CAS CS 365
InClass Note 2

Probability

1. Fair Coin
    a. $Pr(H) = Pr(T) = ½$
    b. Coin flip can be "rigged" → the process is deterministic (person tossing the coin can use the knowledge of coin to predict the outcome with better probability)
    c. Tossing of coin is fair only and only if $Pr(H) = ½$
    d. Suppose we throw the fair coin ten times
        - What is the expected number of heads?

          $X_i = 1$ if the ith toss gives Head
          $= 0$ if it gives Tail
          S_{10} = number of heads in 10 tosses $= X_1 + X_2 + \ldots + X_{10}$

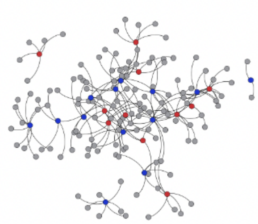
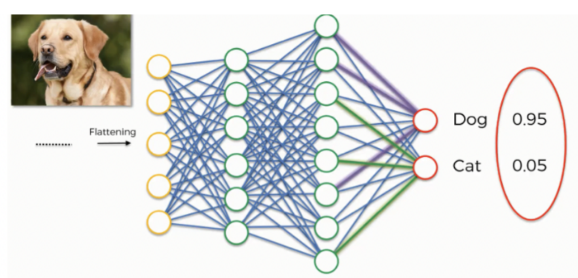
          $E[S_{10}]$
          $= E[\Sigma$ i=1 to i = 10 of $X]$
          $= \Sigma$ i=1 to i = 10 $E[X_i]$

          $E[X_i] = 1 * ½ + 0 * ½$

          Σ i=1 to i = 10 $E[X_i] = 10 * ½ = 5$

          $X_i \to Ber(½)$
          $S_{10} \to$ Binomial (10, ½)

2. Modeling uncertainty

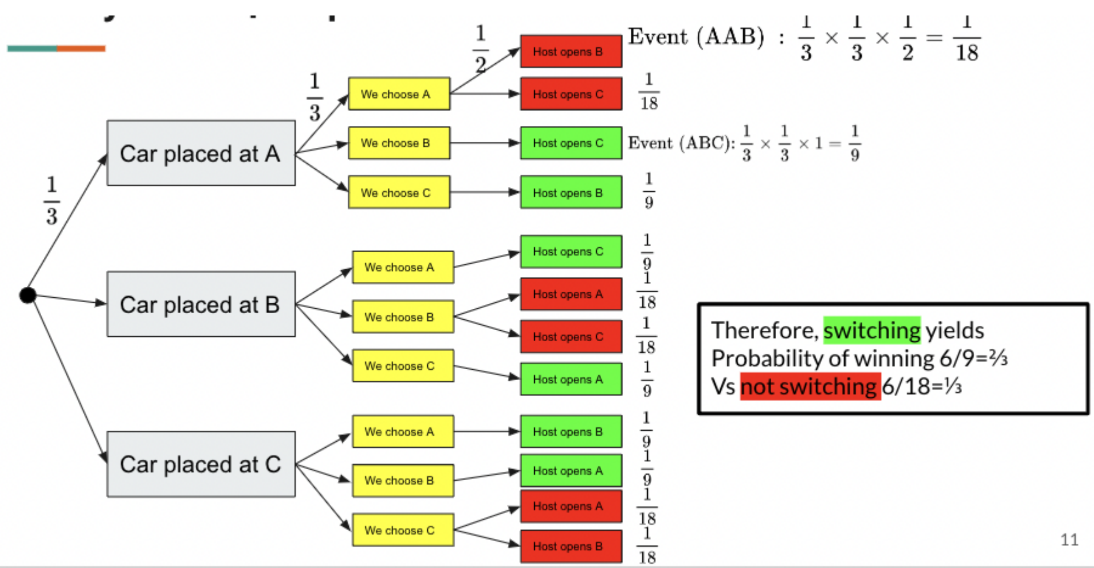
    a. Disease spreading
    b. Information theory, modeling the reliability of numerous complex systems, insurance companies, investments, etc.
3. Reminder
    a. Probability space (Ω, F, P) consists of three elements
    b. Sample space (Ω): set of all possible outcomes

c. Event Space (F): a set of events, an event being a set of outcomes in the sample space
d. Probability (P) : it is a function that assigns each event in the vent space a probability, which is a number between 0 and 1
e. Example:

