WF Scales, ME Sets, and Christoffel Words

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Abstract. With a few exceptions (Chemillier and Truchet 2003), (Chemillier 2004), musical scale theory and combinatorial word theory have remained unaware of each other, despite having an intersection in methods and results that by now is considerable. The theory of words has a long history, with many developments coming in the last few decades; see Lothaire 2002 for an account. The authors thank Franck Jedrzejewski for an initial reference in word theory. The purpose of this paper is to translate between the language of two closely related scale theories and that of the theory of words.

1 Well-Formed Scales

A *scale of N notes* is a subset $\Sigma \subset \mathbb{R}/\mathbb{Z}$ of cardinality N . Geometrically, one can represent a scale Σ as a set of N points around the circle. One can associate to every scale Σ a bijection (natural order of Σ) $\sigma : \mathbb{Z}_N \to \Sigma$ so that $\sigma(k) =$kth note of the scale, $(0 \leq \sigma(0) < \sigma(1) < \cdots < \sigma(N-1) < 1)$.

We say that a scale Σ is *generated* if $\Sigma = \{k\theta \bmod 1, k = 0, \ldots, N-1\}$. The number θ is called a generator of Σ and determines a bijection γ (generation order of Σ)

$$\begin{array}{lcl} \mathbb{Z}_N & \xrightarrow{\gamma} & \Sigma \\ k & \longmapsto & \gamma(k) = k\theta \bmod 1 \end{array}$$

Definition 1. *A generated scale is* well-formed *if the natural order is a multiplicative permutation of the generation order (formally a scale Σ is WF $\Leftrightarrow \gamma^{-1} \circ \sigma \in Aut(\mathbb{Z}_N) \Leftrightarrow \sigma(k) = \{(gk)_{\bmod N}\theta\}$ with $g \in \mathbb{Z}_N^*$).*

Well-formed scales were introduced in Carey and Clampitt 1989 as the scales that share the main properties of the diatonic scale, with equal-interval scales as limiting scales (degenerate well-formed scales). A scale Σ is non-degenerate well-formed if and only if it satisfies Myhill's property, i.e., there are exactly

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two specific interval types of each non-zero generic length (two seconds or steps, two thirds, etc.), demonstrated in Carey and Clampitt 1996. Let $WF(N,g)$ be the class of WF scales of N notes in which g is the multiplier that transforms generational order into natural order. N. Carey and D. Clampitt proved that the scales of $WF(N,g)$ share the following properties:

- Given Σ as defined above, if it is a member of $WF(N,g)$, then the first generated step is $a = \{g\theta\}$ and the last one is $b = 1 - \{(N-g)\theta\}$. They appear $N-g$ and g times, respectively, within the octave.
- g and N are coprime, and the numbers $\{g^{-1} \bmod N, (N-g)^{-1} \bmod N\}$ are the generic lengths (number of steps) of the generator θ, and of $1-\theta$.
- The generator contains $y_g = \left[\frac{g^{-1}_{\text{mod}\,N} \cdot g}{N}\right]$ steps of length a and $x_g = g^{-1}_{\text{mod}\,N} - y_g$ steps of length b.

The scales which share the same number of notes and the same multiplier have therefore all discrete parameters in common. The real lengths of both steps a, b and generator θ may vary within limits given by the following:

Proposition 1. *If $(\Sigma, \sigma) \in WF(N,g)$ is a scale with steps $a = \sigma(1)$, $b = 1 - \sigma(N-1)$ and generator θ, then a, b and θ are verify the following restrictions:*

$$\left.\begin{aligned} y_g a + x_g b &= \theta \\ (N-g)a + gb &= 1 \end{aligned}\right\} (1) \qquad \left.\begin{aligned} 0 < a &< \tfrac{1}{N-g} \\ 0 < b &< \tfrac{1}{g} \\ \tfrac{x_g}{g} < \theta &< \tfrac{y_g}{N-g} \end{aligned}\right\} (2)$$

