Maximally Even Sets
Author(s): John Clough and Jack Douthett
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MAXIMALLY EVEN SETS

John Clough and Jack Douthett

In their 1985 paper on diatonic theory, John Clough and Gerald Myerson give an algorithm for constructing the usual diatonic set (the white-key set) and other sets with the property "cardinality equals variety." They restrict the application of their algorithm to cases where cardinalities of the chromatic and "diatonic" sets are coprime (have no common divisor greater than one). When this restriction is put aside, the algorithm generates a larger world of sets—we call them *maximally even* (ME) sets—which are the subject of this paper.

Many familiar set classes in the 12-pc universe are maximally even: the naked tritone, augmented triad, diminished-seventh chord, whole-tone scale, anhemitonic pentatonic scale, diatonic scale, and octatonic scale. Note that this list includes all the set classes featuring equal division of the octave (except those with cardinalities 0, 1, 12) plus three with unequal division of the octave (cardinalities 5, 7, 8). In analytical discourse, much is made of *oppositions* and *contrasts* among these various set classes (e.g., in analysis of twentieth-century music involving juxtapositions and interactions of whole-tone, diatonic, and octatonic sets). By observing that they are all classes of ME sets, we shift the usual perspective and attend to *similarities* among them.

Previous appeals to the notion of evenness are not limited to descriptions of sets which divide the octave into equal parts. In the *New Grove Dictionary,* William Drabkin defines the diatonic set as one which divides the octave into five whole steps and two half steps with "maximal separation" between the half steps. It is also notable that all the scales treated by Joseph Yasser in his monumental study are reducible to ME sets. But, as far as we know, ME sets have not been previously recognized and studied as a class, either within the 12-note universe or in general.

Part 1 of the paper is devoted to matters of definition and construction, questions of existence and uniqueness, combinatorial properties, and interval content of ME sets. In part 2 we view the usual diatonic set as a member of a special family of ME sets found in chromatic universes of certain sizes, and show that the set of all such "diatonic" sets is the set of ME sets that have exactly one ambiguity. The third and final part deals with generated ME sets, complements, maximally even circles of intervals, and second-order ME sets.

1. Basic Features of ME Sets

Preliminary definitions

We assume that the reader is familiar with concepts of pitch class (pc), pitch-class set (pcset), and the integer model of pitch class, as discussed, for example, in Forte (1973) and John Rahn (1980), and we proceed immediately to definitions based on that model. In these definitions and elsewhere we use the terms *pitch class* and *note* interchangeably.

Definition 1.1. Given a chromatic universe of c pcs (a c-note chromatic universe), we say that c is the *chromatic cardinality*. We represent such a universe by the set $U_c = \{0, 1, 2, \ldots, c-1\}$, and we assume that the integers are assigned to the notes in ascending order.

Throughout the paper it may be convenient to think of chromatic universes based on equal divisions of the 2:1 octave, but this is not a necessary restriction. Any system may be placed in correspondence with our results as long as the intervals are suitably equivalenced.

Definitions 1.2. To indicate a subset of d pcs selected from the chromatic universe of c pcs we write $D_{c,d} = \{D_0, D_1, D_2, \ldots, D_{d-1}\}$. Thus $\{D_0, D_1, D_2, \ldots, D_{d-1}\}$ is a subset of $\{0,1,2, \ldots, c-1\}$. We generally assume $D_0 < D_1 < \ldots < D_{d-1}$. We say that d is the *diatonic cardinality* of $D_{c,d}$. We write $D(c,d)$ to represent the set of all sets with parameters c and d.

The symbol $D_{c,d}$ is the name of a variable whose assigned value is always a particular set with parameters c and d. Where it is necessary to speak of more than one set with parameters c, d in the same context, we also use the symbol $D^*_{c,d}$. Thus $D_{c,d}$ and $D^*_{c,d}$ are variables in $D(c,d)$.

Definitions 1.2 imply $d \leq c$. We assume always that c and d are positive integers; further restrictions on c and d will occasionally be noted.

Unless otherwise stipulated, we assume that the elements of any set of pcs are represented by smallest non-negative integers (mod c): $0, 1, \ , c-1$.

Example 1.1. Let $D_{12,7} = \{0, 2, 4, 5, 7, 9, 11\}$, with (pitch class) C=0. Here $D_{12,7}$ represents the represents the C major diatonic set. The symbol $D(12,7)$ represents the set of *all* seven-note subsets of the twelve-note chromatic universe.

Definition 1.3. Let $D_{c,d} = \{D_0, D_1, D_2, \ldots, D_{d-1}\}$. Then

$$\{j\} + D_{c,d} \stackrel{def}{=} \{j + D_0, j + D_1, j + D_2, \ldots, j + D_{d-1}\},$$

where the elements are reduced (mod c) unless otherwise indicated. We say that $D_{c,d}$ and $D^*_{c,d}$ are *equivalent under transposition* if there exists an integer j such that $D^*_{c,d} = \{j\} + D_{c,d}$.

The above merely generalizes the usual definition of pc transposition so that it applies in a chromatic universe of any cardinality.

We define the term *interval* for *ordered* pairs of pcs only. The notation (C, E♭) means "from C to E♭." If C = *0* C♯ =*1*, etc., we may write (0, 3) instead of (C, E♭). Variable names may also be used in this context: (D_0, D_1), etc. Within a $D_{c,d}$, intervals are measured by chromatic length, *clen*, and by diatonic length, *dlen*. The former is the number of ascending chromatic steps (semitones in the familiar case) from one pc to another; the latter is the number of ascending diatonic steps from one note to the other. Formal definitions follow:

Definition 1.4. Let $D_i, D_j \in D_{c,d}$. Then the *chromatic length* of the interval (D_i, D_j), written clen (D_i, D_j), is the smallest non-negative integer congruent to $D_j - D_i$ (mod c).

Definitions 1.5. Let $D_i, D_j \in D_{c,d}$. Then the *diatonic length* of the interval (D_i, D_j), written dlen (D_i, D_j), is the smallest non-negative integer congruent to $j - i$ (mod d). If dlen $(D_i, D_j) = 1$, the interval is called a *step*.

In this context, we use the terms *diatonic* and *step* broadly. A step is simply the distance from one note of a $D_{c,d}$ to the next higher note. The concepts of step and diatonic length are not restricted to sets conceived as scales, and apply quite generally to all sets. Thus, intervals of any clen may qualify as diatonic "steps."

Example 1.2. In the set {D, F, G, B}={2, 5, 7, 11}, there are four intervals of dlen 2, namely (D, G), (F, B), (G, D), and (B, F). These four intervals have clens 5, 6, 7, and 6, respectively.

Definition 1.6. The *spectrum* of a dlen is the set of clens corresponding to that particular dlen. We write $< I > = \{i_1, i_2, \ldots\}$ to indicate that the spectrum of dlen I is $\{i_1, i_2, \ldots\}$. Thus if S is a set of clens in a chromatic universe of cardinality c, then $< I > = S$ if and only if $D_{(N+I)} - D_N$ (mod c) is an element of S for all N, $0 \leq N \leq d - 1$, and for any s in S there exists an N, $0 \leq N \leq d - 1$, such that $D_{(N+I)} - D_N \equiv s \pmod{c}$, where subscripts are reduced (mod d).

Example 1.3. In the C major set, $<1> = \{1,2\}$, $<2> = \{3,4\}$, $<3> = \{5,6\}$, $<4> = \{6,7\}$, etc. Without ambiguity we may also write, for example, $<C, E> = <2>$ or $<C, E> = \{3,4\}$. Note that in any set $<0> = 0$. The fact that some dlens correspond to more than one clen is related to the notion of "quality" (i.e., major, minor, etc., as in the scalar intervals of traditional harmony).

ME set defined

The notion of a maximally even set is intuitively simple: it is a set whose elements are distributed as evenly as possible around the chromatic circle. The following physical metaphor may make this concept more tangible. Imagine 12 points located equidistantly on the circumference of a circle, as on a clock face, numbered consecutively, 0 through 11. Now imagine an orbit of 4 electrons. Place one electron at the 0 position and require each of the other electrons to occupy one of the 11 remaining positions in such a way that the charge equilibrium is disturbed as little as possible. Since 4 divides 12, there is only one solution: the electrons must occupy positions analogous to the notes of a diminished seventh chord: 0, 3, 6, and 9.

Now suppose there are 7 electrons instead of 4; and again, one of them occupies the 0 position. Since 7 does not divide 12, there is more than one solution; in fact there are 7, just as there are 7 usual diatonic sets that contain any particular pitch class. Or, if we drop the requirement that one of the electrons be in the 0 position, then there are 12 solutions. The electrons are not distributed evenly, but in view of our requirement of minimum disturbance to the charge equilibrium, we can say they are distributed as evenly as possible. It makes sense then to call the sets that represent these distributions *maximally even* sets.

Somewhat more formally, a ME set is a set in which every dlen comes in either one or two clens, as indicated by the following definition.

Definition 1.7. A set of pcs is *maximally even (ME)* if it has the following property: the spectrum of each dlen is either a single integer or two consecutive integers.

We write $M_{c,d}$ to represent a particular $D_{c,d}$ that is maximally even, and $M(c,d)$ to represent the set of all ME sets with parameters c and d. Thus the symbol $M_{c,d}$ is the name of a variable whose assigned value is always a ME set with parameters c and d. Where it is necessary to speak of more than one ME set with parameters c, d, we also use the symbol $M^{*}_{c,d}$. Thus $M_{c,d}$ and $M^{*}_{c,d}$ are variables in $M(c,d)$. Frequently we use the symbol $M_{c,d}$ with no specific assigned value, to mean "any member of $M(c,d)$."

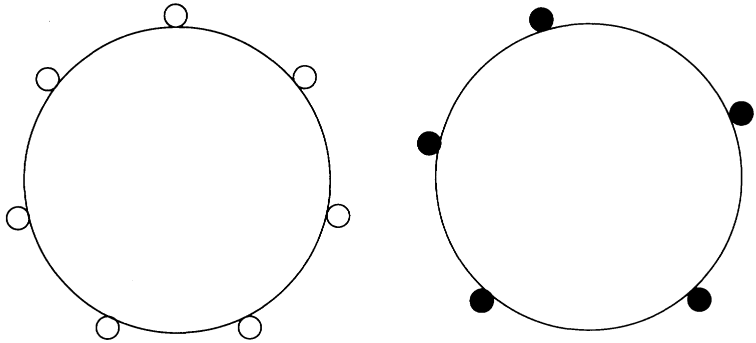
Example 1.4. In the octatonic set $D_{12,8} = \{0, 1, 3, 4, 6, 7, 9, 10\}$, we find the following interval spectrums: $<0> = \{0\}$, $<1> = \{1,2\}$, $<2> = \{3\}$, $<3> = \{4,5\}$, $<4> = \{6\}$, $<5> = \{7,8\}$, $<6> = \{9\}$, and $<7> = \{10,11\}$. Since each of these spectrums is a single integer or two consecutive integers, $D_{12,8}$ is a ME set, and we may write $M_{12,8} = \{0, 1, 3, 4, 6, 7, 9, 10\}$. (It follows easily that all octatonic sets are ME; later we will prove the general principle involved in this assertion.) The symbol $M(12,8)$ represents the set of all ME sets with $c = 12$, $d = 8$. As we will see, this is precisely the set of all octatonic sets.

Here is a method for constructing ME sets, which may provide a more concrete idea of what it means to distribute notes "as evenly as possible": First, choose values for c and d, say 12 and 7, respectively; locate d "white" points equidistantly around the circumference of a circle; and do the same for c−d (in this case 5) "black" points on another circle. These two steps are shown in fig. 1, a and b. Next superimpose one circle on the other so that no two points are in the same location, as in fig. 1c. Finally, assign dlen 1 to all adjacent pairs of white points, and clen 1 to all adjacent pairs of points regardless of color, and "tune" the system as desired—to equal temperament or whatever—as shown in fig. 1d. The white points now represent a ME set with parameters c and d, in this case $M_{12,7}$, and the black points represent a complementary ME set with parameters c and $c-d$, here $M_{12,5}$. (As we shall see, complements of ME sets are themselves maximally even.) A proof that the construction described here generates ME sets will be given in part 3 of the paper.

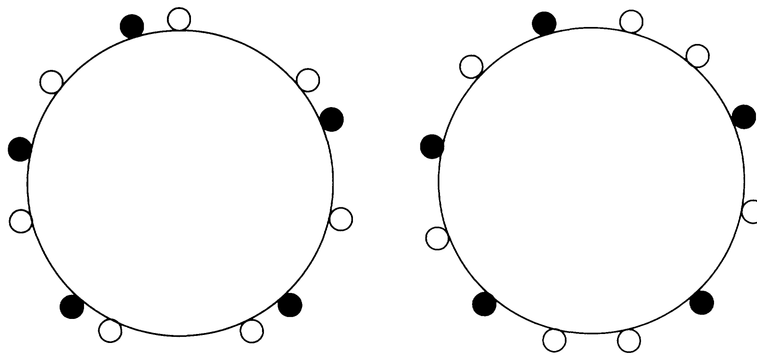
The greatest common divisor of c and d is notated "(c,d)." We establish three classes of ME sets based on (c,d): class A, where $(c,d) = d$; class B, where $1 < (c,d) < d$; and class C, where $(c,d) = 1$. Familiar sets appear as representatives of each class, as shown in figs. 2a, 2b, and 2c. For comparison, fig. 2d is a familiar set that is not maximally even: the set of the melodic minor ascending scale has *three* sizes of dlen 3 (the traditional 4th)—one too many for maximal evenness; that is because *its* half steps do *not* have maximal separation. The significance of this classification will become clear as we discuss the various properties of ME sets.

Existence and uniqueness of $M_{c,d}$ for any c,d

Existence. Although Clough and Myerson do not discuss ME sets as such, they essentially show their existence and uniqueness within



(a) 7-note universe (b) 5-note universe

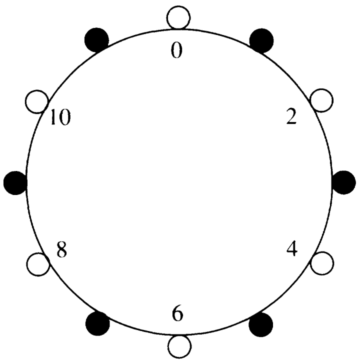


(c) 5- and 7-note universes merged (d) $M_{12,7}$

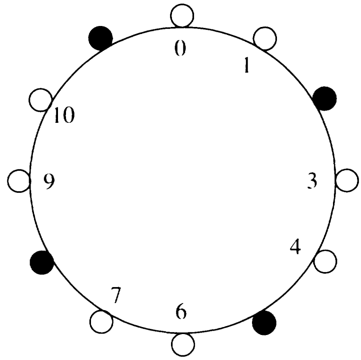
Figure 1. Construction of $M_{12,7}$ by fusion of 5- and 7-note "chromatic" universes

transposition, for any coprime c,d. We first restate without proof Clough and Myerson's existence result (1985, pp. 266–67, theorem 4) in a manner more suitable to our objectives, then work toward generalizing it to all ME sets. Then we follow the same strategy for their uniqueness result.

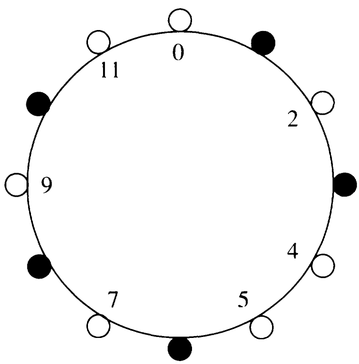
We employ the usual mathematical notation for truncation: $[x] =$ the largest integer not greater than x.



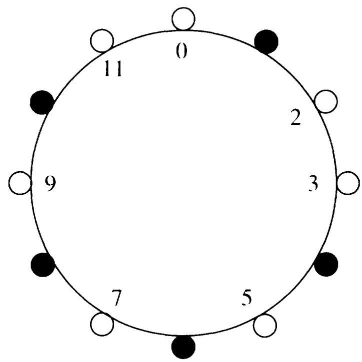
(a) whole-tone set: $c = 12$, $d = 6$;
$(c,d) = d$



(b) octatonic set: $c = 12$, $d = 8$;
$1 < (c,d) = 4 < d$



(c) diatonic set: $c = 12$, $d = 7$;
$(c,d) = 1$



(d) ascending melodic minor: not a ME set; dlen 3 (the 4th) corresponds to more than two clens (4,5,6)

Figure 2. Some familiar ME sets and a familiar non-ME set

Theorem 1.1. Let $(c,d) = 1$, and let

$$D_{c,d} = \left\{ \left\lfloor \frac{0c}{d} \right\rfloor, \left\lfloor \frac{1c}{d} \right\rfloor, \left\lfloor \frac{2c}{d} \right\rfloor, \ldots, \left\lfloor \frac{(d-1)c}{d} \right\rfloor \right\}.$$

Then all nonzero interval spectrums of $D_{c,d}$ are sets of two consecutive integers. Therefore $D_{c,d} \in M(c,d)$.

Ordinarily, the transpositions of a pcset are identified by the transposition operators. For example, the transpositions of $X = \{0, 1, 2\}$ might be labeled as follows: $X_0 = \{0, 1, 2\}$, $X_1 = \{1, 2, 3\}$, $X_2 = \{2, 3, 4\}$, etc. However, our work requires that the transpositions be labeled in "cyclic" order. We will set up the mechanisms for this now, and explain what we mean by "cyclic" as the exposition proceeds.

Definition 1.8. Let c, d, m, N be integers such that $0 \leq m \leq c-1$ and $0 \leq N < d \leq c$. Then the *J-function* with these parameters is expressed by

$$J^m_{c,d}(N) = \left\lceil \frac{cN + m}{d} \right\rceil.$$

Definition 1.9. Let *c, d, m* be as in the above definition. Then the *J-set* with these parameters is the set

$$J^m_{c,d} = \{J^m_{c,d}(0), J^m_{c,d}(1), \ldots, J^m_{c,d}(d-1)\}.$$

The Clough-Myerson algorithm for generating "diatonic" sets, quoted in theorem 1.1 above, is the basis for our J-set. $J^m_{c,d}(N)$ is the $(N+1)$th pc of the J-set $J^m_{c,d}$; that is, $J^m_{c,d}(0)$ is the first pc, $J^m_{c,d}(1)$ is the second pc, etc. The parameters *c* and *d* are the chromatic and diatonic cardinalities. The superscript *m*, called the *mode index*, is a label for a particular transposition; however, setting *m* equal to 0, 1, 2, . . . , etc. does not generate the transpositions of $J^m_{c,d}$ in the "expected" order. Instead it generates them in an order corresponding to their internal cyclic structure. For example,

$$J^0_{12,7} = \{0, 1, 3, 5, 6, 8, 10\} = \text{D}\flat \text{ major set } (\text{C} = 0),$$

$$J^1_{12,7} = \{0, 1, 3, 5, 7, 8, 10\} = \text{A}\flat \text{ major set},$$

$$J^2_{12,7} = \{0, 2, 3, 5, 7, 8, 10\} = \text{E}\flat \text{ major set},$$

etc.

The various transpositions of $J^m_{12,7}$ are generated in an order based on clen 7—the familiar circle of fifths. That is,

$$J^1_{12,7} = \{7\} + J^0_{12,7},$$

$$J^2_{12,7} = \{7\} + J^1_{12,7},$$

.

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$$J^0_{12,7} = \{7\} + J^{11}_{12,7}.$$

This order of generation corresponds to the internal structure of $J^m_{12,7}$, which is also based on clen 7. For example, $J^6_{12,7} = \{0 \cdot 7, 1 \cdot 7, 2 \cdot 7, \ldots, 6 \cdot 7\} \equiv \{0, 7, 2, 9, 4, 11, 6\} = \{0, 2, 4, 6, 7, 9, 11\}$ (mod 12).

For ME sets that do not have a "circle of fifths,"—those with $(c,d) \neq 1$—the situation is somewhat different, but still corresponds to a cyclic aspect of the set. We comment further on this point below in connection with the case $(c,d) \neq 1$, and we take up the question of generators in detail in part 3.

Note that $J^0_{12,7}$ is the "normal order" of the major scale set as defined by Forte (1973, 3–4). In general, the Clough-Myerson algorithm generates sets in normal order. We leave it to the reader to discover the reason for this.

The following lemma and corollaries pertain to the interval of I steps in a J-set—the interval from the $(N+1)$th to the $(N+I+1)$th pc—from $J^m_{c,d}(N)$ to $J^m_{c,d}(N+I)$, where the expressions in parentheses are appropriately reduced (mod d). This is the interval of dlen I.

Lemma 1.1. *Let* c,d,m,N,I be integers such that $0 \leq N < d \leq c$, and $1 \leq I \leq d-1$. Let $cN + m = dq_N + r_N$ and $cI = dq_I + r_I$, where q_N, q_I, r_N, and r_I are integers and $0 \leq r_N < d$ and $0 \leq r_I < d$. Then

$$J^m_{c,d}(N+I)_d - J^m_{c,d}(N) \equiv \left[\frac{cI}{d}\right] + \left[\frac{r_N + r_I}{d}\right] \pmod{c},$$

where the subscript d indicates that the expression $(N+I)$ is reduced (mod d).

Proof. Suppose $N + I < d$. Then

$$\begin{aligned} J^m_{c,d}(N+I)_d - J^m_{c,d}(N) &\equiv J^m_{c,d}(N+I) - J^m_{c,d}(N) \\ &\equiv \left[\frac{c(N+I)+m}{d}\right] \\ &\quad - \left[\frac{cN+m}{d}\right] \pmod{c}. \end{aligned}$$

Now suppose $N + I \geq d$. Then

$$\begin{aligned} J^m_{c,d}(N+I)_d - J^m_{c,d}(N) &\equiv J^m_{c,d}(N+I-d) - J^m_{c,d}(N) \\ &\equiv \left[\frac{c(N+I)+m}{d}\right] - \left[\frac{cN+m}{d}\right] - c \\ &\equiv \left[\frac{c(N+I)+m}{d}\right] \\ &\quad - \left[\frac{cN+m}{d}\right] \pmod{c}. \end{aligned}$$

In either case

$$J^m_{c,d}\,(N + I)_d - J^m_{c,d}(N) \equiv \left\lceil \frac{c(N + I) + m}{d} \right\rceil - \left\lceil \frac{cN + m}{d} \right\rceil$$

$$\equiv \left\lceil \frac{dq_N + r_N + dq_I + r_I}{d} \right\rceil - \left\lceil \frac{dq_N + r_N}{d} \right\rceil$$

$$\equiv \left\lceil \frac{dq_I + r_I}{d} \right\rceil + \left\lceil \frac{r_N + r_I}{d} \right\rceil$$

$$\equiv \left\lceil \frac{cI}{d} \right\rceil + \left\lceil \frac{r_N + r_I}{d} \right\rceil \pmod{c}. \bullet$$

Corollary 1.1.

$$J^m_{c,d}\,(N + I)_d - J^m_{c,d}\,(N) \equiv \left\lfloor \frac{cI}{d} \right\rfloor, \left\lceil \frac{cI}{d} \right\rceil + 1 \pmod{c}.$$

Proof. This follows easily since

$$\left\lceil \frac{r_N + r_I}{d} \right\rceil = 0,1. \bullet$$

Corollary 1.2.

$$clen(J^m_{c,d}(N), J^m_{c,d}\,(N + I)_d) = \left\lfloor \frac{cI}{d} \right\rfloor, \left\lfloor \frac{cI}{d} \right\rfloor + 1.$$

Proof. This follows from corollary 1.1 and definition 1.4. •

Corollary 1.2 tells us that for given I, c, and d, the choices of N and m can affect clen$(J^m_{c,d}\,(N), J^m_{c,d}(N + I)_d)$ by at most 1. We now generalize theorem 1.1, showing the existence of ME sets for any fixed c,d.

Theorem 1.2. For any integer m, $J^m_{c,d} \in M(c,d)$.

Proof. The fact that $J^m_{c,d}$ is maximally even follows easily from the definition of ME set and corollary 1.2. •

Uniqueness. The next problem we face is a more difficult one. Each of the ME sets pictured in Fig. 2 has a J-representation. For example, the octatonic set $\{0, 1, 3, 4, 6, 7, 9, 10\} = J^m_{12,8}$, where $m = 0, 1, 2$, or 3. Is this true in general? For any $M_{c,d}$, does there exist an m, $0 \le m < c$, such that $M_{c,d} = J^m_{c,d}$? Intuitively, it seems there should be essentially one way to choose d pcs from a universe of c pcs in order to satisfy maximal evenness. Indeed that is the case, but we must look closely at the inherent properties of ME sets to prove it.

As we shall see, a ME set with $(c,d) \neq 1$ is "generated" by a smaller ME set with $(c',d') = 1$. It makes sense then to focus our attention first on the case $(c,d) = 1$. We will show that, for fixed c and d, such ME sets are equivalent under transposition. This will allow us to show that J-representations exist for all such ME sets.

The next lemma is proved in Clough and Myerson.

Lemma 1.2. For any $D_{c,d}$ and I, $1 \leq I \leq d-1$, the sum of the clens of the intervals of dlen I is cI.

Lemma 1.3. For any c,d with $(c,d) = 1$, $D_{c,d}$ has no nonzero single-element spectrum.

Proof. Let $(c,d) = 1$ and $D_{c,d} = \{D_0, D_1, \ldots, D_{d-1}\}$. Suppose $D_{c,d}$ has a nonzero single-element spectrum $< I > = \{k\}$. Since there are d intervals with clen k, the sum of the clens of the intervals is dk. But this sum is also cI (lemma 1.2). Thus $dk = cI$. But then d must divide I since $(c,d) = 1$. This cannot be since $1 \leq I \leq d-1$. We conclude that there is no nonzero single-element spectrum.●

Lemmas 1.4 and 1.5 below are proved in Clough and Myerson. The second of these is their uniqueness result (1985, p. 266, theorem 3), and the theorem following is our extension of that result. Each of these lemmas and theorems is preceded by necessary definitions borrowed from their work.

Definition 1.10. Let $D_{c,d} = \{D_0, D_1, \ldots D_{d-1}\}$; let X be a sequence of pcs, $D_{n_1}, D_{n_2}, \ldots, D_{n_k}$, where each term of the sequence is an element of $D_{c,d}$; and let I be a dlen, $0 \leq I \leq d-1$. Then we say that the sequence $D_{n_1+I}, D_{n_2+I}, \ldots, D_{n_k+I}$ (subscripts reduced mod d) is a *diatonic transposition* of X by dlen I.

