ON ALLEN FORTE'S THEORY OF CHORDS

ERIC REGENER

1. *Atonality and a universal theory of chords*

A theory of harmony may reasonably be considered as one whose analytical units are unordered sets of notes, or "chords," under octave-equivalence and transpositional equivalence. Classical harmonic theory enriches this definition by assigning special meanings to certain notes of these sets on contextual or theoretical grounds: the bass and the root, for instance. It also provides valuable extensions of this basic material in its use of "functional" descriptions, which relate realizations of given chords to a reference note, the tonic or local tonic, and of harmonic "progressions," which describe commonly-used sequences of chords. The classical theory is limited, however, in the kinds of chords it can describe. Configurations which cannot be construed in terms of chords "built up in thirds" must have some elements explained in contrapuntal terms, as "non-harmonic tones" of certain specified types, or in some other heuristic way. This century has seen the growth of a repertoire which has yielded to this theory only with difficulty, if at all: that of so-called "atonal" music. For this music a new theory certainly seems to be called for, one in which any set of notes whatsoever can be given an objective harmonic description.

In his recent book, *The Structure of Atonal Music* [1973], Allen Forte gives us the outcome of a number of years' work on this problem. What he presents is a more-or-less systematic description of all possible sets of notes under octave-equivalence and transpositional equivalence, adding the relations of enharmonic and, surprisingly, inversional equivalence. He does not use any of the apparatus of the "functional" description, as is quite natural. As means of relating sets of notes, he uses similarity of interval content, common tones (which he calls "invariants") and abstract inclusion relationships. The choice of musical examples and the word "atonal" in the title make it clear that Forte is trying to construct a harmonic theory specifically for music which cannot reasonably be treated with the classical theory on one hand and is free, on the other hand, of explicit systematic procedures such as twelve-tone row construction. At the same time, he seems to want to exclude music which does not use most of the twelve pitch-classes most of the time: this would appear to be the sense of the unfamiliar word "paratonal" in his Preface [see p. ix].

In the sense that Forte apparently intends it to apply specifically to a repertoire which is not considered amenable to any other variety of harmonic analysis, this might be deemed a kind of theory of last resort, to be applied when all else fails. Indeed, "atonal" is commonly used in a negative sense, to indicate the apparent absence, or using intentional words, the attempted avoidance of certain recognizable "tonal" characteristics [cf. Perle 1972, p. 1]. But surely this is an invidious definition, since it depends on the present limitations of our theoretical tools: a new analytical technique may render some works "tonal" which we previously considered "atonal," or claimed to be so. In the logical extension, we might be able to eliminate altogether the category of "atonal" music and with it the necessity for Forte's theory.

Forte himself clearly does not subscribe to this negative definition. He seems to believe, rather, that his analytical method applies specifically to what might be called "atonal harmonic style" and to that of no other music. This belief may be deduced from his definition of an "atonal composition" as "any composition that exhibits the structural characteristics that are discussed, and exhibits them throughout" [p. ix]. But this definition depends just as crucially as the previous one on the limitations of the theory involved. We might well declare, for example, that Forte's theory is inapplicable to tonal music because his notation nowhere distinguishes a set of pitches from its inversion (though he does make the distinction informally in certain cases). This situation could be remedied, however, by a relatively simple change of notation. Would Forte's heuristic definition of "atonal" then expand to cover tonal music as well? Or has he deliberately abstained from distinguishing pitch-sets from their inversions in his basic notation, in order to limit the scope of what he considers "atonal"? In this case, he would be claiming that atonal music is set apart from tonal music, *by definition,* by the identity of a pitch-set and its inversion for almost all analytical purposes. But after Cone's telling criticism [1967] of the latter proposition as presented by Forte himself, among others, surely no one should be naive enough to assume it without careful thought, and certainly not as an analytical norm.

By his explicit and utter avoidance of conventional terminology and techniques, even by way of identification or any but the most incidental comparison, Forte gives the appearance of wanting to set up his system as a kind of antithesis to conventional harmonic theory, as if there were no possibility of common ground between the two. But this position is contradicted by the music itself. In the "atonal" works of Schoenberg, the tonal references are often made explicit by a sung text (for example, in the last song of *Pierrot*), and there is hardly a work of Berg in which a tonal interpretation is not essential to the proper understanding of the music. Indeed, many of these works, such as Berg's *Lyric Suite* (particularly in the last movement), deal directly with the relation of tonality to its negation, a relation which can only be grasped by openly admitting both sides to the discussion. As Adorno says [1959, pp. 78–79] in an afterword to the *Georgelieder* of Schoenberg, which Forte singles out [p. ix] as beginning the epoch of "atonal" music: "If [they] explode tonality, they are nevertheless interwoven throughout with its fragments. This relationship is not that of an impure style, however, but is the field of tension within which the composition lives."

The decision that a piece is "atonal" is an analytical decision, after all! It is difficult to see on what basis we can deny ourselves the techniques of classical analysis in any given case before we have satisfied ourselves of their irrelevance *by attempting to apply them.* Moreover, the tools of the "new" analysis can be made to illuminate the "old" music, a paradigm being Schoenberg's famous observation [1950, p. 111] that the second phrase of the last movement of Beethoven's Quartet op. 135 is closely related to the retrograde inversion of the first phrase.

In a universal, objective theoretical model, we should be able to approach any music whatsoever with the largest possible assortment of techniques and with no preconceptions whatever as to what we are to meet. In the domain of harmonic analysis, we should be able to classify chords objectively and without bias toward classical nomenclature, since the latter may not apply. It is precisely this purpose which Forte's classification and others like it satisfy so well, whether or not they are primarily intended to do so. To justify using this scheme as a basis for a general theory, it must be possible to reformulate the classical theory in the new terms or similar ones. It will be shown later that this reformulation is not only possible, but easy.

A theory of harmony differs from a theory of chords, presumably, in that it deals with relationships based on succession in time ("progressions"). Since Forte's theory does not deal with temporal relationships in any such systematic way, it therefore seems appropriate to consider it basically a theory of chords. Our criticism, then, with a universal theory of chords as backdrop, will center on the most general, objective, and useful ways to treat Forte's theoretical material.

