A PARTITION PROBLEM POSED BY MILTON BABBITT
(Part I)

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> Its function might be more clearly expressed for all cases by substituting numbers for letter notation.*
>
> —Friedrich Wilhelm Marpurg

INTRODUCTION

The mathematical problems that arise in twelve-tone theory can be separated into two categories, one "combinatorial" and the other "structural"; these labels reflect different emphases rather than exclusive approaches. On the one hand, a combinatorial problem usually calls for the specification of an algorithm which is designed to compute the number of objects of a certain type which possess a particular shared property. A typical example of such a problem would be the following:

(i) Let G be a finite abelian group and j a positive integer. Devise a procedure to compute and systematically list the number of (G,j) chords derivable from G.

In a recent paper Edwin Hewitt and G. D. Halsey [5] have dealt with this and related questions at great length.[1] Indeed (i) is closely related to certain classification problems involving the theory of finite abelian groups.[2]

* Translation by Alfred Mann.

[1] References will be found on the last page of this paper.

[2] For example, given a finite abelian group G, find all of its tessellations (factorizations). Tessellations of finite abelian groups have been of interest since 1942

On the other hand, a structural problem deals with the abstract, or categorical, background out of which combinatorial problems emerge. For instance, one may ask given a family of (G,j) chords which are solutions to (i), how many of them determine subgroups of G and of what type? In other words structural problems illuminate the setting within which one wishes to deal with more concrete compositional and theoretical issues.

Since the choice of category and combinatorial means is intended to reflect corollary music-theoretic issues, it is necessary to exercise some degree of caution in their selection. This raises the question as to how best to deal with partition problems of the type that Babbitt has posed, especially since they are quite different from the questions dealt with in [5].

It is our belief (partly confirmed by our work) that the partition problems in twelve-tone theory properly belong to the study of combinatorial algorithms and the analysis of matrices defined over an abstract ring. These are among the most important and active areas of current mathematical research, and it is our feeling that an understanding of their structure should provide powerful tools for dealing with distinctly twelve-tone problems.

Since what we present here is a preliminary report of ongoing research, we have abandoned the traditional theorem-proof exposition for a more informal mode of presentation. This is especially important in light of the fact that a number of aspects of the theory we shall present have yet to be adequately formalized. Our major concern has been to give a precise mathematical meaning to the process of aggregate formation. In this we feel that we have been successful. In our view the process of forming an aggregate is nothing more than a particular way of forming tableaux out of matrix elements, and then arranging these in block constructions called "mosaics".

As partition problems of this type do not seem to have been dealt

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when Hajós used them to solve a famous problem of Minkowski. Although we cannot find all of the tessellations of finite abelian groups Hewitt and Halsey were able to prove, with the aid of already established theorems, we can find "all tessellations in all groups of any possible interest for musical composition". This leads them directly to the Hajós groups which arise in the solution of Minkowski's problem. (See Hajós, G. "Über einfache und mehrfache Bedeckung des n-dimensional Raumes mit einem Wurfelgitter", *Math. Z.*, 47, 427–67 (1942) and Fuchs, L. *Abelian Groups*, Publishing House of the Hungarian Academy of Sciences, Budapest (1958); London: Pergamon Press (1960), reprinted (1967).)

with in previous mathematical literature it has been necessary to continually modify old concepts or introduce new ones. This is especially true in those aspects of the theory that deal with the problem of generalized aggregates.

At each point in this exposition we have tried to indicate which concepts are in need of further refinement—where certain ideas need to be developed—or where more formal proofs of statements need to be supplied. Nevertheless, we believe this work to be mathematically correct and that the presentation here given will suggest, at least in broad outline, a coherent mathematical way of dealing with the problem of generalized aggregates in twelve-tone theory.

I. *The Basic Problem*

We shall presume familiarity with the concepts and notation of two of Milton Babbitt's articles: "Twelve-Tone Invariants as Compositional Determinants" [2] and "Set-Structure as a Compositional Determinant" [1]. Assistance with unfamiliar combinatorial concepts can be sought in Berge [3]; likewise, material on matrices over a ring can be found in Chapter II of Bourbaki [4]. Our basic construction can be stated and understood without recourse to any one particular mathematical framework. Indeed, in many respects our characterization reduces to the formalization of procedures and techniques that composers and twelve-tone theorists have been using for decades.

