CAS CS 365
Lec 20

Vector Calculus

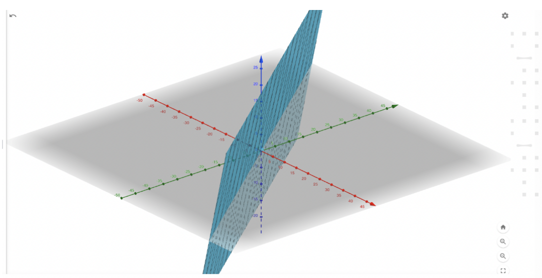
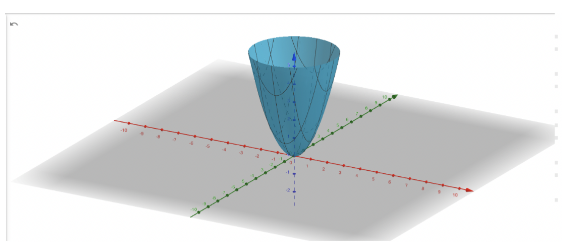
1. Plotting f: R^2 → R
    a. Consider a vector p = [x,y]
    b. How do we plot functions of p such as the following:
    c.

$$z = [4,3]p = 4x + 3y$$

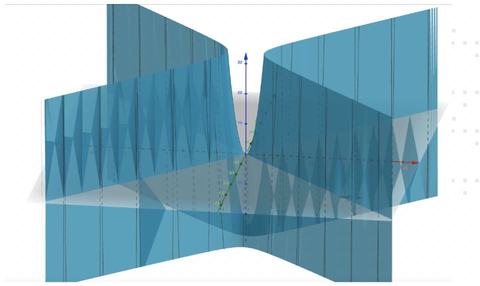
$$z = p^T p = x^2 + y^2$$

$$z = p^T A p = [x,\ y]\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 + y^2$$

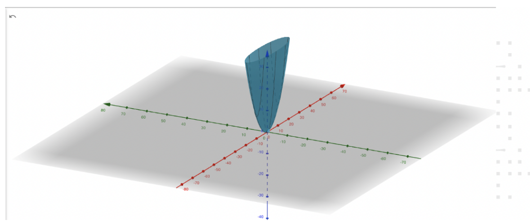
$$z = p^T A p = [x,\ y]\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + y^2$$

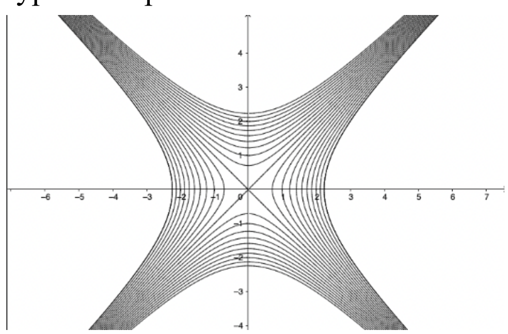
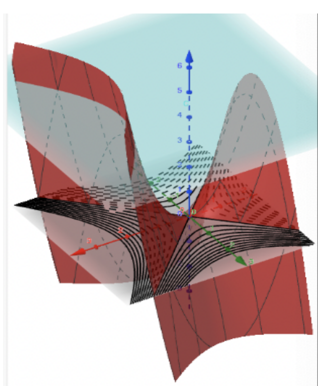
    d. 4x + 3y
        i. 
    e. x^2 + y^2
        i. 
        ii. Getting level curves equal to circle → x^2 +y^2 = c^2

f. x^2-y^2

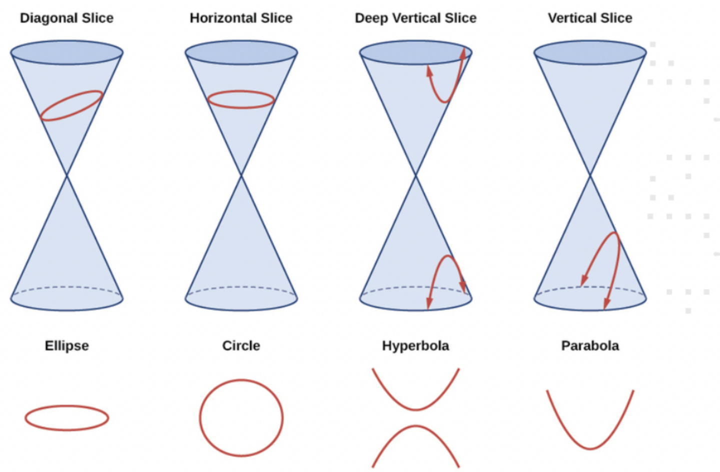
i. 

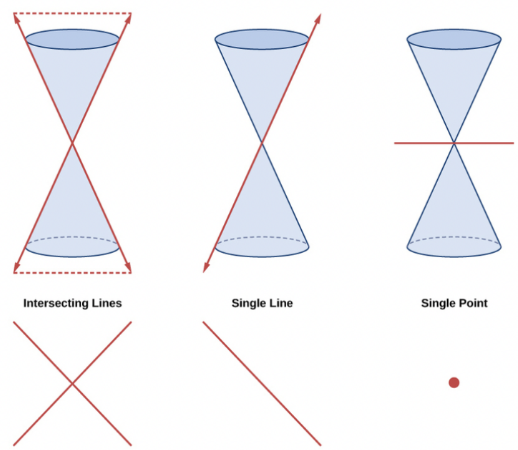
g. $0.1x^2 + 2y^2$

i. 