Proof. The equations (1) are clear. For the inequalities, we have the following limiting cases:

1. If we take θ big enough so that b becomes 0, we lose g notes, obtaining a scale of pattern $\underbrace{aa\ldots a}_{N-g}$ This is a well-formed degenerate scale of length $N-g$, hence $a = \frac{1}{N-g}$ and the generator has generic length $y_g : \theta = \frac{y_g}{N-g}$
2. If we reduce θ so that $a = 0$, we lose $N-g$ steps and we get a well-formed degenerate scale of length g. In this scale $b = \frac{1}{g}$ and $\theta = \frac{x_g}{g}$. ∎

In accordance with proposition 1, the set of scales $WF(N,g)$ can be represented geometrically as the segment of the line given by the equations (1) contained in the parallelepiped defined by the inequalities (2):

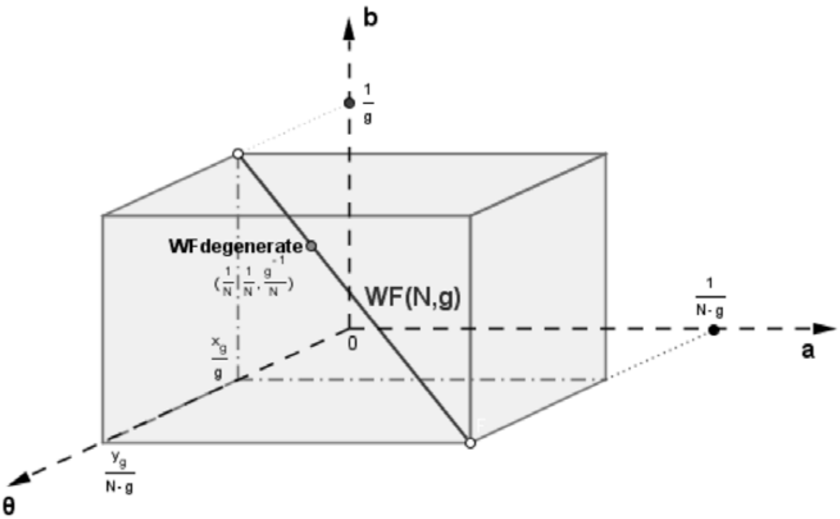

Fig. 1.

2 Christoffel Words

In this section we introduce a pair of geometrical definitions for the word corresponding to the step patterns of well-formed scales. These words are the rational degenerate cases of Sturmian words (infinite aperiodic words with minimal complexity).

Given a real number α such that $0 \leq \alpha < 1$, we define the *(lower) mechanical word* as the infinite word:

$$\begin{aligned} s_\alpha : \mathbb{N} &\longrightarrow \quad \mathbb{Z}_2 \\ n &\longmapsto \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor \end{aligned}$$

In figure 2 we observe the geometrical interpretation of mechanical words.

The line $y = \alpha x$ determines a succession of points of coordinates $P_n = (n, \lfloor \alpha n \rfloor)$ (the succession of points with integer coordinates which are closer to the line and bellow it). $s_\alpha(n) = 0$ (resp. $s_\alpha(n) = 1$) iff the segment $\overline{P_n P_{n+1}}$ is horizontal (resp. diagonal).

If $\alpha = \frac{p}{n} \in \mathbb{Q}$, the mechanical word s_α is periodic and its minimal period is n letters long. In the following lines we will identify each periodic word with its minimal period.

Definition 2. Christoffel words *are rational mechanical words* $s_{\frac{p}{n}}$.

