A Diatonic Chord with Unusual Voice-Leading Capabilities

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Diatonic set theory, as established by John Clough and others (see Clough 1979, 1980, Carey and Clampitt 1989, Carey 1998), applies the tools of standard set theory of 12-tone ET to the heptatonic set of seven tones of the diatonic scale. The two universes differ from each other in a number of ways other than simple cardinality. Although 12 is nearly twice as big as 7, the fact that 7 is prime and 12 composite contributes to a number of subtle differences between them. Every positive integer less than 7 is a unit mod 7, thus every diatonic interval generates the entire set. In the set of 12 tones, there are only four units, 1, 5, 7, and 11. Further, because of tuning, the geographies, if you will, of the sets also differ. In the equal-tempered 12-tone landscape, every place looks like everyplace else. In the diatonic scale, each generic span is inhabited by several different specific intervals. Because of this, the terrain is everywhere distinct, contributing to the phenomenon of gravitational asymmetry and of tonality.

In this paper, I would like to call attention to a trichord in the ordinary diatonic that is endowed with unusual voice-leading capabilities. Recently, neo-Riemannian theory has uncovered voice-leading properties of the common chord, by positioning the triad on the cusp between standard set theory and diatonic set theory. This fluidity of approach gives the theory considerable explanatory power, as it accords with the repertoire for which it has proven particularly successful, namely, highly chromatic music that is on or near the disputed border between tonal and atonal repertories. It has led, at the same time, to some misunderstanding. Thus, in considering "parsimonious voice leading," it has struck some as arbitrary that the L and P operations move by ic 1, but that the R transform involves ic 2. This discrepancy is a problem when viewed solely through the lens of standard set theory, where ic 1 and ic 2 are distinct objects. In diatonic set theory, these objects belong to the same class, namely the class of diatonic steps. Indeed, one of the most profound insights of diatonic theory is its recognition of the generic interval. Blurring distinctions is one of the most important things the theory does. Specific information becomes encoded within the generic. Here is an idea how powerful this may be. Select any diatonic scale. Choose any note from the scale. Go up a third higher in that scale from your starting note. Now go up another third from there. Certainly you're a fifth higher than where you started, but most likely you're a *perfect* fifth higher than where you started. Thus, if we know how to use it, the code allows me to extract specific intervallic information – the perfect fifth – from the generic information we put in – the two non-specific thirds. (Unless you began on the seventh degree!)

It will be within the realm of diatonic set theory that this paper unfolds, however it will be useful to work into the main topic with a fuller examination of the *ordinary* triad. I would like to separate out the information conveyed by both set theories, standard and diatonic, regarding the common chord. It is easy to gloss over the fact that

T. Klouche and T. Noll (Eds.): MCM 2007, CCIS 37, pp. 449–463, 2009.

both theories assign the major triad and the minor triad to a single one of their classes. This is almost, you might say, a lucky accident. Example 1 should help to sort out the issues.

Inversion of the triad

In standard set theory

Triad	I(7)	I(8)	
7	7	8	
3	4	5	
0	0	1	
m	M	M	etc.

Why Major and minor triads belong to the same class in standard set theory. (Dualist model)

In diatonic set theory

Triad	I(4)	I(5)
4	4	5
2	2	3
0	0	1
?	?	?

Transposition of the triad

In standard set theory:

7	8	9	a	b
3	4	5	6	7
0	1	2	3	4
m	m	m	m	m

In diatonic set theory

4	5	6	0	1
2	3	4	5	6
0	1	2	3	4
?	?	?	?	?

Why Major and minor triads belong to the same class in diatonic set theory. (*Stufen* model)

Example 1.

