CAS CS 365
Lec 13

Graphs

1. Goals of this week
    a. $G(n,p)$ model
    b. First and second moment for phase transitions
    c. Exercises on $E[x]$, $Var[x]$ where X is a RV in G
2. Isolated Nodes
    a. There exist a p^* value that if $p > p^*$ in $G(n,p)$, there are no isolated nodes
    b. $E[x] = n(1-p)^{(n-1)}$
        i. $X1 = 1$ if node i is isolated, 0 otherwise
        ii. $X = X1 + \ldots + Xn$
        iii. 1st moment method by using Markov's inequality
            1. $Pr(X > 1) < E[x]$ and if $E[x]$ goes to 0 as $n \to$ infinity, $Pr(X > 0)$ is 0
            2. Therefore, $Pr(X == 0) = 1 - O(1)$
    c. Asymptotic
        i. For small p, $1-p = e^{-p}$
           $(1-p) = e^{\ln(1-p)}$
           $= e^{(-p + p^2/2 + \ldots)}$ by using Taylor Series, which is $e^{(-p)}$
        ii. Lim as $n \to$ infinity, $(1+1/n)^n$
           $= e^{(n \cdot \ln(1+1/n))} = e^1 = e$
    d. $E[x] = n(1-p)^{(n-1)} = n \cdot e^{(-p(n-1))}$
       $= n \cdot e^{(-c\ln(n)/n \cdot (n-1)}$
       $= n \cdot e(-c\ln(n))$
       $= n \cdot n^{(-c)}$
       $= n^{(1-c)}$
    e. So if $c > 1$, then $E[x] = o(1)$, so by Markov's inequality,
       $Pr(X > 1) <= E(x) = O(1)$
       Therefore, $X = 0$ with probability $1 - O(1)$
    f. $Z = 0$ if $1-1/n = 1 - O(1)$
       $= n^2$ if $1/n$
    g. $E[z] = n$
    h. If $p < p^* = \ln(n)/n$, then $X>0$ with high probability
       $Pr(X=0) <= Pr(|X-E[X]| >= E[X]) <= Var(X)/E[X]^2$ as $n \to$ infinity by using Chebyshev inequality
    i. $Var(X) = Var(X1 + \ldots + Xn)$
       $= \sum i = 1$ to n $Var(Xi) + \sum$ (i does not equal j) $Cov(Xi, Xj)$

$\le \sum_{i=1 \text{ to } n} E(X_i) + \sum_{(i \text{ does not equal } j)} Cov(X_i, X_j)$
$= E[X] + \sum_{(i \text{ does not equal } j)} Cov(X_i, X_j)$

** proof **
$X_i = 1$ with probability $(1-p)^{(n-1)}$
$= 0$ with probability $1-(1-p)^{(n-1)}$

$E[X_i] = (1-p)^{(n-1)}$
$Var[X_i] = (1-p)^{(n-1)} * (1-(1-p)^{(n-1)}) \le (1-p)^{(n-1)} = E[X_i]$ since $(1-(1-p)^{(n-1)})$ is at most 1

**End proof **

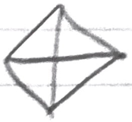
$$\frac{Var(X)}{E[x]^2} \le \frac{1}{E[x]} + \frac{\sum Cov()}{E[x]^2}$$

j. $Cov(X_i, X_j)$
$= E(X_i * X_j) - E[X_i] * E[X_j]$
$= E[X_i * X_j] - (1-p)^{(n-1)} * (1-p)^{(n-1)}$
$= E[X_i * X_j] - (1-p)^{2(n-1)}$
$= Pr[X_i = X_j = 1] - (1-p)^{2(n-1)}$
$= (1-p)^{(2(n-2)+1)} - (1-p)^{2(n-1)}$
$\le (1-p)^{(2(n-2)+1)}$

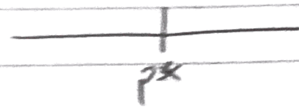
k. $\sum_{(i \text{ does not equal } j)} Cov[X_i, X_j]$
$= n(n-1)*((1-p)^{(2(n-2)+1)} - (1-p)^{2(n-1)})$
$\le n*(n-1) * (1-p)^{(2(n-2)+1)}$

l. Now, we have to prove $\sum Cov()/E[x]^2$ goes to 0
$$\frac{(n*(n-1) * (1-p)^{(2(n-2)+1)})^2}{n^2(1-p)^{(2(n-2))}} \text{ goes to } 0$$

3.