Remark 1. A second way of generating Christoffel words is known as *cutting sequences*. Let $\{Q_n\}_{n \in \mathbb{Q}}$ be the intersection of the line $y = \alpha x$ with the integer grid (the set of lines $\{x = n, y = m\}_{n,m \in \mathbb{Z}}$). We will say that $Q_n = (x_n, y_n)$ is horizontal (resp. vertical) if y_n (x_n) is an integer (if both coordinates are integer numbers, we replace Q_n by two points, the first one horizontal, the second one vertical). We define the *cutting sequence* of the line $y = ax$ as the word

$$C_\alpha(n) = \begin{cases} 0 \text{ if } Q_n \text{ is horizontal} \\ 1 \text{ if } Q_n \text{ is vertical} \end{cases}$$

It can be proven (see Lothaire 2002) that $C_\alpha = S_{\frac{\alpha}{\alpha+1}}$, and therefore Christoffel words can be generated also as cutting sequences.

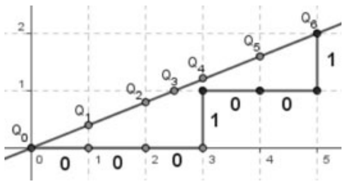

Fig. 2. Cutting sequence $C_{\frac{2}{5}}$

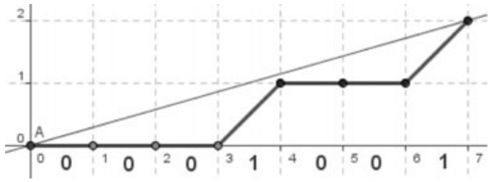

Fig. 3. Mechanical word $S_{\frac{2}{7},0}$

We define the slope of a Christoffel word as the slope of the line that generates it as a cutting sequence. If w is a Christoffel word of length n and slope $\frac{p}{q}$, then $p = |w|_1$, $q = |w|_0$ and w can be generated also as a mechanical word with a line of slope $\frac{|w|_1}{|w|_0 + |w|_1} = \frac{|w|_1}{|w|} = \frac{p}{n}$.

3 Well-Formed Classes and Christoffel Words, Duality

In this section we identify the step pattern of well-formed scales with Christoffel words. An identification between dualities obtaining in both domains can also be established.

Let $\Sigma \in WF(N, g)$ be a well-formed scale with steps a, b, and generator θ. We call the *step pattern* of Σ the finite binary word w given by the following characteristic function:

$$w_\Sigma(i) = \begin{cases} 0 \text{ if the step } \sigma(i) - \sigma(i-1) = \{g\theta\} = a \\ 1 \text{ if the step } \sigma(i) - \sigma(i-1) = 1 - \{(N-g)\theta\} = b \end{cases}$$

Proposition 2. *A scale Σ is non-degenerate well-formed of the class $WF(N, g)$ if and only if its step pattern is a Christoffel word.*

Proof. Let us compute the step pattern of Σ :

$$\sigma(i) - \sigma(i-1) = \{(i+1)g_{\text{mod } N}\theta\} - \{ig_{\text{mod } N}\theta\} =$$
$$= \{((i+1)g_{\text{mod } N} - ig_{\text{mod } N})\theta\} \quad = \begin{cases} \{g\theta\} \\ \{(N-g)\theta\} \end{cases}$$

and therefore

$$w(i) = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow (i+1)g_{\mathrm{mod}\,N} - ig_{\mathrm{mod}\,N} = \begin{cases} g \\ g-N \end{cases}$$
$$\Leftrightarrow [\frac{(i+1)g}{N}] - [\frac{ig}{N}] = \begin{cases} 0 \\ 1 \end{cases}$$

Thus the step pattern of Σ is:

$$w(i) = [\frac{(i+1)g}{N}] - [\frac{ig}{N}]$$

This is exactly the definition of a Christoffel word of slope $\frac{g}{N-g}$. ■

By definition, the Christoffel word of a given slope is unique. Therefore, two scales have the same pattern if and only if they are in the same class $WF(N, g)$. Hence, these classes can be interpreted as equivalence classes modulo scale patterns. The only exceptional case is the degenerate scale; it is the unique scale of $WF(N, g)$ (the only point of the segment defined in proposition 1) with a different pattern (steps a and b are indistinguishable).

