① We are given that $b_1$ has a unique remainder $r_1$ and $b_2$ has a unique remainder $r_2$ when divided by $a$. Therefore, $b_1 = aq_1 + r_1$ and $b_2 = aq_2 + r_2$. When $(b_1 + b_2)$ is divided by $a$, it is same as $b_1/a + b_2/a$. Since we know that $b_1$ is $aq_1 + r_1$ and $b_2$ is $aq_2 + r_2$ when divided by $a$, $b_1 + b_2 = aq_1 + r_1 + aq_2 + r_2$. Therefore, $b_1 + b_2 = a(q_1 + q_2) + r_1 + r_2$, where $q_1 + q_2$ is the quotient and $r_1 + r_2$ is the remainder. From the fact that we divide $b_1$ and $b_2$ by $a$, $r_1 < a$ and $r_2 < a$. However, this fact that both $r_1$ and $r_2$ are less than $a$ does not always mean that $(r_1 + r_2)$ is less than $a$. Therefore, there are two Cases: $r_1 + r_2 \geq a$ and $r_1 + r_2 < a$. This covers all the cases that $r_1 + r_2$ can be. When $r_1 + r_2 < a$, $r_1 + r_2$ divided by $a$ will give the quotient $0$ and cannot be factored by $a$. Therefore, in that case, $(b_1 + b_2)$ divided by $a$ will give remainder $(r_1 + r_2)$. When $r_1 + r_2 \geq a$, $r_1 + r_2$ cannot be greater than $2a$ because each $r_1 < a$ and $r_2 < a$. Therefore, if $r_1 + r_2$ is greater than or equal to $a$, it is $a \leq r_1 + r_2 < 2a$. Since $r_1$ and $r_2$ added is greater than or equal to $a$, $(r_1 + r_2)/a$ has quotient of $1$ and remainder of $r_1 + r_2 - a$. Therefore, $(r_1 + r_2)/a$ becomes $a(1) + r_2 + r_1 - a$. Then, $b_1 + b_2$ divided by $a$ is $a(q_1 + q_2 + 1) + r_2 + r_1 - a$. Since $r_2 + r_1 - a$ cannot be divided by $a$ once again, $r_2 + r_1 - a$ is the remainder if $r_2 + r_1 \geq a$. Finally, if $r_2 + r_1 \geq a$, then the remainder is $r_2 + r_1 - a$ and if $r_2 + r_1 < a$, then the remainder is $r_2 + r_1$.

(2) (a) For any $c$, the ceiling of $\lceil c \rceil$ is unique. It is given that $c$ can be represented by $d-z$, where $d$ is an integer and $z$ is a value from 0 to 1 (0 being inclusive and 1 being exclusive). Since $z$ is 0 or always positive $c$ is always less than or equal to $d$. ($c$ if $c+0=d$, $c=d$ and if $c+$ positive value $=d$, $c<d$, so $c \le d$ always). At the same time, $c$ is always greater than $d-1$, because $z$ is never greater or equal to 1. Therefore, $d-1 < c \le d$.

Proof by contradiction

↳ $\lceil c \rceil$ is not unique. This means that there exists at least two $\lceil c \rceil$ for one value of $c$. This means that there can exist more than one $d$ such that $d-1<c\le d$. Then, there can exist $x$ and $y$ such that $x-1<c\le x$ and $y-1<c\le y$. Since $y$ and $x$ are both greater than or equal to $c$, they can substitute each other and become $x-1<c\le y$ and $y-1<c\le x$. Here, these two equations imply that $x-1<y$ and $y-1<x$. This means that $x-y<1$ and $y-x<1$. Since $y$ and $x$ are both integers ($d$ was an integer) and $x-y$ has to be less than 1, it can only be 0. If $x-y=0$, $x$ has to equal $y$. However, $x$ and $y$ have to be different since there can exist more than one $d$ such that $d-1<c\le d$. Here is contradiction.

Since there cannot exist two or more $d$ such that $d-1<c\le d$, there must be only one integer $d$, which means that $d$ is unique. Since $\lceil c \rceil = d$ and $d$ is unique, $\lceil c \rceil$ is also unique.

② (b) For all a and b, a and b can be represented by $a = d - z$ and $b = m - n$ where d and m are integers and z and n are values within 0 and 1 (0 is inclusive and 1 is exclusive). $\lceil a+b \rceil = \lceil d+m-z-n \rceil$.

- Lemma: $\lceil d+m \rceil = d + \lceil m \rceil$ when d is an integer and m is any number.
Given that we have $\lceil m \rceil$, m is $m = x - n$, where x is an integer and n is a value from $0 \le n < 1$. So, x minus a positive number or 0 is m, which means that m is always less than or equal to x. $m \le x$. Also, since z is less than 1, m is always greater than $x-1$. $x-1 < m \le x$, where $\lceil m \rceil = x$. If you add an integer d to all sides, it becomes $x+d-1 < m+d \le x+d$. By the definition of ceiling, $\lceil m+d \rceil = x+d$. However, since $x = \lceil m \rceil$, $\lceil m+d \rceil$ is same as $\lceil m \rceil + d$. $\lceil m+d \rceil = \lceil m \rceil + d$.

