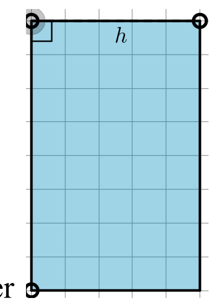
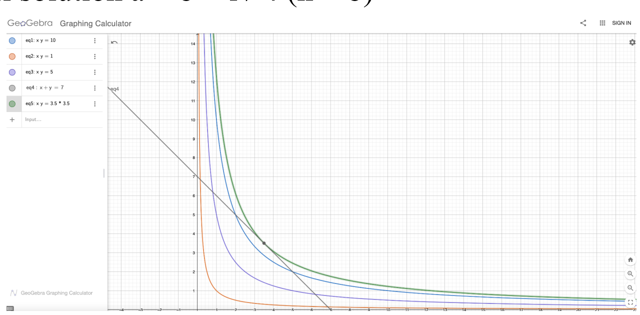
CAS CS 365
Lec 23

Vector Calculus (cont.)

1. Parallelogram of maximum area
    a. Find parallelogram of maximum area with a given perimeter
    b. $$\boxed{\begin{aligned} \max_{a,b,h}\ & ah \\ 2a + 2b &= \ell \\ h &\leq b \\ a, b, h &\geq 0 \end{aligned}}$$
    c. Clearly, given a, b, h = b is an obvious solution
    d. Thus, we get the following equivalent problem:



    e. Find parallelogram of maximum area with a given perimeter
    f. $$\boxed{\begin{aligned} \max_{a,b}\ & ab \\ 2a + 2b &= \ell \\ a, b &\geq 0 \end{aligned}}$$
2. Optimal solution a = b = l / 4 (h = b)
    a. 

3. Transportation problem
    a. Minimize the cost of goods transported from
        i. A set of m sources to… a set of n destinations
        ii. Subject to the supply and demand of the sources and destination respectively

b. Given
    i. $a_1,...,a_m$: units to transfer from sources
    ii. $b_1,...,b_m$: units to receive by destinations
    iii. c_{ij} : cost of transferring a unit from source i to destination j
c. Find the quantities xij to be transferred from source i to destination j for i = 1,...,m j = i,...,n
    i.

$$\boxed{\begin{aligned} &\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ &\sum_{j=1}^{n} x_{ij} = a_i\,, \quad i = 1,\ldots,m \\ &\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,\ldots,n \\ &\quad x_{ij} \geq 0 \end{aligned}}$$

Lec 23

$f: \mathbb{R}^n \to \mathbb{R} \quad (\nabla f)^{1\times n}$
$f: \mathbb{R}^n \to \mathbb{R}^{n^2} \quad (\nabla f)^{n^2 \times n}$
input output

$f(x) = xx^T \quad x \in \mathbb{R}^n$
$f: \mathbb{R}^n \to \mathbb{R}^{n\times n}$

n^2 elements

$$f(x_1, \dots, x_n) = f(x) = \begin{bmatrix} x_1^2 & & x_1 x_j \\ & x_2^2 & \\ & & \ddots \\ & & & x_n^2 \end{bmatrix}$$

$$f(x_1+h, \dots, x_n) = \begin{bmatrix} (x_1+h)^2 & \cdots & (x_1+h)x_n \\ \vdots & & \\ (x_1+h)x_n & & \end{bmatrix}$$ → first row changes

↓

first column changes

$$\lim_{h\to 0} \frac{f(x_1+h, \dots, x_n) - f(x_1, \dots, x_n)}{h} = \lim_{h \to 0} \frac{1}{h} \begin{bmatrix} (x_1+h)^2 - x_1^2 & hx_2 & \cdots & hx_n \\ \vdots & & & \\ hx_n & & 0 & \end{bmatrix}$$

$$= \lim_{h\to 0} \begin{bmatrix} \frac{(x_1+h)^2 - x_1^2}{h} & x_2 & \cdots & x_n \\ \vdots & & & \\ x_n & & 0 & \end{bmatrix}$$