In Carey and Clampitt 1996 the notion of duality between classes of WF scales was established as follows. Given a class of WF scales with parameters N and g, the multiplicative inverse of $g \bmod N$ is the span (generic length) of the generating interval. In the dual class, the notions "generic lengths of the generators" and "number of steps of each type" exchange roles.

Definition 3. *Given a well-formed class $WF(N, g)$, its dual class is $WF(N, g^{-1} \bmod N)$.*

Every Christoffel word w of length n and slope $\frac{p}{q}$ decomposes in a unique way as $w = x \cdot y$, with (x, y) a Christoffel pair (see proposition 6) The lengths of the subwords x and y are, respectively $q^{-1} \bmod n$ and $p^{-1} \bmod n$.

Definition 4. *The dual word of w is the only Christoffel word w^* of length n and slope $\frac{p^{-1} \bmod n}{q^{-1} \bmod n}$.*

The lengths of the subwords that comprise the Christoffel pair associated with w^* are given by (p, q). Thus, with duality the notions "frequencies of 1's and 0's" and "length of the associated Christoffel pair" exchange roles.

The Christoffel word of the dual class $WF(N, g^{-1} \bmod N)$ is the Christoffel word of slope $\frac{g^{-1} \bmod N}{N-g^{-1} \bmod N} = \frac{g^{-1} \bmod N}{(N-g)^{-1} \bmod N}$. This word coincides, by definition, with the dual word w^* of the step pattern of the original class $WF(N, g)$.

Conclusion 1. *Duality between WF scale classes and duality between their related step patterns (viewed as Christoffel words) coincide.*

4 Christoffel Words, Maximally Even Sets and Musical Modes

In this section we relate Christoffel words with the patterns of maximally even sets. These sets were introduced in Clough and Douthett 1991 to describe the scales in which notes are distributed as evenly as possible within a chromatic universe. (Geometrically, one can consider the problem of distributing d black points and $c-d$ white ones around the circle, so that both colors are as mixed as possible.) A d note scale in a universe of c notes is a maximally even set if it is equivalent modulo rotations to the subset $J_{d,c} = \{[k\frac{c}{d}]; k = 0, \ldots, d-1\} \subset \mathbb{Z}/c\mathbb{Z}$. The set of maximally even sets of d notes in a universe of size c is denoted by $ME(c,d)$.

The *characteristic set* of a well-formed scale can be defined as:

$$X_\Sigma = \{i \in \mathbb{Z}/N\mathbb{Z} \text{ so that.} w_\Sigma(i) = 1\} = \left\{\begin{matrix}\text{Place of the } b \text{ steps} \\ \text{of } \Sigma \text{ in scale order}\end{matrix}\right\}$$

N. Carey and D. Clampitt have computed the characteristic set of a WF scale Σ of N notes and multiplier g and have shown that it is conjugated with the set $J^0_{g,N} = \{[k\frac{N}{g}]; k = 0, \ldots, g-1\}$ (see Carey 1998). If we denote by T_k the translation of pc-sets (that is, $T_k X = \{(x+k) \bmod N \text{ with } x \in X\}$) and by I the inversion ($IX = \{-x \bmod N, \text{x} \in X\}$), we can reformulate their results as follows:

Proposition 3. *Let $\Sigma \in WF(N,g)$ be a well-formed scale. The characteristic set of Σ is given by the expression:*

$$X_\Sigma = T_{-1}\left\{\frac{kN + k(-N)_{\mathrm{mod}\, g}}{g}; k = 0, \ldots, g-1\right\}$$

Proposition 4. *The characteristic set X_Σ of the scale $\Sigma \in WF(N,g)$ verifies*

$$T_1 X_\Sigma = I J^0_{g,N}$$

Corollary 1. *The characteristic set of a class $WF(N,g)$ is maximally even.*

Proof. It is a consequence of the properties of ME sets that are proved in (Clough and Douthett 1991): ME sets are invariant under inversion I and rotations T_k. ∎

