SETS, INVARIANCE AND PARTITIONS

Daniel Starr

This paper examines certain fundamentals of twelve-tone theory and stems from previous attempts to deal with its higher-level aspects. I develop here a calculus of unordered pitch-class sets. While unordered sets have been studied before, notably by Hanson,[1] Martino,[2] and Forte,[3] the present study considers different aspects of the problem and emphasizes modular arithmetic as a basis of twelve-tone operations.

Much recent music, whether it is twelve-tone, less-than-twelve-tone, "collection music," or something similar, has been based on a small, conceptually simple set of operations, which have been used differently from work to work. Such operations can be considered axiomatic, whereas a particular set or row occurring in a specific work is the result of a unique compositional process, and acquires significance specifically from its context. It is perhaps meaningful to say that such an object only possesses attributes which are demonstrated by the transformations applied to it in those contexts. I find it both fruitful and intuitive to conceive of an *operation-object* duality, in which "operation" is a concept subject to general discussion, while "object" arises from the discussion of specific works. Thus, to approach various general aspects of twelve-tone or related types of music, I stress what I consider the *operations* that we apply to sets, rows, partitions, etc., rather than on those *objects* themselves, or, for that matter, their classification, which is the topic most often considered.

We often demand that operations be "intuitive" or "audible." The cycle-of-fifths transform, for example, has occasionally been advocated one way or the other. But a row, set, subject or theme is usually not liable to justification, but rather, to development. In other words, the "justification" of an object is not an a priori question; that important issue arises in the challenge of composing a work incorporating it.

Apart from making definitions and demonstrating their implications, I have supplied the reader with hand-calculation methods and tables which show how the properties of the twelve-tone system might be translated, albeit at a primitive level, into a *technique.* As it is traditionally conceived, technique is an aspect of one's compositional craft which has become familiar to the point where it no longer requires conscious effort. The translation of principle into technique is arbitrary, or at least a personal thing: I do not raise the issue to polemicize or to advocate a "correct" way of doing things, but rather to show by example how theory might be connected to practice, and to suggest certain technical questions that might be answered.

The implications of this discussion are only suggested in occasional problems which exemplify certain concepts and properties. Although writings on more advanced topics already exist,[4] the present work is intended to clarify certain underlying definitions in those writings. The reader will forgive me if I begin with the definition of pitch-classes.

We can take an octave of chromatic space from C to B and number the pitches upwards from zero to eleven. These numbers can serve as pitch labels much as letter-names like "C" and "D" stand for pitches that might occur in any register. In other words, they stand for classes of pitches one or more octaves distant from each other and can be thought of as being registrally indeterminate. We will then call the integers 0 through 11 *pitch-classes* (abbreviated as "PCs").

These numbers can also be seen to form a *mod-12 arithmetic system* in which only these twelve number-objects exist. Special *modular arithmetic operators,* similar to the familiar "+," "–," and "X" of integer arithmetic, let us perform a modified type of arithmetic in which the result of any operation is guaranteed to be one of the original twelve objects. The mod-12 arithmetic system is thus *closed* and is a *homomorphic image* of integer arithmetic—that is, it is a finite microcosm of the integer arithmetic system.[5]

The basis of mod-12 arithmetic is an arithmetic operation which projects a number from the world of integer-arithmetic into the more restricted mod-12 system. Such an operation which collapses a whole system into a simpler one is called a *homomorphism.* The projection of a three-dimensional object onto a plane, or the casting of a shadow, make good analogies. The specific operation that generates the mod-12 system is that of *taking a number mod-12*—that is, adding or subtracting enough twelves to or from that number to bring it into the range of 0 to 11 inclusive. Thus to take 35 mod-12, we subtract two twelves making 11; to take –7 mod-12, we add twelve to yield 5. Any multiple of twelve automatically yields zero. It is, in fact, the same operation that we use when confronted with a 24-hour clock to deduce that "20 hours" means "8 p.m."

We can write the mod-12 homomorphism as the function

$$y = (x)\text{mod-}12$$

where x is any integer and y will range from 0 to 11. We express x as the sum $12a + y$, where a is an integer and $0 \leqq y \leqq 11$. This, in effect, is accomplished by dividing x by 12 to obtain the remainder. Note, however, that a negative x will produce both negative quotients and remainders, which can be corrected by adding 12 to the remainder.

$$\begin{array}{r|l} & \;\;-2 \\ \cline{2-2} 12 & -27 \\ & \underline{+24} \\ & \;\;-3 \end{array} \qquad (-27)\text{mod-}12 = 9 = (-3 + 12)$$

Returning to the concept of numbering the notes of the chromatic scale, we see that the operation of taking a number mod-12 can be seen to be the projection of a pitch from any octave of chromatic space into our "reference octave" labelled 0 through 11. Mod-12 numbers can be reduced further with respect to *moduli* smaller than 12. The function y = (x)mod-6, for instance, gives us integers in the range 0 to 5, and y = (x)mod-2 gives either 0 or 1 which tells us whether the number x is even or odd.

For any integer n, (n)mod-12 will, by definition, be in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. This is the *mod-12 universe*, which we will simply call U. Adding two members of U can generate new numbers not belonging to U (for example 8 plus 10 makes 18). The same is true of subtraction and multiplication: 2 − 3 = −1 and 5 × 3 = 15. We can, however, redefine the usual arithmetic operations so that operations on members of U produce another member of U. We define *modular addition, modular negation,* etc., as follows with the use of encircled operator symbols:

$$a \oplus b = (a + b)\text{mod-}12$$
$$\ominus\, a = (-a)\text{mod-}12$$
$$a \ominus b = (a - b)\text{mod-}12$$
$$a \otimes b = (a \times b)\text{mod-}12$$

Here are a few examples:

$$11 \oplus 1 = (11 + 1)\text{mod-}12 = (12)\text{mod-}12 = 0$$
$$7 \oplus 6 = (7 + 6)\text{mod-}12 = (13)\text{mod-}12 = 1$$
$$\ominus\, 9 = (-9)\text{mod-}12 = 3$$
$$5 \otimes 7 = (5 \times 7)\text{mod-}12 = (35)\text{mod-}12 = 11$$

The operators $\oplus$, $\ominus$, and $\otimes$ have many properties in common with the more familiar operators of high school algebra and may be treated in much the same way. The operators $\oplus$ and $\otimes$, in particular, are commutative and associative:

$a \oplus b = b \oplus a$ for any a and b in U
$a \otimes b = b \otimes a$ for any a and b in U
$(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for any a, b, and c in U
$(a \otimes b) \otimes c = a \otimes (b \otimes c)$ for any a, b, and c in U

The operator $\oplus$ distributes over $\otimes$ as follows:

$(a \otimes b) \oplus (a \otimes c) = a \otimes (b \oplus c)$ for any a, b, and c in U

There are both additive and multiplicative identities:

$$a = 1 \otimes a,\ 0 = 0 \otimes a,\ a = a \oplus 0 \qquad \text{for any a in U}$$

The cancellation principle also applies:

$$\begin{array}{ll} \text{if } a = b, \text{ then } a \oplus c = b \oplus c & \text{for a, b, c in U} \\ \text{if } a = b, \text{ then } a \otimes c = b \otimes c & \text{for a, b, c in U} \\ \text{if } a \oplus c = b \oplus c, \text{ then } a = b & \text{for a, b, c in U} \end{array}$$

But note that the following does not hold:

$$\text{NO!!!} \qquad \text{if } a \otimes c = b \otimes c, \text{ then } a = b \qquad \text{NO!!!}$$

If $a = 1$ and $b = 7$, it would follow that $a \otimes 2 = b \otimes 2 = 2$. This would imply that $1 = 7$. Because of this problem we have not defined a "modular division operator," since division by two, for example, would have multiple values, as both $2 \oslash 2 = 7$ and $2 \oslash 2 = 1$. Were we not, however, in the mod-12 system, but rather in one based on a prime "modulus," this problem would not arise and "modular division" would be definable.

Note that modular negation and multiplication by 11 are equivalent operations:

$$\begin{array}{l} a \otimes 11 = \ominus\, a \\ a \otimes (0 \ominus 1) = \ominus\, a \\ (a \otimes 0) \ominus (a \otimes 1) = \ominus\, a \\ 0 \ominus a = \ominus\, a \end{array}$$

The modular "multiplication table" is given in Table 1. The rows and columns marked with asterisks include all 12 members of U, whereas the others merely leap-frog through shorter sequences of 2, 3, 4, or 6 PCs. This occurs because PCs 1, 5, 7, and 11 are relatively prime to 12.

Let us now consider a *linear modular equation* that is analogous to the linear equations of high school algebra:

$$y = (a \otimes x) \oplus b$$

If we restrict the coefficient a to 1, 5, 7, or 11, then y could take on the value of any member in U. We will call such an equation a *twelve-tone operation* since it *preserves all twelve PCs.* It turns out that quadratic and cubic modular equations are either degenerate, in that their ranges fail to include all PCs in U, or else they are linear equations in disguise,

Table 1

			*				*		*				*
	⊗	0	1	2	3	4	5	6	7	8	9	10	11
	0	0	0	0	0	0	0	0	0	0	0	0	0
*	1	0	1	2	3	4	5	6	7	8	9	10	11
	2	0	2	4	6	8	10	0	2	4	6	8	10
	3	0	3	6	9	0	3	6	9	0	3	6	9
	4	0	4	8	0	4	8	0	4	8	0	4	8
*	5	0	5	10	3	8	1	6	11	4	9	2	7
	6	0	6	0	6	0	6	0	6	0	6	0	6
*	7	0	7	2	9	4	11	6	1	8	3	10	5
	8	0	8	4	0	8	4	0	8	4	0	8	4
	9	0	9	6	3	0	9	6	3	0	9	6	3
	10	0	10	8	6	4	2	0	10	8	6	4	2
*	11	0	11	10	9	8	7	6	5	4	3	2	1

Table 2

⊗	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

with the higher order terms summing to a constant. For example:

$$y = (6 \otimes x \otimes x) \oplus x = 7 \otimes x$$
$$y = (2 \otimes x \otimes x \otimes x) \oplus (6 \otimes x \otimes x) \oplus (11 \otimes x) = 7 \otimes x$$
$$y = (8 \otimes x \otimes x \otimes x) \oplus (5 \otimes x) = 1 \otimes x = x$$

We will henceforth narrow the discussion to twelve-tone operations (abbreviated as TTOs), which are necessarily linear modular equations, and we will notate them as pairs $\langle a,b \rangle$ where a = 1, 5, 7, or 11, and b is any PC in U, forming an equation $y = (a \otimes x) \oplus b$. There are 48 of them.

TTOs are familiar to us as operations in the twelve-tone system: $\langle 1,n \rangle$ is a transposition by n semitones, $\langle 11,0 \rangle$ is inversion, $\langle 5,0 \rangle$ is the cycle-of-fourths transform, and $\langle 7,0 \rangle$ is the cycle-of-fifths transform. The last two, like inversion, are multiplicative operations and can therefore be used in chromatic space so as to preserve "contour" (or spacing, in the case of a chord). This is analogous to the way in which pitch-class inversion is used to generate "mirror inversion" (see Example 1).