When the actual value (or values) of a pitch class (or collections of pitch classes) is essential, we shall speak of the element or collection of elements belonging to the "prime set" $(0,1,2,\ldots,11)$. When this is not the case, we shall speak of an integer or subset of integers in the ring Z_{12}. In a similar way we can speak of the transposition T_j (where j is the transposition number), the retrograde R, inversion I, and compositions of these; or we may speak simply of the group G of automorphisms g_j which define the basic twelve-tone theoretic transformations. We shall denote hexachords by $H = (h_1,h_2,\ldots,h_6)$ and trichords by $\overline{T} = (t_1,t_2,t_3)$.

As it was originally proposed, what we shall call *Babbitt's Partition Problem* can be stated as follows:

> "Given an array of four forms of an arbitrary twelve-tone set, how many ways can the array be decomposed entirely into four-part, aggregate forming partitions?"

This paper constitutes Part I of a general solution to a major portion of this problem. In particular we will elaborate a procedure that will allow us to obtain sufficient conditions for the partition problem to have a solution. Part II will deal with the implications of Part I, and will offer detailed proofs of all statements and theorems.

For the purposes of Part I, then, we can informally state the Partition Problem as follows:

> (*) Find an algorithm to compute the number 'n' of all four-part aggregate-forming partitions obtainable from an array of four transformations of an arbitrary twelve-tone set.

We note the following about this formulation.

1) The object of the algorithm is to *compute the number* of a particular class of partitions.
2) It is *not* specified by (*) that the value 'n' represent the decomposition of an array uniformly into four-part partitions.

We will now develop a more formal characterization of (*) in two parts. First, we introduce the following notation. Set $A = Z_{12}$ and let $M_n(A)$ denote the ring of $n \times n$ matrices over A. In particular, we restrict ourselves to where $n = 4$, in which case the *first part* of the partition problem can be stated as follows:

> (**) Find an algorithm to compute the number k of all matrices belonging to $M_4(A)$ which have the property that their rows and columns add up to twelve.

Here we exclude 0 as a matrix element. However, repetitions among ring elements are allowed, and addition is taken in the usual sense. A typical matrix satisfying (**) is:

$$\begin{bmatrix} 5 & 4 & 2 & 1 \\ 5 & 3 & 2 & 2 \\ 1 & 2 & 4 & 5 \\ 1 & 3 & 4 & 4 \end{bmatrix} = M.$$

Note the similarity of (**) to the problem of computing magic squares; however here we do not require that the sum of the diagonal elements be twelve.

Given a matrix M which satisfies (**), each row or column of M represents a *partition,* in the sense of Berge, of the integer 12 into four parts. There are fifteen such partitions:

$(9\ 1^3)$, $(8\ 2\ 1^2)$, $(7\ 3\ 1^2)$, $(7\ 2^2\ 1)$, $(6\ 4\ 1^2)$, $(6\ 3\ 2\ 1)$, $(6\ 2^3)$, $(5^2\ 1^2)$, $(5\ 4\ 2\ 1)$, $(5\ 3^2\ 1)$, $(5\ 3\ 2^2)$, $(4^2\ 3\ 1)$, $(4^2\ 2^2)$, $(4\ 3^2\ 2)$, (3^4)

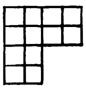
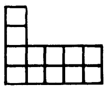
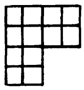
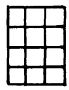
where n^r means the integer n taken r times.

Given such a matrix M we can take any row or column and form what we shall call a *blank tableau*. By a blank tableau we simply mean a tableau with the property that its positions are considered empty until explicitly filled. A position in a blank tableau will be called a *place*.

By a *tableau* we mean a blank tableau whose places have been filled. For example, the matrix

$$M = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 4 & 1 & 4 & 3 \\ 2 & 5 & 2 & 3 \\ 2 & 5 & 2 & 3 \end{bmatrix}$$

yields the blank tableaux

$T_1 =$  $T_2 =$  $T_3 =$  $T_4 =$ 

from its columns (C_1,C_2,C_3,C_4). Note from the blank tableau T_1 we can form the tableau whose places are filled by pitch classes:

$T_1 =$

0	10	1	2
9	11	8	7
3	5		
6	4		

Note also that T contains all of the pitch classes. Tableaux which have this property will play an important role in what follows.