We shall also suggest some reforms of notation and of formal treatment. Presumably it is never too late to propose reforms; some apology nevertheless seems in order for a new criticism of a theoretical framework which has been around, after all, ever since Forte [1964]. The appearance of this theory in book form with few essential changes may be provoking, ironically, just that extensive discussion which should have followed the publication of the earlier article. In any case, our new proposals will have to stand on their merits.

It must not be thought, either, that we intend to disparage Forte's "theory of atonal music" by calling it a "theory of chords." We take the position that Forte's theoretical apparatus seems potentially so general as to serve as the basis for a much wider application than he apparently intends to give it. As we have seen, another motivation is the difficulty, indeed the undesirability, of defining the scope to which it is to be restricted. To these considerations we shall later add the question of the relevance of Forte's concepts in terms of the repertoire he chooses to analyze.

An initial remark on terminology: we shall use the term "pitch-class set" exclusively to refer to a specific set of pitch-classes. This affirmation might seem superfluous, were it not that Forte uses the term for a set of pitch-classes *under transposition-equivalence* (what Babbitt [1955] and Martino [1961] call a "source set" and Howe [1965] calls a "pitch structure"). We shall call this latter construction a *chord,* in the same sense that a dominant seventh "chord" is a chord. A set of pitch-classes which "forms" a given chord will be called a *representation* of that chord, embodying, it is hoped, a sort of type-token distinction. Thus we may say that the pitch-class set [D, F, B] (or [2, 5, 11]) is a representation of the chord called the diminished triad. The diminished triad is a chord, not a pitch-class set. This distinction tends to be obscured by the commonplace use of pitch-class normal form for chords. It is one we are forced to make because it is difficult to talk about ordinary sets of pitch-classes, as we shall have to in discussing common-note relationships later on, when the term "pitch-class set" has been defined to mean something else!

The use of the common term "chord" seems justifiable because the implication of transposition-equivalence seems to fit. If we do not go so far as to use "note" consistently instead of "pitch-class," as Cone [1965] suggests, it is because it is not so natural to assume that a "note" is independent of octave placement as it is to assume that a "chord" is independent of transposition or musical context. For another thing, the term "pitch-class" seems fairly well established in the literature by now.

2. *Sets of pitch-classes*

Forte identifies each possible chord by an arbitrary number designating its position in a table included as an appendix. The order in this table of the chords having a given number of notes is determined by a lexical ordering on the values of the interval-content function: the chords containing the most intervals of a semitone come first, and within these, the chords with the most intervals of two semitones, and so forth. In addition, if two chords share the same interval-content function, one of them is shifted to the end of the table, and inversions are simply omitted. One can understand that Forte wants to keep the same identification numbers as in his earlier article [1964], in which he did not distinguish between chords with identical interval content. But this complex classification renders unnecessarily tedious one of the two principal uses of the table, namely finding out which identification number corresponds to a given set of pitch-classes. Either one must calculate the normal form, not always a simple matter, compare it with the normal form of the inversion to see which one is included in the table, and then search essentially the whole table to locate it; or else, one must compute the interval content in addition, in which case there are only two places in the table where the set may be found, and these two places will have to be distinguished on the basis of the normal form.

The other use of the table is finding out which sets Forte is discussing, since in his text he usually refers to the sets only by their identification numbers. Thus, if one is unwilling to memorize an arbitrary number for each of the 220 sets that Forte identifies, one is obliged to keep a thumb in the appendix practically throughout the book. This convention is not only deleterious to concentration, but obscures what is often important material by treating the chords themselves at second remove. Take the following sentence, for example: "As remarked several times before, pc set 4–28 and its complement 8–28 are sets with special attributes" [p. 101]. Forte's disinclination to say "diminished seventh chord" can be understood, but it is indefensible that such a simple statement should be meaningless without the appendix.

The solution to the problem is quite simple, of course: *use meaningful names*. It would be a great improvement, at the very least, to cite the normal form of a set instead of a meaningless identification number. But even here it is not the specific pitch-classes in the normal form which interest us, but rather the intervals between them. An appropriate way to write a pitch-class set under transposition-equivalence, for purposes of identification at the very least, is in what we shall call *interval notation* (a form of which, I am told, appears in Chrisman [1969]). Taking a "closed-position" representation of the chord (one in which the pitch-class numbers are in increasing order, or any pitch representation spanning less than an octave), we write in sequence the *directed* intervals between successive pitch-classes, including that between the last note of the set and the first. The pitch-class set [0, 3, 10] in interval notation is (372), for example, where the interval 3 appears between pitch-classes 0 and 3, the interval 7 between 3 and 10, and 2 between 10 and 0, always counting semitones in the "upward" direction (this is the sense of the word "directed": thus the last interval is necessarily 2 and not 10). The sum of the numbers in this presentation is always exactly 12 if one is sure to take directed intervals. Any circular permutation of the numbers represents an equivalent pitch-class set, and vice versa. Which one of these represents the ordinary normal or "prime" form described by Forte and others can be quickly determined by the following algorithm:

Algorithm P. Consider a nonempty collection of chords in interval notation, no two identical and all having the same number of notes n.

1. Retain all the chords whose last interval is largest and reject the others. If more than one remains, set the index k to 1 and proceed.
2. Of the remaining chords, retain all those whose k'th interval is smallest and reject the others. If more than one remains, increase k by 1 and repeat this step.

Note that the algorithm does not require all the sets to be equivalent. If the collection consists of all the different circular permutations of a given chord, the chord remaining on completion of the algorithm will be the pitch-class normal form if the first pitch-class is taken to be 0 and the intervals of the chord are added in sequence.

For example, the normal form of the trichord (372) mentioned above is (237), whose prime form as a pitch-class set is [0, 2, 5]. Again, the pentachord [0, 2, 5, 8, 9] corresponds to (23313) in interval notation. After step 1 of the algorithm, the remaining chords are (31323) and (13233) as well as the one given. Clearly the last named is the normal form, and the pitch-class normal form is [0, 1, 4, 6, 9]. This is Forte's set 5-32. Even in a complicated example such as this one, the normal form can be deduced at a glance by a simple series of comparisons and without the repetitive arithmetic required by other methods.

If the algorithm is applied to a collection of different sets already in normal form, it induces an easily remembered ordering among them. For hexachords, this ordering proceeds from the chromatic hexachord (111117) to the whole-tone hexachord (222222). This ordering would solve the problem of finding the sets in the table, particularly if inversions were included.