In standard set theory, major triads and minor triads belong to the same set class (Forte's 3-11) because major and minor triads may be converted into one another through inversion. (See the top part of the Example.) In defining diatonic set classes, John Clough recognizes transpositional equivalence only. Thus, they belong to the same diatonic set class not because of their inversional relationship, but because diatonic transposition converts these triad types into one another. (See the very bottom of the Example.) As we diatonically transpose a triad by step, we visit three major triads, three minor triads, and one diminished triad, in some order. The question marks below the triads in the example indicate the fact that, purely within the context of diatonic theory, we do not know which types these triads are without assigning actual pitch class names. The situation here is analogous to one in which we try to find our way around the piano keyboard with the black keys covered up. Both of these set theories, then, place major and minor triads into a single class but for different reasons. As we just saw, in the case of atonal set classes, major and minor triads belong to the same class because they are inversions of each other. This scheme emphasizes a Riemannian or dualist perspective on the triads. Major and minor triads belong to the same class because each is the reflection of the other. In Clough's diatonic set classes, major and minor belong to the same class because of rotation. This scheme emphasizes a *Stufen* theory view of the triads. The difference between the two schemes is apparent with respect to their categorization of the diminished triad. In atonal set theory, although major and minor triads are members of 3-11, the diminished triad is not. In the 12-tone universe, neither transposition nor inversion will transform a major or minor triad into a diminished triad, which is associated with the Forte's set class 3-10. In the diatonic, on the other hand, the major and minor triads are related to diminished triads by both transposition and inversion, and all three belong to Clough's diatonic set, 223. The top part of Example 1 makes it clear that the inversion in standard set theory of some 037 is always some 047, however, in diatonic theory, the inversion of some triad is, simply, some triad.

In standard set theory, major and minor triads are only related by inversion. In diatonic set theory, they are related by both transposition and inversion, but only transposition matters in the definition of the diatonic set classes. At least part of the reason for this choice, on Clough's part, is that the great majority of diatonic sets exhibit (diatonic) inversional symmetry. Clough's decision not to employ inversional equivalence

Five Diatonic Trichords

Generated Trichords

Example	Set class	Interval Vector
CDE	115	[210]
CEG	223	[021]
CDG	133	[102]

All-interval Trichords

Example	Set class	Interval Vector
CDF	124	[111]
CEF	142	[111]

Example 2.

in defining diatonic sets has significance for only a few asymmetric diatonic sets, of which there are four; two trichords and their tetrachordal complements. These "non-retrogradable" sets – particularly the trichords – are worthy of new consideration for their unique properties.

As we see in Example 2, the triad, diatonic class 223, is one of five diatonic trichords. Three of them are, in a sense, generated. Diatonic thirds, for example, generate the ordinary triad. Clough's descriptor for the triad is 223, where each of the 2's represents a diatonic third, and the 3 represents a diatonic fourth. The set CDE instantiates another generated diatonic trichord, labeled 115. The third type of generated trichord type is 133, an example of which is provided by the chord CDG. Each of these set classes includes the seven transpositions of its sample set.

The three generated trichords have similar diatonic interval vectors. This is a three-place vector, counting the number of steps, thirds, and fifths, that is, it counts the number of unordered diatonic interval classes. Because seven is not a multiple of 3, there are no diatonic trichords with just one interval type: all seconds, or all thirds, or all fifths. Each of the generated set classes has two intervals of one type – the one we are considering as its generator, one of another type, and none of the third. Thus, the 115, or CDE, type has two seconds, one third, and no fifths. The usual triad, CEG, has two thirds, one fifth, and no seconds. The third type, such as CDG, has two fifths, one second, and no thirds. Inversional symmetry in these sets is guaranteed by the presence of generators. Two out of three intervals are alike, which ensures that the transpositions of each of the generated trichords includes its own inversions. (Each may be abstracted as *abb*. Inverting yields *bba*, which is a rotation of *abb*.) In short, the issue of whether or not to include inversional equivalence in the definition of diatonic set classes is moot with respect to these three types.

Transpositions of 124:

F	G	A	B	C	D	E
D	E	F	G	A	B	C
C	D	E	F	G	A	B

Transpositions of 142:

F	G	A	B	C	D	E
E	F	G	A	B	C	D
C	D	E	F	G	A	B

Example 3.