From a musical point of view, tableaux (T) obtainable from columns (C_1,C_2,C_3,C_4) of a matrix M satisfying (**) can be thought of as occurring in *temporal succession,* that is as ordered sequences (T^k) evolving in *time*. However, successions of tableaux which are homogeneous with respect to certain shared properties can, under proper circumstances, be regarded as equivalent, and capable of undergoing permutation without significant alteration of their structure. Therefore, it would be advantageous to construct a framework within which to deal with families of tableaux without regard to their order. That is, we would like to be able to consider collections of tableaux, in the same way we consider families of pitch classes in the theory of all-combinatorial hexachords. This is the motivation behind what we call a *mosaic*.

Specifically, suppose that M is a solution to (**) and that T_k ($k = 1, \ldots, 4$) is a blank tableau generated by the k^{th} *column* of M. Note that there are a total of m ($m = 1,2, \ldots, 12$) places; hence let t^k_m denote the position corresponding to the m^{th} place of T_k where we are assuming that the sequence of places $t^k_1 \cdots\cdots, t^k_{12}$ are assigned by starting in the upper left-hand corner of T_k, and proceeding horizontally across a row until one runs out of positions, returning to the next row, and so on, until all positions have been exhausted).

For example:

$$T_1 = \begin{array}{|c|c|c|c|} \hline t^1_1 & t^1_2 & t^1_3 & t^1_4 \\ \hline t^1_5 & t^1_6 & t^1_7 & t^1_8 \\ \hline t^1_9 & t^1_{10} & \multicolumn{2}{c}{} \\ \cline{1-2} t^1_{11} & t^1_{12} & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

Suppose T_k is a blank tableau and let T^k be the *tableau* obtained from T_k by the mapping $f: A \rightarrow T_k$ which assigns to each place t^k_m an element $f(a)$ of the ring A. Then we can form $r \times n$ dimensional *rectangular arrays* $T^k_{r,n}$ by considering the mapping $F: T^k \rightarrow T^k_{r,n}$ defined by F: $f(a) \rightarrow t^k_{i,j}$ if $t^k_{i,j} = t^k_m$ for some integer m (where $t^k_{i,j}$ is the element in the i^{th} row and j^{th} column of $T^k_{r,n}$ corresponding to the place t^k_m of T_k) or $t^k_{i,j} = *$ otherwise (where we use * as a dummy variable to indicate the absence of a place t^k_m in the blank tableau T_k from which $T^k_{r,n}$ is derived).

For instance, the tableau T_2 from the previous example can be filled and arranged into the array:

$$\begin{bmatrix} 5 & * & * & * & * \\ 4 & * & * & * & * \\ 2 & 1 & 10 & 0 & 6 \\ 7 & 8 & 11 & 9 & 3 \end{bmatrix} \qquad \begin{array}{ll} \text{where } t^2_{1,1} = 5 \text{ but} & \\ t^2 = * \text{ for} & 1 \leq i \leq 2 \\ & 2 \leq j \leq 5 \end{array}$$

The arrays $(T^k_{r,n})$ we will call *partition matrices*. By means of this construction, for each solution M to (**) we can obtain a *block matrix* M*:

$$M^* = \left[\begin{array}{c:c} T^1_{1,1} & T^2_{1,2} \\ \hdashline T^3_{2,1} & T^4_{2,2} \end{array}\right]$$

whose blocks are composed of partition matrices:

$$T^k_{r,n} = \left[\begin{array}{c:c} & * \\ \hdashline (t^k_{i,j}) & \end{array}\right] \qquad \begin{array}{l} k = 1, \ldots, 4 \\ 1 \leq r \leq 9 \\ 1 \leq n \leq 9 \\ t^k_{i,j} = a \text{ or } t^k_{i,j} = * \text{ for} \\ \qquad k_1 \leq i \leq l_1 \\ \qquad k_2 \leq j \leq l_2 \end{array}$$

where the integers (k_1, l_1) and (k_2, l_2) are determined by the blank tableau T_k derived from the k^{th} column of M. Note also nine is the largest integer which can occur in a partition of twelve into four parts.

