CAS CS 365
Lec 22

Vector Calculus (cont.)

1. Chain rule
   a. $$\frac{\partial}{\partial x}(g \circ f)(x) = \frac{\partial}{\partial x} g\left(f(x)\right) = \frac{\partial g}{\partial f}\frac{\partial f}{\partial x}$$
   b. Consider a function f:$R^2 \to R$ of two variables x_1,x_2. Furthermore, suppose that x_1,x_2 are functions of a variable t.
      $$\frac{df}{dt} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1(t)}{\partial t} \\ \frac{\partial x_2(t)}{\partial t} \end{bmatrix}$$
   c. Consider a function f:$R^2 \to R$ of two variables x_1,x_2. Furthermore, suppose that x_1,x_2 are functions of a variable s, t.
      $$Let\, q = [s, t].\ \frac{df}{dq} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1(s,t)}{\partial s} & \frac{\partial x_1(s,t)}{\partial t} \\ \frac{\partial x_2(s,t)}{\partial s} & \frac{\partial x_2(s,t)}{\partial t} \end{bmatrix}$$
2. Jacobian
   a. $$f(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \ldots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$$
   b. The collection of all first-order derivatives of a vector field/vector-valued function f:R^n → R^m is called the Jacobian
   c. $$\boldsymbol{J} = \nabla_{\boldsymbol{x}} \boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}} = \begin{bmatrix} \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\boldsymbol{x})}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_m(\boldsymbol{x})}{\partial x_n} \end{bmatrix},$$
   d. Let $y_1 = -2x_1 + x_2$, $y_2 = x_1 + x_2$. The Jacobian is simply
      $$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$
   e. This example generalizes to the following. Let f(x) = Ax, where A is a m*n matrix, and x is an mx1 vector.

Lec 22

$f: \mathbb{R}^n \to \mathbb{R}^m$ ex) $f(x) = A^{m \times n} \cdot x$

Jacobian

$f: \mathbb{R}^n \to \mathbb{R}^m$

$(x_1, \dots, x_n) \mapsto (y_1, \dots, y_m)$ → function of original vector

$$f(x_1, \dots, x_n) = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$J^{m \times n} = \begin{bmatrix} \left(\frac{dy_1}{dx_1} \cdots \frac{dy_1}{dx_n}\right) \\ \vdots \\ \left(\frac{dy_n}{dx_1} \cdots \frac{dy_n}{dx_m}\right) \end{bmatrix}$$

ex) $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{2 \times 2}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$f(x) = Ax$

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$$J = \begin{bmatrix} \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} \\ \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} \end{bmatrix} = A$$

$y_1 = 3x_1 + 5x_2$

$y_2 = -x_1 + 4x_2$

$\frac{dy_1}{dx_1} = 3 \quad \frac{dy_2}{dx_1} = -1 \quad \frac{dy_1}{dx_2} = 5 \quad \frac{dy_2}{dx_2} = 4$

$f(x) = Ax \quad f: \mathbb{R}^n \to \mathbb{R}^m$

$J = A = \nabla f$

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

$f_i(x) = A_{i1} x_1 + \dots + A_{in} x_n$

$\frac{df_i}{dx_j} = A_{ij}$

$f(x_1, x_2)$, $x_1(t)$, $x_2(t)$ $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$\nabla f = \frac{\partial f}{\partial x}^{1\times 2}$

$$\frac{df}{dt} = \frac{df}{dx}^{1\times2}\left(\frac{dx}{dt}\right)^{2\times1}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix}$$

$\mathbb{R}^2 \to \mathbb{R}$ $\quad$ $\mathbb{R} \to \mathbb{R}^2$

ex) $t \in [0, 2\pi]$
$x : \mathbb{R}^1 \to \mathbb{R}^2$
$x = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2}\frac{dx_2}{dt}$$

ex) $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$
$x_1(t) = \sin(t)$ $\quad x_2(t) = \cos(t)$

$$\frac{df}{dt} = (2x_1 + 2x_2)(\cos(t)) + (2x_1 + 2x_2)(-3\sin(t))$$
$$= 2\sin(t)\cos(t) + 2\cos^2(t) - 6\sin^2(t) - 6\cos(t)\sin(t)$$
$$= -4\sin(t)\cos(t) + 2\cos^2(t) - 6\sin^2(t)$$

ex) $x_1(u,v) = 3u + 4v$ $\quad f : \mathbb{R}^2 \to \mathbb{R}$
$x_2(u,v) = -3u + v$ $\quad x : \mathbb{R}^2 \to \mathbb{R}^2$

$$\frac{\partial f}{\partial \vec{q}} = \frac{\partial f}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \vec{q}}$$

$q = \begin{bmatrix} u \\ v \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -3 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\vec{q}}$

$f(x_1, x_2) = (x_1 + x_2)^2$ $\quad f : \mathbb{R}^2 \to \mathbb{R}$
$f(\vec{x}) = f(A\vec{q})$ $\quad (\nabla f)^{1\times 2}$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \frac{\partial f}{\partial x}$$

$$\begin{bmatrix} 2(x_1+x_2) & 2(x_1+x_2) \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 10(x_1+x_2) \end{bmatrix}^{1\times 2}$$

$$\overset{1\times n}{\left(\frac{\partial z}{\partial t}\right)} = \overset{\text{gradient}}{\overset{1\times m}{\left(\frac{\partial z}{\partial x}\right)}} \overset{m\times n}{\left(\frac{\partial x}{\partial t}\right)} \rightarrow \text{jacobian}$$

$z: \mathbb{R}^m \rightarrow \mathbb{R}$ (scalar field)

$x: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (differential function of n independent variables)

$$\left(\frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_n}\right) \cdot J^{m\times n}$$

$$J^{m\times n} = \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_n} \end{bmatrix}$$

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$

$f'(x) = 3x^2$ $\boxed{\text{Saddle}}$

$f''(x) = 6x$

↳ sign of $f''(x)$ changes