CAS CS 365
Lab 2

Probability review

1. Pairwise independence does not imply mutual independence
   a. X = Flip Coin, Pr(X = heads) = ½
   b. Y = Flip Coin, Pr(Y = heads) = ½
   c. Z = Pr(exactly X or Y are heads (not both)) = ½
      - Pr(X,Z) pairwise independence
      - Pr(X=0, Z=0) = Pr(X=0) * Pr(Z=0) = ¼
      - Pr(X,Y,Z) not mutual independence
      - Pr(X=1,Y=1,Z=0) = ¼ where as Pr(X=1)*Pr(Y=1)*Pr(Z=0) = ⅛
2. Definitions
   a. Expectation
      - When X is a discrete random variable, $\sum_i i \cdot \Pr[X = i]$
      - When X is a continuous random variable, and f(x) is its probability density function,
        - $\int_{-\infty}^{\infty} x \cdot f(x)dx$
   b. Variance
      - $Var[X] = E\left[(X - E[X])^2\right] = E\left[X^2\right] - E[X]^2$
      - Measures how far your data spread out from their average value
   c. Standard deviation
      - Square root of variance
3. Covariance and Correlation
   a. Definitions: $Cov(X, Y) = E[(X - E[x])(Y - E[Y])]$

      $= \sum_x \sum_y (x - E[X])(y - E[Y]) p_{X,Y}(x, y)$

      $Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$
4. Covariance
   a. Some properties
      - Cov(X + c, Y) = Cov(X, Y)
      - Cov(aX + bY, Z) = a * Cov(X,Z) + b * Cov(Y,Z)
   b. Exercise
      - X and Y are two independent N(0,1) random variables and

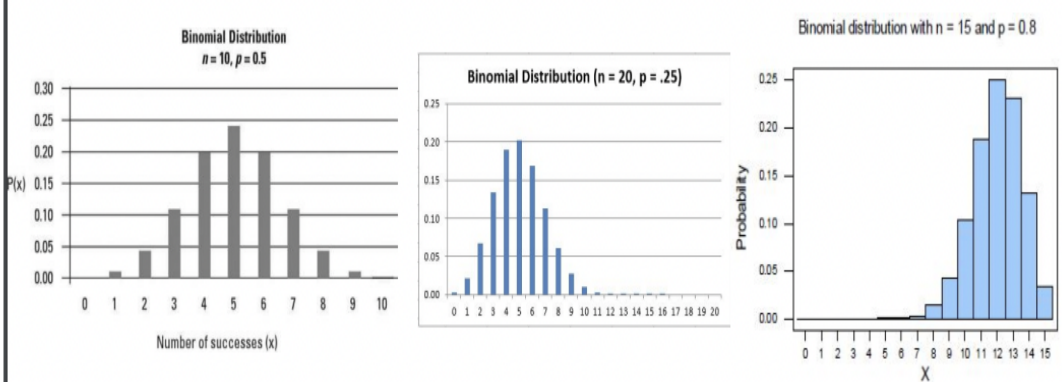
- Z = 1 + X + XY^2
- W = 1 + X
- Find Cov(W,Z)

  Normal distribution:
  E(x) = E(Y) = 0
  Var(x) = Var(Y) = 1

  Cov(1+X, 1+X+XY^2)
  = Cov(X, X + XY^2)
  = Cov(X, X) + Cov(X, XY^2)
  = Var(X) + E[X^2Y^2] - E[X]*E[XY^2]
  = 1 + 1 - 0
  = 2

5. Discrete probability distributions
   a. Bernoulli (p)
      - Discrete
      - Two outcomes: 0, 1
      - Pr(X=1) = p = 1 - q
      - Pr(X=0) = q = 1- p
      - Expectation: E[x] = 1 * p + 0 * (1-p) = p
      - Example: Flip a coin 5 times, what is the probability of having exactly 2 heads
        - ❖ Sample repetitively from a Bernoulli distribution n times
        - ❖ Bernoulli = Binomial with n = 1
        - ❖ Pr [X = k] = nCk * p^k * (1-p)^(n-k)
        - ❖ Expectation = n * p

PDF $\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$



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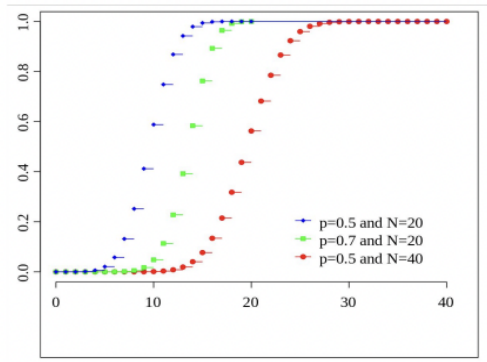
b. Binomial (n, p)

- **CDF:** $F_x(t) = \Pr(X \le T)$



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c. Poisson (lambda)