There is nonetheless a more natural normal form for chords in interval notation. It is defined by the following even simpler algorithm:

Algorithm I. Same assumptions as Algorithm P.

1. Set the index k to 1.

2. Of the remaining chords, retain all those whose k'th element is largest and reject the others. If more than one remains, increase k by 1 and repeat this step.

The normal form defined by Algorithm I might as well be called *interval normal form.* For the example given above, the sets retained after step 2 is executed with $k = 1$ are (33132), (31323), and (32331). After it is executed again, only the first of these remains. The normal form of an n-note chord according to this algorithm is simply that circular permutation having the largest value when considered as an n-digit number (considering an interval of 10 or 11 as a single digit, of course).

The normal order of hexachords induced by Algorithm I begins and ends with the same two chords as before, though the chromatic hexachord now has normal form (711111). It is widely divergent, though, within the hexachords having the same largest interval.

From now on, we shall refer to chords exclusively in interval normal form as defined by Algorithm I, while including references to Forte's identification numbers for clarity.

Interval normal form enables one easily to see certain aspects of internal structure. The inversion of a chord, for example, is that chord with the intervals read cyclically in reverse order, though producing the normal form may take a moment's thought: thus the inversion of the hexachord (432111) is obviously (411123) (Forte's 6–Z40), the inversion of (323211) is (323112) (Forte's 6–Z47), and that of (322131) is (313122) (Forte's 6–31). Inversion-symmetric formations read the same backwards (in some circular permutation) as forwards: e.g., (412221) (Forte's 6–Z26), or (321231) (Forte's 6–Z50). Chords possessing transposition-symmetry (what Howe [1965] unfortunately calls "degenerate" chords) are easily recognized by their periodic structure, and the number of identical transpositions is the number of repetitions of the period: e.g., the twelve transpositions of (313131) (Forte's 6–20) fall into sets of three, and those of (411411) (Forte's 6–7) into identical pairs. A quick glance at (321321) (Forte's 6–30) suffices to show that it is transposition-symmetric at the tritone, but is not inversion-symmetric.

It is even possible to compute the complement of a chord by eye directly from the interval notation: simply replace every string of k consecutive 1's with the interval $k + 2$, and conversely, every interval j larger than 2 by $j - 2$ consecutive 1's. If two intervals larger than 2 are adjacent, the resulting strings of 1's must be separated with a 2. A string of k consecutive 2's is replaced by $k - 1$ consecutive 2's if it appears between two 1's, is left standing if it is between a 1 and an interval exceeding 2, and is replaced by $k + 1$ consecutive 2's if between two larger intervals. The last and the first interval are always considered adjacent, and the integer k

may be equal to 1 or greater throughout. In the two cases not covered by these rules, the complement of (222222) is itself and the complement of (111111 111111), the "total chromatic," is the empty chord.

Though this method takes a bit of practice, it is quite handy. A little thought will make it clear why it works.

We shall need later on a notation for a given representation of a chord S in terms of its interval normal form. Let us say that the symbol T*n*(S) defines the set of pitch-classes consisting of *n*, *n* plus the first interval of S, the resulting note plus the second interval of S, and so on. Thus T0(5124) is [0, 5, 6, 8], while T3(5124) is [3, 8, 9, 11] and T7(5124) is [7, 0, 1, 3] or [0, 1, 3, 7], Forte's normal form for the tetrachord 4–Z29.

3. *Mathematical and formal treatment*

Forte's statement of the normal-form algorithm [pp. 4–5] is technically incorrect, since for a set to be "in best normal order [with] first integer . . . 0" implicitly requires a definition of transposition in addition to the variety of circular permutation he adduces. Forte evidently thinks this definition is important, since it appears in considerable detail on the very next page, but he seems not to realize that he has used it before defining it. He goes on to say [p. 5] that his appendix contains a "complete list of prime forms for the 220 distinct pc sets." This is certainly false, since he has not yet said that he considers inversions equivalent to prime forms, and since he includes only sets having between 3 and 9 constituents.

Later [p. 14], he wants to define interval classes in such a way that the pitch-class interval d is equivalent to $12 - d$. He does this by the following "defined equivalence": "If d is the difference of two pc integers then $d \equiv d'$ mod 12." The perceptive reader may eventually connect the apparently meaningless instance of the primed variable with its sole prior appearance, in the definition [p. 8] of the "inverse of a pitch-class" (*sic*), even though this mention does not constitute a definition. What is more difficult to accept is that the equation as a whole does not have its customary mathematical meaning, namely that d and d′ differ by a multiple of 12. It turns out that Forte has gratuitously redefined the triple-barred equal sign, which has the standard meaning of mathematical congruence and which one would expect to mean just that in just such a context, to refer to the equivalence of interval classes. Worse yet, he has done so uselessly, since the symbol does not reappear on any other page of the book.

It should be noted that Forte, like Howe [1965], apparently uses the term "mod" in a computer-language definition rather than in its correct mathematical usage in conjunction with the sign "≡", as described above. Thus "k mod 12" means, for him, the smallest nonnegative integer which differs from k by a multiple of 12. This usage has the defect that the computer-language definition commonly delivers a negative result for negative

k, which is presumably not what he wants. For a correct elementary definition of the "least nonnegative residue," see Vinogradov [1954, pp. 46–47] or, with apologies for the presumption, Regener [1973, p. 20].

In at least one other case, fundamental material which one would expect to find in a work of this nature is absent. The theorem relating the interval content of a pitch-class set to that of its complement—the extension of the "Babbitt hexachord theorem"—is quite basic to Forte's concept of the set-complex. A proof of this theorem is not really difficult to state in elementary language (as we shall see later), and it is disappointing that Forte does not even make a gesture in that direction.

These errors and omissions are, unfortunately, typical of Forte's writing in this book, particularly blameworthy because they appear in crucial definitions, and surprising in view of the relative clarity of Forte [1964]. Even though what he means may be clear to those in the know, his stumbles are hard to justify. It ought to be expected of a book that *seems* so mathematical that the mathematics be meaningful and correct—that it be mathematics, in short—and that it be there when we have need of it. Any less is a disservice to the many students for whom such a book is an introduction to mathematics as well as to the musical theory involved.

