# A GENERAL THEORY OF COMBINATORIALITY AND THE AGGREGATE

## (Part 2)

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### 4.1 *Partitioning CM-columns*

A CM-column defines a collection of row-segments belonging to different rows. These segments are internally ordered, but their alignment with respect to one another, has not been specified, except so as to define an aggregate. One might divide such a column into sub-columns of fewer than 12 PCs, and thereby specify the vertical alignment and proximity of PCs in greater detail. This process might be shown by inserting dotted vertical lines:

| |
|---|
| B6 |
| 1034 |
| 8A29 |
| 7 |
| 5 |

| | | |
|---|---|---|
| B | | 6 |
| 1 | 03 | 4 |
| | | 8A29 |
| | 7 | |
| 5 | | |

| | |
|---|---|
| | B6 |
| 1034 | |
| 8A | 29 |
| 7 | |
| 5 | |

Each new sub-position will contain an ordered succession of 0 to 12 PCs.

Suppose we were to partition a CM-column in two. There are

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many places where the sub-partition line might be drawn. The number of alternatives will vary with the number of rows and the distribution of PCs in the particular column. Where we are partitioning the $j^{th}$ column of an n-row CM **E**, the exact number will be:[20]

$$\text{part2}(\mathbf{E},n,j) = \prod_{i=1}^{n} (\#E_{i,j} + 1)$$

When all 12 PCs of a column are in a single position, part2(**E**,n,j) has a minimal value of 13. When $n \geq 12$ and there is no more than one PC per column position, part2(**E**,n,j) attains a maximal value of $2^{12} = 4096$. In such cases, one can partition the column so as to isolate any arbitrary set on either side of the sub-partition. As n increases from 1 to 12, part2(**E**,n,j) achieves successively higher maximal values. As a result, there are usually more ways to partition the columns of large CMs than of small ones.

## 4.2 *Begin-sets and End-sets of CM-columns*

When we partition a CM-column in two, we separate two complementary sets from each other. We call the left-hand set of such a partition a *begin-set of the column,* and the right-hand set an *end-set of the column,* extending the begin- and end-set concepts from rows to columns. The following, for example, are begin-sets of the column given in sec. 4.1, since it would be possible to isolate each of them to the left of some partition through the column:

{6,B} {0,1,8,A,B} {8}
{1,5,7,8,B} {1,5,7,8,A}

Similarly, the following are end-sets of the same column:

{6,B} {4,6,9} {5,6,7,9,B}

Note that {6,B} is both a begin-set and an end-set of this column. Unlike rows, CM-columns can have different begin- and end-sets the same size. The total number of begin-sets of the $j^{th}$ column of an n-row CM **E** is part2(**E**,n,j), the number of ways the column can be partitioned in two. Since every begin-set of a column corresponds to a complementary end-set, the numbers of begin- and end-sets of a column are always the same. Since the maximal value of

[20] The expression is to be read, "The product of $(\#E_{i,j} + 1)$ for i from 1 to n."

part2(**E**,n,j) increases with n, there tend to be more begin- and end-sets of columns of large CMs than of small ones.

### 4.3 *Merging CMs*

The rows and columns of two or more CMs may be combined to yield a single larger CM. Thus the following two CMs:

| | | | |
|---|---|---|---|
| $X$: | 0172 | A9B4 | 8536 |
| $T_4(X)$: | 45B6 | 2138 | 097A |
| $T_8(X)$: | 893A | 6570 | 41B2 |

Fig. 24

| | | |
|---|---|---|
| $T_2(M(X))$: | 27104B | 9A6358 |
| $T_2(M(R(X)))$: | 8536A9 | B40172 |

Fig. 25

can be *merged* to form

| | | | | | |
|---|---|---|---|---|---|
| $X$: | 0172 | | A9B4 | | 8536 |
| $T_4(X)$: | 45B6 | | 2138 | | 097A |
| $T_8(X)$: | 893A | | 6570 | | 41B2 |
| $T_2(M(X))$: | | 27104B | | 9A6358 | |
| $T_2(M(R(X)))$: | | 8536A9 | | B40172 | |

Fig. 26

Merging allows us to generate 3-, 5-, and 7-row CMs, which might be difficult to assemble otherwise. A single row can always be treated as a "1-row CM", and can be merged with another CM. Thus, for example, a 2-row CM can be augmented by a third row:

| | | |
|---|---|---|
| $X$: | 0172 | A9B48536 |
| $T_2(I(R(X)))$: | 8B96A354 | 0712 |

Fig. 27

| | | | |
|---|---|---|---|
| $X$: | 0172 | | A9B48536 |
| $T_2(I(R(X)))$: | 8B96A354 | | 0712 |
| $T_A(I(R(X)))$: | | 47526B10839A | |

Fig. 28

### 4.4 *Swapping*

Note that the 5-row CM of Fig. 26 is readily decomposable into its 2- and 3-row constituent CMs, whose segments remain associated as in their initial columnar arrangement. In the case of this CM, however, one can alter the distribution by *swapping* PCs between adjacent columns without changing their order within their respective rows:

| | | | | |
|---|---|---|---|---|
| 0 | 172A9 | | B485 | 36 |
| 45B6 | | 2138 | | 097A |
| 893A | 6 | 57 | 0 | 41B2 |
| 271 | 04B | | 9A63 | 58 |
| | 853 | 6A9B40 | 172 | |

{8,5} swapped
{B,4,0} swapped
{A,9,6} swapped
{1,7,2} swapped

Fig. 29

Thus both row- and column-aggregates have been preserved in the CM. We can operate similarly on the 3-row CM of Fig. 28:

| | | |
|---|---|---|
| 01 | 72A9 | B48536 |
| 8B96A3 | 54 | 0712 |
| 4752 | 6B1083 | 9A |

{9,A} swapped
{4,7,5,2} swapped

Fig. 30

Such swapping is possible when (1) an end-set of one column is the same as some begin-set of the column immediately to its right, and (2) when these sets reside in mutually exclusive collections of rows.[21]

Merging and swapping provide a tool for generating CMs of various sizes and distributions. Note, however, that while any pair of CMs can be merged, it will not always be the case that PCs can be swapped in the resultant CM. Therefore, to obtain a CM of a certain size and distribution through merging and swapping, the initial CMs must be chosen or transformed through the application of row-operations so that the desired swapping is possible.

We noted in sec. 4.2 that the number of distinct begin- or end-sets of a column, given by part2(**E**,n,j), tends to increase with the number of rows in a CM. So, in turn, does the probability that two adjacent columns of an n-row CM will share some set which may be swapped between them. Thus there tend to be more opportunities for swapping among the columns of large CMs than of small ones.

## 4.5 *Overlaid CMs*

Merging and swapping suggest a technique for constructing pairs of CMs which may be stated alone or in contrapuntal combination with each other. To illustrate the idea, we will merge two CMs $\mathbf{E}_1$ and $\mathbf{E}_2$ to form a third CM, $\mathbf{E}_3$:[22]

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\mathbf{E}_3$ | 0 172 | | A 9 | B485 | | 36 | $\mathbf{E}_1$ |
| | 63 | | 584B | 9A | | 271 0 | |
| | BA 49 | | 12 | 0736 | | 8 5 | |
| | 5 8 | | 6370 | 21 | | 94AB | |
| | | 1283 B A | | | 05964 7 | | $\mathbf{E}_2$ |
| | | 7469 50 | | | A B3 821 | | |

Fig. 31

[21] Swapping in 12-row transposition CMs is mentioned in Westergaard, "Toward a Twelve-tone Polyphony", in *Perspectives on Contemporary Music Theory*, ed. Boretz & Cone, New York: W. W. Norton & Co., 1972.

[22] The CM $\mathbf{E}_1$ is derived in sec. 4.6, Fig. 37.

