1.a

Given that $Z = \max(X, Y)$, we can compute the cumulative distribution function (CDF) of Z to find the probability that $Z \leq x$.

Then, we get

CDF of $Z = F_Z(Z \leq x)$
$= F_Z(\max(X, Y) \leq x)$

- If the maximum of X and Y are less than or equal to x, both X and Y should be less than or equal to x.

$F_Z(x) = P(\max(X, Y) \leq x)$
$= P(X \leq x \text{ and } Y \leq x)$

According to the question, X and Y are independent, so we can apply $P(AB) = P(A) \cdot P(B)$

$F_Z(x) = P(X \leq x) \cdot P(Y \leq x)$

Since X and Y share a common density function, $P(X \leq x) = P(Y \leq x)$

$\therefore F_Z(x) = P(X \leq x)^2$

The probability density function $f_Z(x)$ is calculated by computing the derivative of $F_Z(x)$, the CDF.

$\therefore f_Z(x) = \frac{d}{dx} F_Z(x) = \frac{d}{dx}\left(P(X \leq x)^2\right)$
$= 2P(X \leq x) \cdot \frac{d}{dx}\left(P(X \leq x)\right)$

$P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$. If we take the derivative of $P(X \leq x)$, the integral gets cancelled and we get $f(x)$

$\therefore f_Z(x) = 2 f(x) \cdot P(X \leq x)$

1.b

Given that U is a uniform random variable in [0,1], the distribution of $\lfloor 100U \rfloor + 1$ will result to be a uniform distribution.

Since $\lfloor\ \rfloor$ represents the floor function that always rounds the decimal, $\lfloor 100U \rfloor$ will always produce an integer value.

If the value of U has two digits after the decimal point (up until hundredth digit such as 0.02, 0.33, 0.70), 100U will always produce an integer.

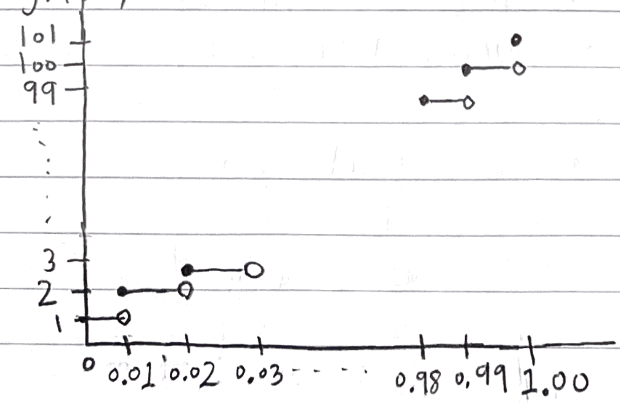
However, if U has values after the hundredth digit (three or more numbers after the decimal point such as 0.007, 0.22222), the value of 100U will contain a decimal and will be rounded down.

This means that no matter what value comes after the hundredth digit for U, the value of $\lfloor 100U \rfloor$ will be identical to the value for U as if those do not exist. (ex) $\lfloor 100(0.7) \rfloor = \lfloor 100(0.703) \rfloor = \lfloor 100(0.7099999) \rfloor = 70$.

Distribution of $\lfloor 100U \rfloor + 1$ is

$$\lfloor 100U \rfloor + 1 = \begin{cases} 1 & \text{when } 0 \le U < 0.01 \\ 2 & \text{when } 0.01 \le U < 0.02 \\ 3 & \text{when } 0.02 \le U < 0.03 \\ \vdots & \\ 100 & \text{when } 0.99 \le U < 1 \\ 101 & \text{when } U = 1 \end{cases}$$

In graph,



1.c

$\lfloor \log U / \log q \rfloor + 1$ has $0 < q < 1$ and U is a uniform random variable in $[0,1]$ means that both $\log U$ and $\log q$ are negative values (since $\log(x)$ is negative when $0 < x \leq 1$).

↳ but U cannot be 0 since $\log 0$ does not exist.

This means that $\lfloor \log U / \log q \rfloor$ is a positive value that is rounded down to the nearest integer.

This suggests that for values $x \geq 1$, $\lfloor \log U / \log q \rfloor + 1$ lies between x and $x+1$ (the value of $\lfloor \log U / \log q \rfloor$ cannot be $x+1$ since the floor function always rounds the value down but can be x)

↳ ex) $3 = \lfloor 3 \rfloor$ but $\lfloor 3.\_\_ \rfloor$ can never equal 4 no matter what value comes after the decimal point.

$\therefore$ $Pr[\lfloor \log U / \log q \rfloor + 1 = x]$ can be written as

$x \leq \log U / \log q + 1 < x+1$ for $\lfloor \log U / \log q \rfloor + 1 = x$

$x - 1 \leq \log U / \log q < x$

$(x-1)(\log q) \geq \log U > x \log q$

↳ since $\log q$ is negative, we flip the signs

By log property,

$\log(q^{x-1}) \geq \log U \geq \log(q^x)$

The logs are all same base, so we can remove "log" from the inequality since $\log x$ is always an increasing function as x increases.

