A Theory of Set-Complexes for Music
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A Theory of Set-Complexes

1.0 Historical Background

For the moment let us regard a "set" as any collection of notes. If we seek historical precedents to justify such arbitrary sets as basic musical components there is no lack of material. Leaving aside the most obvious examples, the traditional chords of tonal music, we find the notion of the set implicit in the late experimental works of Liszt, in the non-functional chords of Debussy's music, in the "constructivist" works of Scriabin (See Perle 1962), and in the iconoclastic atonal music of Schoenberg and his eminent pupils.

In the theoretical literature, too, there are abundant instances of the awareness of non-tonal sets and of their implications for modern music. In 1911 we find Bernhard Ziehn's discussion of the properties of the whole-tone hexachord (Ziehn), and even

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earlier, in 1853, we have K. F. Weitzmann's essay on the augmented triad, an essay which may well have influenced Liszt (Weitzmann, Szelényi). The two editions of Schoenberg's Harmonielehre (Schoenberg) contain many relevant passages, and in his only extended essay on music theory Charles Ives reveals an interest in the properties of collections of notes (Ives, 115). Indeed, it is difficult to say just when the notion of the arbitrary set took root. We do know that Fetis predicted the ascendance of sets functionally independent of the constraints of triadic tonality as early as 1844 (Fétis, 195).

1.1 Purpose of this article

When we consider, even casually, the range of possible pitch-relations in music we are naturally interested in any limiting factors, since they suggest fruitful approaches to the study of syntactic structure. In the case of tonal music we have a very good grasp of limiting factors, but for music with other structural bases there remains much to be done before we can say "we know". *1

This article presents what is believed to be an important limiting factor: the set-complex derived from the inclusion relation. No initial assertion is made that it is the only or even the most important constraint, but merely that, given two sets, one can immediately say whether or not they are in the defined relation. If this decision is considered significant, then of course a great deal more information is made available, as will be shown.

1.2 Synopsis

Before presenting a synopsis I wish to make several brief comments. The reader will find that certain elementary matters are explained with some care in the text or (in some cases) in extended notes to be found at the end of the article. These may be bypassed by the experienced reader, of course. The mathematics employed for that purpose are descriptive. Here formalization serves the usual purpose of avoiding ambiguity, while making it easier to generalize from statements of relations. A sincere effort has been made to give the reader complete information and a minimum of jargon. In the interests of contemporary music theory, I hope that his efforts will be rewarded by new musical insights, that his mathematical sensitivities will not be offended, and that his standards of English syntax will not be seriously violated.

The article begins with certain necessary basic definitions, leading up to an explanation of the relation between pitch-set and interval-set. The idea of interval-content is then presented and elaborated for the purpose of defining a general equivalence relation for pitch-sets. There follows a brief consideration of the correspondence of pitch and interval sets, together with an explanation of relevant combinatory aspects. (Throughout the article combinatory problems and solutions are offered — many for the first time, it is believed.) In order to render the concept of interval-set specific and practicable, complete tables of the distinct interval-sets of the 12-pitch system are given. Reference is made to these tables throughout the article.

All the foregoing may be regarded as introductory to the discussion of unordered pitch-set inclusion which begins the subsequent main part of the paper. This continues with an explanation of the concepts of set-complementation, mappings of complementary sets, and the properties of the inclusion relation. The apparently contradictory concepts of complementation and inclusion are then combined in such a way as to lead to the central idea: the theory of set-complexes.

The remainder of the article elaborates the properties of set-complexes, defines important operations upon them, and applies the theory to the brief analysis of an atonal work, closing with some remarks on implications and problems.

1.3 Pitch-Sets: Ordered and Unordered

The term pitch-set designates any collection of unique pitches (or, more correctly, unique integers representing the residue classes modulo 12, called "pitch-classes" after Babbitt 1955). As customary, the members of a pitch-set will be denoted by the set of integers [0, 1, 2, . . . , 11]. *2 For convenient reference — and since no large-scale analyses are involved — we set 0 = middle C.

If the order of succession of the members of a pitch-set is considered significant we are dealing with an ordered set. For example, if we regard [0, 1, 2] as distinct from [0, 2, 1] we are concerned with ordered sets.

If the order of succession of the elements of a set is not of interest we are concerned with an unordered set. In this case we do not distinguish the set [0, 1, 2] from the set [0, 2, 1]. For example, when we consider simultaneous musical statements of pitch-sets we are usually concerned only with unordered sets. The present paper deals solely with pitch-sets which are unordered. *3

2.0 The Relation between Pitch-Set and Interval-Set

Often a set-theoretic problem in music becomes more meaningful when one considers its combinatory aspects. The term combinatory refers to such questions as: How many?; In how many ways?. Having defined unordered and ordered pitch-sets, we may now ask: How many pitch-sets of each type does the 12-pitch system contain?

The 12-pitch system contains 1,302,060,157 ordered sets. If we wish to study music from the standpoint of sets it is obviously advantageous to reduce that large number. This can be done by deciding to consider only unordered sets, a total of 2^{12} = 4096.*4 A further reduction is made possible when each pitch-set is represented by a set of differences: the set of integers which represents its interval-content. The following three sections (2.1, 2.2, and 2.3) are concerned with this reduction and attendant problems. To begin, we formalize the general relation between the set of 12 pitch-classes (PC's) and the set of interval-classes (IC's).

Let S be the set of PC integers [0, 1, 2, . . . 11], with ordinary addition and subtraction (mod 12), absolute value differences, symbolized $|d|$, and x, y, any two elements of S. Further, let $a = |x-y|$ and $b = x + y'$, where $y' = 12-y$ and is called the inverse of y. Then,

$$a \equiv b \text{ if and only if } a + b = 0 \pmod{12}. \qquad (2.0)$$

The letters a and b represent interval-classes, and the equivalence relation defined by (2.0) partitions S into 7 such classes, comprising the familiar intervals of the chromatic scale.*5 Example 2.0 presents a graph of the partition. By convention, the upper integer represents the class in each case. Since the class containing 0 is trivial we usually speak of 6 instead of 7 interval-classes.

It should be emphasized that the integers in Example 2.0 represent differences; that is, they are the result of a binary operation, subtraction. To understand what this means with respect to the relation between pitch-class and interval-class now under discussion, consider the set of distinct unordered pairs of elements of S which form differences belonging to the same interval-class. For example, in how many ways may the difference 5 be formed from pairs of elements in S? These pairs can be listed in two columns as follows,

0 - 5	5 - 10
1 - 6	6 - 11
2 - 7	7 - 0
3 - 8	8 - 1
4 - 9	9 - 2
	10 - 3
	11 - 4

ERRATUM

Two pages in the present issue (Winter, 1964) are incorrectly numbered and therefore affect a four-page sequence. Kindly make the following changes:

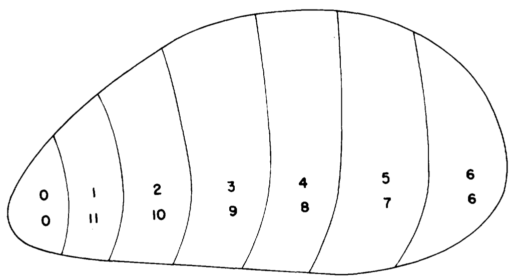
for p. 141 read p. 140

for p. 140 read p. 141

yielding a total of 12 unordered pairs. Each integer in S can appear once and only once in each column and row. A similar listing of pairs which yield the other interval-classes could be made to show that each IC has associated with it 12 pairs — with the exception of IC_6, which has only 6 unordered pairs. In this way the 66 unordered 2-element subsets of S are sorted into 6 interval-classes.

EXAMPLE 2.0

PARTITION FORMED BY EQUIVALENT INTERVAL-CLASSES



2.1 Interval-Content (Interval-Vector)

From the lists of PC pairs in Section 2.0 we concluded that each IC has associated with it 12 pairs, excepting IC_6, which has only 6 pairs. It is evident that we could then list the number of intervals of each class for the total (universal) set of 12 pitches as follows:

IC_1	IC_2	IC_3	IC_4	IC_5	IC_6
12	12	12	12	12	6

For convenience we represent this array in the form of a row vector and will call it an interval-vector.*6

[12 12 12 12 12 6]

The interval-vector in this case represents the interval-content of the universal set.

This leads naturally to the formation of interval-vectors for sets of fewer than 12 elements, an essential theoretical tool, to be discussed in some detail further on. For the moment, it will suffice to point out that since the ordering of a PC set has no effect upon its interval-content, any collection comprising all 12 PC's has the same vector as any other collection, that is, [12 12 12 12 12 6]. Similarly, it can be shown that every

11-note collection must have the same vector: [10 10 10 10 10 5]. Thus, from the standpoint of interval-content, we shall be most interested in the pitch-sets which contain from 2 to 10 elements, since these sets are differentiated with respect to interval-content.

