A RUDIMENTARY GEOMETRIC MODEL FOR CONTEXTUAL TRANSPOSITION AND INVERSION

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ABSTRACT

The dihedral group D_n is, by definition, the (non-Abelian) group of symmetries of the n-sided regular polygon. It is well-known that the group of 12 transpositions and 12 inversions acting on the 12 pitch classes (T/I) is isomorphic to D_{12}, as is the Riemann-Klumpenhouwer *Schritt/Wechsel* group (S/W). It follows that T/I is isomorphic to S/W. However, in Lewin's terms, these two groups are "anti-isomorphic." Also they may be recombined as T/W and S/I, groups whose structures are isomorphic to the Abelian group $Z_2 \times Z_{12}$ (and hence to one another). This paper shows the relationships among these four groups by means of, first, a miniature model consisting of a particular group isomorphic to D_3, its anti-isomorphic group, and their two Abelian recombinations (analogous to the above), all acting on a system of two concentric equilateral triangles; then, second, an expanded model consisting of a particular group isomorphic to D_{12}, its anti-isomorphic group, and their two Abelian recombinations, all acting on a system of two concentric circles with equispaced points located on their circumferences.

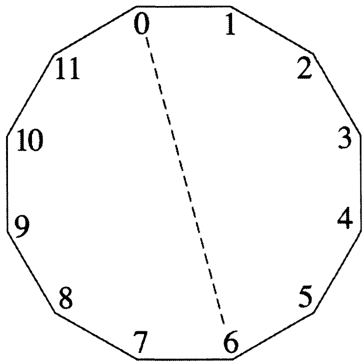
This paper offers a geometric model for some of the central concepts in neo-Riemannian theory, in particular commuting groups (Lewin 1987), anti-isomorphisms (Lewin 1993), and contextual transposition and inversion as exemplified in the *Schritt/Wechsel* group (hereafter, S/W), conceived by Riemann (1880) and reformulated in explicitly group-theoretic terms by Klumpenhouwer (1994). All of these concepts are logically bound up with the familiar group of (non-contextual) transpositions and inversions—the T_n/T_nI group (hereafter, T/I) of atonal set theory—so we begin with a review of that group, then move quickly to construction of the model, which supports connections among T/I, S/W, and two hybrid groups that spring from them.

T/I, consisting of 12 transpositions and an equal number of inversions (traditionally conceived as inversions about pitch classes 0 and 6, followed by transposition), is nicely modeled by the congruence motions of a regular dodecagon (or, perhaps more appropriately, by a circle with 12 equispaced points on the circumference, which must map onto themselves when the circle is moved). Like the inversions and transpositions, these congruence motions also form a group of 24 elements (order 24). The transpositions of T/I correspond to rotations of the dodecagon (or circle), and the inversions of T/I to flips around a particular axis followed by rotations. For example, given the position of the dodecagon as in Example 1, T_1 corresponds to counterclockwise rotation by 30° so that vertex 0 is replaced by vertex 1, and 1 is replaced by 2, etc., while T_1I corresponds to a flip around vertices 0 and 6 (dashed line in Example 1) followed by a rotation (this time clockwise from the viewer's original perspective) again replacing 0 by 1, 1 by 2, etc., with the net result that 0 and 1 exchange, as do 11 and 2, 10, and 3, etc. (Alternatively, the 12 inversions may be defined as flips around the 12 distinct axes.)

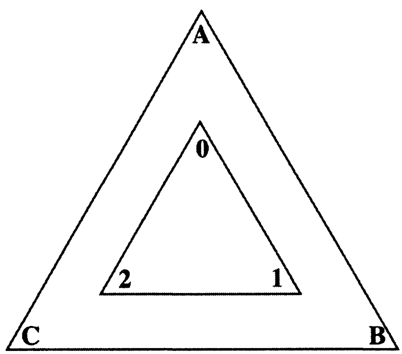
Thus T/I is isomorphic (identical in structure) to the group of congruence motions of the dodecagon, known in mathematics as the *dihedral* group D_{12}. It is well known in music theory that T/I is non-commutative: for any two operations *a, b,* it is not generally true that the result of *a* followed by *b* is the same as the result of *b* followed by *a.*

