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THE STRUCTURE

OF

Introduction

If we consider the twelve-tone series as a means of maintaining an all pitch-class saturated texture, then the all-interval-series (henceforth AIS) may be seen as an extension of the saturation concept to another dimension. Such series have been used most notably by Berg in the LYRIC SUITE and in more recent works by Nono, Stockhausen, Babbitt and others. This paper explores the structure and properties of AISs, their various sub-groups and some of their combinatorial properties. It was found convenient to tabulate certain final and intermediate results with computer programs, but we prefer to keep that aspect of our work in the background, as computer application was not the object of our study. Our results illustrate several interesting properties and provide, in addition, some useful compositional material.

ALL-INTERVAL SERIES

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Basics

An all-interval series is a twelve-tone row among whose eleven successive intervals there are no repetitions, where the interval between two successive pitch-classes (PCs) is the mod-12 difference of the second minus the first. The last note of an AIS is always separated from the first by the interval 6, which is the mod-12 sum of the intervals 1 through 11, no matter what their order. Considering the series as a cycle, with the first note succeeding the last, there are always two 6s and one of each of the other intervals. Thus among the cyclic permutations of an AIS, there is exactly one other form that fits our definition, with eleven distinct intervals. The other cyclic permutations are degenerate, having two 6s and lacking some other interval. A cyclic permutation whose "outer interval" is 6 is rotationally normal. Since the all-interval property is unaffected by the level of transposition, there will always be a transpositionally normal form of any AIS, whose first PC is zero. An AIS that is both transpositionally and rotationally normal is in normal form. There are 3856 such series.

Generation

A simple FORTRAN program is provided in Table 1, which may be used to generate the entire corpus of AISs. It works essentially along the lines of the algorithm described by Bauer-Mengelberg. *1 No detailed analysis of the program is provided, as it is short enough and utilizes so little memory that it can easily be hand-simulated. On an IBM 370/155 computer, it requires about 12 seconds to complete.

Its output is a succession of AISs in ascending alphabetical order by PCs; each series considered as a 12-digit, base-12 number is higher than its predecessors. We can now refer to normal-form AISs by number, an integer between 1 and 3856. The program does not, in fact, generate all 3856 of the series, but stops at number 1928, which is the half-way point. The AISs remaining are the inversions of those already enumerated. The entire list is symmetrical about the pair 1928-1929, with AIS number 3857 - n always being the inversion of AIS number n, which results from the fact that the program in effect examines incrementally numbered pitch permutations, which considered as a set also exhibit this "mirror property."

Closed operations

The set of AISs (henceforth implicitly in normal form) is closed under retrograde (R), inversion (I), and the note-wise multiplication of the series by 5, mod-12 (M). In addition, there is always a w such that the rotation of an AIS w places yields again a rotationally normal AIS. *2 We arbitrarily call this operation Q. One must transpose the results of Q and R so that they are transpositionally normal. For example: *3

P :	0	1	4	9	3	2	A	8	5	7	B	6
R(P):	0	5	1	B	2	4	8	9	3	A	7	6
I(P):	0	B	8	3	9	A	2	4	7	5	1	6
M(P):	0	5	8	9	3	A	2	4	1	B	7	6
Q(P):	0	B	7	5	2	4	8	3	9	A	1	6

Table 2 gives a note-wise description of the operations.

TABLE

1

```
      DIMENSION N(12),I(12), NX(11),IX(11)
      DATA J,K,N/1,12*0,6/,I,NX/6,22*0/,IX/11*0/
C MOVE RIGHT
7     J=J+1
      IF(J.GT.11)GO TO 1
      N(J)=1
C IS N(J) A DUPLICATED NOTE?
4     IF(NX(N(J)).EQ.0)GO TO 2
5     N(J)=N(J)+1
      IF(N(J).EQ.6)GO TO 5
      IF(N(J).GT.11)GO TO 3
      GO TO 4
C CALCULATE I(J), THE INTERVAL
2     I(J)=N(J)-N(J-1)
      IF(I(J).LT.0)I(J)=I(J)+12
C IS I(J) A DUPLICATED INTERVAL?
6     IF(IX(I(J)).EQ.1)GO TO 5
      NX(N(J))=1
      IX(I(J))=1
      GO TO 7
C CALCULATE THE 11TH INTERVAL
1     I(J)=N(12)-N(11)
      IF(I(J).LT.0)I(J)=I(J)+12
      IF(IX(I(J)).EQ.1)GO TO 3
C LAND HERE WHEN AN AIS IS FOUND
      K=K+1
C STATEMENT BELOW IS OPTIONAL—SHORTENS THE TABLE
      IF(K.GE.1929)STOP
      WRITE(6,8)K,N,I
8     FORMAT(I5,2(4X,12I3))
C MOVE LEFT
3     J=J-1
      IF(J.EQ.1)STOP
      NX(N(J))=0
      IX(I(J))=0
      GO TO 5
      END
```

