① (a) For all $n \geq 0$, $f_0 + f_1 + \dots + f_n = f_{n+2} - 1$.

$f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$

Base case: $n = 0$

$$f_0 = f_2 - 1$$
$$0 = 1 - 1$$
$$0 = 0 \checkmark$$

Since we need to prove for all $n \geq 0$, the base case should be 0. when $n = 0$, $f_0 = 0$ and $f_2 = f_1 + f_0 = 1 + 0 = 1$. Therefore, $f_2 - 1$ is 0 and $f_0 = 0$, so $f_0 = f_2 - 1$.

Inductive Step:

Assume $f_0 + f_1 + \dots + f_k = f_{k+2} - 1$ is true from the base case. We need to prove that $f_0 + f_1 + \dots + f_k + f_{k+1} = f_{k+1+2} - 1$, which is $f_0 + f_1 + \dots + f_{k+1} = f_{k+3} - 1$. f_{k+3} is $f_{k+2} + f_{k+1}$, which makes the equation into $f_0 + f_1 + \dots + f_{k+1} = f_{k+2} + f_{k+1} - 1$. Since there exists f_{k+1} in both sides, we can cancel out the f_{k+1} in both sides. $f_0 + f_1 + \dots + f_k = f_{k+2} - 1$, which is the statement that we assumed to be true using inductive hypothesis.

↳ $f_0 + f_1 + \dots + f_k = f_{k+2} - 1$ (inductive hypothesis)

$f_0 + f_1 + \dots + f_{k+1} = f_{k+3} - 1$

$f_0 + f_1 + \dots + \cancel{f_{k+1}} = f_{k+2} + \cancel{f_{k+1}} - 1$

$f_0 + f_1 + \dots + f_k = f_{k+2} - 1$ (True by inductive hypothesis) $\checkmark$

(b) For every $n \geq 2$, $f_n \geq (1.5)^{n-2}$.

Base case: $f_2 = 1 \geq (1.5)^{2-2}$

$1 \geq (1.5)^0$

$1 \geq 1$ ✓

$f_3 = 2 \geq (1.5)^{3-2}$

$2 \geq (1.5)^1$

$2 \geq 1.5$ ✓

We need to prove for all $n \geq 2$, the base case is 2 and 3. f_2 is 1 and $(1.5)^{2-2}$ is 0, so $f_2 = 1 \geq 1$. When $n=3$, f_3 is $f_2 + f_1$, which is 2, and $(1.5)^{3-2}$ is 1.5^1, so $f_3 = 2 \geq 1.5$.

Inductive step:

From base case, assume $f_{k-1} \geq (1.5)^{k-1-2}$ and $f_k \geq (1.5)^{k-2}$ is true from the base case 2 and 3. We need to prove that $f_{k+1} \geq (1.5)^{k+1-2}$ for $k \geq 3$, which is $f_{k+1} \geq (1.5)^{k-1}$. Since f_{k+1} is $f_k + f_{k-1}$ by definition, $f_k + f_{k-1} \geq (1.5)^{k-1}$. f_k itself is greater than or equal to $(1.5)^{k-2}$ and f_{k-1} itself is greater than or equal to $(1.5)^{k-3}$. So, $f_k + f_{k-1}$ is greater than or equal to $(1.5)^{k-2} + (1.5)^{k-3}$. $f_{k+1} = f_k + f_{k-1} \geq (1.5)^{k-2} + (1.5)^{k-3}$. If $(1.5)^{k-2} + (1.5)^{k-3}$ is greater than or equal to $(1.5)^{k-1}$, $f_{k+1} \geq (1.5)^{k-1}$ by transitivity. (If $a \geq b$ and $b \geq c$, $a \geq c$). $(1.5)^{k-2}$ is equivalent to $(1.5)^k (1.5)^{-2}$ and $(1.5)^{k-3}$ is equivalent to $(1.5)^k (1.5)^{-3}$. $(1.5)^{k-2} + (1.5)^{k-3} = (1.5)^k (1.5)^{-2} + (1.5)^k (1.5)^{-3}$, which can be simplified into $(1.5)^k ((1.5)^{-2} + (1.5)^{-3})$. Similarly, $(1.5)^{k-1}$ can be simplified to $(1.5)^k (1.5)^{-1}$. We are proving $(1.5)^{k-2} + (1.5)^{k-3} \geq (1.5)^{k-1}$, which is $(1.5)^k (1.5^{-2} + 1.5^{-3}) \geq (1.5)^k (1.5)^{-1}$. Since $(1.5)^k$ exists in both sides, we can cancel it out. $(1.5^{-2} + 1.5^{-3}) \geq 1.5^{-1}$. Algebracally, it is always true that $0.741 \geq 0.667$, so $f_{k+1} = f_k + f_{k-1} \geq (1.5)^{k-2} + (1.5)^{k-3} \geq (1.5)^{k-1}$. Therefore, $f_{k+1} \geq (1.5)^{k-1}$