$\nabla f = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]$

$\nearrow$ find x that minimizes the function

$\min f(x)$ subject to $x \in Q$

$f: \mathbb{R}^n \to \mathbb{R}$, $Q \subseteq \mathbb{R}^n$

$\hookrightarrow$ subspace of $\mathbb{R}^n$

☆ minimizing $= -\max(-f(x))$ ☆

Case I $Q = \mathbb{R}^n$

unconstrained

Case II $Q \subset \mathbb{R}^n$

Constrained

ex) $f: \mathbb{R}^2 \to \mathbb{R}$

$\min \; 6(x_1 - x_2)^2 + (x_1 + x_2)^2$

such that $x_1 \geq 2x_2$

$x_1^3 + x_2^3 - 2x_1x_2 = 10$

Global min

x^* is global minimum of f on Q if $f(x^*) \leq f(x) \;\forall x \in Q$

Strict global min

$f(x^*) < f(x) \;\forall x \in Q \setminus \{x^*\}$

Local min

$\exists \varepsilon > 0$ such that $\forall x \; \|x - x^*\| \leq \varepsilon$, $f(x) \geq f(x^*)$

Strict local min

$\exists \varepsilon > 0$, such that $\forall x \; \|x - x^*\| \leq \varepsilon$, $f(x) > f(x^*)$

$\min 10 - x \quad x \in [0, 7) \to Q$

$\hookrightarrow$ Suppose there exists a minimum $x^* = 7 - \delta$, for some $\delta > 0$ $(x^* \in Q)$

$x' = 7 - \frac{\delta}{2} \in Q$

$f(x') = 10 - x' = 3 + \frac{\delta}{2}$

$f(x^*) = 3 + \delta > f(x')$

$f: \mathbb{R}^n \to \mathbb{R}$

x^* to be a local minimum is $\nabla f(x^*) = 0$

Proof

Suppose x^* is a local min $\nabla f(x^*) \neq 0$

$x(a) = x^* - a\nabla f(x^*)$, $a > 0$

$$f(x(a)) = f(x^*) + (\underbrace{\nabla f(x^*)}_{\neq 0})(-x^* + x(a))$$
$$= f(x^*) + \nabla f(x^*)(x^* - a(\nabla f(x))^T - x^*)$$
$$= f(x^*) - a(\nabla f(x))^{1 \times n}(\nabla f(x))^T$$
$$= f(x^*) - a\|\nabla f(x^*)\|_2^2 < f(x^*)$$

contradiction

$S = \{x \in Q : \nabla f(x) = 0\}$

is the set of stationary points

If f is continuous and twice differentiable, for any local minimum x^*: $\nabla f(x^*) = 0$, $y^T H_f(x^*) y \geq 0$ $\forall y \in \mathbb{R}^n$

$H_f(x^*)$ is positive semidefinite

$\Downarrow$

eigenvalues ≥ 0

Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$y^T A y = [y_1 \ y_2] \begin{bmatrix} y_1 - y_2 \\ -y_1 + y_2 \end{bmatrix}$$
$$= y_1(y_1 - y_2) + y_2(-y_1 + y_2)$$
$$= (y_1 - y_2)^2 \ \forall y$$

$f(x_1, x_2) = 3(x_1 - x_2)^2 + (x_1 + x_2)^3$

$\nabla f(x) = 0 \Rightarrow \begin{bmatrix} \frac{df}{dx_1} & \cdots & \frac{df}{dx_2} \end{bmatrix} = 0$

$$\frac{df}{dx_1} = 6(x_1 - x_2) + 3(x_1 + x_2)^2$$

$$\frac{df}{dx_2} = -6(x_1 - x_2) + 3(x_1 + x_2)^2$$

$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\nabla f(x^*) = 0)$

$$H_f(x) = \begin{bmatrix} 6 + 6(x_1 + x_2) & -6 + 6(x_1 + x_2) \\ -6 + 6(x_1 + x_2) & 6 + 6(x_1 + x_2) \end{bmatrix}$$

$$\frac{d^2 f}{dx_1^2} = 6 + 6(x_1 + x_2)$$

$$\frac{d^2 f}{dx_1 dx_2} = -6 + 6(x_1 + x_2)$$

$$\frac{d^2 f}{dx_2^2} = 6 + 6(x_1 + x_2)$$