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TABLE

2

Basic Operations

op	notes	normalization constant	w	intervals
I:	$\hat{n}_j \leftarrow \ominus n_j$	none	$\hat{w} \leftarrow w$	$\hat{i}_j \leftarrow \ominus i_j$
M:	$\hat{n}_j \leftarrow 5 \boxtimes n_j$	none	$\hat{w} \leftarrow w$	$\hat{i}_j \leftarrow 5 \boxtimes i_j$
Q:	$\hat{n}_j \leftarrow n_{j \oplus w}$	$\ominus n_{w \oplus 1}$	$\hat{w} \leftarrow \ominus w$	$\hat{i}_j \leftarrow i_{j \oplus w}$
R:	$\hat{n}_j \leftarrow n_{11 \ominus j}$	$\oplus 6$	$\hat{w} \leftarrow \ominus w$	$\hat{i}_j \leftarrow \ominus i_{\ominus j}$

Explanation: n_j and i_j are the jth note and interval of a given AIS in normal form, and $\hat{n}_j$ and $\hat{i}_j$ are the corresponding note and interval of the row resulting from a particular operation. The normalization constant is the interval to which $\hat{n}_j$ must be transposed to normalize the resultant row. The operators $\oplus$, $\ominus$ and $\boxtimes$ correspond to the familiar +, - and x of arithmetic, with the result being taken mod-12.

All four operations are such that if applied an even number of times, one gets what one started with, and all four are commutable with each other. It follows therefrom that there are fifteen distinct composite-operations obtainable by applying from one to four of the basic operations in any order. The so-called M7-operation is the same as the composite MI and need not, therefore, be considered as a basic operation. *4 AISs may then be divided into constellations of sixteen or fewer, allowing for possible duplicate forms, related to each other by some basic or composite operation. It is convenient to display these forms in a Karnaugh-graph, in which adjacent squares are separated by exactly one operation:

		I	IM	M
	P	I(P)	(I(M(P))	M(P)
R	R(P)	R(I(P))	(R(I(M(P)))	R(M(P))
QR	Q(R(P))	Q(R(I(P)))	Q(R(I(M(P))))	Q(R(M(P)))
Q	Q(P)	Q(I(P))	Q(I(M(P)))	Q(M(P))

The graph "wraps around" vertically and horizontally so that the top and bottom rows are adjacent, as are the left- and right-most columns. One can, with this arrangement, refer to a sub-constellation related by certain operations in common. The I-sub-group, for instance, constitutes the middle two columns, the MQ-sub-group is the lower right quarter of the graph, and so forth. One can trace various closed paths through the graph which result from the successive applications of single functions.

Under QR, the ordered content of an AIS, split in two between the notes w - 1 and w, exhibits a useful identity, which is not in fact restricted to the rotation specified by Q, or to AISs at all for that matter. For example:

P:	0	1	3	2	7	A	8	4	B	5	9	6
Q(R(P)):	B	4	8	A	7	2	3	1	0	6	9	5

Note that the QR-form has not been transpositionally normalized. Each partition maps under retrograde into the corresponding partition in the other form. Where w is 6, this identity finds applications in constructing hexachordal combinatorialities (to be discussed later). An analogous phenomenon occurs under QRI in the successive intervals associated with each row. For example:

P: 0 1 3 2 7 A 8 4 B 5 9 6

1 2 B 5 3 A 8 7 6 4 9

[intervals associated with P]

Q(R(I(P))): 1 8 4 2 5 A 9 B 0 6 3 7

7 8 A 3 5 B 2 1 6 9 4

[intervals associated with Q(R(I(P)))]

Source series *5

The basic operations and their composites impose a partition on the set of all AISs, separating them into distinct constellations. Since the members of each constellation have structural similarities to each other, the task of examining the set of all AISs for various characteristics is expedited by calculating an abbreviated table in which each constellation is represented only once. It turns out that this shorter listing of source-AISs (henceforth SAISs) has 267 entries corresponding to constellations of sizes 8 and 16, with each constellation represented by its lowest numbered, and hence alphabetically lowest, member. The test for whether or not an AIS is an SAIS is to generate the (up to) 15 other forms of the series and compare their numbers with that of the series in question. If any of the relative's numbers are lower, the series in question is discarded. The Appendix consists of a listing of the SAISs.

Invariant forms

In constellations containing only eight distinct forms instead of the more prevalent sixteen, there is an operation under which the members of the constellation are invariant. The SAISs exhibit three different invariances: R, QI, and QRMI. *6 No more than one type of invariance is present in any constellation.

R-invariance is present in the familiar "wedge-row";*7

0 B 1 A 2 9 3 8 4 7 5 6

Twenty-two constellations exhibit this invariance. In each case, the tritone occurs in the center of the row; w is 6. While the notes of these rows are a transposition of their own retrograde, the intervals form an 11-element "row" which reproduces itself under retrograde-inversion, since the retrograde operation "inverts" the interval succession. Note that the classical RI-invariance is ruled out in AISs, as it would imply the systematic duplication of intervals.