To compute the interval-content of a given PC set we need only list the differences which each element forms with each other element. This can be done conveniently by computing the differences from left to right and displaying them in a "triangle". For example:

PC set	0	1	2	3
Triangle:		1	2	3
			1	2
				1

Interval-Vector: [321000]

Thus, the differences formed between PC_0 and the other elements are listed in the first row of the triangle; the differences formed by PC_1 are listed in the second row; the differences formed by PC_2 are listed in the third row.

The number of intervals N formed by a PC set S, where the cardinal number of S is M *7 (#(S) = M) is the same as the number of unordered 2-element subsets of S. Table 2.1 summarizes.

Table 2.1

PC set	IC Vector
M	N
1	0
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45
11	55
12	66

When the cardinal number of a PC set is increased by 1, the added note forms additional differences (intervals), one with each of the notes already present. For example, a 3-note set forms 3 differences; when a fourth note is added it then forms 3 new intervals, making a total of 6

In general, the value of any term n_k in column N is

$$n_k = \sum_{i=1}^{k-1} m_i \qquad (2.1)$$

where m is a term in column M.

2.2 Equivalent Pitch-Sets

We now define an equivalence relation for arbitrary pitch-sets A and B. Let v(A) represent the interval-vector of set A, and v(B) represent the interval vector of set B. Then,

$$A \equiv B \text{ if and only if } v(A)=v(B). \qquad (2.2)$$

The defined equivalence has the required reflexive, symmetric, and transitive properties. *5

In less formal language (2.2) states that two pitch-sets are equivalent if they have the same interval-content, and that if they are equivalent they have the same interval-content. The usefulness of this relation can hardly be overestimated. It is necessary, however, to mention two related limitations upon (2.2). First, the sign ≡ does not necessarily mean that A is the same set as B (although it might be). Second, we may know that two sets are equivalent, but not know how the two sets are operationally related. Indeed, as Lewin has shown, there are certain instances in which A = B, yet the two sets cannot be related on the basis of identity, inversion or transposition. (Lewin 1959 and Section 5.1)

3.0 The Distinct Interval-Vectors

The total number of interval-vectors in the 12-pitch system is 200. Of this number three are trivial:

[12 12 12 12 12 6] corresponding to the universal pitch-set,
[10 10 10 10 10 5] corresponding to the 12 distinct 11-note sets, and
[0 0 0 0 0 0] corresponding to the empty set and to the 12 distinct 1-note sets.

The remaining 197 vectors can also be placed in correspondence with the remaining 4070 (= 4096 - 26) unordered pitch sets, and this is done in Section 3.1.

Tables 3.00 through 3.08 contain complete lists of the 197 non-trivial interval vectors.*9 The vectors are numbered consecutively for convenient reference. In each hyphenated pair, the first number indicates cardinality while the second is an ordinal number.*9 Thus, 5-16 designates a set of 5-notes with vector [213211]. The PC numbers represent the normal order (after Martino 1961) of the set. In 23 cases the set has two reduced forms, marked Z.*10 Only the "optimal" normal order is given in the tables.

3.1 The Correspondence of Unordered Pitch-Sets and Interval-Vectors

For the subsequent and main portion of the article it is important to have a conception of the way in which the set of unordered pitch-sets corresponds to the set of interval vectors. The significance of the equivalence relation established in (2.2) then becomes evident.

Normally, each PC set has two basic and distinct forms: a prime form and an inversion. The relation between the two forms can be described as a mapping*8 from P into S, the universal set, under which each element p of P is associated with one and only one inverse element s = p' in S.

table

3.00

DISTINCT 2-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	01	[100000]	12
2	02	[010000]	12
3	03	[001000]	12
4	04	[000100]	12
5	05	[000010]	12
6	06	[000001]	6
			66

3.01

DISTINCT 10-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	0123456789	[988884]	12
2	01234567810	[898884]	12
3	01234567910	[889884]	12
4	01234568910	[888984]	12
5	01234578910	[888894]	12
6	01234678910	[888885]	6
			66

3.02

DISTINCT 3-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	012	[210000]	12
2	013	[111000]	24
3	014	[101100]	24
4	015	[100110]	24
5	016	[100011]	24
6	024	[020100]	12
7	025	[011010]	24
8	026	[010101]	24
9	027	[010020]	12
10	036	[002001]	12
11	037	[001110]	24
12	048	[000300]	4
			220

3.03

DISTINCT 9-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	012345678	[876663]	12
2	012345679	[777663]	24
3	012345689	[767763]	24
4	012345789	[766773]	24
5	012346789	[766674]	24
6	0123456810	[686763]	12
7	0123457910	[677673]	24
8	0123468910	[676764]	24
9	0123567810	[676683]	12
10	0123467910	[668664]	12
11	0123568910	[667773]	24
12	0124568910	[666963]	4
			220

3.04

DISTINCT 4-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	0123	[321000]	12
2	0124	[221100]	24
3	0134	[212100]	12
4	0125	[211110]	24
5	0126	[210111]	24
6	0127	[210021]	12
7	0145	[201210]	12
8	0156	[200121]	12
9	0167	[200022]	6
10	0235	[122010]	12
11	0135	[121110]	12
12	0236	[112101]	24
13	0136	[112011]	24
14	0146	[111120]	24
Z15	0137	[111111]	48
16	0157	[110121]	24
17	0347	[102210]	12
18	0147	[102111]	24
19	0148	[101310]	24
20	0158	[101220]	24
21	0246	[030201]	12
22	0247	[021120]	24
23	0257	[021030]	12
24	0248	[020301]	12
25	0268	[020202]	6
26	0358	[012120]	12
27	0258	[012111]	24
28	0369	[004002	3
			495

3.05

DISTINCT 8-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	01234567	[765442]	12
2	02345678	[665542]	24
3	01234569	[656542]	12
4	01345678	[655552]	24
5	01245678	[654553]	24
6	01235678	[654463]	12
7	01234589	[645652]	12
8	01234789	[644563]	12
9	01236789	[644464]	6
10	012345710	[566452]	12
11	02456789	[565552]	12
12	012346910	[556543]	24
13	02356789	[556453]	24
14	01245679	[555562]	24
Z15	01356789	[555553]	48
16	01246789	[554563]	24
17	012356910	[546652]	12
18	01346789	[546553]	24
19	01345789	[545752]	24
20	01245789	[545662]	24
21	012346810	[474643]	12
22	012357910	[465562]	24
23	012357810	[465472]	12
24	012456810	[464743]	12
25	012467810	[464644]	6
26	012457910	[456562]	12
27	012467910	[456553]	24
28	013467910	[448444]	3
			495

3.06

DISTINCT 5-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	01234	[432100]	12
2	01235	[332110]	24
3	01245	[322210]	24
4	01236	[322111]	24
5	01237	[321121]	24
6	01256	[311221]	24
7	01267	[310132]	24
8	02346	[232201]	12
9	01246	[231211]	24
10	01346	[223111]	24
11	02347	[222220]	24
Z12	01356	[222121]	36
13	01248	[221311]	24
14	01257	[221131]	24
15	01268	[220222]	12
16	01347	[213211]	24
Z17	01348	[212320]	24
Z18	01457	[212221]	48
19	01367	[212122]	24
20	01378	[211231]	24
21	01478	[202321]	24
22	01458	[202420]	12
23	02357	[132130]	24
24	01357	[131221]	24
25	02358	[123121]	24
26	02458	[122311]	24
27	01358	[122230]	24
28	02368	[122212]	24
29	01368	[122131]	24
30	01468	[121321]	24
31	01369	[114112]	24
32	01469	[113221]	24
33	02468	[040402]	12
34	02469	[032221]	12
35	02479	[032140]	12
			792

3.07

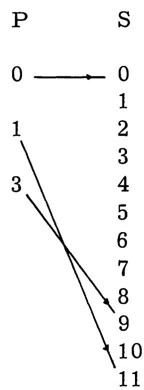
DISTINCT 7-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	0123456	[654321]	12
2	0234567	[554331]	24
3	0345678	[544431]	24
4	0134567	[544332]	24
5	C124567	[543342]	24
6	0145678	[533442]	24
7	0125678	[532353]	24
8	0234568	[454442]	12
9	0245678	[453432]	24
10	0234569	[445332]	24
11	0134568	[444441]	24
Z12	0123568	[444342]	36
13	0234678	[443532]	24
14	0124678	[443352]	24
15	0135678	[442443]	12
16	0134569	[435432]	24
Z17	0134578	[434541]	24
Z18	0124578	[434442]	48
19	0123689	[434343]	24
20	0125789	[433452]	24
21	0145789	[424641]	24
22	0125689	[424542]	12
23	0245679	[354351]	24
24	0246789	[353442]	24
25	0235679	[345342]	24
26	0134579	[344532]	24
27	01257910	[344451]	24
28	0135679	[344433]	24
29	0235789	[344352]	24
30	0135789	[343542]	24
31	0235689	[336333]	24
32	0134689	[335442]	24
33	01246810	[262623]	12
34	01346810	[254442]	12
35	01356810	[254361]	12
			792

table 3.08

DISTINCT 6-NOTE SETS

Set Number	Normal Order	Interval Vector	Number of Pitch-Sets
1	012345	[543210]	12
2	012346	[443211]	24
Z3	012356	[433221]	48
Z4	012456	[432321]	24
5	012367	[422232]	24
Z6	012567	[421242]	24
7	012678	[420243]	6
8	023457	[343230]	12
9	012357	[342231]	24
Z10	013457	[333321]	48
Z11	012457	[333231]	48
Z12	013567	[332232]	48
Z13	013467	[324222]	24
14	013458	[323430]	24
15	012458	[323421]	24
16	014568	[322431]	24
Z17	012478	[322332]	48
18	012578	[322242]	24
Z19	013478	[313431]	48
20	014589	[303630]	4
21	023468	[242412]	24
22	012468	[241422]	24
Z23	023568	[234222]	24
Z24	013468	[233331]	48
Z25	013568	[233241]	48
Z26	013578	[232341]	24
27	013469	[225222]	24
Z28	013569	[224322]	24
Z29	013689	[224232]	36
30	013679	[224223]	12
31	013589	[223431]	24
32	024579	[143250]	12
33	023579	[143241]	24
34	013579	[142422]	24
35	0246810	[060603]	2
			924