A block matrix M* so constructed we call a *mosaic*. In other words

$$\left[\begin{array}{cccc:ccccc} 0 & 10 & 1 & 2 & 5 & * & * & * & * \\ 9 & 11 & 8 & 7 & 4 & * & * & * & * \\ 3 & 5 & * & * & 2 & 1 & 10 & 0 & 6 \\ 6 & 4 & * & * & 7 & 8 & 11 & 9 & 3 \\ \hdashline 3 & 9 & 11 & 8 & 7 & 4 & 6 & * & * \\ 6 & 0 & 10 & 1 & 2 & 5 & 3 & * & * \\ 4 & 7 & * & * & 8 & 11 & 9 & * & * \\ 5 & 2 & * & * & 1 & 10 & 0 & * & * \end{array}\right] = M^*$$

is a·mosaic constructed from the blocks corresponding to the tableaux derived from the columns of the matrix M of our initial example. Note we observe that successions of tableaux can be said to be equivalent with respect to certain shared characteristics if their corresponding mosaics are homogeneous with respect to those characteristics.

We are now in a position to state the *second part* of the partition problem.

Let g_k $(k = 1, \ldots, 4)$ be a sequence of any four transformations of the ring A by elements of the group G. Note that this sequence generates a 4×12 matrix $(g_{i,j})$:

$$M(G) = \begin{bmatrix} g_{1,1} \ldots\ldots\ldots\ldots g_{1,12} \\ g_{2,1} \ldots\ldots\ldots\ldots g_{2,12} \\ g_{3,1} \ldots\ldots\ldots\ldots g_{3,12} \\ g_{4,1} \ldots\ldots\ldots\ldots g_{4,12} \end{bmatrix}$$

For instance,

$$\begin{bmatrix} 0 & 10 & 1 & 2 & 5 & 3 & 9 & 11 & 8 & 7 & 4 & 6 \\ 9 & 11 & 8 & 7 & 4 & 6 & 0 & 10 & 1 & 2 & 5 & 3 \\ 3 & 5 & 2 & 1 & 10 & 0 & 6 & 4 & 7 & 8 & 11 & 9 \\ 6 & 4 & 7 & 8 & 11 & 9 & 3 & 5 & 2 & 1 & 10 & 0 \end{bmatrix}$$

is generated by the transformations S, T_9IS, RT_9IS, RS.

Suppose that M is a solution to (**). Let T_k be the blank tableau generated by the k^{th} *column* of M.

Definition: We will say that the blank tableau T_k is *properly filled* by elements of the matrix $(g_{i,j})$ if there is a mapping $f: (g_{i,j}) \rightarrow T_k$ such that segments of $(g_{i,j})$ are mapped onto rows or columns of T_k in such a way that T_k contains all of A.

In terms of the partition problem given at the beginning of this section, the notion of proper filling simply corresponds to the idea that each blank tableau must be filled by all of the pitch classes and filled in such a way that distinct places contain distinct classes.

Suppose that $\mathbf{M}(M)$ denotes the collection of all matrices of $M_4(A)$ which satisfy (**). (Note $|\mathbf{M}(M)| = k$ where $|\quad|$ denotes cardinality.) Let $\mathbf{T}(T_k)$ denote the collection of all blank tableaux (T_k) generated by the *columns* of elements of $\mathbf{M}(M)$. For a given sequence g_k of transformations and matrix $(g_{i,j})$ let $\nu(T_k)$ denote the number of all T_k in $\mathbf{T}(T_k)$ that can be properly filled by elements of $(g_{i,j})$. Then the *second part* of the partition problem states:

(***) Find an algorithm for computing $\nu(T_k)$.

Note that since there are at most four blank tableaux corresponding to each M in $\mathbf{M}(M)$, then $|\mathbf{T}(T_k)| \leq 4k$ hence we trivially have

$$\nu(T_k) \leq 4k$$

Since the number $\nu(T_k)$ is identically the number "n" of aggregate-forming partitions mentioned in (*), we will define a blank tableau T_k to be a *solution* of the partition problem if it can be properly filled by elements of a fixed matrix $(g_{i,j})$.