There are fifteen constellations exhibiting the QI-invariance. As in R-invariance, w must again equal 6, which implies, one notes, that the position of the 6 is not sufficient information to determine the properties of an AIS. For example:

2 1 4 0 5 3 9 A 7 B 6 8

Among the fifteen constellations exhibiting QRMI-invariance, w takes on an even value and the inner tritone occurs between PCs 3 and 9 for rows in normal form. The explanation for this is that, following the rules in Table 2, the interval-wise description of the QRMI-composite may be constructed as follows:

$$i_j \quad 5 \boxtimes i_{\ominus(j \oplus w)}$$

which becomes an equality when invariance exists under the operation:

$$i_j = 5 \boxtimes i_{\ominus(j \oplus w)}$$

Two examples follow with brackets connecting the pairs of intervals which map to each other under M in a symmetrical pattern:

Figure (a)	w = 4	notes:	0 1 4 9 3 2 A 8 5 7 B 6
		intervals:	(6) 1 3 5 6 B 8 A 9 2 4 7
Figure (b)	w = 10	notes:	0 1 5 7 2 B A 8 4 9 3 6
		intervals:	(6) 1 4 2 7 9 B A 8 5 6 3

The intervals 3, 6 and 9 map into themselves under M; $3 = 5 \oplus 3$, $6 = 5 \boxtimes 6$ and $9 = 5 \boxtimes 9$. For these intervals, then,

$$i_j = 5 \boxtimes i_{\ominus(j \oplus w)} = i_{\ominus(j \oplus w)}$$

So that

$$j = \ominus(j \oplus w)$$

which we may re-write as

$$j \oplus j = w = 2 \boxtimes j$$

from which it follows that $\ominus w$, and hence w are even.

Now consider the sums of the intervals on either side of the interval 6. Each participating M-pair will contribute some multiple of 6 to the sum, since

$$x \oplus (5 \otimes x) = (x \otimes 1) \oplus (5 \otimes x) = (1 \oplus 5) \otimes x = 6 \otimes x$$

Thus the sum of all M-pair contributors will be either 0 or 6. Adding this to the 3 or 9 at the center of either M-nest yields again a 3 or 9 as a grand sum, for which reason, the wth note of a normal-form QRMI-invariant AIS must be either 3 or 9.

Tritone pairs and nests

The chromatic scale may be divided into six pairs of notes a tritone apart from each other:

0-6	1-7	2-8
3-9	4-A	5-B

We connect these pairs of notes with brackets as they lie in an AIS to produce a tritone-nest (TN):

0 4 A B 7 5 2 9 8 1 3 6

AISs related inversionally, by M, or by MI, will have the same TN. Thus, TNs may be used to partition the set of all AISs into constellations containing at least four members. Since it has already been determined that invariance only occurs under R, QI, and QRMI, the four TN-related forms of each AIS must be distinct. Some TN-constellations will be larger, as in the case of the AISs possessing the R-invariance, all of which are constrained to have a TN of the following symmetrical shape:

x x x x x x x x x x x x

So that the TN-constellation containing them must include all 176 such AISs (and as it turns out, no others).

While it is possible to generate $12!/(2^6 \times 6!) = 10{,}395$ distinct TNs, only 204 of them can be found in the entire corpus of AISs. In order for a TN to generate an AIS, the first and last notes must be bracketed to form the outer tritone. Then of the nine remaining adjacent unbracketed pairs, exactly one pair must be bracketed to form the internal tritone:

x x x x x x x x x x x x

Additional brackets may be added provided that (1) no more adjacent pairs are connected, and (2) no two brackets of the same span may have both feet adjacent:

x x x x

as this would imply the duplication of the interval between either pair of consecutive feet.

Using this construction method, it is possible to generate 230 distinct tritone-nests, which implies that 26 of them, or about one ninth of the total, generate no AISs. Table 2a gives nine TNs which under the operation Q, R, and QR generate the 26 fruitless TNs generable with the above algorithm. We were unable to find any systematic explanation of these nine nests, without which a very powerful generalization about AIS structure would result.

Hand-generation of AISs possessing invariances

The program presented earlier generates an exhaustive roster of AISs, in the course of which there is a considerable amount of trial and error or backtracking, for which reason it is not feasible to perform the calculation by hand. AISs were not therefore readily available to earlier generations of 12-tone composers, except for the R-invariant rows,[*8] which one can construct by hand from various hexachords, the amount of backtracking being minimal. Knowing, however, that QI- and QRMI-invariances also exist, it becomes no longer difficult to hand-calculate rows of these types. This puts us in a position of being able to hand-calculate 416 AISs or about one ninth of the entire corpus.