The only way to be sure that such material is correct is to *take it to a mathematician,* in all humility, *before publication,* unless one is really sure of being able to write good mathematics unaided without elementary mistakes! By explaining technically interesting material correctly, we may be able to advance and broaden our field by opening it to serious mathematicians.

In another sense, the less mathematics we use, the better. Special terminology and formal treatment are not ends in themselves, after all, in a work which purports to be oriented primarily toward musical analysis. In such a context, the mathematical treatment should act like the lighting in a Brecht play: it should illuminate and clarify, but without drawing attention to itself, and we should never have doubts about its working properly.

Returning to Forte, it must be said in his favor that if his expository powers and his mathematics seem weak at times, his documentation is extraordinarily accurate. There is hardly a number in the book which is in error, be it in a table, a calculation, or a pitch-class designation in a musical example. This accuracy is clearly due to a superior knowledge of what to do with a computer, especially in view of the fact that he writes all his computer programs himself, and it is to be applauded.

4. *Extension to diatonic chords*

It might be mentioned at this point that the concepts and notations developed both here and in Forte's book apply equally well to diatonic situations, substituting the value 7 for 12 and diatonic for chromatic degrees. The present author has shown elsewhere [1973, see for example pp. 156–159] that the same mathematical formalism applies naturally to both. Thus there are seven diatonic note-classes (Kassler's term), namely C, D, E, F, G, A, and B, irrespective of register or of the applied accidental (whether natural, flat, sharp, double-sharp or whatever), just as there are twelve tempered pitch-classes, irrespective of enharmonic equivalence. The interval of an augmented prime, which corresponds to the application of an accidental, has precisely the same formal function in the diatonic case as the diminished second has in the case of tempered intonation: in one case we have diatonic equivalence, in the other case enharmonic equivalence. Numbering the seven diatonic note-classes in order from zero to six, we can describe and classify diatonic intervals and chords in just the same way as we have been discussing. The triad, [C, E, G] or [0, 2, 4], for example, is represented in diatonic interval notation as (223). There are, in fact, exactly five different diatonic three-note chords, namely: (511), (421), (412), (331), and (322), using the conventions described earlier for interval normal form and normal order. The last of these chords is the interval normal form of any triad. Any one of these chords can be made to correspond to any chromatic trichord whatsoever by suitable (perhaps bizarre) choice of accidentals.

For diatonic analysis, another interesting entity is the open-ended scale of fifths, containing all the note-classes of the notation (the so-called "letter names") under octave-equivalence, but respecting neither enharmonic nor diatonic equivalence. We shall write the "interval" representation of a chord in this scale in terms of the distance in fifths between the notes, taking the "flattest" note first and proceeding to the "sharpest" one. Since this scale, being the most general of all [cf. Regener 1973, p. 172], has no cyclic equivalence as do the other two, there will perforce be one interval less than the number of notes. The interval numbers will be included in brackets < > with commas separating the numbers. For example, Fig. 1 shows that the notated pitch-class chord <2, 4> having successive intervals of 2 and 4 fifths, an incomplete dominant seventh, corresponds to trichord (624) (Forte's 3–8) and to diatonic chord (412), and the German sixth chord <4, 6> to (624) and (322). In Fig. 2 we see the major triad <1, 3>, which corresponds to (543) and (322), and

2 4

. . . B♭ F C G D A E B F♯ C♯ G♯ D♯ A♯ . . .

4 6

Fig. 1

1 3

... D♭ A♭ E♭ B♭ F C G D A E B F♯ C♯ ...

3 8

Fig. 2

a distorted triad <3,8>, which corresponds to the same two chords. This example shows that a notated pitch-class chord is not unambiguously determined by its diatonic and chromatic representations.

The utility of the fifth-scale is shown by the fact that in diatonic harmony the nature of a chord and the direction of its resolution are often determined by the interval with the largest span in fifths, the sum of all the elements of our *ad hoc* notation (in Fig. 1, a tritone and an augmented sixth, respectively). A general study of diatonic harmony might consider the possible chromatic representations of diatonic chords, deriving constraints based on spans in the fifth-scale. This material could then be used to advantage in formalizing a theory of chromatically altered diatonic chords, such as forms a prominent feature of Schoenberg's *Structural Functions of Harmony* [1954].

5. *Interval content and common notes*

We now take up the question of interval content. In this area, there is something to be said for *not* making an interval d equivalent to $12 - d$, the complementary interval with respect to the octave: that is, for always considering directed pitch-class intervals, as in Howe [1965], who presents much of this material quite clearly. If we tabulate the directed intervals from every note in a given pitch-class set with n elements to every other note *including itself,* we get a total of (obviously) n^2 intervals. These break down as follows: n intervals of zero (as many as there are notes in the set), twice as many intervals of six semitones as in the ordinary interval content, and the same number of intervals with $12 - d$ as with d semitones.

We shall say that the value of the *directed interval-content function* of a chord P and a pitch-class interval d is the number of directed pitch-class intervals of size d in P. The set of values of this function, displayed within square brackets for $d = 0, 1, \cdots, 11$, will be called the *directed interval-content vector* (making a concession to computer-programming terminology). This "long form" of interval content is periodic for transposition-symmetric chords: the directed interval-content function for the hexachord (313131) takes the values

[6303 6303 6303] ,

that for the hexachord (321 321) is

[622422 622422] ,

and for (222222) we have

[606060 606060] .

The vectors for (51141) (Forte's 5–7) and its complement (411311), respectively, are

[531013 431013] and [753235 653235] ,

illustrating the fact that a special case need not be made for the interval 6 in the statement of the theorem relating interval content in complementary chords [cf. Forte, p. 77], provided we use the directed interval-content function. The difference in content for each interval is the difference in number of elements between the chord and its complement. If the set has p elements, this difference is $(12 - p) - p = 12 - 2p$.

Forte and others point out that the content of an interval *d* in a given pitch-class set is the same as the number of pitch-classes in common between the set and its transposition up (or down) *d* semitones. It is for this reason that all the interval-content vectors shown thus far have been inversion-symmetric. Thus, in our interval-content function we have essentially compared the set to each of its twelve transpositions. This observation suggests that we generalize the idea of the interval-content function so as to be able to compare pitch-class sets representing two *different* chords.

