Mathematical and Musical Properties of Pairwise Well-Formed Scales

David Clampitt

Ohio State University
clampitt.4@osu.edu

The short paper below presents the definition of the *pairwise well-formed scale* concept, and a few of the significant mathematical and musical features entailed by that definition. The verifications that are easily available are supplied here; for the more difficult proofs which are here omitted, the reader is directed to my dissertation (Clampitt 1997). While the definition itself is quite abstract, the body of implications and equivalences that constitute the theory include several musically attractive properties. With the significant exception of one structural subcategory, all other pairwise well-formed scales participate in *modulating cycles* that generalize the *maximally smooth cycles* defined in Cohn 1996 and intersect with the *Cohn functions* defined in Lewin 1996.

I take this opportunity to celebrate the recent discovery of a strong relationship between musical scale theory and a well-developed branch of mathematics, *algebraic combinatorics on words*. Some translation between these domains will be done here; the paper by Manuel Domínguez (2008) (together with Thomas Noll and myself) provides more details. In particular, I will place the discussion of pairwise well-formed scales in a word-theory context. As far as I can tell, this study has not been pursued by the word combinatorialists, but the study is perhaps sufficiently interesting and mathematics is so vast as to warrant caution in any such assertion. Such a statement cannot be made about the first-order concept of well-formed scales, upon which the notion to be considered is built. The theory of well-formed scales may be mapped into the theory of Christoffel words and Sturmian words. The question is open as to whether parts of the existing mathematical theory may be mapped back to music theory to create new music-theoretical meaning. There are strong suggestions that this may be the case.

The objects being considered here have interpretations in the domain of musical scales, and might also be interpreted as rhythmic patterns. Pairwise well-formed scales abound in world music, including the Japanese *In*-scale or hemitonic pentatonic, so-called gypsy or Hungarian minor, the octatonic-minus-one, and, in the microtonal world, the diatonic scale under just intonation. For musical purposes, one may wish to preserve this level of concreteness. For example, two distinct concrete musical instantiations of the same word are:

Just scale: C 9/8 D 10/9 E 16/15 F 9/8 G 10/9 A 9/8 B 16/15 (C)
$\langle a \quad b \quad c \quad a \quad b \quad a \quad c \rangle$
ma-grama: 4 sa 3 ri 2 ga 4 ma 3 pa 4 dha 2 ni
$\langle a \quad b \quad c \quad a \quad b \quad a \quad c \rangle$

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The intervals for the just scale are given as frequency ratios, and for ma-grama as a number of *srutis*, understood as the band of sound *below* the given scale step, which are identified for our purposes as units of an underlying division of the octave into 22 parts. (See the discussion of early Indian heptatonic scales in Clough, Douthett, Ramanathan, and Rowell 1993.)

The mathematical properties of pairwise well-formed scales emerge most clearly, however, at the level of abstraction which is the domain of combinatorics on words. By *words* are meant, for now, strings of symbols, of finite length, over a finite alphabet A. Subwords are called *factors*, and words are combined via concatenation, so if z, q, and r are words, with $z = qr$, then q and r are factors of z. We define a set C of circular words. Let C = $\{ w \mid w : Z_N \to A$, where N is the length of $w\}$. (It will turn out that $A=\{a,b,c\}$ for our purposes.) The dihedral group D_N acts on C by permuting the arguments of w cyclically/by retrogression: $\mathrm{Rot}_m(w)(j) := w(j+m)$, and $\mathrm{Retro}_m(w)(j) := w(m-j)$, for m from 0 to $N-1$. The *word classes* of C will be the orbits under this action. In word theory (where words are not assumed to be circular), two words x and y are said to be *conjugate* if there exist words u and v such that $x = uv$ and $y = vu$. It should be clear that x and y are conjugate if and only if they are cyclic shifts of each other, and the *conjugacy class* of a word w is the orbit of w under the rotations Rot_m. (See Lothaire 2002 for further discussion of these definitions.)