To indicate the cardinality of a set S (i.e., the number of distinct elements in S), we write $|S|$.

Definition 1.11. For any subset S of $D_{c,d}$, and any sequence of pcs including each element of S at least once, if the d distinct diatonic transpositions of the sequence (by dlens $0,1, \ldots, d-1$) form $|S|$ distinct sequences of clens, then $D_{c,d}$ is said to have property *cardinality equals variety (CV).*

In the following definition, we adopt Clough and Myerson's term honoring John Myhill, the late, eminent logician to whom their work owes much.

Definition 1.12. If all interval spectrums of $D_{c,d}$ are sets of two integers, except $< 0 > = \{0\}$, we say that $D_{c,d}$ has *Myhill's property (MP).*

Lemma 1.4. Myhill's property implies CV.

Definition 1.13. If $< 1 >$ is a set of two consecutive integers, then we say that $D_{c,d}$ is *rounded.*

Lemma 1.5. Let $(c,d) = 1$ and $D_{c,d}$ and $D^*_{c,d}$ be rounded sets with CV. Then $D_{c,d}$ and $D^*_{c,d}$ are equivalent under transposition.

Definition 1.14. If each interval of $D_{c,d}$ has a spectrum consisting of consecutive integers, we say that $D_{c,d}$ has the *consecutivity property (CP).*

Theorem 1.3. For any fixed c and d, $(c,d) = 1$, all $M_{c,d}$ are equivalent under transposition.

Proof. Since $(c,d) = 1$, there are no single-element spectrums except $<0> = \{0\}$ (lemma 1.3). Thus all $M_{c,d}$ must have both CP and MP. It follows that they must be rounded. Since these sets have MP, they must have CV (lemma 1.4). We have now established that (for c, d as stated) all $M_{c,d}$ are rounded and have CV. It follows that they are equivalent under transposition (lemma 1.5).●

We now prove one more lemma that will enable us to show that all ME sets with $(c,d) = 1$ have J-representations.

Lemma 1.6. Let $1 \le d \le c$, $0 \le m \le c - 1$, and j be any non-negative integer. Then

$$J_{c,d}^{(m+jd)_c} = \{j\} + J_{c,d}^{m},$$

where the subscript c following the parenthesized expression indicates reduction (mod c).

Proof. Let $j = 1$ and suppose $m + d < c$. Then

$$\begin{aligned} J_{c,d}^{(m+d)_c}(N) &= J_{c,d}^{m+d}(N) \\ &\equiv \left\lceil \frac{cN+(m+d)}{d} \right\rceil \\ &\equiv 1+\left\lceil \frac{cN+m}{d} \right\rceil \\ &\equiv 1+J_{c,d}^{m}(N) (\text{mod } c). \end{aligned}$$

It follows that

$$J_{c,d}^{(m+d)_c} = \{1\}+J_{c,d}^{m}.$$

Now suppose $m + d \ge c$ and $1 \le N \le d - 1$. Then

$$\begin{aligned} J_{c,d}^{(m+d)_c}(N) &= J_{c,d}^{m+d-c}(N) \\ &\equiv 1+\left\lceil \frac{c(N-1)+m}{d} \right\rceil \\ &\equiv 1 + J_{c,d}^{m}(N-1) (\text{mod } c). \end{aligned}$$

If $N = 0$, then

$$\begin{aligned} J_{c,d}^{(m+d)_c}(0) &\equiv \left\lceil \frac{m+d-c}{d} \right\rceil \\ &\equiv 1+\left\lceil \frac{m-c}{d} \right\rceil \\ &\equiv 1+\left\lceil \frac{m-c}{d} \right\rceil + c \end{aligned}$$

$$\equiv 1 + \left[\frac{c(d-1)+m}{d}\right]$$

$$\equiv 1 + J^{m}_{c,d}\ (d-1)\ (\text{mod } c).$$

Thus

$$J^{(m+d)_c}_{c,d} = \{1\} + J^{m}_{c,d}.$$

Then by induction we see that

$$J^{(m+jd)_c}_{c,d} = \{j\} + J^{m}_{c,d}. \bullet$$

Theorem 1.4. For any fixed c and d, $(c,d) = 1$, and any $M_{c,d}$, there exists an m, $0 \leq m \leq c-1$, such that $M_{c,d} = J^{m}_{c,d}$.

Proof. We know that for any fixed c and d, $(c,d) = 1$, all $M_{c,d}$ are equivalent under transposition (theorem 1.3). Also $J^{0}_{c,d}$ is a ME set (theorem 1.2). Thus $M_{c,d} = \{j\} + J^{0}_{c,d}$ for some integer j. It follows that $M_{c,d} = J^{m}_{c,d}$ where $m \equiv jd$ (mod c) (lemma 1.6). Further, since m is reduced (mod c), we have $0 \leq m \leq c-1$. ●

Now we are ready to relax the condition $(c,d) = 1$. We will show that $M_{c,d}$ with $(c,d) = k \neq 1$ is "generated" by $M_{c',d'}$ with $(c',d') = 1$. To get a preliminary idea of what "generated" means in this case, see fig. 3, which shows how the ME set $J^{7}_{15,9}$ is generated by the smaller ME set $J^{2}_{5,3}$.

To see how this structure is reflected in the definition of the J-set, consider $J^{m}_{15,9}$, as m is set to 0,1,2, etc.:

$$J^{m}_{15,9} = \{0, 1, 3, 5, 6, 8, 10, 11, 13\},\ m = 0, 1, 2;$$

$$J^{m}_{15,9} = \{0, 2, 3, 5, 7, 8, 10, 12, 13\},\ m = 3, 4, 5;$$

$$J^{m}_{15,9} = \{0, 2, 4, 5, 7, 9, 10, 12, 14\},\ m = 6, 7, 8;$$

$M_{15,9}$ has "no circle of fifths"; that is, it cannot be generated by a single clen. However $M_{5,3}$ may be generated by clen 2 or 3 (e.g., $\{0\cdot 2, 1\cdot 2, 2\cdot 2\} = \{0, 2, 4\}$, an element of $M(5,3)$). $J^{m}_{15,9}$ is "generated" by transpositions of $J^{m'}_{5,3}$ for some fixed mode index m': $J^{m}_{15,9}$ is generated by $J^{0}_{5,3}$ for $m = 0, 1, 2$; by $J^{1}_{5,3}$ for $m = 3, 4, 5$; etc. (The reader may verify that this is the case by completing the table of sets for $J^{m}_{15,9}$ above.) It is in this sense that the J-set captures the cyclic structure of ME sets when $(c, d) \neq 1$.

We will soon generalize theorems 1.3 and 1.4, but we must develop some more tools first.

Lemma 1.7. Suppose $(c, d) = k \neq 1$ for some $M_{c,d}$. Then $< jd/k > = \{jc/k\}$ for all j, $1 \leq j \leq k-1$.

Proof. Assume $< jd/k > \neq \{jc/k\}$. The sum of the clens of the intervals of dlen jd/k is $c(jd/k)$ (lemma 1.2). But $c(jd/k) = d(jc/k)$. Since

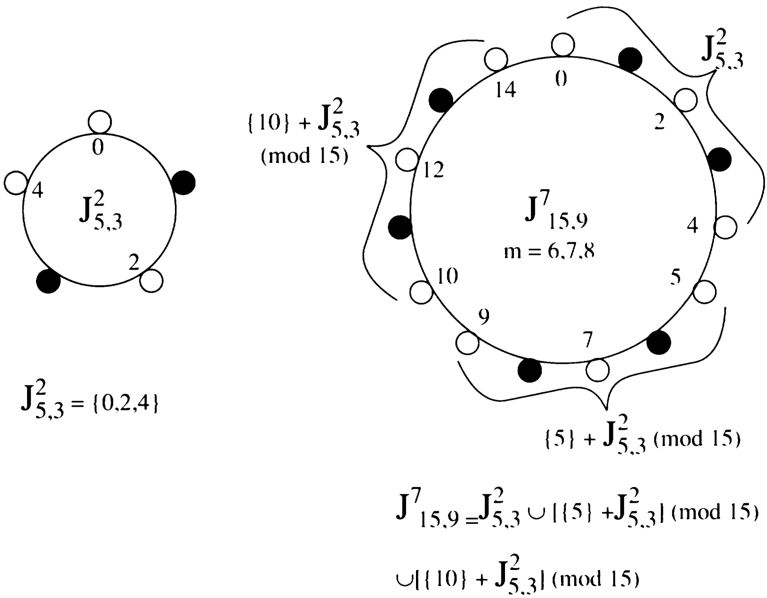


Figure 3. $J^m_{15,9}$, $m = 6, 7, 8$, as generated by $J^2_{5,3}$

there are d intervals of dlen jd/k, if one interval has a clen less than (or greater than) jc/k there must be at least one interval whose clen is greater than (or less than) jc/k, contradicting the assumption that $M_{c,d}$ is a ME set. Thus $< jd/k > \; = \{jc/k\}$. ●

Lemma 1.8. Let $M_{c,d} = \{D_0, D_1, \ldots, D_{d-1}\}$ and suppose $(c, d) = k$. Let $c' = c/k$ and $d' = d/k$, $M_{c',d'} = c' = c/k$ and $d' = d/k$, $Mc',d' = \{D_0, D_1, \ldots, D_{d'-1}\}$, *and* $M_j = M_{c,d} = \cup_{j=0}^{k-1} M_j$.

Proof. We know that $< jd/k > \; = \{jc/k\}$ (lemma 1.7). Thus $D_{jd'} = jc' + D_0$, $D_{1+jd'} = jc' + D_1, \ldots, D_{d'-1+jd'} = jc' + D_{d'-1}$, where $0 \leq j \leq k-1$. It follows that $M_{c,d} = \{D_0, D_1, \ldots, D_{d'-1}, c' + D_0, \ldots, c' + D_{d'-1}, \ldots\ldots, (k-1)c' + D_0, \ldots, (k-1)c' + D_{d'-1}\} = \cup_{j=0}^{k-1} M_j$. ●

We see that $M_{c,d}$ is completely determined by $M_{c',d'}$. The notation $M_{c',d'}$ suggests that $M_{c',d'}$ is a ME set with chromatic and diatonic cardinalities c' and d'. This is fact the case, but we must enter one more lemma before we prove it.

Lemma 1.9. For any $D_{c,d}$ and any associated dlen I, if $<I> = \{k_I\}$ or $\{k_I, k_I + 1\}$, then

$$k_I = \left[\frac{cI}{d}\right].$$

Proof. Suppose there are h intervals of dlen I such that $D_{(N+I)_d} - D_N \equiv k_1 (\text{mod } c), 0 \leq N \leq d-1$. Then there are $d-h$ intervals of dlen I such that $D_{(N+I)_d} - D_N \equiv k_I + 1 \, (\text{mod } c)$. Thus the sum of the clens of these intervals is $hk_I + (d-h)(k_I + 1)$. But this sum is also cI (lemma 1.2). Thus

$$hk_I + (d-h)(k_I + 1) = cI$$

$$k_I = \frac{cI}{d} - \frac{d-h}{d}$$

$$= \left[\frac{cI}{d}\right],$$

since $\frac{cI}{d} - \frac{d-h}{d}$ is an integer and $0 \leq \frac{d-h}{d} < 1$. ●

Lemma 1.10. Let $M_{c,d} = \{D_0, D_1, \ldots, D_{d-1}\}$ and suppose $c' = c/(c,d)$, $d' = d/(c,d)$ and $D_{c',d'} = \{D_0, D_1, \ldots, D_{d'-1}\}$. Then $D_{c',d'}$ is an element of $M(c',d')$.

Proof. We must show

$$D_{(N+I)_{d'}} - D_N \equiv \left[\frac{c'I}{d'}\right], \left[\frac{c'I}{d'}\right] + 1 \ (\text{mod } c') \text{ for all } 0 \leq N \leq d'-1$$

and $1 \leq I \leq d'-1$. First note that since d' divides d there is a non-negative integer n such that $(N+I)_{d'} = (N+I-nd')_d$. (The sum on the left is reduced (mod d'); the sum on the right is reduced (mod d).) Then

$$D_{(N+I)d'} - D_N \equiv D_{(N+I-nd')_d} - D_N$$

$$\equiv \left[\frac{c'(I-nd')}{d'}\right], \left[\frac{c'(I-nd')}{d'}\right] + 1 \text{ (lemma 1.9)}$$

$$\equiv \left[\frac{c'I}{d'}\right] - nc', \left[\frac{c'I}{d'}\right] - nc' + 1 \ (\text{mod } c).$$

But since c' divides c we have

$$D_{(N+I)_{d'}} - D_N \equiv \left[\frac{c'I}{d'}\right], \left[\frac{c'I}{d'}\right] + 1 \ (\text{mod } c').$$

It follows that $D_{c',d'}$ is a ME set. ●

We are now in a position to prove the generalizations of theorems 1.3 and 1.4.

Theorem 1.5. For any $M_{c,d}$ there exists an m, $0 \le m \le c - 1$, such that $M_{c,d} = J^{m}_{c,d}$.

Proof. Let $(c,d) = k$, $c' = c/k$, and $d' = d/k$. Then $M_{c',d'}$ as in lemma 1.10 is a ME set. Since $(c',d') = 1$ there exists an m', $0 \le m' \le c' - 1$ such that $M_{c',d'} = J^{m'}_{c',d'}$ (theorem 1.4). It follows that $M_{c,d} = \cup_{j=0}^{k-1} M_j$ where $M_j = \{jc'\} + J^{m'}_{c',d'}$ (lemma 1.8). Now let $m = km'$. Clearly $0 \le m \le c - 1$. Also $J^{m}_{c,d}(N + d'j) = jc' + J^{m'}_{c',d'}(N)$ when $0 \le j \le k - 1$ and $0 \le N \le d' - 1$. Thus

$$\begin{aligned} J^{m}_{c,d} &= \cup_{j=0}^{k-1} (\{jc'\} + J^{m'}_{c',d'}) \\ &= \cup_{j=0}^{k-1} M_j \\ &= M_{c,d}. \bullet \end{aligned}$$

Theorem 1.6. If $D_{c,d} = \{j\} + M_{c,d}$, then $D_{c,d} \in M(c,d)$; that is, any transposition of a ME set is ME set.

Proof. If $D_{c,d}$ is a transposition of a ME set, then

$$\begin{aligned} D_{c,d} &= \{j\} + M_{c,d} \\ &= \{j\} + J^{m}_{c,d} \end{aligned}$$

for some integer m, $0 \le m \le c - 1$ (theorem 1.5). Then

$$D_{c,d} = J^{(m+jd)_c}_{c,d} \text{ (lemma 1.6).}$$

By theorem 1.2, it follows that $D_{c,d} \in M(c,d)$. $\bullet$

Theorem 1.7. For any fixed c and d, all $M_{c,d}$ are equivalent under transposition.

Proof. Let j be as in lemma 1.8. Let $M_{c,d}$ and $M^{*}_{c,d}$ be two ME sets and suppose $(c,d) = k$. Let $c' = c/k$ and $d' = d/k$. Then

$$M_{c,d} = \cup_{j=0}^{k-1} M_j$$

and

$$M^{*}_{c,d} = \cup_{j=0}^{k-1} M^{*}_j$$

where $M_j = \{jc'\} + J^{m'}_{c',1d'} \pmod{c}$ and $M^{*}_j = \{jc'\} + J^{m'}_{c',2d'} \pmod{c}$ (lemmas 1.8 and 1.10, and theorem 1.4).

Therefore

$$\begin{aligned} M^{*}_j &= \{jc'\} + J^{m'}_{c',2d'} \pmod{c} \\ &= \{i\} + \{jc'\} + J^{m'}_{c',1d'} \pmod{c} \\ &\quad \text{for some integer } i \text{ (theorem 1.3)} \end{aligned}$$

$$= \{i\} + M_j .$$

It follows that

$$M^*_{c,d} = \cup_{j=0}^{k-1} M^*_j$$
$$= \cup_{j=0}^{k-1}(\{i\} + M_j)$$
$$= \{i\} + \cup_{j=0}^{k-1} M_j$$
$$= \{i\} + M_{c,d} .$$

Thus $M_{c,d}$ and $M^*_{c,d}$ are equivalent under transposition. ●

Theorems 1.6 and 1.7 show that for any choice of c and d, there is essentially *one* $M_{c,d}$, in the following sense: all ME sets with the same c and d are related by transposition, and conversely any transposition of a particular $M_{c,d}$ yields another (possibly identical) ME set with the same c and d. That is to say, the ME property is invariant under transposition. But what is the effect of *inversion* on ME sets? Before answering this question we give the usual formal definition for inversion of a set of pcs.

Definition 1.15. $D_{c,d}$ and $D^*_{c,d}$ are related by *inversion* if there exists an integer j such that $D_{c,d} = \{j\} - D^*_{c,d}$.

The next theorem shows that ME sets are inversionally symmetrical: the inversion of a ME set is a ME set.

Theorem 1.8. For fixed c, d, if $D_{c,d}$ and $M_{c,d}$ are related by inversion, then $D_{c,d} \in M(c,d)$.

Proof. If $D_{c,d}$ and $M_{c,d}$ are related by inversion, then by definition there exists an integer j such that $D_{c,d} = \{j\} - M_{c,d}$. Since $M_{c,d}$ is ME there exists an integer m, $0 \le m \le c - 1$ such that $M_{c,d} = J^m_{c,d}$ (theorem 1.5). Then

$$D_{c,d} = \{j\} - J^m_{c,d}$$
$$= \{j\} + \left\{c - \left\lfloor\frac{0c+m}{d}\right\rfloor, c - \left\lfloor\frac{1c+m}{d}\right\rfloor, c - \left\lfloor\frac{2c+m}{d}\right\rfloor, \ldots, c - \left\lfloor\frac{(d-1)c+m}{d}\right\rfloor\right\}$$
$$= \{j\} + \left\{\left\lceil\frac{cd-m}{d}\right\rceil, \left\lceil\frac{c(d-1)-m}{d}\right\rceil, \ldots,\right.$$

$$\left[\frac{c\cdot 2-m}{d}\right], \left[\frac{c\cdot 1-m}{d}\right]\Bigg\}$$

$$= \{j\}$$

$$+ \left\{\left[\frac{c\cdot 0-m}{d}\right], \left[\frac{c\cdot 1-m}{d}\right], \ldots,\right.$$

$$\left.\left[\frac{c(d-1)-m}{d}\right]\right\}$$

$$\text{(i.e., } \left[\frac{cd-m}{d}\right] \equiv \left[\frac{c\cdot 0-m}{d}\right] \pmod{c}\text{)}$$

$$= \{j\} + J_{c,d}^{(-m)_c}$$

$$= J_{c,d}^{(-m+jd)_c} \text{ (lemma 1.6).}$$

Thus $D_{c,d} \in M(c,d)$ (theorem 1.2). ●

Combinatorial properties and symmetry

We now return to the counting problem stated informally above in connection with the electron analogy. For a particular choice of c and d, how many distinct ME pcsets are there, and how many of them include a particular pc? We will show that the answer to the first question depends upon c and (c,d), and the answer to the second question depends upon d and (c,d).

As previously noted, we write $|S|$ to represent the cardinality of S. Thus $|M(c,d)|$ means "the number of distinct pcsets in $M(c,d)$."

It is clear from theorem 1.7 that for any choice of c and d, $M_{c,d}$ is unique within transposition. And theorem 1.8 tells us that inversions of ME sets are not distinct from transpositions. Thus we need consider only transpositions. We will first demonstrate that $\frac{c}{(c,d)}$ is an upper bound for $|M(c,d)|$, and then show that it is also a lower bound, the conclusion being obvious.

Since any $M_{c,d}$ has a J-representation, $J_{c,d}^m$, where $0 \leq m \leq c-1$ (theorem 1.5), we need only consider the set of J-sets $\{J_{c,d}^0, J_{c,d}^1, \ldots, J_{c,d}^{c-1}\}$. Any J-set with parameters c and d must be in this set. Hence theorem 1.5 establishes c as an upper bound for the number of distinct $M_{c,d}$. We now proceed to further restrict this upper bound.

Lemma 1.11. Let $0 \leq j \leq \frac{c}{(c,d)} - 1$. Then

$$J_{c,d}^{j(c,d)} = J_{c,d}^{j(c,d)+1} = \ldots = J_{c,d}^{j(c,d)+(c,d)-1}.$$

Proof. We need to show that corresponding elements of these sets are equal. Choose any N, $0 \le N \le d-1$, and i, $0 \le i \le (c,d)-1$. Then

$$J_{c,d}^{j(c,d)+i}(N) = \left\lceil \frac{cN+j(c,d)+i}{d} \right\rceil .$$

We have the following chain of inequalities:

$$\frac{cN+j(c,d)}{d} < \frac{cN+j(c,d)+1}{d} < \ldots < \frac{cN+(j+1)(c,d)-1}{d} .$$

It follows that corresponding elements have the following relationship:

$$\left\lceil \frac{cN+j(c,d)}{d} \right\rceil \le \left\lceil \frac{cN+j(c,d)+1}{d} \right\rceil \le \ldots \le \left\lceil \frac{cN+(j+1)(c,d)-1}{d} \right\rceil .$$

Now we will show that the first and last members in this chain of inequalities are equal.
Let

$$c' = \frac{c}{(c,d)} \text{ and } d' = \frac{d}{(c,d)} .$$

Thus

$$\frac{cN+j(c,d)}{d} = \frac{c'N+j}{d'} .$$

Let $c'N + j = d'q + r$, where $0 \le r \le d'-1$.
Then

$$\left\lceil \frac{c'N+j}{d'} \right\rceil = \left\lceil \frac{d'q+r}{d'} \right\rceil = q .$$

Also

$$\begin{aligned} \left\lceil \frac{cN+(j+1)(c,d)-1}{d} \right\rceil &= \left\lceil \frac{c'N+j+((c,d)-1)/(c,d)}{d'} \right\rceil \\ &= \left\lceil \frac{d'q+r+((c,d)-1)/(c,d)}{d'} \right\rceil \\ &= q + \left\lceil \frac{r+((c,d)-1)/(c,d)}{d'} \right\rceil . \end{aligned}$$

But $0 \le r \le d'-1$, and $0 \le \dfrac{(c,d)-1}{(c,d)} < 1$. Adding these inequalities we get

$$0 \le r + \frac{(c,d)-1}{(c,d)} < d' .$$

Thus

$$0 \le \frac{r+((c,d)-1)/(c,d)}{d'} < 1 .$$

It follows that

$$q + \left[\frac{r + ((c,d) - 1)/(c,d)}{d'}\right] = q + 0 = q\,.$$

Hence $J_{c,d}^{j(c,d)+i}(N) = q$ for all i, $0 \le i \le (c,d) - 1$. Since N was chosen arbitrarily, we conclude that

$$J_{c,d}^{j(c,d)} = J_{c,d}^{j(c,d)+1} = \ .\ .\ . = J_{c,d}^{j(c,d)+(c,d)-1}\,.\bullet$$

Corollary 1.3. $|\, M(c,d)\, | \le \frac{c}{(c,d)}$.

Proof. Let j be as in lemma 1.11. Then, for each j, $0 \le j \le \frac{c}{(c,d)} - 1$, there are (c,d) equivalent J-sets, each with a distinct mode index m, $0 \le m \le c - 1$. Further, for i, $0 \le i \le (c,d) - 1$, we have

$$\{J_{c,d}^{j(c,d)+i}\}_{i,j} = \{J_{c,d}^{0}, J_{c,d}^{1}, \ .\ .\ .\ .\ , J_{c,d}^{c-1}\}$$

with no duplication of superscripts modulo c in either set. Thus all distinct ME sets are represented in both sets (theorem 1.5). But for a fixed j_0, $0 \le j_0 \le \frac{c}{(c,d)} - 1$, the sets $J_{c,d}^{j_0(c,d)+i}$ are equivalent for all i, $0 \le i \le (c,d) - 1$ (lemma 1.11). Since j_0 can take any value between 0 and $\frac{c}{(c,d)} - 1$, inclusive, there can be at most $\frac{c}{(c,d)}$ distinct pcsets in $M(c,d)$. $\bullet$

The next three lemmas and corollary establish $\frac{c}{(c,d)}$ as a lower bound for $|\, M(c,d)\, |$. Lemma 1.12, from elementary number theory, is given without proof.

Lemma 1.12. Suppose c, d, and m are positive integers and $(c,d) = 1$. Then there exists a positive integer q and an integer N, $0 \le N < d$, such that $cN - dq = -m$.

Lemma 1.13. Let $J_{c,d}^{m} = \{D_0, D_1, \ .\ .\ .\ , D_{d-1}\}$ be a J-set such that $0 \le m \le c - 1$. Then $0 \le D_0 < D_1 < \ .\ .\ . < D_{d-1} \le c - 1$.

Proof. Clearly $0 \le D_0 < D_1 < \ .\ .\ . < D_{d-1}$. Thus we need only show $D_{d-1} \le c - 1$.

$$\begin{aligned} D_{d-1} &= J_{c,d}^{m}(d-1) \\ &= \left[\frac{c(d-1)+m}{d}\right] \\ &\le \left[\frac{c(d-1)+(c-1)}{d}\right] \\ &= c + \left[\frac{-1}{d}\right] \\ &= c - 1.\bullet \end{aligned}$$

Lemma 1.14. Let m_1 and m_2 be integers such that $0 \leq m_1 < m_2 \leq \frac{c}{(c,d)} - 1$. Then $J_{c,d}^{m_1(c,d)} \neq J_{c,d}^{m_2(c,d)}$.

Proof. Let $c' = \frac{c}{(c,d)}$ and $d' = \frac{d}{(c,d)}$. Then $0 < m_2 - m_1 \leq c' - 1$,

$$J_{c,d}^{m_1(c,d)} = \left\{\left[\frac{m_1}{d'}\right], \left[\frac{c'+m_1}{d'}\right], \left[\frac{2c'+m_1}{d'}\right], \ldots, \left[\frac{(d-1)c'+m_1}{d'}\right]\right\},$$

and

$$J_{c,d}^{m_2(c,d)} = \left\{\left[\frac{m_2}{d'}\right], \left[\frac{c'+m_2}{d'}\right], \left[\frac{2c'+m_2}{d'}\right], \ldots, \left[\frac{(d-1)c'+m_2}{d'}\right]\right\},$$

where the elements of the sets are arranged in order of magnitude (mod c) (lemma 1.13). If any of the corresponding elements differ, then the sets cannot be equal. Now there exist integers N and q, $0 \leq N < d'$, $q \geq 0$, such that $c'N - d'(q + 1) = -m_2$ (lemma 1.12). Thus $c'N = d'q + (d' - m_2)$. Then

$$\begin{aligned}
\left[\frac{c'N+m_2}{d'}\right] &= \left[\frac{d'q+(d'-m_2)+m_2}{d'}\right] \\
&= q + 1 \\
&\geq \left[q + \frac{d'-m_2+m_1}{d'}\right] + 1 \text{ (i.e., } d'-m_2 + m_1 < d') \\
&= \left[\frac{d'q+(d'-m_2)+m_1}{d'}\right] + 1 \\
&= \left[\frac{c'N+m_1}{d'}\right] + 1.
\end{aligned}$$

Thus $J_{c,d}^{m_2(c,d)}(N) \geq J_{c,d}^{m_1(c,d)}(N) + 1$, and we have found two corresponding elements that differ by at least 1. It follows that $J_{c,d}^{m_1(c,d)} \neq J_{c,d}^{m_2(c,d)}$. ●

Corollary 1.4. $|M(c,d)| \geq \frac{c}{(c,d)}$.

