ALTERNATIVE INTERPRETATIONS OF SOME MEASURES FROM *PARSIFAL*

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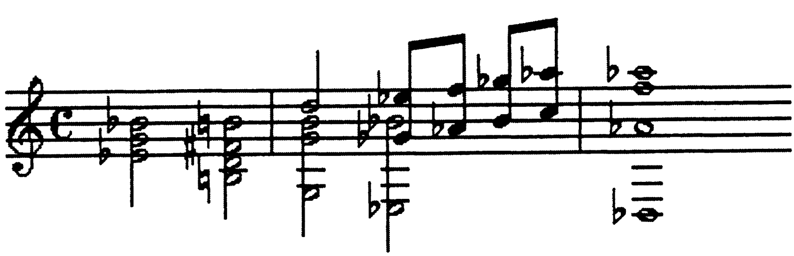
The analytical potential of the hexatonic organization of the consonant triads was demonstrated in a recent article by Richard Cohn (1996). This was accomplished by providing the consonant triad with a status independent of the contexts of diatonic scales, tonal centers, or indeed of triadic roots, a status that is, however, anything but free-floating; rather, a position established within highly constrained spaces carved out by *maximally smooth hexatonic cycles*, defined by Cohn and rehearsed below. Cohn illustrates the range of application of this framework of ideas with, among other late-Romantic examples, various transformations of the Grail motive from Wagner's *Parsifal*. In this paper, I hope to thicken the hexatonic analytical description by superimposing on Cohn's framework two complementary interpretations. I suggest that the correlation of all three perspectives addresses how one navigates, in the absence of any tonal center or of any other sort of privileged point of reference. These interpretive complements might be applied to any of Cohn's examples (and could be extended by analogy to the passages involving dominant and half-diminished seventh chords discussed in some of the companion papers to this one), but I will focus on one of the Grail transformations, where the alternative interpretations pay an extra dividend.

The passage in question is the chromatic-enharmonic form of 'Grail' near the end of the opera, shown in Example 1. After Parsifal has taken the Grail from the shrine and kneels before it, the Grail gradually begins to glow with a soft light ("Allmähliche sanfte Erleuchtung des Grales").[1] The chromatic-enharmonic 'Grail' motive creates a harmonic disturbance within a larger tonal context. The motion in Act III, mm. 1098–1100, from E♭ major to D♭ major takes place within a plagal I-V-IV-I progression in A♭ major, (the tonality of the opera as a whole), beginning with the arrival on A♭ in m. 1088 (after which the diatonic form of 'Grail' appears), and ending in m. 1102 with the plagal cadence.[2] A hexatonic analysis applies to mm. 1098–99, preceding the completion of what Lewin (1984, 346) refers to as "the 'modulation' from V to IV." Example 1 reproduces Cohn's Example 5, with Figure 1 below it showing his interpretation of the passage, with a slight notational adjustment. Where Cohn labels his transformations T_1 and T_3, I label the same transformations R_1 and R_3, since I prefer to reserve T_i for the transpositions of the usual T_N/I_N group of operations. The upper and lower case letters refer to major and minor triads, respectively. I will give a sketch of Cohn's analytical framework with reference to this example.

The triadic space asserted by Cohn arises from a formal property that harmonic triads share with pentatonic and diatonic sets, and with sets that are the complements of harmonic triads. All of these set classes have the ability to support maximally smooth cycles. Two pitch-class sets are in a *maximally smooth* relation if there exists a pitch-class transposition or inversion mapping one set to the other that leaves all but one pitch class of a set invariant and moves the remaining pitch class by interval class 1. It should be clear that this definition is symmetric: X and Y are in the relation if and only if Y and X are in the relation. Under this definition, for example, pairs of sets from set class 3-2, e.g., {C, D♭, E♭} and {C, D, E♭}, are related in a maximally smooth way. A *maximally smooth cycle* is a cycle embracing at least three distinct sets such that adjacent sets in the cycle are in a maximally smooth relation, that is, an ordered set of pitch-class sets each of which is in a maximally smooth relation with both the