Referring back to the TN for R-invariant AISs (see the previous section), we note that the second half of the series must have the same unordered content as the first half, transposed by the interval 6, and that since R-invariance causes all the other intervals to be paired with their complements in the other half of the series, one must begin the generation process by selecting from the following hexachords, all of which contain interval-classes 1-5, but not 6. All four such hexachords do, in fact, generate AISs:

interval vector	pitches
*543210	012345
*343230	01235A
323430	012459
*143250	01358A

One simply "fiddles" with the pitch-ordering of the hexachord chosen until no interval-class is repeated between successive notes. The hexachord 012345 might, for example, be permuted to get 214053, whose interval-class-succession is 13452. The remainder of an R-invariant series is produced by retrograding and transposing by the interval 6: 214053-9B6A78.

The hand-algorithm for QI-invariant AISs differs in that we must select a hexachord that is disjunct from some inverted form of itself:

interval vector	pitches
*543210	012345
443211	012346
342231	012357
*343230	01235A
323421	012458
322431	012489
143241	01357A
223431	013589
*143250	01358A

The hexachords marked with an asterisk here and in the previous table are common to both tables, which might be a compositionally useful relationship between different rows generated from the same hexachord. Referring to the previous example, in which we arrived at the permutation 214053 derived from the first hexachord on the list, a related QI-invariant row can

TABLE 2a

1 x x x x x x x x x x x x

2 x x x x x x x x x x x x

3 x x x x x x x x x x x x

4 x x x x x x x x x x x x

5 x x x x x x x x x x x x

6 x x x x x x x x x x x x

7 x x x x x x x x x x x x

8 x x x x x x x x x x x x

9 x x x x x x x x x x x x

also be generated from the same first half, by following these notes with another group of six obtained by subtracting 214053 from 11, yielding the row 214053-9A7B68, a QI-invariant AIS. Note, however, that the "fiddling" required for this class of invariance is a bit more involved than with the R-invariant class. Not only must the permutation exhibit each of PCs 1-5, but the constant with respect to which we invert the hexachord must be such that a tritone results between the two hexachords.

The procedure for calculating the QRMI-invariant AISs involves constructing an M-nest as shown above in figures (a) and (b). First decide on an even value of w, placing the interval 6 in that position. This splits the pitch sequence into two even-lengthed parts, at the centers of which we place the intervals 3 and 9. The first and last notes are then assigned to some tritone-pair. For example:

let W = 4

notes: 7 n n n n n n n n n n 1

intervals: i 3 i 6 i i i 9 i i i

The remaining M-related interval pairs can then be inserted to form symmetrical nests on either side of the interval 6, taking care that no duplicated pitches are implied by the placement of the intervals:

notes: 7 0 3 4 A 5 9 B 8 6 2 1

intervals: 5 3 1 6 7 4 2 9 A 8 B

Swapping relationships

Glancing through a complete listing of the AISs, one notices many instances where two series are essentially the same except for the displacement of a pair or handful of notes. For example:

0 1 4 9 3 B A 8 5 7 2 6

0 1 4 9 3 2 A 8 5 7 B 6

A computer search shows that more than half of all AISs possess some pair of notes which, if swapped, yield another series in which the all-interval property is preserved, and which is not necessarily related by other operations or by a common tritone-nest. (Note, however, that the swapping of tritone-related pairs does preserve the nest). The performed search was extended to examine the effect of swapping the first and last notes of a series, which turned out in many cases to yield AISs that were neither rotationally nor transpositionally normal. It is possible that some similar re-ordering scheme could, possibly in conjunction with other operations, link all AISs together by chains of relations, which would be an elegant description of the generation of AISs and would doubtless provide interesting compositional devices for pitch-ordering.

Let us, however, make an observation about swap-related AISs. If the swapping of two notes retains the all-interval property, it follows that a corresponding re-ordering of the intervals has occurred. There are two cases to consider: (1) where the swapped notes are adjacent (N.B.: the first and last notes are "adjacent"), and (2) where there are intervening notes. The swapped notes are labelled A and B and the reordered intervals K, L, M and N. D is the interval $B \ominus A$:

case (1) A and B are adjacent:

notes: ... x x x A B x x x ...
intervals: ... i i K D M i i ...

case (2) intervening notes:

D
notes: ... x x x A x . . . x B x x ...
intervals: ... i i K L i . . i M N i ...