Proof. Lemma 1.14 implies that for $0 \leq i,j \leq \frac{c}{(c,d)} - 1$, $J_{c,d}^{i(c,d)} = J_{c,d}^{j(c,d)}$ if and only if $i = j$. Hence there must be at least $\frac{c}{(c,d)}$ distinct pcsets in $M(c,d)$. ●

Theorem 1.9. For fixed c,d, $|M(c,d)| = \frac{c}{(c,d)}$, and

$$M(c,d) = \{J_{c,d}^{0}, J_{c,d}^{(c,d)}, J_{c,d}^{2(c,d)}, \ldots, J_{c,d}^{c-(c,d)}\}.$$

Proof. The fact that there are $\frac{c}{(c,d)}$ distinct pcsets in $M(c,d)$ follows from corollaries 1.3 and 1.4. Their J-representation follows from lemma 1.14. •

Counting problems like the above will be familiar to students of atonal set theory, where the distinct pcsets generated by a specified group of operations are often called the "forms" of a pcset. Thus, if the group of operations includes (i) transposition and (ii) inversion about a fixed axis followed by transposition, there are three forms of the octatonic set. If we restrict the group of operations to (i) above, there are still three forms, since inversions of this particular set are not distinct from transpositions. Following John Rahn (1980, 90–91), we call the number of distinct operations (within the specified group) that map a set into itself the "degree of symmetry." Dividing the degree of symmetry of a set into the number of operations in the group then yields the number of distinct pcsets corresponding to the set. Since ME sets are inversionally symmetrical, it makes no difference whether we choose (i), or (i) and (ii) above as the group of operations; the number of distinct pcsets corresponding to a particular $M_{c,d}$ is the same in either case.

We now turn to the second question raised above. Let us first restate it in more precise terms: Suppose we have before us all of the ME sets for some given c and d. We know from theorem 1.9 that there are $\frac{c}{(c,d)}$ such sets. Now we select a particular pc and ask, how many of these sets include the selected pc? Theorem 1.10 provides the answer, but first we need to have some terminology to discuss "pcsets in $M(c,d)$ which include p." We call the selected pc a "tonic," and we make the following definition:

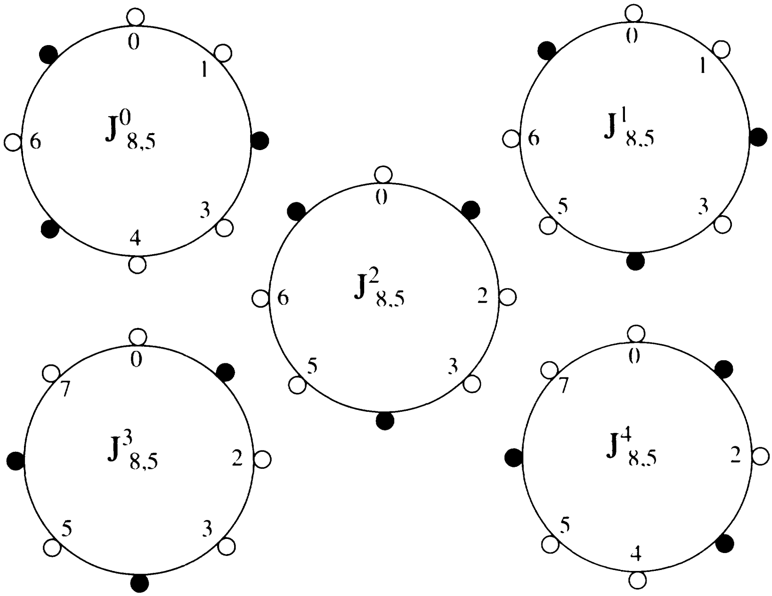
Definition 1.16. A *tonic* ME (TME) set $T^p_{c,d}$ is an $M_{c,d}$ which includes a specified pitch-class p. We write $T^p(c,d)$ to represent the set of all ME sets with parameters c,d that contain the pitch-class p.

As in the case of $M_{c,d}$, the symbol $T^p_{c,d}$ is the name for a variable whose assigned value is always a pcset of certain description. As an example of TME sets, the five sets corresponding to $T^0_{8,5}$ are shown in fig. 4a. All of these sets are members of $M(8,5)$ of course. There are three additional sets in $M(8,5)$ which do not include $p = 0$; these are shown in fig. 4b.

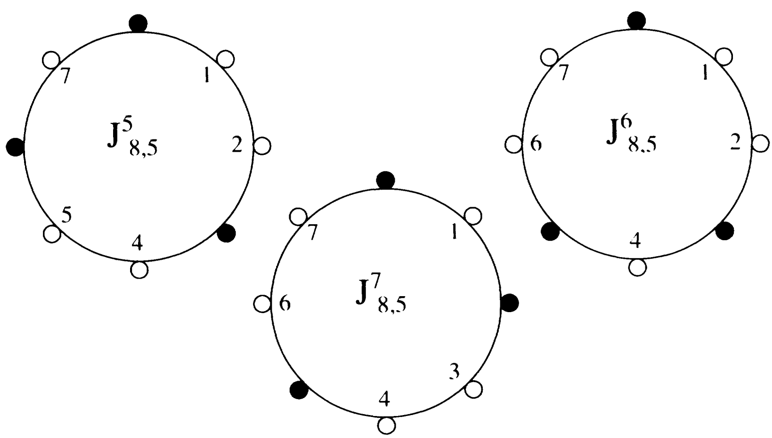
Lemma 1.15. For any fixed c,d, $|T^0(c,d)| = \frac{d}{(c,d)}$, and

$$T^0(c,d) = \{J^0_{c,d}, J^{(c,d)}_{c,d}, J^{2(c,d)}_{c,d}, \ldots, J^{d-(c,d)}_{c,d}\}.$$

Proof. Consider $M(c,d) = \{J^0_{c,d}, J^{(c,d)}_{c,d}, J^{2(c,d)}_{c,d}, \ldots, J^{c-(c,d)}_{c,d}\}$. By lemma 1.13, 0 is an element in the sets in $M(c,d)$ if and only if



(a) Five sets corresponding to $T^0_{8,5}$



(b) Three additional sets in $M(8,5)$

Figure 4. The eight sets of $M(8,5)$

$J_{c,d}^{j(c,d)}(0) = 0$ (where $0 \le j \le \frac{c}{(c,d)} - 1$). But $J_{c,d}^{j(c,d)}(0) = \left\lceil \frac{j(c,d)}{d} \right\rceil$, implying $J_{c,d}^{j(c,d)}(0) = 0$ if and only if $j(c,d) < d$. We may conclude $J_{c,d}^{j(c,d)}(0) = 0$ if and only if $0 \le j \le \frac{d}{(c,d)} - 1$. Thus $|T^0(c,d)| = \frac{d}{(c,d)}$, and

$$T^0(c,d) = \{J_{c,d}^{0}, J_{c,d}^{(c,d)}, J_{c,d}^{2(c,d)}, \ldots, J_{c,d}^{d-(c,d)}\}. \bullet$$

Theorem 1.10. For any fixed c,d and any pc p, $|T^p(c,d)| = \frac{d}{(c,d)}$, and

$$T^p(c,d) = \{J_{c,d}^{(pd)_c}, J_{c,d}^{((c,d)+pd)_c}, J_{c,d}^{(2(c,d)+pd)_c}, \ldots, J_{c,d}^{(d-(c,d)+pd)_c}\}.$$

Proof. By lemma 1.15, $T^0(c,d) = \{J_{c,d}^{0}, J_{c,d}^{(c,d)}, J_{c,d}^{2(c,d)}, \ldots, J_{c,d}^{d-(c,d)}\}$. If each of these sets in $T^0(c,d)$ is transposed by p, the resulting sets remain distinct and each of them contains p (since each contained 0 before the transposition). Similarly, if $M_{c,d} \notin T^0(c,d)$, then its transposition by p will not be in $T^p(c,d)$. Thus the set of ME sets that contain p is

$$\begin{aligned} T^p(c,d) &= \{\{p\} + J_{c,d}^{0}, \{p\} + J_{c,d}^{(c,d)}, \{p\} + J_{c,d}^{2(c,d)}, \ldots, \{p\} + J_{c,d}^{d-(c,d)}\} \\ &= \{J_{c,d}^{(pd)_c}, J_{c,d}^{((c,d)+pd)_c}, J_{c,d}^{(2(c,d)+pd)_c}, \ldots, J_{c,d}^{(d-(c,d)+pd)_c}\} \bullet \end{aligned}$$

(lemma 1.6).

Consider the familiar case $M(12,7)$. Let C = pc 0. Theorem 1.10 tells us there are seven sets in $T^0(12,7)$—the seven diatonic (major-scale) sets which include C; there are of course five additional sets in $M(12,7)$ which *exclude* C. Note that the seven members of $T^0(12,7)$, each a diatonic set with C marked as "tonic," correspond to the seven modes, or more precisely, "modal scales." It is easy to see why this is true. The diatonic set consists of seven contiguous pcs on the circle of fifths. Therefore choosing a particular "tonic," say C, and rotating a seven-pc segment around the circle so that it always includes the tonic, is similar to choosing a particular diatonic set and rotating the "tonic" so that each pc in turn becomes the "tonic." These two rotation procedures are equivalent in that each produces the set of seven modal scales, conceived as a set of pcs with a particular pc marked as "tonic."

Note that if $(c,d) = 1$, $\frac{d}{(c,d)} = d$. Thus if c and d are coprime there are d distinct pcsets in $T^p(c,d)$ for any p. This particular case of theorem 1.10 is a special case of Clough and Myerson's "cardinality equals variety" result. In their terms, given $D_{c,d}$ with $(c,d) = 1$ and each interval spectrum a two-element set, if we list the d possible *scalar* orderings of $D_{c,d}$:

$$D_0, D_1, \ldots, D_{d-1};$$
$$D_1, D_2, \ldots, D_0;$$
$$\vdots$$
$$D_{d-1}, D_0, \ldots, D_{d-2},$$

these d orderings constitute a generic "line class" which must, as they show, appear in d "species." The special case of $M_{12,7}$ is discussed by Richmond Browne (1981), who describes the "unique interval contexts" of the scale degrees.

Interval content

In traditional music theory, the interval-content symmetry of complementary intervals in the diatonic set is well-known: there are two each of minor 2ds and major 7ths, five each of major 2ds and minor 7ths, . . . , and one each of the augmented 4th and diminished 5th. This symmetry does not depend upon the inherent symmetry of the diatonic set; as the following lemma shows, it holds quite generally for any set conceived as "diatonic" (i.e., having intervals measured in "steps").

Lemma 1.16. For any $D_{c,d}$, and associated dlen I ($I \neq 0$), the number of intervals in $<I>$ of clen k equals the number of intervals in $<d - I>$ of clen $c - k$.

Proof. Let dlen $(D_i, D_j) = I$ and clen $(D_i, D_j) = k$, where $i \neq j$. Now consider the interval from D_j to D_i. Clearly dlen (D_i, D_j) + dlen $(D_j, D_i) = d$ and clen(D_i, D_j) + clen $(D_j, D_i) = c$. Such a symmetry exists for each pair of pcs and the lemma follows immediately. ●

Because of the regular structure of ME sets, their interval contents may be computed directly from the parameters c and d. In order to show this, we first establish limits for clens in a particular $<I>$, given that $<I>$ is consistent with ME sets ($<I>$ consists of an integer or two consecutive integers). The interval-content formula follows as theorem 1.11.

Theorem 1.11. For any $M_{c,d}$ and associated dlen I, suppose $cI = dq + r$, where q and r are integers such that $q \geq 0$ and $0 \leq r < d$. Then there are r intervals in $<I>$ of clen $\left\lfloor \frac{cI}{d} \right\rfloor + 1$, and r intervals in $<d - I>$ of clen $c - \left(\left\lfloor \frac{cI}{d} \right\rfloor + 1\right)$. Hence there are $d - r$ intervals in $<I>$ of clen $\left\lfloor \frac{cI}{d} \right\rfloor$ and $d - r$ intervals in $<d - I>$ of clen $c - \left\lfloor \frac{cI}{d} \right\rfloor$.

Proof. From lemma 1.9 we know that the clen of any interval in $<I>$ is $\left[\frac{cI}{d}\right]$ or $\left[\frac{cI}{d}\right] + 1$. Let h be the number of intervals in $<I>$ of clen $\left[\frac{cI}{d}\right] + 1$. Then there will be $d - h$ intervals in $<I>$ of clen $\left[\frac{cI}{d}\right]$. The sum of these clens is

$$h\left(\left[\frac{cI}{d}\right] + 1\right) + (d-h)\left[\frac{cI}{d}\right] = cI \text{ (lemma 1.2).}$$

Thus

$$h = cI - d\left[\frac{cI}{d}\right].$$

Now let $cI = dq + r$, where q and r are integers, $q \geq 0$, $0 \leq r \leq d - 1$. Then

$$\begin{aligned} h &= cI - d\left[\frac{cI}{d}\right] \\ &= dq + r - d\left[\frac{dq+r}{d}\right] \\ &= r. \end{aligned}$$

Thus there are r intervals in $<I>$ of clen $\left[\frac{cI}{d}\right] + 1$ and hence $d-r$ intervals in $<I>$ of clen $\left[\frac{cI}{d}\right]$. From lemma 1.16 it follows immediately that there are r intervals in $<d - I>$ of clen $c - \left(\left[\frac{cI}{d}\right] + 1\right)$ and $d - r$ intervals in $<d - I>$ of clen $c - \left[\frac{cI}{d}\right]$. ●

The information available through theorem 1.11 is more complete than that given by the usual metrics of atonal set theory, which give the multiplicity of each clen in a set. Theorem 1.11 gives the multiplicity of each clen *as associated with the various dlens,* from which the overall multiplicity of each clen may easily be counted. We now define a function that "organizes" the information provided by theorem 1.11.

Definition 1.17. DFUNC(X, k, I) = the numbers of intervals of clen k and dlen I in the set X.

Example 1.5. DFUNC$(M_{12,7}, 2, 1) = 5$ tells us that there are five intervals of clen 2 and dlen 1 (major 2ds) in the usual diatonic set.

Fig. 5 shows the intervals in $M_{8,5}$, as counted by DFUNC. Complementary intervals are indicated by nested brackets.

Note that theorem 1.11 applies only to ME sets. It is, of course, possible to profile *any* pcset by means of DFUNC, and the more complete profile provided by DFUNC seems potentially advantageous. For example, DFUNC is sufficient to distinguish between any two Z-related sets in the 12-pc universe; whether this holds for Z-related pairs (and triples, etc.) in any chromatic universe is an open question.

DFUNC ($M_{8,5}$, 0,0) = 5
DFUNC ($M_{8,5}$, 1,1) = 2
DFUNC ($M_{8,5}$, 2,1) = 3
DFUNC ($M_{8,5}$, 3,2) = 4
DFUNC ($M_{8,5}$, 4,2) = 1
DFUNC ($M_{8,5}$, 4,3) = 1
DFUNC ($M_{8,5}$, 5,3) = 4
DFUNC ($M_{8,5}$, 6,4) = 3
DFUNC ($M_{8,5}$, 7,4) = 2

Figure 5. The interval profile of $M_{8,5}$ = {0,1, 3, 4, 6}

The spectrum of a set

As we will see (particularly in part 2 of this paper), it is sometimes useful to distill the information of DFUNC; for certain purposes we need to know only how many dlens correspond to each clen.

Definition 1.18. The *spectrum* of a set $D_{c,d}$, written *spec*($D_{c,d}$), is the set of all elements in the interval spectrums $<I>$, $0 \leq I \leq d-1$, of $D_{c,d}$, including their multiplicity. That is, spec($D_{c,d}$) $= \cup_{I=0}^{d-1} <I>$. Since multiplicity is included, spec($D_{c,d}$) is a *multiset.*

Example 1.6. Let $D_{12,7}$ = {0, 1, 2, 4, 6, 8, 10}, the "whole tone plus one" set. Then the interval spectrums of $D_{12,7}$, tabulated so as to highlight the symmetries of complementary intervals, are as follows:

$$<0> = \{0\}$$

$$<1> = \{1, 2\} \qquad <6> = \{10, 11\}$$

$$<2> = \{2, 3, 4\} \qquad <5> = \{8, 9, 10\}$$

$$<3> = \{4, 5, 6\} \qquad <4> = \{6, 7, 8\}.$$

The spectrum of the set, spec($D_{12,7}$) = {0, 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 10, 10, 11}. To simplify this notation we write spec($D_{12,7}$) = {0, 1, 2_2, 3, 4_2, 5, 6_2, 7, 8_2, 9, 10_2, 11}. Note that spec($D_{12,7}$) reflects several ambiguous clens. For example, $<1>$ = {1, 2}, $<2>$ = {2, 3, 4}; so clen 2 corresponds to both dlen 1 and dlen 2. We discuss ambiguities more fully in part 2.

We now set forth a few additional concepts which will allow us to show how the spectrum of a set is associated with other properties of the set.

Definition 1.19. $C_c = \{0, 1, 2, \ldots, c-1\}$, a multiset.

Note that while U_c (the chromatic universe) and C_c contain the same elements, they are not identical. U_c is a *set,* whereas C_c is a *multiset* in which each element has multiplicity one.

Definition 1.20. If $<1> = \{1, 2\}$, we say that the set is *reduced.* The above definition is borrowed from Clough and Myerson.

Lemma 1.17. Suppose c and d are positive integers such that $\frac{c}{2} < d \leq c$. Let $S = \cup_{I=10}^{d-1}\left\{\left[\frac{cI}{d}\right], \left[\frac{cI}{d}\right] + 1\right\}$ be a set (not a multiset) where the elements are reduced (mod c). Then $S = U_c = \{0, 1, 2, \ldots, c-1\}$.

Proof. The proof is in three parts:

(i) $0 = \left[\frac{c \cdot 0}{d}\right]$ (trivial).

(ii) either $c-1 = \left[\frac{c(d-1)}{d}\right]$ or $c-1 = \left[\frac{c(d-1)}{d}\right] + 1$.

(iii) $\left[\frac{cI}{d}\right] + 2 \geq \left[\frac{c(I+1)}{d}\right]$ for all I, $0 \leq I \leq d-2$.

Note that since $c \geq d$,

$$\left[\frac{c \cdot 0}{d}\right] < \left[\frac{c \cdot 1}{d}\right] < \left[\frac{c \cdot 2}{d}\right] < \ldots < \left[\frac{c(d-1)}{d}\right] \leq c-1.$$

So if

$$\left[\frac{cI}{d}\right] + 2 \geq \left[\frac{c(I+1)}{d}\right], \text{ and } \left[\frac{c(d-1)}{d}\right] + 1 \geq c-1,$$

then all the integers from 0 to $c-1$, inclusive, will also be in S. Thus $S = \{0, 1, 2, \ldots, c-1\}$.

(i) When $I = 0$, $\left[\frac{cI}{d}\right] = \left[\frac{c \cdot 0}{d}\right] = 0$. Thus $0 \in S$.

(ii) If $c = d$ and $I = d-1$, then

$$\begin{aligned}\left[\frac{cI}{d}\right] &= \left[\frac{c(d-1)}{d}\right] \\ &= \left[\frac{c(c-1)}{c}\right] \\ &= c-1.\end{aligned}$$

If $\frac{1}{2}c < d < c$, then when $I = d - 1$ we have

$$\begin{aligned}\left\lceil\frac{cI}{d}\right\rceil + 1 &= \left\lceil\frac{c(d-1)}{d}\right\rceil + 1\\ &= \left\lceil\frac{dc - d - (c-d)}{d}\right\rceil + 1\\ &= c + \left\lceil\frac{-(c-d)}{d}\right\rceil\\ &= c - 1 \text{ (since } \tfrac{1}{2}c < d < c).\end{aligned}$$

In either case $c - 1 \in S$.

(iii)

$$\begin{aligned}\left\lceil\frac{cI}{d}\right\rceil + 2 &\geq \left\lceil\frac{cI + (c-d)}{d}\right\rceil + 1 \quad \text{(since } d > \tfrac{c}{2})\\ &= \left\lceil\frac{cI + c}{d}\right\rceil\\ &= \left\lceil\frac{c(I+1)}{d}\right\rceil . \bullet\end{aligned}$$

From the uniqueness property of ME sets, it follows that for any $M_{c,d}$, $M^*_{c,d} \in M(c,d)$, spec$(M_{c,d})$ = spec$(M^*_{c,d})$. We are now able to show the following relationships among spec$(M_{c,d})$ and other attributes of $M_{c,d}$.

Theorem 1.12. For any $M_{c,d}$, the following are equivalent:
(1) $\frac{c}{2} < d$.
(2) spec$(M_{c,d})$ includes C_c.
(3) $M_{c,d}$ is reduced or is the entire chromatic set $(d = c)$.

Proof.
(1) ⇒ (2). Let S be as in lemma 1.17. Suppose spec$(M_{c,d})$ does not include C_c. By lemma 1.9 we see that $\left\lceil\frac{cI}{d}\right\rceil \in \langle I \rangle$, $0 \leq I \leq d - 1$. By lemma 1.17 we see that S contains all chromatic lengths. Thus any clen not in spec$(M_{c,d})$ must be of the form $\left\lceil\frac{cI}{d}\right\rceil + 1$. Then $cI = qd + 0$, since there are no clens of $\left\lceil\frac{cI}{d}\right\rceil + 1$ in $\langle I \rangle$ (theorem 1.11). But then

$$\left\lceil\frac{cI}{d}\right\rceil + 1 = q + 1$$

$$= q + 1 + \left\lceil \frac{c-d}{d} \right\rceil \text{ (since } \tfrac{1}{2}c < d \le c)$$

$$= \left\lceil \frac{qd+c}{d} \right\rceil$$

$$= \left\lceil \frac{cI+c}{d} \right\rceil$$

$$= \left\lceil \frac{c(I+1)}{d} \right\rceil \in \, <I+1>,$$

implying $\left\lceil \frac{cI}{d} \right\rceil + 1 \in \, <I + 1>$, contradicting our assumption. Thus if $\frac{1}{2}c < d \le c$, then spec($M_{c,d}$) includes C_c.

(2) ⇒ (3). We will assume that $M_{c,d}$ is not reduced and show that if spec($M_{c,d}$) includes C_c, then $M_{c,d}$ must be the entire chromatic set. If $M_{c,d}$ is not reduced, then $<1> \neq \{1, 2\}$ and thus either $1 \notin \, <1>$ or $<1> = \{1\}$ (lemma 1.9; note $\left\lceil \frac{c \cdot 1}{d} \right\rceil \ge 1$). If $1 \notin \, <1>$, then $\left\lceil \frac{c \cdot 1}{d} \right\rceil \ge 2$, implying $d \le \frac{1}{2}c$ (contradiction). Thus $<1> = \{1\}$. But if $<1> = \{1\}$, then $M_{c,d}$ must be the entire chromatic set.

(3) ⇒ (1). If $M_{c,d}$ is the entire chromatic set or $M_{c,d}$ is reduced, then $1 \in \, <1>$, implying $\left\lceil \frac{c \cdot 1}{d} \right\rceil = 1$. But then $\frac{1}{2}c < d \le c$. ●

Corollary 1.5. $M_{c,d}$ is reduced if and only if $\frac{1}{2}c < d < c$.

Proof. Clearly $M_{c,d}$ is the entire chromatic set if and only if $c = d$. The corollary follows immediately from theorem 1.12, (1) and (3). ●

2. Ambiguities, Tritones, and Diatonic Sets

In the past two decades there have been several approaches to a deeper understanding of the diatonic set. In addition to the work of Clough and Myerson cited above, there are well-known contributions by Gerald Balzano (1980), Benjamin Boretz (1970), Richmond Browne (1981), Carlton Gamer (1967), Robert Gauldin (1983), and Peter Westergaard (1975). More recently, Stephen Dembski (1988) has presented a rational reconstruction of the tonal system as part of his paper on a generalized step-class system, and Jay Rahn (1991) has discussed the coordination of interval sizes in seven-note collections.

Closely related to our efforts here are Norman Carey and David Clampitt's (1989) work and Eytan Agmon's (1989) work. In this part of the paper and in the third and final part, we seek to advance the inquiry into the essence of diatonicism. We will not attempt to review all of the important work cited above, but we will point out the more salient connections, particularly those between Agmon's work and ours.

Clough and Myerson show that the usual diatonic set is one of an infinite class of "special" sets with the following properties: (1) "cardinality equals variety"—a "line class" including n different pcs appear in n species, (2) "partitioning"—chord species map (unambiguously) onto chord genera, and (3) the "deep-set" property as studied by Gamer (1967, 32–59). They offer a definition of this class of sets based on *Myhill's property* together with the "generalized circle of fifths," and show that, in terms of the relationship between c and d, there are just two families of "special" sets: family A, where $c = 2d - 1$, and family B, where $c = 2d - 2$ with d odd. $M_{8,5}$ and $M_{12,7}$, discussed above and shown in figs. 4 and 2c, respectively, are members of family B. For comparison, fig. 6 shows the two ME sets of family A with $d = 5$ and $d = 7$: $M_{9,5}$ and $M_{13,7}$.