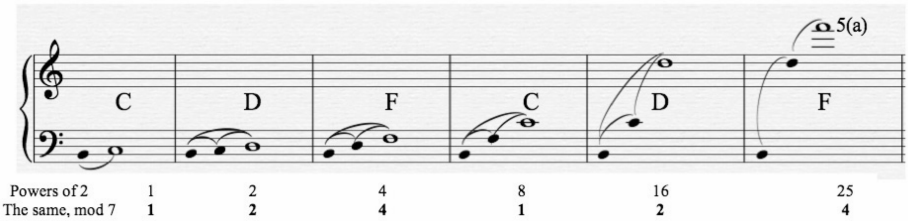
The two other types of diatonic trichords are my main focus today. They exhibit fewer symmetries than the generated trichords because these sets are, in fact, all-interval diatonic trichords. That is, each contains exactly one step, one third, and one fifth. The chord CDF is a member of one such set class, namely 124. Note that it contains exactly one step, CD, one third, DF, and one fifth, FC. If we invert this set around the C-F axis it becomes CEF. Thus, inversions of the 124 set are NOT found among its transpositions. Because inversion is not invoked in the definition of diatonic set classes, the set class that includes CEF belongs to the fifth trichord type; 142, as Clough calls it.

The seven transpositions of trichords 124 and 142 are shown in Example 3. Although these two set classes, 124 and 142, are not generated in the additive sense of the three generated trichords, 115, 223 and 133 (the triad and its two cousins) they can be recursively derived through multiplication[1]. Example 4 shows how this works.

Powers of 2:	2^0	2^1	2^2	2^3	2^4	2^5	2^6	4(a)
	1	2	4	8	16	32	64	
2^n modulo 7	**1**	**2**	**4**	**1**	**2**	**4**	**1**	
Squares:	1^2	2^2	3^2	4^2	5^2	6^2	7^2	4(b)
	1	4	9	16	25	36	49	
Quadratic residues:	**1**	**4**	**2**	**2**	**4**	**1**	**(0)**	

Example 4.

If we begin with any non-zero element modulo 7 and double it, a cycle is generated that repeats after three elements. Beginning with 1 we multiply by 2 to get 2. Multiply by 2 again to get 4. Multiply by 2 once more, and get 8, which again equals 1, modulo 7. The construction yields the set 1, 2, and 4, one of our trichords. That is, powers of 2 modulo 7 belong to the set (1,2,4). Musically, we can put it this way (shown in Example 5a):



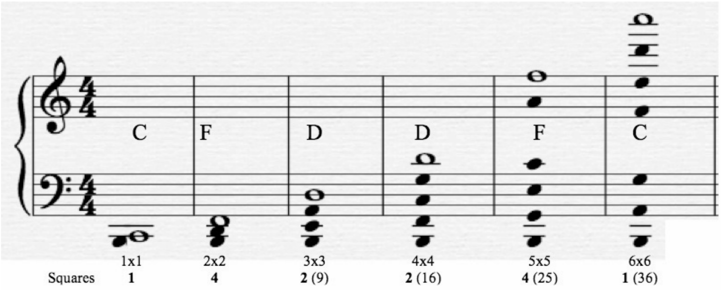
Example 5a.

Begin with the interval of a second. Double it by adding on another second. This forms a third. Doubling the third yields a fifth, and doubling the fifth, yields a ninth, which, reduced modulo the octave is once again a step, returning us to our starting point. This three-element multiplicative cycle is highly significant with respect to the structure of diatonic sequences.

[1] The 14 elements of the two set classes 124 and 142 are orbits of a single pitch class under an affine transformation f(x) = 2x + k mod 7. Mazzola (1990, 121) calls such orbits *circle chords.*

These same numbers, 1, 2, and 4, form the set of quadratic residues modulo 7 and the set containing 3, 5, and 6 are the quadratic non-residues. Any positive integer squared is congruent to 1, 2, or 4 modulo 7 as we see in Example 4b.

Example 5b provides a musical realization of this information:



Example 5b.

