① f) $\forall x (\neg W(x) \rightarrow (S(x) \vee V(x)))$

g) $\exists x (\neg W(x) \wedge \neg S(x) \wedge \neg V(x))$

h) $\forall x (\neg W(x) \rightarrow (S(x) \vee V(x)))$

i) $(x = \text{Ingrid}) \wedge S(x) \wedge W(x)$

j) $\exists x ((x \neq \text{Ingrid}) \wedge S(x))$

k) $\forall x ((x \neq \text{Ingrid}) \rightarrow S(x))$

② e) Proposition

False

For all male patients, if he had migraines, then he had fainting spells and if he had fainting spells, he had migraines.

f) Proposition

True

For all male patients, if he has migraines and has fainting spells, then he was not given the medicine

g) Proposition

True

There exists male patient such that he was given medication and he did not have fainting spells and did not have migraines.

h) Proposition

False

For all male patients, if he had given medicine, he either had fainting spells or had migraines.

③ d) $\forall x\,(P(x) \to M(x))$

Negation: $\neg \forall x\,(P(x) \to M(x))$

Apply: $\exists x\, \neg (P(x) \to M(x))$

$\exists x\, \neg (\neg P(x) \vee M(x))$

$\exists x\, (P(x) \wedge \neg M(x))$

English: There exists some patients that took the placebo and did not have migraines..

e) $\exists x\,(M(x) \wedge P(x))$

Negation: $\neg \exists x\,(M(x) \wedge P(x))$

Apply: $\forall x\, \neg(M(x) \wedge P(x))$

$\forall x\,(\neg M(x) \vee \neg P(x))$

English: Every patient was either not given placebo or did not have migraines.

④ b) $\neg(\forall x\, \exists y\,(P(x,y) \wedge Q(x,y)))$

$\exists x\, \forall y\, \neg(P(x,y) \wedge Q(x,y))$

$\exists x\, \forall y\, (\neg P(x,y) \vee \neg Q(x,y))$

c) $\neg(\exists x\, \forall y\,(P(x,y) \to Q(x,y)))$

$\forall x\, \exists y\, \neg(P(x,y) \to Q(x,y))$

$\forall x\, \exists y\, \neg(\neg P(x,y) \vee Q(x,y))$

$\forall x\, \exists y\,(P(x,y) \wedge \neg Q(x,y))$

d) $\neg(\exists x\, \forall y\,(P(x,y) \leftrightarrow P(y,x)))$

$\forall x\, \exists y\, \neg(P(x,y) \leftrightarrow P(y,x))$

$\forall x\, \exists y\, \neg(P(x,y) \to P(y,x) \wedge P(y,x) \to P(x,y))$

$\forall x\, \exists y\, \neg((\neg P(x,y) \vee P(y,x)) \wedge (\neg P(y,x) \vee P(x,y)))$

$\forall x\, \exists y\,(\neg(\neg P(x,y) \vee P(y,x)) \vee \neg(\neg P(y,x) \vee P(x,y)))$

$\forall x\, \exists y((P(x,y) \wedge P(y,x)) \vee (P(y,x) \wedge P(x,y)))$

⑤ f) $\forall x \exists y (y < x)$

g) $\forall x \exists y (x \neq 0 \rightarrow y = \frac{1}{x})$

h) $\forall x \exists y \forall z ((x \neq 0 \rightarrow y = \frac{1}{x}) \wedge (y \neq z \rightarrow z \neq \frac{1}{x}))$

⑥ f) $\forall x ((x \neq \text{Josephine}) \rightarrow B(\text{Josephine}, x))$

g) $\exists x \forall y (B(\text{Nancy}, x) \wedge ((y \neq x) \rightarrow \neg B(\text{Nancy}, y)))$

h) $\exists x \exists y \exists z ((x \neq y) \wedge \neg B(z, x) \wedge \neg B(z, y))$

(7) 1. $(x \to Y) \land (Z \to T)$ premise

2. $x \to Y$ Simplification 1

3. $Z \to T$ Simplification 1

4. $x \land Z$ Hypothesis

5. x Simplification 4

6. Z Simplification 4

7. Y Modus Ponens 2, 5

8. T Modus Ponens 3, 6

9. $Y \land T$ Conjunction 7, 8

$\therefore (x \land Z) \to (Y \land T)$ Hypothesis Elimination 4, 9

Given that $x \to Y$ and $z \to T$ (line 1), $x \to Y$ and $z \to T$ are both true (line 2&3). Take x and z (line 4). Here, both x and z are true (line 5 and 6). If x is true, it leads to y (line 7) and if z is true, it leads to T (line 8). Therefore if both x and z are true, it leads to both Y and T (line 10), which can be written as:

$$(x \land z) \to (Y \land T).$$

(8) (1) $B \subseteq A$ Premise

(2) $C \subseteq B$ Premise

(3) $\forall b \in B\,(b \in A)$ Definition of $\subseteq$ (1)

(4) $\forall c \in C\,(c \in B)$ Definition of $\subseteq$ (2)

(5) $(b \in B) \rightarrow (b \in A)$ Universal Instantiation (3)

(6) $(c \in C) \rightarrow (c \in B)$ Universal Instantiation (4)