Example 1. Wagner, *Parsifal*, Act 3, mm. 1098–1100

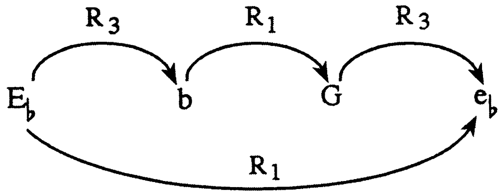


Figure 1. Cohn's analysis

preceding and succeeding pitch-class sets, and predecessor and successor are distinct. Set class 3-2 does not support maximally smooth cycles, then, because each trichord in 3-2 may stand in a maximally smooth relation with only one other trichord. The only nontrivial set classes that permit the formation of such cycles (the trivial examples being the singletons and their eleven-note complements) are the aforementioned set classes 3-11, 5-35, 7-35, and 9-11.[3]

Restricting our attention to the harmonic triads, the maximally smooth cycles partition the twenty-four triads into four disjoint sets of six triads each, which Cohn calls the four hexatonic systems and labels with the cardinal points of the compass. A maximally smooth cycle through the Western hexatonic system is shown below:

$$G \xrightarrow{R_1} g \xrightarrow{R_1} E\flat \xrightarrow{R_1} e\flat \xrightarrow{R_1} B \xrightarrow{R_1} b \xrightarrow{R_1} (G)$$

As Cohn explains, the motivation for the term "hexatonic" is that the set of pitch classes embraced by the six triads of such a cycle is a member of set class 6-20, referred to as hexatonic by analogy with the octatonic set. An economy of materials sets the triadic maximally smooth cycles of triads apart from the non-triadic cycles: six triads constituted by six pitch classes, as opposed to longer cycles involving all of the pitch classes.

The triads of each cycle form a system, moreover, in the sense of a Generalized Interval System, or GIS, as defined by Lewin (1987, 26). Under the transformational point of view that Lewin adopts, Generalized Interval Systems are equivalent to *simply transitive group actions on a set.* A group G of permutations of a set S (operations on S) acts simply transitively on S when, for any pair of elements s and t in S, there is a unique element g of G such $g(s)=t$. The elements of the group G are the transpositions of the GIS; the intervals from s to t and s′ to t′ are the same if and only if $g(s)=t$ and $g(s')=t'$.[4]

The GIS asserted by Cohn for each hexatonic system, for example, takes the six triads of the hexatonic system as the elements of the set S, and takes the cyclic group of order 6 induced by the maximally smooth cycle as the group that acts in a simply transitive way on the set of triads.

Intervals in this GIS are moves along the maximally smooth hexatonic cycle (henceforth, hexatonic cycle). The group is *cyclic* because it has a single generator, either R_1 or R_5. The transpositions combine as elements of the cyclic group modulo 6: $R_mR_n = R_nR_m = R_{(m+n)\text{mod}6}$. It is therefore possible to write R_4 as $R_{(-2)}$ and R_5 as $R_{(-1)}$, etc., so that if we class together an index and its mod 6 complement, the smaller of the values reflects the number of tones displaced. An interval in this GIS, again, measures the number of moves along the cycle (successive applications of R_1), while the notion of interval class (IC) measures the number of pitch classes that change, or, equivalently, total net pitch-class displacement, measured in semitones. Thus, IC 1 = $\{R_1,R_5\}$, one pitch class displaced, two common tones; IC 2 = $\{R_2,R_4\}$, two pitch classes displaced, one common tone; IC 3 = $\{R_3\}$, three pitch classes displaced, zero common tones. Cohn refers to triads related under the symmetrical (because involutional) R_3 transformation as "hexatonic poles," and gives this relationship special attention, by virtue of its unique musical and formal properties.

This GIS arises in a natural and reasonable way, and has a simple structure. The GIS is commutative, because the binary operation in the associated group is commutative (i.e., it doesn't matter whether one performs R_i first or R_j). The only counterintuitive aspect of this group is that the transpositions with odd indices reflect a nonstandard notion of interval: the "interval" from the B-major triad to the B-minor triad is the same as the "interval" from the B-minor triad to the G-major triad, for example. The basic notion of an interval is satisfied, however, because the group is simply transitive on the set of triads: for each ordered pair of triads, there is precisely one transposition that maps one to the other. The action of the group on the triads is shown below, using the permutation notation that decomposes a permutation into a product of disjoint cycles.[5] In this notation, each parenthetical expression represents a cyclic permutation, and it does not matter in what order the cycles are written. R_2, for example, sends E♭ to B, B to G, and G back to E♭, and similarly for the minor triads.

