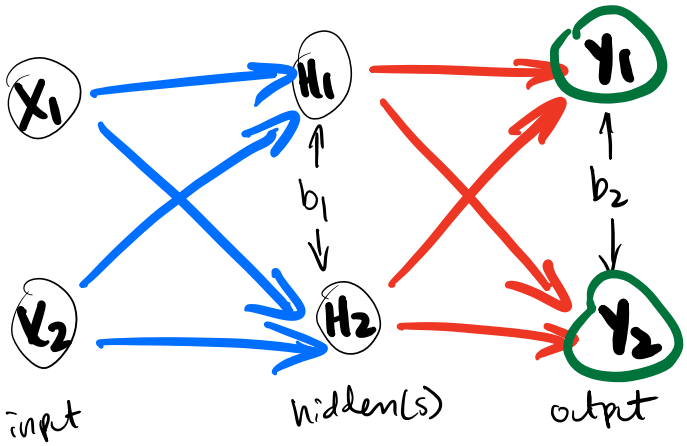
Example of Backpropagation



Values, weights & biases → outputs

Let,

Given $X_1 = 0.48$, $X_2 = 0.84$, $LR = 0.5$

X_1) $W_{X_1H_1} = 0.2$, $W_{X_1H_2} = 0.3$, $W_{H_1Y_1} = 0.1$, $W_{H_1Y_2} = 0.5$, $b_1 = 0.12$, $b_2 = 0.29$ } initial weights

X_2) $W_{X_2H_1} = 0.7$, $W_{X_2H_2} = 0.5$, $W_{H_2Y_1} = 0.6$, $W_{H_2Y_2} = 0.4$

Activation function: sigmoid = $\frac{1}{1+e^{-x}}$

Forward propagation

H_1^*: $X_1 * W_{X_1H_1} + X_2 * W_{X_2H_1} + b_1 = (0.48)(0.2) + (0.84)(0.7) + 0.12 = 0.096 + 0.588 + 0.12 = 0.804$

H_2^*: $X_1 * W_{X_1H_2} + X_2 * W_{X_2H_2} + b_1 = (0.48)(0.3) + (0.48)(0.5) + 0.12 = 0.144 + 0.24 + 0.12 = 0.504$

→ Pass the 1st set of outputs through the activation function for this layer

$output(H_1) = \frac{1}{1+e^{-H_1^*}} = \frac{1}{1+e^{-(0.804)}} = 0.6908$, $output(H_2) = \frac{1}{1+e^{-H_2^*}} = \frac{1}{1+e^{-0.504}} = 0.623$

Y_1^*: $H_1 * W_{H_1Y_1} + H_2 * W_{H_2Y_1} + b_2 = (0.6908)(0.1) + (0.623)(0.6) + 0.29 = 0.7328$

Y_2^*: $H_1 * W_{H_1Y_2} + H_2 * W_{H_2Y_2} + b_2 = (0.6908)(0.5) + (0.623)(0.4) + 0.29 = 0.8846$

$output(Y_1) = \frac{1}{1+e^{-.732}} = 0.675$, $output(Y_2) = \frac{1}{1+e^{-.8846}} = 0.708$

→ Calculating total error

$E_{total} = \sum \frac{1}{2}(target - output)^2$ → $E_1 + E_2$

$= \frac{1}{2}(T_1 - Y_1)^2 + \frac{1}{2}(T_2 - Y_2)^2$

$= \frac{1}{2}(0.48 - 0.675)^2 + \frac{1}{2}(0.84 - 0.708)^2$

$= .019 + .0087 = .0261$

← **Backward Pass**: updating weights, modifying error at weight level

Updating weights preceding final output layer

$$\text{error at } W_{H_1Y_1} = \frac{\partial E_{total}}{\partial W_{H_1Y_1}} = \frac{\partial E_{total}}{\partial y_1} * \frac{\partial y_1}{\partial y_1^*} * \frac{\partial y_1^*}{\partial W_{H_1Y_1}}$$

$$E_{total} = \tfrac{1}{2}(T_1 - y_1)^2 + \tfrac{1}{2}(T_2 - y_2)^2$$

$$\frac{\partial E_{total}}{\partial y_1} = 2\left(\tfrac{1}{2}\right)(T_1 - y_1)(-1) + 0$$