Suppose A and B are arbitrary sets of pitch-classes (*not* under transposition equivalence), and let **M**(A, B) be the number of pitch-classes in common between them. Now if P and Q are chords in interval normal form, then

$$\mathbf{M}(\mathrm{T0}(\mathrm{P}),\ \mathrm{T}d(\mathrm{Q}))$$

is precisely the "interval function" defined by Lewin [1959], allowing for the change of notation. For a given value of *d*, this expression is the number of pitch-classes in common between the "zero" transposition of P and the transposition of Q up *d* semitones. Now we display the values of this function for $d = 0, 1, \cdots, 11$. If $\mathrm{P} = \mathrm{Q}$, this is the directed interval-content vector of P. Otherwise, we have something which cannot reasonably be called an interval-content function any more: we shall call it the *common-note function between* P *and* Q. We shall use the term *configuration* for a representation of two chords P and Q at a certain relative transposition *d*.

Let us define $\mathbf{n}(Q)$ to be the number of elements in the chord Q. If $\mathbf{n}(Q) \leqslant \mathbf{n}(P)$, that is, Q has no more elements than P, then in any configurations having $\mathbf{n}(Q)$ common notes, we shall find Q to be fully contained in P. If there are none, then Q cannot be so contained.

For example, between the hexachord (621111) (Forte's 6–2) and the trichord (831) (Forte's 3–3), the common-note vector is

$$[321110\ 111232].$$

At transposition $d = 0$ we compare [0, 6, 8, 9, 10, 11] with [0, 8, 11] and find 3 common pitch-classes; at transposition $d = 1$ we compare the same set of six notes with [1, 9, 0] and find the two notes 0 and 9 in common, and so forth. At transpositions $d = 0$ and 10, the trichord is contained in the hexachord. At transposition 5, and only there, no pitch-classes are in common, and the trichord is therefore contained in the complementary hexachord (611112) exactly once. On the other hand, the common-note vector between the same trichord and the hexachord (321222) (Forte's 6–33) is

$$[211121\ 221221]:$$

the trichord (831) is contained neither in this hexachord nor in its complement. Looking at the interval-content function above for the hexachord (313131), we see that this chord is apparently contained in its own complement at the three different transposition levels 2, 6, and 10. These all result in identical configurations, however, because of the transposition-symmetry.

We shall now use our notation to compute the number of common pitch-classes between the complements of two given sets. A special case of this result provides the simple proof of the Babbitt hexachord theorem which we promised earlier.

Suppose that A and B are any sets of pitch-classes and A′ and B′ are their complements: A′ is the set of pitch-classes not in A, and the same for B. Obviously every pitch-class of A which is not in common with B must be in common with B′, that is,

$$\mathbf{M}\ (A, B') = \mathbf{n}\ (A) - \mathbf{M}\ (A, B).$$

What is $\mathbf{M}(A', B')$? We write this as $\mathbf{M}(B', A')$ and substitute B′ for A and A′ for B′ in the previous formula. The result is

$$\begin{aligned} \mathbf{M}\ (A', B') &= \mathbf{M}\ (B', A') = \mathbf{n}\ (B') - \mathbf{M}\ (B', A) \\ &= 12 - \mathbf{n}\ (B) - (\mathbf{n}\ (A) - \mathbf{M}\ (A, B)), \end{aligned}$$

substituting the earlier formula directly, and therefore, finally,

$$\mathbf{M}\ (A', B') = 12 - \mathbf{n}\ (A) - \mathbf{n}\ (B) + \mathbf{M}\ (A, B).$$

If B is the transposition of A up d semitones, then $\mathbf{M}$ (A, B) is the content of interval d in A and

$$\mathbf{M}\ (\mathrm{A}', \mathrm{B}') = 12 - 2 \cdot \mathbf{n}\ (\mathrm{A}) + \mathbf{M}\ (\mathrm{A}, \mathrm{B})$$

is its content in A′. This corollary is the extension of the Babbitt hexachord theorem; for hexachords in particular, $\mathbf{n}(\mathrm{A}) = 6$ and

$$\mathbf{M}\ (\mathrm{A}', \mathrm{B}') = \mathbf{M}\ (\mathrm{A}, \mathrm{B}).$$

It should be pointed out that the set of values representing the common-note function is subject to circular permutation, in just the same way as the interval set. The present author in his own work applies Algorithm I to the common-note vector to reduce it to a normal form, whose exact relationship to the original positions of the two sets can be reconstructed by adjoining a transposition number. The common-note vector between a set and itself (its long-form interval content) is necessarily in this normal form to begin with.

Lewin [1959] presents some fundamental results on the nature of the common-note vector. Among many other things, it is not known under what conditions this vector is inversion-symmetric, that is, can be read the same forwards as backwards in some circular permutation. Evidently this will be the case if both of the sets are themselves inversion-symmetric; but this is not a sufficient condition, as is shown by the two examples above which employ the trichord (831). The common-note vectors shown between this trichord and the hexachords (321222) and (621111) are both inversion-symmetric, as is that between the two hexachords,

[433433 423232],

but none of the three chords is the same as its inversion. Might this even be a compositionally or analytically useful property?

Forte uses the term "invariant" for a note in common between a pitch-class set and one of its transpositions or inversions, in supposed analogy with the mathematical notion of invariants under transformation. This analogy is not quite accurate, and is evidently unsuitable for the natural generalization to common notes between forms of nonequivalent chords.

6. *Similarity relations between chords*

Forte defines four similarity relations between chords having the same number of notes, one with respect to inclusion properties and three with respect to interval content [pp. 47–48]. Two n-note chords are "in the relation $\mathrm{R_p}$" if their common-note function takes on the value $n - 1$ at some transposition level (this is not Forte's definition, but a technically correct one based on our terminology). Comparing the interval-content vectors $[a_1 a_2 \cdots a_6]$ and $[b_1\ b_2 \cdots b_6]$ (in short form, of course) of the two pitch-class sets A and B taken individually, he says that they are "in the relation R_0" if the contents a_i and b_i are different for each of the six different interval-classes i; "in the relation R_1" if $a_i = b_i$ for four of the i and in addition $a_j = b_k$ (which implies $a_k = b_j$) for the other two interval classes j and k; and "in the relation R_2" if $a_i = b_i$ for exactly four of the i and R_1 does not hold (these definitions as well are restated).

