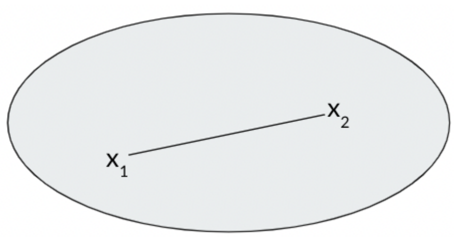
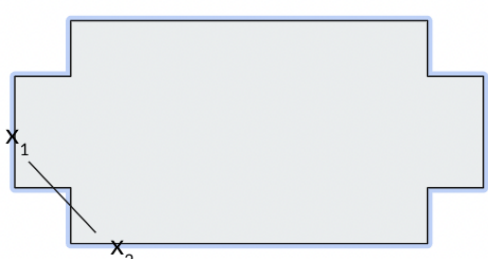
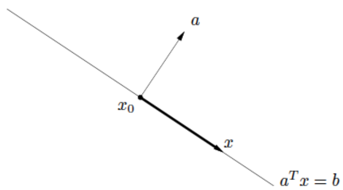
CAS CS 365
Lec 25

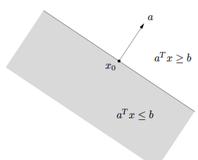
Optimization

1. Line
    a. Suppose x_1, x_2 are two points in R^n. Points of the form
    $y = \theta x_1 + (1-\theta)x_2,\ \theta \in \mathbb{R}$ form the line passing through x_1, x_2
2. Affine set
    a. Definition: A set C is affine if the line through any two distinct points lines in C.
        i. The idea generalizes to more than two points. An affine combination of k points $x_1,..,x_k$ in C is
        $\theta_1 x_1 + \ldots + \theta_k x_k$ where $\theta_1 + \ldots + \theta_k = 1$
    b. Claim: An affine set contains every affine combination of its points
    c. The solution set $\{x|A^{mxn}x^{nx1}=b^{mx1}\}$ is an affine set.
    d. If C is an fine set, and x_0 is in C, then the set
    $V = C - x_0 = \{x - x_0 \mid x \in C\}$ is a subspace
3. Convex vs non-convex set
    a. A set C is convex if the line segment between any two points in C lies in C, i.e., for any x1, x2 in C and for any, $0 \leq \theta \leq 1$
    $\theta x_1 + (1-\theta)x_2 \quad \in C$
    b. Convex → local minimum is global minimum
    c. 

    d. Convex
    e. 

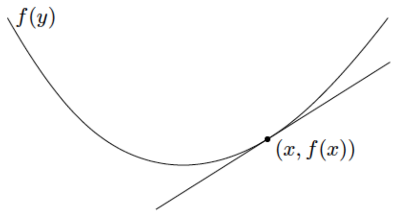
    f. Non-convex

4. Hyperplanes
    a. $$\boxed{\begin{aligned} &a^T x = b, \\ &where \quad a \in \mathbb{R}^n,\ a \neq 0, b \in \mathbb{R} \end{aligned}}$$
    b. b offset of the hyperplane from 0
    c. 

5. Halfspaces
    a. A hyperplane divides R^n into two halfspaces
    b. Halfspaces are convex but not affine
    c. 

6. Convex function
    a. A function F: Rn → R is convex if its domain dom(f) is convex and if for all x, y in dom(f), and theta in [0,1] $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$
        i. It is strictly convex if the inequality is strict for all theta in (0,1)
        ii. f is concave if -f is convex
    b. 

7. Convex function, 1st order condition
    a. Suppose f is differentiable. Then f is convex if its domain is a convex set and $f(y) \geq f(x) + \nabla f(x)(y-x)$

b. 

c. $f(y) \geq f(x) + \nabla f(x)(y - x)$

8. Convex function, 2nd order condition
   a. Assuming f is twice differentiable. f is convex iff f’s domain is convex and the Hessian is positive semidefinite

$$x^T \frac{\partial^2 f(x^\star)}{\partial x^2} x \geq 0, \quad \text{for all } x \in \mathbb{R}^n$$

   b.