R_0: ( )	R_3: (E♭ b) (e♭ G) (B g)
R_1: (E♭ e♭ B b G g)	R_4: (E♭ G B) (e♭ g b)
R_2: (E♭ B G) (e♭ b g)	R_5: (E♭ g G b B e♭)

Cohn's analysis, (displayed above in Figure 1), incorporates the two extremes in his GIS, the ICs 1 and 3, drawing attention to the way the symmetrical hexatonic polar relationships frame an adjacency on the hexatonic cycle, while an adjacency on the cycle also describes the overall move from E♭ major to E♭ minor.

There are two other Generalized Interval Systems that arise in a natural way for a space of hexatonic triads. On one hand, moves along the cycle may be described as alternations of the neo-Riemannian transformations *Parallel* and *Leittonwechsel*, designated herein as P and L, respectively, following Brian Hyer (1989, 1995). On the other hand, the elements of the cycle are related by transposition or inversion operators. The latter is implicit in the definition of the maximally smooth cycle (set-class consistency condition), while Cohn takes the maximum common-tone and minimum voice-leading-interval aspects shared by these two neo-Riemannian operations as his point of departure. The notion of a maximally smooth cycle acquires its force precisely by virtue of these constraints. The groups entailed by these constraints are perhaps not immediately evident, however.

Consider the inversions from the usual T_N/I_N group that link elements of the maximally smooth cycle through the triads of the Western hexatonic system. Under this description, a tour through the cycle is not a six-fold repetition of moves of the same type but twice through the succession $\langle I_9, I_5, I_1 \rangle$, with G or e♭ as initial inputs.

$$\mathrm{G} \xrightarrow{I_9} \mathrm{g} \xrightarrow{I_5} \mathrm{E}\flat \xrightarrow{I_1} \mathrm{e}\flat \xrightarrow{I_9} \mathrm{B} \xrightarrow{I_5} \mathrm{b} \xrightarrow{I_1} (\mathrm{G})$$

The reason for assigning priority to this path through the six triads, however, is the existence of the maximally smooth cycle, which has now been pushed to the background. Moreover, these operations do not merely inhabit the full group of twenty-four T_N/I_N operations, but also coexist in a tightly organized group of six operations. Any pair of the three inversion operations involved in the cycle generates a subgroup of order 6, (forming the smallest group containing the two generating operations), as one can verify by using the rules for combining transpositions and inversions. The elements of this group are T_0, T_4, T_8, I_1, I_5, I_9, acting upon consonant triads, and restricted, for our purposes, to the six triads of the Western hexatonic system. With this restriction, the group acts in a simply transitive way on the set of six triads, as the decomposition into cyclic permutations makes clear:

T_0: ( )	I_1: (E♭ e♭) (G b) (B g)
T_4: (E♭ G B) (e♭ g b)	I_5: (B b) (G e♭) (E♭ g)
T_8: (E♭ B G) (e♭ b g)	I_9: (G g) (E♭ b) (B e♭)

The notion of interval that this GIS presumes has a different perceptual basis from Cohn's. The T_N/I_N hexatonic GIS tracks *which* pitch classes move or are exchanged. Again, the notion of interval may seem counterintuitive. For example, the interval between B-minor and G-major triads is the same as the interval between E♭-minor and E♭-major triads, since the transposition between them (in either direction) is I_1, in both cases. What the ear attends to in this GIS with triads that lie the interval I_1 apart is the exchange between pitch classes G and F♯/G♭: this is what distinguishes e♭ from E♭ and b from G, and is one of the pitch classes exchanged between the hexatonic poles B and g. Figure 2 rewrites Cohn's network interpretation of the 'Grail' motive in terms of the T_N/I_N hexatonic GIS. As in Cohn's network, the intervals between interior elements, b and G, and between the outer elements, E♭ and e♭, have the same label. In Cohn's network, the triad pairs are connected by identically-labeled arrows because they are adjacent along the hexatonic cycle, while here the commonality or invariant involves the pitch classes exchanged, G and F♯/G♭. The initial and final moves between hexatonic poles, however, now bear different labels. To complete the comparison between the networks in Figures 1 and 2, note that in the latter all relationships are reciprocal, represented by double arrows. In Figure 1, the hexatonic polar relationships might have been labeled with double arrows (since R_3 is its own inverse), but not the interior and exterior moves. Finally, the T_N/I_N GIS is non-commutative—the order in which operations in the associated non-abelian (i.e., non-commutative) group are taken generally makes a difference—unlike the situation in Cohn's GIS, associated with a cyclic group. As we will see, the existence of yet another hexatonic GIS, one based on neo-Riemannian transformations, is indissolubly linked to the nonabelian character of the T_N/I_N subgroup associated with this GIS.