An abbreviated modular multiplication table shows a relationship between the purely multiplicative operations (see Table 2). Multiplication of PCs within the collection $\{1, 5, 7, 11\}$ is *closed* (no other PCs are generated), and an identity $x \otimes x = 1$ holds, which is otherwise not generally true.

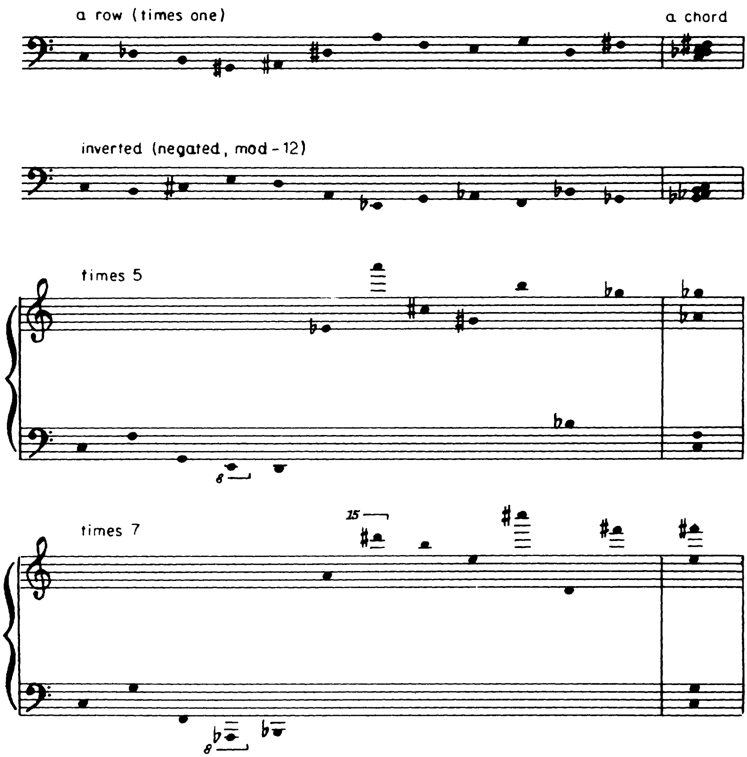
Although the cycle-of-fourths and cycle-of-fifths transforms are less well known than the other operations, they have been used not only in conjunction with rows, but also, for instance, in the music of Bártòk and in jazz, where there is a functional equivalence between chromatic bass lines and basses that move by fifths.

If we had two TTOs $F = \langle a,b \rangle$, and $G = \langle a_2,b_2 \rangle$ which we were to apply in succession, we would call the net result the *composite operation* $G \cdot F$:

$$y = (a_1 \otimes x) \oplus b_1 \qquad (1)$$

and

$$z = (a_2 \otimes y) \oplus b_2 \qquad (2)$$

$G \cdot F$ is a new TTO. Substituting (1) into (2) we get:

a row (times one)
a chord
inverted (negated, mod-12)
times 5
times 7

Example 1

$$z = (a_2 \otimes [(a_1 \otimes x) \oplus b_1]) \oplus b_2$$
$$= (a_2 \otimes a_1 \otimes x) \oplus (a_2 \otimes b_1) \oplus b_2$$

Thus:

$$G \cdot F = < a_2 \otimes a_1, (a_2 \otimes b_1) \oplus b_2 >$$

We can compose a TTO with itself: that is, where $F = < a,b >$, we can calculate the TTO $F \cdot F$ by means of the preceding formula:

$$F \cdot F = < a \otimes a, (a \otimes b) \oplus b >$$
$$= < 1, b \otimes (a \oplus 1) >$$

Thus for any TTO F, $F \cdot F$ is always a transposition.

Note that in a composite $G \cdot F$, the right-hand TTO F is the one applied first, were the composed TTOs to be applied successively to some PC. The operation of composing TTOs is not commutative. In chains of composed TTOs ($H \cdot G \cdot F \ldots$, etc.) it is understood that the individual TTOs are applied in the reverse order from that in which they are written. Strictly speaking, the set of all TTOs comprises a non-Abelian group under the operation of composition, containing various subgroups of different sizes.

Table 3 is a useful guide for composing TTOs by hand. The table gives transposition component for the TTO which results from the composition of any possible pair of TTOs, assuming that the first component (the multiplier) can be figured mentally. For the TTO $G \cdot F$, one finds the row corresponding to the TTO F (the one applied first), and the column corresponding to G (which is applied second). This gives the b-term of a resultant TTO $< a,b >$.

There are three TTOs which can be considered *basic* in that they can be composed with each other simply to generate any of the 48 possible TTOs. We select and label them as follows: $< 1,n >$ is "T_n," $< 11,0 >$ is "I," and $< 5,0 >$ is "M." By composing I and M, the cycle-of-fifths transform ($< 7,0 >$) can be obtained, which can be written as either "IM" or "MI," since the two component operations are commutable with each other. T_n, on the other hand, is commutable with neither I nor M; that is, in general,

$$T_nI \neq IT_n \qquad \text{and} \qquad T_nM \neq MT_n$$

This suggests that we define a *canonical form* for TTOs

Table 3. Transposition components of composed TTOs

SECOND TTO=

	T_n	T_nI	T_nM	T_nIM
FIRST	0123456789AB	0123456789AB	0123456789AB	0123456789AB
P	0123456789AB	0123456789AB	0123456789AB	0123456789AB
T1	123456789AB0	B0123456789A	56789AB01234	789AB0123456
T2	23456789AB01	AB0123456789	AB0123456789	23456789AB01
T3	3456789AB012	9AB012345678	3456789AB012	9AB012345678
T4	456789AB0123	89AB01234567	89AB01234567	456789AB0123
T5	56789AB01234	789AB0123456	123456789AB0	B0123456789A
T6	6789AB012345	6789AB012345	6789AB012345	6789AB012345
T7	789AB0123456	56789AB01234	B0123456789A	123456789AB0
T8	89AB01234567	456789AB0123	456789AB0123	89AB01234567
T9	9AB012345678	3456789AB012	9AB012345678	3456789AB012
TA	AB0123456789	23456789AB01	23456789AB01	AB0123456789
TB	B0123456789A	123456789AB0	789AB0123456	56789AB01234
I	0123456789AB	0123456789AB	0123456789AB	0123456789AB
T1I	123456789AB0	B0123456789A	56789AB01234	789AB0123456
T2I	23456789AB01	AB0123456789	AB0123456789	23456789AB01
T3I	3456789AB012	9AB012345678	3456789AB012	9AB012345678
T4I	456789AB0123	89AB01234567	89AB01234567	456789AB0123
T5I	56789AB01234	789AB0123456	123456789AB0	B0123456789A
T6I	6789AB012345	6789AB012345	6789AB012345	6789AB012345
T7I	789AB0123456	56789AB01234	B0123456789A	123456789AB0
T8I	89AB01234567	456789AB0123	456789AB0123	89AB01234567
T9I	9AB012345678	3456789AB012	9AB012345678	3456789AB012
TAI	AB0123456789	23456789AB01	23456789AB01	AB0123456789
TBI	B0123456789A	123456789AB0	789AB0123456	56789AB01234
M	0123456789AB	0123456789AB	0123456789AB	0123456789AB
T1M	123456789AB0	B0123456789A	56789AB01234	789AB0123456
T2M	23456789AB01	AB0123456789	AB0123456789	23456789AB01
T3M	3456789AB012	9AB012345678	3456789AB012	9AB012345678
T4M	456789AB0123	89AB01234567	89AB01234567	456789AB0123
T5M	56789AB01234	789AB0123456	123456789AB0	B0123456789A
T6M	6789AB012345	6789AB012345	6789AB012345	6789AB012345
T7M	789AB0123456	56789AB01234	B0123456789A	123456789AB0
T8M	89AB01234567	456789AB0123	456789AB0123	89AB01234567
T9M	9AB012345678	3456789AB012	9AB012345678	3456789AB012
TAM	AB0123456789	23456789AB01	23456789AB01	AB0123456789
TBM	B0123456789A	123456789AB0	789AB0123456	56789AB01234
IM	0123456789AB	0123456789AB	0123456789AB	0123456789AB
T1IM	123456789AB0	B0123456789A	56789AB01234	789AB0123456
T2IM	23456789AB01	AB0123456789	AB0123456789	23456789AB01
T3IM	3456789AB012	9AB012345678	3456789AB012	9AB012345678
T4IM	456789AB0123	89AB01234567	89AB01234567	456789AB0123
T5IM	56789AB01234	789AB0123456	123456789AB0	B0123456789A
T6IM	6789AB012345	6789AB012345	6789AB012345	6789AB012345
T7IM	789AB0123456	56789AB01234	B0123456789A	123456789AB0
T8IM	89AB01234567	456789AB0123	456789AB0123	89AB01234567
T9IM	9AB012345678	3456789AB012	9AB012345678	3456789AB012
TAIM	AB0123456789	23456789AB01	23456789AB01	AB0123456789
TBIM	B0123456789A	123456789AB0	789AB0123456	56789AB01234

written as composites of the basic operators in which (1) no basic operator occurs more than once, and (2) T_n, if present, is written on the left and, by convention, is applied last. Given a TTO that is expressed as a composite of basic TTOs that are not in canonical form, we can simplify it by re-writing it as a modular arithmetic expression which can then be reduced to the form $y = (a \otimes x) \oplus b$ by applying algebraic identities. This, in turn, can be written as the canonical expression T_b (if a = 1), or T_bF (where F = "M," "IM," or "I") if a = 5, 7, or 11 respectively. For example, "MT_nI" represents the composite

$$< 11,0 > \cdot < 1,n > \cdot < 5,0 >$$

which defines the following modular equation:

$$\begin{aligned} y &= 5 \otimes ((11 \otimes x) \oplus n) \\ &= (7 \otimes x) \oplus (5 \otimes n) \end{aligned}$$

This, in turn, can be written as the TTO $< 7,5 \otimes n >$ or $T_{5\otimes n}IM$. Table 4 gives the 48 possible TTOs in two notations, along with other information that will be explained below.

A TTO F = $< a,b >$ defines an equation $y = (a \otimes x) \oplus b$. If we solve the equation for x in the following manner

$$\begin{aligned} &y = (a \otimes x) \oplus b \\ &y \ominus b = (a \otimes x) \oplus b \ominus b = a \otimes x \\ &a \otimes (y \ominus b) = a \otimes a \otimes x = 1 \otimes x = x \\ &(a \otimes y) \ominus (a \otimes b) = x \end{aligned}$$

we arrive at the new TTO "F^{-1}" = $< a, \ominus (a \otimes b) >$ which is the *inverse* of F. F^{-1} in effect "undoes" one application of the TTO F.