In the remainder of this paper we will outline a procedure that will enable us to construct blank tableaux which are solutions to the partition problem in the above sense. This procedure will in turn generate an algorithm that will enable us to obtain an estimate of $\nu(T_k)$. Indeed, it yields somewhat more, since each such T_k corresponds to a column of some M satisfying (**). Hence we are able to obtain a lower bound on k. The class of blank tableaux that can be constructed in this manner was suggested by an analysis of Milton Babbitt's *Partitions*; hence they will be called *B-constructible*. Our main Theorem can be stated as follows:

Theorem: A sufficient condition for a blank tableau T_k to be a solution to the partition problem is that it be B-constructible.

Corollary: $\nu(T_k)/4 \leq k$

For the purposes of the present paper we shall only give the outline of a proof of the above Theorem. (A detailed proof will appear in Part II.)

II. *The Main Construction*

One approach to the solution of (*) would be, roughly speaking, a step-by-step method which would examine a given array of set forms over all the possibly four-part partitions of twelve together with their respective permutations. We will reject this method at once for two reasons. First, such a method must be inefficient, because the value of n is computed by attempting to *derive* each of the possible solutions a posteriori. But it is clear that some attempts will *fail* to be proper solutions; however, such failures do not (and cannot) signal the termination of a search. This leads directly to the observation that there are some set forms, together with sets (or configurations of such sets) which can be shown to possess *no* uniform decompositions into four-part aggregates. One need only consider a twelve-tone set together with three transpositions T_j of itself such that the value of j is always 0.

The proposed step-by-step method cannot discriminate such cases. Thus we may infer the cardinal point: that is, the value of n will be partially determined by the structure of a set and the structure of the array of forms of the set. As noted above, it is easily established that for some configurations the value of n will be 0. What we shall propose here is a sketch of a method of construction for a class of twelve-tone sets and their corresponding arrays such that n is relatively large. Further, we will try to substantiate the conjecture that the values of n computable from these configurations represent a genuine lower bound on the value of n over the range of all twelve-tone sets.

We begin by considering an all-combinatorial hexachord H_0. It is clear that we can find trichords $\overline{T}_0$ and $\overline{T}_3$ such that $\overline{T}_3 = RT_jI(\overline{T}_0)$ for some j, since an all-combinatorial hexachord is self-invertible. Hence the inversion of an all-combinatorial hexachord into itself induces a pairing of all the elements in the hexachord. Since the elements of each pair can be placed in corresponding order positions of $\overline{T}_0$ and $\overline{T}_3$ we have $\overline{T}_3 = T_jI(\overline{T}_0)$. Now retrograde $\overline{T}_3$ to obtain $\overline{T}_3 = RT_jI(\overline{T}_0)$.

We next construct the set form S_1 from H_0 by generating the complementary hexachord H_6 such that $H_6 = RT_m(H_0)$ where of course $H_0 \cap H_6 = \phi$. That such a complementary H_6 exists is true by definition of the all-combinatorial hexachords. Furthermore, thus constructed, S_1 is a properly derived, degenerate twelve-tone set.

Now construct the set form S_2 where

a) $S_2 = T_k I(S_0)$
b) $H_0(S_1) \cap H_0(S_2) = \phi$
c) $s_6(S_1) = s_1(S_2)$.

That these conditions can always be met comes from the following observations. H_0 of S_1 has been formed by concatenating a trichord $\overline{T}$ with a retrograde inversion of that trichord. H_6 of S_1 has been formed by a retrograde of H_0, so that H_6 is composed of a pair of trichords which are, in order, an inversion and a retrograde of $\overline{T}_0$. Hence H_6 is a complementary inversion of H_0. In the case of an all-combinatorial hexachord H_0 of order two or greater, there may be more than one complementary inversion, hence condition (c) insures that the hexachords H_6 of S_1 and H_0 of S_2 are identical.

