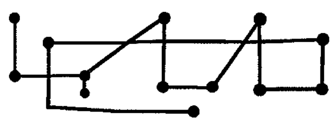
General Equal-Tempered Harmony (Introduction and Part I)



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Introduction

> Men dispute and contend and quarrel on the question, how many fundamental harmonies there are—a contention which, as it occurs to me, is about as illimitable as that on the question, how many genera of plants there are in nature, when nature herself certainly knows nothing of all the genera which have been invented by human ingenuity.
>
> —Gottfried Weber

The commonplace notion that a sonority can be "closely" or "distantly" related to other sonorities evokes a metaphorical metric space, in which individual sonorities are distributed according to what we

will call **chord quality**. Closely and distantly related sonorities are literally close to and distant from one another, respectively, in this sort of spatial model of chord quality; a sonority's quality, in turn, is characterized in terms of its location in that spatial model. From a theoretical point of view, an understanding of the nature of chord quality might take the form of a model of that space's structure and the laws that determine a sonority's position. Let us call this explanatory device **quality space**.

The literature shows two distinct modes of theorizing about chord quality that can be related to the notion of quality space. The first involves the description of functions ("similarity relations") that assign to any pair of chords a numerical index of their similarity, readily interpreted as some kind of distance metric for quality space. The second is concerned with the construction of general classification schemes that organize chords into a hierarchical taxonomy of species, genera, and possibly higher-order divisions. In both the distance- and taxonomy-oriented modes, theorists typically value generality of technique; rather than exhaustively asserting the relationships among all chords, which may seem arbitrary, the theorist will present a relatively simple procedure that generates a whole system.

I showed in "Listening to Similarity Relations" (2001) that the various numerical models generally characterized as pc-set-class similarity relations agree with each other to a high degree, despite major differences in their internal workings. In the present context, it is helpful to think of the numbers put out by similarity relations as estimates of distances between sonorities in quality space. This approach accounts both for the agreement among various similarity relations (since their common assumptions suggest that they are, in principle, estimating distances in a single space) and for their failure to converge completely with one another (since each is only an estimate).

Likewise, it is not difficult to conceive of the pc-set-theoretical tools that produce taxonomic categories superordinate to the pc-set class as modeling structural features of quality space. We might expect, for instance, that the pc sets belonging to a single one of Forte's genera (1988) would lie near one another in quality space, and that the system of genera as a whole can be viewed as a system of overlapping regions in quality space. The same could be said of Forte's pc-set complexes (1973), of any of Morris's set-group systems (1982), or of Hanson's "great categories" (1960). Again, the spatial metaphor can be of assistance in understanding the nature of the differences among all of these systems. For an example of the different ways in which it is possible to describe different regions in the same underlying space, one need only consider the significant ways in which the map of Europe changed over the course of the twentieth century—centuries-old villages in (say) Macedonia or Transylvania have been incorporated into many different states and spheres of influence without significantly changing certain "natural" and "transcendent" national properties of those villages. Such properties tend to be shaped primarily by permanent geographical features such as bodies of water, deserts, or mountain ranges.

Let me be clear about the meaning of the word "natural," which I have also used in "Listening to Similarity Relations" to label what I called the Natural Kinds Hypothesis:

> Similarity relations serve as a model classification scheme for pc-set classes that (a) corresponds to the intuitions of music scholars, (b) can be shown experimentally to correspond to certain judgments of people who are not music scholars, and (c) can be modeled with tools already at hand in pc-set-class theory, which means that similarity relations tell us nothing that we do not already know—not in the intuitive sense, but in terms of the inherent properties of the twelve-tone equal-tempered universe of pitch classes, of which we have a formidable mathematical model already. (Quinn 2001, 109)

I wish to caution the reader away from a strictly Platonic reading of this hypothesis, which does not equal the view that chords arrange themselves "naturally," if that means that they do so without any intellectual intervention whatsoever. Rather, I mean to suggest that a very small number of commitments about the nature of chords and how they relate to one another—commitments built into our common theoretical infrastructure ("the twelve-tone equal-tempered universe of pitch classes" and the other fundamental theoretical entities of pc-set-class theory)—constrain the outcome of whatever intellectual intervention comes between those axiomatic commitments and some distance- or taxonomy-oriented theory of chord quality. Boundaries of political maps are constrained by aspects of geography that are literally "natural," but quality space is structured by a set of assumptions about the nature of chord quality, and we do not need to take a position on whether they are natural or not in order to work out their common theoretical implications.

By way of introducing the commitments that I believe underlie existing theories of chord quality, I propose a simple example that also demonstrates a methodological approach to modeling quality space. One rough-and-ready way to measure the similarity of two chords is to count their common tones. Since we are ultimately interested in a spatial model of qualitative similarity, though, it is helpful to think in terms of distance, and an increasing number of common tones amounts to a *decrease* of distance. Suppose, then, that we measure the distance between two chords by combining the chords, dropping all the common tones, and counting the notes that remain. These remaining notes, the "uncommon tones," make up what is more properly called the **symmetric difference** of the two original chords. In symbols, we write

$$d_1(q, r) = |q \wedge r| = |q| + |r| - 2|q \cap r|,$$

where q and r are the chords in question, the symbols $\wedge$ and $\cap$ take the symmetric difference and intersection, respectively, of the two chords, and vertical bars surrounding a chord indicate the cardinality of that chord. If q is the C-major triad (C, E, G) and r is the E-minor-seventh chord (E, G, B, D), then the symmetric difference $q \wedge r$ is (C, B, D) and $d_1(q, r) = 3$. By way of further example, the chords whose distance from (C, E, G) is 1 are all and only those that are two-note subsets or four-note supersets of the C-major triad.

This measure of distance is sufficient to give us a start on the road to a robust spatial model of chord quality, since it provides the space of all possible chords a minimal but powerfully meaningful geometric structure known as a **metric**. A metric δ over a space has three properties. It is *reflexive*, which means that the distance between a point and itself is 0:

$$\delta(q, q) = 0.$$

It is *symmetric*, which means that the distance from q to r is the same as the distance from r to q:

$$\delta(q, r) = \delta(r, q).$$

Finally, it satisfies the *triangle inequality*, which holds (loosely speaking) that a trip from q to s via a stopover at r is never shorter than a direct trip:

$$\delta(q, s) \leq \delta(q, r) + \delta(r, s)$$

It is elementary to show that d_1 satisfies all three properties, and that it is therefore a metric.

The space that the d_1 metric describes is readily understood in familiar geometric terms. The combinatorial assumption about chords is that they are sets of pitch classes. Let us represent the **characteristic function** of a chord (see Lewin 1959; Starr 1978) in the usual universe of twelve pcs as a 12-tuple

$$q = (q_0, q_1, q_2, \ldots, q_{11}),\ q_i = \begin{cases} 0 \text{ if } i \notin q, \text{ and} \\ 1 \text{ if } i \in q \end{cases}$$

If q is a D-major triad, for example, $q = (0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0)$. We can interpret this as a Cartesian coordinate in 12-dimensional space. Then the space of all chords (interpreted in the usual sense, where doublings are disregarded) is simply the set of vertices of a unit hypercube in 12-dimensional space, and the symmetric-difference metric d_1 calculates taxicab distances in that space.

Two kinds of generalization readily suggest themselves. One concerns the space itself: consider the possibility that a chord can contain more than one instance of a particular pitch class, in which case the coordinates are not limited to 0 and 1. Indeed, we might be interested in the possibility of allowing any positive real number as a coordinate in this space, yielding the possibility of a "fuzzy pc set." The other kind of generalization concerns the metric. We have described a metric that calculates taxicab distances in this chord space; we might consider the possibility of Euclidean distances as well, or any of the other so-called *vector* p*-norms:*

$$d_p(q, r) = \sqrt[p]{\textstyle\sum_i |r_i - q_i|^p}$$

(Here, of course, since the context is arithmetic and not set-theoretic, the vertical bars indicate absolute value and not set cardinality.) So long as $p \geq 1$, d_p will be a metric. When $p = 1$ we have the symmetric-difference metric introduced above, and when $p = 2$ we have the usual Euclidean notion of distance.

The space of the symmetric-difference metric d_1 and its derivatives, which we will refer to generally as **harmonic chord space**, encapsulates some of the views that seem to be axiomatic in the literature on pc-set-class theory. Its underlying assumption, of course, is that chords are understood in a manner we can characterize as *combinatorial*—as sets of pitch classes drawn from an underlying universe. At this stage we do not even need to make any claims about equal temperament or the relative sizes or significances of intervals. Rather, the combinatorial attitude toward the pitch classes constituting the universe is that they are, in Schoenberg's phrase, related only to one another, and the combinatorial attitude toward harmony is that any chord, together with its complement, may be understood simply as a partition of the universe into pitch-class sets.

To get a sense of the implications of the combinatorial conception of chord relations, consider some operation J that maps the pitch class universe onto itself. J can be an interval-preserving operation like a transposition or inversion, or it can be a non-interval-preserving affine transformation like a multiplication operation, or it can be a completely random reshuffling of the pitch classes of the universe. No matter what J is, it affects all distances equally:

$$d_p(p, r) = d_p(\mathrm{J}q, \mathrm{J}r).$$

Because it is fundamentally combinatorial in design, the symmetric-difference metric does not encompass any notion of "voice-leading distance," and (C, E, G) is just as distant from, or similar to, (E, G, B) as it is to (C, D, E) under this measure, simply because both of the latter chords have two common tones with a C-major triad. The fact that one admits of much smoother voice leading to a C-major triad than does the other does not affect the distance. In this sense our nascent model is purely *harmonic*, and entirely distinct from spatial models of voice leading (see, e.g., Cohn 2003; Callender 2004; Callender et al. 2005).

Related to the characterization of the symmetric-difference metric as harmonic in nature are two properties that follow from the role common tones play in its definition. First, given chords q, r, and s, where r and s have the same number of notes, it is the case that q is closer to r than to s under the metric if and only if q has more subsets in common with r than it does with s. Second, the distance between two chords q and r is the same as the distance between their complements, which we will notate Cq and Cr. Both of these properties, which clearly resonate with Forte's set-complex theory and its derivatives, illustrate the inclusional character of the metric's behavior.

Well and good; we have identified a metric over harmonic chord space that shares with pc-set theory certain properties that can be described as combinatorial, harmonic, and inclusional. I want to suggest that these are three of the four basic commitments underlying existing distance- and taxonomy-oriented theories of chord quality; the survey of those theories undertaken in Part 1 will readily establish this. These three commitments add up to the idea that qualitative similarity is intimately tied up with the question of common tones, a connection Regener (1974) thoroughly worked out in his essay on Forte's *Structure of Atonal Music* (1973).

The fourth commitment we can identify, however, is one that sheds light on the apparent weakness of the symmetric-difference metric as a model of quality space. That commitment is to an *abstract* notion of chord quality, in which quality is something that inheres in equivalence classes of chords as much as it does in chords. Indeed, all of the pc-set-theoretic tools I identify as participating in the qualitative program are focused strongly on the problem of relating not chords, but species of chords. (In place of the usual term *pc-set class*, I will use the term **species** to refer to equivalence classes of chords, for reasons to be explained later in this introduction.) The problem that this poses for the symmetric-difference model is that since chord species do not have tones—only chords *per se* do—it is not coherent to speak of common tones among species.