1 Pairwise Well-Formed and Well-Formed Scales

The pairwise well-formed scale concept depends on that of the well-formed scale. At the concrete level, a scale is *well-formed* if it is generated by an interval of constant size and span, that is, if all the notes of the scale may be linked together in a chain where the links are intervals of the same size that span the same number of scale step intervals (see Domínguez, Clampitt, and Noll 2008 for further information).

Here is a definition of well-formedness in terms of multiples of a real number modulo 1, in other words, as fractional parts of multiples of a real number. For θ real and an integer $N>1$, let S = $\{n\theta-\lfloor n\theta\rfloor \mid 0\leq n<N\}$. Then listing the elements of S in order, $0=s_0<s_1<\ldots<s_{N-1}$, we say that S is well-formed if and only if there exists a unit u mod N such that the linear map $\mu : Z_N\to Z_N$: $z\to uz(\mathrm{mod}\ N)$ yields $s_{\mu(z)}=z\theta-\lfloor z\theta\rfloor$. It should be clear that the mathematical definition is equivalent to the ordinary language definition.

A scale is *non-degenerate well-formed* if it is well-formed and its step intervals (differences $s_{i+1}-s_i$, taken mod 1) come in two sizes. The usual diatonic and pentatonic scales are examples of non-degenerate well-formed scales; an equal-interval division of the octave is a degenerate well-formed scale.

By a theorem of well-formed scale theory, there are reciprocal relationships between the multiplicities of the step intervals and the spans of generating intervals (Carey and Clampitt 1996, 71). Define the *intervals of S* to be the ordered pairs of S, that is, the elements of S×S. Define the *span* of an interval (s_i, s_j) to be $(j-i)\mathrm{mod}\ N$, and define the *size* of an interval to be $(s_j-s_i)\mathrm{mod}\ 1$. The *step intervals* are the intervals of span 1. The *multiplicity* of a given interval x is the number of intervals of S that have the same size as x. If S is a non-degenerate well-formed scale of cardinality N, and u is the multiplicity of one of its step intervals, then $N–u$ is the multiplicity of the other step interval, and the spans of the generating intervals are $u^{-1}{}_{\text{mod } N}$ and $-u^{-1}{}_{\text{mod } N}$. For example, in the usual diatonic, the multiplicity of the semitone is 2, the span of the generating perfect fifth is $4 \equiv 2^{-1}{}_{\text{mod } 7}$; the multiplicity of the whole step is 5, the span of the generating perfect fourth is $3 \equiv 5^{-1}{}_{\text{mod } 7.}$

According to this theorem, the structure of a non-degenerate well-formed scale is determined by a matrix of integers, u, $u^{-1}{}_{\text{mod } N}$, $N–u$, and $-u^{-1}{}_{\text{mod } N}$. As these four numbers are themselves already determined by N and u, we may compactly represent a well-formed scale class as wfs(N,u). By convention we choose u to be the multiplicity of the less frequent step interval in a scale of cardinality N. (There must be smaller and larger multiplicities, because gcf(N, u)=1).

Each wfs(N,u) corresponds to a circular word class on two letters. For example, wfs(7,2) is represented by <*a a a b a a b*>, while <*a a b a b a a b a b a b*> corresponds to wfs(12, 5).

Let *w* be a circular word on some alphabet A with *k* letters. Define the *pairwise projections of w* to be the words that result from identifying pairs of letters in *A*. *w* is *pairwise well-formed* if all the pairwise projections of *w* correspond to (non-degenerate) well-formed scale classes (following the definition in Clampitt 1997). If *w* is a word on *k* letters, its pairwise projections are words on *k–1* letters. Since all words corresponding to non-degenerate well-formed scale classes must be on 2 letters, if *w* is pairwise well-formed, it must necessarily be a word on 3 letters. For example the circular word <*a b c a b*> (exemplified by the Japanese *In*-scale or *hira-joshi*, E F A B C (E)) has as its three pairwise projections <*a b a a b*> ($a = c$), <*a a c a a*> ($a = b$), and <*a b b a b*> ($b = c$). Each is well-formed, as the reader may verify, so <*a b c a b*> is pairwise well-formed.