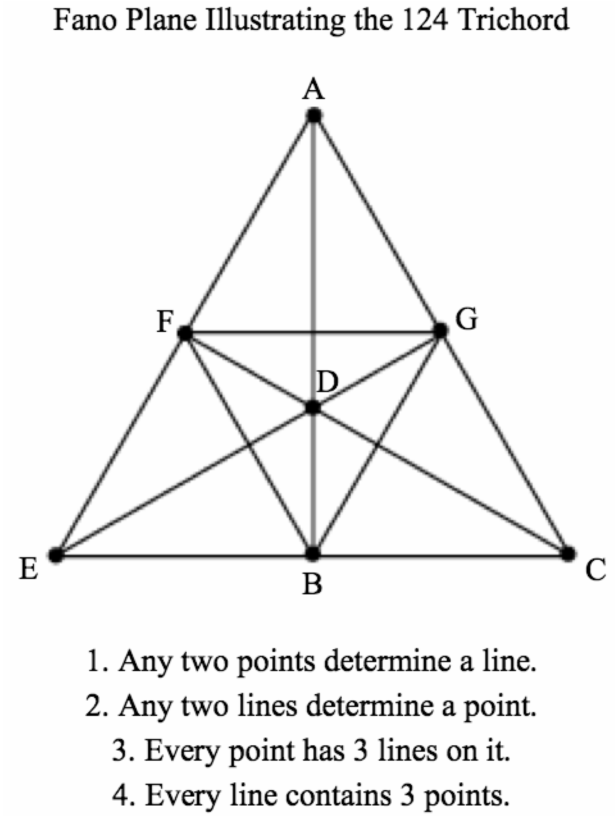
One step is – a step (1 x ic1 = ic1). Two thirds is a fifth (2 x ic2 = ic4). Three fourths is a third (3 x ic3 = ic2). Four fifths is also a third. Five sixths makes a fifth, and six sevenths make a step. While fascinating, I don't know where or if this data finds its place in diatonic practice.

Each set of seven trichords (the 124s and the 142s) constitutes an example of a Steiner triple system. A Steiner triple system is a simple type of block design.[2] In a Steiner triple system, a set of *N* elements is divided into a number of three-element subsets such that each pair of elements belongs to one subset. When *N* is 7, Steiner triple systems require seven three-element subsets. In fact, the transpositions of CDF (or CEF) constitute a Steiner triple system, as a glance back at Example 3 will verify. Now it is the case that each interval appears in exactly one chord, satisfying the definition of a Steiner triple system. This latter point is interesting, and shows how the 124 and 142 chords are different in their voice-leading potential compared to the triad. If you choose an arbitrary interval in the C Major scale, it will appear in two different triads if it is a third, in exactly one triad if you chose a fifth, and in no triad if you chose a second. In the 124 chord, the situation is much more democratic. As a consequence of its status of all-interval trichord, each of the 21 specific intervals of the scale appears in exactly one of the 124 chords.

Steiner triple systems are not uncommon, but require a set whose cardinality is at least 3, and is equivalent to 1 or 3 modulo 6. Thus, scales that support Steiner triples are those with cardinalities 3, 7, 9, 13, 15, 19, 21, and so on. Note that 12 does not support a Steiner triple system. The Steiner triple in the case of 7 is of particular interest to mathematicians in that it is also the finite projective plane of order 2. A projective plane of order n is a figure with $n^2 + n + 1$ points containing $n + 1$ points on each line with particular properties. The projective plan of order 2 may be represented by the figure known as the Fano plane. The particular projective plane of order 2, contains 7 points, because $2^2 + 2 + 1 = 7$. We find the following conditions on the Fano plane:

[2] For more musical block designs see Tom Johnson's article *Networks* in this volume.

1. Any two points determine a line,
2. Any two lines determine a point.
3. Every point has 3 lines on it.
4. Every line contains 3 points.