↳ $f_{k+1} \geq (1.5)^{k-1}$

$f_{k+1} = f_k + f_{k-1} \geq (1.5)^{k-2} + (1.5)^{k-3} \geq (1.5)^{k-1}$

$\cancel{(1.5)^k}(1.5^{-2} + 1.5^{-3}) \geq \cancel{(1.5)^k}(1.5)^{-1}$

$(1.5)^{-2} + (1.5)^{-3} \geq (1.5)^{-1}$ ✓

① (c) For every $n \geq 0$, $f_n \leq 2^{n-1}$

Base case: $n=0$, $f_0 \leq 2^{0-1}$

$0 \leq 2^{-1}$

$0 \leq \frac{1}{2}$ ✓

$n=1$, $f_1 \leq 2^{1-1}$

$1 \leq 2^0$

$1 \leq 1$ ✓

We need to prove for all $n \geq 0$, the base case is 0 and 1. f_0 is 0 and 2^{0-1} is 2^{-1}, which is $\frac{1}{2}$, and $f_0 = 0 \leq \frac{1}{2}$. f_1 is 1 and 2^{1-1} is 2^0, which is 1, and $f_1 = 1 \leq 1$.

Inductive Step:

From base case, assume $f_{k-1} \leq 2^{k-1-1}$ ($f_{k-1} \leq 2^{k-2}$) and $f_k \leq 2^{k-1}$ is true from the base case 0 and 1. We need to prove that $f_{k+1} \leq 2^{k+1-1}$ ($f_{k+1} \leq 2^k$). f_{k+1} is $f_k + f_{k-1}$, since $2^{k-2} \geq f_{k-1}$ and $2^{k-1} \geq f_k$, $f_{k-1} + f_k$ is less than or equal to $2^{k-2} + 2^{k-1}$. $2^{k-1} + 2^{k-2} \geq f_{k-1} + f_k = f_{k+1}$. If we prove that $2^k \geq 2^{k-1} + 2^{k-2}$, 2^k is greater than or equal to f_{k+1} by transitivity. (If $a \geq b$ and $b \geq c$, then $a \geq c$.) 2^{k-1} can be simplified into $2^k(2^{-1})$ and 2^{k-2} can be simplified into $2^k(2^{-2})$. $2^{k-1} + 2^{k-2} = (2^k)(2^{-1}) + (2^k)(2^{-2})$ and it can be simplified into $2^k(2^{-1} + 2^{-2})$. Since 2^k is in both sides of $2^k \geq 2^k(2^{-1} + 2^{-2})$, 2^k can be divided from both sides, leaving $1 \geq (2^{-1} + 2^{-2})$, which is always true since $1 \geq 3/4$. Since $f_{k+1} = f_k + f_{k-1} \leq 2^{k-1} + 2^{k-2} \leq 2^k$, $f_{k+1} \leq 2^k$.

↳ $f_{k+1} \leq 2^k$

$$f_{k+1} = f_k + f_{k-1} \leq \underbrace{2^{k-1} + 2^{k-2} \leq 2^k}_{\downarrow}$$

$$\cancel{(2^k)}(2^{-1}+2^{-2}) \leq \cancel{(2^k)}(1)$$

$3/4 \leq 1$ ✓

① (d) $1+\log_2 X \le n \le 2(1+\log_2 X)$ if $f_n = X$ for some $X > 0$

Since x has to be greater than 0, $f_0 = 0$, so $n=0$ does not get applied. $f_1 = 1$ and $1 > 0$, so this equation gets satisfied from $n \ge 1$.

We need to prove both $1+\log_2 X \le n$ and $n \le 2(1+\log_2 X)$.
For $1+\log_2 X \le n$, you subtract 1 to both sides, giving $\log_2 X \le n-1$. To cancel $\log_2$, you make it powers of 2 and the equation becomes $2^{\log_2 X} \le 2^{n-1}$, $2^{\log_2 X}$ is equal to X, therefore, $x \le 2^{n-1}$, where x is f_n. Since $f_n \le 2^{n-1}$ was proven for all $n \ge 0$ at part c, the inequality $1+\log_2 X \le n$ is true for all $n \ge 1$.