If F is a TTO such that F and F^{-1} are the same, then we say that F is *reflexive*. For instance, all TTOs of the form $< 11,b >$—that is, all inversions—are reflexive, since their inverses have the same value for "b":

$$\begin{aligned} F^{-1} &= < a, \ominus (a \otimes b) > \\ &= < 11, \ominus (11 \otimes b) > \\ &= < 11, 11 \otimes (11 \otimes b) > \\ &= < 11, (11 \otimes 11) \otimes b > \\ &= < 11, b > = F \end{aligned}$$

Table 4. Twelve-Tone Operations

operation		*cycles*	*inverse*
T_0	$\langle 1,0 \rangle$	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	*
T_1	$\langle 1,1 \rangle$	0–1–2–3–4–5–6–7–8–9–10–11	T_{11}
T_2	$\langle 1,2 \rangle$	0–2–4–6–8–10, 1–3–5–7–9–11	T_{10}
T_3	$\langle 1,3 \rangle$	0–3–6–9, 1–4–7–10, 2–5–8–11	T_9
T_4	$\langle 1,4 \rangle$	0–4–8, 1–5–9, 2–6–10, 3–7–11	T_8
T_5	$\langle 1,5 \rangle$	0–5–10–3–8–1–6–11–4–9–2–7	T_7
T_6	$\langle 1,6 \rangle$	0–6, 1–7, 2–8, 3–9, 4–10, 5–11	*
T_7	$\langle 1,7 \rangle$	0–7–2–9–4–11–6–1–8–3–10–5	T_5
T_8	$\langle 1,8 \rangle$	0–8–4, 1–9–5, 2–10–6, 3–11–7	T_4
T_9	$\langle 1,9 \rangle$	0–9–6–3, 1–10–7–4, 2–11–8–5	T_3
T_{10}	$\langle 1,10 \rangle$	0–10–8–6–4–2, 1–11–9–7–5–3	T_2
T_{11}	$\langle 1,11 \rangle$	0–11–10–9–8–7–6–5–4–3–2–1	T_1
M	$\langle 5,0 \rangle$	0, 1–5, 2–10, 3, 4–8, 6, 7–11, 9	*
T_1M	$\langle 5,1 \rangle$	0–1–6–7, 2–11–8–5, 3–4–9–10	T_7M
T_2M	$\langle 5,2 \rangle$	0–2, 1–7, 3–5, 4–10, 6–8, 9–11	*
T_3M	$\langle 5,3 \rangle$	0–3–6–9, 1–8–7–2, 4–11–10–5	T_9M
T_4M	$\langle 5,4 \rangle$	0–4, 1–9, 2, 3–7, 5, 6–10, 8, 11	*
T_5M	$\langle 5,5 \rangle$	0–5–6–11, 1–10–7–4, 2–3–8–9	$T_{11}M$
T_6M	$\langle 5,6 \rangle$	0–6, 1–11, 2–4, 3–9, 5–7, 8–10	*
T_7M	$\langle 5,7 \rangle$	0–7–6–1, 2–5–8–11, 3–10–9–4	T_1M
T_8M	$\langle 5,8 \rangle$	0–8, 1, 2–6, 3–11, 4, 5–9, 10, 7	*
T_9M	$\langle 5,9 \rangle$	0–9–6–3, 1–2–7–8, 4–5–10–11	T_3M
$T_{10}M$	$\langle 5,10 \rangle$	0–10, 1–3, 2–8, 4–6, 5–11, 7–9	*
$T_{11}M$	$\langle 5,11 \rangle$	0–11–6–5, 1–4–7–10, 2–9–8–3	T_5M

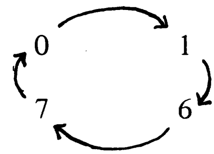
Table 4 (continued)

operation		*cycles*	*inverse*
MI	<7,0>	0, 1–7, 2, 3–9, 4, 5–11, 6, 8, 10	*
T_1MI	<7,1>	0–1–8–9–4–5, 2–3–10–11–6–7	T_5MI
T_2MI	<7,2>	0–2–4–6–8–10, 1–9–5, 3–7–11	T_{10}MI
T_3MI	<7,3>	0–3, 1–10, 2–5, 4–7, 6–9, 8–11	*
T_4MI	<7,4>	0–4–8, 1–11–9–7–5–3, 2–6–10	T_8MI
T_5MI	<7,5>	0–5–4–9–8–1, 2–7–6–11–10–3	T_1MI
T_6MI	<7,6>	0–6, 1, 2–8, 3, 4–10, 5, 7, 9, 11	*
T_7MI	<7,7>	0–7–8–3–4–11, 1–2–9–10–5–6	T_{11}MI
T_8MI	<7,8>	0–8–4, 1–3–5–7–9–11, 2–10–6	T_4MI
T_9MI	<7,9>	0–9, 1–4, 2–11, 3–6, 5–8, 7–10	*
T_{10}MI	<7,10>	0–11–4–3–8–7, 1–6–5–10–9–2	T_7MI
T_{11}MI	<7,11>	0–11–4–3–8–7, 1–6–5–10–9–2	T_7MI
I	<11,0>	0, 1–11, 2–10, 3–9, 4–8, 5–7, 6	*
T_1I	<11,1>	0–1, 2–11, 3–10, 4–9, 5–8, 6–7	*
T_2I	<11,2>	0–2, 1, 3–11, 4–10, 5–9, 6–8, 7	*
T_3I	<11,3>	0–3, 1–2, 4–11, 5–10, 6–9, 7–8	*
T_4I	<11,4>	0–4, 1–3, 2, 5–11, 6–10, 7–9, 8	*
T_5I	<11,5>	0–5, 1–4, 2–3, 6–11, 7–10, 8–9	*
T_6I	<11,6>	0–6, 1–5, 2–4, 3, 7–11, 8–10, 9	*
T_7I	<11,7>	0–7, 1–6, 2–5, 3–4, 8–11, 9–10	*
T_8I	<11,8>	0–8, 1–7, 2–6, 3–5, 4, 9–11, 10	*
T_9I	<11,9>	0–9, 1–8, 2–7, 3–6, 4–5, 10–11	*
T_{10}I	<11,10>	0–10, 1–9, 2–8, 3–7, 4–6, 5, 11	*
T_{11}I	<11,11>	0–11, 1–10, 2–9, 3–8, 4–7, 5–6	*

Similarly, all TTOs of the form $\langle a,0 \rangle$ are reflexive. Table 4 gives the inverses of all TTOs and an asterisk for TTOs which are reflexive.

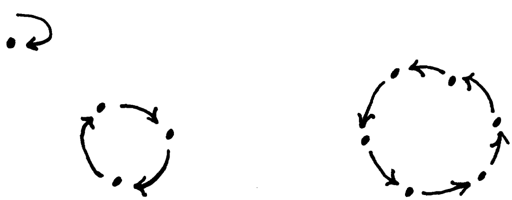
If we apply a TTO successively to some PC, we generate a *cycle.* For example, take the TTO $\langle 5,1 \rangle$ and start with the PC 0:

$$
\begin{aligned}
1 &= (5 \otimes 0) \oplus 1 \\
6 &= (5 \otimes 1) \oplus 1 \\
7 &= (5 \otimes 6) \oplus 1 \\
0 &= (5 \otimes 7) \oplus 1
\end{aligned}
$$

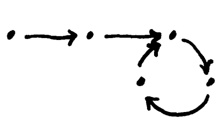


We thus generate the cycle 0–1–6–7–0 . . . which repeats four PCs *ad infinitum.* $\langle 5,1 \rangle$ also generates the cycles 2–11–8–5–2 . . . and 3–4–9–10–3 . . . when we start on other PCs. The TTO F^{-1} will always generate the same cycles as F, but "in reverse order."

The important point to make about cycles is that the PCs in different cycles generated by a TTO are strictly inaccessible to each other. TTOs "map" PCs in patterns like this:



but never like this:



A PC is guaranteed to recur if any TTO is applied to it enough times. Table 4 gives the cycles generated by all 48 TTOs.

Reflexive TTOs can generate cycles of either one or two PCs, but never longer ones. Taking a TTO such as T_7I, T_6, or T_2M—all of which generate cycles of *exactly* 2 PCs—we can construct "invariant rows" by placing the cycles in a symmetrical nest.

T_7I-cycles	0	5	6	10	8	4	3	11	9	1	2	7
T_6M-cycles	0	1	4	2	9	5	11	3	8	10	7	6
T_2M-cycles	0	7	5	4	8	11	9	6	10	3	1	2

These rows have the property that the TTOs which generate them effectively retrograde them.

There are 4,096 distinct unordered combinations of PCs that can be selected from U, including a zero-element combination and a 12-element combination that is the same as U itself. Let us consider a 12-digit binary integer, each of whose digits, or *bits,* corresponds to one of the 12 PCs of U. Specifically, we let the 2^0-bit (lowest order bit) correspond to the PC 0, the 2^1-bit correspond to the PC 1, the 2^2-bit correspond to 2, and so on, with the 2^n-bit always corresponding to the PC n. Given an arbitrary combination of PCs, each bit we set to 1 if the corresponding PC is in the combination, and to 0 if it is absent from the combination. Thus, for instance, the binary integer

0	0	1	1	0	0	1	0	0	0	1	1	bits
11	10	9	8	7	6	5	4	3	2	1	0	PCs

represents a combination of the PCs 0, 1, 5, 8, and 9. There is clearly a one-to-one correspondence between combinations of PCs and 12-bit binary integers, which take on the values zero through 4,095, making 4,096 values in all. We will hereafter use the word *set* loosely to mean a combination of PCs or subset of U.

Two notations for sets suggest themselves: (1) binary notation, as described above,[6] and (2) a list of PCs belonging to a set, which we might enclose within braces:

$$\{0, 1, 5, 8, 9\}$$

Since we are interested here in combinations rather than orderings of PCs, the order of PCs within the braces will be insignificant, so that

$$\{0, 5, 8, 1, 9\} \quad \text{and} \quad \{9, 5, 1, 8, 0\} \text{ etc.}$$

refer to the same set. Binary notation thus has an immediate advantage over brace-notation in that it provides a unique way to notate a set, since it allows no equivalent permutations of the same symbols. In addition, those of us who like to manipulate sets by computer enjoy binary notation, since it corresponds directly to the internal structure of computer memory, and since the correspondence between sets and the integers 0–4,095 lets us evaluate sets by turning them directly into memory addresses or pointers.[7]

We are often interested in the *complement* of a set with respect to U. This will be the set of all PCs in U which do not belong to the set in question. Thus the complement of $\{0, 1, 5, 8, 9\}$ is $\{2, 3, 4, 6, 7, 10, 11\}$ or 110011011100 in binary notation, where complementation amounts to exchanging all "1"s for zeros and vice versa. Where a set has been labelled S, the designation "C(S)" denotes the set which is its complement.

Sets, like individual PCs, can be transformed by a TTO, by applying it to each PC in a set. Thus, for example, the TTO $<5,2>$ transforms the set $\{0, 1, 5, 8, 9\}$ to $\{2, 7, 3, 6, 11\}$. When we apply the TTO F to some set S, the set produced can be written F(S). If S comprises n distinct PCs, so will F(S) comprise n distinct PCs; any pair of PCs in S will correspond to a distinct pair of PCs in F(S)—never to a single PC—which is a corrollary to the property that TTOs preserve all 12 PCs.

The effect of certain TTOs is readily apparent when we see how they transform a set in binary notation. Transpositions—of TTOs of the form $< 1,n >$—rotate the bits n positions:

$$011000101101 \xrightarrow{\quad T_4 \quad} 001011010110$$

Inversions—TTOs of the form $\langle 11,n \rangle$—flip segments of bits around backwards:

$$\begin{array}{c} 011000101101 \\ \big\downarrow T_4I \\ 100011010110 \end{array}$$

If we transform the set $\{4, 3, 5, 9, 11\}$ with the TTO $\langle 11,8 \rangle$ we get the same set that we started with. The set is said to be *invariant* under that particular TTO. $\langle 11,8 \rangle$ sends the PCs of the set to each other so that no new PCs are generated, nor are any lost that already exist:

$$\begin{array}{rcl} 4 & \longrightarrow & 4 \\ 3 & \longrightarrow & 5 \\ 5 & \longrightarrow & 3 \\ 9 & \longrightarrow & 11 \\ 11 & \longrightarrow & 9 \end{array}$$

To indicate this condition we use the phrases, "set S is F-invariant," or "S is invariant under F."