Agmon (1989) offers a mathematical approach to the same class of "special" sets—an approach closely resembling that of Clough-Myerson in some respects and differing from it significantly in others. Agmon measures intervals by clen and dlen, as do Clough and Myerson, and as we do here. He distinguishes between the sets of family A, and those of family B, as defined above and in Clough and Myerson. In defining the class *diatonic,* Agmon quite reasonably rules out the sets of family A on the ground that they are inherently less interesting than those of family B, because the former are generated by the whole-step (clen 2) or its complement, while (for $c > 4$) the latter are generated by a "skip" of more than one diatonic step (e.g., in the usual diatonic set, clen 5 or 7—the perfect 4th or perfect 5th). The question of whether the sets of family A are less interesting has a surprising twist, which we come to in part 3 of the paper. Meanwhile, we accept Agmon's premise and focus on the usual diatonic and other sets of family B.

Some new terminology will be useful in maintaining the distinctions among various kinds of sets. We will call the sets of family B (ME sets with $c = 2d - 2$ and d odd) *diatonic* sets. We will continue to refer to $M_{12,7}$ as the *usual diatonic* set, and we will henceforth refer to other diatonic sets as *hyperdiatonic* sets. Thus our diatonic sets, which include the usual diatonic set plus all hyperdiatonic sets, correspond to Agmon's "diatonic systems." As before, we will refer to the complement of $M_{12,7}$, $M_{12,5}$ as the *usual pentatonic* set.

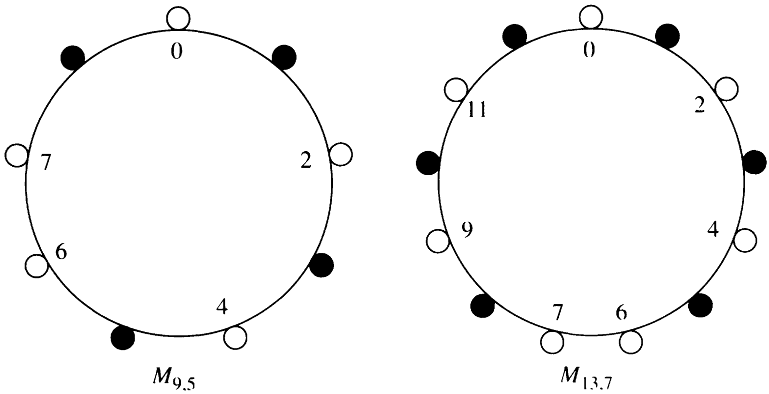


Figure 6. Examples of family A: $c = 2d - 1$

Expanded formal definitions of the above terms will be given later in this part of the paper. Note that with the exceptions of $M_{4,3}$ and $M_{8,5}$, all hyperdiatonic sets are truly "hyper" in that they have *more* elements than the usual diatonic. The definitions of *diatonic length (dlen), diatonic cardinality,* and *diatonic transposition,* and the symbol $D_{c,d}$, are unchanged; as before, these terms and symbols apply broadly to *any* subset of a chromatic universe.

While the Clough-Myerson and Agmon formulations provide much insight, they are fairly complicated, and there is the problem of the "uninteresting" sets with $c = 2d - 1$. This motivates a continued search for formulations of greater intuitive appeal yielding the family of diatonic sets. We take up this quest by investigating the various types of ME sets in terms of *ambiguity,* with particular attention to *tritones.* We then show that the set of diatonic sets is nothing more or less than the set of ME sets with exactly one ambiguity (a tritone). Surprisingly, any one of a number of other conditions may be substituted for "exactly one ambiguity"; within the realm of ME sets, the specified conditions are all equivalent. The theorem enumerating these equivalent conditions is the principal result of this part of the paper. We also provide a method for constructing the sets in question, and a theorem generalizing the well-known fact that, given a naked tritone in the 12-pc universe, any additional note is sufficient to imply a unique transposition of the usual diatonic set.

We first define some terms pertaining to correspondences among dlens and clens. Our definitions are those given by Jay Rahn in an article elsewhere in this journal, reworded to suit the context of our discussion. Our method of counting the number of cases of difference or ambiguity or contradiction in a set also follows that of Rahn.

Definition 2.1. Given D_p, D_q, D_r, $D_s \in D_{c,d}$, if dlen(D_p, D_q) = dlen(D_r, D_s), and clen$(D_p, D_q) \neq$ clen(D_r, D_s), that is a case of *difference.*

Definition 2.2. Given D_p, D_q, D_r, $D_s \in D_{c,d}$, if dlen$(D_p, D_q) \neq$ dlen (D_r, D_s), and clen(D_p, D_q) = clen(D_r, D_s), that is a case of *ambiguity.*

Definition 2.3. Given D_p, D_q, D_r, $D_s \in D_{c,d}$, if dlen$(D_p, D_q) <$ dlen(D_r, D_s), and clen$(D_p, D_q) >$ clen(D_r, D_s), that is a case of *contradiction.*

Examples 2.1. Suppose $D_{c,d}$ = {C, D, E♭, F♯, G, A♭, B} = {0, 2, 3, 6, 7, 8, 11}. Then

(1) dlen(C, E♭) = dlen(D, F♯) and clen(C, E♭) ≠ dlen(D, F♯)—a case of *difference;*

(2) dlen(C, F♯) ≠ dlen(F♯ C,) and clen(C, F♯) = clen(F♯, C)—a case of *ambiguity;*

(3) dlen(E♭, F♯) < dlen(F♯, A♭) and clen(E♭, F♯) > clen(F♯, A♭) —a case of *contradiction.*

Note that each case of difference, ambiguity, or contradiction involves two intervals—that is, two ordered pairs of pcs. The two intervals may have zero, one, or two pcs in common, though in different order positions (all three possibilities are evident in the examples above). It follows immediately from the above definitions and from lemma 1.16 that for each case of difference, ambiguity, or contradiction involving particular dlens and clens, there is a corresponding case involving the complementary dlens and clens.

Examples 2.2. In $M_{12,7}$, $<2>$ = {3, 4}, implying one or more cases of difference. (There are in fact 12 = 3 ·4 cases of difference here—12 pairs of intervals—since there are *three* intervals of dlen 2 with clen 4[major thirds], and *four* intervals of dlen 2 with clen 3 [minor thirds].) The complementary cases arise from $<7-2> = <5>$ = {12 − 3, 12 − 4} = {8, 9}. Complementary cases are not necessarily distinct. For example in $M_{12,7}$, $<3>$ = {5, 6} and $<4>$ = {6, 7}, implying one or more cases of ambiguity. There is in fact a single ambiguity here—that is, a single ambiguous pair of intervals—one interval of dlen 3 with clen 6, and one interval of dlen 4 with clen 6. But these two intervals are mutually complementary.

We now state a number of results connecting the above definitions with ME sets.

Lemma 2.1. No $M_{c,d}$ contains a contradiction.

Proof. By theorem. 1.11, for any dlen I,

$$< I > = \left\{\left[\frac{cI}{d}\right]\right\} \text{ or } \left\{\left[\frac{cI}{d}\right], \left[\frac{cI}{d}\right] + 1\right\}.$$

Also,

$$\left[\frac{c}{d}\right] < \left[\frac{2c}{d}\right] < \left[\frac{3c}{d}\right] < \ldots < \left[\frac{(d-1)c}{d}\right].$$

Thus $\left[\frac{cI}{d}\right] + 1 \leq \left[\frac{c(I+1)}{d}\right]$ for all I, $1 \leq I \leq d-2$. Since no spectrum $< I >$ has an element smaller than $\left[\frac{cI}{d}\right]$ or an element larger than $\left[\frac{cI}{d}\right] + 1$, no element of $< I >$ can be larger than the smallest element of $< I + 1 >$, $1 \leq I \leq d-2$. Thus there can be no contradiction. ●

Lemma 2.2. Let $d \leq \frac{c}{2}$. Then $M_{c,d}$ has no ambiguity.

Proof. If $d \leq \frac{c}{2}$ then

$$\begin{aligned}\left[\frac{c(I+1)}{d}\right] - \left[\frac{cI}{d}\right] &= \left[\frac{cI + c}{d}\right] - \left[\frac{cI}{d}\right] \\ &\geq \left[\frac{cI}{d} + 2\right] - \left[\frac{cI}{d}\right] \\ &= 2.\end{aligned}$$

Thus $\left[\frac{cI}{d}\right] + 1 < \left[\frac{c(I+1)}{d}\right]$ for all I, $0 \leq I \leq d-2$. It follows that there is no ambiguity since the largest possible element in $< I >$ is $\left[\frac{cI}{d}\right] + 1$ and the smallest possible element in $< I + 1 >$ is $\left[\frac{c(I+1)}{d}\right]$ (theorem 1.11). ●

Lemma 2.3. Let $(c,d) = d$. Then $M_{c,d}$ has no ambiguity and no difference.

Proof. By lemma 1.7, $< jd/d > = \{jc/d\}$ for $1 \leq j \leq d-1$. Now let j equal any dlen I, $1 \leq I \leq d-1$. Then $< I > = \{cI/d\}$. Hence there is no difference. Further, since

$$\frac{c \cdot 0}{d} < \frac{c \cdot 1}{d} < \frac{c \cdot 2}{d} < \ldots < \frac{c(d-1)}{d},$$

there can be no ambiguity. ●

As the following two lemmas show, any ambiguity in a ME set involves two consecutive dlens, each with a two-element spectrum.

Lemma 2.4. If c_0 is an element in two distinct spectrums of $M_{c,d}$ (i.e., c_0 is the clen of an ambiguity), then there exists a unique I, $0 \le I \le d-1$ such that $<I> \cap <I+1> = \{c_0\}$. Further

$$c_0 = \left[\frac{cI}{d}\right] + 1 = \left[\frac{c(I+1)}{d}\right].$$

Proof. By lemma 1.9, c_0 must have the form $\left[\frac{cI}{d}\right]$ or $\left[\frac{cI}{d}\right] + 1$, $0 \le I \le d-1$. Further, $\left[\frac{0}{d}\right] < \left[\frac{c}{d}\right] < \left[\frac{2c}{d}\right] < \left[\frac{3c}{d}\right] < \ldots < \left[\frac{(d-1)c}{d}\right] < c$. Thus if c_0 is in two spectrums, then there exists a unique I, $1 \le I \le d-1$, such that $<I> \cap <I+1> = \{c_0\}$, and $c_0 = \left[\frac{cI}{d}\right] + 1 = \left[\frac{c(I+1)}{d}\right]$. ●

Lemma 2.5. For any $M_{c,d}$ and associated dlen I, if $<I> \cap <I+1> \neq \varnothing$ (implying an ambiguity), then $<I>$ and $<I+1>$ are two-element spectrums.

Proof. Assume $c_0 \in <I> \cap <I+1>$. Then $c_0 = \left[\frac{cI}{d}\right] + 1$ (lemma 2.4), implying $<I>$ is a two-element spectrum (theorem 1.11). Now suppose $<I+1>$ is a one-element spectrum. Then $c(I+1) = dq + 0$ (theorem 1.11), implying $cI = dq - c$. Thus, since there is an ambiguity,

$$\left[\frac{cI}{d}\right] + 1 = \left[\frac{c(I+1)}{d}\right] \text{ (lemma 2.4)},$$

$$\left[\frac{dq-c}{d}\right] + 1 = q,$$

and

$$\left[-\frac{c}{d}\right] = -1.$$

This implies $d \ge c$. By definition $d \le c$. Thus $d = c$. But if $d = c$, then $(c,d) = d$, which implies $M_{c,d}$ has no ambiguity (lemma 2.3), contradicting the assumption $<I> \cap <I+1> \neq \varnothing$. Thus $<I+1>$ must be a two-element spectrum. ●

Tritones

We now turn our attention to a particular ambiguity, that with clen $c/2$. This is a special kind of ambiguity—the only one capable of expression by means of a *single pair* of pitch classes, say x and y, spanning clen $c/2$ as the interval from x to y and the interval from y to x. The familiar case is, of course, the tritone of $M_{12,7}$, where the spectrums of the traditional 4th and 5th intersect, both including an interval of six semitones. We borrow the traditional term and call any half-"octave" interval a "tritone," regardless of its dlen.

Definition 2.4. A *tritone* is a two-element subset $\{D_i, D_j\}$ of $D_{c,d}$ such that $\text{clen}(D_i, D_j) = \text{clen}(D_j, D_i) = \frac{c}{2}$. *Tritone* is defined only for c even.

Note that tritones are not necessarily ambiguous. A tritone $\{D_i, D_j\}$ is unambiguous if $\text{dlen}(D_i, D_j) = \text{dlen}(D_j, D_i)$ and ambiguous if $\text{dlen}(Di, D_j) \neq \text{dlen}(D_j, Di)$.

Consider the sets $D_{12,4}$ and $M_{12,4}$ pictured in fig. 7. Both have the same chromatic and diatonic cardinalities, but their tritone contents are very different. In one (fig. 7a) $<1> \cap <3> = \{6\}$, and in the other (fig. 7b) $<2> = \{6\}$; the former has one ambiguous tritone and the latter, two unambiguous tritones. So we see that for pcsets in general, knowing c and d does not tell us how many tritones the set has or whether they are unambiguous or ambiguous. If we restrict ourselves to ME sets, however, we can compute this information from c and d, as the following lemmas show.

Lemma 2.6. If both c and d are even, then $M_{c,d}$ has precisely $d/2$ tritones, all of which are unambiguous, and $< d/2 > = \{c/2\}$.

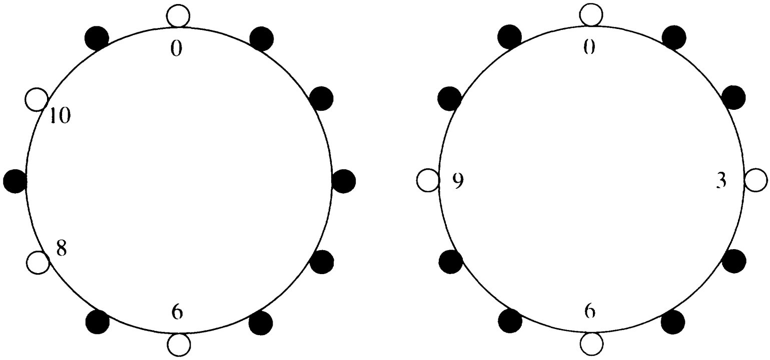
Proof. Since there are d elements in $M_{c,d}$, there can be at most $d/2$ distinct tritones. Now consider $I = d/2$: $cI = c(d/2) = (c/2)d + 0$. Thus all the intervals of dlen $d/2$ have clen $\left\lceil \frac{c(d/2)}{d} \right\rceil$ (theorem 1.11); that is, there are d intervals of clen $\left\lceil \frac{c(d/2)}{d} \right\rceil = c/2$. Thus there must be at least $d/2$ distinct tritones. It follows that there are exactly $d/2$ tritones, and $< d/2 > = \{c/2\}$.

Further, since $< d/2 > = \{c/2\}$ is a single-element spectrum, clen $c/2$ cannot be ambiguous (lemma 2.5). ●

Lemma 2.7. If $M_{c,d}$ has a tritone ambiguity (implying c is even), then d is odd, $< \frac{d-1}{2} > \cap < \frac{d+1}{2} > = \{c/2\}$, and $c/2$ belongs to no other spectrums.

Proof. Assume $M_{c,d}$ has an ambiguity. If d is even, then there are no ambiguous tritones (lemma 2.6). Thus d is odd. Since $M_{c,d}$ has an ambiguity, $d > c/2$ (lemma 2.2). Consider $I = \frac{d-1}{2}$.

$$c\left(\frac{d-1}{2}\right) = d\left(\frac{c}{2} - 1\right) + \frac{c}{2}.$$



(a) $\{0,6,8,10\}$
$<1> \cap <3> = \{6\}$:
one ambiguous tritone

(b) $\{0,3,6,9\}$
$<2> = \{6\}$:
two unambiguous tritones

Figure 7. Ambiguous and unambiguous tritones

Since $d > c/2 > 0$, the element $\left[\frac{c(d-1)/2}{d}\right] + 1$ is a member of $< \frac{d-1}{2} >$ (theorem 1.11). But

$$\left[\frac{c(d-1)/2}{d}\right] + 1 = c/2 + \left[-\frac{c}{2d}\right] + 1$$
$$= c/2 \text{ (since } d > c/2).$$

Thus $c/2 \in < \frac{d-1}{2} >$. Further, $\left[\frac{c(d+1)/2}{d}\right] \in < \frac{d+1}{2} >$ (theorem 1.11). Also

$$\left[\frac{c(d+1)/2}{d}\right] = c/2 + \left[\frac{c}{2d}\right]$$
$$= c/2 \text{ (since } d > c/2).$$

Hence $c/2 \in < \frac{d+1}{2} >$. It follows that $< \frac{d-1}{2} > \cap < \frac{d+1}{2} > = \{c/2\}$, and $c/2$ is in no other spectrum (lemma 2.4). ●

Lemma 2.8. Let c be even, and d odd.

(i) If $d \leq \frac{c}{2}$, then $M_{c,d}$ has no tritones.

(ii) If $\frac{c}{2} < d < c$, then $M_{c,d}$ has $d - \frac{c}{2}$ distinct tritones, all of which are ambiguous.

Proof. Assume $\{D_i, D_j\}$ is a tritone. Then there are $d - 2$ other elements in $M_{c,d}$. Since d is odd, so is $d - 2$ Hence the number of elements on one side of $\{D_i, D_j\}$ must differ from the number of elements on the other. Thus dlen $(D_i, D_j) \neq$ dlen (D_j, D_i) and hence $\{D_i, D_j\}$ is ambiguous.

(i) Assume $d \leq \frac{c}{2}$ and $M_{c,d}$ has a tritone. Then, as shown above, the tritone must be ambiguous. But if there is an ambiguity then $\frac{c}{2} < d$ (lemma 2.2), contradicting the assumption that $d \leq \frac{c}{2}$. It follows that there are no tritones.

(ii) As shown above, any tritones must be ambiguous. It remains to show how many tritones there are when $\frac{c}{2} < d < c$. We know from lemma 2.7 that if there are ambiguous tritones, then

$$< \frac{d-1}{2} > \cap < \frac{d+1}{2} > = \left\{\frac{c}{2}\right\}$$

and there are no other spectrums containing $c/2$. Also

$$c\left(\frac{d-1}{c}\right) = d\left(\frac{c}{2} - 1\right) + \left(d - \frac{c}{2}\right),$$

and

$$\left\lceil\frac{c(d-1)/2}{d}\right\rceil = \frac{c}{2} - 1 .$$

It follows that there are $\left(d - \frac{c}{2}\right)$ intervals with clen$\left(\frac{c}{2} - 1\right) + 1 = c/2$ (theorem 1.11). ●

Lemma 2.9. Let c be even and d odd. Then if $M_{c,d}$ has an ambiguity, it has an ambiguous tritone.

Proof. Since there is an ambiguity, $d > \frac{c}{2}$ (lemma 2.2). If c is even and d is odd, then clearly $c \neq d$. Thus $\frac{c}{2} < d < c$, implying there are $c - \frac{d}{2}$ tritones, all of which are ambiguous (lemma 2.8). ●

Corollary 2.1. Let c be even. Then if $M_{c,d}$ has an ambiguity, it has a tritone.

Proof. Lemma 2.6 covers the case with both c and d even, and lemma 2.9 the case with c even and d odd. ●

This covers some of the properties of tritones in conjunction with ME sets. From lemma 2.9 we see that if c is even and d odd, then the clen of the "first" ambiguity is in $< \frac{d-1}{2} > \cap < \frac{d+1}{2} >$. Might this also be true if both c and d are odd? Since c odd implies there are no tritones, we make the following definition.

Definition 2.5. Suppose c and d are odd and

$$<\frac{d-1}{2}> \cap <\frac{d+1}{2}> \neq \varnothing.$$

Then we say that $D_{c,d}$ contains a *pseudotritone.*

Note that pseudotritones are, by definition, ambiguous. Also, unlike tritones, they are not associated, by definition, with a particluar clen. In fact, a pseudotritone may correspond to more than one clen in the same set, as in the next example.

Example 2.3. The two sets pictured in fig. 8 have the same chromatic and diatonic cardinalities. In both cases, $\frac{d-1}{2} = 2$ and $\frac{d+1}{2} = 3$. Now in fig. 8a, $<2> \cap <3> = \{3, 6\}$, indicating the presence of pseudotritones. On the other hand in fig. 8b, $<2> \cap <3> = \varnothing$, indicating there are no pseudotritones. Note that fig. 8a is not a ME set, and fig. 8b is a ME set. The following lemma shows that these sets exemplify the general situation: pseudotritones occur only in non-ME sets.

Lemma 2.10. No ME set has a pseudotritone.

Proof. If $c = d$, then $(c,d) = d$ and there are no ambiguities (lemma 2.3) and hence no pseudotritones. Assume $d < c$. The largest element in $<\frac{d-1}{2}>$ is $\left\lceil \frac{c(d-1)/2}{d} \right\rceil + 1$. If a pseudotritone exists, then

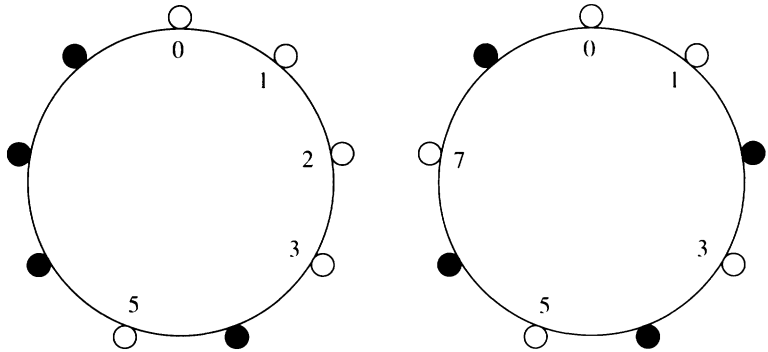
$$\left\lceil \frac{c(d-1)/2}{d} \right\rceil + 1 = \left\lceil \frac{c(d+1)/2}{d} \right\rceil \text{ (lemma 2.4).}$$

But

$$\begin{aligned}\left\lceil \frac{c(d-1)/2}{d} \right\rceil + 1 &= \frac{c+1}{2} + \left\lceil \frac{d-c}{2d} \right\rceil \\ &\leq \frac{c+1}{2} - 1 \text{ (since } d \neq c, \text{ implying } \frac{d-c}{2d} < 0) \\ &= \frac{c-1}{2}.\end{aligned}$$

Further,

$$\begin{aligned}\left\lceil \frac{c(d+1)/2}{d} \right\rceil &= \frac{c+1}{2} + \left\lceil \frac{c-d}{2d} \right\rceil \\ &\geq \frac{c+1}{2} + 0 \text{ (since } \frac{c-d}{2d} > 0) \\ &= \frac{c+1}{2}.\end{aligned}$$



(a) $D_{9,5} = \{0,1,2,3,5\}$
$<2> = \{2,3,5,6\}$
$<3> = \{3,4,6,7\}$
$<2> \cap <3> = (3,6)$
(pseudotritones)

(b) $M_{9,5} = \{0,1,3,5,7\}$
$<2> = \{3,4\}$
$<3> = \{5,6\}$
$<2> \cap <3> =$ null set
(no pseudotritones)

Figure 8

It follows that $\left[\frac{c(d-1)/2}{d}\right] + 1 \neq \left[\frac{c(d+1)/2}{d}\right]$ and hence $< \frac{d-1}{2} > \cap < \frac{d+1}{2} > = \varnothing$ whenever both c and d are odd. •

Thus, for ME sets, if c is even and d odd, then the "first" ambiguity is in $< \frac{d-1}{2} > \cap < \frac{d+1}{2} >$, but if both c and d are odd, then $< \frac{d-1}{2} > \cap < \frac{d+1}{2} >$ is always empty.

The above results concerning tritones may be summarized as follows (note that pseudotritone is defined only for the case where both c and d are odd):

1. If both c and d are even, then $M_{c,d}$ has $d/2$ tritones, all of which are unambiguous; and $< d/2 > = \{c/2\}$ (lemma 2.6).

2. If c is even, d odd, and $d \leq c/2$, then $M_{c,d}$ has no tritones (lemma 2.8).

3. If c is even, d odd, and $c/2 < d < c$, then $M_{c,d}$ has $d - c/2$ tritones, all of which are ambiguous; and $< (d - 1)/2 > \cap < (d + 1)/2 > = \{c/2\}$ (lemma 2.8).

4. If both c and d are odd, then tritone is not defined for $M_{c,d}$, and $M_{c,d}$ has no pseudotritones; that is, $< (d - 1)/2 > \cap < (d + 1)/2 > = \varnothing$ (lemma 2.10).

5. If c is odd and d even, then neither tritone nor pseudotritone is defined for $M_{c,d}$.

It is clear from the above that if a ME set contains one or more tritones, then either all of them are ambiguous or all are unambiguous.

As we demonstrate in the next section, the case of $M_{c,d}$ with c is even, d odd, and $c/2 < d < c$ (e.g., the usual diatonic set; see case 3 above) is of special interest. We need to prove one more lemma regarding ambiguities for this case.

Lemma 2.11. $M_{c,d}$ has an odd number of ambiguous clens if and only if c is even, d is odd, and $\frac{c}{2} < d < c$.

Proof.

($\Rightarrow$) If $M_{c,d}$ has an ambiguity, then $\frac{c}{2} < d < c$ (lemmas 2.2 and 2.3). Further, $<I> \cap <I+1> \neq \varnothing$ if and only if $<d-I> \cap <d-(I+1)> \neq \varnothing$ (lemma 1.16), implying $M_{c,d}$ has an odd number of ambiguous clens if and only if for one of these clens, $<I> = <d-(I+1)>$ (or equivalently $<I+1> = <d-I>$) and $<I> \cap <I+1> \neq \varnothing$. But if $<I> = <d-(I+1)>$ then $I = d-(I+1)$, implying $I = \frac{d-1}{2}$. Hence d must be odd. Further, if c is odd, then $<\frac{d-1}{2}> \cap <\frac{d+1}{2}> = \varnothing$ (lemma 2.10), implying by lemma 1.16 an even number of ambiguous clens (contradiction). Thus c must be even.

($\Leftarrow$) If c is even, d is odd, and $\frac{c}{2} < d < c$, then there is an ambiguous tritone (lemma 2.8), and $<\frac{d-1}{2}> \cap <\frac{d+1}{2}> = \left\{\frac{c}{2}\right\}$ (lemma 2.8, ii). It follows that $<d-\frac{d-1}{2}> \cap <d-\frac{d+1}{2}> = \left\{c-\frac{c}{2}\right\} = \left\{\frac{c}{2}\right\}$ is also an ambiguous tritone. But $<d-\frac{d-1}{2}> \cap <d-\frac{d+1}{2}> = <\frac{d+1}{2}> \cap <\frac{d-1}{2}>$, so these two pairs of interval spectrums are one and the same. Thus if c is even, d is odd, and $M_{c,d}$ has an ambiguity, then it has an ambiguous tritone (lemma 2.9) and all other ambiguities occur in pairs (lemma 1.16). It follows that $M_{c,d}$ has an odd number of ambiguous clens. ●

Diatonic sets

This section culminates in a theorem circumscribing the set of diatonic sets. We begin by stating, without proof, two lemmas which are easily proved via elementary number theory.