The hexatonic cycle proceeds by the application, in alternation, of the neo-Riemannian P and L transforms.[6] The description of the cycle in these terms forms an abstract hemiolic relationship with the cycle of T_N/I_N operations, since P and L alternate three times in the course of the cycle while the pitch-class inversions proceed through two cycles of length three: $\langle I_9, I_5, I_1 \rangle$. The neo-Riemannian operations are contextually defined inversions, in that they provide rules for inverting triads that depend upon the triadic configuration itself for their definitions.[7] Thus, P inverts by preserving the endpoints of the constituent perfect fifth, altering the third of the triad, while L inverts by preserving the constituent minor third, that is, moving by a semitone the root of a major triad or the fifth of a minor triad. Taking a dualist description of triads, L may be defined as the exchange of Riemannian roots between distinct triads with two common tones that form a minor third.[8]

P and L define permutations of the set of six triads in the Western

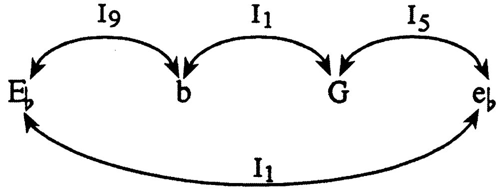


Figure 2. T_N/I_N GIS interpretation

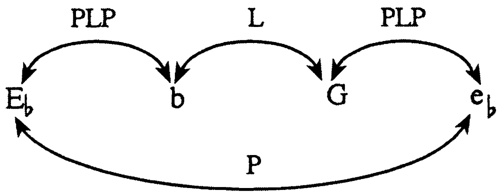


Figure 3. P/L GIS interpretation

hexatonic system (or of those in any hexatonic system, since the operations are contextually defined). Together, P and L generate a group of six operations, the P/L group, which can be defined on the twenty-four consonant triads. Note that P, L, and PLP are their own inverses, while PL and LP are inverses of each other. Now, restricting this group to the six triads of our hexatonic system, the group acts in a simply transitive way, and another non-commutative GIS arises. The action of the P/L group is shown below:

E: ( )	LP: (E♭ G B)(e♭ b g)
P: (E♭ e♭)(G g)(B b)	PL: (E♭ B G)(e♭ g b)
L: (E♭ g)(G b)(B e♭)	PLP: (E♭ b)(G e♭)(B g)

This GIS also has a determinate perceptual basis, based upon the triadic configuration. The GIS description using contextual neo-Riemannian transformations captures how the triadic constituents are altered. Figure 3 again rewrites Cohn's transformational graph, now in terms of the neo-Riemannian hexatonic group.[9] The upper part of the figure parallels that of Figure 1, but here the composition of the three contextual operations is not the same as the interior operation. Note also the double arrows for all operations, as in Figure 2. The R_3 and PLP transforms are similar in effect, because both entail the alteration of all three constituents of the triad on which they act. Contrary to the descriptions in both Figures 1 and 2, however, the interior and exterior transforms are different, because although only one pitch class moves in both instances, and the same pair of pitch classes is being exchanged, in the interior operation one endpoint of the perfect fifth is altered, while in the exterior or composite operation, the third of the triad moves.

The point of invoking any GIS is that its consistency serves as a warrant that something coherent can be heard and/or said in the intervallic or transformational terms that the GIS presumes, assuming that the GIS is well-grounded in musical reality. The types of transformations discussed so far are all natural and reasonable; I have attempted to make explicit the perceptual bases of each of the GISs. The three GISs associated with the same set of six triads mutually reinforce each other, each one placing in the foreground one transformational aspect, but acquiring its force in part by the other aspects that are pushed to the background. Superimposing the interpretations of Figures 2 and 3 on that of Figure 1 yields a "multiple description" or "multiple versions of relationship" in the sense of Bateson 1979, who indicates the power of multiple descriptions in various domains, including binocular vision, neurophysiological processes, and epistemology.