We will then swap PCs between the columns of $\mathbf{E}_3$ to produce a fourth CM $\mathbf{E}_4$ (Fig. 32), restricting ourselves to swapping PCs into initially empty positions. This allows the swapping, in effect, to define partitions of the original columns of $\mathbf{E}_1$ and $\mathbf{E}_2$ into sets which are either (1) swapped leftwards, (2) swapped rightwards, or (3) allowed to remain in the same column subsequent to merging. This partitioning is shown below with dotted lines:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 172 | A | 9 | B485 | | | 36 | $\mathbf{E}_1$ |
| | 63 | | 584B | 9 | A | 271 | 0 | |
| BA | 49 | | 12 | 0736 | | 8 | 5 | |
| 5 | 8 | | 6370 | 21 | | | 94AB | |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 172 A | 9 | B485 | | 36 | $\mathbf{E}_4$ |
| | 63 | 584B | 9 | A 271 | 0 | |
| BA | 49 | 12 | 0736 | 8 | 5 | |
| 5 | 8 | 6370 | 21 | | 94AB | |
| 1283 | B | A | | 05964 | 7 | |
| 7469 | 50 | | A | B3 | 821 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1283 | B | A | | 05964 | 7 | $\mathbf{E}_2$ |
| 7469 | 50 | | A | B3 | 821 | |

Fig. 32

To begin with, $\mathbf{E}_1$ and $\mathbf{E}_2$ form columnar aggregates when each is stated alone, but if the dotted sub-partitions are incorporated into their temporal structure, their contrapuntal combination, $\mathbf{E}_4$, will also make columnar aggregates, though in different places.[23]

[23] 12-row overlaid CMs produced by folding, cliques, or cycles possessing high fragmentations (see sec. 5.1) are illustrated in Babbitt, "Set Structure as a Compositional Determinant", in *Perspectives on Contemporary Music Theory,* ed. Boretz & Cone, New York: W. W. Norton and Co., 1972, and Martino, "The Source Set and its Aggregate Formations", *Journal of Music Theory,* 5/2 (1961).

## 4.6 *Folded CMs*

Given one CM, it is possible to generate another CM with twice as many rows by merging the given CM with some transform of itself and then swapping PCs. For example, the CM of Fig. 33 can be subjected to $T_BI$, $T_5M$, and $T_2$ to give the CMs of Figs. 34–36, which have essentially the same internal structure:

| | | |
|---|---|---|
| $X$: | 0172A9 | B48536 |
| $R(X)$: | 63584B | 9A2710 |

Fig. 33

| | | |
|---|---|---|
| $T_B(I(X))$: | BA4912 | 073685 |
| $T_B(I(R(X)))$: | 586370 | 2194AB |

Fig. 34

| | | |
|---|---|---|
| $T_5(M(X))$: | 5A4372 | 01968B |
| $T_5(M(R(X)))$: | B86910 | 2734A5 |

Fig. 35

| | | |
|---|---|---|
| $T_2(X)$: | 23940B | 16A758 |
| $T_2(R(X))$: | 857A61 | B04932 |

Fig. 36

Various 4-row CMs result when the original CM is merged with one of its transforms and PCs are swapped. Depending on which transform is used, different swappings will be possible.

Thus we can *fold* the original CM *on* $T_BI$—that is, merge it with its transform under $T_BI$—which permits considerable swapping:

step 1:

| | | | | |
|---|---|---|---|---|
| $X$: | 0172A9 | | B48536 | |
| $R(X)$: | 63584B | | 9A2710 | |
| $T_B(I(X))$: | | BA4912 | | 073685 |
| $T_B(I(R(X)))$: | | 586370 | | 2194AB |

Fig. 37

step 2:

| | | | | |
|---|---|---|---|---|
| $X$: | 0172 | A9 | B485 | 36 |
| R($X$): | 63 | 584B | 9A | 2710 |
| $T_B$(I($X$)): | BA49 | 12 | 0736 | 85 |
| $T_B$(I(R($X$))): | 58 | 6370 | 21 | 94AB |

Fig. 37 (*cont.*)

We can fold on $T_5M$:

step 1:

| | | | | |
|---|---|---|---|---|
| $X$: | 0172A9 | | B48536 | |
| R($X$): | 63584B | | 9A2710 | |
| $T_5$(M($X$)): | | 5A4372 | | 01968B |
| $T_5$(M(R($X$))): | | B86910 | | 2734A5 |

step 2:

| | | | |
|---|---|---|---|
| 0172 | A9 | B48536 | |
| | 63584B | 9A | 2710 |
| 5A43 | 72 | 01 | 968B |
| B869 | 10 | 27 | 34A5 |

Fig. 38

And finally, we can fold on $T_2$, which unlike the other two operations provides little opportunity for swapping:

| | | | | |
|---|---|---|---|---|
| $X$: | 0172A9 | | B48536 | |
| R($X$): | 63584B | | 9A2710 | |
| $T_2$($X$): | | 23940B | | 16A758 |
| $T_2$(R($X$)): | | 857A61 | | B04932 |

Fig. 39

One might wish to fold a CM on an operation allowing as much swapping as possible. This potential can be evaluated by inspecting the begin- and end-sets of the columns of the CM to be folded for matches in set-class. To continue the example, we find that a 6-PC begin-set in the first column of the original CM maps to an end-set of the same column under $T_B I$:

begin-set — end-set

| | | | |
|---|---|---|---|
| $X$: | 0172 | A9 | |
| R($X$): | 63 | 584B | |

$$\{A,9,5,8,4,B\} = T_B(I(\{0,1,7,2,6,3\}))$$

Thus, when we fold $T_B I$, {A,9,5,8,4,B} can be swapped in the resultant 4-row CM:

| | | |
|---|---|---|
| 0172 A9 | | |
| 63 584B | | |
| | BA49 12 | |
| | 58 6370 | |

| | | |
|---|---|---|
| 0172 | A9 | |
| 63 | 584B | |
| BA49 | 12 | |
| 58 | 6370 | |

Similarly, an 8-PC begin-set of the same column maps to an end-set under $T_5 M$, and can therefore be swapped:

begin-set — end-set

| | | |
|---|---|---|
| $X$: | 0172 | A9 |
| R($X$): | 6358 | 4B |

$$\{A,9,6,3,5,8,4,B\} = T_5(M(\{0,1,7,2,6,3,5,8\}))$$

| | | |
|---|---|---|
| 0172A9 | | |
| 63584B | | |
| | 5A4372 | |
| | B86910 | |

| | | |
|---|---|---|
| 0172 | A9 | |
| | 63584B | |
| 5A43 | 72 | |
| B869 | 10 | |

Note that in this example, the begin- and end-sets in the original column overlap, with {6,3,5,8} being common to both sets.

Thus in choosing an operation on which to fold a CM, we know that one which maps sizeable begin-sets to end-sets in each column will guarantee lots of swapping, and hence a relatively even distribution of PCs in the resultant CM. "Sizeable" in this context means "close in size to 6", since the swapping of nearly entire aggregates would have little effect on the distribution of PCs. Most likely, one would not enumerate the begin- and end-sets of all CM-columns in searching for an "optimal" folding operation. Rather, one would be satisfied with the best of several guesses made on the behavior of one or two key columns, though an exhaustive search can be readily accomplished with a computer program. Since swapping possibilities tend to be more numerous in larger CMs, the choice of an operation on which to fold a CM becomes less critical as the size of a CM increases—at least as far as distribution is concerned.[24]

The 4-row CM of Fig. 37 can be folded again into an 8-row CM, this time on $T_9I$:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $X$: | 01 | 72 | A9 | | B48 | 5 | 3 | 6 |
| R($X$): | 6 | 3 | 5 | 84B | | 9A | 2 | 710 |
| $T_B$(I($X$)): | B | A49 | 1 | 2 | 073 | 6 | 8 | 5 |
| $T_B$(I(R($X$))): | 5 | 8 | | 6370 | | 21 | 94A | B |
| $T_9$(I($X$)): | 9827 | | B0 | | A51 | 4 | 6 | 3 |
| $T_9$(R(I($X$))): | 3 | 6 | 4 | 15A | | 0B | 7 | 289 |
| $T_A$($X$): | A | B50 | 87 | | 926 | 3 | 1 | 4 |
| $T_A$(R($X$)): | 4 | 1 | 362 | 9 | | 78 | 05B | A |

Fig. 40

Similarly, the 6-row CM of Fig. 16* can be folded into a 12-row CM on $T_BI$ (see Figs. 41a and 41b).[25]

### 4.7 *Constructing 3-row CMs from 2-row CMs*

A 3-row CM can generally be formed from two 2-row CMs through a merging and swapping procedure. Suppose that we have two row-

* See *Perspectives of New Music,* Vol. 16, No. 1, for Figs. 1–23.—Ed.