Lemma 2.12. Let c and d be integers such that $1 \leq d \leq c$. Then there are precisely (c,d) values of I, $0 \leq I \leq d-1$, such that d divides cI.

Remark. In number theory, a is said to "divide" b if and only if a and b are integers, $a \neq 0$, and there is an integer x such that $b = ax$. We write "$a \mid b$" when a divides b, and "$a \nmid b$" otherwise.

Lemma 2.13. If $(a, c) = 1$ and $(b,c) = 1$, then $(ab, c) = 1$.

We now state some properties of spec$(M_{c,d})$. The next lemma shows that the cardinality of spec$(M_{c,d})$ is a function of c and d.

Lemma 2.14. $\mid \text{spec}(M_{c,d}) \mid \; = 2d - (c,d)$.

Proof. From theorem 1.11 we know that $< I >$ is a single element spectrum if and only if $d \mid cI$; else $< I >$ is a two-element spectrum. Thus

$\mid$ spec$(M_{c,d} \mid \; =$ (number of single element spectrums)
$+ 2 \cdot$ (number of two-element spectrums)
$=$ (number of values of I such that $d \mid cI$)
$+\ 2 \cdot$ (number of values of I such that $d\ (\nmid)\ cI$), $0 \leq I \leq d-1$.

But the number of values of I such that $d \mid cI$ is (c,d) (lemma 2.12). There are d possible values of I. Thus the number of values of I such that $d\ (\nmid)\ cI$ is $d - (c,d)$. Therefore

$$\begin{aligned} \mid \text{spec}(M_{c,d}) \mid &= (c,d) + 2(d-(c,d)) \\ &= 2d-(c,d). \bullet \end{aligned}$$

Lemmas 2.15 and 2.16, given without proof, follow directly from lemma 2.4 and the definitions of ME set, ambiguity, and spec$(D_{c,d})$. Note that, by lemma 2.4, the elements of spec$(M_{c,d})$ have multiplicity one or two.

Lemma 2.15. There are n ambiguous clens in $M_{c,d}$ if and only if there are n elements of multiplicity 2 in spec$(M_{c,d})$.

Lemma 2.16. If $\mid$ spec$(M_{c,d}) \mid \, > c$, then $M_{c,d}$ has at least one ambiguous clen.

Lemma 2.17. If $\mid$ spec$(M_{c,d}) \mid \; \geq c + 1$, then C_c is properly included in spec$(M_{c,d})$.

Proof. Recall that C_c is the multiset $\{0, 1, 2, \ldots, c-1\}$. If $\mid$ spec$(M_{c,d}) \mid \; \geq c + 1$, then clearly $M_{c,d}$ has at least one ambiguity. Thus $\frac{c}{2} < d$ (lemma 2.2), implying C_c is included in spec$(M_{c,d})$ (theorem 1.12). Since $\mid$ spec$(M_{c,d}) \mid \, > \, \mid C_c \mid \; = c$, C_c is properly included in spec$(M_{c,d})$. $\bullet$

The usual diatonic set has the following important property: each of the possible clens $\{0, 1, 2, \ldots, c-1\}$ appears exactly once in its spectrum, except for clen 6, which is represented twice. The dual appearance of a clen in the spectrum signals the presence of an ambiguity. Thus spec$(M_{12,7})$ is the smallest spectrum including all clens while containing at least one ambiguity. The following definition formalizes this property, which is similar, though not identical, to Agmon's "efficiency" condition.

Definition 2.6. $\underline{M}(c)$ (read "min c") is the set of ME sets whose spectrums are the smallest sets that properly contains C_c. That is, $M_{c,d} \in \underline{M}(c)$ if and only if both of the following hold true:

(1) C_c is properly included in spec$(M_{c,d})$.

(2) For all $M^*_{c,d}$ such that C_c is properly included in spec $(M^*_{c,d})$, $|\text{spec}(M_{c,d})| \leq |\text{spec}(M^*_{c,d})|$.

We write M_c to indicate an element in $\underline{M}(c)$.

Example 2.3. $\underline{M}(12) = M(12,7)$.

This example tells us that $\underline{M}(12)$ consists of precisely all the major-scale sets.

Note that for a particular choice of c, $\underline{M}(c)$ is not necessarily restricted to a unique d. For example, $|\text{spec}(M_{15,11})| = |\text{spec}(M_{15,12})| = 21$, and $\underline{M}(15) = M(15,11) \cup M(15,12)$ since the set spectrums of $M_{15,11}$ and $M_{15,12}$ are the smallest multisets which properly contain C_{15}. If $c \equiv 3 \pmod 6$, the situation with $\underline{M}(c)$ is complex. For example, $\underline{M}(3) = \varnothing$, $\underline{M}(9) = M(9,7)$, $\underline{M}(15) = M(15,11) \cup M(15,12)$, $\underline{M}(21) = M(21,13)$. We conjecture that when $c \equiv 3 \pmod 6$ there is no finite set of algorithms that describe $\underline{M}(c)$. More specifically, we conjecture that if $3 \mid c$ and $5 \nmid c$, then $\underline{M}(c) = M(c, (c+5)/2)$; if $3 \mid c$, $5 \mid c$, $7 \nmid c$, and $9 \nmid c$, then $\underline{M}(c) = M(c, (c+9)/2)$; if $3 \mid c$, $5 \mid c$, $7 \nmid c$, and $9 \mid c$, then $\underline{M}(c) = M(c, (c+7)/2) \cup M(c, (c+9)/2)$, etc. It would appear that $\underline{M}(c)$ depends on how c factors into powers of odd primes.

Fortunately, if c is even or if $c \equiv 1,5 \pmod 6$, then $\underline{M}(c)$ *is* restricted to a unique d, as we show in theorem 2.1 below. In preparatioin for this theorem, we state the following lemma without proof.

Lemma 2.18. Let a, b, c, x, and y be integers such that $ax + by = c$. Then (x, y) divides c.

Theorem 2.1. Let $c \geq 4$.

(i) If c is not congruent to 0 (mod 4), then $\underline{M}(c) = M(c, c/2 + 1)$ and $|\text{spec}(\underline{M}_c)| = c + 1$.

(ii) If c is not congruent to 2 (mod 4), then $\underline{M}(c) = M(c, c/2 + 2)$ and $|\text{spec}(\underline{M}_c)| = c + 3$.

(iii) If c is not congruent to 1, 5 (mod 6) (or equivalently if c is odd and c is not congruent to 0 (mod 3)), then $\underline{M}(c) = M(c, (c+3)/2)$ and $|\text{spec}(\underline{M}_c)| = c + 2$.

Proof. In general, to prove $\underline{M}(c) = M(c,d)$ for a particular value of d, we first compute $|\text{spec}(M_{c,d})|$. We then show that there is no d^* such that $c < |\text{spec}(M_{c,d^*})| < |\text{spec}(M_{c,d})|$. This implies $M(c,d)$ is included in $\underline{M}(c)$. Finally, we show that if $M_{c,d^{**}} \in \underline{M}(c)$, then $M_{c,d^{**}} \in M(c,d)$.

(i) Assume $c \equiv 0 \pmod 4$. Then $(c/2, c/2 + 1) = 1$ and $(2, c/2+1) = 1$, implying $(c, c/2 + 1) = 1$ (lemma 2.13). Thus

$$\begin{aligned} |\operatorname{spec}(M_{c,c/2+1})| &= 2(c/2+1)-1 \text{ (lemma 2.14)} \\ &= c + 1. \end{aligned}$$

Since C_c is included in $\operatorname{spec}(M_{c,c/2+1})$ (lemma 2.17) and $|\operatorname{spec}(M_{c,c/2+1})| = c + 1$, we conclude that $M_{c,c/2+1} \in \underline{M}(c)$. Thus $M(c, c/2 + 1)$ is included in $\underline{M}(c)$. Further, if $M_{c,d} \in \underline{M}(c)$, then $|\operatorname{spec}(M_{c,d})| = c + 1$. Thus $2d - (c,d) = c + 1$ (lemma 2.14). But then (c,d) divides d, (c,d), and c; hence (c,d) must divide 1, implying $(c,d) = 1$. It follows that

$$\begin{aligned} 2d-1 &= c + 1 \\ d &= c/2 + 1. \end{aligned}$$

Hence $M_{c,d} \in M(c, c/2 + 1)$, implying $\underline{M}(c)$ is included in $M(c, c/2 + 1)$. We conclude that $\underline{M}(c) = M(c, c/2 + 1)$.

(ii) Assume $c = 2 \pmod 4$. Then $(c/2 + 2) - c/2 = 2$, implying $(c/2, c/2 + 2)$ divides 2 (lemma 2.18). Thus $(c/2, c/2 + 2) = 1,2$. But $c/2$ is odd, implying $(c/2, c/2 + 2) = 1$. Further, $c/2 + 2$ is odd, implying $(2, c/2 + 2) = 1$. It follows that $(c, c/2 + 2) = 1$ (lemma 2.13). Now we compute $|\operatorname{spec}(M_{c,c/2+2})|$.

$$\begin{aligned} |\operatorname{spec}(M_{c,c/2+2})| &= 2(c/2 + 2)-1 \text{ (lemma 2.14)} \\ &= c + 3. \end{aligned}$$

Suppose $|\operatorname{spec}(M_{c,d})| = c + 1$. Then

$$2d-(c,d) = c + 1 \text{ (lemma 2.14)}.$$

Thus since (c,d) divides d, (c,d), and c, we conclude that (c,d) divides 1; hence $(c,d) = 1$. It follows that

$$\begin{aligned} 2d-1 &= c + 1 \\ d &= c/2 + 1. \end{aligned}$$

But since $c \equiv 2 \pmod 4$, d is even, implying $(c,d) \geq 2$ (contradiction). Now suppose $|\operatorname{spec}(M_{c,d})| = c + 2$. Since C_c is included in $\operatorname{spec}(M_{c,d})$ (lemma 2.17), $M_{c,d}$ must have two ambiguous clens (lemma 2.15). Hence d must be even (lemma 2.11). Also

$$2d-(c,d) = c + 2 \text{ (lemma 2.14)}.$$

But since $c \equiv 2 \pmod 4$, we have $c + 2 \equiv 0 \pmod 4$, and since d is even, $2d \equiv 0 \pmod 4$. Thus $(c,d) \equiv 0 \pmod 4$, implying 4 divides

(c,d). But then 4 must divide c, implying $c \equiv 0 \pmod 4$ (contradiction). It follows that $M(c, c/2 + 2)$ is included in $\underline{M}(c)$.

Let $M_{c,d} \in \underline{M}(c)$. Then $|\operatorname{spec}(M_{c,d})| = c + 3$. Thus

$$2d - (c,d) = c + 3.$$

Since (c,d) divides d, (c,d), and c, it must divide 3. It follows that $(c,d) = 1,3$. If $(c,d) = 3$, then

$$2d - 3 = c + 3$$
$$d = c/2 + 3.$$

But since $c \equiv 2 \pmod 4$, $c/2 + 3$ must be even. Thus 2 divides both c and d, contradicting the assumption that $(c,d) = 3$. Hence $(c,d) = 1$. But then

$$2d - 1 = c + 3$$
$$d = c/2 + 2.$$

Thus $M_{c,d} \in M(c, c/2 + 2)$, implying $\underline{M}(c)$ is included in $M(c, c/2 + 2)$. It follows that $\underline{M}(c) = M(c, c/2 + 2)$.

(iii) Assume $c \equiv 1,5 \pmod 6$. Then $2((c + 3)/2) - c = 3$. Thus $(c, (c + 3)/2)$ divides 3 (lemma 2.18), implying $(c, (c + 3)/2) = 1,3$. But if c is not congruent to 0 (mod 3), 3 does not divide c. Thus $(c, (c + 3)/2) = 1$. Then

$$\begin{aligned} |\operatorname{spec}(M_{c,(c+3)/2})| &= 2((c+3)/2) - (c, (c+3)/2) \\ &= c + 3 - 1 \\ &= c + 2. \end{aligned}$$

Since c is odd, $M_{c,d}$ must have either no ambiguous clens or an even number of ambiguous clens (lemma 2.11). Thus we cannot have $|\operatorname{spec}(M_{c,d})| = c + 1$ (lemma 2.15). We conclude that $Mc,(c + 3)/2$ is included in $\underline{M}(c)$.

Now assume $M_{c,d} \in \underline{M}(c)$. Then

$$|\operatorname{spec}(M_{c,d})| = c + 2$$
$$2d - (c,d) = c + 2 \text{ (lemma 2.14)}.$$

Since (c,d) divides d, (c,d), and c, $(c,d) = 1,2$. If $(c,d) = 2$, then c must be even (contradiction). Hence $(c,d) = 1$. Then

$$2d - 1 = c + 2$$
$$d = (c+3)/2.$$

Thus $M_{c,d} \in M(c, (c + 3)/2)$, implying $\underline{M}(c)$ is included in $M(c, (c + 3)/2)$. It follows that $\underline{M}(c) = M(c, (c + 3)/2)$. ●

The case $c \equiv 0 \pmod 4$, covered in part 1 of theorem 2.1 above, is of particular importance as we shall see.

The presence of exactly two half steps (intervals of clen 1) is an important characteristic of the usual diatonic set. The next lemma shows that ME sets with two half steps occur only under a particular condition.

Lemma 2.19. $M_{c,d}$ has precisely two intervals of clen 1 if and only if $c = 2(d-1)$.

Proof. It is clear that clen 1 can only be associated with dlen 1, for if clen 1 is associated with, for example, dlen 2, then

$$\left[\frac{c \cdot 2}{d}\right] = 1$$

$$\frac{2c}{d} < 2$$

$$c < d \text{ (contradiction).}$$

($\Rightarrow$) If $M_{c,d}$ has an interval of clen 1, then $1 \in <1>$, implying $\left[\frac{c \cdot 1}{d}\right] = 1$. Thus

$$\frac{c}{d} < 2$$

$$c < 2d.$$

Hence, by theorem 1.11,

$$c \cdot 1 = d + (d-2)$$

$$c = 2(d-1).$$

($\Leftarrow$) If $c = 2(d-1)$, then

$$c \cdot 1 = d + (d-2),$$

implying $M_{c,d}$ has precisely two intervals associated with dlen 1 that have clen 1. Since clen 1 can only be associated with dlen 1, if follows that there are precisely two intervals of clen 1.●

Theorem 2.2 below characterizes the class of ME sets with exactly one ambiguity by stating ten equivalent conditions. Conditions 1–4 deal with ambiguity; conditions 5–7 are different ways of stating the relationship between chromatic and diatonic cardinalities; condition 8 deals with "half steps"; conditions 9 and 10 deal with the spectrum of the set.

Theorem 2.2. For $M_{c,d}$ the following are equivalent:

(1) $M_{c,d}$ has precisely one ambiguity.
(2) $M_{c,d}$ has precisely one ambiguous clen.
(3) $M_{c,d}$ has precisely one ambiguous tritone.

(4) $M_{c,d}$ has precisely one tritone and $d \neq 2$.
(5) $c = 2(d-1)$ and $(c,d) = 1$.
(6) $c = 2(d-1)$ and d is odd.
(7) $c = 2(d-1)$ and $c \equiv 0 \pmod 4$.
(8) $M_{c,d}$ has precisely two intervals of clen 1, and $c \equiv 0 \pmod 4$.
(9) $M(c,d) = \underline{M}(c)$, and $c \equiv 0 \pmod 4$.
(10) $|\operatorname{spec}(\overline{Mc,d})| = c + 1$

Proof. We first show that conditions 7 and 8 are mutually implicative. We then show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 9 \Rightarrow 10 \Rightarrow 1$.

(7 $\Leftrightarrow$ 8) This follows directly from lemma 2.19.

(1 $\Rightarrow$ 2) From the definition of ambiguity it is clear that if $M_{c,d}$ has more than one ambiguous clen, it has more than one ambiguity. Thus if $M_{c,d}$ has precisely one ambiguity, it has precisely one ambiguous clen.

(2 $\Rightarrow$ 3) If $M_{c,d}$ has precisely one ambiguous clen, there is an odd number of ambiguous clens, implying c is even and d odd. (lemma 2.11). Hence there is a tritone ambiguity (lemma 2.9).

Next we assume that there are two (or more) ambiguous tritones, and show that this entails at least two ambiguous clens. Consider the two tritones $\{D_f, D_h\}$ and $\{D_g, D_i\}$ (see fig. 9). Clearly clen $(D_f, D_g) = \text{clen}(D_h, D_i)$. Call this clen c_0: that is $\text{clen}(D_f, D_g) = \text{clen}(D_h, D_i) = c_0$. Since the tritones are ambiguous, their clens correspond to two distinct dlens, say $< I_1 >$ and $< I_1 + 1 >$ (lemma 2.4). Without loss of generality we may assume $\text{dlen}(D_f, D_h) = I_1$ and dlen $(D_g, D_i) = I_1 + 1$. Now let dlen $(D_f, D_g) = I_0$ and $\text{dlen}(D_h, D_i) = I_2$. Then

$$
\begin{aligned}
I_2 &= [\text{dlen}(D_f, D_g) + \text{dlen}(D_g, D_i)] - \text{dlen}(D_f, D_h) \\
&= [I_0 + (I_1 + 1)] - I_1 \\
&= I_0 + 1.
\end{aligned}
$$

But then c_0 is a member of $< I_0 >$ intersect $< I_0 + 1 >$, implying a second ambiguous clen. Thus if there is precisely one ambiguous clen, then there is precisely one ambiguous tritone.

(3 $\Rightarrow$ 4) Assume there is precisely one ambiguous tritone. Then d must be odd (lemma 2.7). Hence $d \neq 2$ and $M_{c,d}$ can contain no unambiguous tritones (lemma 2.8). It follows that $M_{c,d}$ contains precisely one tritone.

(4 $\Rightarrow$ 5) Assume $M_{c,d}$ has precisely one tritone and $d \neq 2$. If d is even then $M_{c,d}$ has $d/2$ tritones (lemma 2.6). Since $M_{c,d}$ has precisely one tritone, $d/2 = 1$, implying $d = 2$, contradicting the assumption.

Thus d must be odd; hence $(2, d) = 1$. Further, since there is precisely one tritone, $d - \frac{c}{2} = 1$ (lemma 2.8). It follows that $(c2, d = 1$ and $c = 2(d-1)$. Since $(2, d) = 1$, $\left(2\left(\frac{c}{2}\right), d\right) = (c,d) = 1$ (lemma 2.13).

(5 ⇒ 6) Assume $(c, d) = 1$ and $c = 2(d - 1)$. Then d must be odd, else $(c,d) \geq 2$.

(6 ⇒ 7) Assume d is odd and $c = 2(d - 1)$. Then $d - 1$ is even and hence $4 \mid 2(d - 1)$. It follows that $c \equiv 0 \pmod 4$.

(7 ⇒ 9) Assume $c = 2(d - 1)$ and $c \equiv 0 \pmod 4$. Then $d = \frac{c}{2} + 1$, implying $M(c,d) = M(c, \frac{c}{2} + 1) = \underline{M}(c)$ (theorem 2.1) and $c \equiv 0 \pmod 4$.

(9 ⇒ 10) Assume $M(c,d) = \underline{M}(c)$ and $c \equiv 0 \pmod 4$. Then $|\operatorname{spec}(\underline{M}_c)| = c + 1$ (theorem 2.1). Thus $|\operatorname{spec}(M_{c,d})| = c + 1$.

(10 ⇒ 1) Assume $|\operatorname{spec}(M_{c,d})| = c + 1$. Then C_c is properly included in $\operatorname{spec}(M_{c,d})$ (lemma 2.17). Since $|C_c| = c$, and every element of C_c has multiplicity 1, there must be precisely one element of $\operatorname{spec}(M_{c,d})$ that has multiplicity 2. Thus there must be precisely one ambiguous clen (lemma 2.15). It follows that c is even and d is odd (lemma 2.11), implying the ambiguous clen is a tritone (lemma 2.9). We must now determine how many tritone ambiguities there are.

Since $|\operatorname{spec}(M_{c,d})| = c + 1$, we know that $2d - (c,d) = c + 1$ (lemma 2.14). Further, since (c,d) divides c and d, (c,d) must divide 1. Hence $(c,d) = 1$. It follows that

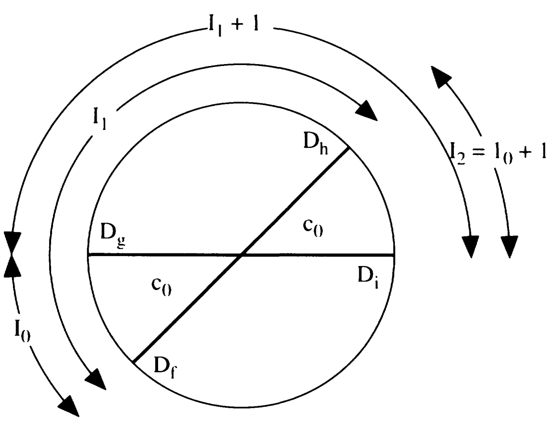


Figure 9

$$2d-1 = c + 1$$

$$d-c/2 = 1.$$

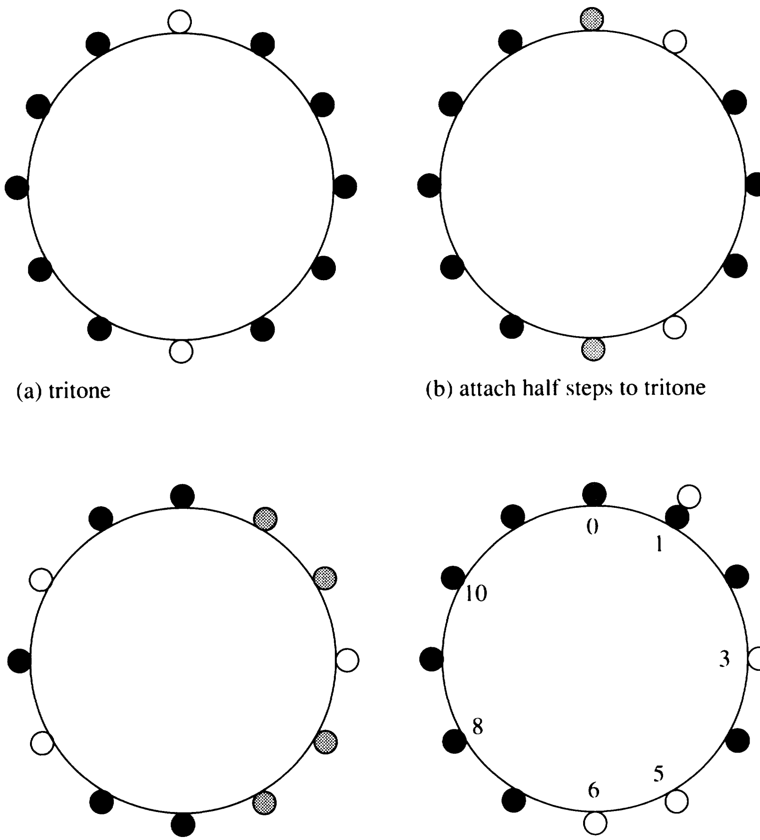
Thus there is precisely one (tritone) ambiguity (lemma 2.8). •

Definitions 2.7. If a set satisfies the conditions of theorem 2.2, we say that it is a *diatonic set.* If a diatonic set has $d = 7$, we say that it is a *usual diatonic set;* otherwise we say that it is a *hyperdiatonic set.*

Together with the uniqueness property of ME sets, theorem 2.2 tells us that each universe whose cardinality is congruent to 0 (mod 4) contains an essentially unique diatonic set—that is, unique within transposition in its respective universe. The most remarkable feature of theorem 2.2 is what might be called "the power of the single ambiguity": ME sets with just one ambiguity exist *only* in universes where the chromatic cardinality is congruent to 0 (mod 4). We feel that characterizing the usual diatonic set as the unique ME set with a lone ambiguity in the 12-pc universe has strong intuitive appeal.

As in the case of any ME set, diatonic sets may be constructed by means of the Clough-Myerson algorithm (theorem 1.1) by simply "plugging in" an appropriate pair of parameters c and d. They may also be constructed by the method of superimposing circles given following example 1.4. Fig. 10 shows another graphic method based on the single tritone and the requirement for maximal separation of the two "half steps": The tritone is first put in place (fig. 10a). Now, for the tritone to be ambiguous, both half-steps must be located on the same "side" of the tritone, and, for maximal evenness, they must be as far apart as possible within this half-octave; it follows that the half-steps must be "attached" to the tritone (fig. 10b). Finally, whole steps are filled in as necessary (fig. 10c). It is easy to see how this construction implies the tetrachordal structure of the set (fig. 10d). Fig. 11 shows the same procedure used to construct the hyperdiatonic set with $c = 16$, $d = 9$. Naturally all of these constructions also produce the complements of diatonic sets, which we will study in the next part of the paper.

It is well known that a tritone plus one other pc is sufficient to determine a unique transposition of $M_{12,7}$. For example, if we choose the tritone {B/C♭, E♯/F} and then select any other pc, say D♯, these three pcs imply the F♯/G♭ major scale and no other major scale. The construction procedure shown in fig. 10 provides insight as to why this is true: In fig. 10b the half steps might have been placed on the left side of the tritone. This would have yielded the other instance of $M_{12,7}$ with the same tritone—a set sharing only the two pcs of the tritone with its "companion" $M_{12,7}$. We will soon show that this is a feature of diatonic sets in general.