The question of meaningful terminology makes itself felt here, too. Not only do the numerical subscripts in the names R_0, R_1 and R_2 have no numerical meaning, but their sequence is actually contrary to the sense of the definitions. Forte himself remarks that "R_1 represents a closer relationship than R_2," with the result that the three relations follow the order: least similarity—great similarity—not as great similarity. One may also feel that the case for musical meaningfulness is not exactly strong. As Clough [1965] points out in his criticism of Forte's earlier article, any such definition is arbitrary; one would like Forte to motivate his choices more than he does.

A close study of Forte's examples in this section does reveal some fascinating relationships, though some spadework may be necessary. For example, he presents a list [p. 56] of eleven "transitive R_1 quintuples of cardinal 6," by which he means collections of five hexachords, any pair of which is in the relation R_1. Examination of the nine different chords involved in these collections (with the ever-necessary help of the appendix) shows them to be five of the six possible hexachords containing the "diminished seventh" tetrachord (3333), together with the complements of the four which are not self-complementary. In each of the collections, the self-complementary hexachord, (321321) (Forte's 6–30), combines with one representative from each of the other four pairs. There seems to be an omission in the table here, incidentally, since only eleven collections appear out of the sixteen (2^4) one expects to find.

These chords nearly exhaust the possibilities of hexachords containing four intervals of 3 semitones. In the complements of the chords containing (3333), these four intervals are arranged as two "diminished triad" trichords (633). Table 1 displays all these chords, along with (332121) (Forte's 6–27), the last of those containing (3333), which has a slightly different interval structure since the interval between the two "extra" pitch-classes is also 3 semitones. Note that the inclusion relations can be verified at sight from the interval notation. They are even more readily seen on a Krenek diagram, drawing the chords as polygons inscribed between the vertices of a regular dodecagon. It may also be noted that all the chords are inversion-symmetric except those which are self-complementary (or better, "inversion-complementary").

It should be mentioned that Forte's "maximal similarity" relation R_1 applies to any pair of chords related by so-called "circle-of-fifths inver-

TABLE 1. HEXACHORDS WITH LARGE CONTENT OF INTERVAL 3

Hexachord	*Forte's number*	*Complement*	*Forte's number*	*Interval content*
(333111)	6–Z42	(512121)	6–Z13	[324222]
(332112)	6–Z45	(421212)	6–Z23	[234222]
(331221)	6–Z28	(323121)	6–Z49	[224322]
(321312)	6–Z29	(321231)	6–Z50	[224232]
(321321)	6–30	(312312)	6–30	[224223]
(332121)	6–27	(331212)	6–27	[225222]

sion," since the effect of this operation on the interval content is to exchange the content of intervals 1 and 5 while leaving the rest of the values unaltered. This operation is certainly of debatable musical significance; its appearance in this context is inevitable, though, since its mathematical effects in the pitch-class domain are so similar to those of ordinary inversion [see Howe 1965, p. 55].

A useful tool for presenting similarity relations is what I shall call the *partition function* V(P, Q, k) of two chords P and Q and an integer k. Its value for a given k is the number of configurations of P and Q having k notes in common. V(P, Q, 0) is the number of configurations with no common pitch-classes, V(P, Q, 1) is the number with a single pitch-class in common, and so forth. For a given hexachord H, for example, V(H, H, 6) is greater than 1 only for the transposition-symmetric chords; for two hexachords H and J, Forte's relation R_p holds if and only if V(H, J, 5) is nonzero and V(H, J, 6) is zero (i.e., H and J are not the same set). Displaying the values of V(H, J, k) for $k = 0, 1, 2, \cdots$, up to the number of notes in whichever chord has fewer, we have the *partition vector,* which may conveniently be written between two double vertical bars. Taking one of our earlier examples, the partition vector of P = (621111) and Q = (831) is || 1632 ||, showing the trichord to be contained at two different transposition levels in the hexachord (since V(P, Q, 3) is 2) and at one transposition level in its complement (since V(P, Q, 0) is 1). The same trichord fits neither with the hexachord (321222) nor with its complement, since the partition vector in this case is || 0660 ||.

It should be clear that the sum of the numbers in the partition vector is 12, since there are always 12 configurations of two sets even if some be identical. Note that the partition vector is *not* subject to circular permutation.

The word "partition" is a mathematical term referring to a way of expressing a given positive integer as a sum of other positive integers, without regard to the order of the constituents or "parts." For two chords P and Q the sum of the values of the common-note function is $\mathbf{n}(P) \cdot \mathbf{n}(Q)$, since each note of P is in common with each note of Q in exactly one of the twelve configurations. These values, considered without regard to order, form a partition of this sum into no more than 12 parts, no part greater than the smaller of $\mathbf{n}(P)$ and $\mathbf{n}(Q)$. It is this partition which the partition function tabulates.

An appropriate generalization of Forte's relation R_p between two chords P and Q might be a function whose value is the largest possible number of notes in common between representatives of P and of Q. This is the largest k for which the partition function $\mathbf{V}(P, Q, k)$ is nonzero, or alternatively, the largest element of the directed common-note vector. Let us call this function the "degree of inclusion" $\mathbf{d}(P, Q)$. It is defined for any two chords P and Q. If Q has fewer constituents than P, then $\mathbf{d}(P, Q)$ can equal $\mathbf{n}(Q)$ if and only if Q can be contained in P, and the expression $\mathbf{V}(P, Q, \mathbf{d}(P, Q))$ gives the number of configurations for which this is possible.

Forte's relation R_p, on the other hand, is defined only when P and Q are different chords having the same number n of elements, and holds if and only if $\mathbf{V}(P, Q, n-1)$ is nonzero, that is, if the degree of inclusion is $n-1$. Our "degree of inclusion" seems to extend its domain of applicability in a worthwhile way, at the same time enriching the range of possible outcomes.