(7) $c \in C$ hypothesis

(8) $c \in B$ Modus Ponens (6),(7)

(9) $c \in A$ Modus Ponens (8),(5)

(10) $c \in C \rightarrow c \in A$ hypothesis elimination (7),(9)

(11) $\forall c \in C\,(c \in A)$ Universal Generalization (10)

(12) $C \subseteq A$ Definition of $\subseteq$ (11)

(13) $\forall a \in A\ Q(a) \equiv \text{True}$ Premise

$\therefore$ $\forall c \in C\ Q(c) \equiv \text{True}$ Definition of subset

Given B is a subset of A (line 1) and C is subset of B (line 2), every elements in B are also in A and every elements in C are also in C (lines 3-4). From this information, we can assume that if an element is in B, it is also in A and if an element is in C, it is also in B (lines 5-6). Therefore, for every elements in C, those elements are also in B and those elements in C that are also in B are also in A (lines 7-10). This means that every elements in C are in A and, therefore, C is a subset of A (lines 11-12). Since for all elements in A Q(a) is true as given (line 13), every elements in C, which are also in A, gives Q(C) True for all all elements in C.

9 1 $f: A \to B$ is surjective Premise

2 $g: B \to C$ is surjective Premise

3 $\forall b \in B\ \exists a \in A\ (f(a) = b)$ Definition of surjectivity 1

4 $\forall c \in C\ \exists b \in B\ (g(b) = c)$ Definition of surjectivity 2

5 $\forall c \in C\ \exists a \in A\ (g(f(a)) = c)$ Rules of equality 3, 4

6 $h(x) = g(f(x))$ Premise

7 $\forall c \in C\ \exists a \in A\ (h(a) = c)$ Definition of h 6

$\therefore$ $H: A \to C$ is surjective Definition of surjective 7

Given that $f: A \to B$ and $g: B \to C$ are surjective (line 1 and 2), there exists elements in A that correspond to every elements in B when elements in A are put into function f (line 3) and there exists elements in B that correspond to every elements in C when elements in B are put into function g (line 4).
If we put the function f (which takes the elements of A and outputs all elements of B) as inputs of function g and put the elements of A that output all elements of B to function f, $g(f(a))$ will be equal to $g(B)$, which will result in all elements of C. Since $h(x) = g(f(x))$ by given (line 6), function h can be written as $h(a) = c$, for some elements in A and all elements in C. Therefore, function h is surjective.

10 Prove $D \cap E = \emptyset$

↳ Proof by Contradiction

$D \cap E = \emptyset$ is False $\equiv D \cap E \neq \emptyset$

1 $D \cap E \neq \emptyset$	Hypothesis
2 $\exists x\,(x \in (D \cap E))$	Definition of $\neq, \emptyset$ 1
3 $z \in (D \cap E)$	Existential Instantiation 2
4 $(z \in D) \land (z \in E)$	Definition of $\cap$ 3
5 $z \in D$	Simplification 4
6 $z \in E$	Simplification 4
7 $z \in (C \cap A)$	Definition of D 5
8 $z \in (C \cap B)$	Definition of E 6
9 $(z \in C) \land (z \in A)$	Definition of $\cap$ 7
10 $(z \in C) \land (z \in B)$	Definition of $\cap$ 8
11 $z \in C$	Simplification 9
12 $z \in A$	Simplification 9
13 $z \in C$	Simplification 10
14 $z \in B$	Simplification 10
15 $z \in (A \cap B)$	Conjunction 12, 14
16 $A \cap B = \emptyset$	Premise
17 $z \in \emptyset$	Equality 15, 16
18 $z \in (\emptyset \cap C \cap C)$	Conjunction 17, 11, 13
19 $z \in \emptyset$	Definition of $\emptyset, \cap$ 18
20 False	
21 $\exists x\,(x \in (D \cap E)) \rightarrow$ False	hypothesis elimination 19, 20
22 $\neg(\exists x\,(x \in (D \cap E))) \lor$ False	Conditional Identity 21
23 $\neg(\exists x\,(x \in (D \cap E)))$	Identity Laws 22
24 $\forall x\,(x \notin (D \cap E))$	De Morgan's Law 23
$\therefore (D \cap E) = \emptyset$	Definition of $\emptyset$

Suppose for the purpose of contradiction, that $D \cap E = \emptyset$ is False. This means that $D \cap E \neq \emptyset$ (line 1). This means that there exists an element in $D \cap E$ (line 2). Let z be in $D \cap E$ (line 3). This means that z is in both D and E (lines 4-6). D is $C \cap A$ and E is $C \cap B$ by definition (lines 7-8). This means that z is in C and A and z is in C and B (lines 9-10). Then, z is in C, z is in A, and z is in C, z is in B (lines 11-14). We can then state that z is in A and B. by conjunction (line 15). However, $A \cap B = \emptyset$ by definition (line 16), so z is in $\emptyset$ (line 17). When we combine that z is in $\emptyset$, z is in C, z is in C, z is in $\emptyset$ by laws of $\cap$ (line 18-19). Since z is in $\emptyset$, there cannot be an element z, which shows that z in $D \cap E$ is false and there cannot be an element in $D \cap E$ (line 20-21). Therefore, since no element is in $D \cap E$, $D \cap E$ is empty set.