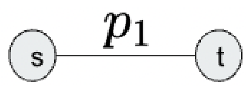
Example 2.29. Consider a sample space containing three elements $\Omega = \{\clubsuit, \heartsuit, \maltese\}$. The event space is then $\mathcal{F} = \Big\{\emptyset, \{\clubsuit\}, \{\heartsuit\}, \{\maltese\}, \{\clubsuit, \heartsuit\}, \{\heartsuit, \maltese\}, \{\clubsuit, \maltese\}, \{\clubsuit, \heartsuit, \maltese\}\Big\}$. One possible $\mathbb{P}$ we could define would be

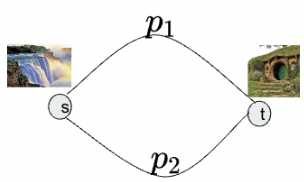
$$\mathbb{P}[\emptyset] = 0, \quad \mathbb{P}[\{\clubsuit\}] = \mathbb{P}[\{\heartsuit\}] = \mathbb{P}[\{\maltese\}] = \frac{1}{3},$$
$$\mathbb{P}[\{\clubsuit, \heartsuit\}] = \mathbb{P}[\{\clubsuit, \maltese\}] = \mathbb{P}[\{\heartsuit, \maltese\}] = \frac{2}{3}, \quad \mathbb{P}[\{\clubsuit, \heartsuit, \maltese\}] = 1.$$

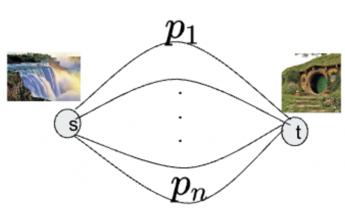
f. space a probability, which is a number between 0 and 1.

4. Monty-hall problem
   a. Suppose you're on a game show, and you're given the choice of three doors
      - Behind one door is a car; behind the other, goats
      - You pick a door, Say 1, and the host, who knows what's behind the doors, opens another door, say 3, which has a goat
      - He then says to you, "Do you want to pick door 2?" Is it to your advantage to switch your choice?
   b. Assumptions:
      - Car is placed uniformly at random (uar) behind a door
      - Out initial guess is also uar
      - The host opens a door with a goat. When there exist two such doors, i.e., our guess is the car, he chooses uar
   c. Steps
      - 

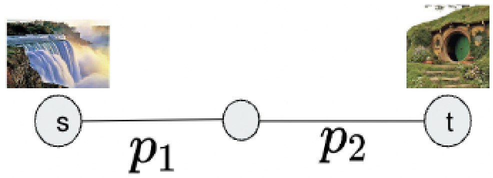
5. Transfer water
    a. Consider a water source s and a destination village t
    b. Each pipe i has probability of failure P_i, Pipes fail independently
    c. Question: What is the probability we cannot get water from s to t? In other words, when is the village t not reachable from the water source s?



    d.  
    e. Exercise 1:
        - 

        - Clearly, there is no path if both pipes fail
        - Since they are independent, the probability of this event is the product of the probabilities of the individual events
        - Thus, failure probability is $p_1 * p_2$
    f. Exercise 2:
        - 

        - Clearly, there is no path if all pipes fail
        - Since they are independent, the probability of this event is the product of the probabilities of the individual events
        - Thus, failure probability is $p_1 * p_2 * ... * p_n$
        - E_i = pipe i fails
        - $Pr(E_1 \cap E_2 \cap \ldots \cap E_n) = \Pi \, Pr(E_n)$
        - **** De Morgan's Law ****
        - **$Pr(\text{not } E_1 \cup \text{not } E_2 \cup \ldots \cup \text{not } E_n) = 1 - p_1 * p_2 * ... * p_n$**
    g. Exercise 3:

- 

- Two events are independent: Pr(not E_1 n not E_2) = Pr(not E_1) * Pr(not E_2)
- Pr (failure) = Pr(E_1 U E_2)
  = 1 - Pr(not (E_1 U E_2))
  = 1- Pr(not E_1 n not E_2) = 1 - (1-p_1) * (1-p_2)
  = 1 - (1 - p_1 - p_2 + p_1*p_2)
  = p_1 + p_2 - p_1*p_2

  OR

- Clearly, there is no path if at least one of the pipes fail
- We condition on whether the one of the two pipes (say the first) is broken or not
- Let A_i be the event that pipe i fails
- Then,

$$\begin{aligned}\Pr(A_1 \cup A_2) &= \Pr(A_1) + \Pr(A_2) - \Pr(A_1 \cap A_2)\\ &= p_1 + p_2 - \Pr(A_1)\Pr(A_2)\\ &= p_1 + p_2 - p_1 p_2\end{aligned}$$

h. Exercise 4

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- This is when all pipes are successful (if one pipe is unsuccessful, water will not go through)
- Instead of thinking of the probability that t will not be reachable from s, we think of the probability that it is. Reminder: Pr(not A) = 1- Pr(A)
- The probability of not failing is

$$\Pr\left(\cap_{i=1}^{n} \bar{A}_i\right) = \prod_{i=1}^{n} \Pr\left(\bar{A}_i\right) = \prod_{i=1}^{n}(1 - p_i)$$