The set [0, 11, 9] into which [0, 1, 3] maps is the familiar inversion form. When P is mapped into S under transposition (defined as the addition mod 12 of any integer k in S to every integer p of P) we have a total of 12 transposed forms of P. Similarly, I maps under transposition to produce 12 forms. Thus from the two basic forms a total of 24 distinct pitch-sets is produced, all with the same interval-content, all corresponding to a single interval vector [111000].

This does not mean that for each vector there are 24 unordered pitch-sets, for in 76 cases the inversion (and its transpositions) is not distinct from some form of the prime, and in addition, there are 13 special partitions of the integer 12 (out of a total of 77) which reduce the number of distinct transpositions of the prime. *9 And finally, there are the 19 Z-forms, which yield more than 24 pitch-sets for a single vector. *10 The correct distribution of unordered pitch-sets with respect to interval-vectors is shown in the right-hand column of each vector table (Tables 3.00-3.08). *11 The total number of unordered pitch-sets corresponding to the vectors of cardinality n is the same as the number of unordered subsets of cardinality n contained in a set of 12 elements. *4 The total number of unordered PC sets, including the trivial cases mentioned in Section 3.0, is 4096, as noted earlier. *4 It should be observed that the correspondence of interval vectors to unordered PC sets could be regarded as a one-to-many mapping under the equivalence relation defined in (2.2).

3.2 Similarity Relations for Vectors of the Same Cardinality

To define an appropriate, musically meaningful similarity relation for any two sets of different cardinality is a task beyond the scope of the present paper. (A similarity relation for complementary sets is discussed in Sections 6.1 and 8.2.) It is possible, however, to define a simple yet effective criterion for similarity in the case of any two sets of the same cardinality.

Two interval-vectors are maximally similar if they share four IC's in the same number. For example,

4-4 [211110]
4-7 [201210]

are in the second-order maximum similarity relation. They intersect in four classes: IC_1, IC_3, IC_5 and IC_6, but the first vector is an arrangement of the collection 0, 1, 1, 1, 1, 2, whereas the second is an arrangement of the collection 0, 0, 1, 1, 2, 2.

The musical significance of this distinction between first and second-order maximum similarity will become evident in Section 9.0, where a brief analysis of a composition is undertaken using techniques derived from the concepts being developed here.

It follows that minimum similarity between sets of the same cardinality means that they share no IC's in the same number. For example,

4-4 [211110]

is in the minimum similarity relation to

4-21 [030201]
4-24 [020301]
4-25 [020202]
4-28 [004002]

and only to these.

The defined similarity relations are reflexive and symmetric *5 but not transitive. Therefore they are not equivalence relations.

4.0 Unordered Pitch-Set Inclusion

We can now begin to deal directly with the topic of the present paper: set-theoretic inclusion. First, a general definition:

A set B is said to be contained in a set A or to be a subset of A if every element of B is also an element of A. If, in fact, B is the same set as A, B is called an improper subset of A. This apparently trivial case is very useful, as we shall see.

The five possible inclusion relations for arbitrary sets A and B are represented below in the usual symbols. Note that each relation can be stated in two ways.

		Read
1	$B \subset A$	B is a subset of A
	$A \supset B$	A is a superset of B
2	$B \subseteq A$	B is a subset of A, or B is equivalent to A
	$A \supseteq B$	A is a superset of B, or A is equivalent to B
3	$A \subset B$	A is a subset of B
	$B \supset A$	B is a superset of A
4	$A \subseteq B$	A is a subset of B, or A is equivalent to B
	$B \supseteq A$	B is a superset of A, or B is equivalent to A
5	$A \not\subset B$	A and B are incomparable
	$B \not\supset A$	B and A are incomparable *12

Since we shall not be concerned with the ordering of a particular subset with respect to the ordering of the set in which it is contained (superset) or with respect to other sets of the same cardinality we are restricted to unordered pitch-set inclusion.

4.1 Combinatory Aspects of Unordered Pitch-Set Inclusion

The initial question deals with cardinalities. Given a set S_a with $\#(S_a) = n$, in how many different cardinalities may it be contained $\#(\subset)$ and how many different cardinalities may it contain $\#(\supset)$? Table 4.10 provides the answer to this elementary but nonetheless significant question.

TABLE 4.10

n	$\#(\subset)$	$\#(\supset)$	n	$\#(\subset)$	$\#(\supset)$
3	9	1	8	4	6
4	8	2	9	3	7
5	7	3	10	2	8
6	6	4	11	1	9
7	5	5	12	0	10

In connection with Table 4.10 observe that for any n, the two values sum to 10. By no means does this imply that any set has as many inclusion relations as any other. Observe also that if the unit set and the null set were admitted as values the sum would be 12 in each case.

We next ask how many unordered subsets does a set S, with $\#(S) = n$, contain? If we let $(\$)$ represent the number of subsets of S, then

$$\#(\$) = 2^n. \tag{4.10}$$

The number 2^n is the power set, mentioned earlier.

An example will point up the need to limit (4.0). Let us take a pitch-set S with $\#(S) = 3$: [0, 1, 2]. The subsets of S are 8 $(=2^3)$ in number:

$$S_1 = [0]$$
$$S_2 = [1]$$
$$S_3 = [2]$$
$$S_4 = [0, 1]$$
$$S_5 = [0, 2]$$
$$S_6 = [1, 2]$$
$$S_7 = [0, 1, 2]$$
$$S_8 = \emptyset$$

It should be evident that the unit subsets, S_1, S_2, S_3, the improper subset S_7, and the null set $\emptyset$ are musically insignificant. The number of musically significant unordered subsets of a set S with $\#(S) = n$ therefore is given by

$$\#(\$) = 2^n - (n+2). \tag{4.11}$$

A more complicated answer is required for the last question: In how many different unordered supersets N of cardinality n may an unordered pitch-set of cardinality m be contained? The answer is given by

$$N = C(m', n-m) \tag{4.12}$$

C(m', n-m) is the number of unordered sets of cardinality n-m which can be contained in a set of cardinality m', where m' is the inverse element associated with m (see Section 2.0). Table 4.11 supplies the answer for any n and m.*13 The appropriate intersection of row and column gives the number of sets of cardinality n which can contain a set of cardinality m.

Note that Table 4.11 represents only pitch-set inclusion. Some of the sets may be equivalent — indeed, must be since the number of inclusion relations indicated exceeds the number of distinct interval-sets in several cases.

Table 4.11

Maximum Number of Unordered Supersets of Cardinality n in which a Set of Cardinality m may be contained

m \ n	3	4	5	6	7	8	9	10	11	Totals	
2	10	45	120	210	252	210	120	45	10	1022	$= 2^{10}-2$
3		9	36	84	126	126	84	36	9	510	$= 2^{9}-2$
4			8	28	56	70	56	28	8	254	$= 2^{8}-2$
5				7	21	35	35	21	7	126	$= 2^{7}-2$
6					6	15	20	15	6	62	$= 2^{6}-2$
7						5	10	10	5	30	$= 1^{5}-2$
8							4	6	4	14	$= 2^{4}-2$
9								3	3	6	$= 2^{3}-2$
10									2	2	$= 2^{2}-2$

5.0 Pitch-Set Complementation

The selection of a pitch-set M from the universal set S, $\#(M) \geq \#(S)$, effects a partition of S into two disjoint subsets. Let us call the other subset N. Now, N comprises all the elements which are in S but which are not in M. For example,

if M = [1, 3, 5, 7, 9, 11]
then N = [0, 2, 4, 6, 8, 10].

In this situation M is called the complement of N with respect to S, and N is called the complement of M with respect to S. More simply, we call M the complement of N and N the complement of M, or we say that M and N are complementary. The conventional notation for the partition formed by complementary sets is

(M N), with # (M) ≥#(N).

The complement of M is also symbolized $\overline{M}$. Thus, in our example above, $\overline{M}$ = N.