I tend to view the history of basic (i.e., nonanalytical) research in pc-set theory as being driven by this problem. There is general methodological agreement on the combinatorial construal of chords as pitch class sets, and on the construction of chord species by means of permutations of the underlying pitch class universe. The only debate over the latter point concerns which permutations are to be viewed as "canonical." Substantial differences, however, emerge when it comes to the question of asserting qualitative relations among chord species. This work has proceeded along two lines which we have characterized as oriented toward distance and taxonomy, respectively. While in both of these lines we find theorists identifying quality with various structural features of chords (interval content, subset structure, and transformational symmetries), several influential synthetic papers by Regener (1974) and Lewin (1977; 1979) show that these approaches all reduce, in some sense, to questions about common tones and how they can be applied to species as abstractions of chords.

The present work begins where "Listening to Similarity Relations" left off. Having showed there that methodologically diverse distance-oriented theories of chord quality yield similar results, which I interpreted above as "estimates of distances between sonorities in quality space," I intend ultimately to present a model of quality space that proceeds from the four assumptions about chord quality that I have articulated here: that it is combinatorial, harmonic, inclusional, and abstract. Indeed, it is derived directly from harmonic chord space, which has been a foil for our discussion so far. The question of how to achieve the abstraction from chords to species is answered by means of a technique David Lewin articulated in his first published article (1959) and left untouched until the last article to appear in print during his lifetime (2001)—specifically, the application of the real-valued discrete Fourier transform to harmonic chord space.

Part 1 is concerned with teasing out the common threads in theories of chord quality, which are woven together in two directions, warp and weft. The warp runs through the two modes of theorizing about chord quality, oriented toward distance and taxonomy. By drawing on the overarching metaphor of quality space, we will show that higher-order taxonomic categories or **genera** are organized around privileged, highly symmetric chord species (**prototypes**), which are quite distant from one another in quality space. The taxonomic categories are structured by virtue of chords being qualitatively close ("similar") to the prototypes in quality space, and the structure of the taxonomy as a whole is productively viewed in terms of **affinities** or distance relationships in quality space: **intrageneric affinities** describe the internal structure of genera, and **intergeneric affinities** describe the structural relationships of genera to one another.

The weft, on the other hand, runs through the various features of chord species that theorists have identified with quality: interval content, subset structure, and transformational symmetries. As we will see, theories proceeding from these different starting points end up asserting the same prototypes. Part 2 (which will appear with Part 3 in the next issue of this journal) works out a theory of generic prototypes that proceeds from Clough and Douthett's theory of maximally even sets (1991). Several new results are presented in connection with that theory, including some that shed light on an error I committed in "Listening to Similarity Relations." That error, which is called the **Intervallic Half-Truth** in this work, was the blithe association of the six evident qualitative genera in twelve-tone equal temperament with the six interval classes, an association that is not possible in most other equal-tempered pitch class universes. Part 2 provides a corrective to that error by reinterpreting the genera as associated with the six types of maximally even sets up to complementation. Throughout this work we will give qualitative genera names of the form $\mathbf{F}(c, d)$, where c is the number of pcs in the universe, and d is the cardinality of the ME set prototypical of the genus. (Although this nomenclature will be slightly uncomfortable in Part 1, its utility will be amply demonstrated in Part 2.) These genera are what I have characterized as "natural kinds" of chords; yet since it is methodologically misleading to think of them as "natural" in any meaningful, theory-independent way—rather, they arise from the four basic assumptions about chord quality that I have identified above—I will abandon this term for the most part and refer to them as **qualitative genera**.

In Part 3 we clear the slate and build a model of quality space on the basis of Lewin's 1959 technique, which is in turn based on the discrete Fourier transform. After showing how the space embodies all of the common threads among theories of chord quality teased out in Part 1, and a brief demonstration of the analytical potential of the model, we will finally deliver the punch line by showing that the space is easily derived from harmonic chord space, the space of the symmetric-difference metric, by folding it up in a particular way.

One of the products of this work, of course, is a distance-oriented similarity measure that agrees to a great extent with the many that are already on the table. In "Listening to Similarity Relations" I strongly suggested that we do not need any more of those. But my argument here is not that one ought to use my similarity relation instead of any of the others in any kind of practical analytical setting, if only because there are so few examples of such practice in the analytical literature. Rather, I am adopting a dialectical strategy here. By starting from the questionable proposition that that there is some kind of "reality" of Natural Kinds lurking behind the agreement among theories of harmonic chord quality, and that each of those theories diverges somehow from that reality, we are able to develop a single framework from which certain seemingly diverse theories can be derived individually. Once that is accomplished, the fictional realism about quality space, which serves as a kind of temporary support during the construction of the framework, is no longer necessary. For the goal is not, in fact, to recover any ultimate truths about chord quality, or to close off approaches to chord quality that proceed from other assumptions (such as those involving voice leading or acoustic consonance), but only to pull together the theoretical implications of a single set of assumptions, and to show how what I earlier called the "warp and weft" of harmonic pc-set theory are woven together so surprisingly tightly as to virtually guarantee that a mirage of Natural Kinds will appear. If the spatial structure they weave is also a fruitful place to stage analytical investigations, so much the better.

AUTHOR'S NOTE

The argument of this work has been substantially reframed since its initial submission, which was virtually identical to my dissertation. Dmitri Tymoczko kindly presented me with a detailed and incisive critique in the summer of 2005, and the editors shared with me the colloquy between Benjamin Boretz and Robert Morris prior to its publication in volume 43/2–44/1 of *Perspectives* (pp. 216–9). All three of these scholars have taken issue with the Platonist spirit embodied in what, in "Listening to Similarity Relations," I called the Natural Kinds Hypothesis. I have been persuaded by their arguments, and have reframed my own arguments accordingly. For this reason, the Boretz/Morris colloquy should not be read against the work published here, at least not without the mediation of my dissertation.

I am deeply in debt to Tymoczko, Morris, and Boretz for their commentary. Among the many other people who have left fingerprints on this work through correspondence and conversation, special thanks are due to Emmanuel Amiot, Michael Buchler, Clifton Callender, Norman Carey, David Clampitt, Richard Cohn, Jack Douthett, Daniel Harrison, Carol Krumhansl, Panayotis Mavromatis, Arthur Samplaki, Ciro Scotto, and Joseph Straus.

Finally, I would like to acknowledge David Lewin's inspiration and encouragement. In November 2002, days after having the idea that his very first article would help me to complete the agenda of "Listening to Similarity Relations," I excitedly wrote him about it; his response was simply "Yes, I enjoy thinking about that too." Hoping he would live long enough to tell me whether I was thinking about it in the right way once I'd fleshed out that idea, I waited too long to ask. So it's with sadness and more the usual measure of humility that I add the traditional disclaimer: any remaining errors are entirely my responsibility.

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PART 1: PROTOTYPES, AFFINITIES, AND THEORIES OF CHORD QUALITY

There are certain highly characteristic chord species so ubiquitous as to have familiar names in relatively widespread use: chromatic clusters; quartal or quintal chords; Perle's interval cycles (whole-tone scales, augmented triads, and diminished-seventh chords) and combinations of these (Messiaen's modes of limited transposition). Each of these plays a special role in various kinds of pc-set theory; each is associated with a unique type of intervallic profile, and each has a relatively limited repertoire of abstract subsets and supersets. Taxonomic theories of atonal harmony typically place such pc-set classes in different genera (which, in turn, are often characterized with reference to those pc-set classes), and similarity relations generally agree that these landmarks are all distant from one another. Even treatments of "twentieth-century harmony" that do not participate in the pc-set-theoretic tradition (e.g., Hanson 1960; Persichetti 1961, and the last few chapters of many tonal-harmony books) end up focusing on these types of chords and sonorities.

At the level of analytical discourse, we are accustomed to hearing about Skryabin's "mystic chord" as a close relative of the whole-tone scale and diatonic collections (Callender 1998), of harmonies in Stravinsky's *Sacre* as being nearly octatonic (van den Toorn 1987, especially 207–11). Neo-Riemannian theory has opened our eyes to the close relationship between (on the one hand) major and minor triads and (on the other) the augmented triad and hexatonic scale (Cohn 2000). Boretz (1972) described relations among diatonic seventh chords in the *Tristan* Prelude with respect to the structural properties of the diminished-seventh chord.

By comparing arbitrary chords to a limited number of "highly characteristic" types, we engage implicitly in the same sort of categorization that we do at the most basic levels of cognition. Cognitive scientists today generally agree that we mentally structure categories in terms of **prototypes**, central members of a category whose other members resemble the prototype(s) to a certain degree. (Instructed to think of a chair, you probably would not instantly come up with a beanbag or a porch swing—but those would be more likely than a white tiger or a candy bar.) In a standard work on the subject, Lakoff (1987, particularly 16–57) gives a brief survey of modern thinking about cognitive categorization, some of the highlights of which will be reviewed here.

Lakoff's intent is to problematize what he calls the classical theory of categories, which holds that a category has sharp boundaries determined by some combination of necessary and sufficient conditions. This is the sort of category that classical sets model, of course. Lakoff begins his discussion with Wittgenstein (1953), attributing to him several revolutionary ideas about categories. For Wittgenstein, a category (his well-known example is the category of games) has unclear and extensible boundaries that are not drawn by necessary and sufficient conditions, but by family relationships—similarities of many different kinds and degrees. At the same time, Wittgenstein allows that one can distinguish between good and bad examples of a category; to return to a previous example, a beanbag is not a good example of the category of chairs, even though it bears family relationships to other members of the category, and in particular to good examples. Lakoff observes that the challenge posed to philosophy by Wittgenstein's conception is that the classical theory of categories *qua* sets has no room for good and bad examples, and identifies Zadeh's (1965) theory of fuzzy sets as a first formal attempt to deal with that challenge (see also Quinn 1997; 2001).

A great body of empirical work by social scientists in the 1960s and 1970s established Wittgenstein's model as a useful point of departure for modeling certain cross-cultural features of human thought. One of the most influential studies was by Berlin and Kay (1969), who presented convincing evidence that, in Lakoff's words,

- Basic color terms name basic color *categories*, whose central members are the same universally. For example, there is always a psychologically real category RED, with focal red as the best, or "purest," example. . . .
- Languages form a hierarchy based on the number of basic color terms they have and the color categories those terms refer to. . . .

  black, white
  red
  yellow, blue, green
  brown
  purple, pink, orange, gray (25)

That is, languages having fewer color terms than those listed invariably have a term low on the list only if they have all of the terms higher on the list; no language has a word for *brown* without also having a term for *red*. Moreover, there was evidence to suggest that certain "focal" colors were better examples, cross-culturally, of these universal color categories than others. Subsequent work by neuroscientists on the perception of color in macaques led to the development by Kay and McDaniel (1978) of a hierarchical model of color categorization based on Zadeh's fuzzy sets. The model, which was based on the sensitivity of retinal cells to specific wavelengths, successfully accounted for large parts of the linguistic hierarchy discovered by Kay and Berlin, especially as far as the focal colors were concerned.