This enrichment might reduce the necessity for Forte's other similarity relations R_0, R_1, and R_2, which present mathematical difficulties [see pp. 52–53] above and beyond their debatable musical significance. It is noteworthy, though, that two chords P and Q which are in relation R_1 often have the same *self-partition functions* $\mathbf{V}(P, P, k)$ and $\mathbf{V}(Q, Q, k)$. For example, the eight chords occupying the first four lines of Table 1 all have partition vector || 0062301 || (remember that the interval content for interval 6 is doubled in the directed common-note vector); the chords on the fifth line, whose common-note vector is

[622422 622422],

have a different partition vector, || 0080202 ||, even though they are in relation R_1 to the other four, and those on the last line have partition vector || 0080121 ||. There are no other chords having the last two vectors named, while four other chords share the vector || 0062301 ||, namely (513111) (Forte's 6–5), whose common-note vector is

[642223 432224],

along with its circle-of-fifths inversion (412311) (Forte's 6–18), and the ordinary inversions of these two chords. In general, if two pairs of chords have the same partition function, the corresponding common-note vectors are obviously permutations of each other. The relation of "having the same partition function" is an equivalence relation on pairs of chords (the two chords of a pair may be the same, as in the above examples), and divides the totality of possible pairs of chords into equivalence classes. It has the transitivity properties that Forte seeks in vain in his own relations [pp. 52–53], and might even have useful compositional applications.

The number of different partition functions is relatively restricted. For example, there are 676 partitions of the number 36 into no more than 12 parts, with no part exceeding the value 6. But to the 3240 (that is, 81•80/2) different pairs of hexachords correspond only 70 different partition functions, of which 14 cover the combinations of the 80 hexachords with themselves. The number of pairs sharing the same function (the size of the equivalence-classes mentioned above) ranges from 1 to 490 (for the class || 0044400 ||) ; fifty-one of these classes comprise 20 pairs or less. Pairs of chords which occupy small classes might well have other special properties of interest.

7. *The set-complex and general inclusion relations*

We turn now to the set-complex, the last and most elaborate of Forte's theoretical constructions. Divesting the definition of Forte's supererogatory notation, a chord S is a member of the "set-complex K(T) about the chord T" provided that S is contained in, or contains, either T or the complement of T. In addition, S is forbidden to have the same number of elements as T or its complement, and both S and T are restricted to a number of elements between 3 and 9, inclusive. Finding that the number of chords in some of his set-complexes is too vast to be useful, he defines as well the "subcomplex Kh(T)." To belong to this subcomplex, a chord S must bear the inclusion relation *both* to T and to its complement.

These collections can perhaps best be described in terms of the *general inclusion lattice* of pitch-class sets. In graph-theoretical terms, this lattice is constructed by writing down a point for every possible chord, and then connecting with a line segment the points corresponding to each pair of chords P and Q for which $\mathbf{d}(P, Q) = \mathbf{n}(Q) = \mathbf{n}(P) - 1$: that is, for which P can contain Q and has exactly one more element than Q. (Obviously this is a theoretical construction, in that writing down the entire lattice would produce rather an obscure picture. We can display small portions of it, though.) For all such pairs, $\mathbf{V}(P, Q, \mathbf{n}(Q))$ is evidently nonzero. An arbitrary chord Q can be contained in a chord P if and only if a *direct path* exists from Q to P: that is, a sequence of line segments connecting Q through successively larger chords to P. This fact follows from the transitivity of the inclusion relation.

Now let us define the *sublattice* $\mathbf{S}(Q, P)$ *between* Q *and* P to be the collection of chords which lie on direct paths between Q and P: that is, all the sets which contain Q and are contained in P (including Q and P, of course). Letting Φ symbolize the "empty chord," with no pitch-classes, and Ω the collection of all twelve pitch-classes, we see that $\mathbf{S}(\Phi, \mathrm{P})$ is the collection of chords contained in P, while $\mathbf{S}(\mathrm{Q}, \Omega)$ is the collection of chords which contains Q, and $\mathbf{S}(\Phi, \Omega)$ is the collection of all possible chords. If P does not contain Q, the sublattice is empty (i.e., contains no chords) by definition.

Forte's set-complex K(T) is simply described as the union of the four sublattices $\mathbf{S}(\Phi, \mathrm{T})$, $\mathbf{S}(\Phi, \mathrm{T}')$, $\mathbf{S}(\mathrm{T}, \Omega)$, and $\mathbf{S}(\mathrm{T}', \Omega)$ (ignoring his strictures on chords with fewer than 3 or more than 9 elements), where T′ symbolizes the complement of T: that is, the collection of all chords which can be contained in T or T′, or which can contain T or T′.

In order to describe the subcomplex Kh(T), we suppose without loss of generality that T has no more than six elements. The collection Kh(T) consists of the sublattice $\mathbf{S}(\mathrm{T}, \mathrm{T}')$, to which are adjoined the intersection of $\mathbf{S}(\Phi, \mathrm{T})$ and $\mathbf{S}(\Phi, \mathrm{T}')$ as well as the intersection of $\mathbf{S}(\mathrm{T}, \Omega)$ and $\mathbf{S}(\mathrm{T}', \Omega)$. That is, it consists of the chords containing T and contained in T′ as well as those contained in both T and T′ or else containing both these chords. If T′ does not contain T, then $\mathbf{S}(\mathrm{T}, \mathrm{T}')$ is empty and Kh(T) consists of the other two collections alone. This condition prevails whenever T is a hexachord of Forte's "Z-type" (i.e., not a combinatorial hexachord) and for one exceptional case: the pentachord (61221) (Forte's 5–Z12), the single pentachord which is not contained in its complement. Forte remarks [p. 101] that there are no hexachords in this subcomplex, but does not mention the reason.

The notation of the lattice and the sublattice, besides its attractive generality and its obvious applicability to musical problems, benefits from a well-developed mathematical theory. In fact, it would be quite interesting to apply the techniques of Rota [1968] to this lattice. This would be a decidedly sophisticated mathematical problem, though, along with such things as the question of deriving formulas for the number of pairs of sets having a given partition function and the problems mentioned before with relation to the common-note function.

8. *Conclusion*

If we recommend a new concept to replace Forte's set-complex, something which he is obviously proud of and has spent an immense amount of time and energy working on, it is because the relation of the set-complex to musical reality seems somewhat debatable. The symmetry of the set-complex in terms of sets and their complements leads to an emphasis on hexachords, which stand in the middle. Forte actually codifies this emphasis in his "Rule 1" for determining which set-complex applies best to a given piece [p. 113]. He makes it clear, what is more, that his choice does not arise from any observation about musical structure, but from theoretical considerations. But it is certainly not proven that hexachords are specially important in atonal music, and Forte's own examples certainly do not go far toward such a proof.