Often a set is invariant under more than one operation. It follows that if S is both F-invariant and G-invariant, where both F and G are TTOs, then S will also be invariant under F · G and G · F, two possible composites of F and G. In addition, a set which is F-invariant will be invariant under F · F and F · (F · F) or under F^{-1}, $F^{-1} \cdot F^{-1}$, etc.

We will now prove the "*invariance theorem*." If some set S consisted of all the PCs belonging to a cycle generated by some TTO F, then S would clearly be F-invariant. F would merely cause the PCs of S to "march around" the cycle. The same would be true of a set formed by combining two or more F-cycles: each cycle would reproduce itself within the bounds of the set. Thus (1) if S comprises an F-cycle or a union of intact F-cycles, then S must be F-invariant.

We will also show that the converse is true. If any set S which is invariant under a TTO F, S must be the union of one or more intact F-cycles, for if S contained only part of a certain F-cycle, the application of F to S would generate at least one of the PCs missing from the incomplete cycle, and S would not be F-invariant. Therefore (2) if S is F-invariant, S is the union of complete F-cycles.

In combining (1) and (2) we conclude that S is F-invariant *if and only if* S is the union of complete F-cycles.

Binary notation can often graphically show the invariance properties of a set. The set $\{2, 3, 6, 7, 10, 11\}$ can readily be seen to be $< 1,4 >$-invariant (transpositionally invariant) since its bits in binary notation can be rotated to re-form the same pattern:

1 1 0 0 1 1 0 0 1 1 0 0 (1 1 0 0)

Inversionally ($< 11,n > -$) invariant sets always form pairs of symmetrical "nests" when pairs of 1s and 0s are properly connected in binary notation:

0 1 1 0 1 1 1 0 0 1 1 1 T_7I-invariance

1 1 0 1 0 0 0 0 0 0 1 0 T_9I-invariance

0 0 1 1 0 1 0 1 1 0 0 1 I-invariance

0 0 1 1 0 1 1 0 0 1 1 1 T_2I-invariance

Note that when the "index of inversion" is even (n, when we are inverting with the TTO T_nI), two bits in the pattern must be tied to themselves, as in the last two examples. This is because an "even inversion" always generates five 2-PC cycles and two 1-PC cycles which are a tritone apart (see Table 4). When a set is T_nI-invariant, the zero-bit (on the far right) will always be tied to the n-bit (n + 1 places from the right).[8] Thus n, if unknown, can be determined visually from an inversion pattern in binary notation.

To find invariances under TTOs of the form T_nM and T_nIM, we must compare a set in binary notation with its transform under M. For example:

$$S = \{0,1,11,8,10,3\} \qquad \xrightarrow{M} \qquad \{0,5,7,4,2,3\}$$

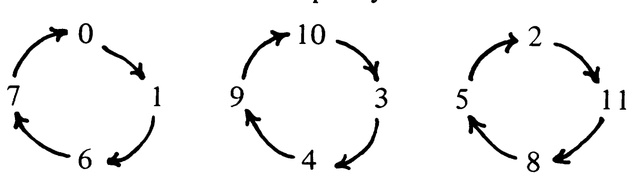
$$\underbrace{110100001011}_{\text{pattern}} \longrightarrow \underbrace{0000101}\underbrace{11101(00001011)}_{\text{pattern}}$$

Here we see that a rotation of the same bit-pattern is produced, showing that S is T_8M-invariant. In the following example, we obtain a rotated mirror of the original pattern resulting from a T_9MI-invariance. Note that the zero bit corresponds to the 9-bit in the transformed set:

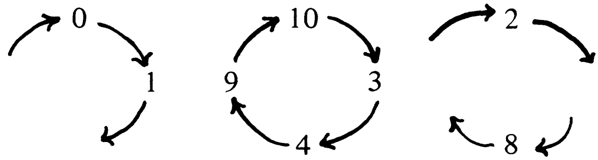
$$S = \{0,9,1,4\} \xrightarrow{\quad M \quad} \{0,9,5,8\}$$

$$\underbrace{001000010011}_{\text{pattern}} \qquad\qquad \underbrace{001100100001(00)}_{\text{pattern}}$$

The invariance theorem suggests an approach to the question of "partial" invariance. When we subject some set to a TTO, how many "new" PCs are generated and how many will remain?[9] Consider the three T_1M-cycles:



Consider also a set S including fragments of these cycles:



$\{0,1,2,3,4,8,9,10\} = S$

Clearly, three new PCs will "enter" S and three will "leave" it when we apply T_1M to it, because S contains three T_1M-cycle fragments: -0–1-, -2-, and -8-. The complete cycle -3–4–9–10- included in S will always be there no matter how many times we apply T_1M, since it is a complete cycle. It will therefore generate no new PCs.

There are certain 6-PC sets for which there exist one or more TTOs which transform them into their complements.[10] The set $\{0, 2, 4, 7, 9, 10\}$, for example, is effectively complemented by the TTO T_3I, whose effect is apparent when we view the set in binary notation. Note that ones are always tied to zeros and vice versa:

0 1 1 0 1 0 0 1 0 1 0 1

There are sets which complement under TTOs of all four multiplicative types, although no succinct nest patterns can be drawn for those involving TTOs of the form T_nM or T_nIM. The class of TTOs utilizable to complement a 6-PC set includes those TTOs which generate invariant rows, in which only cycles of exactly 2 PCs are generated, but also includes those TTOs whose cycles all contain an even number of PCs—for example T_3, T_6, T_1IM, etc. This phenomenon might alternatively be described in terms of invariance: 0, 2, 4, 7, 9, 10, for example, might be said to be T_3IC-invariant.

A set and its complement comprise a *partition* of U, breaking it into two parts. A partition can be written as a "set of sets":

$$\{\{0,1,5,8,9\}, \{2,3,4,6,10,11\}\}$$

We will also consider partitions that break U into 3 or more sets:

$$\{\{0\}, \{5, 1, 3\}, \{2, 4, 6, 7, 8, 9, 10, 11\}\}$$

Each PC occurs exactly once in exactly one component set of such a partition. The cycles generated by a TTO constitute in effect a partition of U. T_1M, for instance, generates the following partition:

$$\{\{0,1,6,7\}, \{2,5,8,11\}, \{3,4,9,10\}\}$$

We say that this is *the partition imposed by* T_1M.

We can take a partition and break up one or more of its component sets into smaller sets. Thus given the partition

$$\{\{0,1,6,7,8\}, \{5,2,3\}, \{10,11,9,4\}\}$$

we might produce one of the following:

$$\{\{0\}, \{1,6,7,8\}, \{5,2,3\}, \{10,11\}, \{9,4\}\}$$
$$\{\{0,1,8\}, \{6,7\}, \{5,3\}, \{2\}, \{10,9,11,4\}\}$$
$$\{\{0,1,6,7,8\}, \{5\}, \{2,3\}, \{4,10\}, \{9,11\}\}$$

All of which are *finer* partitions than the original, which could be regenerated by merging the appropriate components of the new partitions.

There is a *finest partition,* in which U is broken up into 12 1-PC sets. On the other hand, all partitions are finer than the partition in which U occurs intact as a sole partition component. We define any partition to be finer than itself.

Fineness is a *partial ordering,* meaning that only certain pairs of partitions have a fineness relation between them. Some pairs of partitions, instead of being nested within each other, cut across each other and are therefore not related through fineness. For example, B below is finer than A, but C is not, although both B and C have the same number of component sets of the same size:

A. $\{\{0,1,2,3\}, \{4,5,6,7\}, \{8,9,10,11\}\}$

B. $\{\{0,1\}, \{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}, \{10,11\}\}$

C. $\{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}, \{11,0\}\}$

Partitions, which are actually collections of sets, can like individual sets be considered invariant. We say that *a partition* *is F-invariant* when all the component sets of the partition are F-invariant. The following partition, for instance, is M-invariant:

$$\{\{0,9,3\},\ \{1,5,2,10\},\ \{6,4,8,7,11\}\}$$

since by the invariance theorem, all the component sets are unions of one or more M-cycles and are hence individually invariant. Moreover, the partition imposed by M is finer than the partition at hand. In general, the partition imposed by some TTO F is finer than any partition which is F-invariant.

This is useful to us when considering rows which "maintain segmental content" under some operation. Breaking up a row into several segments in effect creates a partition, and any TTO which imposes a finer partition than the one created by segmenting the row will render the contents of the segments invariant. Continuing with the most recent example, we can thus create a row whose segments are M-invariant:

prime form:	1	2	5	10	9	3	0	4	7	8	6	11
transformed by M:	5	10	1	2	9	3	0	8	11	4	6	7

We will now define a kind of set which is conceptually the opposite of a cycle. Consider a TTO whose cycles all have the same number of PCs—T_1M, for example:

0–1–6–7–   11–8–5–2–   3–4–9–10–

If we were to select one PC from each of the T_1M-cycles, we could form sets like the following which we will call T_1M-*cross-sections:*

$$\{0,2,3\} \qquad \{7,2,4\} \qquad \{6,8,10\}$$

Note that these sets are not even partially T_1M-invariant. Applying T_1M to any of them generates 3 new PCs.

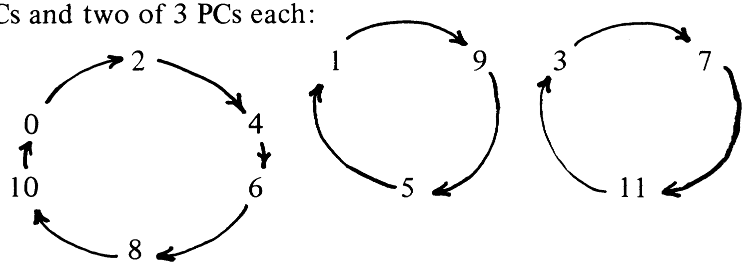
Cross-sections allow us to generate all 12 PCs of U with successive applications of the TTO with respect to which they have been extracted. For example:

U	0	1	6	7	0	1	6	T_1M-cross-sections
	2	11	8	5	2	11		↓
	3	4	9	10	3	4		

T_1M-cycles

⟶   ⟶

The cycles generated by a TTO must be of equal length in order for a cross-section to re-generate U. In some cases, however, something like a cross-section set can be employed instead. The TTO T_2IM, for example, generates one cycle of 6 PCs and two of 3 PCs each:



Sets like these can be formed by taking one PC from each of the shorter cycles and two from the long cycle, which are "half-way-around" from each other. For example:

{1,3,0,6}

short cycles:	1–9–5–	1–9–5 . . . .
	3–7–11–	3–7 . . .
long cycle:	0–2–4–	6–8–10– . . .
	6–8–10–	0–2–4 . . .
	U	U

The long cycle is thus broken in half so that it functions like the two shorter cycles generated by the same TTO. Suffice to say that it is not possible to find cross-sections of all TTOs.

