U(x)$

$U = span(b_1, b_2)$

$\pi_U(x) = \lambda_1 b_1 + \dots + \lambda_m b_m$ for some $\lambda_1 \dots \lambda_m$

$= \begin{bmatrix} | & & | \\ b_1 & \cdots & b_m \\ | & & | \end{bmatrix}\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_m\end{bmatrix}$

$= B \times \lambda$

$(I - P)^2 = (I - P)(I - P)$
$= I^2 - 2P + P^2$
$= I^2 - 2P + P$
$= I^2 - P = I - P$

Projection property

$X^2 = X$