A MATHEMATICAL MODEL OF THE DIATONIC SYSTEM

Eytan Agmon

INTRODUCTION

In this article a mathematical model of the diatonic system is presented, that is, the notion "diatonic system" is defined in mathematical terms. We believe the proposed model to be of considerable descriptive power, and potentially at least, of considerable explanatory power as well. That is to say, the model not only successfully captures the intuitive sense of "diatonic system," but also opens possibilities towards answering deeper questions regarding the diatonic system, such as the source of its remarkable persistency through centuries, if not millennia, of human musical history (Western musical history, at any rate).

In Agmon (1986) it is argued at length that the model represents a "cognitive" theory of diatonicism.[1] While this point is crucial from a broader theoretical perspective, we shall not pursue it again here except for pointing out that the empirical domain which the model describes is not a "perceptual" domain of pitches, but a more abstract domain of mental representation. In particular, to a considerable (and possibly surprising) extent, the model is independent of the question of how it may or may not be implemented in terms of pitches, so that the whole issue of intonation (including, for example, the status of 12-tone equal temperament) is for the most part irrelevant. A theory of intonation, that is, a theory that establishes a correspondence between the "cognitive" entity defined here as "diatonic system," and the entities known as "pitches," is outlined in Agmon (1986).

Recently two studies have appeared of which the subject matter is closely related to ours. Brinkman (1986) describes a method by which "notes" and "intervals," in the traditional sense, are represented as pairs of integer classes mod (12,7) (that is, pairs of integers, where the first integer is a reduced residue mod 12, 0, 1, . . . , 11, and the second integer is a reduced residue mod 7, 0, 1, . . . , 6). Brinkman's concern, however, is primarily practical: devising a system of representation that is useful for computer processing of musical data. The deeper implications of this "binomial" system, be they of mathematical or empirical nature, are not considered.[2]

Clough and Myerson (1985) do not present a "binomial" system as such, but attempt to relate, mathematically, two interpretations of the diatonic scale which they term "specific" and "generic." Specifically, the diatonic scale is interpreted as a partial set of integer classes mod 12 (for example, 0, 2, 4, 5, 7, 9, 11), while generically the diatonic scale is interpreted as a full set of integer classes mod 7 (0, 1, . . . , 6). The authors find some remarkable connections between the two approaches, which lead them, in turn, to some powerful and non-trivial generalizations regarding "diatonic systems" and their properties.

Even a superficial comparison between Clough and Myerson's work and the model proposed here will reveal that the two are related (and even closely so) from the mathematical point of view. Yet while these relations are striking (particularly since the essential ideas presented here were developed independently of Clough and Myerson's paper), they reveal little that is common in terms of underlying approach. For further discussion, see the Appendix.

The central portion of this paper is occupied by a general formal definition of "diatonic system." As we shall see, the generic "diatonic system" allows for an indefinitely large number of specific "diatonic systems," of which the familiar diatonic system is one. It is then shown that an additional constraint on "diatonic system" reduces the indefinitely large number of specific "diatonic systems" to unity, this unique "diatonic system" being the familiar diatonic system. To provide a motivation for the general formal definition, we shall begin with an informal discussion of the mathematical properties of the familiar diatonic system.

We take it as a truism that the familiar diatonic system consists of sets of elements we call "scale steps," and a set of elements we call "diatonic intervals." Note that there are seven alternative sets of scale steps, corresponding to the seven "modes," against a single shared set of diatonic intervals.

In Agmon (1986), Brinkman (1986), and other places, it has been noted that scale steps and diatonic intervals can be represented as pairs of integer classes, mod (12,7), in the sense already informally defined. This is demonstrated in Figure 1, using as a set of scale steps the "major," or "Ionian" mode. Later we shall have some specific things to say regarding the correspondence between integer-pair and conventional representations of diatonic intervals and scale steps, as depicted in Figure 1; for the time being, however, let us simply accept this correspondence as given.

What are the mathematical properties of a set of integer-pair classes mod (12,7) we call a set of scale steps, and the set we call diatonic intervals? To begin with, the two sets are interdependent. Given a set of scale steps, one may derive the set of diatonic intervals by subtracting scale steps two at a time from each other, mod (12,7). For example, given the Ionian mode (see Figure 1), one may subtract scale step (5,3) from scale step (4,2), mod (12,7) as follows:

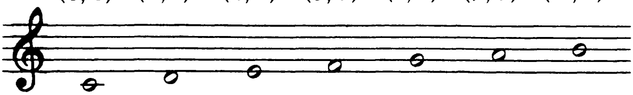
$$(4,2)-(5,3) \text{ mod } (12,7)=(4-5 \text{ mod } 12,\ 2-3 \text{ mod } 7),$$

yielding the result (11,6) which is a diatonic interval. One may subtract in this fashion any scale step from any scale step; the set of all such differences yield, with some repetition, the complete set of diatonic intervals. Note that the subtraction operation corresponds to the intuitive sense of "diatonic interval" which is the "distance"–that is, difference–between two scale steps. Thus, in the specific example given before, the distance between $\hat{4}$ and $\hat{3}$ in major was found to be a major seventh. Had the subtraction operation been carried out in a different order, one would have obtained the result (1,1), or a minor second. A pair of diatonic intervals like (11,6) and (1,1) which add up to (0,0) mod (12,7) are known as "inversionally related."

Deriving the set of diatonic intervals from a given set of scale steps implies that the latter is somehow conceptually prior to the former. This relation, however, may be reversed. As we shall see, given the set of diatonic intervals, one may derive a set of scale steps from it. And although it is possible to proceed either way, the latter approach not only works out more elegantly from the formal point of view, but to our mind at least, it is more satisfying intuitively as well.

How, then, are we given the set of diatonic intervals? Consider Figure 2. The perfect fourth, or integer-pair class (5,3), is multiplied in turn by thirteen different integers from a set of thirteen consecutive integers

scale steps	(0, 0)	(2, 1)	(4, 2)	(5, 3)	(7, 4)	(9, 5)	(11, 6)
staff notation							
letter-name notation	C	D	E	F	G	A	B
sol-fa syllables	Do	Re	Mi	Fa	Sol	La	Ti
integer notation (Schenker)	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$	$\hat{7}$



diatonic intervals

(11, 6)	major seventh	(M7)
(10, 6)	minor seventh	(m7)
(9, 5)	major sixth	(M6)
(8, 5)	minor sixth	(m6)
(7, 4)	perfect fifth	(P5)
(6, 4)	diminished fifth	(d5)
(6, 3)	augmented fourth	(A4)
(5, 3)	perfect fourth	(P4)
(4, 2)	major third	(M3)
(3, 2)	minor third	(m3)
(2, 1)	major second	(M2)
(1, 1)	minor second	(m2)
(0, 0)	perfect prime	(P1)

Figure 1

between -6 and 6, mod (12,7). For example, (5,3) is multiplied by 3, mod (12,7) as follows:

$$3 \cdot (5,3) \bmod (12,7) = (3 \cdot 5 \bmod 12,\ 3 \cdot 3 \bmod 7) = (3,2).$$

The set of thirteen mod (12,7) products $-6\cdot(5,3)$, $-5\cdot(5,3)$, . . . , $6\cdot(5,3)$ yield, as can be seen, the set of thirteen diatonic intervals. Clearly, we could have obtained the same thirteen diatonic intervals by using, in place of the perfect fourth (5,3), the inversionally related perfect fifth (7,4). Either way, the perfect fifth and fourth have a special status within the diatonic system as *cyclic generators.*

Note in Figure 2 that diatonic intervals are arranged equidistantly from (0,0) in inversionally related pairs. Since we shall want to obtain the set of diatonic intervals as differences between pairs of scale steps, this is an obvious necessary property.