2 Some Properties of Pairwise Well-Formed Scales

Let g_a, g_b, and g_c be the multiplicities of the letters *a*, *b*, and *c* (i.e., the step interval multiplicities) in a pairwise well-formed scale of cardinality N. Then gcd(g_a, N) = 1, gcd(g_b, N) = 1, and gcd(g_c, N) = 1. This follows immediately from the requirement that each must serve in turn as a step interval multiplicity in a well-formed scale; recall from above that such multiplicities are necessarily units mod N.

Pairwise well-formed scales are of odd cardinality. This is an immediate consequence of the previous result. Since all three multiplicities must be coprime with N, if N were even, g_a, g_b, g_c would all be odd. But the sum of 3 odd integers is odd, so N would turn out to be odd.

Pairwise well-formed scales exhibit *trivalence*: In a pairwise well-formed scale of cardinality N, intervals of a given non-zero generic span mod N come in 3 specific sizes (Clampitt 1997). (Trivalence does not characterize pairwise well-formedness: e.g., C D-flat E G A-flat (C), <13314>, is trivalent, but not pairwise well-formed.)

3 Classification of Pairwise Well-Formed Scales

The heptatonic scale C D E-flat F-sharp G A-flat B (C), so-called Hungarian minor or gypsy minor, modes of which appear in Indian, Arabic, and Jewish music, is a pairwise well-formed scale with a unique structure, represented by the word <*abacaba*>. The concrete instance of this word as Hungarian minor is not the only one available within the 12-pitch-class system—C D-flat E-flat E F G A-flat (C) also has this structure—but perhaps because it may be understood as a deformation of the usual diatonic, or because of the related fact that it supports a number of harmonic triads, the so-called Hungarian minor is the most prominent representative of this word-class as a scale.

All pairwise well-formed scales are either of the so-called Hungarian or gypsy minor type, that is, of the class expressed by w = <*abacaba*>, or else they have two step intervals of the same multiplicity. We call pairwise well-formed scales of the <*abacaba*> type *singular* and all others *non-singular*. *Only* singular pairwise well-formed scales have 3 distinct step multiplicities (1, 2, 4); in non-singular pairwise well-formed scales the step interval multiplicities are (m, m, n), with $m \neq n$. Singular pairwise well-formed scales are self-similar in that all interval cycles of span d, d non-zero modulo 7, have the same structure (that is, correspond to the word <*abacaba*>).

Considered as scales at the concrete level, singular pairwise well-formed scales may be generated (in the case where the unique interval $c = a+b$). Non-singular pairwise well-formed scales can not be generated. Non-singular pairwise well-formed scales participate in cycles that generalize Cohn's maximally smooth cycles. That is, there exist cycles of length at least 3 where adjacent elements are inversionally related and differ by a single note. Singular pairwise well-formed scales cannot participate in such cycles; they are transformationally frozen in a way that non-singular pairwise well-formed scales are not.

Parallel to the definition of a maximally smooth cycle in Cohn 1996, I define a generalization, the *Q-relation* and *Q-cycle*. Two pitch-class sets are in a Q-relation if there exists a transposition or inversion mapping one set onto the other that leaves all but one pitch class invariant and moves the remaining pitch class by any interval class, where the moving pitch class *slides* between fixed pitch classes, rather than *leaping over* them. For example, {01267} and its inversion (about the 3/4 axis){01567} are Q-related (2 slides to 5), whereas {01267} and its inversion (about the 0/1 axis){0167e} are not (2 leaps over stationary pitch classes to 11). For Q-cycles, let us first stipulate sets of cardinality greater than 3, because trichords with three distinct step sizes are trivially pairwise well-formed and support cycles of Q-related sets, but require some special attention precisely because it is so easy to construct such cycles. For sets of cardinality greater than 3, a Q-cycle is a cycle of length greater than 2 where adjacent sets in the cycle are Q-related.