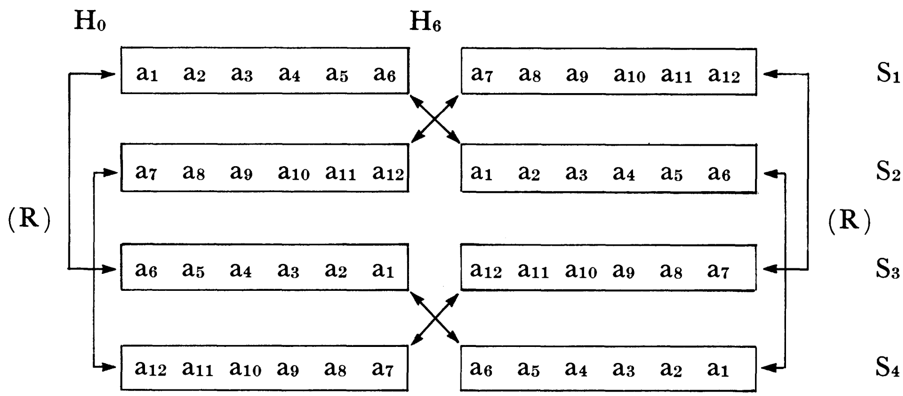
We next form the set form S_3 by retrograding S_2, and then form S_4 by retrograding S_1. Note that since H_0 of S_1 is identical with H_6 of S_2, and H_6 of S_1 is identical with H_0 of S_2, we can observe the following relations among the hexachords of S_1, S_2, S_3, and S_4:

(1) $H_0(S_1) = H_6(S_2)$; (2) $H_6(S_1) = H_0(S_2)$;
(3) $H_0(S_1) = R(H_0(S_3))$; (4) $H_6(S_1) = R(H_6(S_3))$;
(5) $H_0(S_2) = R(H_0(S_4))$; (6) $H_6(S_2) = R(H_6(S_4))$.

Note that all of the operations in the above construction have analogues in the ring A. That is, in principle it is possible to describe all of the music-theoretic transformations together with their chordal counterparts purely in terms of permutations of the ring elements. A full proof of our basic Theorem satisfying all of the requirements of mathematical rigor would appeal precisely to such a reformulation. In Part II we will present a recasting of the above procedure entirely from a ring-theoretic point of view. As a consequence of this procedure we are able to derive a 4×12 matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_{12} & a_{11} & a_{10} & a_9 & a_8 & a_7 \\ a_{12} & a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{bmatrix} = M(A)$$

which displays the noted properties of the set constructions which can be summarized in the following chart.



Since we begin with a specific hexachord H_0 the process just outlined is entirely constructive, and can in fact be described recursively. Furthermore, one can proceed from M(A) to a blank tableau T_k by beginning at the left of M(A) and assigning successive values in the rows of M(A) to corresponding places t^k_m in T_k. For example, blank tableaux corresponding to $(4\ 3^2\ 2)$ and $(7\ 3\ 1^2)$ can be filled:

a_1	a_2	a_3	a_4
a_7	a_8	a_9	
a_6	a_5		
a_{12}	a_{11}	a_{10}	

a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_{11}						
a_{12}						
a_8	a_9	a_{10}				

Given a sequence of four blank tableaux T_1, T_2, T_3, and T_4 one can then proceed to the construction of tableaux and of mosaics using the procedure previously outlined.

We now observe the following properties of any matrix of sets generated in this fashion.

(I) *Any* four-part partition can be found in such an array by the following procedure.

1) Divide S_1 into disjunct segments according to the normal representation of the specified partition.
2) Find the conjunct segment of S_4 with content equivalent to the *second* segment of S_1.
3) Find the conjunct segment of S_2 with content equivalent to the *third* segment of S_1.

4) Find the conjunct segment of S_3 with content equivalent to the *fourth* segment of S_1.

The combination of the first segment of S_1 with the selected segments of S_2, S_3, and S_4 will yield an aggregate under the specified partition. Moreover, the segments of S^2, S_3, and S_4 which *precede* the selected segments will yield a three-part aggregate, and the segments of S_1, S_2, S_3, and S_4 which *follow* the selected segments will yield two disjunct aggregates of two, three, or four parts.

(II) Given an array M(A) with mosaic M*, M(A) will properly fill M* only under the condition that every partition matrix $T^k_{r,n}$ of M* be derivable from a column C_k of some solution M to (**) with the property that C_k contain two elements whose sum is six. That is, there are two elements t_i and t_j belonging to C_k such that $t_i + t_j = 6$.