24 CMs derived by folding are shown in Martino, *op. cit.*

25 The same CM is derived differently in Fig. 15.

step 1:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X$: | 01 | | 72 | | A9 | | B4 | | 85 | | 36 | |
| $T_4(R(X))$: | A7 | | 90 | | 83 | | 12 | | 6B | | 54 | |
| $T_4(X)$: | 45 | | B6 | | 21 | | 38 | | 09 | | 7A | |
| $T_8(R(X))$: | 2B | | 14 | | 07 | | 56 | | A3 | | 98 | |
| $T_8(X)$: | 89 | | 3A | | 65 | | 70 | | 41 | | B2 | |
| $R(X)$: | 63 | | 58 | | 4B | | 9A | | 27 | | 10 | |
| $T_B(I(X))$: | | BA | | 49 | | 12 | | 07 | | 36 | | 85 |
| $T_7(I(R(X)))$: | | 14 | | 2B | | 38 | | A9 | | 50 | | 67 |
| $T_7(I(X))$: | | 76 | | 05 | | 9A | | 83 | | B2 | | 41 |
| $T_3(I(R(X)))$: | | 90 | | A7 | | B4 | | 65 | | 18 | | 23 |
| $T_3(I(X))$: | | 32 | | 81 | | 56 | | 4B | | 7A | | 09 |
| $T_B(I(R(X)))$: | | 58 | | 63 | | 70 | | 21 | | 94 | | AB |

Fig. 41a

step 2:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X$: | 0 | 1 | 7 | 2 | A | 9 | B | 4 | 8 | 5 | 3 | 6 |
| $T_4(R(X))$: | A | 7 | 9 | 0 | 8 | 3 | 1 | 2 | 6 | B | 5 | 4 |
| $T_4(X)$: | 4 | 5 | B | 6 | 2 | 1 | 3 | 8 | 0 | 9 | 7 | A |
| $T_8(R(X))$: | 2 | B | 1 | 4 | 0 | 7 | 5 | 6 | A | 3 | 9 | 8 |
| $T_8(X)$: | 8 | 9 | 3 | A | 6 | 5 | 7 | 0 | 4 | 1 | B | 2 |
| $R(X)$: | 6 | 3 | 5 | 8 | 4 | B | 9 | A | 2 | 7 | 1 | 0 |
| $T_B(I(X))$: | B | A | 4 | 9 | 1 | 2 | 0 | 7 | 3 | 6 | 8 | 5 |
| $T_7(I(R(X)))$: | 1 | 4 | 2 | B | 3 | 8 | A | 9 | 5 | 0 | 6 | 7 |
| $T_7(I(X))$: | 7 | 6 | 0 | 5 | 9 | A | 8 | 3 | B | 2 | 4 | 1 |
| $T_3(I(R(X)))$: | 9 | 0 | A | 7 | B | 4 | 6 | 5 | 1 | 8 | 2 | 3 |
| $T_3(I(X))$: | 3 | 2 | 8 | 1 | 5 | 6 | 4 | B | 7 | A | 0 | 9 |
| $T_B(I(R(X)))$: | 5 | 8 | 6 | 3 | 7 | 0 | 2 | 1 | 9 | 4 | A | B |

Fig. 41b

operations F and G allowing us to form 2-row CMs **D** and **E** based on the same row $P$:

| | | |
|---|---|---|
| $P$: | $D_{1,1}$ | $D_{1,2}$ |
| $F(P)$: | $D_{2,1}$ | $D_{2,2}$ |

| | | |
|---|---|---|
| $P$: | $E_{1,1}$ | $E_{1,2}$ |
| $G(P)$: | $E_{2,1}$ | $E_{2,2}$ |

By the row-column theorem, we know that $[D_{1,1}] = [D_{2,2}]$ and $[E_{1,1}] = [E_{2,2}]$. $[D_{1,1}]$ and $[E_{1,1}]$ will both be begin-sets of $P$, but their sizes may be different. Let us assume that $D_{1,1}$ is shorter than $E_{1,1}$, so that there is a segment $E^*$ of $P$ such that $E_{1,1} = D_{1,1}|E^*$.

We now merge **D** and **E** and swap twice, so as to isolate an entire instance of $P$ in one position of the CM:

| | | | | |
|---|---|---|---|---|
| $P$: | $D_{1,1}$ | $D_{1,2}$ | | |
| $F(P)$: | $D_{2,1}$ | $D_{2,2}$ | | |
| $P$: | | | $D_{1,1} \mid E^*$ | $E_{1,2}$ |
| $G(P)$: | | | $E_{2,1}$ | $E_{2,2}$ |

| | | | | |
|---|---|---|---|---|
| $P$: | $D_{1,1}$ | $D_{1,2}$ | | |
| $F(P)$: | $D_{2,1}$ | | $D_{2,2}$ | |
| $P$: | | $D_{1,1}$ | $E^*$ | $E_{1,2}$ |
| $G(P)$: | | | $E_{2,1}$ | $E_{2,2}$ |

first swapping

| | | | | |
|---|---|---|---|---|
| $P$: | | $D_{1,1} \mid D_{1,2}$ | | |
| $F(P)$: | $D_{2,1}$ | | $D_{2,2}$ | |
| $P$: | $D_{1,1}$ | | $E^*$ | $E_{1,2}$ |
| $G(P)$: | | | $E_{2,1}$ | $E_{2,2}$ |

second swapping

This CM can then be decomposed into a "1-row CM" of $P$, and a 3-row (CM of $P$, $F(P)$, and $G(P)$:

| | | | |
|---|---|---|---|
| $P$: | $D_{1,1}$ | $E^*$ | $E_{1,2}$ |
| $F(P)$: | $D_{2,1}$ | $D_{2,1}$ | |
| $G(P)$: | | $E_{2,1}$ | $E_{2,2}$ |

Fig. 42

We could have arrived alternatively at the same CM by merging a 1-row CM of G($P$) with **D**, though only an operation G producing a CM **E** as described above would result in any swapping.[26]

As an example, we construct a 3-row CM from two 2-row CMs based on the row $X$:

| | | |
|---|---|---|
| $X$: | 0172 | A9B48536 |
| $T_2(I(R(X)))$: | 8B96A354 | 0712 |

Fig. 43a

| | | |
|---|---|---|
| $X$: | 0172A9 | B48536 |
| $T_6(M(X))$: | 6B5483 | 12A790 |

Fig. 43b

| | | | |
|---|---|---|---|
| $X$: | 0172 | A9 | B48536 |
| $T_2(I(R(X)))$: | 8B96A354 | 0712 | |
| $T_6(M(X))$: | | 6B5483 | 12A790 |

Fig. 43c

Note that in this example, further swapping can fill one of the empty positions in the resultant 3-row CM. In general, a variety of 3-row CMs can be generated from a row for which there exist many 2-row CMs.

## 4.8 *CMs Produced by Segmenting the Row in Three*

In constructing 2-row CMs, we were concerned with the properties of a row broken into two complementary segments. By considering a row instead as a concatenation of three segments, we can show how properties of that segmentation can result in CMs of three, four, or more rows.

Let $P = P_1 \mid P_2 \mid P_3$. This segmentation in effect defines a succession of unordered sets: $[P_1] - [P_2] - [P_3]$. If there is an operation G such

[26] The same class of CMs is mentioned in Martino, *op. cit.*

that a segmentation of $G(P)$ defines the same set succession as $P$, then we say that $P$ is *segmentally invariant under* G. Example:

$$X = 0172 \mid A9B4 \mid 8536$$
$$T_2(M(X)) = 2170 \mid 4B9A \mid 6358$$

On the other hand, it is possible for an operation to exchange the contents of two segments, while a third segment remains invariant. Example cont'd:

$$T_B(I(X)) = BA49 \mid 1207 \mid 3685 = [P_2] - [P_1] - [P_3]$$

R, of course, will always exchange the first and third segments, while the middle segment remains invariant. To continue the example, $T_B(I(R(X)))$ defines the set-succession $[P_3] - [P_1] - [P_2]$, which is a cyclic permutation of the segmental content of $P$:

$$T_B(I(R(X))) = 5863 \mid 7021 \mid 94AB$$

The systematic segmental structure of a row can result in various CMs. If $[P_3]$ is invariant under some operation F, then the following CM exists:

| | | | |
|---|---|---|---|
| $P$: | $P_1$ | $P_2$ | $P_3$ |
| $R(P)$: | $[P_3 \mid P_2]$ | $[P_1]$ | |
| $F(R(P))$: | | $[P_3]$ | $[P_2 \mid P_1]$ |

Fig. 44

Example:

| | | | |
|---|---|---|---|
| $X$: | 017 | 2A9B4 | 8536 |
| $R(X)$: | 63584B9A2 | 710 | |
| $T_9(I(M(R(X))))$: | | 3685 | 1207BA49 |

Fig. 45

This is an instance of the class of CMs defined by Fig. 42 (sec. 4.7) in which $P$ can form a 2-row CM with either of the other two rows in the CM.