Next, we need to prove $n \le 2(1+\log_2 X)$. Here, we divide both sides by 2, and the equation becomes $n/2 \le 1+\log_2 X$. Subtract both sides by 1, $n/2 - 1 \le \log_2 X$. Do powers of 2, $2^{(n/2-1)} \le 2^{\log_2 X}$, which makes $2^{(n/2-1)} \le X$, where X is f_n for $n \ge 1$. $2^{(n/2-1)} \le f_n$. Since we proved from part b that $f_n \ge (1.5)^{n-2}$ for $n \ge 2$, if $(1.5)^{n-2}$ is greater than or equal to $(2^{(n/2-1)})$ for $n \ge 2$, $f_n \ge 2^{(n/2-1)}$ for $n \ge 2$.
If we simplify equation $(1.5)^{n-2} \ge 2^{(n/2-1)}$

$$\log_2 1.5^{(n-2)} \ge \log_2 2^{(n/2-1)}$$

$$(n-2)(\log_2 1.5) \ge \frac{n}{2} - 1$$

$$(\log_2 1.5)n - 2\log_2 1.5 + 1 \ge \frac{n}{2}$$

$$2(\log_2 1.5)(n) - 4\log_2 1.5 + 2 \ge n$$

$$2(\log_2 1.5)(n) - 4\log_2 1.5 + 2 - n \ge 0$$

$$n(2\log_2 1.5 - 1) - 4\log_2 1.5 + 2 \ge 0$$

$$n(2\log_2 1.5 - 1) \ge 4\log_2 1.5 - 2$$

$$n \ge \frac{2\cancel{(2\log_2 1.5 - 1)}}{\cancel{2\log_2 1.5 - 1}}$$

$$n \ge 2.$$

From the previous simplification, $(1.5)^{n-2} \geq 2^{(n/2-1)}$ for all $n \geq 2$. So, since $f_n \geq (1.5)^{n-2}$ and $(1.5)^{n-2} \geq 2^{(n/2-1)}$ for all $n \geq 2$, $f_n \geq 2^{(n/2-1)}$ for all $n \geq 2$ by applying transitivity. Now, we proved $f_n \geq 2^{(n/2-1)}$ for all $n \geq 2$, so we only need the case $n=1$ to be proven. $f_1 = 1$ and, $2^{1/2-1} = 2^{-1/2} \approx 0.707$. Since $1 \geq 0.707$, $f_n \geq 2^{(n/2-1)}$ when $n=1$. Since n can only be natural numbers, except 0, since the equation $f_n \geq 2^{(n/2-1)}$ passes for $n=1$ and $n \geq 2$, it passes for all $n \geq 1$.
Therefore,

$$1+\log_2 x \leq n \leq 2(1+\log_2 x) \text{ is true for all } n \geq 1$$

(2) For base case, we have to consider having no posters at all, which means that $n=0$. If $n=0$, there is only one way: not posting it, which is $1=2^0$. Also, when there is one poster, there exists only two ways: posting it or not posting it, $2=2^1$.

The claim is that for $n \in \mathbb{N}$ different posters, there are total of 2^n possible sets of posters on the wall.

Through mathematical induction, we assume that for k posters, there are 2^k ways of posting them on the wall. For $k+1$ posters, we need to prove that there are 2^{k+1} ways of posting them. Going back to the k posters, $k+1$ posters mean that you are adding 1 poster on whether you should hang it or not, on top of already existing k posters, which have 2^k ways to be posted or not. For any poster, you have a choice to add it or not. If you choose not to add the poster, then there are still subset of k posters. In other words, if you choose not to add the poster, there are 2^k ways. If you choose to add the poster, you are adding the poster to the already existing subsets of k posters. Each subset is just changed to add the poster instead of creating new subsets, which keeps the number of subsets of k posters the same, which is 2^k. Since if you choose to not add the poster returns 2^k ways and choosing to add the poster also gives 2^k ways, if you add them, it is 2^k+2^k ways, which is equivalent to $2(2^k)$. $2(2^k)$ is 2^{k+1}. For $k+1$ posters, there are total of 2^{k+1} possible sets of posters I can hang on the wall.