Cross-sections can be used to make rows that form *combination matrices* under a repeated TTO.[11] The successive 4-PC segments of the row 0–1–7–2/10–9–11–4/8–5–3–6, for instance, are T_4-cross-sections. From this row, we can construct a *cycle of rows* which is a matrix of PCs in which transforms of the row run across and T_4-cycles run vertically:

0	1	7	2	10	9	11	4	8	5	3	6
4	5	11	6	2	1	3	8	0	9	7	10
8	9	3	10	6	5	7	0	4	1	11	2
	U				U				U		

The matrix has the property of two-dimensional saturation, with three rows and three larger columns comprising the twelve-tone universe U. Such matrices can be generated by any TTO whose cycles all have the same number of PCs.[12]

If we examine the fineness relation between partitions imposed by different TTOs, we will find, for example, that the $<5,0>$-partition is finer than the $<1,4>$-partition:

$<1,4>$-partition $\{\{0,4,8\}, \{1,5,9\}, \{2,6,10\}, \{3,7,11\}\}$
$<5,0>$-partition $\{\{0\}, \{4,8\}, \{1,5\}, \{9\}, \{1,10\}, \{6\}, \{3\}, \{7,11\}\}$

We say that $<5,0>$ is *stronger* than $<1,4>$. In general, if the partition imposed by a TTO G is finer than the one imposed by a TTO F, we can say that *G is stronger than F.*

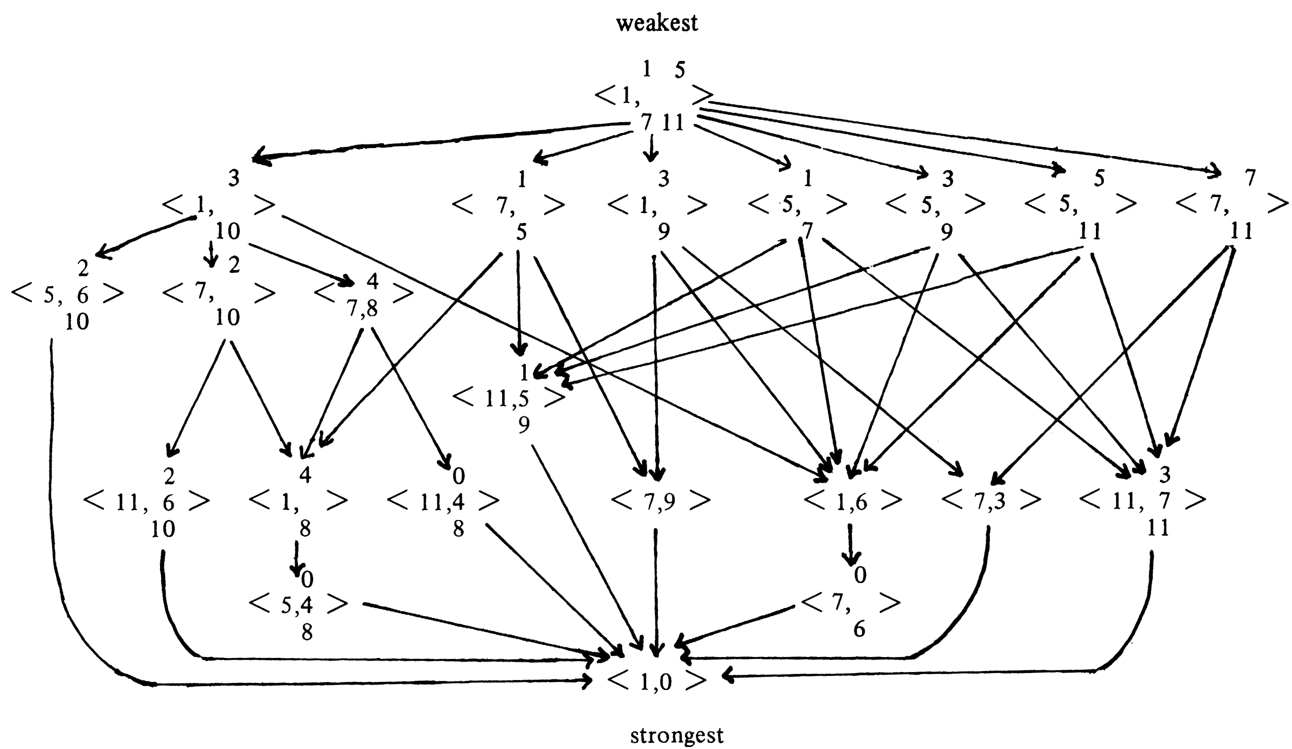
Note that strength is a relation that may exist between TTOs, and, as such, is distinct from fineness, which is a relation defined over partitions. A fineness relation between two partitions imposed by TTOs implies a strength relation between them, although it is possible to speak of fineness in more general contexts in connection with arbitrary partitions which may not necessarily be imposed by TTOs.

Table 5 gives a "strength-lattice" showing the strength relationship between all the TTOs. Since a TTO and its inverse generate the same partition, either has the same strength relationship with a third TTO. Table 5 combines TTOs and their respective inverses into single nodes (along with several trios and a quartet of TTOs which may be treated analogously).

Like fineness, strength is only a partial ordering—only certain pairs of TTOs have a strength relationship. $<5,4>$ and $<11,8>$, for example, have no strength relation (see Table 5), since their imposed partitions cut across each other. Both, however, are stronger than $<1,2>$.

Strength relations are applicable to the manipulation of rows. We continue the previous example in which we generated a cycle of rows based on $<1,4>$. Referring to Table 5, we note that $<5,0>$, $<5,4>$, and $<5,8>$ are all stronger than $<1,4>$, which means that our matrix-columns taken as unordered sets are invariant under all three of these TTOs. We can, for example, subject the entire combination matrix to $<5,0>$:

Table 5. Strength Lattice of TTOs



0	5	10	11	2	9	7	8	4	1	3	6
8	1	7	6	10	5	3	4	0	9	11	2
4	9	3	2	6	1	11	0	8	5	7	10

This columnar invariance suggests an ambiguity that we might exploit in a realization of the PCs into notes: In Ex. 2a the original matrix is "articulated" by placing the three rows above each other in non-overlapping octaves of chromatic space, whereas Ex. 2b articulates the rows of the transformed matrix by placing them in different "instruments"—which might be taken to mean different timbres, locations, etc.—but does so while stating the same notes and verticalities as (a). There are thus two ways to "hear" (b): either as three registrally articulated rows or three instrumentally articulated rows which happen to be a transform of the "registral rows."[13]

Given a set S_0, we can generate a collection of related sets by subjecting S_0 to the various TTOs. Since there are 48 TTOs, we can generate up to 48 related sets (the original one included), but fewer will result where S_0 is invariant under one or more operations. We will call such a collection of related sets a *set-class* (abbreviated "SC"). The set $\{0,1,3,5,6\}$, for example, generates the following SC:

$\{0,1,3,5,6\}$	$\{1,2,4,6,7\}$	$\{2,3,5,7,8\}$
$\{3,4,6,8,9\}$	$\{4,5,7,9,10\}$	$\{5,6,8,10,11\}$
$\{0,6,7,9,11\}$	$\{0,1,7,8,10\}$	$\{1,2,8,9,11\}$
$\{0,2,3,9,10\}$	$\{1,3,4,10,11\}$	$\{0,2,4,5,11\}$

Whatever TTO is applied to one of these sets, another one already in the collection will be produced.

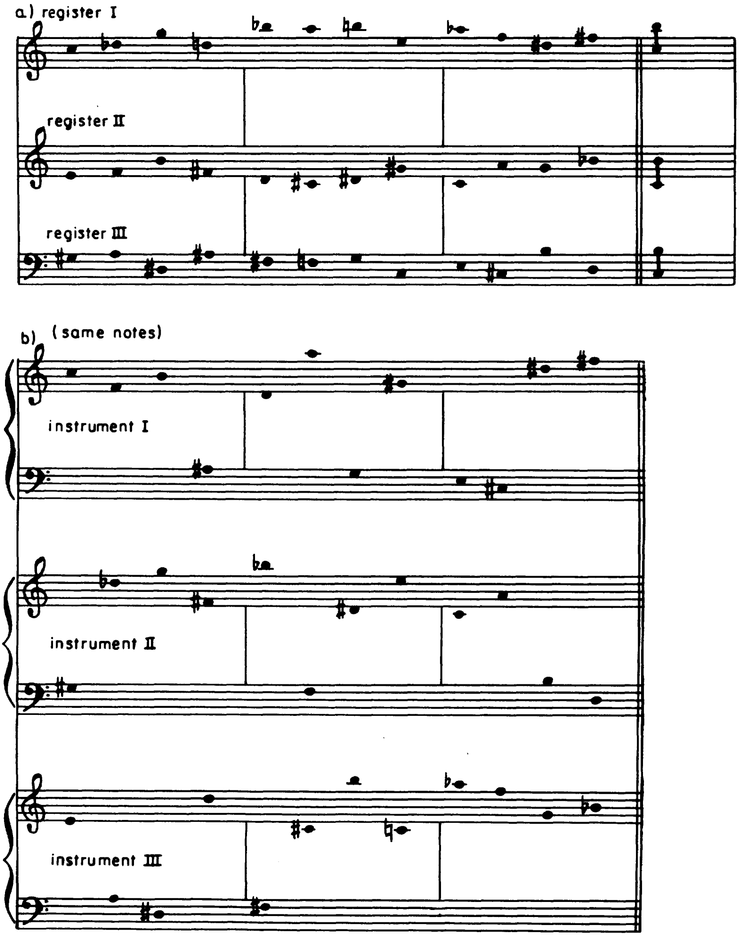
Sets in the same SC will display the same pattern in binary notation, and they will all be invariant under the same number of TTOs. Continuing the above example, we find that

$\{0,1,3,5,6\}$ is invariant under T_6I, M, and T_6IM

$\{1,2,4,6,7\}$ is invariant under T_8I, T_8M, and IM

$\{2,3,5,7,8\}$ is invariant under $T_{10}I$, T_4M, and T_6IM, etc.

Having found that some set S is F-invariant, we may want to know what invariances the set $S' = G(S)$ has where G is some arbitrary TTO. $S' = G(S)$ can be re-written as $G^{-1}(S') = S$.


a) register I
register II
register III
b) (same notes)
instrument I
instrument II
instrument III

Example 2

Both sides can then be subjected to F, giving $F(G^{-1}(S')) = F(S) = S$, and again to G, giving $G(F(G^{-1}(S'))) = G(S) = S'$. At this point we have formulated that $S' = G(S)$ is $(G \cdot F \cdot G^{-1})$-invariant. We can calculate exactly what this composite TTO is. Let $F = \langle a_1,b_1 \rangle$ and $G = \langle a_2,b_2 \rangle$. We know that $G^{-1} = \langle a_2 \ominus (a_2 \otimes b_2) \rangle$, which describes the equation $y = (a_2 \otimes x) \ominus (a_2 \otimes b_2)$. Building on that, $F \cdot G^{-1}$ is described as follows:

$$\begin{aligned} y &= (a_1 \otimes [(a_2 \otimes x) \ominus (a_2 \otimes b_2)]) \oplus b_1 \\ &= (a_1 \otimes a_2 \otimes x) \ominus (a_1 \otimes a_2 \otimes b_2) \oplus b_1 \end{aligned}$$

Finally, $G \cdot F \cdot G^{-1}$ can be written

$$\begin{aligned} y &= a_2 \otimes [(a_1 \otimes a_2 \otimes x) \ominus (a_1 \otimes a_2 \otimes b_2) \oplus b_1] \oplus b_2 \\ &= (a_1 \otimes x) \ominus (a_1 \otimes b_2) \oplus (a_2 \otimes b_1) \oplus b_2 \end{aligned}$$

Therefore

$$G \cdot F \cdot G^{-1} = \langle a_1, \ominus (a_1 \otimes b_2) \oplus (a_2 \otimes b_1) \oplus b_2 \rangle.$$

The multiplicative component of the TTO $G \cdot F \cdot G^{-1}$ is a_1, is the same as in F, the TTO under which the original set was invariant. We have, in effect, simply calculated a new transposition level, taking the place of a_2 in the original invariance. We have thereby proven a theorem which states that for any set S and any TTO G, if S is invariant under some TTO $\langle a,b \rangle$, there is always a b' such that G(S) is invariant under $\langle a,b' \rangle$. Furthermore, b' will take on the value specified in the expression derived above.