If we add the restriction that F exchanges the contents of $P_1$ and $P_2$, then a 4-row CM can be formed by merging the CM of Fig. 44 with a 1-row CM of $F(P)$, and then swapping segments:

| | | | | |
|---|---|---|---|---|
| $P$ | $P_1$ | | $P_2$ | $P_3$ |
| $R(P)$: | $[P_3 \mid P_2]$ | | $[P_1]$ | |
| $F(R(P))$: | | | $[P_3]$ | $[P_1 \mid P_2]$ |
| $F(P)$: | | $[P_2 \mid P_1 \mid P_3]$ | | |

| | | | | |
|---|---|---|---|---|
| $P$: | $P_1$ | | $P_2$ | $P_3$ |
| $R(P)$: | $[P_3]$ | $[P_2]$ | | $[P_1]$ |
| $F(R(P))$: | | $[P_3]$ | $[P_1]$ | $[P_2]$ |
| $F(P)$: | $[P_2]$ | $[P_1]$ | $[P_3]$ | |

Fig. 46

Example:

| | | | | |
|---|---|---|---|---|
| $X$: | 0172 | | A9B4 | 8536 |
| $R(X)$: | 6358 | 4B9A | | 2710 |
| $T_B(I(R(X)))$: | | 5863 | 7021 | 94AB |
| $T_B(I(X))$: | BA49 | 1207 | 3685 | |

Fig. 47

The CM of Fig. 46 might have been derived by folding a 2-row CM of $P$ and $R(P)$ on the operation F, but the derivation shown here points out an iterative overlaying process which might be compositionally relevant:

(1) start with $P$
(2) add $F(R(P))$ to make 2-row counterpoint
(3) merge in $R(P)$ and swap segments to yield Fig. 44
(4) merge in $F(P)$ and swap to yield Fig. 46

If $P$ is segmentally invariant under some other operation G, then $F(P)$, $R(P)$, and $F(R(P))$ will also be segmentally invariant under G. Thus, for example, the second and fourth rows of Fig. 47 may be subjected to $T_2M$:

| | | | | |
|---|---|---|---|---|
| $X$: | 0172 | | A9B4 | 8536 |
| $T_2(M(R(X)))$: | 8536 | A9B4 | | 0172 |
| $T_B(I(R(X)))$: | | 5863 | 7021 | 94AB |
| $T_9(I(M(X)))$: | 9AB4 | 7021 | 5863 | |

Fig. 48

A row may, of course, be broken into four or more segments, in which case segmental invariances and rearrangements may be exploited in an analogous fashion.

## 4.9 *Inspecting Middle Segments of the Row*

We will now consider a different kind of segmental property. Let $P$ and $F(P)$ be two rows which have been broken into three segments each. We let $P = P_1 \mid P_2 \mid P_3$ and $F(P) = Q_1 \mid Q_2 \mid Q_3$. It need not be the case that $P$ and $F(P)$ are broken in the same places. It is possible that the union of the two middle segments $[P_2] \cup [Q_2]$ is an aggregate:

aggregate

$$P = P_1 \mid P_2 \mid P_3$$
$$F(P) = Q_1 \mid Q_2 \mid Q_3$$

aggregate

Example:

aggregate

$$Y = 01 \mid 4392A85 \mid 7B6$$
$$T_9(I(Y)) = 985 \mid 607B1 \mid 42A3$$

aggregate

If this is the case, then the union of the "outer" segments, $[P_1] \cup [P_3] \cup [Q_1] \cup [Q_3]$ will also be an aggregate, since it is the remainder of what had been two aggregates to begin with. In general, if some "middle" segment of $P$ is in the same SC as the complement of some other middle segment of $P$, then there will be an operation F such that an aggregate will be formed by segments of $P$ and $F(P)$ as shown.

Given such rows $P$ and $F(P)$, we can immediately construct a 4-row CM:

| | | | | |
|---|---|---|---|---|
| $P$: | $P_1$ | $P_2$ | | $P_3$ |
| $F(P)$: | $Q_1$ | $Q_2$ | | $Q_3$ |
| $R(P)$: | $[P_3]$ | | $[P_2]$ | $[P_1]$ |
| $F(R(P))$: | $[Q_3]$ | | $[Q_2]$ | $[Q_1]$ |

Fig. 49

Examples:

| | | | | |
|---|---|---|---|---|
| $Y$: | 01 | 4392A85 | | 7B6 |
| $T_9(I(Y))$: | 985 | 607B1 | | 42A3 |
| $R(Y)$: | 6B7 | | 58A2934 | 10 |
| $T_9(I(R(Y)))$: | 3A24 | | 1B706 | 589 |

Fig. 50

| | | | | |
|---|---|---|---|---|
| $Y$: | 01 | 4392A8 | | 57B6 |
| $T_3(I(M(Y)))$: | 3A | 70651B | | 2489 |
| $R(Y)$: | 6B75 | | 8A2934 | 10 |
| $T_3(I(M(R(Y))))$: | 9842 | | B15607 | A3 |

Fig. 51

Here, we perform a folding-like operation on the rows $P$ and $F(P)$, which by themselves do not necessarily comprise a CM. The CM of Fig. 49 could be produced by folding a 2-row CM of $P$ and $R(P)$ on F, but we will now suggest operations on the segments of that CM to produce other CMs that would be unattainable through folding.

If $[P_3] \cup [Q_3]$ is invariant under some operation G, then the lower two rows of Fig. 49 may be replaced by $G(R(P))$ and $G(F(R(P)))$.

Thus, for example, $T_8M$ may be applied to the CM of Fig. 50, since [6B7] ∪ [3A24] is invariant under that operation:

| | | | | |
|---|---|---|---|---|
| $Y$: | 01 | 4392A85 | | 7B6 |
| $T_9(I(Y))$: | 985 | 607B1 | | 42A3 |
| $T_8(M(R(Y)))$: | 237 | | 90A65B4 | 18 |
| $T_5(I(M(R(Y))))$: | BA64 | | 13782 | 905 |

Fig. 52

Alternately, $[Q_2]$ can be swapped across the middle partition of Fig. 49 as follows:

| | | | | |
|---|---|---|---|---|
| $P$: | $P_1$ | $P_2$ | | $P_3$ |
| $F(P)$: | $Q_1$ | | $Q_2$ | $Q_3$ |
| $R(P)$: | $[P_3]$ | | $[P_2]$ | $[P_1]$ |
| $F(R(P))$: | $[Q_3]$ | $[Q_2]$ | | $[Q_1]$ |

Fig. 53

It is now the case that if $[Q_1] \cup [P_3]$ is invariant under some operation H, then $F(P)$ and $R(P)$ can both be replaced by $H(R(P))$ and $H(F(R(P)))$. Thus, for example, the CM of Fig. 51 can be transformed by subjecting its middle two rows to $T_4M$ to yield a CM which would be unattainable through folding:

| | | | | |
|---|---|---|---|---|
| $Y$: | 01 | | 4392A8 | 57B6 |
| $T_7(I(Y))$: | 76 | 34A59B | | 2081 |
| $T_4(M(R(Y)))$: | AB35 | 862170 | | 94 |
| $T_3(I(M(R(Y))))$: | 9842 | | B15607 | A3 |