- Therefore, the right answer is

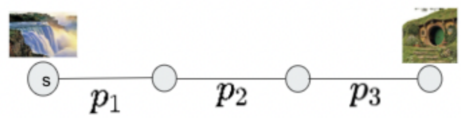
$$1 - \prod_{i=1}^{n}(1 - p_i)$$

OR

- Pr(sending water from s to t) = Pr(No Pipe fails) = Pr(not E_1 n not E_2 n … n not E_n)
- Π Pr(not E_i) = Π (1-Pr(E_i))

6. Reminders: independent events, conditional probability
    a. Intuitive two events A, B are dependent if A's occurrence or non-occurrence provides us with some information about event B
    b. Formally, A and B are independent events if and only iff
    $$\boxed{\Pr(A \cap B) = \Pr(A)\Pr(B)}$$
    c. By rearranging, we get
    $$\boxed{\Pr(A) = \frac{\Pr(A \cap B)}{\Pr(B)}}$$
    d. Recall that by the law of conditional probability
    $$\boxed{\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}}$$
    e. Therefore, when A and B are independent events Pr(A) = Pr(A|B) and Pr(B) = Pr(B|A)
7. Reminders: Law of total probability
    a. Let Ω be a probability space. Let B_1,...,B_m be a partition of Ω. Then,
    $$\boxed{\Pr(A) = \sum_{i=1}^{m} \Pr(A \cap B_i) = \sum_{i=1}^{m} \Pr(B_i)\Pr(A|B_i)}$$
8. Reminder: chain rule
    a. Chain rule:
    $$\Pr(A_1 \cap \ldots \cap A_n) = \Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_2A_1)\ldots\Pr(A_n|A_{n-1}..A_1)$$
    b. In our case the events are mutually independent, so this simplifies to the product of the individual probabilities of the events A_i
    c. Pairwise Independent
        - Events A, B, C
        - P(A n B) = P (A n C) = P (B n C)
    d. Mutually Independent
        - Events A, B, C
        - P(A n B n C) = P(A) * P(B) * P(C)
9. Reminder: conditional probability + Law of total probability → Bayes rule

a. $$\boxed{\Pr(B_i|A) = \frac{\Pr(B_i \cap A)}{\Pr(A)} = \frac{\Pr(B_i)\Pr(A|B_i)}{\sum_{j=1}^{n} \Pr(B_j)\Pr(B_j|A)}}$$

10. Example n = 3
    a. 

$$1-(1-p_1)(1-p_2)(1-p_3) = 1-(1-p_1)(1-p_2-p_3+p_2p_3)$$

$$= 1-(1-p_2-p_3+p_2p_3-p_1+p_1p_2+p_1p_3-p_1p_2p_3)$$

    b. $= p_1+p_2+p_3-p_1p_2-p_1p_3-p_2p_3+p_1p_2p_3$
11. Example n = 4
    a. 

    b. $1 - (1-p_1)(1-p_2)(1-p_3)(1-p_4)$
    $= 1 - (1-p_1-p_2+p_1p_2)(1-p_3-p_4+p_3p_4)$
    $= 1- \dots$
    $= p_1 + p_2 + p_3 + p_4$
    $- p_1p_2 - p_1p_3 -p_1p_4 -p_2p_3 -p_2p_4 -p_3p_4$
    $+ p_1p_2p_3 + p_1p_2p_4 + p_2p_3p_4$
    $- p_1p_2p_3p_4$
12. Reminder: Inclusion exclusion
    a. Another convenient way to write the IE formula is the following
    b. $$\mathbf{P}\left(\bigcup_{i=1}^{n} A_i\right) = S_1 - S_2 + S_3 - \ldots + (-1)^{n-1} S_n$$
    where
    $$S_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}).$$
    c. In our setting, due to the independence of the events A_i, we can write the following expression:
    $$\Pr(\cup A_i) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \subseteq [n], |I|=k} \prod_{i \in I} \Pr(A_i)$$
    let's write down some terms
    $$\begin{aligned}\Pr(\cup A_i) &= p_1 + \ldots + p_n \\ &-(p_1p_2 + \ldots + p_{n-1}p_n) \\ &+(p_1p_2p_3 + \ldots + p_{n-2}p_{n-1}p_n) \\ &- \ldots\end{aligned}$$