① 8.2.1.c

If x is real number and $x \leq 3$, then $x^2-7x+12 \geq 0$.

$x^2-7x+12$ Can be simplified into $(x-4)(x-3)$, which has to be greater than or equal to 0. If x equals 3, $x-3$ gives the value of 0, making the value of $x^2-7x+12$ 0, which is greater than or equal to 0. In addition, if x is less than 3, $x-4$ and $x-3$ will always give a negative value. Since they are both negative, the two negative numbers multiplied will always give positive value. Since all positive numbers are greater than 0, the statement is true.

② 8.2.1.d

The product of two odd integers is an odd integer.

Let's assume that integer x and y are odd integers. They can be written in the form $2x+1$, where n is an integer, since any integer multiplied by 2 is even and even + 1 is odd. Let $x=2k+1$ and $y=2j+1$. If we multiply x and y, the value is $(2k+1)(2j+1)$, which is $4kj+2k+2j+1$. This can be rewritten as $2(2kj+k+j)+1$. Similarly, here, since any number multiplied by 2 is even and 1 plus even is always odd, it proves that $2(2kj+k+j)+1$ is always odd, regardless of what k and j are. Therefore, two odd integers multiplied will always result in odd integer.

③ 8.3.1.f

For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational

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Proof by contrapositive

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For every non-zero real number, if $\frac{1}{x}$ is rational, then x is also rational.

Rational numbers mean that it can be written as a ratio of two numbers that do not share a prime factor. Since $\frac{1}{x}$ is rational, it can be written as a/b, where a and b are integers that do not share a common prime factor. x, the reciprocal of $1/x$, is the value attained by swapping the numerator and denominator of $1/x$. Then, x can be represented as b/a. Since the fact that a and b are still integers that do not share a common factor does not change, x can be written as a ratio of two integers. This shows that if $1/x$ is rational, x is also rational. By proof using contradiction, this fact shows that if x is irrational, $1/x$ is also irrational.

④ 8.3.1.f

For every pair of real numbers x and y, if $x+y$ is irrational, then x is irrational or y is irrational.

↓

Proof by contrapositive

↓

For every pair of real numbers x and y, if x is rational and y is rational, $x+y$ is rational.

Rational numbers are numbers that can be written as the ratio of two integers that do not share a prime factor. Let $x = a/b$, $y = c/d$, where b and d are not zero, and a,b,c,d are integers. if we add x and y, it corresponds to $(ad+bc)/bd$. Since a, b, c, and d are all integers, and integer $*$ integer is always integer, ad, bc, and bd are all integers. Let's assume $ad = e$, $bc = f$, and $bd = g$ where e, f, g are all integers. $(e+f)/g$. Integer + Integer is also always integer, $e+f$ is also integer. Let's say $e+f=h$, where h is an integer. Therefore, $(ad+bc)/bd$ is simplified to h/g, where h and g are both integers. Because h and g are both integers, h/g, which is $x+y$, is always rational. This shows that $x+y$ is rational. By proof using contrapositive, it proves that for every pair of real numbers x and y, if $x+y$ is irrational, then x is irrational or y is irrational.

⑤ 8.4.1. a

If a group of 9 kids have won a total of 100 trophies, then at least one of the 9 kids has won at least 12 trophies.

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Proof by Contradiction

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A group of 9 kids have won a total of 100 trophies, and all of the 9 kids has won less than 12 trophies.

If a kid has won less than 12 trophies, then the most trophies he won is 11 trophies. If this applies to all nine kids, every kids has won at most 11 trophies. The maximum number of trophies they earn as a total is 9 * 11, which is 99 trophies. However, the statement says that 9 kids have earned 100 trophies, which is less than 99 trophies, with 99 being the max number of trophies 9 kids can collect. Therefore, the statement is False, which proves that If a group of 9 kids have won a total of 100 trophies, then at least one of the 9 kids has won at least 12 trophies.