$\therefore$ $q^{x-1} \geq U \geq q^x$ ↳ in other words $\log x < \log y$ when $x < y$

Therefore if we apply $P(\lfloor \log U / \log q \rfloor + 1 = x)$, we get $\frac{q^{x-1} - q^x}{1} = q^x \cdot q^{(-1)} - q^x = q^x(q^{-1} - 1)$

$= q^x(\frac{1}{q} - 1) = q^x(\frac{1-q}{q})$

$= q^{x-1}(1-q)$ for $x \geq 1$

Because $x \geq 1$, q^{x-1} never becomes q^{-1}, which shows that $(q^{x-1})(1-q)$ when $x \geq 1$ is a geometric distribution.

The parameter of the geometric distribution is q, which is value from 0 to 1 $\to 0 < q < 1$, as the question states.

(2) (a) $Pr(N=1 \mid Z=0.05)$ when $n=2$

According to Bayes' rule,

$$Pr(N=1 \mid Z=0.05) = \frac{Pr[Z=0.05 \mid N=1] \cdot P(N=1)}{Pr(Z=0.05)}$$

- $Pr(Z=0.05)$ when $n=2$ is

$$Pr[Z=0.05 \mid N=1] \cdot P[N=1] + Pr[Z=0.05 \mid N=2] \cdot P(N=2)$$

- $P[N=1] = P[N=2] = \frac{1}{2}$, as the question states
- we draw N iid uniform RV $\{X_i\}_{i=1,\dots,N}$ in $[0,1]$ and the minimum of those values is Z, which is 0.05.
- We let $Y = \{x_1, \dots, x_n\}$ for $n \geq 1$.

- Since we have to find $Pr[Z=0.05 \mid N=2]$, the probability that $Z=0.05$ given that $n=2$, we need to calculate the pdf of the distribution Y.
- To get the pdf, we take the CDF of the distribution and take the derivative of it.
- The CDF of Y is

$$Pr_Y(z) = Pr(Y \leq z) = Pr(\min(X_1, \dots, x_n) \leq z) = 1 - Pr(\min(X_1, \dots, x_n) \geq Z)$$
$$= 1 - Pr(x_1 \geq Z, \dots, x_n \geq Z)$$

- Since all x_i are independent according to the question,

$$= 1 - Pr(x_1 \geq Z) \cdots Pr(x_n \geq Z)$$
$$= 1 - (1-z)^n$$

Taking the derivative, we get

$n(1-z)^{n-1}$ for all n.

$$\therefore Pr[Z=0.05 \mid N=1] = 1 \cdot (1-z)^0 = 1$$
$$Pr[Z=0.05 \mid N=2] = 2 \cdot (1-Z)^1 = 2(0.95)$$

Going back to Bayes' rule, we get

$$Pr(N=1 \mid Z=0.05) = \frac{Pr[Z=0.05 \mid N=1] \cdot P(N=1)}{Pr[Z=0.05 \mid N=1] \cdot P(N=1) + Pr[Z=0.05 \mid N=2] \cdot P(N=2)}$$

$$= \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + 2(0.95) \cdot \frac{1}{2}} = 0.3448 \text{ when } n=2$$

(b) $Pr[N=1 \mid Z=0.05)$ when $n=10$

$$Pr[N=1 \mid Z=0.05] = \frac{Pr[Z=0.05 \mid N=1] \cdot P(N=1)}{Pr(Z=0.05)}$$

- When $n=10$, $Pr(Z=0.05) = Pr[Z=0.05 \mid N=1] \cdot P(n=1) + Pr[Z=0.05 \mid N=10] \cdot Pr(N=10)$.
- Once again, $Pr[N=1] = Pr[N=10] = 1/2$ as the question states
- From 2(a), we know that

$$Pr[Z=0.05 \mid N=n] = n(1-z)^{n-1}$$

$$\therefore\ Pr[Z=0.05 \mid N=10] = 10 \cdot (0.95)^9$$

$$Pr[Z=0.05 \mid N=1] = 1 \cdot (0.95)^0 = 1$$

Going back to Bayes' Theorem,
$Pr[N=1 \mid Z=0.05]$ when $n=10$ is

$$\frac{Pr[Z=0.05 \mid N=1] \cdot Pr[N=1]}{Pr[Z=0.05 \mid N=1] \cdot Pr[N=1] + Pr[Z=0.05 \mid N=10] \cdot Pr[N=10]}$$

$$= \frac{1 \cdot 1/2}{1 \cdot 1/2 + 10 \cdot (0.95)^9 \cdot 1/2}$$

$$= 0.1369$$