In Figure 3 seven subsets are extracted from the set of thirteen diatonic intervals, arranged cyclically as in Figure 2. In this cyclic representation each subset consists of seven adjacent diatonic intervals and is thus a set of seven products, mod (12,7), of (5,3) and different integers from a set of seven consecutive integers between -6 and 6. It can be easily shown that mod (12,7) differences between any two integer-pair classes belonging to any given subset are diatonic intervals, and that the set of all such differences yield, with some repetition, the complete set of diatonic intervals; thus each of the seven subsets is a set of scale steps. If scale step (0,0) is construed in each case as a "tonic" (or "finalis"), the seven sets of scale steps correspond, as can be seen, to the seven diatonic "modes" (at the right-hand side of Figure 3 each mode is ordered as an "ascending scale").

It may not be obvious at first, but scale steps and diatonic intervals have two other important properties. To bring these properties to light we have devised a series of diatonic-system "imposters" (Figures 4 and 5). In Figure 4 we have two such imposters, A and B. In both cases we have a cyclically generated set of "diatonic intervals," structured symmetrically around (0,0); and in both cases several sets of "scale steps," all having the necessary subtraction property, are extracted from the set of diatonic intervals as subsets. However, the "diatonic system" in A consists of only eleven "diatonic intervals" $-5\cdot(5,3)$, $-4\cdot(5,3)$, . . . , $5\cdot(5,3)$ mod (12,7), and correspondingly, each set of "scale steps" contains only six integer-pair classes. In B there are thirteen "diatonic intervals" and each set of "scale steps" contains seven integer-pair classes; but this "diatonic system" is mod (13,5), and the cyclic generator is (5,2). As a result, in A there are eleven "diatonic intervals" against twelve integer classes mod 12, so that one integer class mod 12, namely 6, is not contained as a first member within the set of "diatonic intervals"; and there is one integer class mod 7 missing as a *second* member within any set of "scale steps." In B the set of "diatonic intervals" contains a full set of integer classes mod 13, 0, 1, . . . , 12 as first

6 times (5, 3) mod (12,7) = (6, 4) = d5
5 " " " " = (1, 1) = m2
4 " " " " = (8, 5) = m6
3 " " " " = (3, 2) = m3
2 " " " " = (10, 6) = m7
1 " " " " = (5, 3) = P4
0 " " " " = (0, 0) = P1
−1 " " " " = (7, 4) = P5
−2 " " " " = (2, 1) = M2
−3 " " " " = (9, 5) = M6
−4 " " " " = (4, 2) = M3
−5 " " " " = (11, 6) = M7
−6 " " " " = (6, 3) = A4

inversionally-related pairs

Figure 2

diatonic intervals, arranged cyclically

(6, 4)
(1, 1)
(8, 5)
(3, 2)
(10, 6)
(5, 3)
(0, 0)
(7, 4)
(2, 1)
(9, 5)
(4, 2)
(11, 6)
(6, 3)

scale steps, arranged scalarly

(0, 0), (1, 1), (3, 2), (5, 3), (6, 4), (8, 5), (10,6) ("Locrian")
(0, 0), (1, 1), (3, 2), (5, 3), (7, 4), (8, 5), (10, 6) ("Phrygian")
(0, 0), (2, 1), (3, 2), (5, 3), (7, 4), (8, 5), (10, 6) ("Aeolian")
(0, 0), (2, 1), (3, 2), (5, 3), (7, 4), (9, 5), (10, 6) ("Dorian")
(0, 0), (2, 1), (4, 2), (5, 3), (7, 4), (9, 5), (10, 6) ("Mixolydian")
(0, 0), (2, 1), (4, 2), (5, 3), (7, 4), (9, 5), (11, 6) ("Ionian")
(0, 0), (2, 1), (4, 2), (6, 3), (7, 4), (9, 5), (11, 6) ("Lydian")