These studies (among studies of other kinds of categories that Lakoff describes) provided an empirical basis for Wittgenstein's characterization of the general structure of categories as conceptual entities. What had not yet been answered was the question of what sorts of categories we tend to form in response to our observation of things in the world. Lakoff details a number of anthropological studies, undertaken by the aforementioned Berlin and his associates, of the ways in which members of different cultures categorize plants and animals and compares them with the taxonomy laid out by Linnaeus, concluding that "the genus was established as that level of biological discontinuity at which human beings could most easily perceive, agree on, learn, remember, and name the discontinuities. . . . Berlin found that there is a close fit at this level between the categories of Linnaean biology and basic-level categories in folk biology" (35). Findings such as this—which suggest that for any taxonomic hierarchy, there is a psychologically *basic level* of categories akin to biological genera—were synthesized by Eleanor Rosch into what has now become the standard view of categories: that we organize categories (which have prototypes, or focal elements) into taxonomic hierarchies with a basic level. Paraphrasing from an important article by Rosch and several of her collaborators (1976), Lakoff (46) characterizes the basic level as, among others,

- The highest level at which category members have similarly perceived overall shapes. . . .
- The highest level at which a single mental image can reflect the entire category. . . .
- The level with the most commonly used labels for category members. . . .
- The first level to enter the lexicon of a language. . . .
- The level at which terms are used in neutral contexts. . . .
- The level at which most of our knowledge is organized.

Lakoff concludes by observing (following Rosch) that we must be wary of giving prototypes too important a role in any theory of mental representation for categories: "Prototype effects, that is, asymmetries among category members such as goodness-of-example judgments, are superficial phenomena which may have many sources" (56). This warning cuts two ways. One the one hand, one must not reason, from the apparent naturalness (or, more neutrally, near-universality) of basic-level categories and their prototypes, to the conclusion that things in the world have inherent properties that sort them into those categories and determine whether or not they are prototypes; see the discussion of the "Myth of Intension" in Quinn (2001). Categories are products of the mind; it is a commonplace among biologists that there are not necessarily "natural kinds" corresponding to taxonomic divisions. On the other hand, one should not take the prototype/basic-level theory of categories to mean that peripheral members of some category are conceptualized with reference to the prototypes of the category—only that prototypes tend to stand at the confluence of the different family relations that constitute the category in the first place; see the discussion of the "Myth of Staggering Complexity" in Quinn (2001).

This is a point of departure for Lakoff, and it is where we leave his particular approach to categories aside, in order to return to the geography of chord quality. Lakoff is primarily concerned with high-level inquiry into the workings of language and the "embodied mind," and the notion that categorization is the very substrate of conceptualization. Zbikowski (2002) provides a rich discussion of the musical issues that fall out of this notion, showing that musical understanding emerges from the interaction of conceptual entities that have a feedback relationship to categorical or taxonomic knowledge—at once grounded in categories shared (as style knowledge) among members of a musical community, and constitutive of these same categories. Work of this kind cannot proceed without a deep theoretical understanding of the nature of the categories themselves, and it is clear that we lack such an understanding for the harmonic categories generally treated under the problematic headings of "post-tonal" or "twentieth-century" harmony. Our motivation is to provide such an understanding, and with it a foundation for higher-level conceptualizations of harmony.

Lakoff's particular manner of framing the issue of categorization (and Zbikowski's success in generating from it a theory of musical understanding) grounds the basic assumptions of our inquiry. There is an evident consensus among theorists that there are basic-level categories of harmony that are hierarchically superior to those of the chord and the species, which we will continue to call **genera**, intending to refer not to the specific theories of Forte (1988) and Parks (1989), but to any theory of harmonic relationships that transcends the species level. Moreover, overwhelming implicit and explicit evidence from the theoretical literature suggests that certain characteristic chord species, including those mentioned in the opening paragraphs of Part 1, are prototypes of genera that have many of the features Rosch attributes to basic-level categories.

We may remain neutral on the question of whether these categories and prototypes are a feature of listener psychology, although some work has been done in this direction (Mavromatis and Williamson 1997; 1999; Kuusi 2000; 2001; Samplaski 2000). Instead, we are interested in the psychology of the discipline of pc-set theory as a whole, which seems to include certain assumptions about the nature of chord quality articulated in the introduction to this study.

§ 1.1. THE INTERVALLIC APPROACH TO CHORD QUALITY

The *locus classicus* of chord quality is often taken to be the interval class vector; Straus, for example, observes that "the quality of a sonority can be roughly summarized by listing all the intervals it contains" (2000, 10). Howard Hanson seems to have been the first to use this principle as the basis for a complete and rigorous pc-set classification system:

> In a broader sense, the combinations of tones in our system of equal temperament—whether such sounds consist of two tones or many—tend to group themselves into sounds which have a preponderance of one of these [interval classes]. In other words, most sonorities fall

into one of the six great categories: perfect-fifth *types*, major-third *types*, minor-third *types*, and so forth. (1960, 27)

A large folding chart provided with Hanson's book enumerates the harmonic species and classifies them into his seven genera (the six he describes, plus one catchall category for pc-set classes without a predominant interval class).

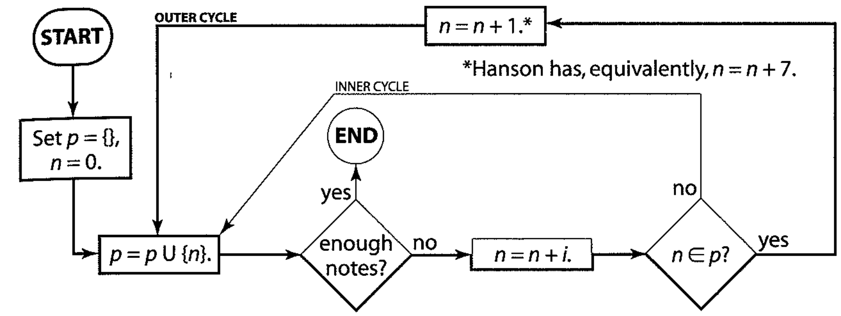
1.1.1. Prototypes

Hanson clearly delineates a set of prototypes for his categories—these are what he calls the projections of the six interval classes. For ic 1 and ic 5, the **projections** are easily defined as those species whose exemplars are contiguous segments of the chromatic scale and circle of fifths, respectively. In projecting the other interval classes, Hanson runs into the problem of the interval cycle: "We have observed," he writes,

> that there are only two intervals which can be projected consistently through the twelve tones, the perfect fifth and the minor second. The major second may be projected through a six-tone series and then must resort to the interjection of a "foreign" tone to continue the projection, while the minor third can be projected in pure form through only four tones.
>
> We come now to the major third, which can be projected only to three tones (1960, 123).

Hanson's rather ingenious solution to the problem concerns the addition of a "foreign tone" into the projection, which introduces a pitch class outside of the just-completed interval cycle and provides a starting point for the continuation of the projection. Invariably, his "foreign tone" is a fifth above the starting point, although a shift of a semitone would work just as well.

It is relatively easy to describe his procedure for generating chordal prototypes as an algorithm, although he does not do so explicitly (for unclear reasons, he uses a different procedure with the tritone, although he ends up with identical results). Example 1 describes the procedure for generating a chord p that is the species prime form of a d-note projection of interval class i. The internal variable n stands for notes added to the chord. The layout of the flowchart clearly shows that Hanson's procedure has the structure of a nested interval cycle—an "inner cycle" of the interval class i being projected, and an "outer cycle" (for Hanson, a 7-cycle, and for us, a 1-cycle) that generates foreign tones in order to make available fresh transpositions of the i-cycle. Example 2 displays the species



EXAMPLE 1: ALGORITHMIC DESCRIPTION OF HANSON'S PROJECTION PROCEDURE

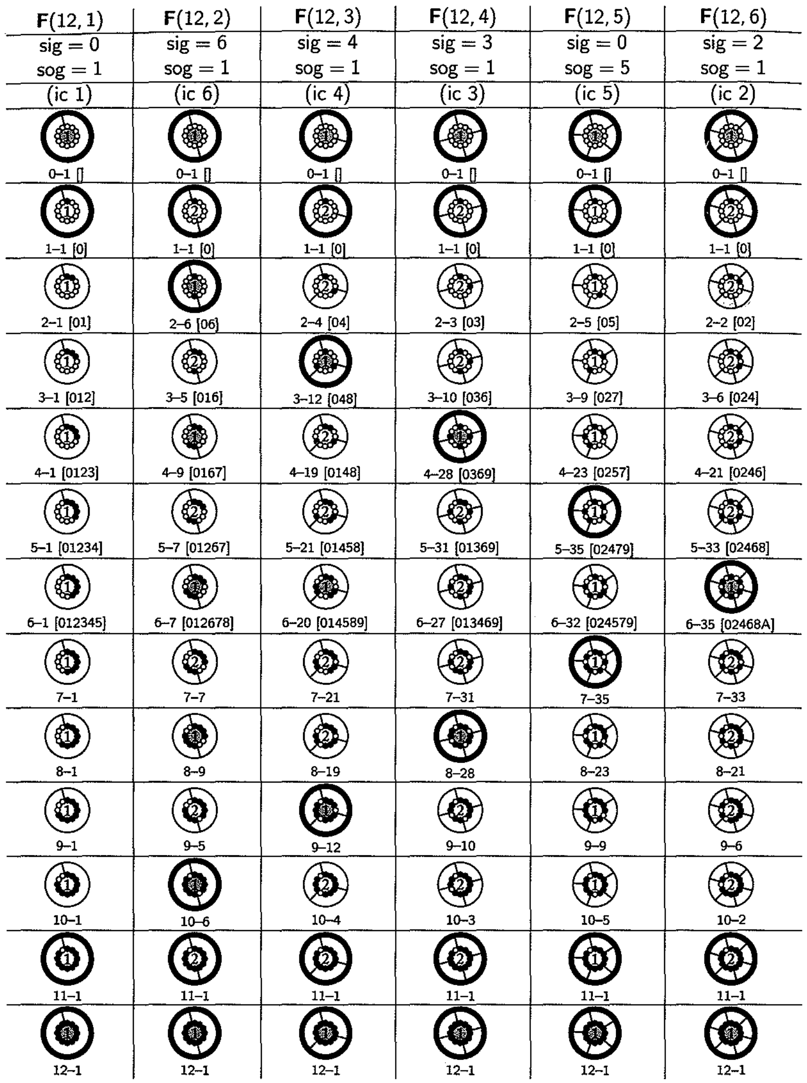
prime form of all chords generated by Hanson's procedure, using both the traditional Forte nomenclature and clockface diagrams. (The clockface diagrams include some additional graphical apparatus that will be explained shortly.) "Trivial" species corresponding to the null pc set, the aggregate, and singletons and their complements are included as well.

As Hanson himself observes, his choice of the perfect fifth as the interval that generates the "foreign tone" is essentially arbitrary. He allows that other solutions will work in individual cases (such as minor thirds in the ic-4 projection), but that only foreign-tone cycles of fifths or semitones will work in all cases. In this connection it is interesting to contemplate Example 3, reproduced from Headlam (1996), which duplicates Hanson's approach to projection, but with foreign tones introduced along a cycle of semitones rather than fifths, as we have done in Example 1. In fact, our algorithm, when allowed to run all the way through the aggregate, generates precisely the same notes as appear in each line of Headlam's figure.