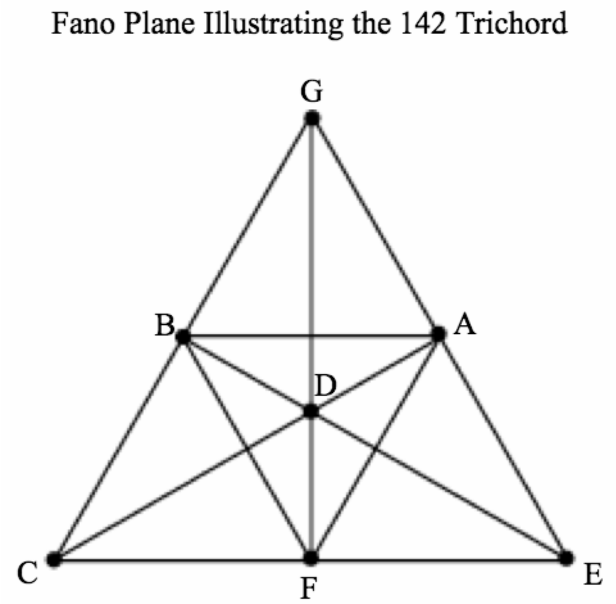


Example 6a.

The points in the Fano plane of Example 6a are labeled with pitch class names in such a way as to demonstrate the diatonic set class 124 as a Steiner triple system. Example 6b does the same thing for the 142 chords.

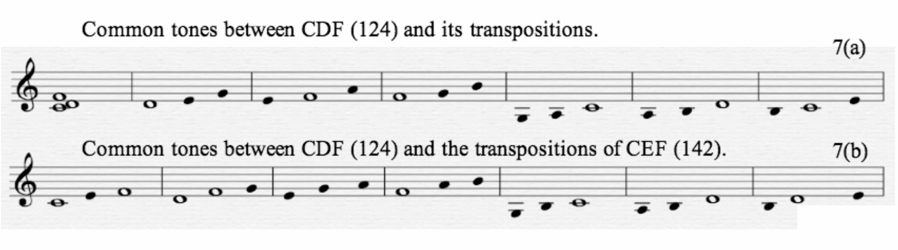
Each figure contains seven lines and seven points. The seven lines consist of the three making up the outer equilateral triangle, the three lines that connect vertices with their opposite sides, and the inner, smaller equilateral triangle. Within the context of the projective plane, this smaller triangle counts as a single line. (Sometimes it is represented as a circle, rather than as a triangle.)

What would the contrapuntal possibilities be for music in which the referential sonority were either of these set classes, or both used in conjunction? In three-voice counterpoint, no restrictions apply to the content of any pair of voices. (Any two points determine a line.) Restricting to a *single* set class, any pair of notes completely determines the third voice. If your counterpoint allows *both* 124 and 142 types, then any given pair of notes allows a choice of two notes for the third voice, one of which completes the unique 124 chord for that pair, the other yielding 142.



Example 6b.

Consider 124 chords with regard to common tones: each 124 chord shares exactly one common tone with each of the others. (Any two lines determine a point.) Example 7a shows the common tones between CDF and each of the other 124 transpositions.



Example 7.

The consistency of common tones does not imply the presence of parsimonious voice leading among the moving, non common tones. Although a set of minimally perturbed motions can be established for each of the connections, I simply prefer to celebrate the demise of voice-leading parsimony and the rise of extravagance. Happy times are here again. Go ahead, little tones, go and leap. Common tones between 124 and 142 types behave in a more complex, but highly regular manner. Example 7b shows the common tones between CDF and the transpositions of the 142 set class. Any given 124 chord shares 2 common tones with three of the 142s, 1 common tone with three others, and has no common tones with one.

In Example 8, the 124 chords are labeled as type A, the 142s as B. CF is arbitrarily chosen as the fourth to serve as the starting point in the transpositions of each chord.

Referential Sonorities for "Amarantha"

124 = "A"
142 = "B"

- A, B: Diatonic all-interval trichords.
- Any pair of notes belongs to a single A chord, and a single B chord.
- A*n* and B(8-*n*) are exact inversions around pc 'D' axis of symmetry.
- Any two A chords (or B chords) share a single common tone.
- Shared elements between types:

A type	B type	shared element
A*n*	B*n*	4th
A*n*	B*n*+1	3rd
A*n*	B*n*+2	---
A*n*	B*n*+3	common tone
A*n*	B*n*+4	common tone
A*n*	B*n*+5	2nd
A*n*	B*n*+6	common tone

Example 8.