$$= -(T_1 - y_1) = -(0.98 - 0.675) = -.195$$

$$y_1 = \frac{1}{1+e^{-y_1^*}}$$

$$\frac{\partial y_1}{\partial y_1^*} = y_1(1 - y_1) = 0.675(1 - 0.675) = .219$$

$$y_1^* = (H_1)(W_{H_1Y_1}) + (H_2)(W_{H_2Y_1}) + b_2$$

$$\frac{\partial y_1^*}{\partial W_{H_1Y_1}} = H_1 = 0.6908$$

$$\therefore \frac{\partial E_{total}}{\partial W_{H_1Y_1}} = (.195)(.219)(.6908) = .0295$$

$$\text{Updating } W_{H_1Y_1} = W_{H_1Y_1} - L.R.\left(\frac{\partial E_{total}}{\partial W_{H_1Y_1}}\right)$$

$$= 0.1 - (0.5)(.0295) = 0.08525$$

$$\text{error at } W_{H_1Y_2} = \frac{\partial E_{total}}{\partial W_{H_1Y_2}} = \frac{\partial E_{total}}{\partial y_2} * \frac{\partial y_2}{\partial y_2^*} * \frac{\partial y_2^*}{\partial W_{H_1Y_2}}$$

$-(T_2 - y_2)$ $\quad$ $y_2(1 - y_2)$ $\quad$ H_1

$= .132 \times = .207 \times = 0.6908$

$= .189$

Updating $W_{H_1Y_2} = 0.5 - (0.5)(.189) = .4055$

$$\text{error at } W_{H_2Y_1} = \frac{\partial E_{total}}{\partial W_{H_2Y_1}} = \frac{\partial E_{total}}{\partial y_1} * \frac{\partial y_1}{\partial y_1^*} * \frac{\partial y_1^*}{\partial W_{H_2Y_1}}$$

$-.195$ $\quad$ $.219$ $\quad$ $H_2 = .623$

$\Pi = -.0266$

$$W_{H_2Y_1} = 0.6 - (0.5)(-.0266) = 0.5867$$

$$\text{error at } W_{H_2Y_2} = \frac{\partial E_{total}}{\partial y_2} * \frac{\partial y_2}{\partial y_2^*} * \frac{\partial y_2^*}{\partial W_{H_2Y_2}}$$

$\Pi = (.132)(.207)(.623)$

$= .0170$

$W_{H_2Y_2} = 0.4 - (0.5)(.0170) = .3915$

Updating weights proceeding initial input layer

weights: $W_{x_1H_1}, W_{x_1H_2}, W_{x_2H_1}, W_{x_2H_2}$

$$\frac{\partial E_{total}}{\partial W_{x_1H_1}} = \left(\frac{\partial E_{total}}{\partial H_1}\right)\left(\frac{\partial H_1}{\partial H_1^*}\right)\left(\frac{\partial H_1^*}{\partial W_{x_1H_1}}\right)$$

$W_{x_1H_1}$

↳ $\frac{\partial E_{total}}{\partial H_1} = \frac{\partial E_1}{\partial H_1} + \frac{\partial E_2}{\partial H_1}$

↳ $\frac{\partial E_1}{\partial H_1} = \left(\frac{\partial E_1}{\partial y_1^*}\right)\left(\frac{\partial y_1^*}{\partial H_1}\right)$ → $W_{H_1Y_1}$

↳ $\frac{\partial E_1}{\partial y_1^*} = \left(\frac{\partial E_1}{\partial y_1}\right)\left(\frac{\partial y_1}{\partial y_1^*}\right)$

$= (.195)(.219) = .0427$

$= (.0427)(0.1) = .00427$

$\frac{\partial E_2}{\partial H_1} = \left(\frac{\partial E_2}{\partial y_2^*}\right)\left(\frac{\partial y_2^*}{\partial H_1}\right)$ → $W_{H_1Y_2}$

↳ $\left(\frac{\partial E_2}{\partial y_2}\right)\left(\frac{\partial y_2}{\partial y_2^*}\right) = (.132)(.207) = .027$

$= (.027)(0.5) = .0135$

$= .00427 + .0135 = .01777$

$\frac{\partial H_1}{\partial H_1^*} = \left(H_1 = \frac{1}{1+e^{-H_1^*}}\right) \Rightarrow H_1(1-H_1) = (0.6908)(1-0.6908) = .2136$