Figure 3

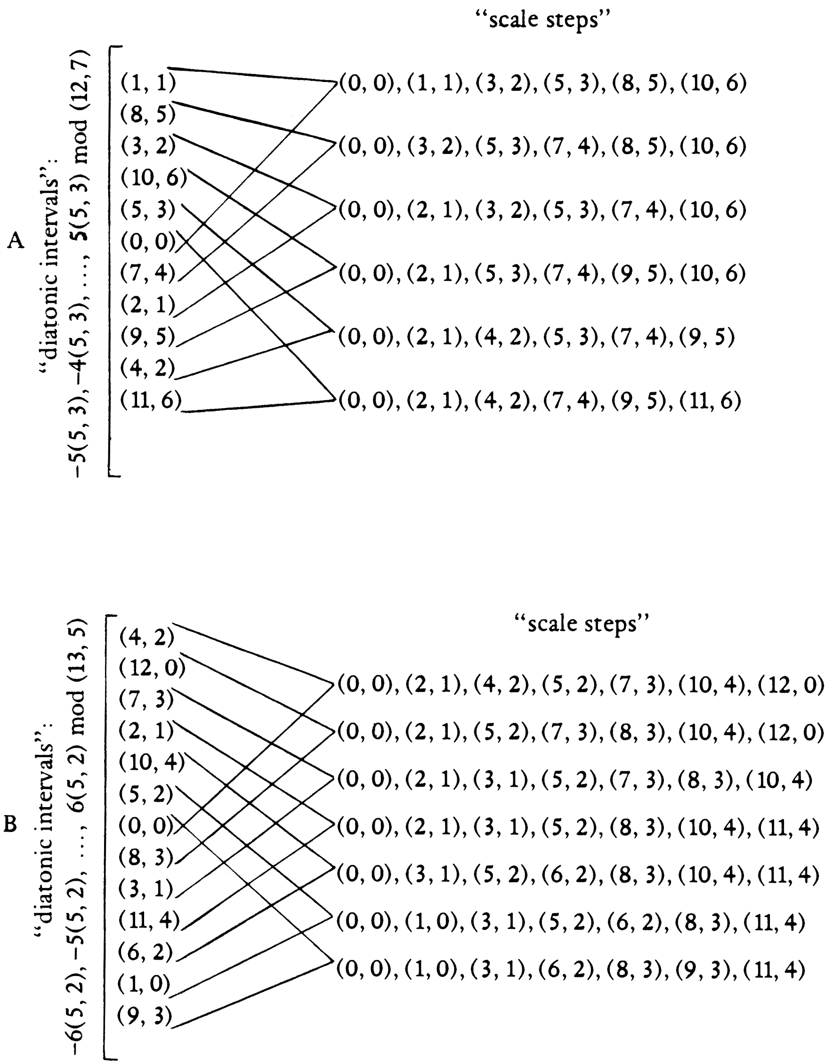

A
"scale steps"
"diatonic intervals": −5(5, 3), −4(5, 3), ..., 5(5, 3) mod (12, 7)
(1, 1)
(8, 5)
(3, 2)
(10, 6)
(5, 3)
(0, 0)
(7, 4)
(2, 1)
(9, 5)
(4, 2)
(11, 6)
(0, 0), (1, 1), (3, 2), (5, 3), (8, 5), (10, 6)
(0, 0), (3, 2), (5, 3), (7, 4), (8, 5), (10, 6)
(0, 0), (2, 1), (3, 2), (5, 3), (7, 4), (10, 6)
(0, 0), (2, 1), (5, 3), (7, 4), (9, 5), (10, 6)
(0, 0), (2, 1), (4, 2), (5, 3), (7, 4), (9, 5)
(0, 0), (2, 1), (4, 2), (7, 4), (9, 5), (11, 6)
B
"scale steps"
"diatonic intervals": −6(5, 2), −5(5, 2), ..., 6(5, 2) mod (13, 5)
(4, 2)
(12, 0)
(7, 3)
(2, 1)
(10, 4)
(5, 2)
(0, 0)
(8, 3)
(3, 1)
(11, 4)
(6, 2)
(1, 0)
(9, 3)
(0, 0), (2, 1), (4, 2), (5, 2), (7, 3), (10, 4), (12, 0)
(0, 0), (2, 1), (5, 2), (7, 3), (8, 3), (10, 4), (12, 0)
(0, 0), (2, 1), (3, 1), (5, 2), (7, 3), (8, 3), (10, 4)
(0, 0), (2, 1), (3, 1), (5, 2), (8, 3), (10, 4), (11, 4)
(0, 0), (3, 1), (5, 2), (6, 2), (8, 3), (10, 4), (11, 4)
(0, 0), (1, 0), (3, 1), (5, 2), (6, 2), (8, 3), (11, 4)
(0, 0), (1, 0), (3, 1), (6, 2), (8, 3), (9, 3), (11, 4)

Figure 4

members, and every set of "scale steps" contains a full set of integer classes mod 5 0, 1, . . . , 4 as second members. But there are seven integer-pair classes within any set of "scale steps," so that two integer classes mod 5 appear twice as second members in every set. In light of these two imposters it is not difficult to see that the familiar diatonic system has the property that the set of diatonic intervals contains a full set of integer classes mod 12 as first members, while a set of scale steps contains a full set of integer classes mod 7 as second members; moreover, the set of diatonic intervals and a set of scale steps contain the minimal number of integer-pair classes necessary in each case.

Note that since the number of diatonic intervals is necessarily odd (recall the symmetrical structure of the set of diatonic intervals noted in connection with Figure 2), the familiar diatonic system has a pair of diatonic intervals sharing the same integer class mod 12, namely 6, as a first member: we have an *even* number of integer classes mod 12, namely twelve, so that the minimal odd number equalling at least twelve is larger by one, namely thirteen. The pair of diatonic intervals sharing 6 as a first member, namely the augmented fourth (6,3) and the diminished fifth (6,4), are known as "enharmonically equivalent." Let us compare the familiar diatonic system in this respect with the diatonic-system imposter depicted in Figure 4B. The latter is a mod (13,5) system; since 13 is an odd number, each integer class mod 13 is contained as a first member in exactly one "diatonic interval," and a pair of enharmonically equivalent "diatonic intervals" does not exist.

In Figure 5 we have another diatonic-system imposter. Here the problem with the earlier imposters is rectified: as with the familiar diatonic system we have thirteen "diatonic intervals" against twelve integer classes mod 12, and seven sets of seven "scale steps," against seven integer classes mod 7. Note, however, that unlike the familiar diatonic system the cyclic generator here is (5,2) rather than (5,3). As a result, the relation between the integer classes mod 12 and the corresponding integer classes mod 7 within the set of "diatonic intervals" and any set of "scale steps" is unlike the relation we find in the familiar diatonic system. At the right-hand side of Figure 5 every set of "scale steps" is ordered with its mod-7 integer classes forming an ascending sequence: $0<1< . . . <6$. But unlike the familiar diatonic system (cf. Figure 3), the corresponding mod-12 integer classes do not form a similar sequence: their value falls and rises in a seemingly random fashion. To use conventional terminology, it is impossible to order a set of scale steps here as an ascending scale. This brings to light another property of the familiar diatonic system: within a set of scale steps the mod-12 integer classes are a *monotone increasing function* of the corresponding mod-7 integer classes; a value increase within the latter entails a value increase within the former. Within the set of diatonic intervals the situation is slightly different. The mod-7 integer classes are a monotone increasing

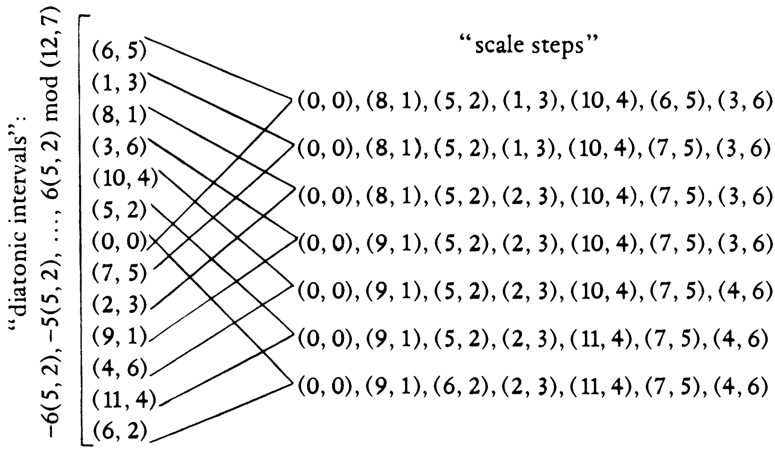

"diatonic intervals":
−6(5, 2), −5(5, 2), ..., 6(5, 2) mod (12, 7)
(6, 5)
(1, 3)
(8, 1)
(3, 6)
(10, 4)
(5, 2)
(0, 0)
(7, 5)
(2, 3)
(9, 1)
(4, 6)
(11, 4)
(6, 2)
"scale steps"
(0, 0), (8, 1), (5, 2), (1, 3), (10, 4), (6, 5), (3, 6)
(0, 0), (8, 1), (5, 2), (1, 3), (10, 4), (7, 5), (3, 6)
(0, 0), (8, 1), (5, 2), (2, 3), (10, 4), (7, 5), (3, 6)
(0, 0), (9, 1), (5, 2), (2, 3), (10, 4), (7, 5), (3, 6)
(0, 0), (9, 1), (5, 2), (2, 3), (10, 4), (7, 5), (4, 6)
(0, 0), (9, 1), (5, 2), (2, 3), (11, 4), (7, 5), (4, 6)
(0, 0), (9, 1), (6, 2), (2, 3), (11, 4), (7, 5), (4, 6)

Figure 5

function of the corresponding mod-12 integer classes, in the sense that a value increase within the latter entails a value increase *or no change in value whatever* within the former.

We have discussed, informally, the mathematical properties of the familiar diatonic system, but have not stated explicitly its most fundamental property. The familiar diatonic system consists of integer pairs (x,y), divided into classes in the following way: each class consists of integer pairs of the form (s+k12,t+l7), where s=0, 1, . . . , 11, t=0, 1, . . . , 6, and k and l run over all integers. We say that any two integer pairs belonging to the same class, for example, (0,0) and (12,7), (−2,1) and (10,8), are equivalent, mod (12,7), or "octave equivalent." Thus, any formal discussion of the diatonic system must begin with the notion of octave equivalence.