The bracket is labelled with the interval between A and B. After swapping, we get the following arrangement:

case (1) x x x B A x x x
$K \oplus D$ $\ominus D$ $M \oplus D$

$\ominus D$
case (2) x x B x . . . x A x x
$K \oplus D$ $L \ominus D$ $M \ominus D$ $N \oplus D$

If the all-interval property is preserved, the intervals must map into each other as follows:

case (1) $\{K, D, M\} = \{K \oplus D, \ominus D, M \oplus D\}$

case (2) $\{K, L, M, N\} = \{K \oplus D, L \ominus D, M \ominus D, N \oplus D\}$

Thus the possibility of note-swapping may be determined by inspecting the intervals. Returning to our previous example, we observe the behavior of the intervals:

```
0 1 4 9 3 B A 8 5 7 2 6
        8 B       7 4

0 1 4 9 3 2 A 8 5 7 B 6
        B 8       4 7
```

As a second example, we take a case where the swapped notes are adjacent:

```
0 1 4 2 A 3 9 8 5 7 B 6
        5 6 B

0 1 4 2 A 9 3 8 5 7 B 6
        B 6 5
```

Here, the swapping causes the series to lose its QI-invariance, which shows that by adding the swapping operation to our hand-calculation repertoire, non-invariant AISs become hand-calculable, as are the QI-, R-, and QRMI-forms from which they may be derived.

Hexachordal Combinatoriality of AISs

Though AISs are highly restricted in their structure, it is still quite possible to construct fruitful combinatorialities among members of source-constellations. Most of the classical combinatorialities are possible as are some others using variants generated under the Q- and M-operations. Hexachordal combinatoriality (henceforth HC) occurs when two related row-forms can be stated as

$$\begin{aligned} P &= A,\ B \\ F(P) &= B,\ A \end{aligned}$$

where A and B are disjunct hexachords. Table 3 shows the resultant HCs where function F is some composite of I, R, and M, and Tn is transposition to the interval n. *9 While the table only shows rows in their relation to P, the same HCs are possible after applying any composite operation or transposition to both participating members.

Where w = 6, the Q operator is effectively the same as R, simply swapping the two hexachords of the row. Thus RI and

TABLE

3

Hexachordal Combinatorialities

F	condition	
P	A = Tn(B)	for some n
i	A = Tn(I(B))	for some n
RI	A = Tn(I(A))	for some n
R(trivial)	A = anything	
R(non-trivial)	A = TN(A)	for some n
M	A = Tn(M(B))	for some n
MR	A = Tn(M(A))	for some n
MI	A = Tn(M(I(B)))	for some n
MRI	A = Tn(M(I(A)))	for some n

QI, MI and QRMI, QRI and I and so forth become interchangeable.

If the operator M or MI generates a HC, the hexachord in question must be self-symmetric under M, which implies that positions 1 and 5 of its interval vector must be the same, because the M operation causes these two figures to swap places. *10 For example:

interval vector	pitches
421242	012567

Where w = 6, the Q operation generates hexachords C and D, which are not necessarily related to hexachords in the prime form of the series.

$$P = A, B$$
$$Q(P) = C, D$$

Three possibilities arise. If $A \neq B \neq C \neq D$, then no HCs involving Q or its composites are possible. If $A = D$ and $B = C$, then Q may be treated again as if it were R, as where $w = 6$. Finally, where $A = C$ and $B = D$, P and Q(P) are hexachordally equivalent, so that Q may be applied to any operation that already generates HCs. The application of M and Q may, however, produce hexachords related by other composite operations. The same "source" hexachord might be produced, but inverted or in its alternate Z-form. *11 It is also possible that, under the Q-operation, an M-related hexachord is generated, in which case P becomes hexachordally equivalent to QM.

Tetrachordal combinatoriality

Tetrachordal combinatoriality (henceforth TC) is more restricted than HC, as only 7 out of 29 tetrachords generate it and no trivial forms exist analogous to the R-type HC which is possible for any row (see Martino). Classical TC is based on transposition in which P is combined with T4(P) and T8(P):

$$P = a, b, c$$
$$T_4(P) = T_4(a), T_4(b), T_4(c)$$
$$T_8(P) = T_8(a), T_8(b), T_8(c)$$

These three forms must be disjunct, which implies that position 4 of the interval vectors of a, b and c must be 0 (see Forte or

Babbitt for discussion of transpositional invariance). Herein lies the restriction to seven tetrachords. One can, of course, substitute any transform of P which is also of the form Tn(a), Tn(b), Tn(c). Table 4 gives the rules for classical TC. *12

Being restricted to tetrachords lacking interval class 4, we can discard many AISs from candidacy for TC with a simple test. Dividing an arbitrary AIS into three four-note segments, we note that if the interval 4 or 8 occurs within a segment, that series must be discarded, for at least one tetrachord contains interval-class 4. For example:

(a) notes: 0 1 3 9 2 A 5 4 7 b 8 6

intervals: 1 2 6 5 8 7 B 3 4 9 A

TC is impossible

(b) notes: 0 1 3 A 2 B 5 8 4 9 7 6

intervals: 1 2 7 4 9 6 3 8 5 A B

While this is a necessary condition, it is not however a sufficient one. If the representative of a constellation fails the test, so do the forms related by composite operations of I, M, and R. It is possible, however, that the Q-form passes the test when the prime form does not and vice versa.