The example also shows more detail regarding common tones between the A and B types.

The multiplicities of the specific types of 124 and 142 chords also follow the pattern 1, 2, and 4. Consider the A types. The chord A4 (FGB) is the only 124 to contain a tritone. Let us call this specific type 'z.' The A3 and A7 chords are the only two to contain a minor second. This is specific type 'y.' The other four, A1, A2, A5, and A6, exemplify type 'x.' There is four xs, two ys, and one z. The same parsing can be done for the 142 chords, but this is not shown in the example. The types x' y' and z' would be found in reverse order on the 142 transpositions. For the 124 chords, the x type, which contains a major second and a perfect fourth, is the most common, and is also the least dissonant from the point of view of traditional consonance and dissonance values. (I am assuming the minor second to be more dissonant than the major second, the augmented fourth more than the perfect fourth, and no significant difference in thirds.) The y type is rarer, occurring twice, and contains the more dissonant minor second, while the unique z type contains the even more dissonant tritone. The correspondence between rarity and dissonance is, in a sense, more clearly articulated here than it is for triads. While it is true that the rarest triad, the diminished triad BDF, is the only one to contain a tritone, the major and minor triads each occur three times, and so are not distinct with respect to multiplicity. This is one of the factors that has allowed for the major and minor triads to be treated as functionally equivalent in diatonic settings, however, the three-way distinction evident in the A and B chords presents novel compositional opportunities.

I composed the song “Amarantha” in order to explore the possibility of employing these chords as the referential sonorities in a diatonic setting, replacing the common triad. “Amarantha” contains chords of other types as well as the 124 and 142 types, however these two provide the predominant sonority. The 124 and 142 chords are also very effective in melodic contexts, and it is possible to combine the melodic and harmonic behaviors of these sonorities as we do with ordinary triads. Each vertical appearance of one of our A or B sonorities is marked below each system of the score.

The annotated score of “Amarantha” is found in the Appendix. The analysis reveals some important features of the song. There is a 3-note motto found throughout the piece which places an A or B chord, starting on a strong beat, in the rhythmic pattern, long-short-long, where the first two of these are half note quarter note. The contour is always a rise and a fall, which can manifest as either low-high-middle or middle-high-low. The first instance of the motto is found in the tenor in measure 2. At the beginning of the measure, the voices form a vertical A6, while the tenor extends the sonority into the melodic realm. The motto next appears as an A2 in the soprano in the beginning of measures 5 and 6. The A3 chord is featured in measure 8. It sounds at the initiation of both strong beats, and appears as the first note of each syllable in the lower voices. The soprano carries the motto with the B3 chord, sharing pitch classes A and E with the A3 chord.

A bit of word painting underlies the setting in measure 18. Beginning with the last eighth of the previous measure, the vertical sonorities interweave the A and B chords in sequence: A4, B6, A5, B7, A6, B1. At the same time, interlocking A and B chords are found in the soprano and alto, together with the pitch contour of the motto. In the soprano, the eighth note (end of m. 17) G initiates an A5 (G C A); the C on the downbeat begins a B6 (C A D), the A an A6, the D a B7. The alto line, F D G E A F imbricates B2, A2, B3, A3. The bass forms voice exchanges with pairs of soprano notes. This pseudo canon, together with the vertical sonorities, serves to highlight the text, “But neatly tangled at the best.” Another example of sequential treatment is found in measures 23-24, beginning on the third beat of measure 23.

The rhythmic aspect of these sets are not greatly exploited here, although one may find rhythmic instances of the 124 (A) set quite frequently, in the form eighth, quarter, half. The first such instance begins on the eighth note C in the fifth measure of the soprano line.