Note that when the dodecagon is rotated or flipped so as to be congruent with itself, there is implicit a persistence of the figure in its *original* state; that is how we judge the "new" figure to be congruent. Thus there are in a sense *two* dodecagons in play—"old" and "new." To see more readily how this duality works, let us use triangles instead of dodecagons. Example 2 shows a system of two equilateral triangles ABC and 012. Though they are merely concentric, not congruent (to facilitate the demonstration in two dimensions), we may consider the two triangles to be effectively congruent, with paired vertices A and 0, B and 1, C and 2. For the position of the triangles in Example 2, we will say that the *value* of A is 0, or the value of 0 is A, and write simply A0. The *state* of the system in Figure 2 is given by the triple of assignments A0, B1, C2 (obviously any two of these assignments implies the third).

Allowing rotations and flips that keep the two triangles "congruent," it is clear that the system has six possible states. Numbered 1–6, these states are shown as columns in the table of Example 3. Operations on the



Example 1



Example 2

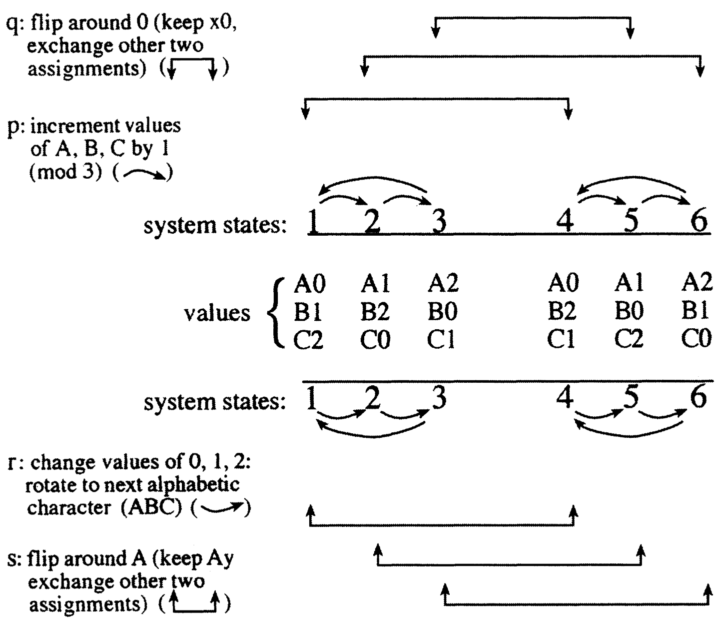
system may now be defined in terms of how they affect the pairings of vertices. We need to define four such operations, as follows:

- *p*, a rotation that increments the values of A, B, C, by one (mod 3), e.g., A0 → A1, B1 → B2, C2 → C0
- *q*, a flip that retains the assignment of 0 and exchanges the other two assignments, e.g., A1 → A2, B2 → B1, C0 → C0
- *r*, a rotation that increments the values of 0, 1, 2, by one alphabetic character (C increments to A), e.g., A0 → B0, B1 → C1, C2 → A2
- *s*, a flip that retains the assignment of A and exchanges the other two assignments, e.g., A1 → A1, B2→ B0, C0 → C2.

The action of these operators on the system is shown in Example 3 by means of curved arrows (rotations) and crooked arrows (flips).

We now describe four different but closely related mathematical groups, G_1, G_2, G_3, G_4, that fall out from these operations. G_1 has the set of elements $\{I, p, p^2, q, qp, qp^2\}$, which includes the *generators* of the group, p and q, and all their distinct products, one of which is I, the *identity* operation that keeps the system in its current state. p^2 means "perform p twice." (Henceforth in this paper combinations of group elements are written with right orthogonal notation: qp means q followed by p—not the reverse.) It is easy to see the effects of these various operations if we think of them in terms of permutations of system states, as shown in Example 4. For example, p corresponds to the permutation (123) (456), meaning that if we repeatedly apply p, state 1 changes to state 2, then to state 3, then back to state 1, etc. Indeed the six permutations corresponding to the six operations also form a group.[1] Also given in Example 4 is the "multiplication table" of G_1, showing how all binary combinations of elements produce an element of the group. Note that G_1 is *not* commutative: $pq \neq qp$ in general.