Fig. 54

## 4.10 *Canonic CMs*

Up to this point, we have not constructed CMs in which rowforms were duplicated. Any row, however, can generate a family of *canonic* CMs, which contain multiple instances of a single row-form. These CMs can be derived from a 2-row CM of $P$ and $R(P)$, which is based on a segmentation of $P$ into $P_1$ and $P_2$:

| | | |
|---|---|---|
| $P$: | $P_1$ | $P_2$ |
| $R(P)$: | $[P_2]$ | $[P_1]$ |

Fig. 55

We merge this CM with a 1-row CM comprising an additional instance of $P$, and swap segments:

| | | | |
|---|---|---|---|
| $P$: | $P_1$ | | $P_2$ |
| $P$: | | $P_1 \mid P_2$ | |
| $R(P)$: | $[P_2]$ | | $[P_1]$ |

| | | |
|---|---|---|
| $P_1$ | $P_2$ | |
| | $P_1$ | $P_2$ |
| $[P_2]$ | | $[P_1]$ |

Fig. 56

By splitting the row into more segments, we can generate larger canonic CMs:

| | | | | |
|---|---|---|---|---|
| $P$: | $P_1 \mid P_2$ | $P_3$ | | |
| $P$: | | $P_1$ | $P_2 \mid P_3$ | |
| $P$: | | | $P_1$ | $P_2 \mid P_3$ |
| $R(P)$: | $[P_3]$ | $[P_2]$ | | $[P_1]$ |

Fig. 57

Example:

| | | | | |
|---|---|---|---|---|
| $X$: | 0172A9B4 | 8536 | | |
| $X$: | | 0172 | A9B48536 | |
| $X$: | | | 0172 | A9B48536 |
| $R(X)$: | 6358 | 4B9A | | 2710 |

Fig. 58

Because of the segmental invariances of the row $X$ (see sec. 4.8), the first and second rows of Fig. 58 can be subjected to $T_B I$ and $T_2 I$, respectively:

| | | | | |
|---|---|---|---|---|
| $T_B(I(X))$: | BA492107 | 3685 | | |
| $T_2(I(X))$: | | 2170 | 453A69B8 | |
| $X$: | | | 0172 | A9B48536 |
| $R(X)$: | 6358 | 4B9A | | 2710 |

Fig. 59

## 4.11 *Free CMs and Complementary CMs*

If there is no more than one PC in any position of a CM, we say that it is a *free* CM (Figs. 14 and 41b, for example). Free CMs necessarily have twelve or more rows, and any column of a free CM can be partitioned into any pair of complementary begin- and end-sets. Any single PC can be swapped between adjacent columns of a free CM, but it is not true that any *set* can be swapped, since an arbitrary set will not automatically reside in mutually exclusive collections of rows.

A free CM can often be split into a pair of smaller CMs which might have been merged to form it. The free CM of Fig. 41b, for example, contains the rows $X$, $T_4(X)$, and $T_8(X)$ which stand alone to form the CM of Fig. 10. It will follow then that the remaining nine rows of Fig. 41b also comprise a CM, which can be extracted by performing the sequence of swaps shown in Figs. 60 a–c. This isolates the three rows mentioned in three columns of Fig. 60c so that they can be "unmerged" from the other rows. Thus we arrive at a new CM by *complementing* the collection of rows comprising a given CM with respect to the collection defined by a free CM based on the same row.[27] Similarly, the 8-row CM of Fig. 61 is the complement of the CM of Fig. 37 with respect to the same free CM. Since a single row can be regarded as a 1-row CM, it is always possible to "swap out" any row of a free CM to yield a CM one row smaller, as in Fig. 62.

[27] It would always be possible to place a collection of rows in the n diagonal positions of a CM, no matter what row-forms they happened to be, and thus construct an n-row CM. The fragmentation (sec. 5.1) would, of course, be minimal, and the CM would be trivial, as no counterpoint would be involved. The point is that non-trivial CMs generally yield non-trivial complements with respect to free CMs.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X$: | 0 | 1 | 7 | 2 | A | 9 | B | 4 | 8 | 5 | 3 | 6 |
| | A | 7 | 9 | 0 | 8 | 3 | 1 | 2 | 6 | B | 5 | 4 |
| $T_4(X)$: | 4 | 5 | B | 6 | 2 | 1 | 3 | 8 | 0 | 9 | 7 | A |
| | 2 | B | 1 | 4 | 0 | 7 | 5 | 6 | A | 3 | 9 | 8 |
| $T_8(X)$: | 8 | 9 | 3 | A | 6 | 5 | 7 | 0 | 4 | 1 | B | 2 |
| | 6 | 3 | 5 | 8 | 4 | B | 9 | A | 2 | 7 | 1 | 0 |
| | B | A | 4 | 9 | 1 | 2 | 0 | 7 | 3 | 6 | 8 | 5 |
| | 1 | 4 | 2 | B | 3 | 8 | A | 9 | 5 | 0 | 6 | 7 |
| | 7 | 6 | 0 | 5 | 9 | A | 8 | 3 | B | 2 | 4 | 1 |
| | 9 | 0 | A | 7 | B | 4 | 6 | 5 | 1 | 8 | 2 | 3 |
| | 3 | 2 | 8 | 1 | 5 | 6 | 4 | B | 7 | A | 0 | 9 |
| | 5 | 8 | 6 | 3 | 7 | 0 | 2 | 1 | 9 | 4 | A | B |

Fig. 60a

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X$: | | 01 | 72 | | | A9 | B4 | | | 85 | 36 | |
| | A | 7 | 9 | 0 | 8 | 3 | 1 | 2 | 6 | B | 5 | 4 |
| $T_4(X)$: | | 45 | B6 | | | 21 | 38 | | | 09 | 7A | |
| | 2 | B | 1 | 4 | 0 | 7 | 5 | 6 | A | 3 | 9 | 8 |
| $T_8(X)$: | | 89 | 3A | | | 65 | 70 | | | 41 | B2 | |
| | 6 | 3 | 5 | 8 | 4 | B | 9 | A | 2 | 7 | 1 | 0 |
| | B | A | 4 | 9 | 12 | | | 07 | 3 | 6 | 8 | 5 |
| | 14 | | | 2B | 3 | 8 | A | 9 | 50 | | | 67 |
| | 7 | 6 | 0 | 5 | 9A | | | 83 | B | 2 | 4 | 1 |
| | 90 | | | A7 | B | 4 | 6 | 5 | 18 | | | 23 |
| | 3 | 2 | 8 | 1 | 56 | | | 4B | 7 | A | 0 | 9 |
| | 58 | | | 63 | 7 | 0 | 2 | 1 | 94 | | | AB |

Fig. 60b

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X$: | | | 0172 | | | | A9B4 | | | | 8536 | |
| | A | 79 | | 0 | 8 | 31 | | 2 | 6 | B5 | | 4 |
| $T_4(X)$: | | | 45B6 | | | | 2138 | | | | 097A | |
| | 2 | B1 | | 4 | 0 | 75 | | 6 | A | 39 | | 8 |
| $T_8(X)$: | | | 893A | | | | 6570 | | | | 41B2 | |
| | 6 | 35 | | 8 | 4 | B9 | | A | 2 | 71 | | 0 |
| | B | A4 | | 9 | 12 | | | 07 | 3 | 68 | | 5 |
| | 14 | | | 2B | 3 | 8A | | 9 | 50 | | | 67 |
| | 7 | 60 | | 5 | 9A | | | 83 | B | 24 | | 1 |
| | 90 | | | A7 | B | 46 | | 5 | 18 | | | 23 |
| | 3 | 28 | | 1 | 56 | | | 4B | 7 | A0 | | 9 |
| | 58 | | | 63 | 7 | 02 | | 1 | 94 | | | AB |

Fig. 60c

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| A | 79 | 0 | 8 | 31 | 2 | 6 | B5 | 4 |
| 2 | B1 | 4 | 0 | 75 | 6 | A | 39 | 8 |
| 6 | 35 | 8 | 4 | B9 | A | 2 | 71 | 0 |
| B | A4 | 9 | 12 | | 07 | 3 | 68 | 5 |
| 14 | | 2B | 3 | 8A | 9 | 50 | | 67 |
| 7 | 60 | 5 | 9A | | 83 | B | 24 | 1 |
| 90 | | A7 | B | 46 | 5 | 18 | | 23 |
| 3 | 28 | 1 | 56 | | 4B | 7 | A0 | 9 |
| 58 | | 63 | 7 | 02 | 1 | 94 | | AB |

Fig. 60d

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $T_4(X)$: | 45 | B6 | | 21 | 3 | 809 | 7 | A |
| $T_8(X)$: | 8 | 93 | A6 | 5 | 70 | 4 | 1 | B2 |
| $T_4(R(X))$: | A | 7 | 908 | 3 | 12 | | 6B | 54 |
| $T_8(R(X))$: | 2B | 1 | 4 | 07 | 5 | 6A | 39 | 8 |
| $T_7(I(X))$: | 76 | 05 | | 9A | 8 | 3B2 | 4 | 1 |
| $T_3(I(X))$: | 3 | 28 | 15 | 6 | 4B | 7 | A | 09 |
| $T_7(I(R(X)))$: | 1 | 4 | 2B3 | 8 | A9 | | 50 | 67 |
| $T_3(I(R(X)))$: | 90 | A | 7 | B4 | 6 | 51 | 82 | 3 |