Before turning to the central, formal portion of this article, let us return for a moment to Figure 1. In what sense are conventional and integer-pair representations of scale steps and diatonic intervals interchangeable? A set of scale steps embodies a one-to-one correspondence between a partial set of integer classes mod 12, and the full set of integer classes mod 7—each integer class of a set of seven integer classes mod 12 corresponds to exactly one integer class mod 7, and vice versa. Clearly, the integer classes mod 7 alone suffice to specify uniquely any given scale steps. One may therefore use the mod-7 integer classes as a kind of *notational shorthand* for representing scale steps. Indeed, letter-name notation, Sol-Fa syllables, and Schenker's integer notation can all be construed as isomorphical with a mod-7 system of integer classes; and disregarding register, staff notation can also be construed in the same way (that is, one may establish a one-to-one correspondence between *staff positions* and integer classes mod 7). Note that in all of the above systems and the mod-12 integer classes are implicitly understood; musicians simply *know* that the "semitonal structure" of (say) the major scale is "1, 1, ½, 1, 1, 1, ½."

The set of diatonic intervals embodies the following correspondence between the full set of integer classes mod 12 and the full set of integer classes mod 7: each integer class mod 12 except 6 corresponds to exactly one integer class mod 7, while each integer class mod 7 except 0 corresponds to exactly *two* integer classes mod 12. In this case, conventional nomenclature takes into account the mod-7 *and* mod-12 integer classes as well as other considerations. The mod-7 integer classes are translated in an obvious way into one of seven interval *types,* prime, second, third, etc., while the corresponding mod-12 integer classes, along with other considerations, are translated into one of five *qualities:* perfect, major, minor, augmented, and diminished. Thus, (0,0) is a "perfect" prime because it is necessarily connected with the fundamental notion of octave, or mod (12,7), equivalence. Seconds, thirds, sixths, and sevenths come in two qualities each, "major," and "minor"; clearly, within each type the interval with the larger mod-12 integer class is major, while the other is minor. The fourth (5,3) and fifth (7,4) are "perfect" because of their special status within the diatonic system as cyclic generators. And finally, the fourth (6,3) and fifth (6,4) are "augmented" and "diminished," respectively, because the fourth (5,3) and fifth (7,4) are "perfect," not "minor" and "major," respectively.

A GENERAL FORMAL DEFINITION OF "DIATONIC SYSTEM"

Preliminary Definitions and Notations

1. *Octave Equivalence.* Let a and b be two integers, $a>b>1$, such that

(1) $$\text{g.c.d.}(a,b)=1$$

(that is, a and b are coprime, their greatest common divisor is 1).

In what follows we shall be dealing with pairs of integers (x,y) divided into classes in the following way: each class consists of pairs of integers of the form $(s+ka,t+lb)$, where $s=0, 1, \ldots, a-1$; $t=0, 1, \ldots, b-1$; and k and l run over all integers. Any two pairs of integers $(s+ia,t+jb)$ and $(s+i'a,t+j'b)$ belonging to the same class (i, j, i', and j' here are fixed integers) will be said to be equivalent (or congruent), mod (a,b), or "octave equivalent," and we shall write:

$$(s+ia,t+jb) \equiv (s+i'a,t+j'b) \bmod (a,b).$$

Clearly, there are ab different classes of octave equivalent integer pairs. We shall use the notation (s,t) mod (a,b) to represent an arbitrary class.

We shall use the following operations involving integer-pair classes mod (a,b):

$$(s,t)-(s',t') \bmod (a,b) \equiv (s-s' \bmod a, t-t' \bmod b);$$
$$n(s,t) \bmod (a,b) \equiv (ns \bmod a, nt \bmod b).$$

2. *Quintic Class.* A class (q,r) mod (a,b), $0<q\leq a-1$, $0<r\leq b-1$, will be referred to as "quintic" if it satisfies:

(2) $$\text{g.c.d.}(a,q)=1, \text{ and g.c.d.}(b,r)=1.$$

Note that the quintic class has the property that it is a cyclic generator of all the classes mod (a,b), in the sense that for any class (s,t) there exists an integer n such that $n(q,r) \equiv (s,t) \bmod (a,b)$ (this follows from the "Chinese remainder theorem" of elementary number theory; see Dickson 1939, p. 16).

3. *Diatonic Intervals.* Let N be a positive integer smaller than ab/2. We shall refer to the set of 2N+1 classes n(q,r) mod (a,b), n=0, $\pm 1, \ldots, \pm N$, as a set of diatonic intervals of order N.

Note that the same set of diatonic intervals is obtained by the quintic class $-(q,r) \equiv (a-q,b-r)$ mod (a,b).

4. *Scale Steps.* Given a set of diatonic intervals of order N (relative to some q,r), we shall refer to a subset of this set as a set of scale steps if all differences $(s,t)-(s',t')$ mod (a,b) of any two members of the subset belong to the given set of diatonic intervals, and moreover, the set of all such differences yield (with possible repetition) the complete given set of diatonic intervals.

As is easily verified, a set of scale steps exists, and moreover, there are N+1 such sets, which are of the form m(q,r) mod (a,b), where m runs over N+1 consecutive integers between $-N$ and N.

In what follows we shall be interested in a set D of diatonic intervals and a corresponding set S of scale steps having certain properties. We shall say that a set D of diatonic intervals and a corresponding set S of scale steps are "efficient" if they jointly satisfy:

(1) For any $s=0, 1, \ldots, a-1$, there exists at least one t satisfying $0 \leq t \leq b-1$, such that (s,t) is in D, and for any $t=0, 1, \ldots, b-1$, there exists at least one s satisfying $0 \leq s \leq a-1$, such that (s,t) is in S;

(2) D and S as above have the minimal number of elements.

For a set of diatonic intervals and a corresponding set of scale steps to be efficient, a, b, and N cannot be arbitrary; in fact, we have the following

PROPOSITION 1. **Necessary and sufficient conditions for a set of diatonic intervals and a corresponding set of scale steps to be efficient are the following:**

$$b=\frac{a+1}{2}+1 \text{ (a is odd), or} \tag{3}$$
$$b=(a/2)+1 \text{ (a is even);}$$

$$N=b-1. \tag{4}$$

Proof. We shall show first that (3) and (4) are necessary conditions for efficient diatonic intervals and scale steps. By definition, a set D of diatonic intervals contains 2N+1 elements n(q,r) mod (a,b), n=0, $\pm 1, \ldots, \pm N$. Clearly, to fulfill the first efficiency requirement 2N+1 must equal at least a; selecting now the minimal 2N+1 (the second efficiency requirement) we get:

(5) $$2N+1=a \text{ (a is odd), or}$$
$$2N+1=a+1 \text{ (a is even).}$$

Given a set D of diatonic intervals, a corresponding set S of scale steps contains N+1 elements m(q,r) mod (a,b), where m runs over N+1 consecutive integers between −N and N. Clearly, to fulfill the first efficiency requirement N+1 must equal at least b; selecting now the minimal N+1 (the second efficiency requirement) we get:

(6) $$N+1=b.$$

The two requirements (3) and (4) of Proposition 1 follow directly from (5) and (6), which proves that (3) and (4) are necessary conditions for efficient diatonic intervals and scale steps. To show that (3) and (4) are also sufficient conditions for efficient diatonic intervals and scale steps it suffices to show that (3) and (4) imply (5) and (6), which is trivial. This concludes the proof of Proposition 1.

From now on a set D of diatonic intervals and a corresponding set S of scale steps are always assumed to be efficient.