Using this formula, we can calculate TTOs under which any transform of a set is invariant without having to examine the set itself. While all the arithmetic operators in the formula are modular, such formulas can generally be evaluated with ordinary arithmetic operators—on a hand-calculator, for instance—and only the final result need be taken mod-12. The set $\{0,1,3,5,6\}$, for example, is invariant under $\langle 11,6 \rangle$, $\langle 5,0 \rangle$ and $\langle 7,6 \rangle$. If we transform the set with the TTO $\langle 5,2 \rangle$, the resultant set would have a corresponding set of three invariances which could be calculated as follows: Corresponding to $\langle 11,6 \rangle$ there would be a TTO

$$\begin{aligned} &\langle 11,[-(11 \times 2) + (5 \times 6) + 2] \text{ mod-}12 \rangle \\ &= \langle 11,[10] \text{ mod-}12 \rangle = \langle 11,10 \rangle. \end{aligned}$$

Corresponding to $\langle 5,0 \rangle$, there would be a TTO

$$< 5,[-(5 \times 2) + (5 \times 0) + 2] \text{ mod-}12 >$$
$$= < 5,[-8] \text{ mod-}12 > = < 5,4 >.$$

And corresponding to $< 7,6 >$, there is a TTO

$$< 7,[-(7 \times 2) + (5 \times 6) + 2] \text{ mod-}12 >$$
$$= < 7,[18] \text{ mod-}12 > = < 7,6 >$$

As a convenience, Table 6 gives a matrix of adjusted transposition operators for all invariances (table rows) under all possible transforms (columns). Thus to adjust the invariance $< 11,6 >$ for a set transformed by the TTO $< 5,2 >$, we would select the table row labelled T_6I (corresponding to $< 11,6 >$), and the column T_2M ($< 5,2 >$) giving an adjusted transposition of 10, telling us that the transformed set would be $< 11,10 >$-invariant.

We may inspect the "begin- and end-sets" of a row in binary notation to find recurring SCs. For this purpose, we will look at these sets both "as they stand" and also transformed by $< 5,0 >$ (see Table 7). The same bit-pattern in the 6-PC sets can be matched up as follows:

6-PC begin-set 0 1 1 0 1 0 0 0 0 1 1 1
6-PC end-set (0 1 1 0 1 0)0 0 0 1 1 1 0 1 1 0 1 0
(under $< 5,0 >$)

This shows that the end-set can be obtained by subjecting the begin-set to $< 5,6 >$. Therefore, by putting the original row in counterpoint with its transform under $< 5,6 >$, we obtain a combination matrix exhibiting "columnar saturation."[14]

0	1	7	2	10	9	|	11	4	8	5	3	6	original row
6	11	5	4	8	3	|	1	2	10	7	9	0	transformed by $< 5,6 >$
		U							U				

Since there were no matches of set-class between begun- and end-sets of other sizes, we know that no other TTO will generate such multi-dimensional saturation (unless we allow retrograde forms of the row!).

The reader will note that this is a case where the 6-PC set in question is T_6MC-invariant. These complementary sets happen to be in the same class, which, however, is not always the case with 6-PC set-complement pairs.

Binary-notated sets can be read simply as binary integers which range from 0 to 4,095. This suggests that a simple, albeit arbitrary, ordering might be imposed on sets. Unless

Table 6. Table of Invariance Adjustments

N.B. To save space, "A" designates 10 and "B" designates 11

TRANSFORMATION=	T_n	T_nI	T_nM	T_nIM
INVAR n=	0123456789AB	0123456789AB	0123456789AB	0123456789AB
P	000000000000	000000000000	000000000000	000000000000
T1	111111111111	BBBBBBBBBBBB	555555555555	777777777777
T2	222222222222	AAAAAAAAAAAA	AAAAAAAAAAAA	222222222222
T3	333333333333	999999999999	333333333333	999999999999
T4	444444444444	888888888888	888888888888	444444444444
T5	555555555555	777777777777	111111111111	BBBBBBBBBBBB
T6	666666666666	666666666666	666666666666	666666666666
T7	777777777777	555555555555	BBBBBBBBBBBB	111111111111
T8	888888888888	444444444444	444444444444	888888888888
T9	999999999999	333333333333	999999999999	333333333333
TA	AAAAAAAAAAAA	222222222222	222222222222	AAAAAAAAAAAA
TB	BBBBBBBBBBBB	111111111111	777777777777	555555555555
I	02468A02468A	02468A02468A	02468A02468A	02468A02468A
T1I	13579B13579B	B13579B13579	579B13579B13	79B13579B135
T2I	2468A02468A0	A02468A02468	A02468A02468	2468A02468A0
T3I	3579B13579B1	9B13579B1357	3579B13579B1	9B13579B1357
T4I	468A02468A02	8A02468A0246	8A02468A0246	468A02468A02
T5I	579B13579B13	79B13579B135	13579B13579B	B13579B13579
T6I	68A02468A024	68A02468A024	68A02468A024	68A02468A024
T7I	79B13579B135	579B13579B13	B13579B13579	13579B13579B
T8I	8A02468A0246	468A02468A02	468A02468A02	8A02468A0246
T9I	9B13579B1357	3579B13579B1	9B13579B1357	3579B13579B1
TAI	A02468A02468	2468A02468A0	2468A02468A0	A02468A02468
TBI	B13579B13579	13579B13579B	79B13579B135	579B13579B13
M	084084084084	084084084084	084084084084	084084084084
T1M	195195195195	B73B73B73B73	519519519519	73B73B73B73B
T2M	2A62A62A62A6	A62A62A62A62	A62A62A62A62	2A62A62A62A6
T3M	3B73B73B73B7	951951951951	3B73B73B73B7	951951951951
T4M	408408408408	840840840840	840840840840	408408408408
T5M	519519519519	73B73B73B73B	195195195195	B73B73B73B73
T6M	62A62A62A62A	62A62A62A62A	62A62A62A62A	62A62A62A62A
T7M	73B73B73B73B	519519519519	B73B73B73B73	195195195195
T8M	840840840840	408408408408	408408408408	840840840840
T9M	951951951951	3B73B73B73B7	951951951951	3B73B73B73B7
TAM	A62A62A62A62	2A62A62A62A6	2A62A62A62A6	A62A62A62A62
TBM	B73B73B73B73	195195195195	73B73B73B73B	519519519519
IM	060606060606	060606060606	060606060606	060606060606
T1IM	171717171717	B5B5B5B5B5B5	5B5B5B5B5B5B	717171717171
T2IM	282828282828	A4A4A4A4A4A4	A4A4A4A4A4A4	282828282828
T3IM	393939393939	939393939393	393939393939	939393939393
T4IM	4A4A4A4A4A4A	828282828282	828282828282	4A4A4A4A4A4A
T5IM	5B5B5B5B5B5B	717171717171	171717171717	B5B5B5B5B5B5
T6IM	606060606060	606060606060	606060606060	606060606060
T7IM	717171717171	5B5B5B5B5B5B	B5B5B5B5B5B5	171717171717
T8IM	828282828282	4A4A4A4A4A4A	4A4A4A4A4A4A	828282828282
T9IM	939393939393	393939393939	939393939393	393939393939
TAIM	A4A4A4A4A4A4	282828282828	282828282828	A4A4A4A4A4A4
TBIM	B5B5B5B5B5B5	171717171717	717171717171	5B5B5B5B5B5B

Table 7

row = 0 1 7 2 10 9 11 4 8 5 3 6

Number PCs	*Begin-set*	*End-set*	*End-set Under <5,0>*
3	{0,1,7}	{3,5,6}	{1,3,6}
4	{0,1,2,7}	{3,5,6,8}	{1,3,4,6}
5	{0,1,2,7,10}	{3,4,5,6,8}	{1,3,4,6,8}
6	{0,1,2,7,9,10}	{3,4,5,6,8,11}	{1,3,4,6,7,8}

in binary notation . . .

3	000010000011	000001101000	000001001010
4	000010000111	000101101000	000001011010
5	010010000111	000101111000	000101011010
6	011010000111	100101111000	000111011010

same pattern rotated
by 6 positions

two sets are the same, they will have a greater-than or less-than relation to each other in their binary form. This amounts to a *total ordering*, since there is always a relation between two arbitrary sets, as opposed to a *partial ordering* like fineness or strength. The sets of a SC may be stacked in an ascending order with the smallest set in binary notation coming first and the largest set coming last. We will arbitrarily define the "least" set–i.e., the first one on the list–as the *normal form* of the SC. This can be used to represent the SC, since many of its properties are shared with the other sets in the class.[15] The particular ordering that we have defined will also have a useful corollary for sets written in brace notation. If we stack the PCs of two sets of the same size in ascending order, the set which comes first alphabetically will turn out to be the "lesser set" as defined above.

Using the normal forms of the various SCs, we can assemble a short table giving the properties of all SCs, and hence, of all sets. This, of course, was the accomplishment of Hanson, Forte, Martino, and others, each of whom constructed such a table to suit a particular collection of definitions of similarity, ordering and operation, which in turn served some larger goal. Hanson's set tabulation system was notable in that he did not connect it, as most of the others have done, with atonal or twelve-tone music, but rather with his own concept of "harmony." None of these systems can be considered more correct than another, unless we have something specific in mind that the system is supposed to achieve. Nevertheless, there is a common thread running through them which consists of basic concepts such as "operation," "equivalence" and "ordering," and a collection of passive objects upon which one operates.

Table 8 gives a set-class-table that follows the definitions given in this paper. It contains fewer sets than most other tables since the cycle-of-fifths operation is treated as part of the system here. The author has had extensive experience with the table in hand-calculations, and in its implementation into various computer programmes. The SCs are put in ascending order (alphabetical) of their respective normal forms.

In most papers treating this topic, set-class concepts are introduced at the onset of the discussion, with the immediate objective of "classifying chords." The present discussion, however, has focused on the nature of the operations that "chords" or sets might be subjected to. Usually, the set-class concept is tied up with that of *interval content,* which I will treat only tangentially here. Interval content as a set-classification scheme conflicts with the inclusion of fourths and fifths transforms as basic 12-tone operators, since these operations alter certain intervals. Since, however, the interval content is in fact changed, it provides us with an aural correlate to TTOs of the form $<5,n>$ and $<7,n>$ that is perhaps more realistic than the contour expansion idea discussed earlier.