An advantage of invoking the two non-commutative GISs is that subtle relationships to the D♭-major point of arrival are manifested. Cohn marks the shift at this point by merely noting an exit from the hexatonic system. Indeed, we must leave the confines of the given hexatonic GIS, but the hexatonic groups are subgroups of larger groups that act on all twenty-four triads. Clearly, the T_N/I_N group of order 6 is a subgroup of the whole group of transpositions and inversions. The P/L group is also a subgroup of what, following Klumpenhouwer 1994, we might call the *Schritt/Wechsel* group, of twenty-four contextually defined neo-Riemannian operations. The full Riemannian group can also be understood as the R/L group, generated by the Relative (R) and Leittonwechsel (L) operations, as shown in Cohn 1997, 29–36. One can see that the alternation of L and R embraces all twenty-four triads, since the composition of L followed by R transposes a triad by a perfect fifth, up or down depending on whether it is major or minor. This is Riemann's *Quintschritt*. One can form a GIS consisting of the R/L group acting on the full set class of the consonant triads. Implicitly, whenever a GIS based upon one of the smaller groups is invoked (e.g., the P/L group), those transformations take place within the larger space as well. However, one can include the D♭-major harmony by extending into the larger GIS without losing contact with the hexatonic framework. This can be achieved by applying Lewin's notion of *modulation of a transformational system*. In preparation for the extension to the larger group, Figure 4 reorganizes the neo-Riemannian relationships of Figure 3, placing major and minor triads on different axes.

One of the results in Lewin's general transformational theory is that "when a system modulates by an operation A, the transformation $f' = AfA^{-1}$ plays the same role in the modulated system that f played in the original . . . ."[10] The notion of system in this context is very general, and a transformation is any mapping of a set into itself. In this instance, the hexatonic GIS associated with the P/L group is the system under consideration, in particular the network presented in Figure 4, and the operations L, P, LP, PLP are transformations. Lewin's f′ is what we may call the conjugate of f by A. When Figure 4 is extended to include the D♭-major harmony of m. 1100, the extension is a modulation of the existing network; the new relationships are conjugates by R (R is its own inverse, i.e., $R = R^{-1}$) of the existing ones: not only are the relationships between

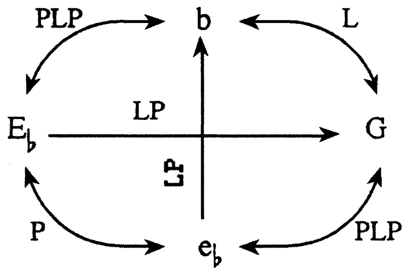


Figure 4. Network of relationships under the P/L GIS interpretation

b and D♭ and between e♭ and D♭ conjugates of relationships in Figure 4, but also PL = R(LP)R, and $(RP)^2$ is self-conjugate.[11] The extended network is shown in Figure 5.

All but the first three chronologically adjacent relationships shown in Figure 5 are in the modulated system: as stated above, PL = R(LP)R, and a relationship not shown, the interval from E♭ to D♭, is the conjugate of the interval from D♭ to E♭. The network presents some of the logic behind the status of D♭ as a temporary goal of tonal motion. Most significantly, the reciprocal relation between the chronologically adjacent harmonies b and G in the old system parallels that between the chronologically adjacent harmonies e♭ and D♭ in the modulated system: L is transformed into RLR. As well, e♭ stands in the same relation to G (in the old system) as b does to D♭ (in the new): here, the PLP relation is replaced by its conjugate RPLPR. B minor and E♭ minor act as pivots in the system-modulation. The old system is the GIS of the Western hexatonic triads, with the P/L group. The new system, also a Generalized Interval System, includes the three triads of the Klingsor music (see Act I, mm. 628–32), the minor triads b, g, and e♭, that are shared with the Western hexatonic system, and includes D♭, along with F and A. The group of the new system, obtained from the old GIS through conjugation by R, shares the three contextual transpositions of the old group, but has three new contextual inversions, RPR, RLR, and RPLPR. On the poetic-dramatic level, Lewin has suggested that in other contexts D♭ is a substitution for the D major of the spear motive. In Lerdahl's reading, the presence of two of the Klingsor triads within this ultimate statement of the Grail motive reflects Parsifal's appropriation of the magic power of the spear, and the Klingsor triads are connected to D♭ by the system modulation.