Viewed as a complete system, Hanson's projections have five properties that make them particularly attractive as a set of prototypes for harmonic genera:

The Unique-Prototype Property (UPP): Each of the six genera has one, and only one, prototypical species of any given cardinality. This feature makes Hanson's system quite tidy, but it is not necessarily a requirement of a conceptually robust taxonomy.

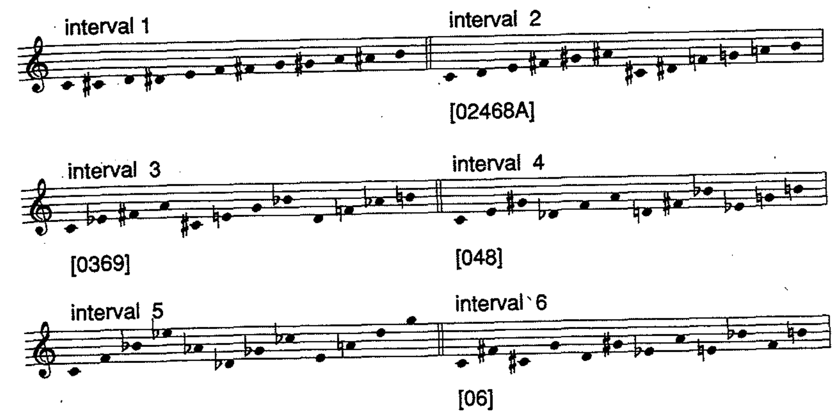
The Unique-Genus Property (UGP): Each of Hanson's projections is a prototype of one, and only one, genus. In contrast to the UPP, this would seem to be necessary for conceptual robustness;



EXAMPLE 2: HANSON'S PROJECTIONS; TENTATIVE PROTOTYPES OF THE QUALITATIVE GENERA F(12, *N*)

after all, a taxonomy that cannot unambiguously classify its prototypes necessarily cannot be trusted to classify anything else.

The Intrageneric Inclusion Property (IIP): Each generic prototype (abstractly) includes all smaller prototypes of the same genus. In



EXAMPLE 3: FIGURE 1.1 FROM HEADLAM (1996)

the broader context of manifold theories of harmony, many of which privilege inclusion relations (even tonal theory does this in several important ways), this is a desirable feature.

The Prototype Complementation Property (PCP): The complement of any generic prototype is another prototype of the same genus. Many aspects of pc-set theory (especially those connected with Forte's work) and twelve-tone theory are concerned with complementation. Even tonal theory depends on complementation to some extent, when it comes to distinguishing harmonic functions in terms of quasi-complementary relationships *within* diatonic collections.

The Prototype Familiarity Property (PFP): Many of these prototypes are chord species of the sort discussed in the introduction to this paper, species that have familiar names and important roles in a wide range of analytical applications. This is an important part of what makes harmonic genera function as basic-level categories. In Example 2, some of the clockface diagrams have thicker circles surrounding them, and some have shaded interiors. The former are maximally even, and the latter are transpositionally invariant; all are familiar.

Hanson characterizes the constitutive feature of his genera as the "preponderance" of some interval class. This does not tell the whole story, though, since—to cite one example—while the whole-tone collection (6–35) has exactly as many instances of ic 4 as the hexatonic collection (6–20), only the latter is a prototype of the ic-4 genus, thanks to some hand-waving on Hanson's part as concerns the foreign-tone technique: in a footnote, he discounts foreign-tone generators for the ic-4 cycle that would result in whole-tone formations on the grounds that whole-tone formations properly belong in the family of ic-2 projections. No detailed and general treatment of the issue is to be found in Hanson's book.

card.	ic 2	ic 3	ic 4	ic 6
2	2–2 [02]^{p}	2–3 [03]^{p}	2–4 [04]^{p}	2–6 [06]^{p}
3	3–6 [024]^{p}	3–10 [036]^{pd}	3–12 [048]^{p}	3–5 [016]^{p}
				3–8 [026]
				3–10 [036]^{d}
4	4–21 [0246]^{p}	4–28 [0369]^{pd}	4–19 [0148]^{p}	4–9 [0167]^{p}
			4–24 [0248]	4–25 [0268]
				4–28 [0369]^{d}
5	5–33 [02468]^{pd}	5–31 [01369]^{pd}	5–21 [01458]^{p}	5–7 [01267]^{p}
			5–33 [02468]^{d}	5–15 [01268]
				5–19 [01367]
				5–28 [02368]
				5–31 [01369]^{d}
				5–33 [02468]^{d}
6	6–35 [02468A]^{pd}	6–27 [013469]^{p}	6–20 [014589]^{p}	6–7 [012678]^{p}
			6–35 [02468A]^{d}	6–30 [013679]
				6–35 [02468A]^{d}

(^{p}) indicates a maxpoint that is also a Hanson projection; (^{d}) indicates a "duplicate" maxpoint of several ics.

EXAMPLE 4: ERIKSSON'S MAXPOINTS

Notwithstanding Hanson's lack of interest in treating this issue with more rigor, his projections are, in many ways, ideal prototypes of intervallically constituted genera, since each projection is what Eriksson (1986) calls a **maxpoint**, a chord species "containing the maximum number for its size of at least one interval class" (96). The maxpoints of ics 1 and 5 correspond one-to-one with Hanson's projections; maxpoints of the other ics are tabulated in Example 4. Eriksson's treatment of maxpoints occurs in the context of his own development of a taxonomic theory for harmonic species, and the maxpoints are his initial candidates for generic prototypes (he refines the system later in his paper). Immediately apparent from the proliferation of chord species in the ic-4 and ic-6 columns of Example 4 is the fact that as a set of prototypes, maxpoints do not have UPP. Nor, for that matter, do they have UGP—species marked with a superscript *d* are maxpoints of more than one interval. All of the offenders are either projections of ic 3 that also happen to have the maximum number of tritones, or projections of ic 2 that also happen to have the maximum number of both major thirds and tritones.

One way to improve the situation would be to change the definition of a maxpoint. Isolating certain special maxpoints (call them **supermaxpoints**) that not only maximize some ic, but also have strictly more occurrences of that ic than any other chord species of the same cardinality, reinstates a weak form of UPP. Rather than having multiple prototypes of the same cardinality in certain genera, we now have no prototypes for certain cardinalities in the genera associated with ics 4 and 6; otherwise the supermaxpoints are coextensive with Hanson's projections (see Example 5). At the same time, this maneuver solves the problem of UGP, since each of the multiply affiliated maxpoints marked with a *d* in Example 4 is a supermaxpoint of exactly one interval class. And, in fact, the supermaxpoints are all also projections.

card.	ic 1	ic 2	ic 3	ic 4	ic 5	ic 6
2	2–1 [01]^{p}	2–2 [02]^{p}	2–3 [03]^{p}	2–4 [04]^{p}	2–5 [05]^{p}	2–6 [06]^{p}
3	3–1 [012]^{p}	3–6 [024]^{p}	3–10 [036]^{p}	3–12 [048]^{p}	3–9 [027]^{p}	
4	4–1 [0123]^{p}	4–21 [0246]^{p}	4–28 [0369]^{p}		4–23 [0257]^{p}	
5	5–1 [01234]^{p}	5–33 [02468]^{p}	5–31 [01369]^{p}		5–35 [02479]^{p}	
6	6–1 [012345]^{p}	6–35 [02468A]^{p}	6–27 [013469]^{p}		6–32 [024579]^{p}	

EXAMPLE 5: THE "SUPERMAXPOINTS"

Noting problems akin to those that could be circumvented by isolating the supermaxpoints, and after an extensive discussion of abstract inclusion relations among maxpoints (clearly related to IIP), Eriksson eventually abandons the idea of using chord species as prototypes at all. Rather, his "model" for each of his seven genera (he calls them *regions*) is a partial ordering of interval classes, ordered according to their frequency of occurrence in a given species of chord. Example 6, adapted from his Example 6, lists the models for the seven genera he eventually asserts. The highly suggestive (and innovative) move from intervallic maxpoints to regional models is not without problems—for example, the Petrushka chord 6–30 [013679] is a maxpoint of ic 6, but its interval vector, $\langle 2, 2, 4, 0, 2, 3\rangle$, resembles the genus III model of high ic-3 and -6 content much more than it does the genus VI model, which specifically provides for low ic-3 content.

	I	II	III	IV	V	VI	VII*
typical ics	1	2, 4, 6	3, 6	4	5	6	2
	2				2		
↓	3, 4			1, 3, 5	3, 4	1, 5	1, 5, 3, 4
atypical ics	5, 6	1, 3, 5	1, 2, 4, 5	2, 6	1, 6	2, 3, 4	6

*Only M-invariant chord species may belong to this genus.

EXAMPLE 6: MODELS OF RELATIVE IC MULTIPLICITY IN ERIKSSON'S SEVEN "REGIONS" (GENERA)

1.1.2. Intrageneric Affinities

In related work on a similarity-oriented theory, Michael Buchler (2001) suggests that in order to compare two chord species on the basis of their ic content, it is beneficial to contextualize the interval class vector, as he puts it, by means of "tools that take account of what is minimally and maximally possible in a given cardinality" (264), judging ic content relative to such possibilities. Buchler calls the relativized measure of ic content the *degree of saturation.* A maxpoint for some ic is, in Buchler's terms, fully saturated with that ic. Setting aside for the moment the considerations that lead Eriksson to supplement the theory of maxpoints with ic-vector "models" in asserting his generic prototypes, we observe that Buchler's generalization of maxpoints into degrees of saturation provides a fuzzification of the prototype idea. A prototype is a very good example of a category; other members of a category may be ranked in terms of their own goodness-of-example as well. We will use the term **intrageneric affinity** to refer to this goodness-of-example relationship.

To the extent that interval content determines the intrageneric affinities of intervallically constituted genera, one way to figure the affinity (degree of membership) of a chord in such a genus might be to determine the degree to which the pc-set class is saturated with the relevant ic (Buchler's footnote 11 may be taken to suggest something along these lines). Yet this sort of classification system would have as generic prototypes *all* maxpoint pc sets, and thus inherit the UPP-and UGP-related problems discussed above—the same problems that Hanson and Eriksson seek to avoid, each in his own way. Recognizing this problem, Buchler defines what he calls the "maximal cyclic fragmentation condition" (270), which is met by all and only the maxpoints singled out as prototypes under Hanson's system of projections (indeed, Buchler's condition is precisely that used implicitly by Hanson). Buchler's work is the background for a similarity relation, though, and not a classification

scheme, and he does not offer any suggestions as to how one might go about fuzzifying the "maximal cyclic fragmentation condition" in a way that would model intrageneric affinities for a system of genera constituted by this specialized notion of interval class saturation. For that matter, neither Hanson nor Eriksson makes any finer distinction between members of a genus than between prototypes and nonprototypes.

If the intrageneric affinities of a genus are the degrees to which various chord species exemplify the genus, and if the best examples of a genus are its prototypes, then a natural way to measure them with extant tools of pc-set theory—and to provide the "finer distinctions" we do not get from Buchler, Hanson, or Eriksson—is to consider the similarity of a chord to generic prototypes using fuzzy similarity relations that are based on interval content, such as Morris's SIM and ASIM relations (1979), Isaacson's IcVSIM (1990) and his various ISIM$_n$ relations (1996), and Scott and Isaacson's ANGLE (1998). Scott and Isaacson provide a technical overview of the mathematical connections among these relations, but all essentially measure the degree to which two ic vectors have the same profile—in stark contrast to the "original" similarity relations from Forte (1973), which neither come in degrees nor treat the ic vector as the sort of thing that can have a shape, instead focusing on the yes-or-no question of whether corresponding entries in two ic vectors are equal.