$\frac{\partial H_1^*}{\partial W_{x_1H_1}} = X_1 = 0.48$

$\therefore (.01777)(.2136)(0.48) = .00182$

Updating $W_{x_1H_1} = 0.2 - (0.5)(.00182) = .199$

Updating W_{x,H_2}

$$\frac{\partial E_{total}}{\partial W_{x,H_2}} = \left(\frac{\partial E_{total}}{\partial H_2}\right)\left(\frac{\partial H_2}{\partial H_2^*}\right)\left(\frac{\partial H_2^*}{\partial W_{x,H_2}}\right) \qquad W_{x,H_2}$$

$$\hookrightarrow \frac{\partial E_{total}}{\partial H_2} = \frac{\partial E_1}{\partial H_2} + \frac{\partial E_2}{\partial H_2}$$

$$\hookrightarrow \left(\frac{\partial E_1}{\partial y_1^*}\right)\left(\frac{\partial y_1^*}{\partial H_2}\right)$$

$$= \left(\frac{\partial E_1}{\partial y_1}\right)\left(\frac{\partial y_1}{\partial y^*}\right)\left(W_{H_2 y_1}\right)$$

$$= (.195)(.119)(.6) = .0256$$

$$\frac{\partial E_2}{\partial H_2} = \left(\frac{\partial E_2}{\partial y_2}\right)\left(\frac{\partial y_2}{\partial y_2^*}\right)\left(W_{H_2 y_2}\right)$$

$$= (.132)(-.207)(.4) = .0109$$

$$= .0256 + .0109 = .0365$$

$$\frac{\partial H_2}{\partial H_2^*} = H_2(1-H_2) = (0.623)(.377) = .235$$

$$\frac{\partial H_2^*}{\partial W_{x,H_2}} = X_1 = 0.48$$

$$\therefore (.0365)(.235)(.48) = .00412$$

Updating $W_{x,H_2} = 0.3 - (0.5)(.00412) = .298$

Updating $W_{X_2H_1}$

$$\frac{\partial E_{total}}{\partial W_{X_2H_1}} = \left(\frac{\partial E_{total}}{\partial H_1}\right)\left(\frac{\partial H_1}{\partial H_1^*}\right)\left(\frac{\partial H_1^*}{\partial W_{X_2H_1}}\right) \quad X_2 H_1$$

$$\frac{\partial E_{total}}{\partial H_1} = \frac{\partial E_1}{\partial H_1} + \frac{\partial E_2}{\partial H_1}$$

$$\left(\frac{\partial E_1}{\partial y_1}\right)\left(\frac{\partial y_1}{\partial y_1^*}\right)(W_{H_1Y_1}) + \left(\frac{\partial E_2}{\partial y_2}\right)\left(\frac{\partial y_2}{\partial y_2^*}\right)(W_{H_1Y_2})$$

$$= -.0177$$

$$\frac{\partial H_1}{\partial H_1^*} = H_1(1-H_1) = .2136$$

$$\frac{\partial H_1^*}{\partial W_{X_2H_1}} = X_2 = .84$$

$$\therefore (-.0177)(.2136)(.84) = -.00317$$

Updating $W_{X_2H_1} = 0.7 - (0.5)(-.00317)$

$$= 0.698$$

Updating $W_{X_2H_2}$

$$\frac{\partial E_{total}}{\partial W_{X_2H_2}} = \left(\frac{\partial E_{total}}{\partial H_2}\right)\left(\frac{\partial H_2}{\partial H_2^*}\right)\left(\frac{\partial H_1^*}{\partial W_{X_2H_2}}\right)$$

$$= -.0256 \quad .235 \quad .84$$

$$\therefore \Pi = -.00505$$

$W_{X_2H_2} = 0.5 - (0.5)(-.00505) = 0.497$

Summary:

initial values:

$x_1 = 0.98$, $x_2 = 0.84$

$b_1 = 0.12$, $b_2 = 0.29$

Forward propagation

$H_1^* = 0.804$, $\text{sigmoid}(H_1^*) = 0.6908$

$H_2^* = 0.504$, $\text{sigmoid}(H_2^*) = 0.623$

$Y_1^* = 0.7328$, $\text{sigmoid}(Y_1^*) = 0.675$

$Y_2^* = 0.6846$, $\text{sigmoid}(Y_2^*) = 0.708$

Backward propagation

$W_{H_1Y_1} = 0.1 \Rightarrow .08525$

$W_{H_1Y_2} = 0.5 \Rightarrow .4055$

$W_{H_2Y_1} = 0.6 \Rightarrow 0.5867$

$W_{H_2Y_2} = 0.4 \Rightarrow 0.3915$

$W_{X_1H_1} = 0.2 \Rightarrow 0.199$

$W_{X_1H_2} = 0.3 \Rightarrow 0.298$

$W_{X_2H_1} = 0.7 \Rightarrow 0.698$

$W_{X_2H_2} = 0.5 \Rightarrow 0.497$