The system modulation I have asserted coincides with the "modulation" from V to IV in the larger plagal progression described in Lewin 1984 and in Lerdahl 1994. Lerdahl takes E♭ minor to be a pivot chord in the traditional sense. Lewin also takes the E♭-minor and B-minor har-

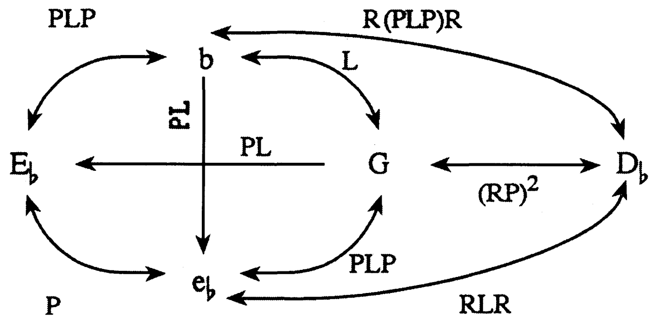


Figure 5. Extended network, including relationships conjugated by R (relative transform)

monies to be pivots, in different ways. E♭ minor is recognized as pivotal in (traditional) Riemannian terms because while in the context of A♭ major it has Dominant function, sharing root and fifth with the initial E♭ major, in relation to D♭ it is Subdominant parallel, i.e., relative minor of G♭. In Lewin's analysis of mm. 1098–1100 as a chromatic transformation of the diatonic "Dresden Amen" version of the Grail motive, however, the B-minor harmony is a different kind of pivot, an enharmonic pivot mediating between motivic Stufen space and Riemann space, an enharmony that Lewin considers crucial to the drama as a whole.

Alternatively, this passage may be viewed through the lens of the T_N/I_N GIS. Again, the first four harmonies stay within a hexatonic system, with the T_N/I_N subgroup $\{T_0, T_4, T_8, I_1, I_5, I_9\}$ constituting the group of intervals for this system. Figure 6 presents the passage in terms of this GIS. Again, D♭ participates in a modulation of systems, where relationships in the hexatonic system—I_1 and I_5,—are replaced by their conjugates by I_6: $I_{11} = I_6I_1I_6$, $I_7 = I_6I_5I_6$. T_6 commutes with I_6, so T_6 is self-conjugate, and T_4 and T_8 exchange places under conjugation by I_6.[12]

The close relationship between the two non-commutative hexatonic GISs should now be explored. From Lewin's theory of Generalized Interval Systems, every non-commutative GIS with a nonabelian group G simply transitive on a set X has associated with it a unique non-commutative GIS with a group H (distinct from G) simply transitive on the same set X. The elements of G and H commute, (i.e., considered as permutations of X, gh=hg, for all g, h in G, H, respectively) and G and H are (anti-)isomorphic. An anti-isomorphism is a mapping **i** that is one-to-one and onto from G to H such that $\mathbf{i}(g_1g_2)=\mathbf{i}(g_2)\mathbf{i}(g_1)$, for all elements g_1, g_2 in G.[13] Each group serves as the group of interval-preserving operations for the other GIS, where an interval-preserving operation A is one such that int(x,y) = int(A(x), A(y)). The P/L hexatonic GIS and the T_N/I_N hexatonic GIS are the complementary GISs that Lewin's theory demands. Moreover, the larger systems that include them, the GISs associated with the R/L group and with the full T_N/I_N group, both acting on the complete set of triads in set class 3-11, are similarly complementary.[14]

A comparison of Figures 5 and 6 makes apparent the reciprocal relationship between the two non-commutative hexatonic Generalized Interval Systems. For example, if we call int_1 the interval function of the Riemannian P/L GIS and int_2 the interval function of the T_N/I_N GIS, we have $\mathrm{int}_1(E\flat, b) = \mathrm{int}_1(I_1(E\flat), I_1(b)) = PLP$, and conversely, for example, $\mathrm{int}_2(E\flat, e\flat) = \mathrm{int}_2(PLP(E\flat), PLP(e\flat)) = I_1$.

The extension of the network to include D♭ implicitly invokes the larger GIS, either that associated with the L/R group or that associated with the full T_N/I_N group, but it has been suggested that this has been done without losing contact with the original hexatonic context, by virtue of the system-modulation invoked. The following discussion will attempt to make that idea more explicit.

The mapping which takes every element of a group to its conjugate by some element is an automorphism of that group, that is, an isomorphism or structure-preserving mapping of the group with itself. Such automorphisms are called inner automorphisms. For example, consider the inner automorphism of the L/R group that maps every element F in the group to the element $RFR^{-1} = RFR$. An automorphism maps a subgroup onto a subgroup: the P/L subgroup is mapped onto another subgroup of order 6 by the inner automorphism defined by R, the subgroup $R\{P/L\}R^{-1}$ = {E, LP, PL, RPR, RLR, RPLPR} (because RLPR = PL and RPLR = LP). While the P/L group is simply transitive when its action is restricted to any of the four hexatonic subsets of the complete set of triads, the $R(P/L)R^{-1}$ subgroup acts in a simply transitive way when restricted to any subset of triads of the type {D♭, e♭, A, b, F, g}, which I will call Soderberg-type sets.[15]