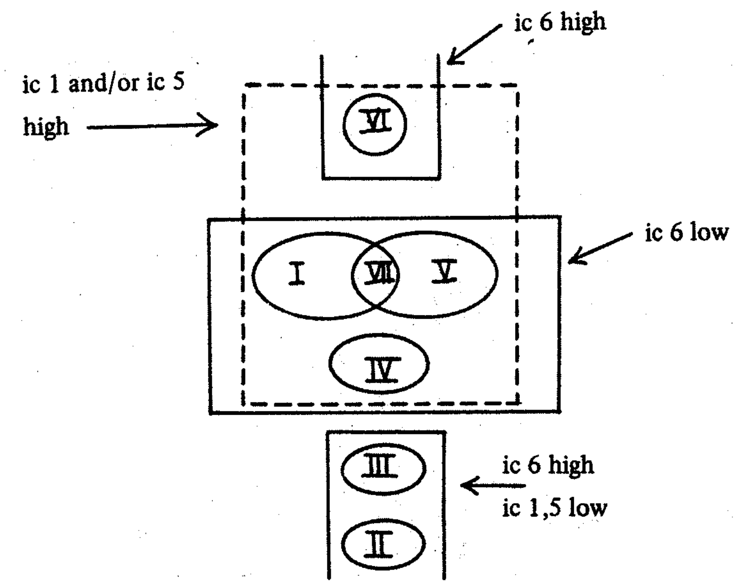
Fuzzy similarity relations do not directly suggest a system of prototypes, although they do suggest genera (see Quinn 2001). The discussions we have undertaken so far, in connection with the intensional characterization of the aforementioned similarity relations as being oriented toward interval content, suggest that Hanson's system of prototypes, or something like it, could form the basis of a generic taxonomy, with similarity relations predicting (or modeling, if you prefer) the intrageneric affinities of the genera. A more purely extensional approach might derive from the technique of multidimensional scaling, which treats dissimilarity measurements among objects as distances in a multidimensional space, then finds a distribution of the objects in the space that provides a good fit with the data. Highly dissimilar objects will be far apart, and highly similar objects will be close together. The application of multidimensional scaling to data from chordal similarity relations tends to produce distributions in which certain chords are at the "edges" of the distribution—thereby being as far away from one another as possible—and these chords tend to be the prototypes we have been discussing. Cognitively oriented work on multidimensional scaling of chord similarity data has been conducted by Mavromatis and Williamson (1999), Samplaski (2000), and Kuusi (2001).

Another approach might proceed from Eriksson's prototypical models of ic-vector shape. Block and Douthett (1994), who do not cite Eriksson's article, present a related, but more general approach (see, e.g., p. 22: "a composer may wish to find a family of sets that have certain intervals suppressed or eliminated and at the same time have others emphasized"). They develop a general structure (a "weight vector") that corresponds to such situations, and a procedure derived from vector algebra for determining how well a particular ic vector exemplifies this weighting. Many of the examples in their article consist of a table of chord species arranged according to goodness-of-example, and although they do not take the step of isolating prototypical ic-vector shapes—this was Eriksson's major achievement—it is the case that the ten such examples they adduce are headed up by chord species corresponding to Hanson's projections.

While any particular instance of these two approaches will likely produce similar taxonomies (see Quinn 2001), such an ad-hoc model would have pragmatic rather than explanatory value, and not much pragmatic value at that: it would be useful for answering questions that only straw men are asking ("how well is genus g exemplified by species s?") without shedding any light on interesting, abstract questions about why so many superficially different theories converge at the basic level of chord quality. If we seek more interesting questions, rather than answers to less interesting ones, we should be highly suspicious of any such ad-hoc approach to answering questions about intrageneric affinities.

Without committing to any particular approach yet, we will use the term **quality space** to describe an idealized spatial distribution of chord species, especially one that reflects the intrageneric affinities of the qualitative genera we have been describing. The prototypes of each qualitative genus lie in a particular "population center" of quality space, and the spatial position of an arbitrary chord reflects, by its distance from these various remote regions, its affinity to each of the qualitative genera. Our ultimate goal is a theory of chord quality that describes quality space more specifically, and that makes the prediction that multidimensional scaling of data generated by any of the particular procedures described above will produce spatial distributions that converge on our model of quality space.

Eriksson's graphic depiction of his "regions," reproduced here as Example 7, is particularly suggestive in this regard—we may interpret his genera somewhat literally as regions of quality space. A peculiarity of Eriksson's layout, and of his system of genera, is the relationship of his genera I, V, and VII. In the graphic layout, he suggests that VII is constituted by the overlap of peripheral parts of I and V; and in his description of the intervallic model of genus VII he includes the caveat that such chord species must contain equal numbers of ic 1 and ic 5, which is tantamount to asserting that they must be invariant under M, the circle-of-fifths transform. These points, which we will revisit a bit later (section 1.2.2), throw into question the status of his genus VII as an independent "population center," since he seems to characterize it more as disputed territory between the regions of genera I and V. Under this reading, we are left with (more or less) the usual six intervallically constituted genera.



EXAMPLE 7: ERIKSSON'S GRAPHIC REPRESENTATION OF HIS GENERA

1.1.3. Intergeneric Affinities

We have been imagining a fuzzy taxonomy for chord species in which there are six genera roughly corresponding to the six interval classes. Each genus is endowed with a structure described by its intrageneric affinities, which depend on the notion of generic prototypes (best examples of genera) and the idea that the similarity of an arbitrary chord to the prototypes of a genus is equivalent to the goodness-of-example of the chord to the genus. We now consider the question of how genera are related to one another, or how one might go about drawing analogies between genera. Our case study will involve the genera associated with ics 1 and 5.

Two basic observations will get us going. First: any prototype of one of these two genera, when subjected to the M_5 (circle-of-fourths) or M_7 (circle-of-fifths) transforms, yields a prototype of the other genus; this is the case irrespective of whether one chooses Hanson's projections, Eriksson's maxpoints or ic-vector models, our supermaxpoints, or Buchler's saturated chords to serve as prototypes. Second: suppose we have two chords, and their degree of similarity is s (as measured by any of the similarity relations mentioned above); transform both chords by M (by which we mean "either M_5 or M_7"), and the degree of similarity between the transformed chords is also s. Choosing an example at random, let p be an exemplar of the species 4–27 [0258], and let q be an exemplar of the species 6–13 [013467]. Buchler's SATSIM reports their degree of similarity as 0.261, and Scott and Isaacson's ANGLE gives 0.117. Transforming p by M yields an exemplar of 4–12 [0236]; q becomes an exemplar of 6–50 [014679] under either transformation. Asked about the similarity of these latter two chords, SATSIM gives 0.261 again, and ANGLE gives 0.117 again.

We will not attempt to prove here why intervallically constituted similarity relations behave in this way (see Morris 2001, ch. 4), but only to show that the two observations just made allow us to make an interesting analogy between the two genera in question. Since the intrageneric affinities of the genera are determined by similarity relations involving their respective prototypes, and since the prototypes of the two genera are swapped under certain transformations, and since the similarity relations determining intrageneric affinities seem to be invariant under those same transformations, we can draw the following general conclusion about the relationship among these two genera: The degree to which a species (e.g., 6–3 [012356]) exemplifies one genus is the same as the degree to which its M_5-transform (e.g., 6–25 [013568]) exemplifies the other genus. This conclusion pertains to what we will call the **intergeneric affinities** between the two genera in question.

In a sense, this use of the word *affinities* is closely allied to the eponymous doctrine of medieval theory, which concerns the relationship of different, but functionally identical, notes in different tetrachords. Each tetrachord has notes called *protus*, *deuterus*, *tritus*, and *tetrardus;* these qualitative labels (intragenerically) describe the positions of those particular notes in, say, the *graves* tetrachord. Affinities among the tetrachords are described (intergenerically) by identity among the qualitative names and related, coincidentally, by transpositions through fourths or fifths—the *protus* of the *graves* tetrachord has an affinity to the *protus* of the *finales* tetrachord a fourth higher, and to the *protus* of the *superiores* tetrachord a fifth above that.

The existence of intergeneric affinities between the two genera we have studied raises the more general issue of affinities between any two genera. The M_5 and M_7 operators relate the genera in question because each "expands" members of ic 1 into members of ic 5, and further expands members of ic 5 into members of ic 1, all the while leaving all other intervals invariant (up to interval class). To generalize this to other genera, we might briefly consider a multiplicative operator that expands members of ic 1 into other intervals, but here we face Hanson's original problem: the interval cycle. Suppose, for example, we invent an operator that transforms every pc c into $2c$ (mod 12). Example 8 contrasts this "M_2" operator with the usual M_5 operator; the problem is that while M_5 and M_7 (like the usual transposition and inversion operators) are one-to-one mappings of the pitch class universe onto itself, this "M_2" is a many-to-one mapping. Any two pcs separated by a tritone map to the same pc under "M_2," and no pcs map to an odd pc under this operator.

M_5c	0	5	A	3	8	1	6	B	4	9	2	7
c	0	1	2	3	4	5	6	7	8	9	A	B
"M_2c"	0	2	4	6	8	A	0	2	4	6	8	A

EXAMPLE 8: MULTIPLICATION BY 2 AS AN EPIMORPHISM

In an unpublished manuscript (2000), Daniel Harrison notes this problem—which is endemic to the generalization of multiplicative operators in the twelve-pc universe—and presents a suggestive work-around. His N operator, which is a replacement for "M_2," multiplies pitch class integers by slightly more than 2 ($2 + 1/6$, to be exact), then rounds down to the nearest integer. Example 9 details the mapping. Just at the point where "M_2" starts to repeat itself (pc 6), duplicating the even whole-tone collection, the extra $1/6$ in Harrison's multiplier is amplified past the point of being rounded away, and N maps the remaining six pcs to the odd whole-tone collection.

Harrison's solution is suggestive for two reasons. First, when his N is compounded on itself "N^2," it acts as a substitute for "M_4" (also an epimorphism); furthermore, its inverse (N^{-1}) acts as a substitute for "M_6," and its inverse compounded on itself (N^{-2}) acts as a substitute for "M_3." In this way, N serves as a generator of a series of one-to-one mappings that simulate all the other types of multiplication we would need to

⋮

$N^{-3}c$	0	7	3	A	6	2	9	5	1	8	4	B	
$N^{-2}c$	0	3	6	9	1	4	7	A	2	5	8	B	= "M_3"
$N^{-1}c$	0	6	1	7	2	8	3	9	4	A	5	B	= "M_6"
c	0	1	2	3	4	5	6	7	8	9	A	B	
Nc	0	2	4	6	8	A	1	3	5	7	9	B	= "M_2"
N^2c	0	4	8	1	5	9	2	6	A	3	7	B	= "M_4"
N^3c	0	8	5	2	A	7	4	1	9	6	3	B	

⋮

EXAMPLE 9: HARRISON'S N OPERATOR

investigate intergeneric affinities in the six-genus taxonomy we have been exploring.