When w is 4 or 8, the tetrachords map into each other cyclically under Q. If in such a case another row-form X can be derived which is the remaining cyclic form, a TC will result. Table 5 gives the rules for such combinatorialities. These combinatorialities are not restricted to the narrow selection demanded by the classical TCs, as the placement of w violates the conditions shown in the above paragraph. A larger number of SAISs can generate them, and if P can generate such a TC, so can Q(P).

Even where w has some value other than 4 or 8, Q-involved TCs are still possible where the same trio of tetrachords is reproduced under Q or one of its composites. In general, where

P = a, b, c

Q(P) = d, e, f

and F is some composite of M and I, there are six useful cases to consider:

TABLE

4

Classical Tetrachordal Combinatorialities

combination	condition	
$T_n(I(P)) = P$	$T_n(I(a)) = a$	for some n
$T_4(P)$	$T_n(I(b)) = b$	
$T_8(P)$	$T_n(I(c)) = c$	

NB: The tetrachords involved here are self-symmetric under inversion

$T_n(R(I(P))) = P$	$T_n(I(a)) = c$	for some n
$T_4(P)$	$T_n(I(c)) = a$	
$T_8(P)$	$T_n(I(b)) = b$	

NB: same tetrachords as above

$T_6(R(P)) = P$	$T_6(a) = c$	
$T_4(P)$	$T_6(c) = a$	
$T_8(P)$	$T_6(b) = b$	
$T_6(P) = P$	$T_6(a) = a$	
$T_4(P)$	$T_6(b) = b$	
$T_8(P)$	$T_6(c) = c$	

NB: In the latter two combinations only two tetrachords will work

$T_t(M(P)) = P$	$T_t(M(a)) = a$	for some t
$T_4(P)$	$T_t(M(b)) = b$	
$T_8(P)$	$T_t(N(c)) = c$	
$T_t(M(R(P))) = P$	$T_t(M(c)) = a$	for some t
$T_4(P)$	$T_t(M(b)) = b$	
$T_8(P)$	$T_t(M(a)) = c$	
$T_t(I(M(P))) = P$	$T_t(I(M(a))) = a$	for some t
$T_4(P)$	$T_t(I(M(b))) = b$	
$T_8(P)$	$T_t(I(M(c))) = c$	
$T_t(M(R(I(P)))) = P$	$T_t(I(M(a))) = c$	for some t
$T_4(P)$	$T_t(I(M(b))) = b$	
$T_8(P)$	$T_t(I(M(c))) = a$	

TABLE 5

Q-involved TCs

combination		conditions	
P		$a = T_6(b)$	
Q(P)		$b = T_6(a)$	
T_6(R(P))		$c = T_6(c)$	
	NB: c has the 'P/R' property		
T_t(R(I(P)))		$a = T_t(I(b))$	for some t
Q(P)		$b = T_t(I(a))$	
R(P)		$c = T_t(c)$	
	NB: c has the 'RI/I' property		
T_t(M(P))		$a = T_t(M(b))$	for some t
R(Q(P))		$b = T_t(M(a))$	
P		$c = T_t(M(c))$	
	NB: c has the 'MR' property		
T_t(M(R(I(P))))		$a = T_t(I(M(b)))$	for some t
Q(P)		$b = T_t(I(M(a)))$	
P		$c = T_t(I(M(c)))$	
	NB: c has the 'MRI' property		

(1) a = F(d) F(Q(P)) ≡ P
b = F(e)
c = F(f)

(2) a = F(e) same situation as where w = 4
b = F(f)
c = F(d)

(3) a = F(f) same situation as where w = 8
b = F(d)
c = F(e)

(4) a = F(f) F(Q(R(P))) ≡ P
b = F(e)
c = F(d)

(5) a = F(e) F(Q(R(P))) behaves like Q where w = 8
b = F(d)
c = F(f)

(6) a = F(d) F(Q(R(P))) behaves like Q where w = 4
b = F(f)
c = F(e)

Conclusions

The possibility of a non-backtracking generation algorithm is still to be realized, as is some function or ensemble of functions which would allow any AIS to be mapped into any other. We have observed that many AISs exhibit the multiple order properties described in Batstone MOF, Babbitt TRE, and elsewhere, whereby a transformation of P may be partitioned such that P results when the partitions are re-ordered. Here the invariant AISs present a rather obvious example, in which there is only one partition. A rather intriguing case is SAIS 44 *13 (see Appendix), which generates a transposition of itself when one "leap-frogs" through it, no matter how many notes are skipped:

P:	0 1 4 2 9 5 B 3 8 A 7 6
every 2nd note:	1 2 5 3 A 6 0 4 9 B 8 7
every 3rd note:	4 5 8 6 1 9 3 7 0 2 B 8

every nth note, et cetera. . .