↳ set is □, ☆ and you are choosing whether to add ◇

Don't Add ◇	Add ◇
∅	∅, ◇
□, ☆	□, ☆, ◇
□	□, ◇
☆	☆, ◇
$]\ 2^2$	$]\ 2^2$ (size of set is still 2^k

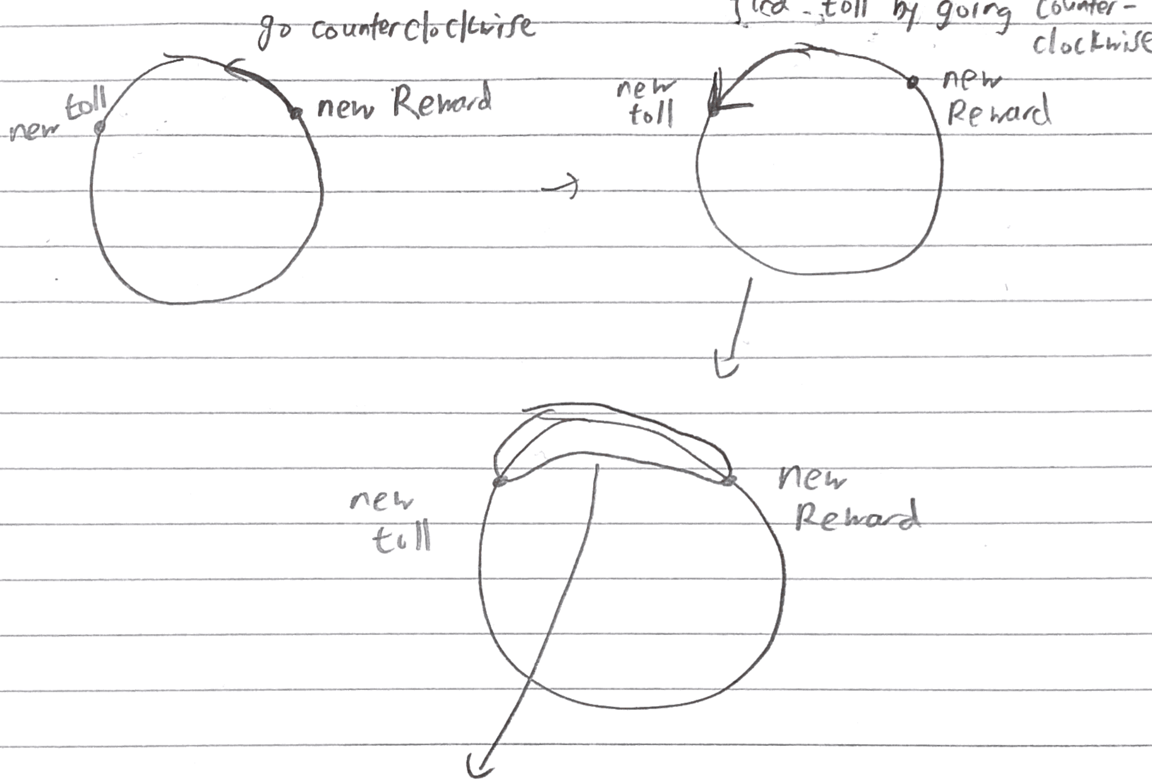
$2^2+2^2=2^2\cdot 2=2^3=2^{k+1}$

(3) There is a circular road.

Base case: If $n=0$, there are no toll booths and no reward booths, you never need to pay $1, so you never need to stop, which means that you are constantly moving.

Inductive case:

For k, $k \geq 0$, there exists a starting point that allows you to walk without stopping. We need to prove that for $k+1$ toll booths and $k+1$ reward booths, there exists a starting point that allows you to move constantly without stopping. Having $k+1$ booths means that there exists 1 more reward booth and 1 more toll both from k booths, which is already assumed to be true with our inductive step. The only way you stop is when you encounter a toll booth with no money, which means that you visit a toll booth prior to visiting a reward booth. In other words, if you pass a reward bouth before you encounter a toll booth, you will never need to stop, since reward bouth pays you the bill to pass the toll booth. Therefore, if you add a toll bouth and a reward booth to the k existing booths, if you enter a reward bouth before the toll booth, you are all set. This can be done by choosing a starting place that is at a counterclock wise direction of the reward booth, that is before the toll booth. So, if you add a random reward bouth on a circle, go counterclockwise direction until you find a tool bouth (since two booths cannot coexist at a same location). Once you find a tool booth, choose your starting location as anywhere between the two booths in the counterclockwise direction of reward booth. This way, since you move clockwise direction, you will hit the new added reward booth first, which will provide the fee to pass the new added toll booth. And since the rest k booths have already been assumed to be true, there always exists a starting point to the circle that will make you go continuously.



there exists a starting point that makes you move constantly.