Note that when a is odd, the set D of diatonic intervals has the property that for any given s=0, 1, . . . , a−1, there exists exactly one t satisfying $0 \leq t \leq b-1$ such that (s,t) is in D. Similarly, for the corresponding set S of scale steps, for any given t=0, 1, . . . , b−1, there exists exactly one s satisfying $0 \leq s \leq a-1$ such that (s,t) is in S (since S is a subset of D it is also clear that different scale steps will have a different s). When a is even the property holds for the set of scale steps. However, in the complete set of diatonic intervals {(s,t)} the situation is slightly different. We have the following

PROPOSITION 2. **When a is even, the set of diatonic intervals {(s,t)} contains exactly two diatonic intervals having the same first member; these diatonic intervals are (a/2,r) and (a/2,b−r).**

Proof. From Proposition 1 N = a/2, so that D is given by the set n(q,r) mod (a,b), n=−(a/2), −(a/2)+1, . . . , a/2. Clearly, nq mod a, n= −(a/2)+1, −(a/2)+2, . . . , a/2, is a complete set of residues mod a, whereas $-(a/2)q \equiv (a/2)q$ mod a. Furthermore, since q is prime to a (which is even by hypothesis) q is odd, and therefore $\pm(a/2)q \equiv$ a/2 mod a. Finally, from Proposition 1 we have a/2=b−1, and thus $(a/2)r \equiv (b-1)r \equiv b-r \pmod b$, and similarly, $-(a/2)r \equiv r \pmod b$. This proves Proposition 2.

We shall refer to the pair of diatonic intervals (a/2,r) and (a/2,b−r) as "enharmonically equivalent." As we shall see, in our model an odd a will be ruled out, so that every possible "diatonic system" will contain a pair of enharmonically equivalent diatonic intervals.

We shall now prove another proposition regarding efficient diatonic intervals, which will be of use later.

PROPOSITION 3. **For any $t=1, 2, \ldots, b-1$, there exists a pair s and s′ ($s \neq s'$) satisfying $1 \leq s,s' \leq a-1$ such that (s,t) and (s′,t) are in D.**

Proof. We can write the 2N diatonic intervals $n(q,r)$ mod (a,b), $n = \pm 1, \pm 2, \ldots, \pm N$, as a set of N pairs: $|n|$ (q,r) and $(-b+|n|)$ (q,r) mod (a,b). It is clear that mod (a,b) each pair is of the form $\{(s,t), (s',t)\}$, with $s \neq s'$, and that different pairs have a different t in this representation.

We shall now consider a set of (efficient) diatonic intervals of a special kind. We shall say that a set of diatonic intervals is "coherent" if it has the following property: for any pair of diatonic intervals (s,t) and (s′,t′) such that $s > s'$, the relation $t \geq t'$ holds. If S is a set of scale steps which corresponds to a coherent set D of diatonic intervals, we shall call it coherent as well. It is easy to see (using the fact that S is a subset of D) that a coherent set of scale steps has the property that for any pair of scale steps (s,t) and (s′,t′) such that $s > s'$, the relation $t > t'$ holds.[3]

For a set of diatonic intervals to be coherent, (q,r) cannot be arbitrary. In fact, we have the following

THEOREM. **Necessary and sufficient conditions for a set of (efficient) diatonic intervals to be coherent are the following:**

(7) (a) a is odd:

$$(q,r)=(2,1) \text{ or } (a-2,b-1);$$

(b) a is even:

$$(q,r)=\left(\frac{a}{2}+1, \frac{b+1}{2}\right) \text{ or } \left(\frac{a}{2}-1, \frac{b-1}{2}\right).$$

To prove the theorem we shall first prove a

LEMMA. **A necessary condition for a set of diatonic intervals to be coherent is the following:**

CONSECUTIVITY PROPERTY (CP).[4] **For any pair of different diatonic intervals (s,t) and (s′,t′) such that $t=t'$ (see Proposition 3), the relation $s-s' = \pm 1$ holds.**

Proof. Consider a pair of different diatonic intervals (s,t) and (s′, t′) such that $t=t'$, and suppose the set D of diatonic intervals to which they belong is coherent. Now suppose by contradiction that CP does not hold, and $|s-s'|>1$. Then there exists another diatonic interval (s″,t″) such that $s>s''>s'$ or $s<s''<s'$. It is easy to see that our coherence condition cannot hold simultaneously for both pairs of diatonic intervals (s,t) and (s″,t″), and (s″,t″) and (s′,t′), which contradicts our coherence assumption. This proves the lemma.

Proof of the theorem. From CP and the definition of diatonic intervals (relative to q,r) it follows that if $n_1 r \equiv n_2 r \pmod b$, $-N \leq n_1, n_2 \leq N$, the relation $n_1 q \equiv n_2 q \pm 1 \pmod a$ holds. That is:

(8) $(n_1-n_2)r \equiv 0 \pmod{b}$ implies $(n_1-n_2)q \equiv 1$ or $a-1 \pmod{a}$.

By Proposition 1 $N=b-1$, and therefore $|n_1-n_2|<2b$. It therefore follows from (8) that $n_1-n_2=\pm b$, and we have

(9) $$bq \equiv 1 \text{ or } a-1 \pmod{a}.$$

Assuming that a is odd we have by Proposition 1 $b=\frac{a+1}{2}+1$. By substitution in (9) we get

$$q(\frac{a+1}{2}+1) \equiv 1 \text{ or } a-1 \pmod{a},$$

from which it is easily concluded that

(10) $$q=2 \text{ or } a-2 \text{ (a is odd)}.$$

By CP we may order the set of diatonic intervals as follows:

$$(0,0),\ (1,t_1),\ (2,t_1),\ (3,t_2),\ (4,t_2),\ \ldots,\ (a-2,t_{b-1}),\ (a-1,t_{b-1}).$$

Since the set $\{t\}$ is a complete set of residues mod b, and assuming that the above set of diatonic intervals is coherent, it follows that $t_1=1$, $t_2=2$, $\ldots$, $t_{b-1}=b-1$. Since $t_1=1$, $(2,1)$ is a diatonic interval, therefore $(2,1) \equiv n(q,r) \bmod (a,b)$ for some n, $-N\leq n\leq N$. By (10) $q=2$ or $a-2$. If $q=2$, then necessarily $n=1$, whence $r=1$; similarly, if $q=a-2$, $r=b-1$. This proves that (7a) is a necessary condition for coherent diatonic intervals when a is odd.

Assuming now that a is even we have by Proposition 1 $b=(a/2)+1$. By substitution in (9) we get

$$q(\frac{a}{2}+1) \equiv 1 \text{ or } a-1 \pmod{a},$$

from which it is easily concluded that

$$q=(a/2)+1 \text{ or } (a/2)-1 \text{ (a is even)}.$$

It can be easily proven that r and $b-r$ (see Proposition 2) are consecutive integers, using a line of argument similar to that used in proving CP. Therefore, $r-(b-r)=\pm 1$, or

$$r=(b+1)/2 \text{ or } (b-1)/2 \text{ (a is even)}.$$

It can be checked easily that $(q,r)=(\frac{a}{2}+1,\frac{b-1}{2})$ or $(\frac{a}{2}-1,\frac{b+1}{2})$ does not yield coherent diatonic intervals. This proves that (7b) is a necessary condition for coherent diatonic intervals when a is even.

We will now show that (7a) and (7b) are also sufficient conditions for coherent diatonic intervals.