⑥ If a relation is transitive, aRx and xRy means that aRy. If we apply this definition here, $a|b$ and $b|c$ means $a|c$, where $a|b$ means a divides b. If a divides b, b/a can be written as q. Since $b/a = q$, $b = qa$. Similarly, $b|c$ can be written as k. Since $c/b = k$, $c = bk$. Since $b = qa$, if we apply this to c, $c = (qa)k$, which can be writen as $c = (qk)a$. Note that q and k are integers since a can divide b and b can divide c means that there are no remainders. Since q and k are integers, and integer * integer is integer, $q*k$ is an integer. Let's say $q \cdot k = z$. Then, $a = zc$. Since there are no remainders here with z being an integer, a divides c. This is equivalent to $a|c$. Therefore, if $a|b$ and $b|c$, $a|c$.

Collaborator: Junhui Cho

(7)(a) Proof by contradiction. r_1 and r_2 are both remainders after the division of b by a and $r_1 \neq r_2$. If $r_1 \neq r_2$, either $r_2 > r_1$ or $r_1 > r_2$. Without losing generality, let's say $r_1 > r_2$. If $r_2 \neq r_1$, this also means that there is q such that $q_1 \neq q_2$. This means that if you divide b by a, there should be two different qs and rs. Therefore, $b = aq_1 + r_1$ and $b = aq_2 + r_2$. Since b itself is same, we can write it as $aq_1 + r_1 = aq_2 + r_2$. Then, we can subtract r_2 by both sides, $aq_1 + r_1 - r_2 = aq_2$. Then, we subtract both sides by aq_1. $r_1 - r_2 = aq_2 - aq_1$. Since a is also the same, the right side can be simplified into $r_1 - r_2 = a(q_2 - q_1)$. From the question, it is given that there are no integer multiples of x that are greater than 0 and less than x. From $r_1 - r_2 = a(q_2 - q_1)$, $r_1 - r_2$ is a multiple of a since q_2 and q_1 are integers that are different. (So $q_2 - q_1$ is never 0). Therefore $r_1 - r_2 \geq a$. However, the question states that $a > r_1$ and $a > r_2$. (a is greater than the remainders). Since $a > r_1$ and $r_2 > 0$, a should be greater than $a > r_1 - r_2$. This contradicts with the information that $r_1 - r_2 \geq a$. Therefore, the statement that $r_1 \neq r_2$ is false, and therefore, by proof using contradiction, if r_1 and r_2 are both remainders after division of b by a, then $r_1 = r_2$.

⑦(b) From the equation $b=qa+s$, if we subtract both sides by qa, it becomes $s=b-qa$. The task is to prove that there exists the set S such that has at least one remainder s in the set. By definition s has to be 0 or positive, so $s \geq 0$. Since $s=b-qa$, $b-qa \geq 0$. Even though a has to be greater than 0, b can be positive, zero, or negative. First, let's assume that b is positive. $b-qa \geq 0$ can be written as $b \geq qa$. In other words, as long as b is greater than or equal to qa, the requirement is met and there exists remainder s in set S. Q can be any integer. If $b \geq qa$ and b is positive, the equation is always true when $q=0$, because $b \geq 0$ with b being positive. A positive value is always greater than or equal to 0. Similarly, if $b=0$, and $q=0$, $0 \geq 0$ is true since 0 is greater than or equal to zero. If b is negative, $b-qa \geq 0$ still has to be true for any q. If $q=b$, then it can be simplified to $b-ba \geq 0$, where b is negative and a is positive by law. It can be simplified into $b(1-a) \geq 0$. Considering b is negative, $(1-a)$ has to equal 0 or negative to satisfy the condition. Since a is a positive integer $(a>0)$, the lowest value of a is 1. Then, $(1-a)$ becomes 0 and $b \cdot 0 \geq 0$ is true because a number multiplied by 0 is 0 and 0 is always greater than equal to 0. Also, if a is greater than 1, $(1-a)$ becomes negative always. If $(1-a)$ is negative and b is negative, $(1-a) * b$ is always going to be positive. A positive value is always greater than or equal to 0. Therefore, for any b (whether b is positive, 0, or negative), there exists q that makes remainder s greater than or equal to 0, so the set S always contains at least one remainder.