By this Lemma, since $m+d$ is an integer, $\lceil a+b \rceil = d+m+\lceil -z-n \rceil$. On the right side, $\lceil a \rceil + \lceil b \rceil$ becomes $\lceil d-z \rceil + \lceil m-n \rceil$, which means $d+m+\lceil -z \rceil + \lceil -n \rceil$. Since $d+m$ is on both sides, we can take out $d+m$. So, we need to prove that $\lceil a+b \rceil = \lceil a \rceil + \lceil b \rceil$ or $\lceil a \rceil + \lceil b \rceil - 1$, which now is $\lceil -z-n \rceil = \lceil -z \rceil + \lceil -n \rceil$ or $\lceil -z \rceil + \lceil -n \rceil - 1$. Since z and n cannot exceed 1 each, $z+n$ cannot be greater than 2. There are two cases: $0 \le z+n < 1$ and $1 \le z+n < 2$. Since $\lceil -z-n \rceil$ is same as $\lceil -(z+n) \rceil$, if $0 \le z+n < 1$, $\lceil -(z+n) \rceil$ is a negative value between 0 and 1 (0 is inclusive and 1 is exclusive), that is ceiled, such that $-z-n = x-y$, where x is integer and y is value between 0 and 1 (0 is inclusive and 1 being exclusive). Here, x is 0, and $y = z+n$. Therefore, $\lceil -z-n \rceil = 0$ when $0 \le z+n < 1$. On the right side, $\lceil -z \rceil$ and $\lceil -n \rceil$ are all values from $0 \le z < 1$ and $0 \le n < 1$, so, $-z = 0 - z$ and $-n = 0 - n$. So, $\lceil -z \rceil$ and $\lceil -n \rceil$ are both 0 and $0+0$ is 0. Since $0 = 0$, this case passes. At the second case, if $1 \le z+n < 2$, then $-z-n$ can be represented by $p-q$ where p is a integer and $0 \le q < 1$. In this case, $-z-n = -1 + (-z-n+1)$. Therefore, $\lceil -z-n \rceil = -1$ and since $\lceil -z \rceil + \lceil -n \rceil = 0$, $\lceil -z \rceil + \lceil -n \rceil - 1 = -1$ and in this case, $\lceil -z-n \rceil = \lceil -z \rceil + \lceil -n \rceil - 1$. Finally, if z and n is $0 \le z+n < 1$, $\lceil a+b \rceil = \lceil a \rceil + \lceil b \rceil$, and if z and n is $1 \le z+n < 2$, $\lceil a+b \rceil = \lceil a \rceil + \lceil b \rceil - 1$.

③ Proof by Contradiction

$\hookrightarrow \exists n \in \mathbb{N}, \sum_{i=0}^{n} 2^i \neq 2^{n+1}-1.$

Since there exists $n$ such that $\sum_{i=0}^{n} 2^i \neq 2^{n+1}-1$, there is at least one $n$ in natural numbers that $\sum_{i=0}^{n} 2^i \neq 2^{n+1}-1$. Let's call the set that has all collections of $n$ that doesn't qualify $\sum_{i=0}^{n} 2^i \neq 2^{n+1}-1$ as $S$. Since $n$ is a natural number, there exists an integer (positive) $k$ such that is the lowest number in set $S$. First of all, if $n=1$, $2^0+2^1=2^2-1$ $(2+1=4-1)$ and if $n=2$, $2^0+2^1+2^2=2^3-1$ $(1+2+4=8-1)$, and if $n=3$, $2^0+2^1+2^2+2^3=2^4-1$ $(1+2+4+8=16-1)$ are all true, so $n=1,2,3$ is not inside $S$ and therefore is not $k$. Knowing that $k>3$, $\sum_{i=0}^{k} 2^k \neq 2^{k+1}-1$ can be expanded to $1+2+4+8+\dots+2^k \neq 2^{k+1}-1$. $2^{k+1}$ is equal to $2(2^k)$ and therefore $1+2+4+8+\dots+2^k \neq 2^k(2)-1$. If we divide all sides by 2, the new equation is $\frac{1}{2}+1+2+4+\dots+2^{k-1} \neq 2^k-1/2$, where $2(\frac{1}{2}+1+2+4+\dots+2^{k-1})$ equals $(1+2+4+8+\dots+2^k)$ and $2(2^k-1/2)$ equals $(2^{k+1}+1)$. Since $1+2+4+8+\dots+2^k$ doesn't equal $2^{k+1}-1$, the value when both sides are divided by 2 must also not equal each other

$\hookrightarrow \frac{1}{2}+1+2+4+\dots+2^{k-1} \neq 2^k-\frac{1}{2}$. However, if this is true, $k$ is not the lowest number in $S$ that the equation is false. It is $k-1$ that is the lowest. Therefore, here is contradiction.

$\hookleftarrow$

Since $\exists n \in \mathbb{N}, \sum_{i=0}^{n} 2^i \neq 2^{n+1}-1$ is false, $\forall n \in \mathbb{N}, \sum_{i=0}^{n} 2^i = 2^{n+1}-1$ is always true.