If a is odd it follows from (7a) and Proposition 1 that $bq \equiv 1$ or $a-1$

(mod a); it follows that CP holds, and we may order the set of diatonic intervals as follows:

$$(0,0),\ (1,t_1),\ (2,t_1),\ (3,t_2),\ (4,t_2),\ \ldots,\ (a-2,t_N),\ (a-1,t_N).$$

Assuming $(q,r)=(2,1)$ we have $t_1=1$. On the basis of Proposition 1, $n(q,r)$ mod (a,b), $n=2, 3, \ldots, N$, is the set $\{(4,2), (6,3), \ldots, (a-1,b-1)\}$, and therefore $t_2=2$, $t_3=3$, $\ldots$, $t_N=b-1$. Similarly, assuming $(q,r)=(a-2,b-1)$ we have $t_N=b-1$. On the basis of Proposition 1 $n(q,r)$ mod (a,b), $n=2, 3, \ldots, N$, is the set $\{(a-4,b-2), (a-6,b-3), \ldots, (1,1)\}$, and therefore $t_{N-1}=b-2$, $t_{N-2}=b-3$, $\ldots$, $t_1=1$. This proves that (7a) is a sufficient condition for coherent diatonic intervals when a is odd.

If a is even it follows from (7b) and Proposition 1 that $bq \equiv 1$ or $a-1$ (mod a), and therefore CP holds here as well. From Proposition 2 $(a/2,r)$ and $(a/2,b-r)$ are diatonic intervals. Assuming $(q,r)=(\frac{a}{2}+1,\frac{b+1}{2})$ we have $r>b-r$, and we may order the set of all diatonic intervals as follows:

$$(0,0),\ (1,t_1),\ (2,t_1),\ (3,t_2),\ (4,t_2),\ \ldots,\ (\tfrac{a}{2}-1,b-r),\ (a/2,b-r),$$
$$(a/2,r),\ (\tfrac{a}{2}+1,r),\ \ldots,\ (a-4,\ t_{b-2}),\ (a-3,t_{b-2}),\ (a-2,t_{b-1}),\ (a-1,t_{b-1}).$$

From Proposition 1 it follows that $n(q,r)$ mod (a,b), $n=2, 4, 6, \ldots, N$, $(q,r) = (\frac{a}{2}+1,\frac{b+1}{2})$, is the set $\{(2,1), (4,2), (6,3), \ldots, (a/2,b-r)\}$, and therefore $t_1=1$, $t_2=2$, $\ldots$, $t_{N/2}=b-r$. By definition, the set of diatonic intervals consists of inversionally related pairs, and referring to the set of diatonic intervals as ordered above, it is clear that these pairs are $(1,t_1)$ and $(a-1,t_{b-1})$, $(2,t_1)$ and $(a-2,t_{b-1})$, $\ldots$, $(a/2,b-r)$ and $(a/2,r)$. Therefore, $t_{b-1}=b-1$, $t_{b-2}=b-2$, $\ldots$, $t_{(N/2)+1}=r$. A similar argument may be developed assuming $(q,r)=(\frac{a}{2}-1,\frac{b-1}{2})$, which proves that (7b) is a sufficient condition for coherent diatonic intervals when a is even. This concludes the proof of our theorem.

From now on we shall use (q,r) and (q',r') to denote the specific quintic classes yielding coherent diatonic intervals, as follows:

(12) $\quad (q,r)=(2,1)$ (a is odd) or $(\frac{a}{2}-1,\frac{b-1}{2})$ (a is even),

$\quad\quad (q',r')=(a-2,b-1)$ (a is odd) or $(\frac{a}{2}+1,\frac{b+1}{2})$ (a is even).

Clearly, $(q',r')=(a-q,b-r)$, so that (q,r) and (q',r') yield the same set of diatonic intervals.

We shall now develop an argument showing that the case a is odd is structurally less interesting than the case a is even. This will lead us to propose an additional constraint on (q,r), whereby the case a is odd is ruled out altogether.

By definition, a set S of scale steps may be ordered as follows:

$$m(q,r),\ m(q,r)+(q,r),\ m(q,r)+2(q,r),\ \ldots,\ m(q,r)+N(q,r),$$

all mod (a,b), and where m is a fixed integer between 0 and $-N$. We shall refer to this ordering of S as "cyclic." If S is coherent (and efficient), we may also order it as follows:

$$m(q,r),\ m(q,r)+(s_1,t_1),\ m(q,r)+(s_2,t_2),\ \ldots,\ m(q,r)+(s_N,t_N),$$

all mod (a,b), and were $(0,0)<(s_1,t_1)<(s_2,t_2)< \ldots <(s_N,t_N)$ (m is the same fixed integer as before). We shall refer to this ordering of S as "scalar." Clearly, the cyclic and scalar orderings of S will be distinct if and only if $n(q,r) \not\equiv (s_n,t_n)$ mod (a,b) for a least some values of n, $1 \leq n \leq N$. This will not be the case if $N(q,r)<(a,b)$, because it then follows that $n(q,r) \equiv (s_n,t_n)$ mod $(a,b)=n(q,r)$, for *all* values of n, $1 \leq n \leq N$. That is, a necessary (and clearly also sufficient) condition for the cyclic and scalar orderings of S to be distinct is

$$N(q,r)>(a,b).$$

On the basis of Proposition 1, and assuming, as in (12), $(q,r)<(a/2,b/2)$, it follows that

(13) $$(q,r)>(2,1).$$

Since (q,r) always equals (2,1) when a is odd, the cyclic and scalar orderings of a set S of scale steps will be indistinct when a is odd. Clearly, this is a structurally less interesting case than the case a is even. We therefore propose (13) as an additional constraint on (q,r), thereby ruling out the case a is odd (or any other case where the cyclic and scalar orderings of S are indistinct) altogether.

From now on we shall refer to a set of diatonic intervals and a corresponding set of scale steps satisfying conditions (3), (4), (12), and (13), as a diatonic system.

Let us now search for specific values for a, b, q, r, and N, yielding a diatonic system. On the basis of (3), (4), and (12), a is even, it follows that all the following relations are necessary:

(14) $$a=2b-2=2q+2=4r=2N.$$

The smallest positive integer a for which b, q, r, and N are all positive integers satisfying (13) and (14) is 8; the next smallest a is larger by 4, etc. We therefore have the following possible solutions for (14) under the constraint (13):

$$a=8+4n,\ b=5+2n,\ q=3+2n,\ r=2+n,\ \text{and}\ N=4+2n,$$

where n is any positive integer or 0. It is easily verifiable that a, b, q, and r as above satisfy conditions (1)−(2), and therefore for any n there exists a (unique) diatonic system DS_n; the familiar diatonic system is DS_1, with a=12, b=7, q=5, r=3, and N=6, as discussed informally in the preceding section. The set $\{DS\}=DS_0, DS_1, \ldots, DS_\infty$, contains all possible diatonic systems.

As an example of a diatonic system other than DS_1, consider DS_0, the smallest possible diatonic system (Figure 6). In this diatonic system a=8, b=5, q=3, r=2, and N=4, so that the set of diatonic intervals contains nine elements n(3,2) mod (8,5), n=−4, −3, . . . , 4; a set of scale steps contains five elements m(3,2) mod (8,5), where m runs over five consecutive integers between −4 and 4 (Figure 6A). In Figure 6B the diatonic intervals of DS_0 are translated into conventional nomenclature. Since (0,0) is necessarily tied with the fundamental notion of octave equivalence (which is actually "sixth equivalence" in this case), it is a "perfect" prime, as in the familiar diatonic system. The diatonic intervals (1,1) and (2,1) are a "minor" and "major" second, respectively, also as in the familiar diatonic system. Note, however, that (7,4) and (6,4), which are inversionally related to (1,1) and (2,1), respectively, are a "major" and a "minor" fifth, respectively. Unlike the familiar diatonic system, a "fifth" here is not a cyclic generator (it is not a "quintic class"), and therefore it is not "perfect." The pair of cyclic generators (3,2) and (5,3) are a "perfect" third and fourth, respectively; and finally, the pair of enharmonically equivalent diatonic intervals (4,2) and (4,3) are an "augmented" third and "diminished" fourth, respectively.