5.1 Complement-Mapping

In order to approach the central "set-complex" property described in Section 7.0 it is necessary to understand the relation between set-theoretic complementation and set-theoretic inclusion. The present section therefore shows in some detail that any set N can be mapped into (or onto in the case of hexachords) its complement M(M≥N). *8 Under such a mapping N thus becomes a subset of M, with the general result that there is a clearly defined relation between set-theoretic complementation and set-theoretic inclusion. In subsequent sections we will examine that relation and its musical implications. The reader who finds arithmetic and symbolic material tedious would do well to skip ahead to Section 6.0, accepting on faith that complement-mapping is possible in all cases.

The discussion of complement-mapping divides naturally into two parts: the four cases involving into-mappings and the single case of onto-mapping (hexachords).

To determine an effective mapping, it is obviously insufficient merely to set up an arbitrary matching or correspondence of pitch-numbers. The subset N of M into which N is transformed must in fact be an equivalent set according to (2.2).

Perhaps the simplest mapping which preserves interval-content is effected by transposition. If this mapping is described (in the usual way) as the addition (mod 12) of some integer k to each element of N, it can be shown that the only possible values of k are those of the empty interval-classes in the interval-vector of N. Therefore, if the vector contains at least one representative of each class the mapping under transposition is not effective.

A solution, and one which accounts for the mapping of all sets of 2, 3, 4, and 5 notes (with one exception in the latter case) is found in the double mapping IT: inversion followed by transposition. Since we know that interval-content is preserved under the I-mapping, we have only to determine the requirements on k, the transposition number for the subsequent T-mapping. Let us consider an example of an effective mapping under I followed by T. For convenience, only the relevant subset of the complement is listed, not the entire complement.

S		I		T (k=9)
0	→	0	→	9
1	→	11	→	8
2	→	10	→	7
5	→	7	→	4
6	→	6	→	3

In the example the value of k can only be 9, as indicated. The mathematical basis of the solution to the general case is as follows.

Let s_i, s_j be any two elements of S. (Possibly $s_i = s_j$)

Let s_i' be that element of I which is the inverse element associated with s_i.

Let t_i be any element of T (the image of s'_i under T). Then,

$s_i' + k = t_i$ by definition of T-mapping

and $s_i + t_i = k$ since $s_i + s'_i = 0 \pmod{12}$.

Thus, if

$x = s_i + s_j$, where x is an element of the set [0, 1, 2, . . . , 11], then x is not a value of k; because, if

$$s_i + t_i = k$$

and

$$s_i + s_j = k$$

then

$$t_i = s_j \quad (\text{and } t_j = s_i),$$

which contradicts the requirement of effective mapping of S into T, the complement of S.

As a result of this negative requirement, k can have any value except one which represents a sum of s_i and s_j in S.

We have shown the requirement on values of k which permit effective complement-mapping, where the cardinality of the set being mapped is 2, 3, 4, or 5. It remains to be shown that every such set contains at least one value of k.

Observe that there are eleven possible effective values of k (k = 1, k = 2, k = 3, . . . k = 11). Observe, further, that the cardinal number of the set of sums of a set of n elements is

$$n + (n-1) + (n-2) + \ldots + (n - n)$$

If $\#(H_n)$ designates the cardinality of the set of sums of a set of n elements, then

$$\#(H_2) = 3$$
$$\#(H_3) = 6$$
$$\#(H_4) = 10$$

and therefore these sets cannot exclude all values of k, hence must permit at least one value.

Since $\#(H_5) = 15$ it is possible that some 5-note set may not admit a value of k; that is, the sums may exhaust the set of transposition numbers. It can be shown that only one 5-note pitch-set has this property, 5-12: [0, 1, 3, 5, 6]. *14 Two mappings can be defined which map this set into its complement:

(1) $S \; \alpha \; I \; \beta \; Y_1$

$0 \rightarrow 0 \rightarrow 4$
$1 \rightarrow 11 \rightarrow 11$
$3 \rightarrow 9 \rightarrow 9$
$5 \rightarrow 7 \rightarrow 7$
$6 \rightarrow 6 \rightarrow 10$

(2) $S \; \alpha \; I \; \beta \; Y_2$

$0 \rightarrow 0 \rightarrow 8$
$1 \rightarrow 11 \rightarrow 11$
$3 \rightarrow 9 \rightarrow 9$
$5 \rightarrow 7 \rightarrow 7$
$6 \rightarrow 6 \rightarrow 2$

Observe that the basis of both mappings is complementation. The first mapping α produces, as before, a set I of the same interval-content as S, but two elements of I are not elements of the complement of S. The second mapping β supplies the required transformation: $0 \longrightarrow 4$, $6 \longrightarrow 10$. As shown, the mapping may also carry 0 onto 8 and 6 onto 2, while, as before, the other elements remain stationary (fixed) from the first mapping (identity mapping).

Finally, let us consider the "onto" complement-mapping in the case of hexachords. In Martino 1961, Table I, complement-mapping of hexachords is displayed in terms of the defined operations of the 12-tone "system" and Babbitt's combinatorial principle. Eight of the sets listed do not possess any of the three special combinatorial properties and are called by Martino R types: An R type hexachord maps onto its complement trivially. Hexachord-mapping may be defined in other ways and without reference to 12-tone theory or practice. For example, the two successive mappings described below will map any hexachord onto its complements.

(1) Let S represent any hexachord. Then, $\alpha : S \rightarrow I$ by the rule, for every $s \in S$, $\alpha(s) = 12-s$

(or, more simply $\alpha(s) = s'$)

(2) The second mapping maps I onto V by the rules,

(a) for every $i \in I$, $\beta(i) = i$ if $i \notin S$, but

(b) if $I \in S$, $\beta(i) = v$

and $\beta(i') = v'$.

In this way, the elements of I which are not elements of the complement of S are paired off as inverses and matched with inverse pairs in the complement of S to complete the mapping. For example:

S	α	I	β	V
0	$\rightarrow$	0	$\rightarrow$	6
1	$\rightarrow$	11	$\rightarrow$	11
3	$\rightarrow$	9	$\rightarrow$	9
4	$\rightarrow$	8	$\rightarrow$	2
7	$\rightarrow$	5	$\rightarrow$	5
8	$\rightarrow$	4	$\rightarrow$	10

Thus any set N can be mapped into or onto its complement M ($M \geq N$). Under such mapping N becomes a subset of M, and we have a clearly defined relation, one with significant musical implications, between set-theoretic complementation and set-theoretic inclusion.

6.0 Interval-Set Inclusion

A pitch-set N is said to be an intervallic subset of a set M if N can be mapped into (or onto) M under some mapping. This notion of intervallic inclusion makes it possible to deal with the properties of the inclusion relation at a further level of abstraction, and renders description of the precise mapping of N unnecessary. Indeed, from the standpoint of general theory we are more interested in the "potential" pitch-set inclusion relation than in whether or not it exists in a particular case. An example will make this clear.

N = [0, 1, 2] M= [4, 5, 6, 7, 8]

N is an intervallic subset of M, since N can be mapped into M under T, where k = 4 (or k=5 or 6)

Thus, the interval-set of N, 3-1, is a subset of the interval-set of M, 5-1.

Observe that if N is an intervallic subset of M, then M must contain at least the same interval-classes as N and in the same cardinal number. This condition is necessary, but not sufficient. For example, 3-6 $\not\subset$ 4-25, yet 4-25 fulfills the condition described.

Henceforth inclusion is to be understood as "intervallic" inclusion, with literal pitch-set inclusion as a special case.

6.1 Complementary Interval-Sets

Because of the relation between inclusion and complementation explained in Section 5.1 we are particularly interested in the intervallic properties of a set and its complement. This can now be discussed in terms of interval-vectors.

Complementary interval-vectors (vectors of complementary PC sets) have the following remarkable relation: *15

Suppose that we have any two complementary pitch-sets M and N with # (M) $\geq$ #(N), and their corresponding interval-vectors, v(M) and v(N). Then

if the cardinal number of any IC_i in v(N), where i=1, 2, 3, 4, or 5, is k, in v(M) it is (M - N) + k.

For i = 6, set (M -N) = (M-N) / 2.

By arranging the differences (M-N) in vector form we obtain a characteristic "difference"-vector for each pair of complementary set-magnitudes. These are summarized in the table below:

Complementary Set Magnitudes	Difference-Vectors
6 - 6 = 0	[000000]
5 - 7 = 2	[222221]
4 - 8 = 4	[444442]
3 - 9 = 6	[666663]
2 - 10 = 8	[888884]

The difference-vectors are useful in a number of ways. For example, given the vector of any set N, one can compute the vector of $\bar{N}$ directly (without knowing the pitch-set $\bar{N}$) by adding or subtracting the approppropriate difference-vector.

For Example,

5-6	[3 1 1 2 2 1]
Difference Vector	[2 2 2 2 2 1]
7-6	[5 3 3 4 4 2]

The musical significance of the difference-vector resides in the fact that it represents a constant difference for all classes, a scaling factor. To use our previous sets, this means that 5-6 has the same relative distribution of intervals as 7-6, its complement. The difference between the two sets lies in cardinality or "scale", not in distribution. Seen in this light it is evident that complementary sets are very similar in intervallic structure, despite the difference in cardinal number — an interpretation which is important for subsequent portions of the article, especially for Section 8.2.

