4) a) $Z = X + Y \qquad p_{X,Y} = p_X \cdot p_Y$

show that $p_Z = p_Y * p_X$

$$p_Z(z) = \int_x p_X(x)\, p_Y(z-x)\, dx$$

$$= \int_y p_Y(y)\, p_X(z-y)\, dy$$

substitution : $x = z - y$

$$\frac{dx}{dy} = -1 \rightarrow dx = -1\,dy$$

$$p_Z(z) = \int_{x_1}^{x_2} p_X(x)\, p_Y(z-x)\, dx = -\int_{y_1}^{y_2} p_X(z-y)\, p_Y(y)\, dy = \int_{y_2}^{y_1} p_X(z-y)\, p_Y(y)\, dy$$

$$\int_x p(x)\, p_Y(z-x)\, dx = \int_y p_Y(y)\, p_X(z-y)\, dy$$

$$p_Z = p_Y * p_X = p_X * p_Y$$

b) additive: $C_p^Z = C_p^X + C_p^Y$

$$C_{X+Y}(w) = \log \mathbb{E}(e^{w(X+Y)})$$

$$= \log(\mathbb{E}(e^{wX}) \cdot \mathbb{E}(e^{wY}))$$

$$= \log \mathbb{E}(e^{wX}) + \log \mathbb{E}(e^{wY})$$

$$= C_X(w) + C_Y(w) = C_Z(w)$$