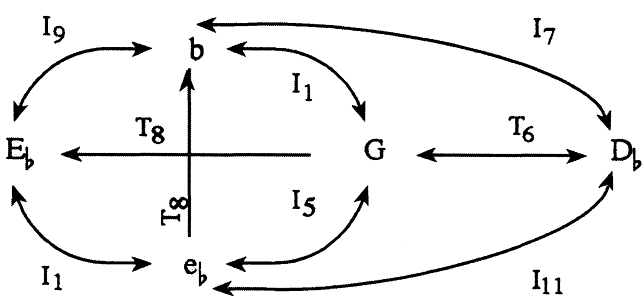


Figure 6. Extended network with T_N/I_N labels, including relationships conjugated by I_6

Unlike the contextually defined P/L subgroup, the subgroup that is simply transitive on the Western hexatonic system, $\{T_0, T_4, T_8, I_1, I_5, I_9\}$, is not simply transitive on all of the hexatonic systems. In addition to the Western system, the orbits of this subgroup, as they are called, are the Eastern hexatonic system, consisting of the triadic set {a, A, f, F, c♯, C♯}, and two Soderberg-type sets. Conjugating the subgroup by I_6 yields the isomorphic subgroup $\{T_0, T_4, T_8, I_3, I_7, I_{11}\}$, whose orbits are the Northern and Southern hexatonic systems, and two Soderberg-type sets, including the set {D♭, e♭, A, b, F, g}.[16] Thus, the system of relations that include the D♭ harmony involve complementary GISs that are projections of the hexatonic GISs, and whose sets overlap in three triads. Of these, e♭ and b function in the passage as pivots between the two systems, in the various ways we have observed.

While this modulation-by-conjugation is a special case, the three alternative analytical approaches may be engaged whenever a hexatonic analysis is appropriate. One further direction for generalization is to the triadic chain generated by the alternation of P with R, forming the eight triads of an octatonic set (see Cohn 1997, 35). This group, which also stands as a subgroup of the L/R or *Schritt/Wechsel* group acting on the entire 3-11 set class, can be characterized by three GISs that are analogous to the three hexatonic GISs introduced above. Generalizing further, whenever it is reasonable to invoke a cycle where adjacencies along the cycle are inversionally related, three GIS descriptions are possible. For example, the cycles involving dominant and half-diminished seventh chords give rise to such triple descriptions.

Do the hexatonic cycles have a privileged status, as Cohn 1996 suggests? I believe so, for in addition to the defining properties of the maximally smooth cycle, the balance between the "forces of unity (six triads) and diversity (four cycles)" that Cohn adduces, and the over-determined nature of the triad itself, as discussed by Cohn, there are two other formal features of the cycle that set it somewhat or completely apart. The maximally smooth hexatonic cycle shares with the parsimonious octatonic triadic cycle the property that the same T_N/I_N subgroups that yield GISs acting on the triads also yield GISs acting on the underlying sets of pitch classes.[17] This situation does not obtain with the Soderberg-type sets, for example, which involve six triads but embrace all twelve pitch classes, nor with the group acting on the full set class. Finally, uniquely, the number of interval classes in a hexatonic GIS matches the number of constituents of a triad, lending force to the cyclic GIS description: the pitch classes are turned over in one half-cycle.