Interval content is based on the *interval-class* concept. In chromatic space, we measure intervals as distances up or down from one note to another. In modular space, however, there is no "up" or "down," since PCs are indeterminate in their registral placement. We cannot say, for example, that a vertical alignment of PCs C and E is a major third or minor sixth, as neither is necessarily above the other. Since the choice between octave-compliments of intervals is irresolvable in modular space, they can be merged and treated as if they were the same, giving us *six interval-classes* (abbreviated "ICs") in place of eleven intervals. We say that the unordered PC-pair $\{a,b\}$ belongs to the IC $a \ominus b$ or $b \ominus a$, whichever is the smaller of the two. Thus $\{1,2\}$ and $\{6,5\}$ belong to IC 1; $\{1,5\}$ and $\{8,4\}$ belong to IC 4, etc.

In an n-PC set there are $\frac{n}{2}(n-1)$ possible pairs of PCs, and hence the same number of intervals to be evaluated. A 6-PC set, for example, has 15 PC-pairs. By *interval content,* we generally mean the frequency distribution of ICs among the PC-pairs in a set. The "chromatic hexachord" $\{0,1,2,3,4,5\}$, for example, exhibits the following IC distribution or interval content:

IC	1	2	3	4	5	6
frequency	5	4	3	2	1	0

In this case there are more minor second/major sevenths than any other IC, there are no tritones, etc.

ICs, when subjected to TTOs, behave, in effect, the same way that PCs do. If we subject a PC-pair $\{m,n\}$ (whose IC is the lesser of $m \ominus n$, or $n \ominus m$) to some TTO $<a,b>$, a new PC pair $\{(a \otimes m) \oplus b, (a \otimes n) \oplus b\}$ is produced whose IC is the lesser of the following:

Table 8. Table of Set-Classes

# pcs	sc #	normal form	interval content	Forte-numbers as is	Forte-numbers under <5,0>	binary pattern	invariances, if any			
2	1	01	100000	1	5	000000000011	T1I			
2	2	02	010000	2	2	000000000101	T2I	T2M	IM	
2	3	03	001000	3	3	000000001001	T3I	M	T3IM	
2	4	04	000100	4	4	000000010001	T4I	T4M	IM	
2	5	06	000001	6	6	000001000001	T6	I	T6I	M
							T6M	IM	T6IM	
3	1	012	210000	1	9	000000000111	T2I			
3	2	013	111000	2	7	000000001011				
3	3	014	101100	3	11	000000010011				
3	4	015	100110	4	4	000000100011	M			
3	5	016	100011	5	5	000001000011	T6IM			
3	6	024	020100	6	6	000000010101	T4I	T4M	IM	
3	7	026	010101	8	8	000001000101	IM			
3	8	036	002001	10	10	000001001001	T6I	M	T6IM	
3	9	048	000300	12	12	000100010001	T4	T8	I	T4I
							T8I	M	T4M	T8M
							IM	T4IM	T8IM	
4	1	0123	321000	1	23	000000001111	T3I			
4	2	0124	221100	2	22	000000010111				
4	3	0125	211110	4	14	000000100111				
4	4	0126	210111	5	16	000001000111				
4	5	0127	210021	6	6	000010000111	T2I	T2M	IM	
4	6	0134	212100	3	26	000000011011	T4I			
4	7	0135	121110	11	11	000000101011	M			
4	8	0136	112011	13	13	000001001011	T6IM			
4	9	0137	111111	29	15	000010001011				
4	10	0139	112101	12	27	001000001011				
4	11	013A	122010	10	10	010000001011	T1I	TAM	T3IM	
4	12	0145	201210	7	20	000000110011	T5I			
4	13	0147	102111	18	18	000010010011	IM			
4	14	0148	101310	19	19	000100010011	T8M			
4	15	0149	102210	17	17	001000010011	T1I	T4M	T9IM	
4	16	0156	200121	8	8	000001100011	T6I	M	T6IM	
4	17	0167	200022	9	9	000011000011	T6	T1I	T7I	T1M
							T7M	IM	T6IM	
4	18	0246	030201	21	21	000001010101	T6I	T6M	IM	
4	19	0248	020301	24	24	000100010101	T4I	T4M	IM	
4	20	0268	020202	25	25	000101000101	T6	T2I	T8I	T2M
							T8M	IM	T6IM	
4	21	0369	004002	28	28	001001001001	T3	T6	T9	I
							T3I	T6I	T9I	M
							T3M	T6M	T9M	IM
							T3IM	T6IM	T9IM	
5	1	01234	432100	1	35	000000011111	T4I			

Forte-numbers

# pcs	sc #	normal form	interval content	as is	under <5,0>	binary pattern	invariances, if any		
5	2	01235	332110	2	23	000000101111			
5	3	01236	322111	4	29	000001001111			
5	4	01237	321121	5	14	000010001111			
5	5	01245	322210	3	27	000000110111			
5	6	01246	231211	9	24	000001010111			
5	7	01247	222121	36	36	000010010111	IM		
5	8	01248	221311	13	30	000100010111			
5	9	01249	222220	11	11	001000010111	T4M		
5	10	0124A	232201	8	34	010000010111	T2I		
5	11	01256	311221	6	20	000001100111			
5	12	01258	212221	38	18	000100100111			
5	13	01259	212320	37	17	001000100111	T2I		
5	14	01267	310132	7	7	000011000111	IM		
5	15	01268	220222	15	15	000101000111	T2I	T8M	T6IM
5	16	01346	223111	10	25	000001011011			
5	17	01347	213211	16	35	000010011011			
5	18	01356	222121	12	12	000001101011	T6I	M	T6IM
5	19	01359	122311	26	26	001000101011	M		
5	20	01367	212122	19	19	000011001011	T6IM		
5	21	01369	114112	31	31	001001001011	T6IM		
5	22	01379	122212	28	28	001010001011	IM		
5	23	01458	202420	21	21	000100110011	M		
5	24	01478	202321	22	22	000110010011	T8I	T8M	IM
5	25	02468	040402	33	33	000101010101	T8I	T8M	IM
6	1	012345	543210	1	32	000000111111	T5I		
6	2	012346	443211	2	33	000001011111			
6	3	012347	433221	36	47	000010011111			
6	4	012348	432321	37	48	000100011111	T4I		
6	5	012356	433221	3	25	000001101111			
6	6	012357	342231	9	9	000010101111	T2M		
6	7	012358	333231	40	11	000100101111			
6	8	012359	333321	39	24	001000101111			
6	9	01235A	343230	8	8	010000101111	T3I	M	T3IM
6	10	012367	422232	5	18	000011001111			
6	11	012368	332232	41	41	000101001111	T6IM		
6	12	012369	324222	42	29	001001001111	T3I		
6	13	012378	421242	38	6	000110001111	T3I		
6	14	012456	432321	4	26	000001110111	T6I		
6	15	012458	323421	15	31	000100110111			
6	16	012459	323430	14	14	001000110111	T4M		
6	17	01245A	333321	10	46	010000110111			
6	18	012467	332232	12	12	000011010111	IM		
6	19	012468	241422	22	22	000101010111	T8M		
6	20	01246A	242412	21	34	010001010111			
6	21	012478	322332	17	17	000110010111	IM		

Forte-numbers

# pcs	sc #	normal form	interval content	as is	under <5,0>	binary pattern	invariances, if any
6	22	01247A	234222	45	45	010010010111	T2I T2M IM
6	23	012489	322431	16	16	001100010111	T4M
6	24	012568	322332	43	43	000101100111	T6IM
6	25	012569	313431	44	19	001001100111	
6	26	012678	420243	7	7	000111000111	T6 T2I T8I T2M
							T8M IM T6IM
6	27	013467	324222	13	50	000011011011	T7I
6	28	013469	225222	27	27	001001011011	T9IM
6	29	01346A	234222	23	23	010001011011	T4I TAM T6IM
6	30	013479	224322	49	49	001010011011	T4I T4M IM
6	31	013569	224322	28	28	001001101011	T6I M T6IM
6	32	013679	224223	30	30	001011001011	T6 IM T6IM
6	33	014589	3C363C	20	20	001100110011	T4 T8 T1I T5I
							T9I M T4M T8M
							T1IM T5IM T9IM
6	34	02468A	C60603	35	35	010101010101	T2 T4 T6 T8
							TA I T2I T4I
							T6I T8I TAI M
							T2M T4M T6M T8M
							TAM IM T2IM T4IM
							T6IM T8IM TAIM
7	1	0123456	654321	1	35	000001111111	T6I
7	2	0123457	554331	2	23	000010111111	
7	3	0123458	544431	3	27	000100111111	
7	4	0123467	544332	4	27	000011011111	
7	5	0123468	453432	9	24	000101011111	
7	6	0123469	445332	10	25	001001011111	
7	7	012346A	454422	8	34	010001011111	T4I
7	8	0123478	533442	6	20	000110011111	
7	9	0123479	444342	12	12	001010011111	T4I T4M IM
7	10	0123567	543342	5	14	000011101111	
7	11	0123568	444342	36	36	000101101111	T6IM
7	12	0123569	435432	16	32	001001101111	
7	13	012356A	444441	11	11	010001101111	M
7	14	0123589	434442	18	38	001100101111	
7	15	0123678	532353	7	7	000111001111	T6IM
7	16	0123679	434343	19	19	001011001111	IM
7	17	0124568	443532	13	30	000101110111	
7	18	0124569	434541	17	37	001001110111	T6I
7	19	0124589	424641	21	21	001100110111	T4M
7	20	012458A	344532	26	26	010100110111	M
7	21	0124678	442443	15	15	000111010111	T8I T8M IM
7	22	012467A	344433	28	28	010011010111	IM
7	23	012468A	262623	33	33	010101010111	T2I T8M T6IM
7	24	0125689	424542	22	22	001101100111	T2I T8M T6IM