The combinatoriality of trichords, dyads, and unequal partitions have been omitted for reasons of space. An intriguing line of inquiry would be to find the restrictions needed to generate combinatorialities whose vertical aggregates could be ordered so that AISs always result. The topics of swapping and TNs might also be further explored, and not, for that matter, restricted to the context of AISs or even to ordered pitch-collections. *14

The AISs tend to comprise a microcosm of the set of all rows, representing most of the combinatorial properties generally available as well as some of the more exotic ordering relations. It should be noted that the intervals of an AIS sum to 66, which is the average of the highest and lowest possible values, 121 and 11 respectively, that the interval succession of any 12 pitch-row can assume.

AISs tend to present an interesting extension to Babbitt's time-point system of rhythmic serialism, for an AIS used for such purposes could create an 11-element "row" of distinct durational values from one "attack point" to the next.

* * *

APPENDIX

1	1	01327A84B596
2	2	013295A47B86
3	3	01329748B5A6
4	4	013725BA8496
5	5	01372A8B5496
6	6	01372B4A9586
7	7	01372BA49586
8	8	01374298B5A6
9	9	01374B9825A6
10	10	01375298B4A6
11	11	013752A498B6
12	12	0137584A92B6
13	13	013825974BA6
14	14	01382A547B96
15	15	0138475BA296
16	16	01385B7A9426
17	17	01387A4B9526
18	18	0138B5942A76
19	20	0138B954A726
20	21	01392A547B86
21	22	01392A7548B6
22	23	01392A8547B6
23	24	01392B7A5486
24	25	01395A874B26
25	26	01398425A7B6
26	27	01398475A2B6
27	28	013A25B74986
28	29	013A25B84976
29	30	013A28B75496
30	31	013A2B584976
31	32	013A48B95276
32	34	013A87495B26
33	35	013A925B7486
34	36	013A95B27486
35	37	013B274A5986
36	38	013B4287A596
37	39	013B4A725986
38	40	013B4A872596
39	41	013B527A9486
40	42	013B94825A76
41	43	013B985A4726
42	44	01427985B3A6
43	45	014293857BA6
44	46	014295B38A76
45	47	014295B8A376
46	48	0142A37985B6
47	49	0142A39857B6
48	50	0142A5B38796
49	51	0142A93857B6
50	52	0143792A5B86
51	53	0143792A85B6
52	54	0143795A82B6
53	55	014382A57B96
54	57	014392A57B86
55	58	014392A857B6
56	59	0143A2795B86
57	60	0143A85B7926
58	61	014835B92A76
59	62	014872A35B96
60	63	014872B935A6
61	65	0148792B53A6
62	67	0148A5392B76
63	68	014932A57B86
64	69	014932A857B6
65	71	01493BA85726
66	73	0149B52A8376
67	74	0149B82A5376
68	75	014A279853B6
69	76	014A32957B86
70	77	014A3B985726
71	78	014A53B87926
72	79	014A83592B76
73	81	014A9572B386
74	82	014B392A8576
75	83	014B52A37986
76	85	014B9528A376
77	86	014BA3795286
78	87	014BA7359286
79	89	014BA8273596
80	90	014BA8573926
81	91	0152493A8B76
82	95	01524B9387A6
83	97	015287A3B946
84	98	01528A3B9476
85	99	01529738BA46
86	100	01529B4A8736
87	101	01529B7A4386
88	102	01529BA47386
89	103	0152A839B476
90	107	0153924B87A6
91	108	0153A928B746
92	109	0153B4728A96
93	112	0154728A3B96
94	113	015472A39B86
95	114	015472B938A6
96	115	0154A83B2796
97	118	01572A438B96
98	119	01572A8B4396
99	121	01572BA84936
100	122	015738BA4296
101	124	015742A398B6
102	127	01574932A8B6
103	128	01574A3B2986
104	132	0157A3284B96
105	139	01583BA42796
106	142	0158A3942B76
107	144	015A429B8736
108	145	015A42B79836
109	146	015A47329B86
110	148	015A7943B286
111	150	015A82B79436
112	152	015A8B724396
113	153	015A9B742836
114	154	015B297438A6
115	155	015B2A794386
116	158	015B74298A36
117	159	015B7A832496
118	160	015B832497A6
119	161	015B87942A36
120	162	015B8A942736
121	163	017253B48A96
122	164	0172548A3B96
123	165	0172A3548B96
124	166	0172A8549B36
125	167	0172A859B436
126	168	0172A8B35496
127	169	0172A8B43596
128	170	0172A9B48536
129	173	01738A254B96
130	174	0173A2549B86
131	175	0173A8549B26
132	182	017498A53B26
133	183	01749B32A586
134	184	0174B2A35986

