Konvexe Optimierung

5. Übungsserie

Aufgabe 33

$$
\begin{aligned}
h_\delta(x,y) &= \sum_i^n \delta^2 \left(\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2} - 1 \right) \\
&= \delta^2 \left(\sum_i^n \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \sum_i^n 1 \right) \\
&= \delta^2 \left(\sum_i^n \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - n \right) \\
&= \delta^2 \sum_i^n \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \delta^2 n
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x_0} h_\delta(x,y) &= \frac{\partial}{\partial x_0} \sum_i^n \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \delta^2 n && (1)\\
&= \sum_i^n \frac{\partial}{\partial x_0} \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \frac{\partial}{\partial x_0} \delta^2 n && (2)\\
&= \sum_i^n \frac{\partial}{\partial x_0} \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} && (3)\\
&= \sum_i^n \frac{1}{2} \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x_0} \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right) && (4)\\
&= \sum_i^n \frac{1}{2} \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{-\frac{1}{2}} \cdot 2 \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right) \cdot \frac{\partial}{\partial x_0} \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right) && (5)\\
&= \sum_i^n \frac{1}{2} \left(1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2\right)^{-\frac{1}{2}} \cdot 2 \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right) \cdot \frac{\xi_i \cdot \delta}{\delta^2} && (6)\\
&= \sum_i^n \frac{\left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right) \cdot \frac{\xi_i \cdot \delta}{\delta^2}}{\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2}} && (7)\\
&= \sum_i^n \frac{\left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right) \cdot \frac{\xi_i}{\delta}}{\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2}} && (8)\\
&= \sum_i^n \frac{\xi_i^2 x_0 + \xi_i x_1 - \xi_i \eta_i}{\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2}} && (9)\\
& && (10)
\end{aligned}
$$

$$\frac{\partial}{\partial x_1} h_\delta(x,y) = \sum_i^n \frac{\left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right) \cdot \frac{\delta}{\delta^2}}{\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2}} \tag{11}$$

$$= \sum_i^n \frac{\frac{\xi_i x_0 + x_1 - \eta_i}{\delta^2}}{\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2}} \tag{12}$$

$$= \sum_i^n \frac{\xi_i x_0 + x_1 - \eta_i}{\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2}} \tag{13}$$

$$\tag{14}$$

Hessian

$$H(phl) = \begin{bmatrix} \frac{\partial phl}{\partial x_0 \partial x_0} & \frac{\partial phl}{\partial x_0 \partial x_1} \\ \frac{\partial phl}{\partial x_1 \partial x_0} & \frac{\partial phl}{\partial x_1 \partial x_1} \end{bmatrix} \tag{15}$$

$$
\begin{aligned}
\frac{\partial phl}{\partial x_1 \partial x_0} &= \frac{\partial}{\partial x_0} \sum_i^n \frac{\xi_i x_0 + x_1 - \eta_i}{\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2}} \\
&= \sum_i^n \frac{\partial}{\partial x_0} \frac{\xi_i x_0 + x_1 - \eta_i}{\sqrt{1 + \left(\frac{\xi_i x_0 + x_1 - \eta_i}{\delta}\right)^2}} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \frac{\partial}{\partial x_0}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}}}{\left(\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}}\right)^2} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \frac{\partial}{\partial x_0}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}}}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \frac{\partial}{\partial x_0}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}}}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \frac{1}{2}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x_0}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \frac{1}{2}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x_0}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \frac{1}{2}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{\partial}{\partial x_0} 1 + \frac{\partial}{\partial x_0}\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \frac{1}{2}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x_0}\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{\frac{1}{2}} - \left(\xi_i x_0 + x_1 - \eta_i\right) \cdot \frac{1}{2}\left(1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2\right)^{-\frac{1}{2}} \cdot 2\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right) \cdot \frac{\partial}{\partial x_0}\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_0}\left(\xi_i x_0 + x_1 - \eta_i\right) \sqrt{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{\xi_i x_0+x_1-\eta_i}{\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}} \cdot \frac{\xi_i x_0+x_1-\eta_i}{\delta} \cdot \frac{\partial}{\partial x_0}\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\xi_i \sqrt{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i x_0+x_1-\eta_i)^2(\xi_i\delta+\xi_i x_0+x_1-\eta_i)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}} \cdot}{1 + \left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial phl}{\partial x_1 \partial x_1} &= \frac{\partial}{\partial x_0} \sum_i^n \frac{\xi_i x_0 + x_1 - \eta_i}{\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}} \\
&= \sum_i^n \frac{\frac{\partial}{\partial x_1}\left(\xi_i x_0 + x_1 - \eta_i\right)\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i x_0+x_1-\eta_i)^2\left(\frac{\partial}{\partial x_1}(\xi_i x_0+x_1-\eta_i)\delta+(\xi_i x_0+x_1-\eta_i)\right)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}}\cdot}{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
&= \sum_i^n \frac{\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i x_0+x_1-\eta_i)^2(\delta+\xi_i x_0+x_1-\eta_i)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}}\cdot}{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial phl}{\partial x_0 \partial x_0} &= \frac{\partial}{\partial x_0} \sum_i^n \frac{\xi_i^2 x_0 + \xi_i x_1 - \xi_i \eta_i}{\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}} \\
&= \sum_i^n \frac{\xi_i^2\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i^2 x_0+\xi_i x_1-\xi_i\eta_i)(\xi_i x_0+x_1-\eta_i)(\xi_i\delta+\xi_i x_0+x_1-\eta_i)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}}\cdot}{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial phl}{\partial x_0 \partial x_1} &= \frac{\partial}{\partial x_1} \sum_i^n \frac{\xi_i^2 x_0 + \xi_i x_1 - \xi_i \eta_i}{\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}} \\
&= \sum_i^n \frac{\xi_i\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i^2 x_0+\xi_i x_1-\xi_i\eta_i)(\xi_i x_0+x_1-\eta_i)(\delta+\xi_i x_0+x_1-\eta_i)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}}\cdot}{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}
\end{aligned}
$$

$$
H(phl) = \begin{bmatrix} \frac{\partial phl}{\partial x_0 \partial x_0} & \frac{\partial phl}{\partial x_0 \partial x_1} \\ \frac{\partial phl}{\partial x_1 \partial x_0} & \frac{\partial phl}{\partial x_1 \partial x_1} \end{bmatrix} \tag{16}
$$

$$
= \begin{bmatrix}
\sum_i^n \frac{\xi_i^2\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i^2 x_0+\xi_i x_1-\xi_i\eta_i)(\xi_i x_0+x_1-\eta_i)(\xi_i\delta+\xi_i x_0+x_1-\eta_i)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}}\cdot}{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} & \sum_i^n \frac{\xi_i\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i^2 x_0+\xi_i x_1-\xi_i\eta_i)(\xi_i x_0+x_1-\eta_i)(\delta+\xi_i x_0+x_1-\eta_i)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}}\cdot}{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} \\
\sum_i^n \frac{\xi_i\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i x_0+x_1-\eta_i)^2(\xi_i\delta+\xi_i x_0+x_1-\eta_i)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}}\cdot}{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} & \sum_i^n \frac{\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2} - \frac{(\xi_i x_0+x_1-\eta_i)^2(\delta+\xi_i x_0+x_1-\eta_i)}{\delta\sqrt{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}}\cdot}{1+\left(\frac{\xi_i x_0+x_1-\eta_i}{\delta}\right)^2}
\end{bmatrix} \tag{17}
$$