As noted above, one can consider a maximally even set as a distribution of points around a circle. In this context it is convenient to introduce *cyclic words*:

Definition 5. *Two words u and v are conjugated if and only if $u = xy, v = yx$, for some words x, y.*

If one understands words as written circularly, conjugation can be thought of as an equivalence relation — via circular rotations — whose classes are called *cyclic words.*

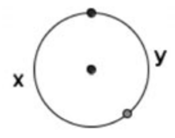

Fig. 4. Conjugation can be viewed as a rotation congruence

Definition 6. *Given a scale $\Sigma \in WF(N,g)$ with pattern w_Σ each of the representatives of the cyclic word $[w_\Sigma]$ is called a* mode of the scale Σ.

In the next table we present the traditional modes of the diatonic scale, which are rotations of the scale pattern associated to $WF(2,7)$.

Mode	*Pattern*	*ME set*
Lydian	0001001	$X_C = \{3,6\}$
Mixolydian	0010010	$T_6 X_C = \{2,5\}$
Aeolian	0100100	$T_5 X_C = \{1,4\}$
Locrian	1001000	$T_4 X_C = \{0,3\}$
Ionian	0010001	$T_3 X_C = \{2,6\}$
Dorian	0100010	$T_2 X_C = \{1,5\}$
Phrygian	1000100	$T_1 X_C = \{0,4\}$

Proposition 5. *Given $g < N$, two coprime integers, there is a natural bijection between the set $ME(N,g)$ and the modes of non-degenerate scales Σ in $WF(N,g)$.*

Proof. One just has to associate, to every set X in $ME(N,g)$, its characteristic word w_X. Every ME set of $ME(N,g)$ is a rotation of X. Hence, by corollary 1, w_X is a rotation of the scale pattern of $WF(N,g)$. ■

Remark 2. Maximal evenness is one of the modern topics in mathematical music theory with influence and applications in other fields:

1. A new formulation of maximal evenness has been given by E. Amiot in (Amiot 2007). The author uses the language of complex analysis, in particular Discrete Fourier Transforms, and shows in a very elegant way the basic properties of ME sets and a equivalence between this formulation and the classic definition.
2. G. Toussaint, F. Gomez-Martín and other authors have studied recently geometric aspects of musical rhythm. In particular they translated the idea of maximizing the evenness between onsets (d) and time-span (c) of the desired rhythm. The resulting rhythms, called Euclidean rhythms, turned out to be particularly attractive when c and d are coprime. For more details see (Damiane and Gomez-Martín 2006), (Toussaint 2005a) and (Toussaint 2005b). For a connection between maximally even sets and many other topics of science, such as calendar design, euclidean algorithm for

computing the g.c.d., spallation neutron source in nuclear physics,etc., see (Damiane and Gomez-Martín 2006).

3. T. Noll introduced the term *Clough word* in (Noll 2007) for the cyclic words whose characteristic set is maximally even with coprime frequencies. In this paper he also established a recursive construction of these kind of words. Considering our previous discussion, it is easy to set a canonical bijection between Christoffel words of frequencies $|w|_0 = N - g, |w|_1 = g$ and Clough words of diatonic length g and chromatic length N.

5 Christoffel Tree and the Monoid $SL(2, \mathbb{N})$

In this section we analyze the algebraic structure of Christoffel words, and its consequences in well-formed scale theory. The decomposition of Christoffel words is extracted from (Lothaire 2002, Chap. 2), which is the central reference for algebraic combinatorics on words. In this book, the demonstrations are based on *standard words*, which are conjugated to Christoffel words (if 0c1 is a Christoffel word, c10 is standard). For a detailed adaptation of the proofs in terms of Christoffel words see (Domínguez 2007). Furthermore, standard words and Sturmian morphisms play very special roles in the description of the modes of a scale (see (Noll, Clampitt and Domínguez 2007)).