7.0 Basic Properties of the Inclusion Relation

In Section 4.0 we considered the five possible ways in which two sets may be associated by the inclusion relation. Here attention is called to the "incomparable" situation. Consider, for example, these two pitch-sets:

6-35: [0, 2, 4, 6, 8, 10]
10-1: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]

It is impossible to define a mapping which will map 6-35 into 10-1, while retaining interval-content. To say this in another way, there is no subset of 10-1 such that its interval vector is the same as the interval vector of 6-35.

Therefore, sets M and N may or may not be in the inclusion relation, and the fact that $\#(M) > \#(N)$ by no means implies that $M \supset N$. In Section 4.1 we examined combinatory aspects of the inclusion relation and were able to answer questions involving "how many?". Now we will begin to answer the more interesting question, "which?". To do this we first consider three interrelated basic properties of the inclusion relation. The last of these provides the conceptual basis for the construction of set-complexes: sets of sets related by inclusion.

Property 1: Transitive Property.

In ordinary language, the transitive property means that if a set M contains a set N_1, and N_1 contains a set N_2, then M contains N_2 by virtue of its inclusion in N_1. In this case, however, we are interested to extend transitivity beyond one subset, to include all possible subsets, thus forming a chain of sets related by inclusion. In symbols ($M \geq N$ in all cases):

$$\text{If } M \supset N_1, \text{ and } N_1 \supset N_2,\ N_2 \supset N_3,\ N_3 \supset \ldots \supset N_n \quad (7.0)$$
$$\text{then } M \supset N_2,\ M \supset N_3,\ M \supset \ldots \supset N_n.$$

An equivalent statement ("dual" statement) of this and the other properties may be made by exchanging $\supset$ for $\subset$ and reversing the relative positions of M and N. (Cf. Section 4.0)

Property 2: "Dual-Complement" Property.

This property associates each set with its complement, as follows:

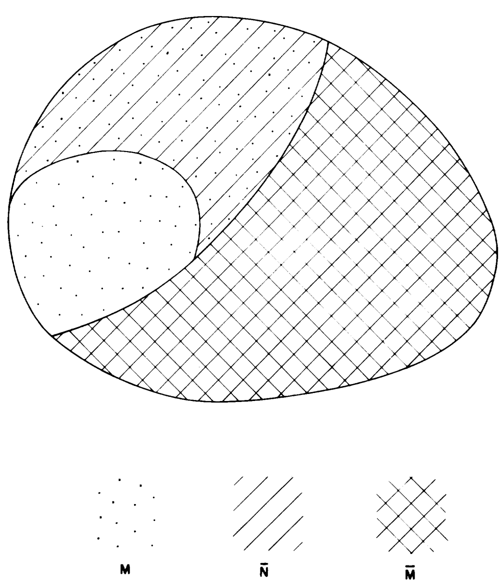
$$\text{If } M \supset N_i, \text{ then } \overline{M} \subset \overline{N_i}, \text{ for all } N_i \text{ such that } N_i \subset M. \quad (7.01)$$

Example 7.00 presents a set-theoretic Venn diagram illustrating this situation. It should be emphasized that by virtue of the transitive property, the "dual-complement" property extends to every subset of a given set and the complement of every subset, as indicated in the statement above.

example

7.00

VENN DIAGRAM OF
INCLUSION PROPERTY 2



Property 3: "Set-Complex" Property.

By extending Properties 1 and 2 to the special case where N = $\overline{M}$ (and N is thus unique) we derive the set-complex property, a property which is central to the present paper. Before stating the property, recall that if N = M, then $N \subset M$ under some mapping (Section 5.1).

(7.02)

If we have two sets M, N, such that $N = \overline{M}$ $(M \geq N)$, then:

For every set M_i, such that $M_i \subseteq M$,

there is a set N_i such that $N_i \supseteq N$,

and $M_i = \overline{N_i}$.

The dual statement is:

For every N_i, such that $N_i \supseteq N$, (7.03)

there is a set M_i such that $M_i \subseteq M$,

and $N_i = \overline{M_i}$.

By property 3 it is possible to construct about each of the distinct sets and its complement a symmetrical array of sets to which they are in the inclusion relation. Such an array will be called a set-complex, and the complementary pair about which the complex is arranged will be called the reference-pair, M/N.

Example 7.01 presents the total set-complex about sets 4-28 and 8-28. Observe that the total complex comprises two subcomplexes, K(M/N) [Read: the complex about M, N.] and K(N/M). These subcomplexes correspond to (7.02) and (7.03), respectively. Observe that K(M/N) exhausts the 2-, 3-, and 4-note sets and their complements, while K(N/M) excludes 5-, 6-, and 7-note sets (and their complements). Compare Tables 8.0 and 8.2.

A complete roster of set-complexes for the 12-pitch system has been compiled by the author with the aid of a computer. Space limitations make it impossible to present this list in its entirety but complexes thought to be of special interest will be displayed in connection with the discussion of certain general characteristics. *21

example

7.01

			5-34			
			5-32			
			5-31			
			5-29			
			5-28			
			5-26			
			5-25			
			5-22			
			5-19			
		6-30	5-18			
		6-29	5-16			
		6-28	5-12			
		6-27	5-10			
		6-23	5-8	all	all	all
	8-28 $\supset$ 7-31	6-13	5-4	4	3	2
K(M/N)						
	4-28 $\subset$ 5-31	6-13	7-4	8	9	10
		6-23	7-8	all	all	all
		6-27	7-10			
		6-28	7-12			
		6-29	7-16			
		6-30	7-18			
			7-19			
			7-22			
			7-25			
			7-26			
			7-28			
			7-29			
			7-31			
			7-32			
			7-34			

				2-6
	4-28 $\supseteq$	4-28	3-10	2-3
K(N/M)				
	8-28 $\subseteq$	8-28	9-10	10-3
				10-6

8.0 Combinatory Aspects of the Set Complexes.

This section and the four which follow consider certain interesting characteristics of the set-complexes. The notion of set-complexes raises a number of provocative questions concerning their properties, and it is necessary to limit the present discussion to what are judged to be the more basic of these

We begin with elementary combinatory information. Table 8.0 gives the exhaustive inclusions for each cardinality, pointing up the fact that with increase of difference in cardinality, inclusion becomes less distinctive. If, for example, a particular 3-note set is a subset of every 7-note set and, indeed, all three-note sets (with the exception of 3-12) are contained in each 7-note set, there appears to be little interest in considering unordered supersets of 3-note sets. (Section 8.2 presents a modification of this view, however.)

In connection with Table 8.0 observe that only hexachords are not exhaustive with respect either to $\subset$ or $\supset$.

Table 8.01 lists the sets for each cardinality of N/M which have the greatest and least number of inclusion relations. For example 3-2 contains 3 subsets and 9-2 is contained in 3 supersets; 3-2 is contained in 151 supersets and 9-2 contains 151 subsets.

The sets with the least number of inclusion relations are precisely those sets which have the greatest possible number of intervals concentrated in the fewest possible interval-classes. Conversely, the sets with the greatest number of inclusion relations have the maximum number of interval-classes with approximately equal distribution in each IC. It is not difficult to understand why this is the case. Observe that the sets in the right-hand column are the familiar augmented triad, diminished 7th chord, whole-tone pentachord, and whole-tone hexachord. *9

8.1 Interval-Content of the Reference Pair M/N

In order to interpret the relation between a member of the reference pair and any set in the complex it is essential to understand the particular characteristics of the former. These characteristics can be described in terms of interval-content. Therefore, we first take up certain elementary and general aspects of the interval-vector.

table 8.0

EXHAUSTIVE INCLUSION RELATIONS

Any (Every) Set of Cardinality		Any (Every) Set of Cardinality			
2	⊂	7	8	9	10
10	⊃	5	4	3	2
3	⊂	7	8	9	10
9	⊃	5	4	3	2
4	⊂		8	9	10
8	⊃		4	3	2
5	⊂			9	10
7	⊃			3	2