Before we bring this formal section of the article to a close, consider the following three propositions, which we submit without proof:

PROPOSITION 4. **Any set of scale steps may be written, mod (a,b), as a series of "whole steps" and "half steps," as follows:**

$$m(q,r),\ m(q,r)+(2,1),\ m(q,r)+2(2,1),\ \ldots,\ m(q,r)+r(2,1),$$
$$m(q,r)+(1,1)+r(2,1),\ m(q,r)+(1,1)+(r+1)(2,1),\ \ldots,$$
$$m(q,r)+(1,1)+(2r-1)(2,1) \equiv m(q,r)+(a-1,b-1),$$

where m is a fixed integer between 0 and N, inclusive.

(For example, the familiar major scale may be written, mod 12,7, as follows:

$$(5,3),\ (5,3)+(2,1),\ (5,3)+2(2,1),\ (5,3)+3(2,1),\ (5,3)+(1,1)+3(2,1),$$
$$(5,3)+(1,1)+4(2,1),\ (5,3)+(1,1)+5(2,1) \equiv (5,3)+(11,6).$$

The "tetrachordal" structure of the familiar major scale follows from this proposition.)

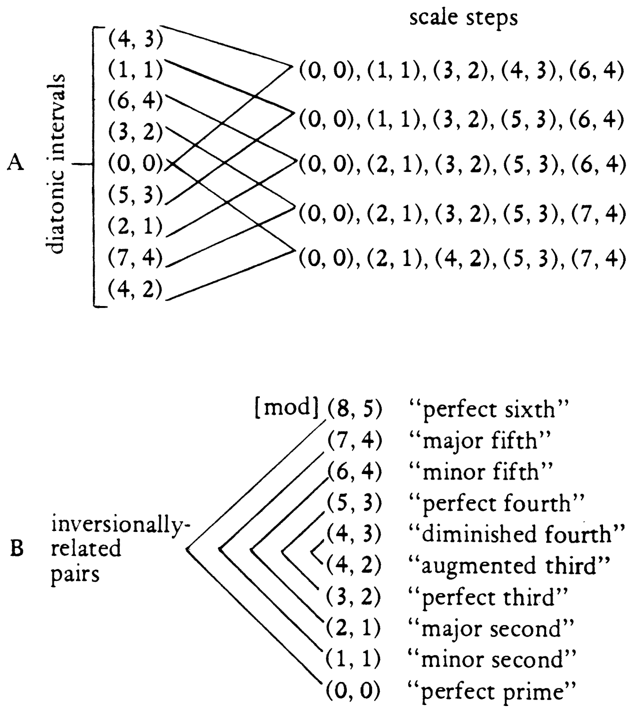

scale steps
A
diatonic intervals
(4, 3)
(1, 1)
(6, 4)
(3, 2)
(0, 0)
(5, 3)
(2, 1)
(7, 4)
(4, 2)
(0, 0), (1, 1), (3, 2), (4, 3), (6, 4)
(0, 0), (1, 1), (3, 2), (5, 3), (6, 4)
(0, 0), (2, 1), (3, 2), (5, 3), (6, 4)
(0, 0), (2, 1), (3, 2), (5, 3), (7, 4)
(0, 0), (2, 1), (4, 2), (5, 3), (7, 4)
B
inversionally-
related
pairs
[mod] (8, 5) "perfect sixth"
(7, 4) "major fifth"
(6, 4) "minor fifth"
(5, 3) "perfect fourth"
(4, 3) "diminished fourth"
(4, 2) "augmented third"
(3, 2) "perfect third"
(2, 1) "major second"
(1, 1) "minor second"
(0, 0) "perfect prime"

Figure 6

PROPOSITION 5. **Given a set of scale steps m(q,r) mod (a,b) where m runs over N+1 consecutive integers between −N and N, the set mq mod a is a "deep scale" in the sense used in Gamer (1967).**

PROPOSITION 6. **If b is a prime, all "chords" (except those of cardinality b) have the property CV (cardinality equals variety), as discussed in Clough and Myerson (1985).**

These three propositions are not quite comparable. Whereas for Proposition 4 to be true, a set of scale steps must be both efficient and coherent, it suffices for a set of scale steps to be efficient for Proposition 5 to hold. As for Proposition 6, it is true even if a set of scale steps is neither efficient nor coherent.

AN ADDITIONAL CONSTRAINT ON "DIATONIC SYSTEM"

According to the definition of "diatonic system" proposed in the preceding section, the set {DS} of all possible diatonic systems contains an infinite number of elements, DS_0, DS_1, . . . , DS_∞. However, we are familiar with only one of these elements: DS_1. A natural question therefore arises: is there something special about the familiar diatonic system which sets it apart from any of the infinite number of other possible "diatonic systems"? Is it possible to constrain the notion "diatonic system" in some additional way so that the set {DS} contain a *single* element, namely DS_1?

As a candidate for such a possible constraint, consider the following:

$$a=12.$$

To be sure, only DS_1 fulfills this additional constraint. But invoking the number twelve in such an *ad hoc* fashion must be severely censured; we are concerned here with some genuine property of the diatonic system, not with numerology.

The property to be presently proposed concerns intonation. Strictly speaking, therefore, it must be presented within the context of a theory of intonation, an undertaking which lies beyond the scope of the present article. It must be therefore noted that the following discussion merely sketches the general idea; for a fuller discussion, see Chapter 3 in Agmon (1986).

It is a well known fact that the "tempered" and "pure" "fifths" lie very close to each other. By "fifth" we mean an interval in the *perceptual* sense, that is, a *log frequency-ratio.* Thus, the tempered "fifth," taken in relation to the "octave" (i.e., log2), is 7/12, or 0.58333 . . . , while the pure "fifth" is $\log_2 3/2$, an irrational number equalling approximately 0.584963. The significance of this fact is enormous: it means that there exists a minimal amount of conflict between two intonational preferences with respect to the diatonic interval (7,4) mod (12,7), the tempered "fifth" 7/12, and the pure "fifth" $\log_2 3/2$. Note that if we multiply 7/12 and $\log_2 3/2$ by 1200 we get the value of the tempered and pure "fifths" in cents: 700 and (approximately) 701.95 cents, respectively.

Clearly, the familiar tempered "fifth" is given by $q'/a=b/a=(5+2n)/(8+4n)$, $n=1$. More generally, the "tempered fifth" of any diatonic system DS_n is given by $(5+2n)/(8+4n)$, $n=0, 1, \ldots, \infty$. It is clear that $(5+2n)/(8+4n)$ tends to 0.5 (that is, 600 cents) when n increases indefinitely. On the basis of this fact it can be shown that the "tempered fifth" of DS_1 is the "purest" fifth that is diatonically possible; see Table 1.

Remarkably enough, 7/12 is not only the best approximation of $\log_2 3/2$ that the expression $(5+2n)/(8+4n)$ makes available, but is also close to $\log_2 3/2$ *independently* of the above expression. In fact, 7/12 is closer to $\log_2 3/2$ than *any* fraction x/y of which the denominator y is smaller than 29, as is shown in Table 2 (in Table 2 each fraction i/j is closer to $\log_2 3/2$, or 701.95 cents, than any fraction x/y with a denominator y smaller than that of the following fraction i/j). Alternatively stated, 12-tone equal temperament offers the purest possible "fifth" obtained through equal division short of dividing the "octave" into no less than 29 equal parts.