G_2 is similarly described in Example 4, though we omit the multiplication table, which the reader may wish to work out as an exercise. In



Example 3. System of dual triangles: generators and system states

elements, G_1	*corresponding elements of permutation group (refer to system states, Ex. 3)*
I	(1) (2) ... (6)
p	(123) (456)
p^2	(132) (465)
q	(14) (26) (35)
qp	(15) (24) (36)
qp^2	(16) (25) (34)

multiplication table, G_1:

		second operand					
		I	p	p^2	q	qp	qp^2
	I	I	p	p^2	q	qp	qp^2
	p	p	p^2	I	qp^2	q	qp
first	p^2	p^2	I	p	qp	qp^2	q
operand	q	q	qp	qp^2	I	p	p^2
	qp	qp	qp^2	q	p^2	I	p
	qp^2	qp^2	q	qp	p	p^2	I

elements, G_2	*corresponding elements of permutation group*
I	(1) (2)...(6)
r	(132) (456)
r^2	(123) (465)
s	(14) (25) (36)
sr	(15) (26) (34)
sr^2	(16) (24) (35)

Example 4. Non-commutative groups G_1 and G_2, based on dual-triangles, and their corresponding permutation groups

view of their parallel descriptions (Example 3) it is not surprising that G_1 and G_2 appear, in terms of their permutations (Example 4), to have the same structure. In fact their structures *are* "the same" in a well-defined mathematical sense. Let us define a map α as follows:

$$\alpha: I \to I \quad p \to r \quad p^2 \to r^2 \quad q \to s \quad qp \to sr \quad qp^2 \to sr^2$$

On the left-hand sides of the six pairs are all the elements of G_1 and on the right-hand sides, all the elements of G_2. Let i(x) denote the image of x (e.g., i(p) = r). Then, under α, i(ab) = i(a)i(b), for any two elements a, b of G_1 (the image of the product equals the product of the images). Thus α is an isomorphism from G_1 to G_2. Either G_1 or G_2 may be taken to describe the congruence motions of *either* ABC *or* 012, if the remaining triangle is fixed in place. In this interpretation G_1 or G_2 becomes simply the abstract group of congruence motions of an equilateral triangle, known in mathematics as the dihedral group D_3. Thus G_1, G_2, and D_3 are all isomorphic to one another. Isomorphic to these, as well, are subgroups of T/I such as $\{T_0, T_4, T_8, I_1, I_5, I_9\}$, discussed by Clampitt (1998). Like all dihedral groups of order 6 or higher, these groups are non-commutative.

We now define a second map β as follows:

$$\beta: I \to I \quad p \to r^2 \quad p^2 \to r \quad q \to s \quad qp \to sr \quad qp^2 \to sr^2$$

Under β, i(ab) = i(a)i(b) no longer holds. Suppose, for example, $a = p$, $b = q$. Then

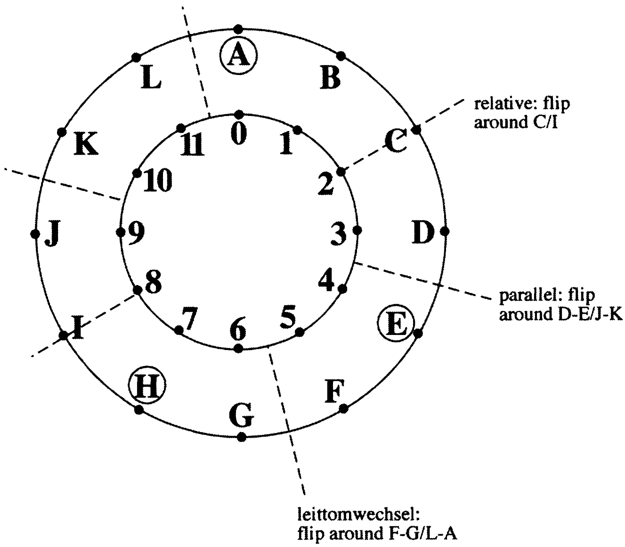
$$\mathrm{i}(pq) = \mathrm{i}(qp^2) = sr^2$$

