1.a

An example of random variable for which chebyshev's inequality is tight is when $x=0$ for probability 0.5, and $x=1$ with probability 0.5. This is bernoulli random variable.

Chebyshev's inequality tells that
$$\Pr(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

In the Bernoulli random variable above,
$$\mu = 0(0.5) + (1)(0.5) = 0.5$$
$$\sigma^2 = p(1-p) = 0.5(0.5) = 0.25$$

When we want to find $\Pr(X \leq 0 \text{ or } X \geq 1)$, the actual probability of it occurring is 1 because no matter what you choose, you are getting 0 with $p=0.5$ and 1 with $p=0.5$.

If we solve in Chebyshev's inequality, we get
$$\Pr(X \leq 0 \cup X \geq 1)$$
$$= \Pr(|X-\tfrac{1}{2}| \geq \tfrac{1}{2}) \leq \frac{Var(X)}{(1/2)^2} \quad \text{where } Var(X) = 0.25$$
$$\therefore \Pr(X \leq 0 \cup X \geq 1) = \Pr(|X-\tfrac{1}{2}| \geq \tfrac{1}{2}) \leq \frac{1/4}{1/4} = 1,$$

which is same as the actual probability, implying that the inequality holds as equality in this example.

1.b

There are total of n chains, which means that there are total of n-1 sending of bits from one chain to another.

Each sending of bit has probability of p to be flipped.

In order for the bit to arrive to An correctly, there must be only even # of flips to occur. This is because if the bit b is flipped to b', it should be flipped again to b, which means that there must be even # of flips.

It is also important that the order (whether the bit is flipped in A_3 or A_4) does not matter, meaning that the problem can be written as binomial distribution.

All we need to do is add all the binomial distribution for even # of flips, which can be done by using sigma notation.

$$\sum_{k=0}^{k=(n-1)/2} {}_{n-1}C_{2k} (p)^{2k} (1-p)^{(n-1-2k)}$$

since we only care about even # of flips, we use 2K and let the upper bound of the notation as $(n-1)/2$.

1.c

A first makes a statement that is true with probability p and then b confirms the statement of A with probability p.

Since we are interested in calculating the probability that A is telling the truth that comes from B, we are looking for Pr(A|B).

Even though B is confirming A's statement, B does it with independently with whatever A says. This is because no matter what A says, B independently confirms that A is correct or wrong.

$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)}$, where A = event that person A tells the truth, B = event that person B confirms the statement from Person A

The case where A and B both happens is when A tells the truth and B confirms A's statement, which happens with $p \cdot p = p^2$.

$\therefore P(A \cap B) = p^2$

When considering all the cases where the statement from A and B arrives as truth, it not only includes the case when they both tell the truth but also is possible when A and B both lie, meaning that $P(B) = p^2 + (1-p)^2$.

$$\boxed{\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{p^2}{p^2+(1-p)^2} = \frac{p^2}{p^2+p^2-2p+1} = \frac{p^2}{2p^2-2p+1}}$$

1.d

Considering X, Y, Z as uniform random variables in $[0,1]$, we want to find $P(X+Y+Z \leq 1)$.

This means that we have to integral $1 \cdot dz \cdot dy \cdot dx$ (integral three times).

Now, we have to figure out the bounds of the integral. We can write z as $z = 1-y-x$ and y as $y = 1-x$. Finally, the bound of x, the last integral, is from $0, 1$ as it is the bound given by the question.

$\therefore$ We get

$$\int_0^1 \int_0^{1-x} \int_0^{1-y-x} 1 \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} 1-y-x \, dy \, dx = \int_0^1 \left[y - xy - \frac{1}{2}y^2\right]_0^{1-x} dx$$

$$= \int_0^1 1-x-x(1-x)-\frac{1}{2}(1-x)^2 \, dx = \int_0^1 1-x-x+x^2-\frac{1}{2}(x^2-2x+1) \, dx$$

$$= \int_0^1 \frac{1}{2}x^2 - x + \frac{1}{2} \, dx$$

$$= \left[\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x\right]_0^1 = \frac{1}{6} - \frac{1}{2} + \frac{1}{2} = 1/6.$$

$\therefore Pr(x+y+z \leq 1) = 1/6.$