As an example of a Q-cycle for pairwise well-formed scales, consider the Japanese hemitonic pentatonic, the *In*-scale, which may be notated as E F A B C (E). Like the usual pentatonic, it can be considered a connected segment of the diatonic cycle of fifths, but unlike the usual pentatonic, this subset embraces the diminished fifth as well: A E B F C. Its cyclic step-interval sequence modulo 12 is <1 4 2 1 4>. It is possible to modulate to an inverted form of the scale either by moving one note down a (chromatic) semitone, (exchanging the positions of adjacent 2 and 1), or by moving one note up a tone, (exchanging the positions of adjacent 4 and 2), and if one assumes 12-note equal temperament, through all 24 members of the Forte set class 5-20: E F A B C $\rightarrow$ E F A B^{b} C $\rightarrow$ E F A B^{b} D $\rightarrow$ . . . $\rightarrow$ B C E F$^{\#}$ G $\rightarrow$ B C E F G $\rightarrow$ (B C E F A). The product of two successive inversions is a T_5-transposition of the original set. All non-singular pairwise well-formed scales exhibit either such cycles (within equal-divisions of the octave) or infinite chains, where adjacent sets in the chain are Q-related.

As an example of an infinite Q-chain, consider the diatonic scale in just intonation. Here the intervals of motion are alternately syntonic commas and larger *limmas*. The just major scale has step intervals (in frequency ratios) as follows:

do		re		mi		fa		sol		la		ti		(do)
	9/8		10/9		16/15		9/8		10/9		9/8		16/15	

If *re* is lowered by a syntonic comma (81/80), this produces an inverted form of the scale: <10/9, 9/8, 16/15, 9/8, 10/9, 9/8, 16/15>. (Because 9/8 x 80/81 = 10/9. Since the intervals are expressed here in frequency ratios, "subtracting a syntonic comma" is expressed as division by 81/80, or multiplication by its inverse.) If we follow this operation by lowering *ti* by a larger *limma* (multiplication by 128/135), the result is an inversion again. The composition of the two operations is a transposition by a just (Pythagorean) perfect fourth. Alternating these two operations forms an infinite Q-chain.

The distinction between cycles and chains only arises for concrete instantiations of the scale. At the word-theory level of description, however, the transformations take place within the class of conjugates of the word and of its reversal. The Japanese pentatonic, for example, is represented by the word <*a b c a b*>. The Q-cycle exchanges adjacent letters *c a* and *b c* (or *c b*), always yielding rotations or retrogressions of the original word: *a b c a b* $\rightarrow$ *a b a c b* $\rightarrow$ *a b a b c* $\rightarrow$ *c b a b a* $\rightarrow$ *b c a b a* $\rightarrow$ *b a c b a* $\rightarrow$ *b a b c a* $\rightarrow$ *b a b a c* $\rightarrow$ *c a b a b* $\rightarrow$ *a c b a b* $\rightarrow$ (*a b c b a*), a cycle containing all ten elements of the word class, under the action of D_5.

References

Carey, N., Clampitt, D.: Self-Similar Pitch Structures, Their Duals and Rhythmic Analogues. Perspectives of New Music 34(2), 62–87 (1996)

Clampitt, D.: Pairwise Well-Formed Scales: Structural and Transformational Properties. Ph.D. diss., State University of New York at Buffalo (1997)

Clough, J., Douthett, J., Ramanathan, N., Rowell, L.: Early Indian Heptatonic Scales and Recent Diatonic Theory. Music Theory Spectrum 15, 36–58 (1993)

Cohn, R.: Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions. Music Analysis 15, 9–40 (1996)

Domínguez, M., Clampitt, D., Noll, T.: Well-formed Scales, Maximally Even Sets, and Christoffel Words. In: Klouche, T., Noll, T. (eds.) MCM 2007. CCIS, vol. 37. Springer, Heidelberg (2008)

Lewin, D.: Cohn Functions. Journal of Music Theory 40, 181–216 (1996)

Lothaire, M.: Algebraic Combinatorics on Words. Cambridge University Press, Cambridge (2002)