Fig. 61

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 7 | 2 | A | 9 | B | 48 | 5 | 3 | 6 |
| 45 | B | 6 | | 2 | 13 | 8 | 0 | 9 | 7 | A |
| 8 | 9 | 3 | A6 | 5 | 7 | 0 | | 41 | | B2 |
| A | 7 | 9 | 08 | 3 | | 12 | 6 | | B5 | 4 |
| 2 | | B1 | 4 | 07 | 5 | 6 | | A3 | 9 | 8 |
| 6 | 35 | 8 | | 4 | B | 9 | A2 | 7 | 1 | 0 |
| B | A | 4 | 9 | 1 | 20 | 7 | 3 | 6 | 8 | 5 |
| 7 | 6 | 05 | | 9 | A8 | 3 | | B2 | 4 | 1 |
| 3 | 28 | | 15 | 6 | | 4 | B7 | | A0 | 9 |
| 1 | 4 | 2 | B3 | 8 | | A | 95 | 0 | 6 | 7 |
| 9 | 0 | A | 7 | B | 46 | 5 | 1 | 8 | 2 | 3 |

Fig. 62

### 5.1 *Fragmentation*

Not all CMs with the same number of rows have the same distribution of PCs among columns. In CMs generated by cliques or cycles of rows, for instance, all CM-positions have the same number of PCs, whereas in other CMs, the number of PCs can vary considerably from position to position. We will say that the more evenly PCs are distributed throughout a CM, the more *fragmented* the CM is. Highly fragmented CMs have roughly the same number of PCs in each position, while relatively unfragmented CMs are characterized by row-segments that are unequal in length, with long unbroken segments in some positions and either short segments or no PCs at all in the others.

To begin with, we will consider the fragmentation of a single row broken into segments, which we can evaluate in terms of ordered pairs. If the PC k follows the PC j in some row-segment, we say that the segment contains the *ordered pair* j:k. A row-segment of n PCs will define n $(n-1)/2$ ordered pairs (henceforth 'OPs'). The 5-PC segment 2A9B4, for example, contains the following 10 OPs:

| | | | |
|---|---|---|---|
| 2:A | 2:9 | 2:B | 2:4 |
| A:9 | A:B | A:4 | |
| 9:B | 9:4 | | |
| B:4 | | | |

An entire row defines 66 OPs.

We can classify the OPs of a CM as lying either (1) within single row-segments, or (2) between pairs of row-segments. Thus, for example, in a row comprising three segments of 3, 4, and 5 PCs each, there will be $3 + 6 + 10 = 19$ OPs within individual segments, and a remainder of 47 OPs between segments. Thus a fraction of 47/66 of the OPs will be between segments, which we will consider to be a measure of the fragmentation of that particular segmentation.

Fragmentation tells us something about the "evenness" of a segmentation. Three segments of 4 PCs each, for instance, will have a fragmentation of 48/66, the highest possible fragmentation for a row segmented in three; while a degenerate segmentation into 0, 0, and 12 PCs will have a fragmentation of zero. All other segmentations will have fragmentations falling between these extremes. To achieve a higher fragmentation, a row must be divided into more segments. When a row is finally broken into twelve 1-PC segments, as in a free CM, a fragmentation of one is achieved. We can interpret this by saying that the ordering of a maximally fragmented row has been incorporated entirely into its segmentation. We could also evaluate the fragmentation of the row-segments comprising a CM-column. There, fragmentation translates as the amount of freedom one has in aligning the segments of a column with respect to one another in a realization of the CM.

An n-row CM is in effect a segmentation of n rows into n segments each. We can calculate the fragmentation of an n-row CM as the average of the fragmentations of all its individual rows—or alternately, of all its columns, which is algebraically the same thing. Thus the fragmentation of an n-row CM **E** is given by the expression

$$1-\left(\frac{1}{66n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\#E_{i,j}(\#E_{i,j}-1)\right)\right)$$

Table II gives the fragmentations of selected CM examples in this paper. (See pp. 76–77.)

Fragmentation gives us a way to look at the effects of merging and swapping. When two CMs are merged, empty matrix positions are created, in effect providing a way to segment the rows further. Swapping PCs into the empty positions increases the fragmentation of the merged CM, and can moreover always raise it higher than the fragmentation of either original CM. Compare, for example, the fragmentations given in Table II of Figs. 24–26 with Fig. 29, and Figs. 27–28 with Fig. 30, in which merging and swapping were performed in succession.

## 5.2 *Association*

The number of rows in a CM and fragmentation are measures of the distribution of PCs and do not take into account how the PCs are related to each other. *Association,* on the other hand, is an overall measure of the tendency of certain PCs to recur with others within CM positions. In the following CM, for example, a quartet of PCs and four pairs of PCs recur together:

| | | | | |
|---|---|---|---|---|
| $X$: | 0172 | A9B4 | | 8536 |
| $T_B(I(R(X)))$: | | 5863 | 7021 | 94AB |
| $T_3(X)$: | 34A5 | 1027 | B869 | |
| $T_2(I(R(X)))$: | 8B96 | | A354 | 0712 |

Fig. 63

This can be tabulated as an *association-set,* which is a set of sets, each of which defines a recurrent collection of PCs:

$$\{\{0,1,7,2\},\{9,B\},\{3,5\},\{8,6\},\{4,A\}\}$$

The union of the member sets of an association-set (henceforth 'AS') is always an aggregate. Table II gives the ASs of selected CM examples in this paper. A PC which does not recur consistently with any other PC will be entered as a one-element (singleton) set in the AS (as in the ASs of Figs. 19 and 31, for example).

Association is minimal in a CM whose AS comprises twelve 1-PC sets, and it is maximal in the degenerate case where the AS contains a single 12-PC set. The amount of association in a CM can be evaluated numerically by counting the number of pairs of PCs contained entirely within members of its AS. Thus in the AS given above, there are 6 + 1 + 1 + 1 + 1 = 10 pair-wise PC associations.

Each member of the AS of an n-row CM can be stated as the intersection of a collection of n CM-positions no two of which are in the same row or column. Example cont'd:

$$\{0,1,2,7\} = [0127] \cap [1027] \cap [7021] \cap [0712]$$
$$\{9,B\} = [89B6] \cap [A9B4] \cap [B869] \cap [94AB]$$
$$\{3,5\} = [34A5] \cap [5863] \cap [A354] \cap [8536]$$
et cetera. . . .

In an n-row CM, there are n! such collections of CM positions, whose intersection is often empty:

$$\emptyset = [0172] \cap [5863] \cap [B869] \cap [0712]$$
$$\emptyset = [\ \ ] \cap [\ \ ] \cap [B869] \cap [8536]$$
et cetera. . . .

The empty set is, however, never entered as a member of an AS. The AS of an n-row CM can, in any case, have no more than n! components. Therefore, 2- and 3-row CMs have at most 2- and 6-member ASs, respectively, which means that they necessarily have some degree of association. Thus small CMs tend to establish "harmonic regions" of PC-associations. CMs of four or more rows can, however, also have significant amounts of association (Figs. 18, 26, and 39, for example). The pattern of association in a CM tends to allow or deny the "motivic identity" of row-segments in a CM—that is, where association is present, the composer is free to treat associated PCs as motivic units.