His emphasis on complementation seems a bit overdone to begin with. It might result from an earlier theoretical emphasis on Schoenberg, for whom of all composers the constant circulation of the twelve pitch-classes is important as such. Forte reaches the extreme of asserting, for example, that in a certain passage a set of four notes is "represented by its complement" [p. 112]. Certainly such relations can be important, but not, one would think, to the extent that the most crucial part of the theory stands or falls by them! The recognition of more general inclusion relations might aid in remedying this problem.

As an illustration, consider the famous melody of the slow movement of Dvorak's Symphony "From the New World." Here a pentatonic melody appears above what we may take, for the sake of argument, to be a strictly diatonic harmonization. Now the pentatonic scale is included in the diatonic scale; it is also true that the pentatonic scale, as a pentachord, is the complement of the diatonic scale. The set-complex conception naturally and irrelevantly favors the latter interpretation, the more so since Forte has established no theoretical vocabulary to pinpoint the nature of the inclusion relationship involved. Worse yet, Rule 1 now requires the analysis to proceed in terms of set-complexes about hexachords, though the music clearly delineates a collection of 5 notes on the one hand and one of 7 notes on the other, and has nothing to do with hexachords. To the objection that the set-complex theory cannot apply since this piece is tonal, it must be answered that it *does* apply, and very straightforwardly too. But the results make no sense; and we are entitled to question whether, in a more complex piece, a less straightforward application of the theory could be clearly seen to make more sense than this. (It would be mischievous to suggest that the "New World" Symphony is atonal by Forte's definition since Forte's theory applies to it.)

Forte is faced with the dilemma of trying to expound a new theory at the same time as he tries to test it on some of the most refractory music in existence. It even seems to be a matter of principle with him not to discuss pieces which might make possible some more elementary demonstrations of the theory's applicability. Noteworthy, for example, is the complete absence of any serious discussion of Bartók's music, which would seem an ideal ground on which to show off the merits of an abstract theory of chords. Such early works of Boulez as the Second Piano Sonata show a use of "harmonic" motives to form complete twelve-tone sets in a way that seems immediately susceptible to treatment by Forte's theory.

In this case, the existence of a reasonably complex structure about which much is already known would provide an excellent frame of reference.

One gets the impression, in a way, that this involved and somewhat untidy theory ended up by building on itself, rather than on sufficiently general considerations of the repertoire it was originally intended to explicate. Nevertheless, one certainly senses a keen appreciation of this repertoire. The cogency of the extremely interesting analysis of the third of Schoenberg's Five Pieces for Orchestra, op. 16 [pp. 166ff.], for example, derives from its layered treatment of motivic elements and not from the set-complex analysis, which stands apart as a practically independent entity. It is too bad that this piece does not place the emphasis on complement relations which would tend to justify Forte's theoretical formulation.

The energy which Forte expends elaborating purely theoretical details seems not to leave him enough time for a really thorough consideration of the methodological context within which he is working. This is certainly unfortunate, since in this reader's opinion the lasting value of the work lies in the possibility of generalizing his techniques to a large-scale harmonic theory embracing tonal as well as atonal music. It is heartening to see from his other work, though it is not apparent within the scope of *The Structure of Atonal Music*, that he too is willing to leave this possibility open.

REFERENCES

Adorno, Theodor W. 1959. "Zu den Georgeliedern." Afterword to Arnold Schoenberg: *Fünfzehn Gedichte aus "Das Buch der Hängenden Gärten" von Stefan George,* Wiesbaden: Insel-Verlag, pp. 76–85.

Babbitt, Milton. 1955. "Some Aspects of Twelve-Tone Composition." *The Score and I.M.A. Magazine* 12, pp. 53–61.

Chrisman, Richard A. 1969. "A Theory of Axis-Tonality for Twentieth-Century Music." Ph.D. dissertation, Yale University.

Clough, John. 1965. "Pitch-Set Equivalence and Inclusion." *Journal of Music Theory* 9:1, pp. 163–171.

Cone, Edward T. 1965. "A Budding Grove." *Perspectives of New Music* 3:2, pp. 38–46.

Cone, Edward T. 1967. "Beyond Analysis." *Perspectives of New Music* 6:1, pp. 33–51.

Forte, Allen. 1964. "A Theory of Set-Complexes for Music." *Journal of Music Theory* 8:2, pp. 136–183.

Forte, Allen. 1973. *The Structure of Atonal Music.* New Haven: Yale University Press.

Howe, Jr., Hubert S. 1965. "Some Combinational Properties of Pitch Structures." *Perspectives of New Music* 4:1, pp. 45–61.

Lewin, David. 1959. "Intervallic Relations between Two Collections of Notes." *Journal of Music Theory* 3:2, pp. 298–301.

Martino, Donald. 1961. "The Source Set and its Aggregate Formations." *Journal of Music Theory* 5:2, pp. 225–273.

Perle, George. 1972. *Serial Composition and Atonality.* 3rd ed. Berkeley: University of California Press.

Regener, Eric. 1973. *Pitch Notation and Equal Temperament: A Formal Study.* Occasional Papers, no. 6: Music. Berkeley: University of California Press.

Rota, Gian-Carlo. 1964. "On the Foundations of Combinatorial Theory. I. Theory of Möbius Functions." *Zeitschrift für Wahrscheinlichkeitstheorie* 2, pp. 340–368.

Schoenberg, Arnold. 1950. *Style and Idea.* New York: Philosophical Library.

Schoenberg, Arnold. 1954. *Structural Functions of Harmony.* New York: W. W. Norton.

Vinogradov, I. M. 1954. *Elements of Number Theory.* New York: Dover Books.

Addendum

At the June 1975 meeting of the Canadian Association of University Schools of Music in Edmonton, Rudolf Komorous of the University of Victoria presented a paper on the harmonic classification scheme of Karel Janecek's *Foundations of Modern Harmony* (*Základy moderní harmonie,* Prague, 1965). In this work, Janecek enumerates all the 352 possible chords of zero to twelve notes, using an interval notation and normal form nearly identical to those defined here (except that the normal order gives smaller intervals precedence over larger ones). He goes on to classify these chords systematically on the basis of consonant and dissonant interval combinations and to develop a theory of tonal and "modern" harmony at the same time straightforward and highly complex. I am grateful to Prof. Komorous for showing me this fascinating book, which seems to answer many of the questions I raise here on the basis of work going back to the 1930's.