That this condition is necessary can be seen from the following example. Clearly

$$M = \begin{bmatrix} 5 & 3 & 2 & 2 \\ 3 & 2 & 5 & 2 \\ 2 & 2 & 3 & 5 \\ 2 & 5 & 2 & 3 \end{bmatrix}$$

is a solution to (**). However there is no way that the matrix

$$M(A) = \begin{bmatrix} 0 & 10 & 1 & 2 & 5 & 3 & 9 & 11 & 8 & 7 & 4 & 6 \\ 9 & 11 & 8 & 7 & 4 & 6 & 0 & 10 & 1 & 2 & 5 & 3 \\ 3 & 5 & 2 & 1 & 10 & 0 & 6 & 4 & 7 & 8 & 11 & 9 \\ 6 & 4 & 7 & 8 & 11 & 9 & 3 & 5 & 2 & 1 & 10 & 0 \end{bmatrix}$$

can properly fill any mosaic containing a partition matrix derivable from a column of M.

The condition that properly filled mosaics contain partition matrices derivable from partitions of the integer twelve into four parts, at least two of which add up to six, expresses one of the deepest and most powerful properties possessed by aggregate forming partitions arrived at by means of the process of B-construction; namely, the property of of invariance with respect to certain twelve-tone operations.

One aspect of this invariance can be expressed as follows. Partition problems in twelve-tone theory relate, not only to decompositions of twelve (the collection of full pitch classes), but also to decompositions of hexachords. Hence, hexachords belonging to equivalent mosaics have the property that if one looks at their "harmonic succession" *in time,* this "harmonic succession" can be presented while at the same time being *constituted of segments of different sizes of different lines.*

Another aspect is that we should like to be certain that transformations of mosaics (at least those that are equivalent in some sense) will not alter their essential mathematical characteristics.

It is not too difficult to demonstrate that if, in the process of forming H_0, you take the inversion of T_0 rather than its retrograde inversion, you will arrive at a 4×12 matrix M(A) which will properly fill some but *not* all of the B-constructible tableaux. The reason for this is that in the derivation of the hexachord H_0 of S_1, the hexachords H_0 of S_3 and S_4 are *not* the retrogrades of the hexachords H_0 of S_1 and S_2 but merely are retrogrades of the *order* of the trichords. In other words:

(III) The class of blank tableaux (T_k) which are solutions to the partition problem is larger than the class of B-constructible blank tableaux, that is, the requirement that a blank tableau be B-constructible is sufficient but *not necessary* for it to be a solution to the partition problem. Hence the inequalities in the previous section are strict.

The requirement that two mosaics M_1^* and M_2^* have partition matrices derivable from blank tableaux which correspond to columns of M representing partitions of the integer 12 into four parts (two of which sum to six) is an equivalence relation in the class of all mosaics which represent a solution.

We therefore end with the following question: is there a *canonic form* which will enable us to determine when two mosaics are in fact transformations of the *same mosaic?* If so, then we shall have a means of determining when two mosaics belong to the same equivalence class, i.e., are homogeneous with respect to shared characteristics.

REFERENCES

[1] Babbitt, Milton. "Set Structure as a Compositional Determinant", in *Perspectives on Contemporary Music Theory*. New York: W. W. Norton & Co. Inc. 1972.

[2] ———. "Twelve-Tone Invariants as Compositional Determinants", in *Problems of Modern Music*. New York: W. W. Norton & Co. Inc. 1960.

[3] Berge, C. *Principles of Combinatorics*. New York: Academic Press 1971.

[4] Bourbaki, Nicolas. *Elements De Mathematique, Algebre*. Paris: Hermann 1943, 1947, 1948, 1971. (Translated as "Elements of Mathematics, Algebra", I, chapters 1–3. Reading: Addison-Wesley 1974.)

[5] Halsey, G. D. and Hewitt, E. "A Group-Theoretic Method in Musical Theory". To appear in *Selecta Mathematica* (Heidelberg) in German translation as "Eine Gruppentheoretische Methode in der Musiktheorie".