But $$\mathrm{i}(p)\mathrm{i}(q) = r^2 s = sr \neq sr^2$$

Under β, however, i(ab) = i(b)i(a), for any a, b in G_1. Following Lewin (1987), we will say that β is an *anti-isomorphism* from G_1 to G_2.[2] Further, although G_1 and G_2 are both non-commutative, they are related to one another as what Lewin calls "commuting groups"—any element of one group commutes with any element of the other. This enables the formation of two new groups via recombination of elements from G_1 and G_2. They are G_3, with generators p and s, and G_4, with generators q and r, as set forth in Example 5. G_3 and G_4 *are* commutative groups: for any two elements a, b, of G_3, $ab = ba$, and similarly for G_4.[3]

elements		*corresponding elements of permutation groups (refer to system states in Ex. 3)*	
G_3	G_4	G_3	G_4
I	I	(1) (2) ...(6)	(1) (2) ...(6)
p	r	(123) (456)	(132) (456)
p^2	r^2	(132) (465)	(123) (465)
s	q	(14) (25) (36)	(14) (26) (35)
sp	qr	(153426)	(152436)
sp^2	qr^2	(162435)	(163425)

Example 5. Commutative groups G_3 and G_4, based on dual-triangles, and their corresponding permutation groups



Example 6. Contextual inversions of the consonant triad, modelled by dual circles

We now expand the machinery constructed above for dual triangles to apply to dual dodecagons. Logically the triangles of Example 2 should become dodecagons, but, with no effect on results, we will use the concentric circles of Example 6 instead, as they are more familiar in music theory.[4] Each of the four groups G_1, G_2, G_3, G_4 expands from a 6-element to a 24-element group. We label these new groups H_1, H_2, H_3, H_4 in correspondence to their counterparts in the system of dual triangles. The groups work just as they did before, so we will reuse *p, q, r, s,* redefining these symbols only slightly to apply in H_1, H_2, H_3, H_4: For *p* and *r* in H_1, H_2, H_3, H_4, values are incremented mod 12 or mod the first 12 letters of the alphabet (A-L), respectively, instead of mod 3 or the first 3 letters, as for G_1, G_2, G_3, G_4. For *q* and *s* in H_1, H_2, H_3, H_4, flips are slightly more complicated as there are 12 possible axes, half of which run "between" points on the circles (all this will be quite familiar to students of atonal set theory). For *q,* we have axes 0/6, 0-1/6-7, 1/7, ... , 5-6/11-0; and for *s,* A/G, A-B/G-H, B/H, ... , F-G/L-A.

Just as G_1, G_2, G_3, G_4 pertain to the six states of the dual-triangle system, so do H_1, H_2, H_3, H_4 pertain to the 24 states of the dual-dodecagon system. But we need not work with the dual-dodecagon system directly. Since its groups are isomorphic to the musical groups we wish to explore, we can move directly to musical interpretations of H_1, H_2, H_3, H_4. Thus, in what follows, "H_1" means "the interpretation of H_1 to apply in the pitch-class system"; similarly for H_2, H_3, H_4.

H_1 has the set of operations $p, p^2, p^3, \ldots, p^{11}, p^{12} (= I)$ (the transpositions), and $q, qp, qp^2, qp^3, \ldots, qp^{11}$ (the inversions). (p is the same as T_1 in T/I, and $p^{12} (= I)$ is the same as T_0.) All of these are precisely the operations of T/I. We preserve our notation from above, instead of shifting to the more familiar music-theoretic notation, in order to show the relation between H_1 and the remaining groups to be discussed. To see how transposition works here, note that p calls for the values of A, B, C, ... , L to be incremented by one. This is essentially what is done in transposing, say, {0, 4, 7} to {1, 5, 8}. Suppose A = 0, E = 4, H = 7 (A0, E4, H7 in shorthand). Then after applying p we have A1, E5, H8.

H_2 has the set of operations $r, r^2, r^3, \ldots, r^{11}, r^{12} (= I)$ and $s, sr, sr^2, sr^3, \ldots, sr^{11}$. The first 12 of these (powers of r) represent, in a very general sense, *contextual* transpositions; the remaining 12 (combinations of s with powers of r) represent, in the same general sense, *contextual* inversions. It will be easier to understand this once we have looked at an example of a specific *context* for contextual transposition and inversion.