## 5.3 *Partition-classes*

In this paper, we have primarily been concerned with the problems of generating a class of CMs from a single, perhaps arbitrary, row. ASs, however, suggest a way that a CM might conversely be used to de-

TABLE II: Fragmentation and Association of Figures in this paper

| *Fig.* | *# Rows* | *Fragmentation* | *# Pairs Associated* | *Association Set* |
|---|---|---|---|---|
| 1 | 7 | .774 | 0 | {all singletons} |
| 2 | 4 | .727 | 6 | {{0,1},{2,8},{3,4},{6,7},{5,B},{9,A}} |
| 4 | 2 | .530 | 31 | {{0,1,3,4,9},{2,5,6,7,8,A,B}} |
| 5 | 2 | .545 | 30 | {{0,1,2,7,9,A},{3,4,5,6,8,B}} |
| 6 | 2 | .485 | 34 | {{0,1,2,7},{3,4,5,6,8,9,A,B}} |
| 7a | 3 | .505 | 21 | {{0,1,2,3,4,9},{5,},{6},{7,8,A,B}} |
| 10 | 3 | .727 | 6 | {{1,2,},{3,8},{4,B},{5,6},{7,0},{9,A}} |
| 11 | 6 | .909 | 0 | {all singletons} |
| 12–15 | 12 | 1.0 | 0 | {all singletons} |
| 16 | 6 | .909 | 0 | {all singletons} |
| 19 | 3 | .727 | 9 | {{7,8,2},{1},{3,4,A},{9},{0,B,6},{5}} |
| 21 | 8 | .939 | 0 | {all singletons} |
| 24 | 3 | .727 | 6 | {{1,2},{3,8},{4,B},{5,6},{7,0},{9,A}} |
| 25 | 2 | .545 | 30 | {{0,1,2,4,7,B},{3,5,6,8,9,A}} |
| 26 | 5 | .654 | 6 | {{1,2},{3,8},{4,B},{5,6},{7,0},{9,A}} |
| 27 | 2 | .485 | 34 | {{0,1,2,7},{3,4,5,6,8,9,A,B}} |
| 28 | 3 | .323 | 34 | {{0,1,2,7},{3,4,5,6,8,9,A,B}} |
| 29 | 5 | .751 | 3 | {{1,2},{4,B}{9,A}, singletons} |
| 30 | 3 | .667 | 10 | {{0,1},{7,2},{A,9},{4,5},{B,8,3,6}} |
| 31 | 6 | .707 | 2 | {{0,7},{1,2}, singletons} |
| 32 | 6 | .810 | 1 | {{1,2}, singletons} |
| 33–36 | 2 | .545 | 30 | {in each case, two 6-PC sets} |

| *Fig.* | *# Rows* | *Fragmentation* | *# Pairs Associated* | *Association Set* |
|---|---|---|---|---|
| 37 | 4 | .787 | 6 | {{0,7},{1,2},{3,6},{9,**A**},{4,**B**},{5,8}} |
| 38 | 4 | .727 | 8 | {{0,1},{7,2},{3,4,5},{**A**},{9},{**B**,8,6}} |
| 39 | 4 | .545 | 12 | {{0,2,9},{1,7,**A**},{6,5,8}.{3,4,**B**}} |
| 40 | 8 | .886 | 0 | {all singletons} |
| 43c | 3 | .566 | 22 | {{0,1,7,2},{**A**,9},{3,4,5,6,8,**B**}} |
| 45 | 3 | .535 | 19 | {{0,1,7},{2,**A**,9,**B**,4},{8,5,3,6}} |
| 47,48 | 3 | .727 | 18 | {{0,1,7,2},{**A**,9,**B**,4},{8,5,3,6}} |
| 50 | 4 | .667 | 13 | {{0,1}, {5,8,9},{6,**B**,7}, {2,3,4,**A**}} |
| 51 | 4 | .667 | 14 | {{0,1}, {3,**A**},{6,5,7,**B**},{2,4,8,9}} |
| 52 | 4 | .667 | 4 | {{0},{1},{9,5},{8},{2,3},{7},{**B**,6},{**A**,4}} |
| 54 | 4 | .667 | 6 | {{0.1},{7,6},{**A**,3},{**B**,5},{9,4},{8,2}} |
| 58,59 | 4 | .545 | 18 | {{0,1,7,2},{**A**,9,**B**,4},{8,5,3,6}} |

fine a collection of rows with the same combinatorial properties, in a fashion which is more specific than the traditional notion of "source sets". We pose the question: given a CM based on some row, what other rows would also yield a CM under the same operations and with the same distribution of PCs?

A class of rows having this property is the *partition-class* generated by the AS of a CM. Any PCs belonging to the same member-set of an AS may be permuted with each other to yield a row in the same partition-class. Consider the AS of the CM of Fig. 63, which is based on the row $X$. We connect the PCs of each AS component with brackets as they lie in $X$:

0 1 7 2 A 9 B 4 8 5 3 6

Permuting the PCs under all or any of the brackets gives us other rows with the combinatorial properties exercised by $X$ in the given CM:

2 1 0 7 4 9 B A 6 3 5 8
1 0 2 7 4 B 9 A 8 3 5 6
1 2 0 7 A B 9 4 6 5 3 8
et cetera . . . .

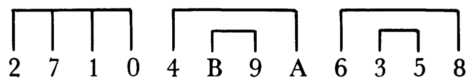
Since each bracket can be permuted independently, the row $X$, as employed in Fig. 63, generates a partition-class of $4! \cdot 2! \cdot 2! \cdot 2! \cdot 2!$ rows. The number of pair-wise PC-associations, in this case 10, gives the maximal number of order inversions by which the rows of a partition-class may differ from one another.[28] A CM can be built on one of these rows with the same operations and distribution of PCs as Fig. 63. Example:

| | | | | |
|---|---|---|---|---|
| $Q$: | 2107 | 49BA | | 6358 |
| $T_B(I(R(Q)))$: | | 3685 | 1027 | 4BA9 |
| $T_3(Q)$: | 543A | 7021 | 968B | |
| $T_2(R(I(Q)))$: | 69B8 | | 435A | 7210 |

Fig. 64

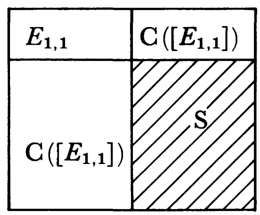
[28] Babbitt, "Twelve-Tone Invariants as Compositional Determinants," *Journal of Music Theory,* 5/1 (1961).

Note that the partition-class defined by this particular AS is redundant in that it includes the row $T_2(M(X))$, which is a transform of the original row:



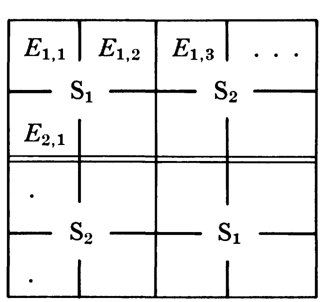

## 5.4 *The Diagonal Theorem*

The diagonal theorem is a corollary to the row-column theorem (sec. 2.7). Consider the set $[E_{1,1}]$ in some n-row CM **E**:




We know from the row-column theorem that $C([E_{1,1}])$ resides in the remaining $n - 1$ positions of both the first row and first column of **E**. The crosshatched area contains the complement of $C([E_{1,1}])$ along with $n - 1$ aggregates. Thus the set $[E_{1,1}]$ is highlighted in region S, since its PCs are more populous there. This effect is strongest for small values of n, for as n increases, the PCs in $[E_{1,1}]$ become a smaller fraction of the total number of PCs in the region S.

We can also place several positions at once in a diagonal relationship to each other. We might, for instance, divide a 4-row CM **E** into diagonal regions as follows:




Here, the regions $S_1$ and $S_2$ may include duplicated PCs, but the frequency of PCs in both $S_1$ regions will be the same, as it will in both $S_2$ regions. The PC-frequencies in the $S_1$ and $S_2$ regions will be complementary with respect to two aggregates. We have chosen diagonal regions in adjacent rows and columns, which is not a necessary condition for the complement relation to hold. The theorem also holds if, for instance, we choose $S_1$ and $S_2$ so that

$$\begin{aligned} S_1 &= E_{1,1},\ E_{1,3},\ E_{3,1}, \text{ and } E_{3,3} \\ &= E_{2,2},\ E_{2,4},\ E_{4,2}, \text{ and } E_{4,4} \end{aligned}$$

and

$$\begin{aligned} S_2 &= E_{1,2},\ E_{1,4},\ E_{3,2}, \text{ and } E_{3,4} \\ &= E_{2,1},\ E_{2,3},\ E_{4,1}, \text{ and } E_{4,3} \end{aligned}$$

in which case, neither region would occupy adjacent CM-rows or -columns. We can define 'diagonal' more generally as follows: Given any collection of columns and any collection of rows in a CM, the positions in the intersection of the column- and row-collections will be in a diagonal relation to the positions in the intersection of the complements of those collections.