Second, these uses of N also happen to simulate Hanson's method for generating projections of ics 2, 3, 4, and 6 (compare Example 9 with Example 3 and the associated discussion). As a consequence, it happens that transforming certain exemplars of Hanson's ic-1 projections by N or any of its several compounds just mentioned will yield exemplars of other Hanson projections. In particular, the prime-form exemplar of any ic-1 projection (regardless of cardinality) will transform into an exemplar of some other projection. Clearly this suggests the beginning of a quite general way to assert intergeneric affinities among our qualitative genera. Unfortunately, the interrelationship of prototypes is only guaranteed when we are dealing with prime forms; for example, N transforms {01234}, an ic-1 projection, into {02468}, an ic-2 projection; but it also transforms {45678}, another ic-1 projection, into {1358A}—which is not only not an ic-2 projection, but is also an ic-5 projection! The mathematical issue at hand is outside the scope of our present inquiry, but the upshot is that any chord can be transformed into *any other* chord of the same cardinality by some combination of N with the usual transposition and inversion operators. While N gets us tantalizingly close to a theory of intergeneric affinities, its explosive interaction with the operators that define chord species tightly circumscribe its usefulness. We will revisit the issue of intergeneric affinities in the next section.

§ 1.2. OTHER APPROACHES TO CHORD QUALITY

In addition to the intervallic approaches we have studied, theorists have explored other ways of qualitatively relating chords. As we will see, much of the ground we have covered so far is intimately connected to these other approaches, despite superficial differences.

1.2.1. The Inclusional Approach

The most widespread alternative approach to chord quality involves the ways in which chords include one another and, by extension, in which chord species abstractly include one another. The large folding chart supplied by Hanson (1960) details abstract inclusion relations among species in a manner that clearly and vividly shows that chords tend to belong to the same genera as their subsets and supersets. Nowhere in his book, however, does Hanson explicitly make this observation, although his general approach (which involves extending the idea of projection from intervals to trichords) is so thoroughly shot through with the idea of inclusion that, to misuse Hanson's own words (272), one "may well ask whether any such detailed analysis went on in the mind of the composer as he was writing the passage. The answer is probably, 'consciously—no, subconsciously—yes'."

Forte's treatment of set complexes of the K and Kh types (1973) represent the first explicit and thoroughgoing study of abstract inclusion relations that has a qualitative character. It is useless as a generic taxonomic system, however, because if a pc-set complex is a genus whose nexus is its prototype, it follows that any chord species whatsoever can be a prototype of some genus. Forte's nested complexes, however, show an interesting parallel with the fuzzy generic structures we have been imagining. Each K-complex, as a category, has a single prototype up to complementation; yet we may interpret the Kh-subcomplex about that same prototype as a class of privileged members of the K-complex—better examples of the genus. In Forte's words, the idea of the Kh-subcomplex is to supply "additional refinement of the set-complex concept in order to provide significant distinctions among compositional sets."

Much later, Forte (1988) used similar principles to develop a quasi-taxonomic system of what he calls *genera* and *supragenera*. There is still considerable overlap among categories at the generic and suprageneric levels, but taxonomic hierarchy is more clearly manifest in this theory, since the supragenera wholly include their respective genera. (At the same time, the number of chord species assigned to just one genus, or even just one supragenus, is disappointingly small, and Forte offers no further means of grading goodness-of-example within either level.)

Introducing the system, Forte announces that "we posit the intervallic content of pitch class sets as the fundamental basis of the genera" (188), although this basis is operative only as far as selecting generic prototypes (which are all trichords) is concerned, and from there the constitutive principle is once again inclusion. Complicating the situation somewhat is the fact that, strictly speaking, not all of his prototypes are trichords—some are *pairs* of trichords that have two out of three ics in common. (For more on the relationship between Forte's genera and his set-complexes, see Morris 1997).

The conceptual bridge between thinking about interval content and thinking about abstract inclusion relations was first solidly erected by Lewin (1977), who, observing that an interval class is simply a species of two-note chord, suggested that the qualitative utility of thinking about interval content might be extended to the consideration of subset content generally. Not long thereafter, responding both to Morris's development of the first graded similarity relation, the ic-based SIM, and to Rahn's subset-based TMEMB, Lewin positioned his own REL as a conceptual generalization of both (Morris 1979–80; Rahn 1979–80; Lewin 1979–80).

Lewin's observations, and other lines of thought that originate with them, may help to explain why the interval-based approach to chord quality converges with the subset-based approaches. On the one hand, similarity relations such as TMEMB and REL that (more or less) simply count common subset species, as well as relations like Castrén's fantastically complicated RECREL (1994), which compares the subset *structures* of chord species rather than just the subsets, produce results that fall in line with those produced by ic-based similarity relations when glimpsed from an extensional, taxonomic point of view (Quinn 2001). On the other hand, while the prototypes of Forte's system of genera are technically trichords or pairs of trichords, he frequently describes the genera in "very informal descriptive terms so that the genera might seem more accessible and familiar to the reader" (Forte 1988, 200). These terms include *whole-tone*, *diminished*, *augmented*, *chroma[tic]*, and *dia[tonic]*, which vividly recall the sorts of prototypes that arise under strictly intervallic approaches. In the related generic system of Parks (1989), similarly "accessible and familiar" terms arise. They are "informal" only because none of the builders of formal chord-quality models seems to have been willing to take them seriously enough to seek the appropriate formalizations instead of offering appeals to intuition, which seem rather vacuous in the otherwise highly rigorous context of pc-set theory.

There are three good reasons to take these accessible labels seriously. The first has to do with prototypes—the prototypes of a qualitative taxonomy constituted by inclusion relations would seem to be precisely the same as those that fall out of an intervallic approach. In particular, we have observed that Hanson's prototypes have IIP and PCP, which amounts to saying that the prototypes of any qualitative genus are Kh-related to each other. We have also observed that they have PFP, which establishes a strong conceptual connection to Forte's and Parks's "accessible and familiar" names for their own genera.

The second concerns intrageneric affinities, the modeling of which we have been delegating to fuzzy similarity relations. It having been established that inclusional similarity relations end up producing largely the same results as intervallic similarity relations, there is good reason to suppose that they would agree as to the intrageneric affinities of genera constituted by our working set of familiar intervallic-cum-inclusional prototypes.

The third involves intergeneric affinities. We have yet to get terribly far with this concept, but our observations concerning the M_5 and M_7 operations, which establish affinities between $\mathbf{F}(12, 1)$ and $\mathbf{F}(12, 5)$—now that we are no longer discussing interval content *per se*, we will henceforth refer to qualitative genera by these symbols, which were silently introduced in Example 2—carry over to the inclusional realm as well. Suppose we have a chord p that includes a chord q; it follows immediately that M_5p includes M_5q. From there it is a short journey to the conclusion that whatever an inclusional similarity relation says about species s and t will be the same as what it says about M_5s and M_5t, something that is borne out by all of the inclusional similarity relations mentioned.

Most existing approaches to qualitative chord taxonomy belong to either the intervallic or inclusional approaches. We have seen how these two types may be linked both technically and conceptually, and how they tend to create similar sorts of genera. This, perhaps, raises the question of which approach is "right." On the face of it, this is rather a silly question, which seems to invite intensional discourse of the sort I have argued fervently against elsewhere. Hanson, for instance, *says* his projections are intervallically conceived, but that does not disallow us from asserting, as we have, that they also have a strong inclusional aspect, and that they seem to reflect maximal evenness and transpositional invariance in some way as well. At the same time, I am going to argue later on that neither the intervallic nor the inclusional approach is "right," and so I do not wish to throw out the question as intensional and therefore trivial. Rather, the question should be viewed as theoretical and therefore explanatory in nature—the theory of chord quality we are developing seems to transcend the distinction between intervallic and inclusional approaches, largely because the intervallic and inclusional aspects of the generic prototypes and affinities converge; neither is necessary or sufficient to explain the other. Furthermore, there are a few loose ends concerning maximal evenness and transpositional invariance (which seem to be contributing factors to the all-important PFP), and considerably more loose ends concerning intergeneric affinities, that we are as yet at a loss to explain.

Our survey will continue and conclude with a study of two sharply distinctive approaches to the construction of genera that will take us away from the intervallic and inclusional approaches, and advance us toward an understanding of said loose ends. These two approaches are truly *sui generis*, so to speak, in terms of both the constitution of genera and the nature of the genera that are thereby constituted. Yet both are concerned with questions about chords evidencing symmetry under certain transformations of the underlying pitch class universe.

1.2.2. MORRIS'S ALGEBRAIC APPROACH

In the tradition of mainline pc-set theory, Morris (1982; 2001) adheres strictly to what Lakoff characterizes as the "classical view" of categories—that the extension of a category is a set, and that the intension of a category is a collection of necessary and sufficient conditions. Morris begins with a survey of the common approaches to gathering chords into species, referring to each such approach as a *set-group system*. He cites four, describing the intensional conditions for species in each system; to paraphrase:

- Under $SG(v)$, two chords belong to the same species if and only if they have the same interval content.
- Under $SG(1)$, two chords belong to the same species if and only if they can be transformed into one another by pc transposition.
- Under $SG(2)$, two chords belong to the same species if and only if they can be transformed into one another by pc transposition and/or pc inversion.
- Under $SG(3)$, two chords belong to the same species if and only if they can be transformed into one another by pc transposition and/or pc inversion and/or the circle-of-fifths transform M.

Morris uses the term *set-group* in a highly general sense; because he does not assert anything like a basic level of classification, there is no way to make a distinction of kind between species and genera. Yet the taxonomic instinct looms large in his theory of set-group systems. Much of this theory is concerned with the ways in which one set-group system can be hierarchically related to another, in the hard-and-fast manner of Linnaean *species*, *genera*, *familiae*, *ordines*, and so forth up the taxonomic ranks.

Generalizing Lakoff's terminology, we might assert a classical view of taxonomy that can be characterized inductively: each taxonomic rank is a classical system of categories, and any two things classified in the same category at a given rank are necessarily classified in the same category at all higher (coarser) ranks. Morris's discussion of the four set-group systems just listed strongly implies a classical view of taxonomy; much of his initial discussion amounts to the observation that $SG(1)$, $SG(2)$, and $SG(3)$ may be successive ranks of a classical taxonomy. His subsequent development of an array of additional set-group systems seems to be motivated by frustration that $SG(v)$ cannot be worked into a classical taxonomy with the other three. In particular, $SG(v)$ is taxonomically superior to both $SG(1)$ and $SG(2)$, but is taxonomically neither superior nor inferior to $SG(3)$. (Counterexamples are 5–12 [01356] and 5–36 [01247], which are not M-related but do have the same interval content; and ics 1 and 5 themselves, which are M-related but clearly do not have the same interval content.) Morris initially suggests a bit of fudging: if we were to decree ic 1 and ic 5 to be identical, redefining "interval content" accordingly, the problem would go away, with $SG(v)$ taking its place in the hierarchy just above $SG(3)$.