To construct such an example, we designate three appropriate positions on the outer circle of Example 6, say, A, E, and H, as representing the structure of the consonant triad. This structure, represented by the triple AEH is now the context for transposition and inversion. Next we define Riemann's Parallel, Relative, and Leittonwechsel on the basis of AEH. They are, respectively, flips around the axes D-E/J-K, C/I, and F-G/L-A. Notice that we have not specified whether AEH represents a major or a minor triad; in fact it represents both. To effect, say, the Parallel transformation on a particular triad, we bring the system to a state where the triad's numbered pitch classes are aligned with AEH (as the consonant triad has no non-trivial symmetries, this state will be unique), execute the required flip around D-E/J-K, and observe the resulting values of A, E, and H. Choose, for example, {0, 3, 7}. Aligning this set with AEH, we have A7, E3, H0. The flip around D-E/J-K produces A0, E4, H7, and the result of the transformation is therefore {0, 4, 7}.

We need not work out the details here, but the remaining nine *Wechsel* may be defined in similar fashion as flips around particular axes, and the twelve *Schritte* follow naturally from these.[5] The very general sense (referred to above) in which H_2 represents contextual transpositions and inversions is simply that H_2 is isomorphic to S/W or to any other group based on the well-defined contextual transposition and inversion of sets in a particular set class of the 12 pitch-class universe. The distinction is that H_2 acts on the system states of the dual-circle model of pitch classes,[6] while, for example, S/W acts on the consonant triads.

As with G_3 and G_4, H_3 and H_4 may now be constructed via recombination of elements from H_1 and H_2. H_3 is based on generators p and s, and H_4 on q and r. In music-theoretic terms, we have mixed operators from T/I and S/W to form new groups T/W and S/I. Like G_3 and G_4, H_3 and H_4 are commutative groups.[7] In summary, we have two kinds of transposition, "ordinary" (non-contextual) transposition and contextual, likewise two kinds of inversion. To form a workable group of order 24, one can pair either kind of transposition with either kind of inversion—whence the four such groups described in this paper.

NOTES

1. Inspecting the permutations corresponding to each element of G_1, one can check that the axioms for a mathematical group are satisfied: closure—all products of elements (i.e., sequences of permutations) are equal to an element of the group; identity—there is an element, I, (namely, the permutation (1) (2) ... (6)) such that $xI = Ix = x$, for any element x; inverses—each of the elements $I, q, qp,$ *and* qp^2 is its own inverse ($I = I^2 = q^2 = (qp)^2 = (qp^2)^2$), and p and p^2 are inverses ($I = pp^2 = p^2p$); associativity—for any three elements $a, b, c,$ we have $(ab)c = a(bc)$.
2. In general, for groups G and H, if and only if there is an isomorphism from G to H, we can define an anti-isomorphism simply by exchanging the positions of inverse-related images (r and r^2 in the present case). Since such manipulation has no effect on abstract group structures, it is significant only where we are concerned with group actions on sets or systems.
3. For the mathematically minded, G_3 and G_4 (and their corresponding permutation groups) are isomorphic to Z_2 X Z_3 (which, since 2 and 3 are coprime, is identical to Z_6, the cyclic group of integers {0, 1, ... , 5} under addition mod 6). Thus, although G_3 and G_4 are described above in terms of *two* generators, each can be generated by a *single* generator (*sp* or *qr*, respectively). Isomorphic to G_3 and G_4 (and hence to Z_6) is the group of intervals underlying the *maximally smooth hexatonic cycle,* as discussed in Clampitt 1998.
4. For our purposes, the congruence motions of the dodecagon become congruence motions of the circle with 12 equispaced points on the circumference—motions which bring the 12 points into coincidence.
5. Just as T/I may be generated by T_1 (or T_{11} or T_5 or T_7) plus *any one* of the 12 inversion operators, so S/W and comparable groups may be generated by a single contextual transposition operator of order 12 plus any one of the contextual inversions.
6. David Lewin pointed out at the 1997 Buffalo Symposium that this model would be adaptable to Babbitt's (1961) conception of a 12-tone row as a set of pitch-class/order-number pairs.
7. Again, for the mathematically minded, H_3 and H_4 (and their corresponding permutation groups) are isomorphic to Z_2 X Z_{12}. However, since 2 and 12 are not coprime, these groups, unlike G_3 and G_4, are not cyclic.