8.01

SETS HAVING GREATEST AND LEAST NUMBERS OF INCLUSION RELATIONS

		Greatest			Least
3-2	[111000]	3	3-12	[000300]	1
9-2	[777663]	151	9-12	[666963]	106
4-15	[111111]	12	4-28	[004002]	3
8-15	[555553]	123	8-28	[448444]	68
5-18	[212221]	25	5-33	[040402]	9
7-18	[434442]	89	7-33	[262624]	44
6-10	[333321]	52	6-35	[060603]	35

table 8.10

MAXIMUM AND MINIMUM NUMBERS OF INTERVALS IN EACH CLASS FOR SETS OF 2 TO 12 NOTES

Cardinal Number of Set	Interval-Class 1 Max	1 Min	2 Max	2 Min	3 Max	3 Min	4 Max	4 Min	5 Max	5 Min	6 Max	6 Min
2	1	0	1	0	1	0	1	0	1	0	1	0
3	2	0	2	0	2	0	3	0	2	0	1	0
4	3	0	3	0	4	0	3	0	3	0	2	0
5	4	0	4	0	4	0	4	1	4	0	2	0
6	5	0	6	0	5	0	6	2	5	0	3	0
7	6	2	6	2	6	2	6	3	6	2	3	1
8	7	4	7	4	8	4	7	4	7	4	4	2
9	8	6	8	6	8	6	9	6	8	6	4	3
10	9	8	9	8	9	8	9	7	9	7	5	4
11	10	10	10	10	10	10	10	10	10	10	5	5
12	12	12	12	12	12	12	12	12	12	12	6	6

table

8.11

PARTITIONS OF INTERVAL-CLASSES INTO DISJOINT MAXIMUM AND MINIMUM SUBSETS

Max	Min
4	1, 2, 3, 5, 6
1, 2	3, 4, 5, 6
3, 6	1, 2, 4, 5
1, 5, 6	2, 3, 4
2, 3, 5	1, 4, 6
2, 4, 6	1, 3, 5

example

8.21

K(6-35)

			8-25	9-12	10-6
			8-24	9-8	10-4
	$\subset$	7-33	8-21	9-6	10-2
6-35					
	$\supset$	5-33	4-21	3-6	10-2
			4-24	3-8	10-4
			4-25	3-12	10-6

Let us consider the maximum number and the minimum number of intervals in each class for sets of each cardinality. These numbers are summarized in Table 8.10. The table indicates a number of remarkable intervallic properties dependent upon the size of the set. Observe, for example, that every 5-note set must contain at least one IC_4, while every 6-note set must contain at least two IC_4. Beginning with sets of cardinality 7 the max and min gradually converge until in sets of cardinalities 11 and 12 they are equal. These sets have a fixed interval-content, as noted earlier in Section 2.1.

Perhaps of more interest is the reciprocal relation shown in Table 8.11 between maximized IC's and minimized IC's in sets which have a max-min characteristic. The table is read from left to right or right to left. For example, the first row reads: "if IC_4 is maximized, then IC_1, IC_2, IC_3, IC_5, and IC_6 are minimized". Observe that in each case there is formed a partition of the six interval-classes and that the partition-types correspond exactly to the possible 2-part partitions of the integer 6 into 2 positive integers. The number of each partition-type is the same as the number of elements in the smallest subset in each case. Here again we see that IC_4 has a unique characteristic: unlike the other even interval-classes it is never in the max subset with an odd interval-class.

8.2 Relations between Sets in a Complex: the Most Significant Subcomplex K_s

Example 8.2 shows the complex about 5-33/7-33, the smallest complex involving those cardinalities (Table 8.01). For many total complexes the number of sets involved is so large as to lose significance. For example, K(3-2/7-2) contains 154 distinct sets. Therefore it would be valuable if we could delimit the total complex so as to define a "most significant" subcomplex, containing those sets and only those sets which are maximally similar to the reference pair. Inspection of the total complex about any hexachordal reference pair suggests a way in which such a significant subcomplex may be discovered for pairs of all magnitudes.

Consider, for example, the total complex about 6-35 (Example 8.21). Each set in the complex is maximally similar to the reference pair. (in the case of the hexachord, M = N, of course.) Each subset, moreover, corresponds to a complementary superset which has an equivalent distribution of intervals and therefore maximally similar intervallic structure.

table

8.2 TOTAL COMPLEX

K(M/N) HERE M IS 7-33
K(N/M) N IS 5-33

				8-27		
				8-25		
				8-24		
			7-34	8-22		
			7-33	8-21		
			7-30	8-19		
			7-28	8-16		
			7-24	8-15		
		6-35	7-15	8-12		
		6-34	7-13	8-11		
		6-22	7-9	8-5		
5-33 $\subseteq$	5-33	6-21	7-8	8-2	all 9	all 10
7-33 $\supseteq$	7-33	6-21	5-8	4-2	all 3	all 2
		6-22	5-9	4-5		
		6-34	5-13	4-11		
		6-35	5-24	4-12		
			5-28	4-15		
			5-30	4-16		
			5-33	4-19		
			5-34	4-21		
				4-22		
				4-24		
				4-25		
				4-27		

				8-25	9-12	10-6
				8-24	9-8	10-4
7-33 $\subseteq$			7-33	8-21	9-6	10-2
5-33 $\supseteq$			5-33	4-21	3-6	2-2
				4-24	3-8	2-4
				4-25	3-12	2-6

8.23 THE MOST SIGNIFICANT SUBCOMPLEX K_S(N)

HERE N IS 5-33

		6-35				
		6-34		8-25	9-12	10-6
		6-22		8-24	9-8	10-4
5-33 $\subseteq$	5-33	6-21	7-33	8-21	9-6	10-2
7-33 $\supseteq$	7-33	6-21	5-33	4-21	3-6	2-2
		6-22		4-24	3-8	2-4
		6-34		4-25	3-12	2-6
		6-35				

Recall, further, that there is no exhaustive inclusion in the case of hexachords, which means that the hexachordal complexes are more distinctive that those of other cardinalities.

To obtain the subcomplex of maximally similar members of a total complex about M and N we have only to observe that, as in the case of the hexachordal complex, the most similar supersets of N are those whose complements are subsets of M. Moreover, the most similar sets are those which are both supersets of N and subsets of M. The models in Table 8.22 illustrate the relation of members of a complex for each pair of complementary cardinal numbers. For instance, where (N) = 5 and (M) = 7, sets of three cardinalities (t, 6, and 7) may be both subsets and supersets.

It is now not difficult to define in set-theoretic terms a subcomplex K_s, the most significant subcomplex of a total inclusion set-complex. Observe that each pair of subcomplexes K(M/N) and K(N/M) contains at most three sets of subsets having the same cardinal number (see Table 8.22). Let us call these sets of subsets P_i, Q_i, and S_i, where the subscript denotes cardinality. The most significant subcomplex K_s about M and N is then defined as

$$K_s(M/N) = P_i Q_i S_i \qquad (8.2)$$

where i = 2, 3, 4, . . ., 10.

To simplify notation we shall write only K_s(N) to designate the most significant subcomplex about M/N.

Example 8.23 shows K_s(5-33). When this is compared with the total complex shown in Example 8.20 the value of K_s as a limiting factor becomes evident. Set 5-33, the familar whole-tone pentad, has the following characteristics: IC_2, IC_4, and IC_6 are maximized, and the odd IC's are minimized. Now observe that whereas the total complex contains all 2-note and 10-note sets, K_s contains only those which feature the maximized IC's. The other subsets of K_s also preserve the characteristics of the reference-pair, and all such sets which possess those characteristics are contained in K_s.

8.3 Set-Theoretic Operations for Complexes

Here we define set-theoretic operations for complexes and consider briefly some of the interesting implications of those relations.

table

8.22

THE RELATION OF CARDINALITIES
IN K(M/N) AND K(N/M)

6-a $\subseteq$	6	7	8	9	10
6-a $\supseteq$	6	7	8	9	10
6-a $\subseteq$	6	7	8	9	10
6-a $\supseteq$	6	7	8	9	10

5-a $\subseteq$	5	6	7	8	9	10
7-a $\supseteq$	7	6	5	4	3	2
7-a $\subseteq$			7	8	9	10
5-a $\supseteq$			5	4	3	2

4-a $\subseteq$	4	5	6	7	8	9	10
8-a $\supseteq$	8	7	6	5	4	3	2
8-a $\subseteq$					8	9	10
4-a $\supseteq$					4	3	2

3-a $\subseteq$	3	4	5	6	7	8	9	10
9-a $\supseteq$	9	8	7	6	5	4	3	2
9-a $\subseteq$							9	10
3-a $\supseteq$							3	2

2-a $\subseteq$	2	3	4	5	6	7	8	9	10
10-a $\supseteq$	10	9	8	7	6	5	4	3	2
10-a $\subseteq$									10
2-a $\supseteq$									2

Intersection*15

One possible extension of set-theoretic intersection to set-complexes is as follows. Let $\sum$ designate a set of set-complexes K_S, where arbitrary M_i has the same cardinal number as arbitrary M_j and M_i is maximally similar to M_j (Section 3.2). By the "intersection of the sets of $\sum$ " is meant the subset consisting of all the sets which belong to every set-complex K_{S_i} in the set of set-complexes $\sum$. This special subset is symbolized by

$$\bigcap_{K_S \subset \sum} K_S \begin{pmatrix} N \\ N \end{pmatrix}$$

It is interesting to note that the complexes about certain maximally similar sets M do not intersect. For example, although maximally similar, 5-27 [122230] and 5-28 [122212] share no set in their respective significant complexes. This points up the uniqueness of the set-complexes, or, to be more specific, the uniqueness of the most significant subcomplexes of the total complexes.