135	185	01748958A326
136	187	01752A9B4836
137	190	0175A29B8436
138	191	01792A854B36
139	203	018253749BA6
140	205	0182A7549B36
141	206	0182A759B436
142	209	0184257BA396
143	210	018425BA3796
144	211	0184A3257B96
145	212	018539247BA6
146	215	01857A93B426
147	217	01857B92A436
148	218	01859BA42736
149	219	0185B32497A6
150	220	0185B92437A6
151	221	0185BA249736
152	222	018752A49B36
153	225	018A2B549736
154	226	018A2B754936
155	228	018A75493B26
156	229	018A93B47526
157	235	018B75A24396
158	236	018B9524A376
159	238	018BA3957426
160	239	018BA4273596
161	241	0192437A5B86
162	243	019243A7B586
163	245	01925B3A8746
164	250	01935A874B26
165	251	01938A754B26
166	254	019425B8A376
167	255	01942857BA36
168	257	01942B58A376
169	258	019435A7B286
170	259	019438A75B26
171	260	019475B8A326
172	261	01947BA35286
173	265	01972B548A36
174	266	0197435A28B6
175	271	01974BA35826
176	275	0197A354B826
177	280	0198B53A2746
178	281	019B3274A586
179	282	019B437A5286
180	283	019B43A75826
181	285	019B4A758326
182	287	01A25B794386
183	289	01A27943B586
184	292	01A29738B546
185	293	01A29B538476
186	294	01A29B547386
187	295	01A325B79486
188	298	01A3297B5846
189	300	01A35B984726
190	301	01A3725B9846
191	303	01A375B29846
192	304	01A3795428B6
193	305	01A37954B286
194	308	01A39B725486
195	310	01A3B5879426
196	311	01A3B9482576
197	314	01A483B92576
198	316	01A49B758376
199	317	01A4B2735986
200	320	01A53B479826
201	321	01A57398B426
202	323	01A58739B426
203	324	01A58793B426
204	329	01A827954B36
205	331	01A83B547926
206	332	01A83B592476
207	337	01A872495B36
208	338	01A872593B46
209	345	01A9725B3846
210	346	01A9B5384726
211	355	01B43A795826
212	357	01B47398A526
213	358	01B482573A96
214	362	01B5249837A6
215	364	01B5843A2796
216	365	01B5843A7926
217	366	01B5873A2496
218	368	01B72548A396
219	379	01B8A3954726
220	383	021495B3A786
221	384	02149785B3A6
222	385	0214B395A786
223	386	0214B39785A6
224	389	0214B93785A6
225	393	0215A784B936
226	395	0215B749A836
227	396	02158 7A83496
228	397	0215B83497A6
229	403	021845A7B936
230	404	0218475AB396
231	405	021849A75B36
232	406	021849A7B536
233	407	02185B3497A6
234	408	0218B39745A6
235	409	0218B53749A6
236	410	0218B93745A6
237	419	021A3B594786
238	422	021A593B4786
239	427	021B583749A6
240	437	023714B985A6
241	438	02371BA58496
242	452	0238471BA596
243	453	02384BA17596
244	456	023854B917A6
245	459	02385971 4BA6
246	467	023A15B74986
247	470	023A7B514986
248	471	023A87B51496
249	472	023A8B715496
250	475	023B417A5986
251	476	023B471A5986
252	484	02517B43A896
253	485	02517BA83496
254	493	025389174BA6
255	495	025389417BA6
256	502	02549173AB86
257	516	025A91734B86
258	520	025B318749A6
259	673	02A5471B3896
260	674	02A547B13896
261	751	0315B87249A6
262	766	03187B5249A6
263	791	0324B715A896
264	844	03427B518A96
265	863	03497B5218A6
266	1019	039124B875A6
267	1020	03912754B8A6

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1 See Bauer-Mengelberg EIR.

2 w takes on values from 2 to 10. If $w = 1$ or 11, the internal and external tritones would be adjacent, duplicating a PC in the series.

3 The symbols 'A' and 'B' denote 10 and 11 respectively, here and in subsequent tables and examples.

4 For a detailed treatment of multiplicative operations on PCs, see Howe CPP.

5 The derivation of source series or "generators" from the 3856 AIS has been discussed by David Cohen in RAIR. Cohen asserts there are "266 independent eleven-interval row generators."

6 R invariance in AISs is discussed in Bauer-Mengelberg EIR and earlier (1940) in Krenek SIC.

7 The AISs of the LYRIC SUITE, Nono's IL CANTO SOSPESO (the so-called wedge-row) and Stockhausen's GRUPPEN all have R-invariance. Stockhausen subjects his row to order-number transpositions, but never by w places.

Berg: 5409728136AB
Nono: 9A8B70615243
Stockhausen: 7385460AB291

8 However, in Krenek, SIC, the series E♭, G♭, D♭, G, C, D, B, B♭, A♭, E, F, A where $w = 3$ is given as an example.

9 After Babbitt SS and Martino SAF.

10 Forte's R_1 similarity includes such an interchange.

11 In Forte TSM, the unordered sets are said to be in the Z-relation if they have the same interval content but do not map into one another under T_t or I.

12 After Martino SAF.

13 This row is mentioned in Lewin CTR and was discovered by Pohlman Mallalieu.

14 Indeed the equations in Forte DRS (pp. 176, 177) describing the relationship between Z-pairs are relevant to the TN concept.

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