(a) tritone

(b) attach half steps to tritone

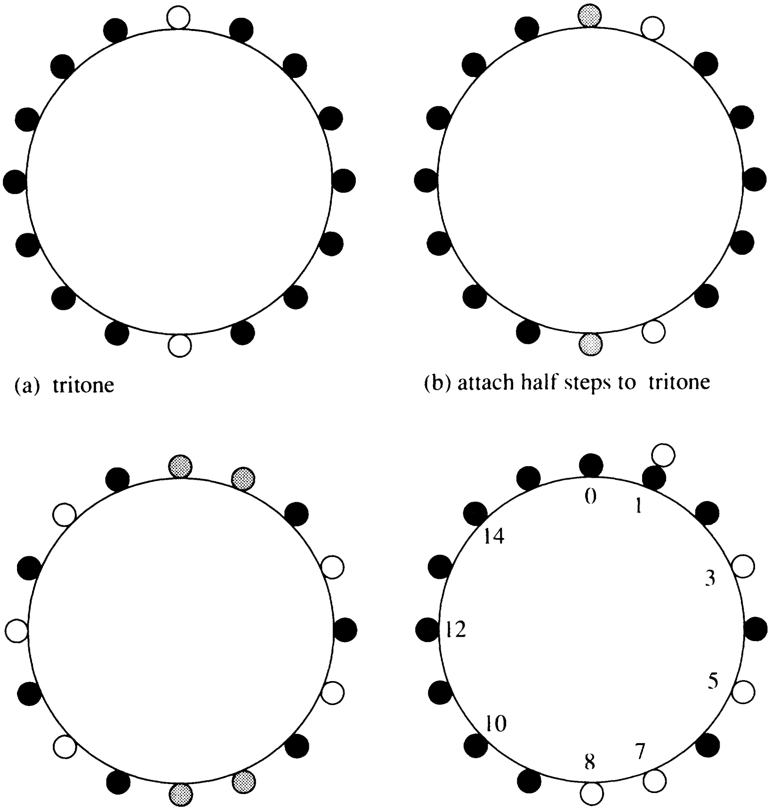
(c) fill in whole steps

(d) tetrachords: union of {1,3,5,6} and {8,10,0,1}

Figure 10. Construction of $M_{12,7}$, the usual diatonic set

Lemma 2.20. Let t be a tritone $\{D_i, D_j\}$. There are exactly two (transpositionally equivalent) diatonic sets which include t.

Proof. By definition and by theorem 2.2, a diatonic set has exactly one tritone. By lemma 2.7 either $\mathrm{dlen}(D_i, D_j) = \frac{d-1}{2}$ and $\mathrm{dlen}(D_j, D_i) = \frac{d+1}{2}$ or these assignments are reversed. The two possibilities imply two distinct (transpositionally equivalent) diatonic sets. ●



(a) tritone

(b) attach half steps to tritone

(c) fill in whole steps

(d) tetrachords : union of {1,3,5,7,8} and {10,12,14,0,1}

Figure 11. Construction of diatonic set $M_{16,9}$

Theorem 2.3. Let t be as in the lemma 2.20, let M_A and M_B be the two diatonic sets which include t, and let D be a 3-element set $\{t\} \cup \{D_k\}$, where D_k is a pc distinct from both D_i and D_j. Then D is included in either M_A or M_B but not both.

Proof. Since c is even, the chromatic universe $\{0, 1, \ldots, c - 1\}$ may be divided into $\frac{c}{2}$ tritones, $\{0, c/2\}$, $\{1, 1 + c/2\}$, $\{2, 2 + c/2\}$, . . . , $\{c/2 - 1, c - 1\}$, one of which is t. Suppose $t = \{0, c/2\}$. Consider any one of the other tritones, say $\{1, 1 + c/2\}$. If pc $1 \in M_A$, then pc $(1 + c/2) \notin M_A$ because if both of these pcs were in

M_A, this would imply that M_A has more than one tritone, contradicting the assumption that it is a diatonic set. Similarly, for any tritone except *t*, if one of the pcs of the tritone is in M_A, the other is not. Now since M_A and M_B are diatonic sets, $|M_A| = |M_B| = c/2 + 1 = 2 + (c/2 - 1)$. Therefore M_A and M_B each contain the two pcs of *t* plus exactly one pc from each of the $(c/2 - 1)$ other tritones. It follows that *D* is in either M_A or M_B but not both. ●

3 Generated Sets; Complements; Interval Circles; and Second-Order ME Sets

Generated ME sets

Extending the familiar concept of the circle of fifths, Clough and Myerson show that any rounded set with $(c,d) = 1$ may be generated by a single clen. May all ME sets be formed in this way? We proceed to explore this question and other related ones.

Definition 3.1. A *g-generated* ME set, $M^g_{c,d}$, is a ME set that can be represented as follows: $M^g_{c,d} = \{j, j + g, j + 2g, \ldots, j + (d - 1)g\}$, where *j* is a non-negative integer, $0 \leq j \leq c - 1$, and $1 \leq g \leq c - 1$.

Examples 3.1. The usual diatonic and pentatonic sets, $M^5_{12,7}$ and $M^5_{12,5}$, are 5-generated (5=perfect 4th).

Lemma 3.1. An integer g, $1 < g < c - 1$, generates $M_{c,d}$ if and only if $c - g$ also generates $M_{c,d}$.

Proof. Let $M^g_{c,d}$ be represented as in the above definition and let $-M^g_{c,d}$ be the inversion of $M^g_{c,d}$. Since $-M^g_{c,d}$ is also a ME set (theorem 1.8), we may write

$$-M^g_{c,d} = \{-j,\ -j + (c-g),\ -j+2(c-g),\ \ldots,\ -j + (d-1)(c-g)\}.$$

Both sets have the same parameters *c* and *d*. Thus they are equivalent under transposition. Hence there exists an integer *n* such that

$$\begin{aligned} M^g_{c,d} &= \{n\} - M^g_{c,d} \\ &= \{n-j,\ n-j + (c-g),\ \ldots,\ n-j \\ &\quad + (d-1)(c-g)\} \\ &= M^{c-g}_{c,d}. \end{aligned}$$

It follows that if *g* generates $M^g_{c,d}$, so does $c - g$. The converse follows immediately since $c - (c - g) = g$. ●

Examples 3.2. The generators of the usual diatonic set are 5 and $12-5=7$. The generators of the usual whole-tone set are 2 and $12-2=10$.

Note that any set of diatonic cardinality 0, 1, $c-1$, or c, is maximally even. In this sense we can regard sets of these cardinalities as *trivially* ME and other ME sets as nontrivially so. This distinction will help to simplify the following treatment of generators.

Lemma 3.2. If g is a generator of $M_{c,d}$ (nontrivial), then g is an unambiguous clen.

Proof. Assume g is an ambiguous clen. Then

$$g = \left[\frac{cI}{d}\right] + 1 = \left[\frac{c(I+1)}{d}\right] \text{ (lemma 2.4).}$$

Further,

$$cI = d\left[\frac{cI}{d}\right] + r_1 \text{ (where } 0 \le r_1 \le d-1)$$

and

$$c(I+1) = d\left[\frac{c(I+1)}{d}\right] + r_2 \ (0 \le r_2 \le d-1).$$

Thus there are r_1 intervals associated with I, whose clen is

$$g = \left[\frac{cI}{d}\right] + 1$$

and $d - r_2$ intervals associated with $I+1$, whose clen is

$$\left[\frac{c(I+1)}{d}\right] \text{ (theorem 1.11).}$$

But then

$$cI = d(g-1) + r_1$$
$$r_1 = cI - dg + d$$

and

$$c(I+1) = dg + r_2$$
$$d - r_2 = dg + d - cI - c.$$

Since g is a generator, at least $d-1$ intervals must have clen g. Hence

$$r_1 + (d-r_2) \ge d-1$$
$$(dg + d - cI - c) + (cI - dg + d) \ge d-1$$
$$d \ge c-1,$$

contradicting the hypothesis that $M_{c,d}$ is nontrivial. It follows that g is an unambiguous clen. ●

Lemma 3.3. Suppose $(c,d) \neq 1$. If $M_{c,d}$ has a generator, then $(c,d) = d$.

Proof. If g generates $M^g_{c,d}$, then all intervals with clen g are associated with the same dlen, say I (lemma 3.2). If I has a two-element spectrum, then there is one interval associated with I that has clen $g \pm 1$. So

$$cI = gd + 1$$

or

$$cI = gd + (d-1) \text{ (theorem 1.11).}$$

In either case, since (c,d) divides c and d, (c,d) must also divide 1, contradicting the hypthesis that $(c,d) \neq 1$. It follows that all intervals associated with I have clen g. Therefore

$$dg = cI \text{ (lemma 1.2).}$$

Now consider two ways of representing $M^g_{c,d}$:

(1) $M^g_{c,d} = \{D_0, D_1, D_2, \ldots, D_{d-1}\}$, where $0 \leq D_0 < D_1 < D_2 < \ldots < D_{d-1} < c$;

(2) $M^g_{c,d} = \{j, j + g, j + 2g, \ldots, j + (d\text{-}1)g\}$, where the elements are reduced (mod c).

Suppose <l> is a 2-element spectrum. Then

$$<I> = \{[c/d], [c/d] + 1\} \text{ (theorem 1.11)}$$

and there exist D_{k_2+1}, D_{k_1}, D_{k_2+1}, D_{k_2} in $M^g_{c,d}$ such that

$$D_{k_1+1} - D_{k_1} \equiv [c/d] + 1 \pmod{c}$$
$$D_{k_1+1} - D_{k_2} \equiv [c/d] \pmod{c}.$$

But since $M^g_{c,d}$ is a generated ME set, there exist integers m_1, n_1, m_2, and n_2 such that

$$D_{k_1+1} \equiv j + m_1 g \pmod{c}$$
$$D_{k_1} \equiv j + n_1 g \pmod{c}$$
$$D_{k_2+1} \equiv j + m_2 g \pmod{c}$$
$$D_{k_2} \equiv j + n_2 g \pmod{c}.$$

Thus

$$D_{k_1+1} - D_{k_1} \equiv (m_1 - n_1)g \equiv [c/d] + 1 \pmod{c}$$
$$D_{k_2+1} - D_{k_2} \equiv (m_2 - n_2)g \equiv [c/d] \pmod{c}.$$

Subtracting the second congruence from the first we get

$$(m_1 - n_1 - m_2 + n_2)g \equiv 1 \pmod{c}.$$

It follows that

$$(g,c) = 1.$$

Clearly I $\leq g$. But since $dg = cI$ and $(g,c) = 1$, g divides I. It follows that $g = I$, implying c $= d$. But then <1> must be a single-element spectrum (theorem 1.11), contradicting the assumption that <1> is a 2-element spectrum. Hence <1> must be a single-element spectrum and thus $(c,d) = d$ (theorem 1.11).●

Lemma 3.4. Let $M_{c,d}$ be a nontrivial ME set and suppose $(c,d) = 1$. Then there are $d - 1$ intervals whose clen is g and whose dlen is $I \neq 0$ if and only if g is a generator of $M_{c,d.}$

Proof.

($\Rightarrow$) Since $(c,d) = 1$, all nonzero interval spectrums have two elements (lemma 1.3). Thus, of the intervals associated with I, $d - 1$ have clen g and one has clen $g \pm 1$, (lemma 3.2). Then for

$$M_{c,d} = \{D_0, D_1, \ldots, D_{d-1}\}$$

there is some D_j such that clen $(D_j, D_{j+I}) = g \pm 1$, where the subscript $j + I$ is reduced (mod d). Now we order $M_{c,d}$ as follows:

$$M_{c,d} = \{D_{j+I}, D_{j+2I}, \ldots, D_{j+(d-1)I}, D_j\},$$

where all subscripts are reduced (mod d). But clen $(D_j, D_{j+I}) = g \pm 1$, implying that the clen from any element other than D_j to the next element must be g. Thus we may write

$$M_{c,d} = \{D_{j+I}, D_{j+I} + g, D_{j+I} + 2g, \ldots, D_{j+I} + (d - 1)g\}.$$

It follows that $M_{c,d}$ is g-generated.

($\Leftarrow$) If g is a generator, then g is unambiguous (lemma 3.2). Hence all intervals of clen g must have the same dlen, say I. Clearly, if g is a generator, then g $\neq 0$ and hence $g \notin$<0>. Then there must be $d - 1$ intervals with clen g and one with clen $g \pm 1$, all with dlen I (lemma 1.3). ●

The direct part of the above lemma holds true for both trivial and nontrivial ME sets. As the following example shows, the converse part of the lemma does not hold for trivial ME sets.

Example 3.3. $M^7_{12,11}$ is generated by clen 7. However, there are not $d - 1 = 10$ intervals of clen 7 associated with one particular dlen. Hence clen 7 must be ambiguous. In fact, $7 \in \langle 6 \rangle \cap \langle 7 \rangle$.

The following lemma and corollary, given without proof, are well-known in number theory.

Lemma 3.5. If $(c,d) = 1$, then there exist unique integers g_1 and g_2, $1 \leq g_1, g_2 \leq c - 1$, such that $dg_1 \equiv 1$ and $dg_2 \equiv -1 \pmod{c}$.

Corollary 3.1. Let c, d, g_1, g_2 be as in the above lemma. Then $g_2 = c - g_1$.

As established in part 1, there are three classes of ME sets based on (c,d): class A, $(c,d) = 1$; class B, $(c,d) = d$; and class C, $1<(c,d)<d$. The following theorem shows that the generator of $M_{c,d}$, if there is one, is dependent upon (c,d).

Theorem 3.1. Let $M_{c,d} = \{D_0, D_1, \ldots, D_{d-1}\}$. Then

A. The following are equivalent.

(1A) $(c,d) = 1$.

(2A) All nonzero interval spectrums have two elements.

(3A) g_1, g_2, as defined in lemma 3.5 exist and are generators.

B. The following are equivalent.

(1B) $(c,d) = d$.

(2B) All interval spectrums have one element.

(3B) c/d and $c-c/d$ are generators.

C. The following are equivalent.

(1C) $1<(c,d)<d$.

(2C) There is at least one nonzero spectrum with one element and one with two elements.

(3C) $M_{c,d}$ has no generator.

Proof.

A. **(1A⇒2A)** Assume $(c,d) = 1$. Then every nonzero spectrum has two elements (lemma 1.3).

(2A⇒3A) Note that $\langle d/(c,d) \rangle = \{c/(c,d)\}$ (lemma 1.7). Since every nonzero spectrum has two elements, we must conclude that $(c, d) = 1$, else there would be a nonzero one-element spectrum, namely $\langle d/(c, d) \rangle$ (lemma 1.7). Hence g_1 and g_2 as in lemma 3.2 exist. Consider g_1 with $dg_1 \equiv -1 \pmod{c}$. There exists an I, $1 \leq I \leq d - 1$, such that $cI = dg_1 + 1$. It follows that, for the set of d intervals associated with I, there are precisely $d - 1$ of them with clen g_1 (theorem 1.11). Thus g_1 is a generator (lemma 3.4). By lemma 3.1 and corollary 3.1, $M_{c,d}$ is also generated by g_2.

(3A⇒1A) Clearly if $dg_1 \equiv -1 \pmod{c}$ or $dg_2 \equiv 1 \pmod{c}$, then $(c, d) = 1$.

B. **(1B⇒2B)** Assume $(c,d) = d$. Then $\langle I \rangle = \{cI/d\}$ (lemma 1.7), implying every spectrum has one element.

(2B⇒3B) Assume every spectrum has one element. Consider the intervals of dlen 1. By theorem 1.11, $c \cdot 1 = dq$. Thus $d \mid c$ and $<1> = \{c/d\}$ (lemma 1.7). It follows that

$$M_{c,d} = \{D_0, D_0 + c/d, D_0 + 2(c/d), \ldots, D_0 + (d - 1)(c/d)\}$$

and hence c/d is a generator. Further, $c - c/d$ is a generator (lemma 3.1).

(3B⇒1B) If c/d and $c - c/d$ are generators, then c/d must be an integer. Thus $(c,d) = d$.

C. Here the method is to show that 1C and 2C imply each other, and that 1C and 3C imply each other.

(1C⇒2C) Assume $1 < (c,d) < d$. Then d does not divide c, and hence $c \cdot 1 = dq + r$, where $1 \leq r \leq d - 1$. It follows that $<1>$ is a two-element spectrum (theorem 1.11). If $I = \frac{d}{(c,d)}$, then $<I>$ is a one-element spectrum (lemma 1.7). Thus $M_{c,d}$ has both one- and two-element nonzero spectrums.

(2C⇒1C) Assume that there is at least one 1-element nonzero spectrum and at least one 2-element spectrum. If $(c,d) = 1$, then there are no 1-element nonzero spectrums (part A); and if $(c,d) = d$, then there are no 2-element spectrums (part B). Hence $1 < (c,d) < d$.

(1C⇒3C) This follows immediately from lemma 3.3.

(3C⇒1C) From parts A and B we know that if $(c,d) = 1$ or $(c,d) = d$, then $M_{c,d}$ is a generated set. Hence if $M_{c,d}$ is not a generated set, then $1 < (c,d) < d$. ●

Theorem 3. 1 will be important in relation to complementary ME sets, considered in the next section. We will also need to consider trivial ME sets.

Theorem 3.2. Trivial ME sets are generated.

Proof. Since neither $M_{c,0}$ nor $M_{c,1}$ has a second element which can be generated with reference to a first element, we may consider any g, $0 \leq g \leq c - 1$, to be a generator. Any clen that is coprime to c is a generator for $M_{c,c-1}$ or $M_{c,c}$. ●

ME complements

In the study of ME sets, as in atonal set theory, the complement of a set with respect to the chromatic universe turns out to be an object of considerable interest. The reader may already have a number of insights regarding the complementation of the usual diatonic and (anhemitonic) pentatonic sets, $M_{12,7}$ and $M_{12,5}$. In this section we will see how some of these insights may be generalized for diatonic sets or all ME sets.

We will usually speak, not of literal complementation (as in the case of white and black keys), but of "abstract" complementation. As in atonal set theory, if each of two set classes includes one of a pair of literally complementary sets, then we say that the two set classes are abstractly complementary. For example, $M(12,7)$ and $M(12,5)$ are abstractly complementary, since they contain, respectively, the C-major set $\{0, 2, 4, 5, 7, 9, 11\}$ and the literally complementary pentatonic set $\{1, 3, 6, 8, 10\}$. Where the terms *literal* and *abstract* are omitted in our discussion, the distinction will be clear from context.

Definition 3.2. The *complement* of $D_{c,d}$, $\mathrm{compl}(D_{c,d})$, is $U_c \backslash D_{c,d}$ (the set consisting of all elements of U_c not in $D_{c,d}$). Note that the cardinality of $\mathrm{compl}(D_{c,d})$ is $c - d$.

We will state without proof two lemma easily proved in elementary number theory, and then prove that the complement of a ME set is itself a ME set.

Lemma 3.6. For any integers p and d there exist integers q and r, $0 \leq r \leq \frac{d}{(c,d)} - 1$, such that

$$c = dq + (c,d)r.$$

Lemma 3.7. Let c, d, x, y, z be integers such that

$$\frac{x}{c} + \frac{y}{d} = \frac{z}{(c,d)}.$$

Then c divides $(c,d)x$ and d divides $(c,d)y$.

Theorem 3.3. $J^m_{c,d}$ and $J^{c-m-1}_{c,c-d}$, $0 \leq m \leq c - 1$, are complementary sets; that is, $J^{c-m-1}_{c,c-d} = \mathrm{compl}(J^m_{c,d})$.

Proof. Let $m = j(c,d) + r$, where $0 \leq r \leq (c,d) - 1$. Then

$$c - m - 1 = (c - (j + 1)(c,d)) + ((c,d) - r - 1)$$
$$\text{(where } 0 \leq (c,d) - r - 1 \leq (c,d) - 1).$$

So $J^m_{c,d} = J^{j(c,d)}_{c,d}$ and $J^{c-m-1}_{c,c-d} = J^{c-(j+1)(c,d)}_{c,c-d}$ (lemma 1.11). Thus we need only show that $J^{j(c,d)}_{c,d}$ and $J^{c-(j+1)(c,d)}_{c,c-d}$ are complementary sets. But since $|J^{j(c,d)}_{c,d}| = d$ and $|J^{c-(j+1)(c,d)}_{c,c-d}| = c - d$, these two sets are literally complementary if and only if they have no elements in common.

Assume, to the contrary, that the two sets have one or more elements in common. By definition the elements of $J^{j(c,d)}_{c,d}$ and $J^{c-(j+1)(c,d)}_{c,c-d}$ are $\left[\frac{cN + j(c,d)}{d}\right]$, $0 \leq N \leq d - 1$, and $\left[\frac{cM + c - (j + 1)(c,d)}{c - d}\right]$, $0 \leq M \leq c - d - 1$, respectively. Then by assumption there exist M and N such that

$$\left[\frac{cN + j(c,d)}{d}\right] = \left[\frac{cM + c - (j + 1)(c,d)}{c - d}\right] = p,$$

for some pitch class p.

Then by lemma 3.6

$$cN + j(c,d) = dp + (c,d)r_1,$$

where $0 \le r_1 \le d/(c,d) - 1$, and

$$cM + c-(j + 1)(c,d) = (c-d)p + (c,d)r_2,$$

where $0 \le r_2 \le d/(c,d) - 1) - 1$, noting here that $(c, c - d) = (c,d)$. Now, adding the remainder inequalities, we get

$$0 \le r_1 + r_2 \le \frac{c}{(c,d)} - 2.$$

With some manipulation we solve $cN + j(c,d) = dp + (c,d)r_1$ for p. Substituting appropriately in $cM + c-(j + 1)(c,d) = (c-d)p + (c,d)r_2$ we obtain

$$\frac{r_2 + r_2 + 1}{c} + \frac{(c/(c,d))N + j + r_1}{d} = \frac{M + N + 1}{(c,d)}.$$

Thus c divides $(c,d)(r_1 + r_2 + 1)$(lemma 3.7). Clearly $(c,d)(r_1 + r_2 + 1) \neq 0$; therefore

$$(c,d)(r_1 + r_2 + 1) \ge c$$

and

$$r_1 + r_2 \ge \frac{c}{(c,d)} - 1,$$

contradicting the previous computation. We conclude that $J^m_{c,d}$ and $J^{c-m-1}_{c,c-d}$ are complementary sets. •

Corollary 3.2. For any $M_{c,d}$, compl$(M_{c,d}) \in$ M$(c,c-d)$.

Proof. This follows directly from theorems 1.2, 1.5, and 3.3. •

Examples 3.4. (i) Compl$(J^2_{12,5}) = J^9_{12,7}$
(ii) Compl$(J^9_{12,8}) = J^2_{12,4}$

What is the relationship between complementation and g-generated ME sets? It is well-known that the usual diatonic and pentatonic sets are both generated by clens 5 or 7. Is it true in general that the complement of a g-generated ME set is also g-generated? The following theorem provides the answer.

Theorem 3.4. $M_{c,d}$ and compl$(M_{c,d})$ are both generated ME sets if and only if $(c,d) = 1$ or $d = 0, \frac{c}{2}, c$.

Proof.

$(\Rightarrow)$ We will prove the contrapositive.

(a) If $(c,d) \neq 1$ and $c/2 < d < c$, then $1 < (c,d) < d$, and $M_{c,d}$ has no generator (theorem 3.1, C).

(b) If $(c,d) \neq 1$ and $0 < d < c/2$, then $c - 0 > c - d > c - c/2$, implying $c/2 < c - d < c$. Thus $1 < (c, c - d) < c - d$ (since $c,d) = (c, c - d)$ and $d < c/2$), implying compl$(M_{c,d}) = M_{c,c-d}$ has no generator (theorem 3.1, C). Thus if $M_{c,d}$ and compl$(M_{c,d})$ are both generated ME sets, then either $(c,d) = 1$ or $d = 0, c/2, c$.

($\Leftarrow$)

(a) If $(c,d) = 1$, then $(c,c - d) = 1$, and both $M_{c,d}$ and compl$(M_{c,d})$ are generated ME sets (theorems 3.1, A and 3.2).

(b) If $d = 0$, or $d = c$, then the corresponding ME sets and their complements are trivially ME, hence generated (theorem 3.2).

(c) If $d = c/2$, then $(c, c-d) = (c,c/2) = c/2$, and both $M_{c,c/2}$ and compl$(M_{c,c/2})$ are generated ME sets (theorem 3.1, B). ●

Having gained some understanding of ME set complements, we are now at a logical place to recall the construction of ME sets by means of points distributed around the circumference of a circle. The following theorem proves the claim that this construction, first presented following example 1.4, yields a ME set and its complement.

Theorem 3.5. Let c and d be positive integers such that $d < c$. Place d white points equidistantly around the circumference of a circle. Then place $c - d$ black points equidistantly around the circumference so that no two points (one white, one black) are in the same location. Assume that intervals from one point to another are measured clockwise around the circumference. Assign clen 1 to every adjacent pair of points, regardless of color. Then both the set of white points and the set of black points represent ME sets in a chromatic universe of cardinality c.

Proof. Clearly if $d = c - d$, then the theorem holds true. If $d \neq c - d$, we may assume without loss of generality that $d > c - d$. The general approach to the proof will be as follows:

(a) Find the smallest clockwise distance from a white to a black point; call this distance δ (fig. 12a).

(b) Call the white point a_0. Moving clockwise around the circle, call the other points $a_1, a_2, \ldots, a_{c-1}$, consecutively. Assume that the distance around the circumference is d. Calculate the distances from a_0 to all points in terms of δ and d (fig. 12b), and label each point with its respective distance from a_0.

(c) Define an order-preserving map σ that sends a_j to j (fig. 12c).

(d) Show that σ maps the black points (and hence the white points) to a ME set.