Union

Set-theoretic union can be defined following the format for intersection above. The "union of the sets of $\sum$ " is symbolized by

$$\bigcup_{K_S \subset \sum} K_S \begin{pmatrix} N \\ N \end{pmatrix}$$

and comprises all the sets which belong to at least one subcomplex K_S in the set $\sum$ of complexes whose reference pairs fulfill the condition of maximum similarity described above.

Difference

In considering two complexes A and B it would be of interest to know which sets are unique to A and which are unique to B. This is given by the set C, where C = A-B, the set of all elements which are in A and which are not in B. Similarly, we may let D = B-A, the set of all sets in B which are not in A. Set C is called the difference of A and B; set D is called the difference of B and A.

Symmetric Difference

The collection of sets unique to two complexes A and B can be defined in set-theoretic language as the symmetric difference, symbolized $A \vartriangle B$, where

$$A \vartriangle B = (A-B)+(B-A)$$

To conclude this brief discussion, we note that set-theoretic operations on complexes and sets of complexes are most meaningful for music when they can be directly interpreted in terms of similarity and dissimilarity. Intersection and symmetric difference therefore appear to be the most significant operations for analysis.

9.0 An Application to Analysis

It appears that the theory of set-complexes might be particularly valuable in studies of atonal music, where general concepts are now needed to carry research beyond the stage of contextual analysis. In this section, therefore, we present an elementary analysis in terms of set-complex theory in order to indicate the way in which this theory can illuminate many aspects of the structure of an individual work.

We restrict the discussion to unordered relations, and questions which would naturally arise in a more refined analysis, for instance, the question of set-succession – will be ignored. *17

The work to be examined is the fourth piece in Webern's Op. 5, Five Pieces for String Quartet (1909), a work comprising thirteen measures. (The reader may wish to consult his score for the musical notation.) We will confine the analysis to the listing of the compositional sets*17 and to the brief discussion of the set-characteristics and significant aspects of structure made evident in the set-complex relations. *18

The thirteen compositional sets in the work divide naturally into three categories: those which emphasize IC_6, those which emphasize IC_4, and those having other characteristics. A list, including locations of first statements follows.

Sets with IC_6 max, IC_4 min	Sets with IC_4 max, IC_6 min or near min
3-5 [100011] Bar 3:5611	3-4 [100110] Bar 4-5:015
4-9 [200022] Bar 2:05611	3-12[000300] Bar 7:2611
5-7 [310132] Bar 4:016711	4-24[020301] Bar 7:24610 (Superset *19)
7-19[434343] Bar 6:014671011	5-30[121321] Bar 7:2461011 (Superset)

Other

3-2 [111000]	no IC_4, IC_6	Bar 2:346
4-8 [200121]		Bar 1:04511
6-34[142422]	IC_2, IC_4, IC_6 near max	Bar 8:24671011 (Superset)
7-13[443542]	IC_4 near max	Bar 8:23461011 (Superset)
7-33[262623]	IC_2, IC_4, IC_6 max	Bar 9:024681011 (Superset)

For the purpose of this analysis it will suffice to concentrate upon seven of the most prominent sets, giving the following information for each: degree of similarity to other compositional sets of the same cardinality (Section 3.2), significant inclusions and exclusions, as revealed by $\cap$ and Δ (Section 8.3).

3-2: This set is not maximally similar to either 3-4, 3-5, or 3-12. It is first given as the set of non-intersecting pitches of 4-8 and 4-9 (See Perle 1962, 16). The set is not contained either in 4-8 or in 4-9, and since it excludes both IC_4 and IC_6 is very distinct from those sets in terms of interval-content.

3-4: This set has first order maximum similarity to 3-5 (see Section 3.2). It is not contained in K_s(7-19), but is one of the two 3-note subsets of 4-9.

The harmonic-contrapuntal structure of the work is remarkably stable from an intervallic standpoint. A vertical or diagonal cross-section taken at any point except the middle section (bars 7-9) will yield either 3-4 or 3-5. This can be interpreted as a fluctuation involving only two interval-classes: IC_4 and IC_6, since 3-4 and 3-5 differ only with respect to these, as remarked. It is probably correct to say that the interacting lines and, to a large extent, the rhythm are controlled by these alternating sets.

4-8: This tetrachord has second-order maximum similarity to 4-9. It is contained in K(7-19), but not K_S(7-19). As noted above, it contains only two 3-note subsets, 3-4 and 3-5, both of which are important compositional sets here.

4-9: Maximally similar to 4-8. It is contained in K_S(7-19). It contains only one 3-note subset, 3-5; any set of three notes selected from 4-9 will have the interval-vector of 3-5. Sets 4-8 and 4-9 thus form a pair analogous to the pair 3-4 and 3-5 since both pairs differ only with respect to the same interval-classes: IC_4 and IC_6.

Example 9.0 is a schematic representation of the modes of occurrence of the five sets just discussed. Note especially the symmetrical statement of the two subsets 3-4 and 3-5 in set 4-8.

The important superset 5-7*19 contains (uniquely) sets 3-4, 3-5, 4-8, and 4-9. Example 9.01 shows how these subsets are expressed in the piece.*20

7-19: This is one of the most prominent melodic sets. It occurs as a solo line in bars 6, 10, and 12. K_S(7-19) contains 3-2, 3-5, and 4-9. Set 5-7 is a member of K(7-19), and this inclusion relation is expressed in the piece just before the statement of 7-19 in bar 6, where 7-19 and 5-7 intersect in the pitch-set 0167, which is easily recognized as 4-9.

Of the many interesting intersections, we select three which have special prominence in the work.

$$\bigcap K_S \begin{pmatrix} 3\text{-}4 \\ 3\text{-}5 \end{pmatrix} = [4\text{-}8,\ 5\text{-}7,\ 5\text{-}30,\ 6\text{-}34,\ 7\text{-}13]$$

$$\bigcap K_S \begin{pmatrix} 3\text{-}4 \\ 3\text{-}5 \end{pmatrix} = [4\text{-}9,\ 7\text{-}19]$$

Thus, both 3-4 and 3-5 are present when any of the sets in the intersection set is stated, whereas only 3-5 is present when either 4-9 or 7-19 occurs.

$$\bigcap K_s \begin{pmatrix} 3\text{-}2 \\ 3\text{-}4 \\ 3\text{-}5 \\ 3\text{-}12 \end{pmatrix} = [6\text{-}34]$$

Set 6-34 occurs only in the middle section. Only the 3-note sets occur both in middle and outside sections and all the 3-note sets intersect in 6-34, as remarked.

$$\bigcap K_s \begin{pmatrix} 4\text{-}8 \\ 4\text{-}9 \end{pmatrix} = [3\text{-}5,\ 5\text{-}7]$$

$$\bigcap K_s \begin{pmatrix} 4\text{-}8 \\ 4\text{-}9 \end{pmatrix} = [3\text{-}4,\ 7\text{-}19]$$

The formal relational statements show that the two "thematic" sets, 4-8 and 4-9, are joined in 3-5 and 5-7, but separated in 3-4 and 7-19. This again reflects the very narrow fluctuation of interval-content which characterizes the work.

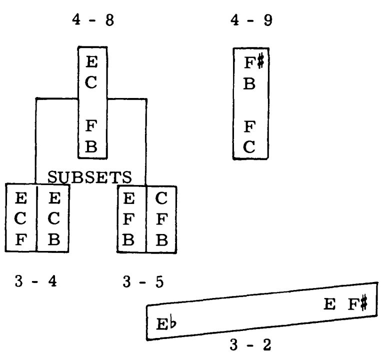
Perhaps the most interesting formal feature of this miniature, and one which is made especially evident through the set-complex analysis, is the structural disjunction which takes place as a result of the "incomparable" relation. This occurs both in the small span, with 3-2 sounding against 4-8 and 4-9 as shown in Example 9.0 and in the larger span: in the middle section (bars 7-9) which has its own set of sets, all interrelated by inclusion, but incomparable with respect to all sets of 4 notes or more in the outside sections.

Continuity and regulated change also are evident both in small and large dimensions. The fluctuation of IC_4 and IC_6 mentioned above is repeated in the contrast between the middle section (IC_4) and the sections bounding it, which emphasize IC_6.

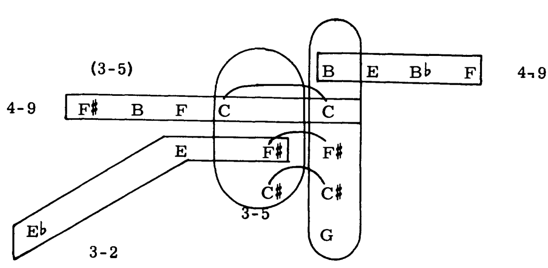
To sum up, we have indicated some of the ways in which the theory of set-complexes can illuminate structure when applied to analysis. In interpreting these relations for specific works it is essential first to consider intervallic content, intervallic similarity and so on at the level of the individual set, as well as intersections of set complexes and inclusion chains at the level of larger successions. It is not difficult to see how a basic analysis of this kind can then be refined to include ordered relations, timbre, rhythm, mode of attack and other factors.

example

9.0



9.01



10.0 Some Implications

The theory of set-complexes has special significance for the study of syntactic structure in the case of atonal music, and work is now in progress to refine and test related research tools, with the expectation that useful general analytic methods and descriptive techniques will result. A number of problems and extensions have suggested themselves, many of which will require the assistance of a computer for efficient solution. It now appears that with continued work in this area we can expect to discover a number of promising hypotheses.