Forte-numbers

# pcs	sc #	normal form	interval content	as is	under <5,0>	binary pattern	invariances, if any			
7	25	0134679	336333	31	31	001011011011	IM			
8	1	01234567	765442	1	23	000011111111	T7I			
8	2	01234568	665542	2	22	000101111111				
8	3	01234569	656542	3	26	001001111111	T6I			
8	4	01234578	655552	4	14	000110111111				
8	5	01234579	565552	11	11	001010111111	T4M			
8	6	0123457A	566452	10	10	010010111111	T5I	T2M	T3IM	
8	7	01234589	645652	7	20	001100111111	T5I			
8	8	01234678	654553	5	16	000111011111				
8	9	01234679	556453	13	13	001011011111	IM			
8	10	0123467A	556543	12	27	010011011111				
8	11	01234689	555553	15	29	001101011111				
8	12	0123468A	474643	21	21	010101011111	T4I	TAM	T6IM	
8	13	01234789	644563	8	8	001110011111	T4I	T4M	IM	
8	14	01235678	654463	6	6	000111101111	T8I	T2M	T6IM	
8	15	01235689	546553	18	18	001101101111	T6IM			
8	16	0123569A	546652	17	17	011001101111	T3I	M	T3IM	
8	17	01236789	644464	9	9	001111001111	T6	T3I	T9I	T3M
							T9M	IM	T6IM	
8	18	01245689	545752	19	19	001101110111	T8M			
8	19	0124568A	464743	24	24	010101110111	T6I	M	T6IM	
8	20	0124678A	464644	25	25	010111010111	T6	T2I	T8I	T2M
							T8M	IM	T6IM	
8	21	0134679A	448444	28	28	011011011011	T3	T6	T9	T1I
							T4I	T7I	TAI	T1M
							T4M	T7M	TAM	IM
							T3IM	T6IM	T9IM	
9	1	012345678	876663	1	9	000111111111	T8I			
9	2	012345679	777663	2	7	001011111111				
9	3	012345689	767763	3	11	001101111111				
9	4	01234568A	686763	6	6	010101111111	T6I	M	T6IM	
9	5	012345789	766773	4	4	001110111111	T4M			
9	6	012346789	766674	5	5	001111011111	IM			
9	7	01234678A	676764	8	8	010111011111	T6IM			
9	8	01234679A	668664	10	10	011011011111	T4I	T4M	IM	
9	9	01245689A	666963	12	12	011101110111	T4	T8	T2I	T6I
							TAI	M	T4M	T8M
							T2IM	T6IM	TAIM	
10	1	0123456789	988884	1	5	001111111111	T9I			
10	2	012345678A	898884	2	2	010111111111	T8I	T2M	T6IM	
10	3	012345679A	889884	3	3	011011111111	T7I	T4M	T3IM	
10	4	012345689A	888984	4	4	011101111111	T6I	M	T6IM	
10	5	012346789A	888885	6	6	011111011111	T6	T4I	TAI	T4M
							TAM	IM	T6IM	

$$((a \otimes m) \oplus b) \ominus ((a \otimes n) \oplus b) = a \otimes (m \ominus n)$$

or

$$((a \otimes n) \oplus b) \ominus ((a \otimes m) \oplus b) = a \otimes (n \ominus m)$$

In other words, instead of computing a new IC from the transformed PCs, the same result can be reached by multiplying the initial IC by a and then taking its octave-complement to get it back within the range of 1 to 6. Furthermore, when operating on an IC with the TTO $< a,b >$, only a has to be considered, since b "drops out of the equation."

TTOs of the form $< 1,b >$ and $< 11,b >$ preserve interval content, while those of the form $< 5,b >$ and $< 7,b >$—i.e., those which are composites including **M**—preserve ICs 2, 3, 4, and 6 while exchanging ICs 1 and 5. Therefore, to continue a previous example, when we subject the chromatic hexachord to the TTO $< 7,0 >$, we get the following IC frequency distribution in the resultant "Pythagorean Hexachord," {0,7,2,9,4,11} :

ICs	1	2	3	4	5	6
frequency	1	5	3	2	5	0

Note that numbers of "chromatic" and "perfect" ICs have been exchanged Thus there are typically two species of IC-distribution among the members of a SC, with the frequencies of ICs 1 and 5 exchanged, unless, of course, those frequencies are the same.

On the other hand, there are IC frequency distributions common to more than one SC. The following sets, for example, share the distribution 4–3–3–2–2–1. They belong, however, to separate SCs, which is readily seen from their different patterns in binary notation:[16]

{0,1,2,3,5,6}	{0,1,2,3,4,7}
000001101111	000010011111

Thus, as defined here, interval content equivalence and SC-equivalence are different set classification schemes which "cut across" each other.[17]

Let us assume that A and B are two complementary sets and that the number of PCs in A is greater than or equal to the number of PCs in B. The *complement theorem* tells us that for each IC n, the larger set (A) will have as many or more instances of IC n than the smaller set. Specifically, where #A and #B denote the numbers of PCs in A and B respectively, there will be (#A – #B) more instances of IC n in A than in B, where $n \neq 6$, and there will be $\frac{1}{2}$(#A – #B) more where n = 6. For example, let us choose the following complementary 7- and 5-PC sets A and B:

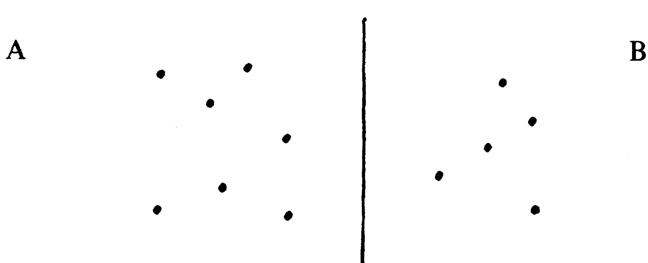
$$A = \{0,1,2,4,5,6,8\} \qquad B = \{3,7,9,10,11\}$$

Their respective interval contents are 4–4–3–5–3–2 and 2–2–1–3–1–1, which means that the frequency of each IC in A is always 2 greater than the corresponding frequency in B, with the exception of IC 6, whose frequency in A is exactly one greater in A than in B.

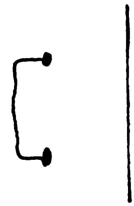
To prove that this is always the case, let us first define *pairs (n,S)* as the set of all unordered PC-pairs $\{s_1,s_2\}$ which belong to some set S and form an instance of IC n. For example, let S = $\{0,1,4,5,7\}$. Pairs (1,S) = $\{\{0,1\}, \{4,5\}\}$, pairs(2,S) = $\{\{5,7\}\}$, pairs(5,S) = $\{\{0,5\}, \{0,7\}\}$, etc. Therefore the size of the set pairs(n,S), which we might write as # pairs(n,S), is the frequency of IC n in S.

Where $n \neq 6$, each PC in S can participate in up to two PC-pairs in pairs(n,S), namely $\{s, s \oplus n\}$ and $\{s \ominus n, s\}$. Since $\{s, s \oplus 6\}$ is the same PC-pair as $\{s \ominus 6, s\}$, each PC can participate in only one PC-pair in pairs(6,S). Thus for $n \neq 6$, pairs (n,U) contains 12 PC-pairs, while pairs(6,U) contains only 6.

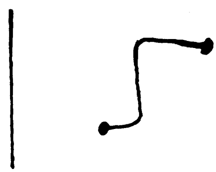
Now let us return to our complementary sets A and B, which we might symbolize with 12 points separated by a line:



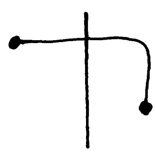
Each of the six or 12 PC-pairs(n,U) will be located either (1) on the left, with both pair-members in A, contributing to the interval-content of A:



or (2) on the right with both pair-members in B, contributing to the interval-content of B:



or (3) with a pair-member on either side, contributing to the interval-content of neither A nor B:



If we restrict ourselves momentarily to the case where n $\neq$ 6, each PC of U participates in two of the 12 PC-pairs in pairs (n,U), making 24 *pair-memberships* of PCs in PC-pairs at large. Since #A $\geqslant$ #B, the pair-memberships in A will outnumber those in B, since their number is proportional to the number of PCs in either set. There will be twice as many pair-memberships in either set as PCs, which means that there will be 2(#A - #B) more pair-memberships in A than in B.

If we prune the pair-memberships of A and B of those pair-memberships in category (3) above–i.e., those pair-memberships which straddle the line and contribute to the interval content of neither A nor B—we subtract the *same number* of pair-memberships from both sides. The pruned A-side would still have twice as many pair-memberships as the pruned B-side —namely 2(#A - #B). The pruned collections of pair-memberships would then only pertain to pairs in categories (1) and (2) above, which reside completely on one side of the line. Therefore, the number of pruned pair-memberships associated with IC n (still $\neq$ 6) in A is exactly twice # pairs(n,A), the frequency of IC n in A. The same is true for B. In a similar fashion we can divide the excess of pair-memberships in A in half to deduce that there are (#A - #B) more instances of IC n in A than in B for n $\neq$ 6.

The same reasoning applies to n = 6, but because # pairs (6,U) = 6 there are only 12 pair-memberships at large. This leads to the conclusion that # pairs(6,A) exceeds # pairs(6,B) by $\frac{1}{2}$(#A - #B).

The complement theorem tells us that when dealing with complementary 6-PC sets A and B, the difference in frequency of each interval-class between A and B will be (#A - #B), which equals zero. Complementary 6-PC sets therefore have the same interval content. This is known as the *hexachord theorem.*

NOTES

1. Howard Hanson, *The Harmonic Materials of Twentieth-Century Music* (New York: Appleton-Century-Crofts, 1960).
2. Donald Martino, "The Source Set and Its Aggregate Formations," *Journal of Music Theory* 5/2 (1961):224–273.
3. Allen Forte, *The Structure of Atonal Music* (New Haven: Yale University Press, 1973).
4. The reader is referred to two articles which this author wrote in collaboration with Robert Morris: "The Structure of All-Interval Series," *Journal of Music Theory* 18/2 (1974):364–389, and "A General Theory of Combinatoriality and the Aggregate," forthcoming in *Perspectives of New Music.*
5. For information concerning modular systems, the reader is asked to consult a modern algebra textbook such as Birkhoff and Bartee, *Modern Applied Algebra* (New York: McGraw-Hill, 1970).
6. A similar notation is used by Chrisman in his article "Identification and Correlation of Pitch-Sets," *Journal of Music Theory* 15 (1971): 58–83.

7. Bo Alphonce of Yale University has arrived at this independently and has incorporated it into his computer programs.
8. See John Rothgeb, "Some Ordering Relationships in the Twelve-Tone System," *Journal of Music Theory* 11/2 (1967):176–197, especially Tables 1 and 2, pp. 180–81.
9. Allen Forte, *op. cit.*, p. 42ff.
10. Forte's 6-PC "z-sets" specifically lack this property.
11. Starr and Morris, "A General Theory of Combinatoriality and the Aggregate," forthcoming in *Perspectives of New Music.*
12. *Ibid.* T_n-cross-sections can also be regarded in terms of moduli lower than 12 (see above). A T_n-cross-section is a set, all of whose member PCs are different when taken mod-n.
13. Strength relations provide us with a rule whereby a cycle of rows in a combination matrix may be substituted: if F is stronger than G, then a G-cycle of rows, with their transforms, may be subjected to F. Strength relations provide a general rule which tells us when substitutions can be made. This rule is more all-inclusive than the rules presented in sections 3.3–3.5 in Starr and Morris, "A General Theory of Combinatoriality and the Aggregate."
14. *Ibid.*
15. In *The Structure of Atonal Music* and in papers, Forte defines "normal form" in a different, but functionally equivalent manner. He recommends a hand-algorithm for calculating the normal form of a set as he defines it. The differences between this author's ordering and Forte's are discussed to some extent by Bo Alphonce in "The Invariance Matrix" (unpublished dissertation, Yale University, 1974). See also Milton Babbitt, "Set Structure as a Compositional Determinant," *Journal of Music Theory* 5/2 (1961):72–94, and Richard Teitelbaum, "Intervallic Relations in Atonal Music," *Journal of Music Theory* 9/1 (1965):72–127.
16. This category of sets was first described by David Lewin in "The Intervallic Content of a Collection of Notes," *Journal of Music Theory* 4/1 (1960):98–101. The term "z-relation" was coined by Forte, *op. cit.*
17. In the systems of Hanson and Forte there is no M operator: SC-equivalence becomes simply a finer partition of all sets than interval-content equivalence.