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$y = \theta x_1 + (1-\theta) x_2 \quad 0 \leq \theta \leq 1$

$\text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \{p \in \mathbb{R}^2 \mid p = \begin{bmatrix}x\\x\end{bmatrix} \; x \in \mathbb{R}\}$

$\theta \begin{bmatrix}x\\x\end{bmatrix} + (1-\theta)\begin{bmatrix}y\\y\end{bmatrix} = \begin{bmatrix}\theta x + (1-\theta) y\\ \theta x + (1-\theta) y\end{bmatrix}$

- any affine set is translated subspace (and make sure 0 is contained)

- $Ax = b$
  - if no solutions, $m > n$
  - if many solutions, $m < n$

Full rank ($A^{n \times n}$)
↳ columns span $\mathbb{R}^n \rightarrow Ax = b$ has one solution

Affine
$C = \{x \in \mathbb{R}^n \mid Ax = b\}$

Let $x_1, x_2 \in C_2$ ($x_1 \neq x_2$) $y =$
We want to prove that for all $\theta \in \mathbb{R}$, $\theta x_1 + (1-\theta) x_2 \in C$
↳ verify that $Ax_1 = Ax_2 = b$

$y = \theta A x_1 + (1-\theta) A x_2 = \theta b + (1-\theta) b$
$= b$

hyperplane:

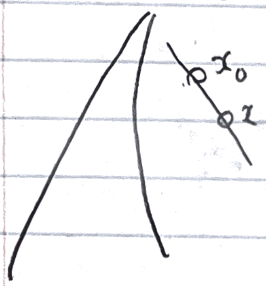
$x_2 = 3x_1 + 5$

$x_2 - 3x_1 = 5$

$$(-3, 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5$$

$a^T x = 5$

$\{x : a^T x = b,\ x \in \mathbb{R}^n\}$



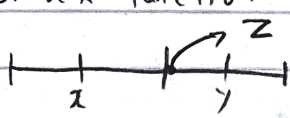
$$\left.\begin{array}{l} a^T x = b \\ a^T x_0 = b \end{array}\right\} a^T(x - x_0)$$

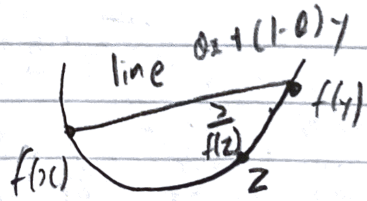
$\underbrace{a^T x \geq b \qquad a^T x \leq b}_{\text{half spaces}}$

convex $\not\to$ affine

affine $\to$ convex

Convex function:





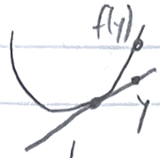
$z = \theta x + (1-\theta)y$

$0 \leq \theta \leq 1$

↳ $\theta f(x) + (1-\theta)f(y) \geq f(\theta x + (1-\theta)y)$

- If opposite happens, the function is concave

- differentiable (1st order)



$f: \mathbb{R}^n \to \mathbb{R}$

$f(x) + (\nabla f(x))(y-x)$

↳ tangent lies below the curve

$$f(y) \geq f(x) + \nabla f(x)(y-x),$$

then f is convex

$\ell(y) = f(x) + \nabla f(x)(y-x)$ ← Taylor approximation at $y = x$

$f(y) \geq \ell(y)$

↳ function ↳ line

→ not convex because $\ell(y) > f(y)$ at $y=n$

- 2nd order

$f(x,y) = \dfrac{x^2}{y}$ $x \in \mathbb{R}$, $y > 0$

$f: \mathbb{R}^2 \to \mathbb{R}$ $\nearrow \frac{\partial f}{\partial x}$ $\nearrow \frac{\partial f}{\partial y}$

$\nabla f(x,y) = \left[\dfrac{2x}{y}, \dfrac{-x^2}{y^2}\right]$

$$H_f(x,y) = \begin{bmatrix} \dfrac{2}{y} & \dfrac{-2x}{y^2} \\ \dfrac{-2x}{y^2} & \dfrac{2x^2}{y^3} \end{bmatrix}$$

→ prove semi-definite

① find eigenvalues, ≥ 0

② $h^T H h$ is always ≥ 0