Assuming that by "species" we mean the taxa of $SG(2)$, which are the just the familiar pc-set classes of Forte (1973), we may choose either $SG(3)$ or any of its superior set-group systems to function as what we have been uniformly calling genera. For reasons that will be immediately evident, let us distinguish between the **algebraic genera** of Morris's set-group taxonomy and the **qualitative genera** that have been our main topic thus far. Even with $SG(3)$ itself, we see something completely different happening in Morris's taxonomy than with all of the other approaches we have discussed to this point. Recall that all of those systems (implicitly or explicitly) include separate qualitative genera for ic 1-type and ic 5-type chord species, the prototypes of which are connected segments of the chromatic scale and the circle of fifths, respectively; then note that by assigning M-related chord species to the same algebraic genus, the $SG(3)$ system asserts algebraic-generic equivalence between pairs of qualitative prototypes. Any $\mathbf{F}(12, 1)$ prototype is the M-transform of the $\mathbf{F}(12, 5)$ prototype of the same cardinality, and vice versa.

Consider now the chord species displayed in Example 10. They are arranged into two columns that correspond to the fuzzy qualitative genera $\mathbf{F}(12, 1)$ and $\mathbf{F}(12, 5)$, respectively. The top row contains prototypes of each genus, and successively lower rows contain species more distant from the prototypes (most similarity relations will agree on this). At the same time, the two entries in each row are M-related to one another, and consequently belong to the same "genus" of $SG(3)$. What

	ic-1 genus	ic-5 genus
prototype	5–1 [01234]	5–35 [02479]
↓	5–2 [01235]	5–23 [02357]
↓	5–3 [01245]	5–27 [01358]
↓	5–9 [01246]	5–24 [01357]
distant	5–33 [02468]	5–33 [02468]

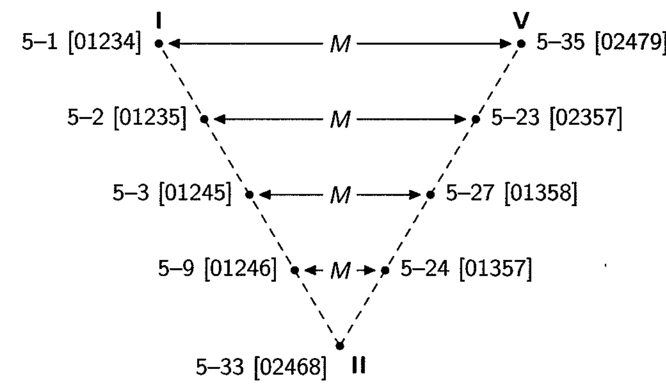
EXAMPLE 10: MORIS'S ALGRBRAIC *SG*(3) GENERA (ROWS) VERSUS QUALITATIVE GENERA (COLUMNS)

this means is that *SG*(3) brings together species that play the *same* role in two *different* qualitative genera that are wholly M-related to one another.

What we are dealing with, of course, is the the same phenomenon we discussed earlier in connection with intergeneric affinities between the qualitative genera. Two chords belong to the same *algebraic* genus of *SG*(3) if and only if there is an intergeneric affinity between them with respect to the two *qualitative* genera in question. In this sense, Morris's algebraic genera are exactly orthogonal to qualitative genera—and in a way that engages an important feature of the system of qualitative genera.

Referring back to Eriksson's map of his genera (Example 7), and pretending that this is in fact a map of quality space, let us locate the chord species of Example 10 on the map. The locations of the prototypes 5–1 [01234] and 5–35 [02479] are where Eriksson has marked "I" and "V," respectively. The species 5–33 [02468], in the bottom row of Example 10, is a prototype of the qualitative genus **F**(12, 6), which, because of that genus's association with ic 2, can therefore be located where Eriksson has marked "II." Now draw two straight lines from 5–33: one to each of the two prototypes 5–1 and 5–35. Distribute the other chord species from the two columns of Example 10 evenly along those two lines, and in the same order. The resulting distances are very roughly those that most similarity relations will estimate for all of the chord species involved.

The arrangement just described, and sketched in Example 11, is consistent with the following assertion: The M operation corresponds to a reflection (flip) about a vertical line running through the center of Example 7. This assertion, in turn, is consistent with two important observations. First, as we have already noted, Eriksson's genus VII consists, by fiat, entirely of M-invariant chord species. In his graphic representation, that genus lies between I and V and directly on the axis of



EXAMPLE 11: M_5 STRUCTURES CERTAIN INTERGENIC AFFINITIES; IT ALSO PRESERVES CERTAIN INTRAGENIC AFFINITIES

reflection corresponding to M. Second, as we have not previously noted, nearly all of the agreed-upon prototypes of the genera associated with ics 2, 3, 4, and 6 are M-invariant. Again, in Eriksson's graphic representation, the centers of these genera lie directly on the axis of symmetry in question. (Oddly, Eriksson does not discuss the circle-of-fifths transform at all, despite the fact that many of his figures display this vivid symmetry.)

We have been emphasizing for some time that M_5 structures intergeneric affinities between **F**(12, 1) and **F**(12, 5), but have just noted that the prototype of the other four genera are all invariant under M_5. Recall now from our initial discussion of intergeneric affinities that the similarity of two chords p and q is exactly the same as the similarity of M_5p and M_5q. Suppose that p is invariant under M_5; then it follows that q and M_5q are each identically similar to p—which leads us immediately to the conclusion that that M_5 preserves the *intra*generic affinities of these other four genera. Thus Example 11 may be taken as a demonstration not only of M_5's role in structuring the intergeneric affinities between **F**(12, 1) and **F**(12, 5), but also of the invariance of the intrageneric affinities of **F**(12, 6) under M_5.

The algebraic genera of *SG*(3)—or, more particularly, the taxonomic relationships between *SG*(2) and *SG*(3)—engage all of the issues raised by qualitative genera, but from an orthogonal point of view. That being the case, we should be particularly interested in Morris's major question: What would *SG*(4) be? Each of the numbered set-group systems relates to its taxonomically inferior rank by adding to the canon of equivalence operators. Therefore such an *SG*(4) could be created by finding some other permutation of pcs (in addition to transposition, inversion, and M) and elevating it to the status of an equivalence operator. In particular, Morris is interested in operators that

> preserve certain interval classes while changing others in the pc sets they transform. In this way, the vectors of the mapped sets will be able to retain some of their features. As a result, the set-groups within the set-groups systems we will develop will be related by a kind of similarity measure. Fortunately, all of these strictures are inter-related; by actualizing one property we will gain the others (1982, 113–4).

Such **nonstandard operators** will only be useful if they do not simply collapse all chords of a single cardinality into one category—Harrison's N operator does this, since, as discussed above, it can be used in combination with transposition and inversion to turn any chord into any other chord of the same cardinality.

Each of Morris's nonstandard operators begins with a partition of the aggregate into identical dyads, then maps each pc in each dyad into the other. For his α, for instance, he partitions the aggregate into six instances of ic 1. There is only one way to do this (up to transposition):

$$\{0,1\}\{2,3\}\{4,5\}\{6,7\}\{8,9\}\{A,B\}.$$

Note that in this partition, the ic-1 dyads are arranged along an "outer" ic-2 cycle. The α operator simply swaps the pcs within each of these dyads; another way to describe this is that each pc in the even whole-tone collection gets transposed up a semitone, and each pc in the odd whole-tone collection down a semitone. Immediately we can see that every prototype species of **F**(12, 6) is necessarily invariant under α; it happens that the prototypes of **F**(12, 3) are also invariant. One would do well to wonder whether these invariances generalize from the particular prototypes of these genera to their intrageneric affinities as a whole, as we saw to be the case with M_5's preservation of the intrageneric affinities of four qualitative genera.

The α operator does not leave any other generic prototypes invariant, but it does something suggestive to the prototypes of **F**(12, 2) and **F**(12, 4)—associated, respectively, with ics 6 and 3. Consider the tetrachordal prototypes, which are 4–9 [0167] and 4–28 [0369]. Thanks to transpositional and inversional invariances, there are only six exemplars of the former, and just three of the latter. All three of the 4–28 chords become 4–9 chords when transformed by α. Since α (like all of Morris's non-

standard operators) is its own inverse, those 4–9 chords—{1278}, {349A}, and {56B0}—transform into exemplars of 4–28 under α as well. The other three 4–9 chords—{0167}, {2389}, and {45AB}—are invariant under α. As a result, this pair of species constitutes a single algebraic genus under *SG*(α). Indeed, the prototypes of these two genera invariably relate to one another via α, with one exception: the hexachordal **F**(12, 2) prototype, 6–7 {012678}, is transformed by α not into Hanson's hexachordal ic-3 projection, but into the closely related Petrushka chord, 6–30 [013679]. (But remember that our elevation of Hanson's projections to prototype status is not infallible, and that 6–30 is a maxpoint of ic 6, among other things.) The obvious question concerning affinities is whether α describes intergeneric affinities between **F**(12, 2) and **F**(12, 4).

op.	partition	genera with prototypes... invariant	swapped
α	{0,1} {2,3} {4,5} {6,7} {8,9} {A,B}	**F**(12, 3) **F**(12, 6)	**F**(12, 2), **F**(12, 4)
β	{0,2}, {1,3}, {4,6}, {5,7}, {8,A}, {9,B}	**F**(12, 3)	
γ	{0,3}, {1,4}, {2,5}, {6,9}, {7,A}, {8,B} {0,3}, {2,5}, {4,7}, {6,9}, {8,B}, {A,1}	**F**(12, 2) **F**(12, 4)	
δ_1	{0} {1} {2} {3} {4,8} {5,9} {6,A} {7,B}	**F**(12, 3)	**F**(12, 1), **F**(12, 4), **F**(12, 5)
δ_2	{0} {1} {2} {7} {B,3} {4,8} {5,9} {6,A}	**F**(12, 3)	
δ_5	{0} {1} {3} {6} {A,2} {4,8} {5,9} {7,B}	**F**(12, 3)	**F**(12, 1), **F**(12, 4), **F**(12, 5)

EXAMPLE 12: MORRIS'S NONSTANDARD OPERATORS

Without delving deeply into Morris's theory, we observe that the other operators are constructed likewise (beginning with other homogeneous partitions of the aggregate), and produce similarly interesting results. The table in Example 12 lists the six nonstandard operators Morris uses to construct set-group systems. As α is based on a partition into ic-1 dyads, β and γ stem from ic-2 and ic-3 partitions, respectively. The aggregate cannot be partitioned into six discrete instances of ic 4, so Morris constructs a family of δ operators based on best-case partitions, of which only three generate unique set-group systems.

It should be observed that the intervals (and corresponding prototypes) preserved under each operator are not necessarily the same as those involved in the partition underlying each operator. We saw that the ic-1 partition underlying α is best understood as being structured by an ic-2 cycle; thus it is the ic-2 and ic-4 prototypes that are preserved. Similarly, the ic-2 partition underlying β is structured by a hexatonic scale, a prototype of the genus **F**(12, 3), all of whose prototypes are β-invariant.