## 5.5 *Linear Aggregates*

We have defined the combination matrix as a contrapuntal juxtaposition of rows. We can also consider the succession of rows that results when the ordering of PCs in each column is fixed as the CM is realized in composition. The 4-row CM of Fig. 2, for example, might be realized as the following succession of four rows:

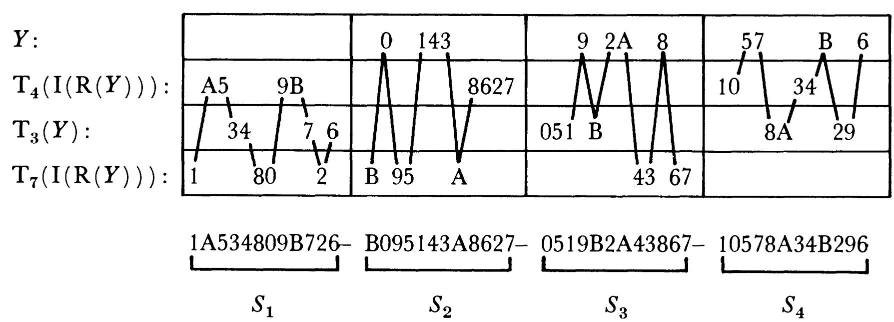



Fig. 65

This is, of course, only one of many possible linearizations of the CM, whose columns could be ordered in many other ways without disturbing the order or content of any of its row-segments. In a free CM, each columnar aggregate can be linearized to form any row. In general, the $j^{th}$ column of an n-row CM **E** can be aligned to form der(**E**,n,j) different linear aggregates (henceforth 'LAs'), where

$$\mathrm{der}(\mathbf{E},n,j) = 12!\ \prod_{i=1}^{n} \frac{1}{(\#E_{i,j})\,!}$$

Note that the four LAs given in Fig. 65 are not the only rows present in the resultant string of 48 PCs. There are in additions three LAs which straddle pairs of adjacent CM-columns:

$S_5$ $S_7$

1A534809B726–B095143A8 267–0519B2A43867–10578A34B296

$S_6$

Such trans-columnar LAs are possible when the complement of an end-set of one column is a begin-set of the column immediately to its right. This implies that it must be possible to partition two adjacent columns into the same pair of complementary begin- and end-sets.[29] Trans-columnar LAs result when the begin- and end-sets of the columns become begin- and end-sets of rows comprising the LAs. In the example, {2, 6, 7} is an end-set of both the first and second column, {0, 1, 5, 9, B} is a begin-set of both the second and third columns, and {1, 0, 5} is a begin-set of both the third and fourth columns:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $Y$: | | | 01 | 43 | | | 9 2A8 | 5 | 7B6 |
| $T_4(I(R(Y)))$: | A59B | | | 8 | 627 | | | 10 | 34 |
| $T_3(Y)$: | 34 | 76 | | | | 051 | B | | |
| $T_7(I(R(Y)))$: | 108 | 2 | B95 | A | | | 4367 | | 8A29 |

Fig. 66

One can interpret the number of trans-columnar LAs as a measure of saturation in a realized CM. Saturation is maximal in a CM which

[29] Using the terminology of Babbitt, "Some Aspects of Twelve-Tone Composition", *The Score & IMA Magazine,* 12 (1955), we would say that trans-columnar LAs are "secondary sets" with respect to the succession of LAs which lie entirely within CM-columns. (See note 5, p. 47 of "Some Aspects . . .")

has been realized so that all columnar LAs are the same row, which is feasible in free CMs.

It is possible to structure the relationship of the LAs of a CM to each other and to the row on which the CM is based. As an example, recall from sec. 3.4 that the structure of the row $X$ is such that in each 4-PC segment, each 4-clique is represented once:

$$(X)\text{mod-}4 = (0132)(2130)(0132)$$

Furthermore, note that the middle segment subtracted from 2, modulo-4 gives the same permutation (0132) as the outer segments. Thus, it will be possible for LAs in the first and third columns of the CM of Fig. 10 to be the same row, while the middle column can be a transform of that row under $T_2I$:

| | | | |
|---|---|---|---|
| $X$: | 0 1 7 2 | A 9B 4 | 8 53 6 |
| $T_4(X)$: | 4 5 B6 | 2 1 38 | 0 9 7A |
| $T_8(X)$: | 89 3 A | 65 7 0 | 41 B 2 |
| | 04189537AB62 | 2A1659B74380 | 04189537AB62 |
| | $S$ | $T_2(I(S))$ | $S$ |

Fig. 67

This kind of structure suggests a compositionally projectable relation between the rows $X$ and $S$.[30] We will not, however, discuss the intricacies of such "derived" rows in this study.

## 6.1 *Summary and Remarks*

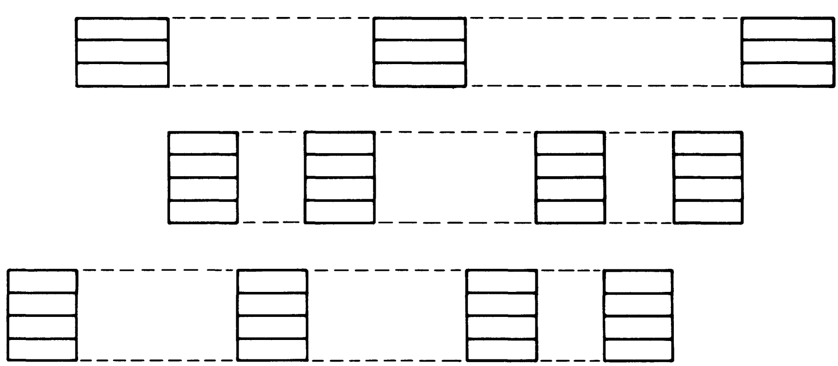
Sections III and IV have presented techniques for constructing CMs and operations on CMs which preserve their defining properties. There are basically three categories of techniques we might use to generate CMs based on an arbitrary row: (1) techniques whereby a CM results from the segmental structure of the row—e.g., constructing 2- and 3-row CMs from begin- and end-set structure, or using cliques or cycles of rows as CMs; (2) techniques whereby a given CM is modified to form a new CM with the same number of rows—e.g., substituting rows or swapping PCs between the columns of a CM; and

[30] Rows like $S$ are called "derived sets" in *ibid.*

(3) techniques in which we operate on given CMs to yield a new CM with a different number of rows—e.g., merging and complementation.

These techniques comprise a recursive system capable of generating CMs of arbitrary size. Category (1) establishes an initial collection of CMs which is guaranteed to be non-empty for any row, since for any row $P$, there are always 2-row CMs of $P$ and $R(P)$, as well as transposition CMs and some of their derivatives. Category (2) augments this collection, but to a limited extent, since only a finite number of row-substitutions or swappings will be possible in any CM. Category (3), however, can be used to augment a collection of CMs *ad infinitum,* since it generates CMs that are larger than those already in the collection. Thus row-combination, like certain formal languages, is a generative system in which the recursive application of simple rules generates a countable infinity of well-formed expressions. While it is not likely that any single composition would comprise more than a finite number of rows, one can always generate an arbitrarily large CM, which might encompass an entire composition.

Such a composition would be a well-formed structure, assembled according to the rules (techniques) of row-combination, which when viewed in this light become much more than a solution to localized problems of counterpoint. The notion of merging suggests musical structures which interrupt one another:



One might similarly dovetail a sequence of CM-events by merging the final column of each CM with the first column of its successor.

Overlaying, on the other hand, suggests means to superimpose processes so that they might unfold simultaneously. Events could be made to "cross-fade" into one another, and local events might be made into "windows" between moments of an event that unfolds over a larger time interval.

Various systematic properties of the twelve-tone system have been discussed that the authors have taken as an invitation to composition. This study is not intended as a manifesto of how music ought to be written, but rather as a response to problems that have already been posed and seem to have been of general concern in recent years. We cannot assume that combinatoriality is the last word concerning the twelve-tone system. Just as the saturation- and row-concepts led to combinatoriality, and combinatoriality leads in turn to merging, swapping, overlaying and so forth, it is not difficult to imagine that those concepts might be recombined to yield further concepts.

We have explored certain implications of an *a priori* decision to use rows and contrapuntal aggregates in composition—a decision that a composer must make for himself, for reasons which would necessarily be outside the domain of this paper. No property or implication of the twelve-tone system or any other system comprises an imperative to adopt it. But since it is explored here, in other writings, and moreover in much recent music, the twelve-tone system might offer us more freedom than a system which is less well understood.