- Poisson distribution is a probability distribution that is used to show how many times an event is likely to occur over a specified period
- Events are independent of each other
- The occurrence of one event does not affect the probability another event will occur
- The average rate (events per time period) is constant.
- Two events cannot occur at the same time

$$\Pr[X = k] = \frac{\lambda^k e^{-\lambda}}{k!}$$

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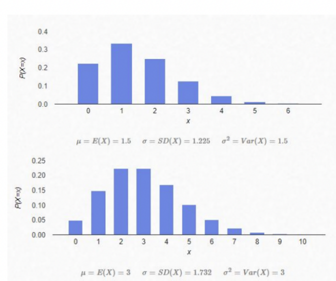

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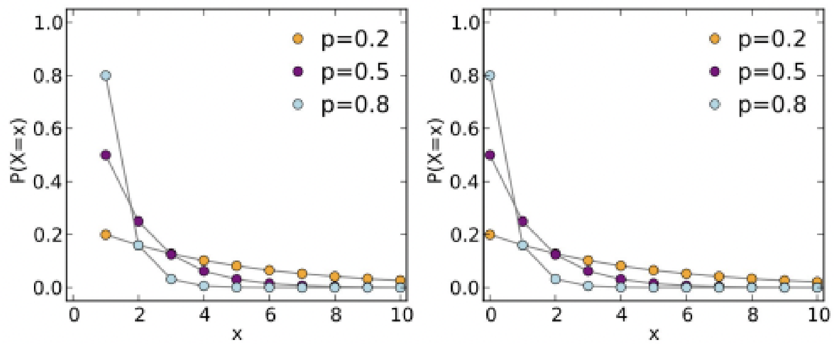
- Example: You are "on call" at work. And every night between 12 and 8 AM you receive about 6 calls from customers
  - ❖ Question: what is the probability you receive a call from 2-4AM?
  - ❖ Definitions: X ~ P(lambda) where X is a random variable with a Poisson distribution, and "lambda" is the mean for the interval
  - ❖ 6 calls over an 8 hour window is about ¾ of a call every hour, our window hours, so our "lambda" value is 2 * ¾ = 1.5

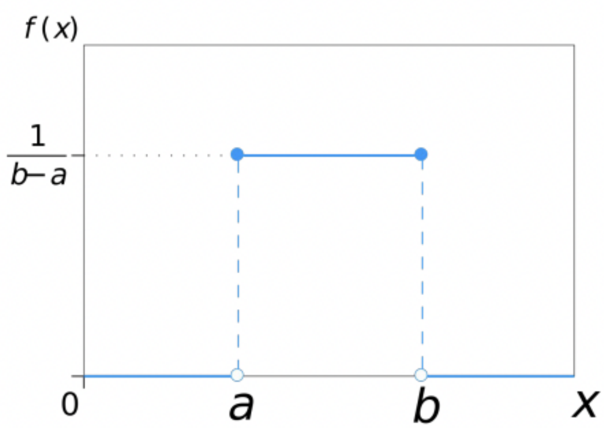
d. Geometric (p)

- Discrete, closely related to Bernoulli
- Probability of N failures before a success
- Pr(X=k) = (1-p)^(k-1)* p
- 

- Expected number = $\sum_{k=1}^{\infty} k * (1-p)^{k-1} p$ = 1/p
- Example: What is the probability I flip a coin 4 times before getting heads?

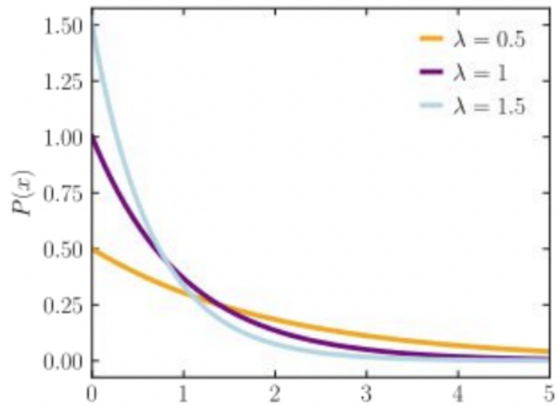
6. Continuous probability distribution

a. Uniform (a, b)

- 

- **E[X]** where P(X=x) = 1/N:
$$\frac{1}{b-a}$$
- Var[x] = (b-a)^2/12

b. Exponential (lambda)

- 

- The exponential distribution is the probability distribution of the time between events in a Poisson point process
- Parameterized by (lambda)
- PDF: Pr[X=x] = (lambda) * e^(-lambda*x)
- Expected value = 1/lambda
- Variance: 1/lambda^2

c. Laplace (μ, b)

- It can be thought of as two exponential distributions (with an additional location parameter) spliced together along the abscissa
- Parameterized by μ (location - shift along the x axis) and b (scaling - maximum peak value)

$$\Pr[X = x] = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

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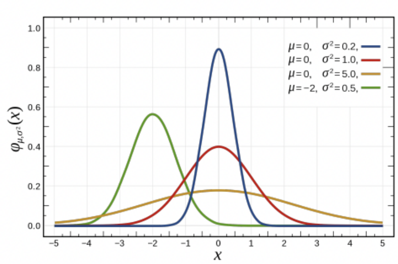
d. Gaussian (μ, o^2) - one dimension

- 

$$\Pr[X = x] = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

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- μ = mean, o = standard deviation
- It is important partly due to CLT