Extending beyond these special problems, it is hoped that the theory presented here will be of use to the historian. Consideration of the special properties of sets as well as the special nature of their complexes and subcomplexes may suggest interesting hypotheses of historical development, style, and so on. (Wilding-White) The complex about the triad, for example, includes some prominent atonal sets (for example, the all interval tetrachord), but excludes three out of six all-combinatorial source hexachords, sets which possess high redundancy under transformation. Finally, in order to interpret statistics, to arrive at an effective evaluation of set selection and relatedness, it is essential to have a knowledge of the structural properties, including combinatory aspects, of set-complexes and the operations defined upon them.

r e f e r e n c e s

1 There is one additional recent exception: 12-tone music. The structure of this music was first studied in depth by Babbitt (see all references above). The author gratefully acknowledges the influence of Babbitt's writings, as well as the important work of Martino and of Lewin.

2 For typographical reasons square instead of curly brackets are used to enclose the elements of sets.

3 Permutation is synonymous with ordered set, and combination is synonymous with unordered set.

4 Perle 1962, 117-118 presents a rather involved explanation of the number 4096. It is the same as the number of unordered subsets of a set of 12 elements, 2^{12}, the so-called power set. The number 4096 is explained more fully as the sum of the coefficients of the binomial expansion, the general form of which is $(x + y)^n$. For this case n = 12.

5 An equivalence relation R defined for any two elements a and b of S has three properties: (1) aRa, the reflexive property; (2) aRb implies bRa, the symmetric property; (3) aRb and bRc implies aRc, the transitive property. An equivalence relation defined upon a set of elements S sorts the elements of S into non-intersecting subsets, or classes, which exhaust S. The set of non-intersecting classes is called a partition of S.

6 This is not a mathematical vector in the rigorous sense. The only operation employed is elementwise subtraction (as in Section 6.1), and the result of the operation is not a member of the set of 197 distinct vectors.

7 For any set S the number of elements S contains is called the cardinal number of S, symbolized # (S), or the cardinality of S. This convenient term helps to avoid confusion, since "number" has several referents in the present article.

8 A rule of correspondence which associates with every member s of a set S one and only one member t of a set T is said to be a mapping from S into T. Consider, for example, the following mapping:

$$\begin{array}{ccc} S & \propto & T \\ 0 & \longrightarrow & 0 \\ 1 & \longrightarrow & 11 \\ 2 & \longrightarrow & 10 \\ & & 8 \end{array}$$

The rule of correspondence α maps S into T ($\alpha: S \rightarrow T$). This mapping is also called one-to-one (1-1) since any element t in T is paired with only one element is S. The mapping is therefore "into and 1-1". The set S is called the domain of the mapping; the elements of T which are paired with elements of S comprise the range of the mapping. If the range of the mapping were the same as T, the mapping would be from S onto, not into, T. The following mapping is both onto and 1-1:

$$\begin{array}{ccc} S & \delta & T \\ 0 & \longrightarrow & 5 \\ 1 & \longrightarrow & 6 \\ 2 & \longrightarrow & 7 \end{array}$$

We will be concerned solely with 1-1 mappings. When S and T represent the pitch-numbers of hexachords we will obviously be dealing with mappings onto; in all other cases we will be interested in mappings from S into T, where S contains fewer elements than T. Mapping is another way of describing a function, one which emphasizes the rule of correspondence rather than the elements which are associated by it.

9 The 13 special partitions of 12 which account for the 13 normal order pitch-sets without distinct I-forms and with fewer than 12 distinct transpositions of P (37 in all) are:

Set Number	Partition	Number of Pitch-Sets	Partition	Set Number
6-7	441111	4		
6-20	333111	6		
6-35	222222	2		
4-9	5511	6	33111111	8-9
4-25	4422	6	22221111	8-25
4-28	3333	4	22221111	8-28
3-12	444	3	222111111	9-12
2-6	66	6	2211111111	10-6
		37		

10 These forms cannot be reduced into a single form by transposition or inversion, or inversion followed by transposition. (See Lewin 1960 and Section 5.1.) They will be referred to as Z-forms. Unless inversion of the Z-form (and inversion followed by transposition) produces redundant sets, the total number of sets corresponding to a single vector is 48 in these cases.

11 The ordering of the distinct interval vectors differs from that of Martino, which was set up to emphasize certain structural features of the 12-tone system, and from that of Hanson, which is biased in favor of triadic tonality. The present ordering merely follows that established in the vectors of 2-note sets: The vector is treated as a number, so that proceeding from top to bottom if i, j, k are three consecutive vectors $i < j < k$.

This uncomplicated ordering has several advantages. First, it is not structurally biased. Second, it groups sets of similar structure more closely together by reference number. Third, and most important for the purpose of this paper, it provides complementary sets (see 5.0) with the same ordinal number. For example, 4-15 is the complement of 8-15.

It should be remarked that the count of distinct sets differs from that of Martino. For practical reasons Martino did not list sets whose cardinal number exceeded 6 (although such a list was implicit). He also gave separate status to the sets designated Z*10, but only for tetrachords and pentachords. On the basis of (2.2) the present lists therefore are somewhat shorter. To be specific,

Present Number		Martino
Z 4-15	corresponds to	14, 16
Z 5-12	" "	16, 18
Z 5-17	" "	5, 7
Z 5-18	" "	21, 23

There were also some minor corrections made in normal form, compared with Martino.

In terms of (2.2) Hanson's count is incorrect, as is Perle's (Perle 1962, 121), since both list all Z-forms as distinct. Thus, for example, both Perle and Hanson list 50 hexachords, although both authors presumably equate a large number of other sets on the basis of interval-content.

Set-theoretic inclusion differs from the usual order relation for numbers (less than $<$, equal to =, or greater than $>$), in that two sets A and B may be incomparable: neither is contained in the other. For any pair of positive integers, however, we can state an order relation between them.

Explanation of the mathematical reasoning for the number N of supersets of cardinality n which may contain a set of cardinality m, given for the general case by

$$N = C(m', n-m).$$

Let us take an example where m = 2. We select a 2-note set: [0, 1]. This selection effects a partition:

0 1 | 2 3 4 5 6 7 8 9 10 11

We now make an exhaustive list of supersets containing [0, 1]:

[0, 1, 2]
[0, 1, 3]
[0, 1, 4]
[0, 1, 5]
[0, 1, 6]
[0, 1, 7]
[0, 1, 8]
[0, 1, 9]
[0, 1, 10]
[0, 1, 11]
and no others.

Observe that this process is the same as answering the question: How many unordered subsets of cardinality n-m (in this case, 3-2=1) may be selected from a set of m' elements (in this case m'=10)? In every instance the union of each of these subsets with the "fixed" set which is to be contained (in this case [0, 1]) yields a list such as that above.

14 In Hanson, 331f. we find a lengthy discussion of what is called the "maverick" sonority, described as ". . . the only sonority in all of the tonal material of the twelve-tone scale which is not itself a part of its own complementary scale."

This description is incorrect. First, the collection in question, Set 5-12, can be mapped into its complement – as shown in the present article. Second, the maverick sonority is not unique. Sets 5-17 and 5-18 map into their complements under IT, however, with single values of k. Hanson completely disregarded 15 hexachords which have the same property, perhaps because he did not establish a clear relation between pitch-set and interval-set.

15 I am indebted to Lewin 1960, 99 and to Babbitt 1961, 81 for the description of this property and for the equation. The latter (from Babbitt) has been modified slightly to include the case of IC_6 for any pair of complementary interval-vectors.

16 Given two sets S and T, the union of S and T (notated $S \cup T$ or $S + T$) is the set comprising all those elements which are either in S or in T.

Given two sets S and T, the intersection of S and T (notated $S \cap T$ or ST) is the set comprising all those elements which are both in S and in T.

17 Perle 1962, 16-19 gives a contextual analysis of this piece, mainly from the standpoint of pitch-sets.

18 Compare my earlier set-theoretic analysis of an atonal work (Forte 1963). This was essentially a contextual analysis without a general background. To a large extent it was the deficiencies of that analysis which led to the present article. It should be remarked that the latter is only a part of a larger undertaking: a general theory of set-relations.

19 The term superset here merely means that the set is not stated as a separate melodic or harmonic set: it exists as the union of its subsets, an "harmonic" collection only.

20 Set 5-7 has the unusual property that no other 5-note set is maximally similar to it. Only one other 5-note set has this property: set 5-33, the whole-tone pentad, which is the complement of an important superset in the work being examined (comprises all of bar 9).

21 The program was written in the language called the Michigan Algorithm Decoder (MAD) and executed by the IBM 7094/7040 DCS installation at the Yale Computer Center.

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