After points have been labeled with their distances from a_0, the sets

$$S_{c,d} = \{0, 1, 2, \ldots, d - 1\},$$

$$S_{c,c-d} = \left\{\delta, \delta + \frac{d}{c - d}, \delta + \frac{2d}{c - d}, \ldots, \delta + \frac{(c - d - 1)d}{c - d}\right\}$$

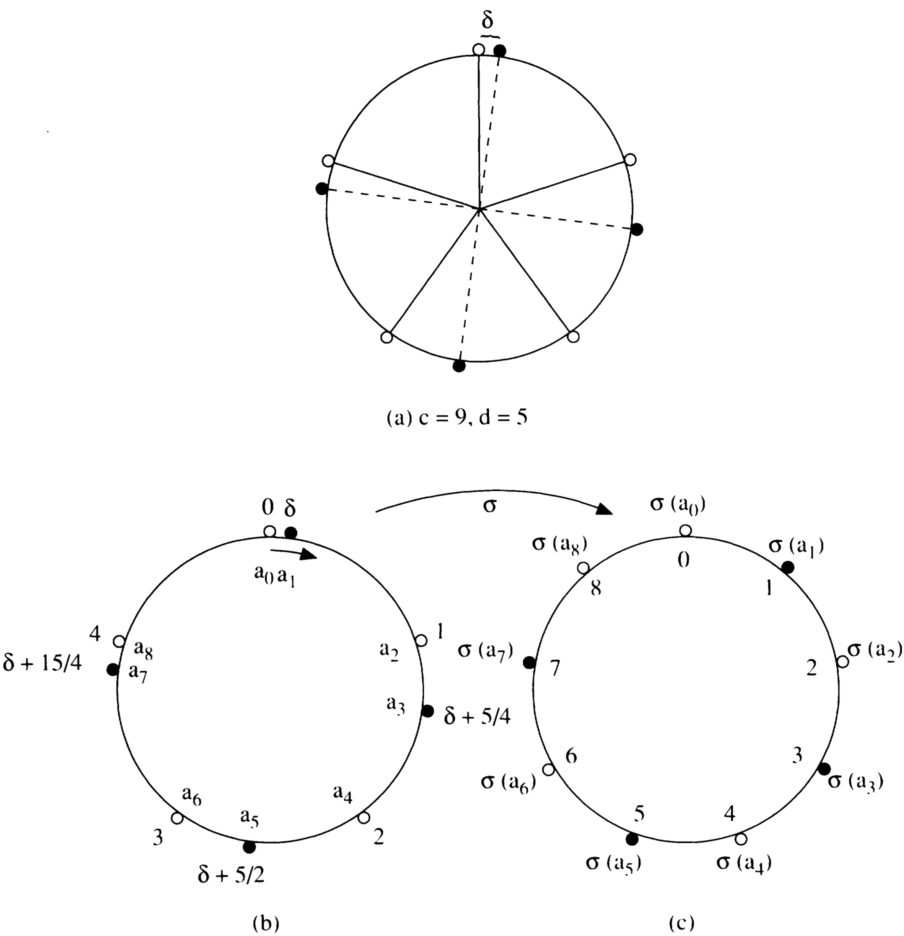


Figure 12. Construction of ME sets with superimposed circles.

represent the sets of white and black points, respectively. Now, if $\delta + \frac{jd}{c - d} > \left[\frac{jd}{c - d}\right] + 1$, then, since $\frac{jd}{c - d} < \left[\frac{jd}{c - d}\right] + 1$, we have $\delta + \frac{jd}{c - d} < \delta + \left[\frac{jd}{c - d}\right] + 1$, implying

$$0 < \left(\delta + \frac{jd}{c - d}\right) - \left(\left[\frac{jd}{c - d}\right] + 1\right) < \delta,$$

contradicting the assumption that δ is the smallest clockwise distance from a white point to a black point. Thus

$$\left[\frac{jd}{c-d}\right] < \delta + \frac{jd}{c-d} < \left[\frac{jd}{c-d}\right] + 1,$$

where $0 \leq j \leq c - d - 1$. Let $S_c = S_{c,d} \cup S_{c,c-d}$. Then

$$S_c = \left\{0, \delta, 1, \ldots, \left[\frac{jd}{c-d}\right], \delta + \frac{jd}{c-d}, \left[\frac{jd}{c-d}\right] + 1, \ldots, c-1\right\}$$

will be the set of points listed in order of magnitude. Label these elements $a_0, a_1, \ldots, a_{c-1}$, respectively. Then

$$S_c = \{a_0, a_1, \ldots, a_{c-1}\}.$$

Now construct a map σ such that

$$\begin{aligned} \sigma\colon S_c &\rightarrow U_c \\ a_j &\rightarrow j, \end{aligned}$$

where U_c is the chromatic universe. Clearly σ is well defined, onto, one-to-one, and preserves order. Further, σ maps $\delta + \frac{jd}{c-d}$ to $\left[\frac{jd}{c-d}\right] + j + 1$.

We now show that $\sigma(S_{c,d})$ and $\sigma(S_{c,c-d})$ are ME sets.

$$\begin{aligned}
\sigma(S_{c,c-d}) &= \left\{\sigma(\delta), \sigma(\delta + \frac{d}{c-d}), \ldots, \sigma(\delta + \frac{(c-d-1)d}{c-d})\right\} \\
&= \left\{\left[\frac{0 \cdot d}{c-d}\right] + 1, \left[\frac{1 \cdot d}{c-d}\right] \right. \\
&\quad \left. + 2, \ldots, \left[\frac{(c-d-1) \cdot d}{c-d}\right] + (c+d)\right\} \\
&= \left\{\left[\frac{0 \cdot c + (c-d)}{c-d}\right], \left[\frac{1 \cdot c + (c-d)}{c-d}\right], \ldots, \right. \\
&\qquad \left. \left[\frac{(c-d-1) \cdot c + (c-d)}{c-d}\right]\right\} \\
&= \{J^{c-d}_{c,c-d}(0), J^{c-d}_{c,c-d}(1), \ldots, J^{c-d}_{c,c-d}(c-d-1\}) \\
&= J^{c-d}_{c,c-d} \in M(c, c-d).
\end{aligned}$$

Thus σ $(S_{c,c-d})$ is a ME set, and, since complements of ME sets are ME sets, $\sigma(S_{c,d})$ must also be a ME set. ●

The reader will recall that $\underline{M}(c)$, the ME set whose spectrum is a smallest proper superset of C_c, played a role in the characterization of diatonic sets. A similar or "inverse" situation exists with respect to the characterization of diatonic set complements.

Definition 3.3. $\overline{M}(c)$ (read "**max-**c") is the set of ME sets whose spectrums are the largest proper subsets of C_c. That is, $M_{c,d} \in \overline{M}(c)$ if and only if both of the following hold true:

(1) Spec($M_{c,d}$) is properly included in C_c.

(2) For all $M^*_{c,d}$ such that spec($M^*_{c,d}$) is properly included in C_c, $|$ spec($M^*_{c,d}$) $| \leq |$ spec($M_{c,d}$) $|$.

We write $\overline{M}_c$ to indicate an element in $\overline{M}(c)$.

Example 3.5. $\overline{M}(12) = M(12,5)$, the usual pentatonic set. Spec($M_{12,5}$) = {0, 2, 3, 4, 5, 7, 8, 9, 10} = $C_{12} \setminus$ {1, 6, 11}.

The question naturally arises: for any particular choice of c, is there a unique d such that $M(c,d— = \overline{M}(c)$? In contrast to the situation with $\underline{M}(c)$, examined previously, there are no problem cases here, as the following theorem shows.

Theorem 3.6. Let $c \geq 2$.

(i) If c = 2, 4, 6, then $\overline{M}(c) = M(c,c/2)$ and $|$ spec($\overline{M}_c$)$| = c/2$.

(ii) If c is odd, then $\overline{M}(c) = M(c,(c - 1)/2)$ and $|$ spec($\overline{M}_c$) $| = c - 2$.

(iii) If c is even and $c \geq 8$, then $\overline{M}(c) = M(c,c/2 - 1)$ and

(a) if $c \equiv 0$ (mod 4), then $|$ spec($\overline{M}_c$)$| = c - 3$, or

(b) if $c \equiv 2$ (mod 4), then $|$ spec($\overline{M}_c$)$| = c - 4$.

Proof. In general, to prove $\overline{M}(c) = M(c,d)$ for a particular value of d, we first compute $|$ spec($M_{c,d}$) $|$. We then show that there is no d* such that d* $\neq$ d and $|$ spec($M_{c,d}$) $| \leq |$ spec($M_{c,d*}$) $| < c$.

(i) We can show by computation that for c = 2, 4, 6, M(c) = $M(c,c/2)$. Further,

$$\begin{aligned} |\text{ spec}(M_{c,\,c/2})\,| &= 2\,(c/2) - (c,\, c/2)(\text{lemma } 2.14) \\ &= c - c/2 \\ &= c/2. \end{aligned}$$

(ii) Assume c is odd. Then $c - 2((c-1)/2) = 1$, implying $(c, (c-1)/2) = 1$ (lemma 2.18). Then

$$\begin{aligned} |\text{ spec}(M_{c,(c-1)/2})\,| &= 2((c-1)/2) - (c,\, (c-1)/2) \\ &= (c-1) - 1 \\ &= c - 2. \end{aligned}$$

If $d > c/2$, then C_c is included in spec ($M_{c,d}$) (theorem 1.12); hence $d < c/2$. Further, $M_{c,d}$ has no ambiguities (lemma 2.2), implying spec($M_{c,d}$) is included in C_c. Let $d = (c - 1)/2 - n$, where $1 \leq n \leq (c - 1)/2$. We will show $|$ spec($M_{c,d}$) $| < c - 2$. Then

$$\begin{aligned} |\text{ spec}(M_{c,d})\,| &= |\text{ spec}(M_{c,\,(c-1)/2-n})\,| \\ &= 2((c - 1)/2 - n) - (c, c - 1)/2 - n)(\text{lemma } 2.14) \\ &\leq (c - 1 - 2n) - 1 \ (\text{since } (c,(c - 1)/2 - n) \geq 1) \\ &\leq c - 4 \ (\text{since } n \geq 1). \end{aligned}$$

Thus $M_{c,d} \notin \overline{M}(c)$. It follows that $\overline{M}(c) = M(c,(c - 1)/2)$.

(iii) Assume c is even and $c \geq 8$.

We can show by computation that $\overline{M}(8) = M(8,8/2 - 1)$ and $|\operatorname{spec}(M_{8,3})| = 8 - 3$. Thus for $c = 8$ we have $\overline{M}(c) = M(c,c/2 - 1)$ and $|\operatorname{spec}(M_{c,\, c/2-1})| = c - 3$.

Now suppose $c > 8$. As in (i) above, $d \leq c/2$ (theorem 1.12) and $M_{c,d}$ has no ambiguities (lemma 2.2). Hence $\operatorname{spec}(M_{c,d})$ is included in C_c. We now consider the two cases (a) $c \equiv 0 \pmod 4$ and (b) $c \equiv 2 \pmod 4$.

(a) If $c \equiv 0 \pmod 4$, then $d = c/2 - 1$ must be odd. Thus $(2, d) = 1$. Further, $d - c/2 = -1$, implying $(c/2, d) = 1$ (lemma 2.18). Thus $(c, d) = 1$ (lemma 2.13), and

$$\begin{aligned} |\operatorname{spec}(M_{c,c/2-1})| &= 2(c/2 - 1) - (c, c/2 - 1) \\ &= (c - 2) - 1 \\ &= c - 3. \end{aligned}$$

(b) If $c \equiv 2 \pmod 4$, then $d = c/2 - 1$ must be even, implying $(c, d) \geq 2$. Also $c - 2(c/2 - 1) = 2$, implying $(c,d) \leq 2$ (lemma 2.18). Thus $(c,d) = 2$ and

$$\begin{aligned} |\operatorname{spec}(M_{c,c/2-1})| &= 2(c/2 - 1) - (c,d) \\ &= (c - 2) - 2 \\ &= c - 4. \end{aligned}$$

In either case

$$|\operatorname{spec}(M_{c,c/2-1})| \leq c - 3.$$

Thus $\operatorname{spec}(M_{c,c/2-1})$ is properly included in C_c (lemma 2.2). Further,

$$|\operatorname{spec}(M_{c,c/2-1})| \geq c - 4.$$

Now we must show that for $d = c/2 - n$, where $2 \leq n \leq c/2$,

$$|\operatorname{spec}(M_{c,c/2-n})| < c - 4.$$

But

$$\begin{aligned} |\operatorname{spec}(M_{c,c/2-n})| &= 2(c/2 - n) - (c, c/2 - n) \\ &\leq (c - 2n) - 1 \text{ (since } (c, c/2 - n) \geq 1) \\ &\leq c - 5 \text{ (since } n \geq 2) \\ &< c - 4. \end{aligned}$$

Hence if $d < c/2 - 1$, then $|\operatorname{spec}(M_{c,d})| < |\operatorname{spec}(M_{c,c/2-1})|$. Now we consider $d = c/2$.

$$\begin{aligned} |\operatorname{spec}(M_{c,c/2})| &= 2(c/2) - (c, c/2) \\ &= c - c/2 \\ &= c/2 \\ &< c - 4 \text{ (since } c > 8). \end{aligned}$$

Thus $|\operatorname{spec}(M_{c,c/2})| < |\operatorname{spec}(M_{c,c/2-1})|$. Since we can show by computation that $\overline{M}(8) = M(8,3)$, we conclude that if c is even and $c \geq 8$, then $\overline{M}(c) = M(c, c/2 - 1)$. ●

We will return to our investigation of the properties of $\underline{M}(c)$ and $\overline{M}(c)$ later. First we explore some further relationships between diatonic sets and their complements.

Recall that diatonic sets have precisely one tritone. Indeed we used this property in the construction of diatonic sets. In the case of diatonic set complements, the clen $c/2 - 2$ (the major 3d in the usual diatonic set) plays a similar role—it is the singular interval of this clen. Lemma 3.8 below shows that ME sets with precisely one interval of this clen are restricted to particular relationships between c and d. In preparation for the lemma, note that up to this point we have been writing $cI = dq + r$ where $0 \leq r < d$, attaching little or no importance to the quantity q; however

$$cI = dq + r = d\left[\frac{cI}{d}\right] + r.$$

Lemma 3.8. Let $c \geq 8$. Then $M_{c,d}$ has precisely one interval of clen $c/2 - 2$ if and only if $(c,d) = 1$ and either $c = 2(d + 1)$ or $c = 2(3d - 1)$.

Proof.

($\Rightarrow$) $c \geq 8$ implies $c/2 - 2 \geq 2$, which in turn implies that clen $c/2 - 2$ is in a nonzero interval spectrum (theorem 1.11); also note that $\left[\frac{cI}{d}\right] > 0$ implies $I \neq 0$.) If $M_{c,d}$ has precisely one interval of clen $c/2 - 2$, then there exists an interval of dlen I such that $c/2 - 2 \in \langle I \rangle$ and $cI = d\left[\frac{cI}{d}\right] + 1$ or $cI = d\left[\frac{cI}{d}\right] + (d - 1)$ (theorem 1.11). In either case since (c,d) divides c and d, (c,d) must divide 1; thus $(c,d) = 1$, and clearly $d \neq c/2$, else $(c,d) \neq 1$. Thus either $d > c/2$ or $d < c/2$. We consider each of these cases in turn.

Case 1. Suppose $d > c/2$. Then $M_{c,d}$ has a tritone ambiguity and $\left\langle \frac{d-1}{2} \right\rangle \cap \left\langle \frac{d+1}{2} \right\rangle = \left\{ \frac{c}{2} \right\}$ (lemma 2.8). Since $(c,d) = 1$ every nonzero spectrum has two elements (lemma 1.3). It follows that $c/2 - 1 \in \left\langle \frac{d-1}{2} \right\rangle$. Hence $c/2 - 2 \in \left\langle \frac{d-3}{2} \right\rangle$, since both $c/2$ and $c/2 - 1$ are in $\left\langle \frac{d-1}{2} \right\rangle$. Then either $\left[\frac{c((d-3)/2)}{d}\right] = c/2 - 2$ or $\left[\frac{c((d-3)/2)}{d}\right] + 1 = c/2 - 2$ (theorem 1.11). We now consider these two subcases in turn.

(*i*) *If* $\left[\frac{c((d-3)/2}{d}\right] = c/2 - 2$, then

$$c((d - 3)/2 = d\left[\frac{c((d - 3)/2)}{d}\right] + d - 1 \text{ (theorem 1.11)}$$
$$= d(c/2 - 2) + d - 1$$
$$3c = 2d + 2$$
$$\leq 2\, c + 2 \text{ (since } d \leq c)$$
$$c \leq 2 \text{ (contradiction).}$$

(*ii*) *If* $\left[\frac{c((d - 3)/2)}{d}\right] + 1 = c/2 - 2$, then

$$\left[\frac{c((d - 3)/2)}{d}\right] = c/2 - 3$$

and

$$c((d - 3)/2 = d\left[\frac{c((d - 3)/2)}{d}\right] + 1 \text{ (theorem 1.11)}$$
$$= d(c/2 - 3) + 1$$
$$3c = 6d - 2.$$

But this would imply 3 divides 2 (contradiction).

Case 2. If $d < c/2$, then there are no ambiguities (lemma 2.2) and no tritones (lemma 2.8). Since $(c,d) = 1$ every nonzero spectrum has two elements (lemma 1.3). Hence $\left\langle\frac{d - 1}{2}\right\rangle$ has two elements and has no tritone. Thus $c/2 - 2 \in \left\langle\frac{d - 1}{2}\right\rangle$. It follows that either $\left[\frac{c((d - 1)/2)}{d}\right] = c/2 - 2$ or $\left[\frac{c((d - 1)/2)}{d}\right] + 1 = c/2 - 2$ (theorem 1.11). We now consider these two subcases in turn.

(*i*) *If* $\left[\frac{c((d - 1)/2)}{d}\right] = c/2 - 2$, then

$$c((d - 1)/2) = d\left[\frac{c((d - 1)/2)}{d}\right] + d - 1 \text{ (theorem 1.11)}$$
$$= d(c/2 - 2) + d - 1$$
$$c = 2(d + 1).$$

(*ii*) *If* $\left[\frac{c((d - 1)/2)}{d}\right] + 1 = c/2 - 2$, then $\left[\frac{c((d - 1)/2)}{d}\right] = c/2 - 3$, and

$$c((d - 1)/2) = \left[\frac{c((d - 1)/2)}{d}\right]) + 1 \text{ (theorem 1.11)}$$
$$= d(c/2 - 3) + 1$$
$$c = 2(3d - 1).$$

($\Leftarrow$) From $(c,d) = 1$ and either $c = 2(d + 1)$ or $c = 2(3d - 1)$, it follows that c must be even and d odd. Further, every nonzero spectrum has two elements (lemma 1.3) and $M_{c,d}$ has no tritones (lemma 2.8). Hence if $c/2 - 2$ is in any interval spectrum, it must be in $\left\langle \frac{d-1}{2} \right\rangle$.

Case 1. Assume $c = 2(d + 1)$. Then

$$\left[\frac{cI}{d}\right] = \left[\frac{c((d-1)/2)}{d}\right]$$
$$= c/2 - 2$$
$$+ \left[\frac{c-4}{c-2}\right] \text{ (since c = 2(d + 1))}$$
$$= c/2 - 2 \text{ (since } c \geq 8).$$

Thus $c/2 - 2 \in \langle (d - 1)/2 \rangle$ (theorem 1.11). Further

$$c((d-1)/2) = d(c/2 - 1) + (d - 1) \text{ (since } c = 2(d + 1))$$
$$= d\left[\frac{c((d-1)/2)}{d}\right] + (d - 1).$$

Thus there is precisely one interval of clen $\left[\frac{c((d-1)/2)}{d}\right]$ (theorem 1.11; also note that $I = (d - 1)/2$). But, as shown above, $\left[\frac{c((d-1)/2)}{d}\right] = c/2 - 2$.

Case 2. Assume $c = 2(3d - 1)$. Then

$$\left[\frac{cI}{d}\right] = \left[\frac{c((d-1)/2)}{d}\right]$$
$$= c/2 - 3 + \left[\frac{6}{c+2}\right] \text{ (since } c = 2(3d$$
$$-1))$$
$$= c/2 - 3 \text{ (since } c \geq 8).$$

Thus $c/2 - 2 \in \langle (d - 1)/2 \rangle$ (lemma 1.3 and theorem 1.11). Further

$$c((d-1)/2) = d(c/2 - 3) + 1 \text{ (since } c = 2(3d - 1))$$
$$= d\left[\frac{c((d-1)/2)}{d}\right] + 1.$$

Thus there is precisely one interval of clen $\left[\frac{c((d-1)/2)}{d}\right] + 1$ (theorem 1.11). But, as implied above, $\left[\frac{c((d-1)/2)}{d}\right] + 1 = c/2 - 2.$ ●

If a set contains precisely one interval of clen $c/2 - 2$, it must contain precisely one interval of clen $c - (c/2 - 2) = c/2 + 2$, and the converse is also true (theorem 1.11). Therefore, in the above lemma and henceforth, whatever is implied by a singular instance of $c/2 - 2$ is also implied by a singular instance of $c/2 + 2$.

Examples 3.6. The case $c = 2(d + 1)$from lemma 3.8 is exemplified by $M(16,7)$; the case $c = 2(3d - 1)$ is exemplified by $M(16,3)$. Each of the ME sets in these classes has exactly one interval of clen $6 = 16/2 - 2$.

We are now in a position to prove the characterization theorem for diatonic set complements. The equivalent conditions enumerated for diatonic set complements are to some extent "complementary" to those enumerated for diatonic sets in theorem 2.2; also, as in the case of diatonic sets, the condition $c \equiv 0 \pmod 4$ is central here.

Theorem 3.7. For $M_{c,d}$ with $c \geq 8$, the following are equivalent:

(1) $M_{c,d}$ is the complement of a diatonic set.
(2) $M_{c,d}$ has precisely one interval of clen $c/2 - 2$ and $c \neq 2(3d - 1)$.
(3) $c = 2(d + 1)$ and $(c,d) = 1$.
(4) $c = 2(d + 1)$ and d is odd.
(5) $c = 2(d + 1)$ and $c \equiv 0 \pmod 4$.
(6) $M_{c,d}$ has precisely two intervals of clen 3, and $c \equiv 0 \pmod 4$.
(7) $M(c,d) = \overline{M}(c)$, and $c \equiv 0 \pmod 4$.
(8) c is even and $\mathrm{spec}(M_{c,d}) = C_c\{1, \frac{c}{2}, c - 1\}$.
(9) $|\,\mathrm{spec}(M_{c,d})\,| = c - 3$.

Proof. We will show $2 \Leftrightarrow 3$, then $1 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8 \Rightarrow 9 \Rightarrow 1$.

(2 $\Leftrightarrow$ 3) This follows directly from lemma 3.8.

(1 $\Rightarrow$ 3) If $M_{c,d}$ is the complement of a diatonic set, then $\mathrm{compl}(M_{c,d}) = M_{c,c-d}$ is a diatonic set. Then $c = 2((c - d) - 1)$, and $(c, c - d) = 1$ (theorem 2.2). Thus $c = 2(d + 1)$ and $(c,d) = 1$.

(3 $\Rightarrow$ 4) Assume $c = 2(d + 1)$ and $(c,d) = 1$. Then d must be odd, else $(c,d) \geq 2$.

(4 $\Rightarrow$ 5) Assume $c = 2(d + 1)$ and d is odd. Then $d + 1$ is even, hence 4 divides $2(d + 1)$. Thus $c \equiv 0 \pmod 4$.

(5 $\Rightarrow$ 6) Assume $c = 2(d + 1)$ and $c \equiv 0 \pmod 4$. It follows that $c = 2d + 2$. But then

$$c \cdot 1 = \left\lfloor \frac{c \cdot 1}{d} \right\rfloor + 2$$

and there must be two intervals associated with dlen 1 whose clen is $\left\lfloor \frac{c \cdot 1}{d} \right\rfloor + 1 = 3$ (theorem 1.11). Since $M_{c,d}$ has no ambiguities (lemma

2.2), there must be precisely two intervals of clen 3 and, by assumption, $c \equiv 0 \pmod 4$.

(6 ⇒ 7) Assume $M_{c,d}$ has precisely two intervals of clen 3, and $c \equiv 0 \pmod 4$. If 3 is an ambiguous clen, then there exists an I, $0 \leq I \leq d - 1$, such that $<I> \cap <I + 1> = \{3\}$ and $\left[\frac{cI}{d}\right] + 1 = \left[\frac{c(I+1)}{d}\right] = 3$ (lemma 2.4).

But since 3 is associated with both I and $I + 1$, and since there are precisely two intervals of clen 3, we have

$$cI = \left[\frac{cI}{d}\right]d + (d - 1)$$

(theorem 1.11)

$$c(I + 1) = \left[\frac{c(I+1)}{d}\right]d + 1.$$

Since $\left[\frac{cI}{c}\right] + 1 = \left[\frac{c(I+1)}{d}\right] = 3$, we have

$$cI = 3d - 1$$

and

$$c(I + 1) = 3d + 1.$$

Putting these two equations together, we get $c = 2$, contradicting the assumption that $c \geq 8$. Hence 3 cannot be an ambiguous clen.

Since both intervals of clen 3 must be associated with the same dlen, say I, then either $\left[\frac{cI}{d}\right] = 3$ or $\left[\frac{cI}{d}\right] + 1 = 3$ (lemma 1.9). If $\left[\frac{cI}{d}\right] = 3$, then

$$\begin{aligned} cI &= \left[\frac{cI}{d}\right]d + d - 2 \text{ (theorem 1.11)} \\ &= 4d - 2. \end{aligned}$$

But since $c \equiv 0 \pmod 4$, 4 divides c, and since 4 divides $4d$, 4 must divide 2 (contradiction). It follows that $\left[\frac{cI}{d}\right] + 1 = 3$. But then

$$\begin{aligned} cI &= \left[\frac{cI}{d}\right]d + 2 \text{ (theorem 1.11)} \\ &= 2d + 2. \end{aligned}$$

Since $\left[\frac{cI}{d}\right] \in <I>$ (lemma 1.9), $2 \in <I>$, implying $I = 1$ or 2. If $I = 2$, then

$$c \cdot 2 = 2d + 2$$
$$c = d + 1.$$

Since $c \geq 8$, we have $d \geq 7$. Hence

$$\left[\frac{c(I+1)}{d}\right] = \left[\frac{c \cdot 3}{d}\right]$$
$$= 3 + \left[\frac{3}{d}\right] \text{ (since } c = d + 1)$$
$$= 3 \text{ (since } d \geq 7).$$

But then 3 would be an ambiguous clen (lemma 1.9) (i.e., $3 \in <2> \cap <3>$, contradiction). It follows that $I = 1$ and $c \cdot 1 = 2d + 2$. Thus $d = c/2 + 1$, and since $c \equiv 0 \pmod 4$, $M(c,d) = \overline{M}(c)$ (theorem 3.6).

(7 ⇒ 8) Assume $c \equiv 0 \pmod 4$ and $M(c,d) = \overline{M}(c)$. Then $d = \frac{c}{2} - 1$ (theorem 3.6). Thus $d < \frac{c}{2}$, implying $M_{c,d}$ has no ambiguities (lemma 2.2) and all elements of spec($M_{c,d}$) have multiplicity one. Then $\left[\frac{c}{d}\right]$ is the smallest element in <1> (lemma 1.9) and thus the smallest nonzero element in spec($M_{c,d}$). But $\left[\frac{c}{d}\right] \geq 2$ since $d < \frac{c}{2}$. It follows that $1 \notin$ spec($M_{c,d}$). Further, no interval spectrum includes $c - 1$ (theorem 1.11) and hence $c - 1 \notin$ spec($M_{c,d}$). Further, since $c \equiv 0 \pmod 4$ and $d = \frac{c}{2} - 1$, c is even, d is odd, and $d < \frac{c}{2}$. Thus $\frac{c}{2} \notin$ spec($M_{c,d}$)(lemma 2.8, i). Since $c \equiv 0 \pmod 4$ and $M(c,d) = \overline{M}(c)$, $|\text{spec}(M_{c,d})| = c - 3$ (theorem 3.6). And since spec($M_{c,d}$) has no elements of multiplicity 2, we conclude that spec($M_{c,d}$) $= C_c \setminus \{1, \frac{c}{2}, c - 1\}$.