From the foregoing remarks it is clear that the choice a=12 is not arbitrary. On the basis of the relationship between the tempered and pure "fifths" the familiar diatonic system DS_1 can be construed as unique within the class of possible diatonic systems.

CONCLUSION

The diatonic system, according to the model proposed here, is unique within the class of possible "diatonic systems;" this fact, of course, may speak strongly in favor of the proposed model. However, like any model, the model proposed here must pass the tests of descriptive and explanatory adequacy. We believe our model captures the intuitive sense of "diatonic system," and thus passes the test of descriptive adequacy. As to explanatory adequacy, that is, the ability of the model to help provide answers for deeper questions regarding the diatonic system, this is a matter which must be raised within a broader theoretical perspective than the one provided in this article.

In closing, then, we might remind ourselves that music theory is an empirical science, not mathematics. We must never allow the luster of mathematics to blind us to the simple truth that the ultimate test of any musical theory is empirical, not mathematical. Of course, this raises the question of exactly what *kind* of empirical science music theory is. But this, again, is a question that must be dealt with elsewhere.[5]

Table 1

system	DS_0	DS_1	DS_2	DS_3	------	DS_n $n \to \infty$
tempered "fifth"	$\frac{5}{8}$	$\frac{7}{12}$	$\frac{9}{16}$	$\frac{11}{20}$		$\frac{1}{2}$
tempered "fifth" in cents	750	700	675	660		600

Table 2

$\frac{i}{j}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{7}{12}$	$\frac{17}{29}$	$\frac{24}{41}$	------
$\frac{1200\,i}{j}$	720 cents	685.71 cents	700 cents	703.45 cents	702.44 cents	

APPENDIX

SOME COMMENTS REGARDING THE RELATIONSHIP BETWEEN CLOUGH AND MYERSON (1985) AND THE PRESENT WORK

The following correspondences exist between the Clough and Myerson (C&M) paper and the present paper:

(1) Correspondences between defined properties.

Partitioning. C&M's partitioning property, translated into present terminology and notation, states that given a set $\{(s,t)\}$ of "diatonic intervals," $0 \leq s \leq a-1$, $0 \leq t \leq b-1$, for any s in $\{s\}$ (the set of all different s's in $\{(s,t)\}$), with possibly one exception, there exists exactly one t such that (s,t) is in $\{(s,t)\}$. If one adds the constraint that the set $\{s\}$ contains *all* a different s's, this is exactly equivalent to the property of efficiency defined here, with respect to the set of diatonic intervals.

Myhill's Property. C&M's MP corresponds exactly to the property described in Proposition 3 here.

Consecutivity Property (CP). An identical property by this name is defined in C&M's paper and the present paper.

(2) Correspondences between mathematical expressions.

Theorem 5. $c=2d-1$ or $2d-2$ in C&M's Theorem 5 corresponds exactly to $b=\frac{a+1}{2}+1$ or $(a/2)+1$ in Proposition 1 here.

Lemma 4. C&M's Lemma 4 contains the relation $d'=(cc'+1)/d$. In present notation the relation reads $q'=(ar'+1)/b$, or $bq'-ar'=1$. Assuming $N=b-1$, this is in fact a sufficient (and, if $a<2b$, also necessary) condition for coherent (but not necessarily efficient!) diatonic intervals, and is thus closely related to conditions (7a) and (7b) in this paper, which apply to coherent *and* efficient diatonic intervals.

As to C&M's derivations, we submit the following analysis:

C&M's point of departure (middle of p. 262) is what one might call an efficient and coherent set of scale steps which are *not* necessarily cyclically orderable (thus, these are not "scale steps" in the sense defined here). Specifically, C&M define a set of residues mod a, 0, 1, . . . , $a-1$, and a set of residues mod b, 0, 1, . . . , $b-1$, $a>b$. They then select a partial set of exactly b different residue classes mod a, and establish a correspondence between this partial set and the full set of b residue classes mod b that is both one-to-one and monotonous, at least in the following sense: given any two scale steps (s,t) and (s',t'), $0 \leq s,s' \leq a-1$, $0 \leq t,t' \leq b-1$, if $s>s'$, then $t+\bar{t} \bmod b > t'$ for some fixed $\bar{t}$, $0 \leq \bar{t} \leq b-1$. C&M are not explicit about these assumptions, which are nonetheless clearly implicit in their discussion.

C&M are now interested in finding out under what conditions their set of scale steps has the property "cardinality equals variety" (CV). Since the generalized notion of a "cycle of fifths" is crucial for this purpose, their more immediate goal is to order their scale steps cyclically. To this end, they introduce two additional assumptions (this time explicitly): MP and a semi-reduced scale. As it turns out (Lemma 4, in conjunction with Theorem 2 later on), C&M's combined assumptions are sufficient conditions for a set of coherent diatonic intervals *which are cyclically orderable.* C&M are thus able to order their set of scale steps cyclically, from which they are finally able to deduce CV.

C&M's Theorem 5 is based on the following assumptions: (1) an efficient and coherent set of scale steps (this, again, is implicitly understood); (2) MP; (3) a *reduced* scale; (4) the partitioning property.

With respect to C&M's derivations, the following should be noted: (1) A set of cyclically generated diatonic intervals and a corresponding set of scale steps are sufficient conditions for CV (the sets need not be efficient nor coherent); (2) A set of efficient scale steps mod (a,b), a and b are coprime, partitioning, and the relation $a<2b$, suffice to imply the relation $b=\frac{a+1}{2}+1$ or $(a/2)+1$ (C&M's $c=2d-1$ or $2d-2$, Theorem 5).

It is a great pleasure to thank Shmuel Agmon for many helpful discussions in the course of preparing this paper. I would also like to thank my referees for some valuable suggestions. An important source of inspiration for the present work was the work of Gerald Balzano, especially Balzano (1982).

NOTES

1. See especially Chapter 1, and Chapter 2, section 2.1. The present article is based on Agmon (1986).
2. Brinkman notes that a similar system is mentioned in Terry Winograd, "Linguistics and the Computer Analysis of Tonal Harmony," *Journal of Music Theory* 12 (1968), 2–49.
3. The property of coherence was first described and discussed in Balzano (1982); the term "coherence" is Balzano's too.
4. "Consecutivity property" is the term used in Clough and Myerson (1985) to refer to the same property described here.
5. See Agmon (forthcoming).

LIST OF WORKS CITED

Agmon, Eytan. 1986. "Diatonicism, Chromaticism, and Enharmonicism: A Study in Cognition and Perception." Ph.D. dissertation, City University of New York.

———. Forthcoming. "Music Theory as Cognitive Science: Some Conceptual and Methodological Issues." *Music Perception.*

Balzano, Gerald J. 1982. "The Pitch Set as a Level of Description for Studying Musical Pitch Perception." In Manfred Clynes, ed., *Music, Mind, and Brain: The Neuropsychology of Music.* New York: Plenum.

Brinkman, Alexander R. 1986. "A Binomial Representation of Pitch for Computer Processing of Musical Data." *Music Theory Spectrum* 8, 44–57.

Clough, John, and Gerald Myerson. 1985. "Variety and Multiplicity in Diatonic Systems." *Journal of Music Theory* 29, 249–270.

Dickson, Leonard E. 1939. *Modern Elementary Theory of Numbers.* Chicago: University of Chicago Press.

Gamer, Carlton. 1967. "Deep Scales and Difference Sets in Equal-Tempered Systems." *American Society of University Composers,* proceedings of the second annual conference, April 1967, 113–122.