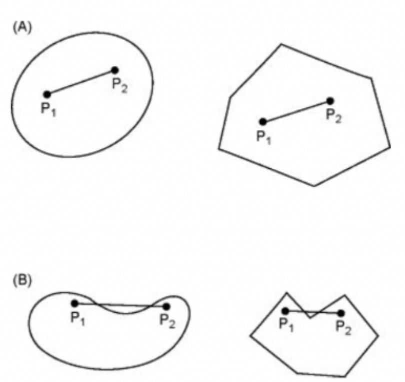
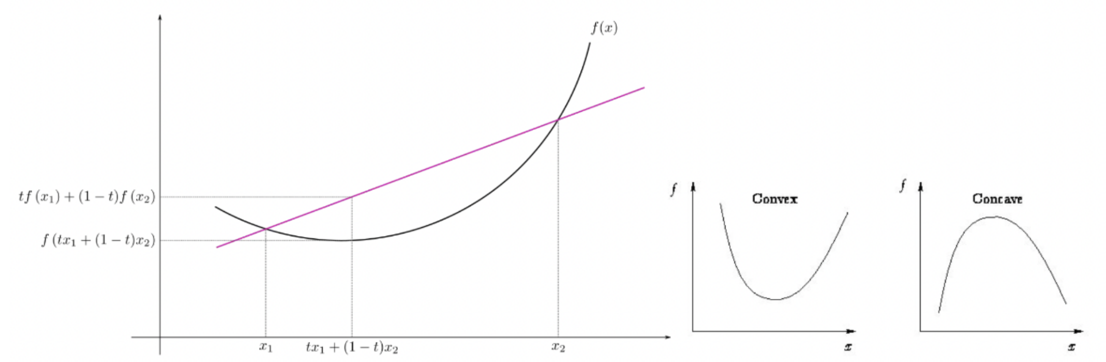
CAS CS 365
Lab 11

1. Gradient descent
    a. Gradient descent is a first-order iterative optimization algorithm for finding a local minimum of a differentiable function
    b. $x_{n+1} = x_n - \gamma \nabla F(x_n),\ where\, \gamma\, is\, step\, size$
    c. Given a function F(x) = x_1^2 + x_2^2
        i. Assume we start from position (1,1) and trying to find a local minimum, which direction should we go?
        ii. Why step size affect the result?
2. Convex Set
    a. A set C is convex if the line segment between any two points in C lies in C
    b. $\forall x_1, x_2 \in C, \forall \theta \in [0,1],\ \theta x_1 + (1-\theta)x_2 \in C$
    c. Some extreme examples
        i. The empty set
        ii. The singleton set
        iii. The complete set
    d. 

3. Prove the convexity of the following sets
    a. The unit ball $\{x : ||x|| \leq 1\}$
    b. Let A be an m by m PSD matrix, for any a >= 0, the set $\{x \in \mathbb{R}^m : x^T Ax \leq a\}$
4. Convex function
    a. A function is convex if its domain is a convex set and

$$\forall x, y \in dom(f), \forall \theta \in [0,1]$$

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

b. 

c. All linear functions are convex and concave
d. For any real value a, e^(ax) is convex
e. x^a is convex when a > 1 or a <= 0, and concave when 0 <= a <= 1

Lab 11

For any $x_1, x_2 \in C \Rightarrow \begin{cases} \|x_1\|_2 \leq 1 & \|x_1\|^2 \leq 1 \\ \|x_2\|_2 \leq 1 & \|x_2\|^2 \leq 1 \end{cases}$

$\forall \theta \in [0,1] \Rightarrow \|\theta x_1 + (1-\theta)x_2\|_2^2 \leq 1$

$\Rightarrow \theta^2 \underbrace{\|x_1\|^2}_{\leq 1} + (1-\theta)^2 \underbrace{\|x_2\|^2}_{\leq 1} + 2\theta(1-\theta)(x_1 x_2) \leq 1$

$\Rightarrow LHS \leq \theta^2 + (1-\theta)^2 + 2\theta(1-\theta)(x_1 x_2) \leq 1$

$\|x_1 - x_2\|^2 \geq 0$

$\hookrightarrow x_1^2 - 2x_1x_2 + x_2^2 \geq 0$

$x_1^2 + x_2^2 \geq 2x_1x_2$

$\hookrightarrow x_1^2 + x_2^2 \leq 2$ from the condition.

Therefore,

$LHS \leq \theta^2 + (1-\theta)^2 + \theta(1-\theta)(2)$

$\leq \theta^2 + 1 - 2\theta + \theta^2 + (\theta - \theta^2)(2)$

$\leq 2\theta^2 - 2\theta + 1 + 2\theta - 2\theta^2$

≤ 1

$$x^T A x = \sum_{i,j} a_j \cdot x_i \cdot x_j$$

$$\begin{cases} x^T A x \leq a \\ y^T A y \leq a \end{cases} \qquad z = \theta x + (1-\theta) y \Rightarrow z^T A z \leq a$$

$$\forall \theta \in [0,1]$$

$$\sum a_{ij} (\theta x_i + (1-\theta) y_i)(\theta x_j + (1-\theta) y_j)$$

$$= \sum a_{ij} (\theta^2 x_i x_j + (1-\theta)^2 y_i y_j + \theta(1-\theta)(x_i y_i - x_j y_j))$$

$$\leq \theta^2 a + (1-\theta)^2 a + \sum \theta(1-\theta) \underbrace{(x_i y_i + x_j y_j)}_{\hookrightarrow \Delta}$$

$$\leq a(\theta^2 + (1-\theta)^2 + 2\theta(1-\theta)$$

$$\leq a$$

$$(x-y)^T A (x-y) \geq 0$$

$$= \sum a_j (x_i - y_i)(x_j - y_j)$$

$$= \sum a_j (x_i x_j - x_j y_i - x_i y_j + y_i y_j)$$

$$\Rightarrow \sum a_{ij} (x_i x_j + y_i y_j) \geq \sum a_j (\Delta)$$