2. Level curves
   a. The level curves of a function f of two variables x, y are the curves with equation $f(x,y) = c$, where c is a constant in the range of f
   b. $x^2 - y^2$
      i. Hyperbolic paraboloid
      ii. 
      iii. 
      iv. $x^2-y^2 = k$
      v. When $k = 0$, $x=y$, $x= -y$

3. Conic sections

a. 

b. 

4. General form of conic sections

a. $$Ax^2+Bxy+Cy^2+Dx+Ey+F = 0$$

b. Identify the values of A and C from the general form

   i. If A and C are nonzero, have the same sign, and are not equal to each other, then the graph may be an ellipse

   ii. If A and C are equal and nonzero and have the same sign, then the graph may be a circle

   iii. If A and C are nonzero and have opposite signs, then the graph may be a hyperbola

   iv. If either A or C is zero, then the graph may be a parabola

c. Example

i.

Conic Sections	Example
ellipse	$4x^2 + 9y^2 = 1$
circle	$4x^2 + 4y^2 = 1$
hyperbola	$4x^2 - 9y^2 = 1$
parabola	$4x^2 = 9y$ or $4y^2 = 9x$

5. Back to our hyperbolic paraboloid

a.

$$f(x,y) = x^2 - y^2 = 0 \Rightarrow (x-y)\cdot(x+y) = 0$$

$$f(x,y) = c \Rightarrow \frac{x^2}{c} - \frac{y^2}{c} = 1 \text{ (Hyperbola!)}$$

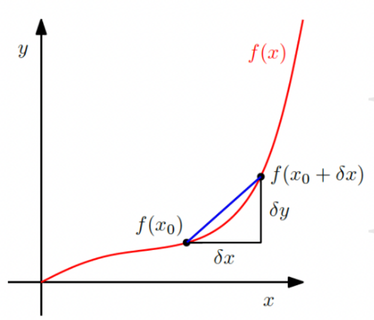
6. A refresher 1: Single variable function
   a. The difference quotient computes the slope of the secant line through two points of y = f(x)
   b.

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

   c. The idea of the derivative f'(x) is that it is the slope of the tangent line at x to the curve
   d.

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

   e.



7. A refresher 2: Single variable function

   a.
$$\text{Product rule:} \quad (f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \tag{5.29}$$
$$\text{Quotient rule:} \quad \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \tag{5.30}$$
$$\text{Sum rule:} \quad (f(x) + g(x))' = f'(x) + g'(x) \tag{5.31}$$
$$\text{Chain rule:} \quad \big(g(f(x))\big)' = (g \circ f)'(x) = g'(f(x))f'(x) \tag{5.32}$$

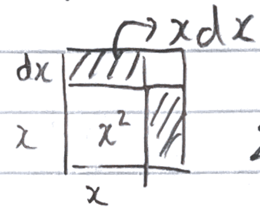
8. Matrix Calculus
   a. Scalar field, a function f that maps vectors to reals $f : \mathbb{R}^n \to \mathbb{R}$
   b. $z = [4, 3]p = 4x + 3y$
   c. $z = p^T p = x^2 + y^2$
   d. Vector field, or vector valued functions $f : \mathbb{R}^n \to \mathbb{R}^m$
   e. Functions of matrices f(A)
      i. f(A) = e^(10*A)
      ii. f(A) = A + A^2
9. Gradient of a scalar field
   a. Partial derivative at x = (x_1,...,x_n)
   b. $\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}, \; i = 1, \ldots, n$
   c. We collect them at the row vector known as the gradient of the function f
   d. $\nabla f(x) = \nabla_x f = \mathrm{grad} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{1 \times n}$
   e. Remark: the gradient collects the slopes in the positive x_i direction for all i = 1…n
10.

Lec 20

$f(x) = x^3 \quad f'(x) = 3x^2$

$f(x) = x^2 \quad f'(x) = 2x$

geographical



$2xdx + (dx)^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x - h = 2x$$

$f(x) = x^n \qquad f'(x) = \frac{df}{dx} = nx^{n-1}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - (x)^n}{h} = \frac{x^n + nx^{n-1}h + \dots) - x^n}{h}$$

$$= \frac{\sum_{k=0}^{n} {}_nC_k \cdot x^{n-k} h^k - x^n}{h} = \frac{{}_nC_1 x^{n-1} \cdot h + {}_nC_2 x^{n-2} h^2 + \dots}{h}$$

$$= \lim_{h \to 0} {}_nC_1 x^{n-1} + {}_nC_2 x^{n-2} h + \dots = {}_nC_1 x^{n-1}$$

$$= n x^{n-1}$$

$f(p) = \|p\|_2^2 \geq 0 \qquad p = \binom{x}{y}$

$f(p) = c^2 = x^2 + y^2$

↳ circle

- gradient of f at $\binom{0}{0}$

$f(x,y) = ye^x + xy + y^2 - 3$

$\frac{\partial f}{\partial x} = ye^x + y \qquad \frac{\partial f}{\partial y} = e^x + x + 2y$

$\nabla f(x,y) = (ye^x + y,\ e^x + x + 2y)$

$\nabla f(0,0) = (0,1)$