The two possible ic-3 partitions (generating γ) are structured by 6–7 [012678], a prototype of **F**(12, 2), and the whole-tone collection, with appropriate invariances ensuing. Each one of the six nonstandard operators leaves some array of generic prototypes invariant, raising the possibility of intrageneric-affinity preservation. Three (α, δ_1, and δ_5) consistently conflate prototypes of different genera into the same set-groups, raising the possibility of intergeneric affinities. (Since each operator is a "root" of M_5, each conflates prototypes of **F**(12, 1) and **F**(12, 5), which is not mentioned in the table.) Insofar as we have had some general success in describing affinities in terms of prototypes and similarity relations, the questions just raised about the relationship between Morris's set-group systems and the affinities of the qualitative genera are empirical ones. The event is that the nonstandard operators do not, on the whole, work as neatly as we might have hoped in the general case, despite their tidy treatment of the prototypes.

1.2.3. Cohn's cyclic approach

As an adjunct to his theory of transpositional invariance, Cohn (1991) develops five set-group systems of his own. Each is based on a special kind of nonstandard multiplicative operator he calls a *CYCLE homomorphism*, q.v. also Starr (1978). The "M_2" operator we discussed above in connection with Harrison's N operator is one of these (CYCLE-6). Cohn is interested in taking advantage of precisely the same characteristic of these operators that Harrison viewed as a problem—specifically, that they do not necessarily preserve cardinality. A simple example, which we have already considered, concerns the transformation of a tritone under CYCLE-6 ("M_2"), which becomes a single pc. From a certain point of view, there is something missing from this statement—one might like to think that the CYCLE-6 transform of a tritone is actually two "copies" of the same pc—but in the usual framework of pc-set theory, which treats chords as classical sets of pitch classes, there is no technical facility for making counting "copies" of a pc in a chord. Lewin (1977) and Morris (1998) have both suggested that such a facility, sometimes known as a *multiset*, might come in handy in certain theoretical contexts, but the important and fascinating issues involved constitute a can of worms best left closed until Part 3.

Example 13 lists the five CYCLE homomorphisms, along with the multiplicative operators they closely resemble. In Cohn's conception, the homomorphisms "act not only upon individual pitch classes, but on all mod-12 pc sets as well, mapping them into set classes within smaller universes" (1991, 15). These smaller universes are represented by the integers modulo n, where n is the number of the CYCLE homomorphism.

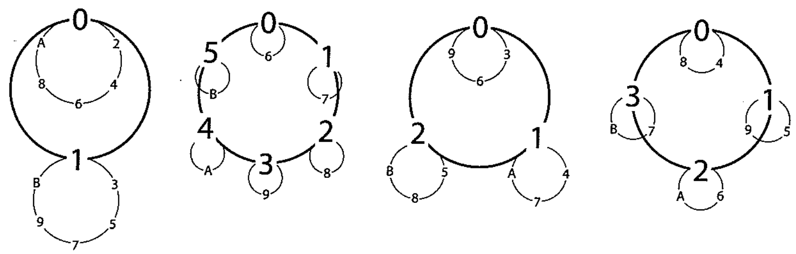
	0	1	2	3	4	5	6	7	8	9	A	B	
"M_2"	0	2	4	6	8	A	0	2	4	6	8	A	
CYCLE-6	0	1	2	3	4	5	0	1	2	3	4	5	(mod 6)
"M_3"	0	3	6	9	0	3	6	9	0	3	6	9	
CYCLE-4	0	1	2	3	0	1	2	3	0	1	2	3	(mod 4)
"M_4"	0	4	8	0	4	8	0	4	8	0	4	8	
CYCLE-3	0	1	2	0	1	2	0	1	2	0	1	2	(mod 3)
"M_6"	0	6	0	6	0	6	0	6	0	6	0	6	
CYCLE-2	0	1	0	1	0	1	0	1	0	1	0	1	(mod 2)
"M_{12}"	0	0	0	0	0	0	0	0	0	0	0	0	
CYCLE-{1,5}	0	0	0	0	0	0	0	0	0	0	0	0	(mod 1)

EXAMPLE 13: COHN'S CYCLE HOMOMORPHISMS AS MULTIPLICATIVE OPERATORS

There is an important, but subtle, shift of perspective contained in Cohn's invocation of smaller universes here; it is most clearly manifest in the mismatch between the numerical labels of the "M_m" operators and their associated CYCLE-n homomorphisms.

Cohn captures this shift of perspective in a diagram adapted here as Example 14. The lighter circles in the diagram represent interval cycles; they also represent the collections of pcs that map identically under the relevant CYCLE homomorphism. The heavier circles represent what Cohn calls "cycles of cycles" (12). This terminology refers to the fact that the CYCLE homomorphisms involve collapsing interval cycles into a representative indicated here with large type; this representative "may be thought of as not only furnishing a label for that cycle, but as vortically inhaling its contents" (14) under the relevant homomorphism. Once the interval cycles are collapsed, it is clear that the cycles are themselves cyclically related to each other; this accounts for the periodic nature of the mappings detailed in Example 13, clarified there with vertical lines. The "cycles of cycles" are Cohn's "smaller universes."

In simple terms, this shift of perspective calls into question the usual interpretation of multiplicative operators and subjects them to a sort of figure-ground reversal—conceptualizing them not as things that expand intervals within the fixed frame of octave equivalence, but as things that preserve intervals while shrinking the universe. Each of the CYCLE homomorphisms, as Cohn uses them, extends the idea of octave equivalence to smaller intervals. CYCLE-2 envisions a world of whole-tone equivalence; CYCLE-3, minor-third equivalence; CYCLE-4, major-third



EXAMPLE 14: ADAPTED FROM COHN (1991), TABLE 5

equivalence; and CYCLE-6, tritone equivalence. Cohn's set-groups are nothing more than chord species of the usual kind—equivalence classes of classical sets of pitch classes under transposition and inversion—but under these novel notions of pitch class equivalence.

Cohn gets to this point, as we have mentioned, through a study of transpositionally invariant chords. Many of these are Hanson projections—all, in fact, besides the French-sixth chord 4–25 [0268] and the Petrushka chord 6–30 [013679]—and are therefore depicted in Example 2, where the interior of their clockface diagrams are colored gray. Inspection of these graphical representations highlight Cohn's point: a transpositionally invariant set exhibits a repeated pattern that is periodic at one of the intervals of equivalence mentioned above. The hexatonic collection 6–20 [014589], for instance, a prototype of what we are calling **F**(12, 3), is periodic at the major third. Since the pattern is periodic modulo 12, its structure can be completely, and more efficiently, described in terms of a nonperiodic pattern in a smaller universe; the mapping from the twelve-pc universe down to the smaller universe is provided by a CYCLE homomorphism.

We are now in a position to discuss the mismatch between our labels for the qualitative genera and the labels for the interval classes associated with them. This is precisely the same as the mismatch noted above between the numerical labels of the "M_m" operators and their associated CYCLE-n homomorphisms. The hexatonic collection is periodic at the major third. This interval being a representative of ic 4, the CYCLE-4 homomorphism provides the mapping into the requisite smaller universe (major-third equivalence). When we characterize CYCLE-4 as "M_3," or when we assert the species 6–20 as a prototype of the **F**(12, 3) genus, we are making reference to the fact that there are three major thirds to the octave. Similar statements could be made about, for example, CYCLE-2, ic-2 equivalence, and the transpositionally invariant whole-tone collection 6–35 [02468A], on the one hand, and "M_6," six whole tones to the octave, and 6–35 as a prototype of **F**(12, 6) on the other. The clockface diagrams in Example 2 are each supplemented with markings that show their division into q equal parts, where q is the index of the qualitative genus **F**(12, q).

Cohn's approach opens up a number of interesting avenues of inquiry in relation to the issues that have been placed on the table so far. The various approaches we have considered all seem to suggest, in one way or the other, that intergeneric affinities (the structural relationships between qualitative genera) have something to do with multiplication and interval cycles. Moreover, the prototypes laid out in Example 2—which, despite the fact that Hanson conceived them intervallically, have strongly resonated throughout our survey of different approaches to chord quality—seem to warrant further inquiry from the point of view of transpositional invariance, given how many of them have that symmetry. Cohn's work appears to tie all of these ideas together, suggesting that chord quality has something to do with equal divisions of the octave, but it raises two important questions of its own.

First, there are five CYCLE homomorphisms and six qualitative genera. CYCLE-{1,5}, which simply collapses all pcs into one category under semitone-or fifth-equivalence, corresponds in some vague sense to **F**(12, 1) and **F**(12, 5), if only conceptually, but these are precisely the genera whose only transpositionally invariant prototypes are trivial—the empty chord and the aggregate—and also in violation of UGP, if trivially so. What the prototypes of these genera have in common is something we have not yet treated adequately: each is a contiguous segment of the chromatic scale in the case of **F**(12, 1), or the circle of fifths in the case of **F**(12, 5).

Second, the concept of a one-to-one mapping between interval classes and divisions of the octave, which may be implied by the strong connection between Cohn's approach and our working set of prototypes derived from Hanson's intervallic approach, is tenuous at best. Our best example of this in the twelve-pc universe has to do with **F**(12, 5)—the octave is not equally divisible into five equal parts. Nonetheless, it is well-known that certain of our prototypes for **F**(12, 5), viz. the pentatonic collection 5–35 [02479] and its complement, the diatonic collection), are *maximally even*; they represent the *best* way to distribute five pcs (or five absences of pcs) equally around pitch-class space. As it happens, these chords can also be generated by ic 5—what may come as a disappointing surprise, as we will see in Part 2, is that this property does not generalize in the obvious way (see Carey and Clampitt 1996).

Having completed our survey of existing approaches to qualitative taxonomy, we will change course now, pursuing the promising idea of equal divisions of a pitch class universe, which is covered by the theory of maximally even chord species. This will be the key to fitting together some of the seemingly incompatible approaches we have covered here.

(Parts 2 and 3 will appear in volume 45/1 of *Perspectives.*)

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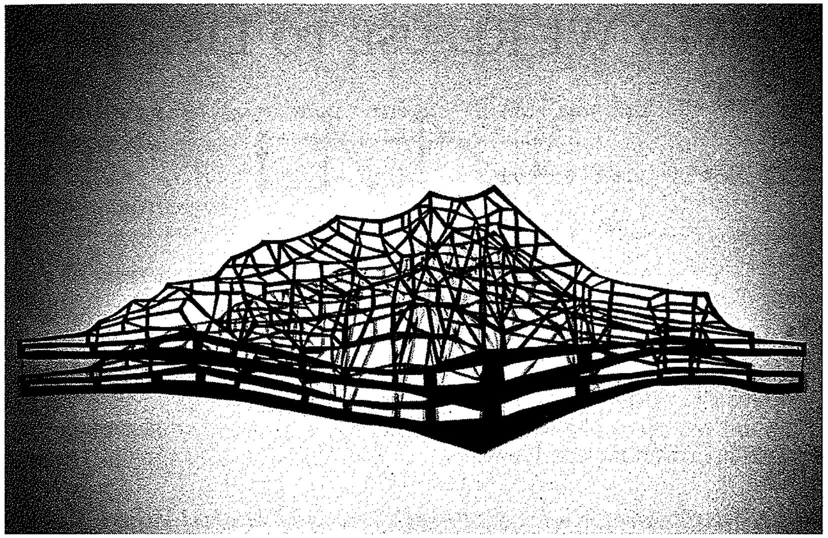

Victoria Haven: *Mirror Mountain* (2004), reflective mylar; 19 by 33 by 1 inches; Howard House Contemporary Art