$K_4 \to 4$ nodes and $\binom{4}{2}$ edges



Let X to the # of K_4's in $G(n,p)$

$X = \sum_{i=1}^{\binom{n}{4}} X_i$ where $X_i \begin{cases} 1 & \text{if the } i\text{-th 4-cliqus or} \\ 0 & \text{otherwise} \end{cases}$ 

$$E[X] = \sum_{i=1}^{\binom{n}{4}} E(x_i) = \binom{n}{4} p^6$$

↳ since $Pr(X_i = 1) = p^6$ for all i

 → 6 edges connected

- what value of p makes $\binom{n}{4} p^6$ go to 0?

↳ threshold value for p^*

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4!} \leq n^4$$

$$\binom{n}{4} p^6 \leq n^4 p^6 \to 0$$

$$n^4 p^6 \ll 1$$

$$p \ll \underline{n^{-2/3}}$$

↳ threshold value to guess

Prove

If $p \ll n^{-2/3}$, then $X=0$ with high probability

$E(X) = \binom{n}{4} p^6 \leq n^4 p^6 \ll 1$

By Markov's inequality, $\Pr(X \geq 1) \leq \underbrace{E[X]}_{\text{goes to } 0}$

$p = o(n^{-2/3})$

$\hookrightarrow p = \dfrac{1}{n^{2/3} q(n)}$ such that $q(n) \to \infty$ as $n \to \infty$

$p = \omega(n^{2/3})$

$\hookrightarrow p = \dfrac{q(n)}{n^{2/3}}$

$$n^4 p^6 = n^4 \left(\frac{1}{n^{2/3} q(n)}\right)^6$$

$$= n^4 \left(\frac{1}{n^4 q(n)^6}\right) = \frac{1}{q(n)^6} \to 0$$

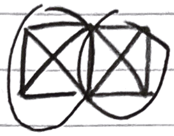
$$Var(X) \leq E(X) + \sum_{i \neq j} Cov(X_i, X_j)$$

$$\frac{Var(X)}{E(X)^2} \leq \frac{E(X)}{E(X)^2} + \frac{\sum_{i \neq j} Cov(X_i, X_j)}{E(X)^2} \to 0$$

$$\leq \frac{1}{E(X)} + \frac{\sum Cov(X_i, X_j)}{E(X)^2} \to 0$$

$\hookrightarrow$ Cov is not 0 when two share two or more nodes

① Share two nodes



$x_i \quad x_j \qquad Cov(x_i, x_j) = E(x_i, x_j) - E(x_i)E(x_j)$

$= p^{11} - p^6 \cdot p^6$

$= p^{11} - p^{12}$

$\binom{n}{6}\binom{6}{2}$

↳ possible 

$\leq n^6$

② Share three nodes

$Cov(x_i, x_j) = E(x_i x_j) - E(x_i)E(x_j)$

$= p^9 - p^{12}$

$$\sum_{i \neq j} Cov(x_i, x_j) \leq n^6(p^{11} - p^{12}) + n^5(p^9 - p^{12})$$

$$\leq n^6 p^{11} + n^5 p^9$$

$$\frac{\sum_{i \neq j} Cov(x_i, x_j)}{E(x)^2} \leq \frac{n^6 p^{11} + n^5 p^9}{n^8 p^{12}}$$

- The covariance of x_i is variance

$$\frac{Var(x)}{E(x)^2} \leq \frac{1}{E(x)} + \frac{n^6 p^{11} + n^5 p^9}{n^8 p^{12}}$$

if $p \gg n^{-2/3}$, then $\frac{Var(x)}{E(x)^2} \to 0$

$p = \frac{q(n)}{n^{2/3}}$ such that $q(n) \to \infty$

$$\leq \frac{1}{q(n)^6} + \frac{1}{n^2 p} + \frac{1}{n^3 p^3}$$

↳ goes to 0