(8 ⇒ 9) $|\text{Spec}(M_{c,d})| = c - 3$ follows immediately from spec($M_{c,d}$) $= C_c \setminus \{1, \frac{c}{2}, c - 1\}$.

(9 ⇒ 1) Assume $|\text{spec}(M_{c,d})| = c - 3$. Then $2d - (c,d) = c - 3$ (lemma 2.14), implying $(c,d) = 1$ or 3, since (c,d) divides both c and d. If $(c,d) = 3$, then

$$2d - (c,d) = c - 3$$
$$2d - 3 = c - 3$$
$$d = \frac{c}{2}.$$

But then $(c,d) = \frac{c}{2}$, implying $\frac{c}{2} = 3$. Thus $c = 6$, contradicting the assumption $c \geq 8$. If $(c,d) = 1$, then

$$2d - (c,d) = c - 3$$
$$2d - 1 = c - 3$$
$$d = \frac{c}{2} - 1.$$

But then compl($M_{c,d}$) = $M_{c,c-d}$, and

$$c - d = c - (c/2 - 1) \qquad \text{(since } d = c/2 - 1\text{)}$$
$$c = 2[(c - d) - 1].$$

Hence c is even, and since $(c,d) = 1$, $(c, c - d) = 1$, implying $c - d$ is odd. Thus $M_{c,c-d}$ is a diatonic set (theorem 2.2). It follows that $M_{c,d}$ is the complement of a diatonic set. ●

In light of theorem 3.7, we now introduce some logical extensions of our terminology for diatonic sets.

Definitions 3.4. We say that the complement of a diatonic set is a *pentatonic* set; the complement of a usual diatonic set is a *usual pentatonic* set; the complement of a hyperdiatonic set is a *hyperpentatonic* set.

Note that with the exception of the usual pentatonic ("black key") sets, which have five notes, the infinitely many *other* pentatonic sets, which we call hyperpentatonic, are *not* 5-note sets. Note also that, in the 12-note chromatic universe, the class *pentatonic* does not include 5-note sets other than the *usual* pentatonic. While these possible confusions are unfortunate, they are outweighed in the present context by the symmetry between definitions for diatonic and pentatonic sets, which reinforces our results on complementation. To avoid one of the pitfalls noted above, we will use the cautionary term *(hyper)pentatonic* instead of *pentatonic.*

The extremal sets $\underline{M}(c)$ and $\overline{M}(c)$ play "complementary" roles in characterizing diatonic and (hyper)pentatonic sets, respectively. From theorems 2.2 and 3.6 we see that if $M_{c,d}$ is a diatonic set or if compl($M_{c,d}$) is a (hyper)pentatonic set, then $M_{c,d} \in \underline{M}(c)$, and compl($M_{c,d}$) $\in \overline{M}(c)$. We now show that the connections among these extremal, diatonic, and (hyper)pentatonic sets are still deeper.

Theorem 3.8. Let $c \geq 8$. Then the following are equivalent:

(1) $M_{c,d} \in \underline{M}(c)$, and compl($M_{c,d}$) $\in \overline{M}(c)$.
(2) $\underline{M}(c)$ is the set of diatonic sets in U_c.
(3) $\overline{M}(c)$ is the set of (hyper)pentatonic sets in U_c.

Proof. It is clear from theorems 2.1 and 3.5 that statement 1 holds if $c \equiv 0 \pmod 4$ and $d = c/2 + 1$, or possibly if $c \equiv 3 \pmod 6$ as theorem 2.1 does not include this case. First we will show that statement 1 does not hold if $c \equiv 3 \pmod 6$.

If $c \equiv 3 \pmod 6$, then c is odd and, by theorem 3.6, $\overline{M}(c) = M(c,(c-1)/2)$. It follows that if $\text{compl}(M_{c,d}) = M_{c,c-d} \in M(c,(c-1)/2)$, then $c - d = (c-1)/2$ and hence $d = (c+1)/2$. But then

$$\begin{aligned} |\text{spec}(M_{c,(c+1)/2})| &= 2((c+1)/2) - (c,(c+1)/2) \text{ (lemma 2.14)} \\ &\leq 2((c+1)/2) - 1 \\ &= c. \end{aligned}$$

Hence $\text{spec}(M_{c,(c+1)/2})$ does not properly contain C_c, implying $M_{c,(c+1)/2} \notin \underline{M}(c)$. Thus statement 1 holds if and only if $c \equiv 0 \pmod 4$ and $d = c/2 + 1$. The equivalences between statements 1 and 2 and statements 1 and 3 follow immediately from theorems 2.2 and 3.6, respectively. •

It is well-known that the usual diatonic and usual pentatonic sets are generated by clens 7 and 5—their cardinalities. Theorem 3.9 below shows that this is a general property of diatonic and (hyper)pentatonic sets.

Theorem 3.9. Any diatonic or (hyper)pentatonic set with parameters c,d may be generated by d. (The cardinality of the set is a generator.)

Proof. If $M_{c,d}$ is a diatonic set or its complement, then $c \equiv 0$ (mod 4), $(c,d) = 1$, and $d = \frac{c}{2} \pm 1$ (theorems 2.2 and 3.6). Then there exists g, $1 \leq g \leq c - 1$, such that $dg \equiv 1 \pmod c$(lemma 3.5), and g is a generator (theorem 3.1, ii). We now show that $g = d = \frac{c}{2} \pm 1$ is a solution to this equation.

$$\begin{aligned} dg &\equiv (\tfrac{c}{2} \pm 1)\ (\tfrac{c}{2} \pm 1) \\ &\equiv c\,(\tfrac{c}{4}) \pm c + 1 \\ &\equiv 1 \pmod c \text{ (since } c \equiv 0 \pmod 4\text{)}. \end{aligned}$$

Thus d is a generator. •

(In addition to the (hyper)diatonic and (hyper)pentatonic sets, the sets generated by Balzano (1980) are also ME sets generated by clens equal to their cardinalities. We leave it to the reader to verify this.)

As theorem 2.3 shows, when $c \equiv 0 \pmod 4$, a tritone plus one additional pc is sufficient to define a unique transposition of a diatonic set. Clearly, the same procedure defines *by exclusion* a unique transposition of a (hyper)pentatonic set. Is there a similar procedure that defines a unique transposition of a (hyper)pentatonic set by *inclusion* (and hence a unique transposition of a diatonic set by *exclusion*)? It is easy to see, though perhaps not widely recognized, that the choice of any two pcs capable of forming a major 3d defines by exclusion a unique transposition of the diatonic set. Suppose we choose, for example, pcs F♯/G♭ and A♯/B♭. These are of course the excluded boundary notes of the C-major set on the circle of 5ths. It is clear that the only diatonic set which excludes both of these pcs is the C-major set. We now show that such a property exists quite generally in the world of diatonic sets and their complementary (hyper)pentatonic sets.

Theorem 3.10. Let $c \equiv 0 \pmod 4$ and let $\{D_i, D_j\}$ be a pcset such that $\mathrm{clen}(D_i, D_j) = c/2 - 2$. Then there is a unique transposition of a (hyper)pentatonic set, say $M_{c,d}$, which includes $\{D_i, D_j\}$, and hence a unique transposition of a diatonic set, $\mathrm{compl}(M_{c,d})$, which excludes $\{D_i, D_j\}$.

Proof. Assume $M_{c,d}$ and $M^*_{c,d}$ are two distinct (hyper)pentatonic sets that each include $\{D_i, D_j\}$. Then $M^*_{c,d}$ is a transposition of $M_{c,d}$ (theorem 1.6) and hence $M^*_{c,d} = \{k\} + M_{c,d}$. Further k is not congruent to 0 (mod c), else $M_{c,d} = M^*_{c,d}$. From $M_{c,d} \supset \{D_i, D_j\}$ it follows that $M^*_{c,d} \supset \{k + D_i, k + D_j\}$. But then

$$\begin{aligned}\mathrm{clen}(k + D_i, k + D_j) &\equiv (k + D_j) - (k + D_i)\\ &\equiv D_j - D_i \pmod c\\ &\equiv \mathrm{clen}(D_i, D_j)\\ &= c/2 - 2,\end{aligned}$$

implying $M^*_{c,d}$ has two intervals of clen $c/2 - 2$ and contradicting the assumption that $M^*_{c,d}$ is a (hyper)pentatonic set (theorem 3.7). ●

Finally, what of the *construction* of (hyper)pentatonic sets? Given appropriate values for c and d, we can construct a (hyper)pentatonic $M_{c,d}$ by using the Clough-Myerson algorithm (theorem 1.1), or by the method of fig. 1, or by superposing intervals of clen d (theorem 3.9). Or we may construct a complementary diatonic set, $M_{c,c-d}$, by means of the method illustrated in figs. 10 and 11—the remaining notes are the desired (hyper)pentatonic set.

The latter method is indirect, but theorem 3.7 suggests a *direct* method along the same lines. Recall from the theorem that a (hyper)-pentatonic set contains one interval of clen $c/2 - 2$ and two intervals of clen 3. Begin the construction with the singular interval $c/2 - 2$. To this interval attach the two intervals of clen 3; one of these should be attached to each of the initial notes, *outside* the interval of $c/2 - 2$ (and *inside* the complementary interval of $c/2 + 2$). Complete the construction by filling in whole steps where intervals larger than clen 3 remain.

The method of construction illustrated in figs. 10 and 11, and adapted to the (hyper)pentatonic case above, provides insight regarding the interval content of diatonic and (hyper)pentatonic sets. In diatonic sets, the various clens appear as follows (we write "#clen j" for "the multiplicity of clen j"):

$$\#\text{clen } j = j + 1, \text{ for odd } j < c/2$$
$$\#\text{clen } j = c/2 - j + 1, \text{ for even } j \leq c/2$$
$$\#\text{clen } j = \#\text{clen } c - j, \text{ for } j > c/2$$

And for (hyper)pentatonic sets:

$$\#\text{clen } j = j - 1, \text{ for odd } j < c/2$$
$$\#\text{clen } j = c/2 - j - 1, \text{ for even } j < c/2$$
$$\#\text{clen } j = 0, \text{ for } j = c/2$$
$$\#\text{clen } j = \#\text{clen } c - j, \text{ for } j > c/2$$

The above may be verified by means of theorem 1.11. Now with reference to fig. 11 (the diatonic case), one can see that after the lone tritone (clen $c/2$) and 2 half-steps (clen 1's) are in place, the filling in of whole-steps (clen 2's) will yield—for odd clens—4 intervals of clen 3, 6 intervals of clen 5, etc., and—for even clens—3 intervals of clen $c/2 - 2$, 5 intervals of clen $c/2 - 4$, etc. And similarly for the (hyper)-pentatonic case. Gamer's (1967) work has helped us to understand the unique multiplicity property of deep scales (i.e., all clens of 1 through $c/2$ appear in different numbers); from theorem 1.11 and the construction method discussed above we gain some insight about the *pattern* of those unique multiplicities for one kind of deep scale—the diatonic—and about the pattern of multiplicities (which are similarly unique except for nonzero clens) in its complement—the (hyper)pentatonic.

Interval circles

Well known in tonal music are "circles" of dlen 2 or its complement, dlen 5 (upward and downward 3ds, respectively; e.g., in the C-major set, C - E - G - B - D - F - A - C). Such circles are ME in the following sense. The seven intervals of dlen 2—three of clen 4 and four of clen 3 (major and minor 3ds)(theorem 1.11)—are distributed as evenly as possible around the circle of 3ds: clen 4's alternate with clen 3's except for one necessary juxtaposition of two clen 3's (see fig. 13a). As a more formal test for maximal evenness, we may reconceive this structure as follows: the circle of 3's and 4's is reinterpreted to give the clens for successive *steps,* thus doubling the size of the original chromatic universe. The resulting set then turns out to be precisely the ME set with $c = 24$ and $d = 7$, pictured on fig. 13b.

In addition to ME circles of dlen 2,5, $M_{12,7}$ has ME circles of dlen 1, 6 (2ds, 7ths, respectively), and 3, 4 (4ths, 5ths, respectively). The circle of 4ths or 5ths is trivially ME in the sense that it has only one clen different from the rest, so the location of this rare interval relative to the other intervals is immaterial; the circle is ME in any event. However the circle of 2ds (7ths) is nontrivially even: it has two rare intervals—half steps—and they are maximally separated. In summary, all interval circles in $M_{12,7}$ are maximally even; one of them and its inversion are trivially so, the others nontrivially so.

The following theorem shows that, quite generally, interval circles in ME sets are themselves maximally even.

Theorem 3.11. Let c, d be fixed such that $c \geq d > I \geq 1$ and let $M_{c,d} = \{D_0, D_1, \ldots, D_{d-1}\}$. Then, for all j, $0 \leq j \leq d-1$, there exists an $M_{cI,d} = \{E_0, E_1, \ldots, E_{d-1}\}$ such that $\text{clen}(D_{jI}, D_{(j+1)I}) = \text{clen}(E_j, E_{j+1})$, where the subscripts are reduced (mod d).

Proof. Since $M_{c,d} \in M(c,d)$ there exists an m, $0 \leq m \leq c-1$, such that $M_{c,d} = J^m_{c,d}$ (theorem 1.5). Then

$$\begin{aligned} \text{clen}(D_{jI}, D_{(j+1)I}) &= \text{clen}(J^m_{c,d}(jI), J^m_{c,d}((j+1)I)). \\ &\equiv J^m_{c,d}((j+1)I) - J^m_{c,d}(jI) \\ &\equiv \left\lfloor \frac{cI(j+1)+m}{d} \right\rfloor - \left\lfloor \frac{cIj+m}{d} \right\rfloor \pmod{c}. \end{aligned}$$

Now consider $J^m_{cI,d} = \{E_0, E_1, \ldots, E_{d-1}\}$. Then

$$\begin{aligned} \text{clen}(E_j, E_{(j+1)}) &= \text{clen}(J^m_{cI,d}(j), J^m_{cI,d}(j+1)) \\ &\equiv \left\lfloor \frac{cI(j+1)+m}{d} \right\rfloor \\ &\quad - \left\lfloor \frac{cIj+m}{d} \right\rfloor \pmod{cI}. \end{aligned}$$

Clearly $\text{clen}(D_{jI}, D_{(j+1)I}) \equiv \text{clen}(E_j, E_{j+1}) \pmod{c}$, and $\text{clen}(D_{jI}, D_{(j+1)I}) \leq \text{clen}(E_j, E_{j+1})$. Hence

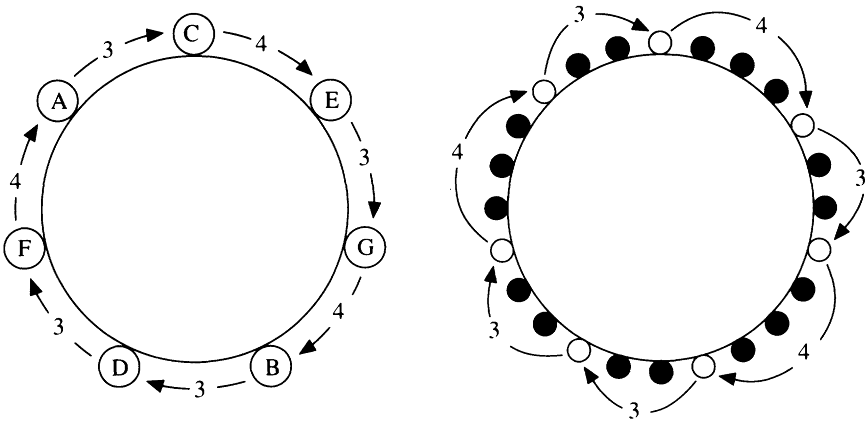
$$\text{clen}(E_j, E_{j+1}) = nc + \text{clen}(D_{jI}, D_{(j+1)I}),$$

where n is a non-negative integer. Now we need only show $\text{clen}(E_j, E_{j+1}) \leq c-1$.

By corollary 1.2, $\text{clen}(E_j, E_{j+1}) = \left\lfloor \frac{cI}{d} \right\rfloor, \left\lfloor \frac{cI}{d} \right\rfloor + 1$. Suppose $d = c$. Then

$$\begin{aligned} \text{clen}(E_j, E_{j+1}) &= \left\lfloor \frac{cI}{d} \right\rfloor \text{ (theorem 1.1)} \\ &= I \\ &\leq d-1 \\ &= c-1. \end{aligned}$$

Thus $\text{clen}(E_j, E_{j+1}) = \text{clen}(D_{jI}, D_{(j+1)I})$.



(a) Circle of 3ds (b) $M_{24,7}$

Figure 13. Isomorphism between traditional circle of 3ds and $M_{24,7}$

Now suppose $d < c$. Then

$$\begin{aligned}
\text{clen}(E_j, E_{j+1}) &= \left\lceil\frac{cI}{d}\right\rceil, \left\lceil\frac{cI}{d}\right\rceil + 1 \\
&\leq \left\lceil\frac{c(d-1)}{c}\right\rceil + \\
&\leq c - 1.
\end{aligned}$$

Again, it follows that $\text{clen}(E_j, E_{j+1}) = \text{clen}(D_{jI}, D_{(j+1)I})$. •

Comparing the sets pictured in fig. 13a and 13b, we find that the latter "collapses" into the former if pcs 14, 17, and 21 are reduced (mod 12) to 2, 5, and 9; however this relationship does not hold in general. We make the following conjecture: Let $M_{c,d}$ and $M_{cI,d}$ be as in theorem 3.11. Now reduce the elements of $M_{cI,d}$ modulo c and call this set $M_{cI,d}$ (mod c). Then $M_{cI,d}$ (mod c) = $M_{c,d}$ if and only if $(d, I) = 1$.

As another example of interval circles in ME sets, consider the "quarter-tone diatonic" set $M_{24,13}$. Since 13 is prime, all dlens will exhaust the set as they "circle." The circle of dlen 6 (or dlen 7) is trivial in the same sense as the circle of 5ths in $M_{12,7}$, but the circles of dlens 1, 2, 3, 4, and 5 (and their complements) are nontrivially ME—each with a different distribution of "major" and "minor" sizes. This scenario is depicted in fig. 14.

I	clens of $(D_O, D_I), (D_I, D_{2I}), (D_{2I}, D_{3I}), \ldots, (D_{12I}, D_O)$												
1	1	2	2	2	2	2	1	2	2	2	2	2	2
2	3	4	4	3	4	4	3	4	4	3	4	4	4
3	5	6	5	6	5	6	5	6	5	6	5	6	6
4	7	7	8	7	7	8	7	8	7	7	8	7	8
5	9	9	9	9	10	9	9	9	10	9	9	9	10
6	11	11	11	11	11	11	11	11	11	11	11	11	12
.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.
12	22	22	22	22	22	22	23	22	22	22	22	22	23

Figure 14. Circles of dlen I in $J^0_{24,13}$

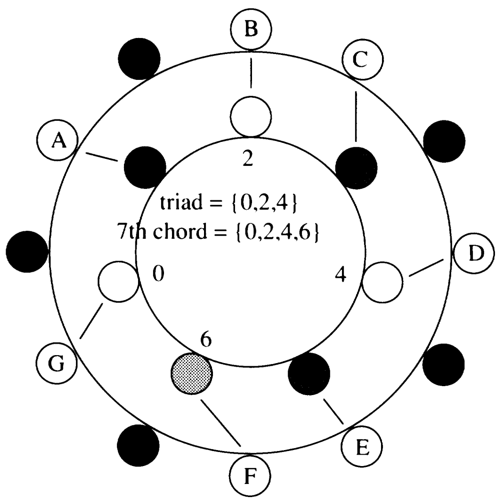
Second-order ME sets

The triads and seventh chords of tonal music are usually defined as subsets of the diatonic set $M_{12,7}$ (with allowances for inflection in the case of the minor mode). If the usual diatonic set is itself conceived as a subset of the 12-note chromatic universe, then triads and seventh chords are subsets of subsets. A question naturally arises here: if, as we have seen, the diatonic set is ME, are the familiar chords, its subsets, also ME? But how shall we interpret this question? If we mean "are the diatonic triads and seventh chords maximally even with respect to the 12-note chromatic?" then the answer is clearly "no." Of the three 3-note diatonic chords (major, minor, and diminished triads) and four 4-note diatonic chords (the correspondences "three" with "3-note," "four" with "4-note" are due to CV; see part 1 above), none is ME in this sense. In fact no 3- or 4-note subset of $M_{12,7}$ is ME in this sense, for $M_{12,3}$ = the augmented triad and $M_{12,4}$ = the diminished-seventh chord, neither of which is a subset of $M_{12,7}$.

Now suppose we put the question differently: if the diatonic set is maximally even with respect to the chromatic universe, are the tertian chords maximally even *with respect to the diatonic set?* To answer this question, think of the diatonic set *as though it were chromatic.* ($M_{12,7}$ is, after all, ME, which is to say that, within the 12-note equal-tempered template, it is the best approximation to 7-note equal temperament.) From this perspective, triads and seventh-chords *are* maximally even. At this level they are in fact instances of $M_{7,3}$ and $M_{7,4}$, respectively—the best approximations to 3- and 4-note equal temperament within the 7-note template.

This tri-level structure is illustrated in fig. 15, where the outer circle represents the 7-note diatonic as small white circles within the 12-note chromatic, and the inner circle represents the triad {0, 2, 4} and the seventh-chord {0, 2, 4, 6} within the 7-note diatonic reconceived as "chromatic." The interval spectrums of these chords with respect to their *seven-note* context are given beneath the figure. Here the unit *dlen* is the distance from one chord note to the next higher; unit *clen* is the upward step in the seven-note set. Maximal evenness is evident from the fact that each dlen is associated with two clens. We call such sets *second-order* ME sets.

Thus the complete roster of first- *and* second-order ME sets consists of the augmented triad, diminished-seventh chord, three diatonic triads, and four diatonic 7th chords—in the view of many theorists, precisely the set of chords capable of harmonic function in tonal music. It is interesting to note in passing that the tertian 9th chords are not second-order ME sets, but the usual pentatonic scale and its dia-



Interval **spectrums** of triad and 7th-chord, with respect to 7-note diatonic:
triad: <1> = {2, 3}, <2) = {, 5}
7th-chord: <1> = {1, 2}, <2>1{3, 4}, <3> = {5, 6}

Interval **vectors** of triad and 7th-chord, with respect to 7-note diatonic:
triad: [0 2 1]
7th chord: [1 3 2]

Figure 15. Triad and seventh chord as second-order ME sets

tonic transpositions are. Also worthy of passing mention is the fact that any particular triad or seventh-chord is either a first- or a second-order ME set but not both, while the usual pentatonic scale is both a first- and a second-order ME set. Does this property carry over to diatonic sets in general? We conjecture that it does.

The generic triads and seventh chords of $M_{12,7}$ are complementary: extract a triad from the diatonic set and a seventh chord remains. The seventh chord is the second-order ME set with cardinality $(d + 1)/2$, and the triad is the second-order ME set with cardinality $(d - 1)/2$. The relationship of these cardinalities to d, the cardinality of their diatonic "universe," is similar to the relationship of d and $c - d$ to c, where the diatonic set is the first-order ME set with cardinality $(c + 2)/2$ and its complementary pentatonic set is the first-order ME set with cardinality $(c - 2)/2$. In each case, the two complementary cardinalities are the smallest integer greater than, and the largest integer less than, 1/2 the "parent" cardinality.

Conceived as second order ME sets, diatonic triads and 7th chords are none other than a 4-note member of family A discussed earlier—the family of "less interesting" sets with $c = 2d - 1$—and its 3-note complement. Thus we are dealing with what may be reasonably described as a quasi-diatonic set and its complement *within* a diatonic set. The usual diatonic set is a member of family B ($c = 2d - 2$, d odd) with respect to the total chromatic; the diatonic *chords* are members of family A *with respect to the usual diatonic set considered as "chromatic."* As members of family A, triads and seventh chords lack a true "circle of 5ths" within the parent diatonic set: their generator with respect to the usual diatonic set is the same as their "step"—a traditional 3d. However they have all the properties noted for this family by Clough and Myerson: cardinality equals variety, partitioning, and the deep-scale property.

A significant consequence of the deep-scaleness of triads and seventh chords is that, by application of the well-known theorem, the number of common tones between two triads or two seventh chords in the same diatonic set is given by the interval vector for those chords within the diatonic set. For example, the interval vector of the triad, given at the bottom of fig. 15, [0, 2, 1], that is, no intervals of 1—2ds or 7ths, two intervals of 2—3ds or 6ths—and one interval of 3—4th or 5th; from the vector, we can read the familiar information that if two triads are related by step they have no common tones, if by 3d two common tones, and if by 5th one common tone. These vectors are precisely the ones given by Gamer for 3-note deep scales in their 7-note equal-tempered context in his classic paper on equal tempered systems (1967, 42). The vectors were later given with their "diatonic" meaning as in fig. 15 in a paper by one of us (Clough, 1979, 48), but their implications for common tones among triads and 7th chords have not been previously noticed, as far as we know.

The structure observed here is remarkable for what it shows regarding relationships between three nested *levels* of pc sets, which might be labeled chromatic,diatonic, and harmonic. The celebrated unique multiplicity property of the diatonic set with its consequences for hierarchical structuring of transpositions of that set within the 12-tone chromatic, as first observed, we believe, by Milton Babbitt (1965), is present as well in triads and seventh chords, considered as subsets within a seven-note world, where the consequences of that property for hierarchical structuring of transpositions may reasonably be regarded as an important feature of tonal harmony.

To generalize the above, chromatic universes which support diatonic sets also support nested arrays of first-order, second-order, and high-order ME sets that are deep scales with respect to their parent collections. We suggest that the concept of diatonic system might well include, as an essential feature, the hierarchical aspects discussed here. Thus, it would encompass the diatonic system as defined by Agmon (our diatonic set), plus one or more additional levels of structure including, in the usual case, the *harmonic* level.

Further investigations along these lines may be expected to yield additional insights regarding tonal music as we know it, and to provide a basis for invention of hypertonal musics in microtonal worlds.

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