Until fairly recently, scale theory was focused on acoustical and psycho-acoustical aspects of musical systems. This has produced a long line of studies from Pythagoras through the dualists that aim to consider musical scales from the perspective of the purity – or lack thereof – of their foundational objects, such as the octaves, fifths, and thirds of the overtone series. Scale theories were concerned with reconciling the irreconcilable postulates of these systems. Recent scale theory has turned to the purely abstract properties of scales: To the patterns that the steps intervals make; to the cyclic properties inherent in prime versus composite cardinality; to the even distribution of its elements in a larger pitch-class universe. This paper might help to show how the turn to the abstract reveals ways in which a very old scale might still hold surprising and new compositional possibilities.

References

Carey, N., Clampitt, D.: Aspects of well-formed scales. Music Theory Spectrum 11(2), 62–87 (1989)

Carey, N.: Distribution modulo 1 and musical scales. Ph.D Dissertation. University of Rochester (1998)

Clough, J.: Aspects of diatonic sets. Journal of Music Theory 23, 45–61 (1979)

Clough, J.: Diatonic interval sets and transformation. Perspectives of New Music 18(1-2), 461–482 (1980)

Mazzola, G.: Die Geometrie der Töne. Birkhäuser, Basel (1990)

Appendix


Amarantha
Norman Carey
poem by Richard Lovelace
motto
S
A
T
A - ma - ran - tha sweet and fair, Ah, braid no more that
A1 - A5 B3 A6 B6 A6 B3 B3 A3 - B7 B5 A5
shin - ing hair! As my cur - i - ous hand or eye
B4 - B3 A4 A7 B7 B3 B2 B7 A7 B4 A5 B3 A1 B7 A7 B7
Hov - er - ing round thee, let it fly! Let it fly as
A7 - - B5 B4 A5 B5 A3 A3 (
A2
A6
B3
B6
B2


10
A1
A2
un - con- fined As its calm rav - ish- er, the wind,
B3
un - con - fined As its calm rav - ish- er, the wind,
un - con- fined As its calm rav - ish-er, the wind,
) B6 A6 - A5 B5 A4 B3 B7 A5 B4 B7
13
A6
Who hath left his dar - ling, the East To wan - ton o - ver that
Who hath left his dar - ling. the East To wan - ton o - ver that
Who hath left his dar - ling, the East To wan - ton o - ver that
B2 A4 A7 B1 A3 A6 B7 - - A6 A3 B3 - A4 A5 B7
16
spi - cy nest. Ev - ery tress must be con- fess'd. But
spi - cy nest. Ev - ery tress must be con - fess'd. But
spi - cy nest. Ev - ery tress must be con- fess'd. But
A5 B4 - B7 A6 B1 A4 A7 B4 B3 A2 A4


18
S
neat - ly tan - gled at the best, Like a clue of gol - den thread Most
A
neat - ly tan - gled at the best, Like a clue of gol - den thread Most
T
neat - ly tan - gled at the best, Like a clue of gol - den thread Most
B6 A5 B7 A6 B1 B1 B2 - A4 A7 A2 A7 - B2 A1 A4 B6 B7 A7 B5 A5
20
S
ex - cel - lent - ly ra - vel - lèd. Do not then wind
A
ex - cel - lent - ly ra - vel - lèd. Do not then wind
T
ex - cel - lent - ly ra - vel - lèd. Do not then wind
A6 B4 A4 B4 A7 A7 B2 - A3 B2 - A6 B3
22
A2
A1
S
up that light In rib - bands, and o - ver -
A
up that light In rib - bands, and o - ver -
T
up that light In rib - bands, and o - ver -
A4 B4 A5 B5 B2 A1 B3 A2 A3 A1 B1 A2 - A7 B7


24
A1
A7
A6
A5
S
cloud in night Like the sun in his ear - ly ray: But
A
cloud in night Like the sun in his ear - ly ray: But
T
A6
cloud in night Like the sun in his ear - ly ray: But
A1 - A6 B6 A7 B7 A7 B6 A7 B1 B7 A6 B6 A6 A6 B2
27
A5
S
shake your head, and scat - ter day!
A1 + B2
A
shake your head, and scat - ter day!
T
shake your head, and scat - ter day!
A1 B1 A5 B5 B6 A1 B1 A2 B2 A3 A1