NOTES

1. This passage is also discussed in Lewin 1984, 345–46; in Lewin 1992, 57–58; in Cook 1994, 5–6 and passim; and in Lerdahl 1994, 140–43. The present paper extends a treatment in Clampitt 1997, 183–91.
2. Lewin (1992, 57) hears a local ii-(V)-I progression in the motion from E♭ minor to D♭ major, the dominant suggested in the upper voices, on the last eighth-note. Lerdahl's tonal reading is similar (1994, 143).
3. Formal and general investigations into precisely which set classes may support such cycles are found in Lewin 1996 and Clampitt 1997.
4. Lewin 1987, 157–59.
5. By a theorem of elementary group theory, every permutation of N objects can be uniquely expressed as the composition of disjoint cyclic permutations. The empty cycle, denoted by ( ), leaves all objects fixed. See Herstein 1964, 66–67.
6. Hyer (1989, 1995) exposed the group structures produced by the neo-Riemannian transformations. According to Hyer, for Riemann "the parallel, relative, and leittonwechsel are always applied to tonic, dominant, or subdominant triads, and are never independently available" (1989, 168). In the neo-Riemannian formulation here, following Hyer and others, these operations may be applied to any triad, without reference to any tonal context. Also, following the practice of Hyer, Lewin, and Cohn, I will use right-functional orthography for Riemannian operations. That is, for these operations the arguments appear before the functions that are applied to their arguments. Thus, the composite operation PL means first apply P, then apply L to the output of P. For T_N/I_N operations, I follow the convention of left-functional orthography.
7. The notion of a contextually defined inversion is primarily due to Lewin. For an understanding of contextually defined operations as a general concept I am indebted to discussions of them that arose in a seminar taught by John Clough, Fall 1996. In particular, Nora Engebretsen, James Jefferis, and Jonathan Kochavi, along with Professor Clough, contributed to an emerging notion of contextual operations.
8. Klumpenhouwer 1994 and Kopp 1995 discuss in depth the relationship between transformations such as P and L and a dualist conception of triads. Klumpenhouwer's interpretation suggests that Riemann's early thinking, especially as expressed in his 1880 *Skizze einer Neuen Methode der Harmonielehre*, was closer to the neo-Riemannian formulation than his later synthesis (or accomodation), although Klumpenhouwer has appropriated Riemann's transformations, as he himself states (1994, [6]), for his own analytical purposes.
9. Compare with Lewin (1992, 57), who interpolates chords between the hexatonic poles to form a chain of alternating P and L transformations.
10. Lewin 1987, 149. See also Lewin 1990, 86.
11. One can see that $(RP)^2$ is self-conjugate, because RP transposes a triad by a minor third, up or down depending on the modality of the argument, so composing RP with itself yields a modally-matched triad a tritone away. Thus $(RP)^4 = E$, the identity operation, so $(RP)^2 = (PR)^2$, and the conjugate of $(RP)^2$, $R((RP)^2)R = R(RPRP)R = RR(PRPR) = (PR)^2$. It is also the case that conjugating the composition of any pair of distinct transformations P, L, and R by the remaining transformation is the same as reversing the order of the composed pair. That is, $A(BC)A = CB$, where A, B, C are distinct. One can see this by observing that composing

any pair of transformations (LP, for example) transposes a given triad, up or down depending on that triad's modality, and reversing the order of the composition reverses the direction of the transposition. These are examples of what John Clough calls "contextual transpositions." Preceding the composition with the other operator changes the mode of the triadic argument (and transposes it, possibly by the zero interval), while following the composition by the same operator (its own inverse) undoes the action of the first, reversing the mode and the direction of the transposition. The net effect is to change the mode of the initial triadic input, or, equivalently, to reverse the order of the composed pair of transformations. Thus, R(LP)R = PL, for example.

12. Figures 5 and 6 are an example of an *isography*; moreover, they exemplify what Klumpenhouwer calls *isomorphic networks* (Lewin 1990, 86). To see this, mentally rotate Figure 5, say, 180° about the major-key axis, then identify I_1 with PLP, T_8 with PL, etc.
13. Lewin 1987, 251–53. As Lewin makes clear, groups that are anti-isomorphic are isomorphic (i.e., have the same structure) and vice versa. See the discussion in Clough 1998.
14. The P/L and the T_N/I_N hexatonic groups exemplify the dual perspectives on the dihedral group of order 6 found in Clough 1998. An explicit anti-isomorphism **i** between the groups follows: $E \leftrightarrow T_0$, $P \leftrightarrow I_1$, $L \leftrightarrow I_5$, $PLP \leftrightarrow I_9$, $LP \leftrightarrow T_4$, $PL \leftrightarrow T_8$. (It is necessary to remember the different functional orthographies for the two groups. For example, $\mathbf{i}(I_1I_5) = \mathbf{i}(T_8) = PL = \mathbf{i}(I_1)\mathbf{i}(I_5)$. The change in orthography masks the anti-isomorphism.)
15. The possible interest of such sets was communicated to me by Stephen Soderberg, in a 1996 email message.
16. As one would expect, the conjugate neo-Riemannian subgroup designated $R\{P/L\}R^{-1}$ has as its orbits all four sets of the Soderberg type.
17. This is a slight *abus de langage*, and a notational abuse as well: the group that acts on the triads of the Western hexatonic system, for example, is conceptually a different group from the one that acts on the underlying pitch-class set from set class 6-20, but the identification is a natural one.