Let G and D be two applications that transforms the set $\{0,1\}^* \times \{0,1\}^*$ (pairs of finite words on the alphabet $\{0,1\}$) into itself, defined by:

$$\begin{array}{rcl} G, D : \{0,1\}^* \times \{0,1\}^* & \longrightarrow & \{0,1\}^* \times \{0,1\}^* \\ (u,v) & \xmapsto{G} & G(u,v) = (u, uv) \\ (u,v) & \xmapsto{D} & D(u,v) = (uv, v) \end{array}$$

Definition 7. *The set of* Christoffel pairs *is the smallest set of pairs of words containing the pair* $(0,1)$ *and closed under* $\{G, D\}$.

By construction, the set of Christoffel pairs can be represented in a tree diagram with root $(0,1)$ and nodes the Christoffel pairs.

The following proposition identifies the set of Christoffel words with the pairs of words in the Christoffel tree:

Proposition 6. *A pair of words is a Christoffel pair if and only if it has one of the following shapes:*

1. $(0, 0^n1)$ *with* $n \in \mathbb{N}$ *(these are the pairs* $G^n(0,1)$*)*
2. $(01^n, 1)$ *with* $n \in \mathbb{N}$ *(these are the pairs* $D^n(0,1)$*)*
3. (x, y) *with* x *and* y *Christoffel words.*

Furthermore, the next application is a bijection that identifies the Christoffel words with the pairs generated by G *and* D :

$$\begin{array}{ccc} \textit{Christoffel pairs} & \xrightarrow{\sim} & \textit{Christoffel words} \\ (x,y) & \longmapsto & x \cdot y \end{array}$$

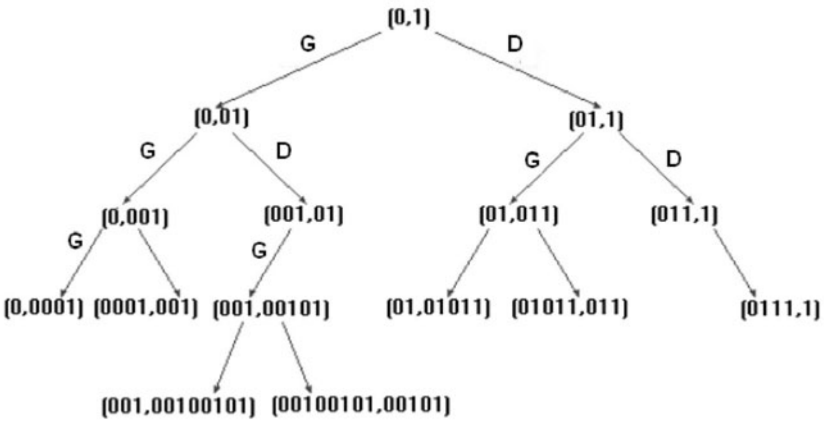

Fig. 5. The tree of Christoffel pairs

Let us consider now the matrices $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and let $SL(\mathbb{N}, 2)$ be the monoid of matrices 2×2 with natural entries and determinant 1. It is well known (see Noll 2007 for a proof) that:

Proposition 7. *$SL(\mathbb{N}, 2)$ is freely generated by $\{R, L\}$.*

Corollary 2. *There is a canonical identification between the set of finite words $\{G, D\}^*$ and the monoid $SL(\mathbb{N}, 2)$.*

For every Christoffel word $w = x \cdot y$ there is a unique path that leads from $(0, 1)$ to (x, y), or equivalently, there is a word $W \in \{G, D\}^*$ such that $W(0, 1) = (x, y)$. This word W is called the *generating word of* w. The associated matrix $A_w \in SL(\mathbb{N}, 2)$, obtained exchanging G by L, D by R and the concatenation by the product of matrices, is called the *incidence matrix* of the word w.

Proposition 8. *The incidence matrix of a Christoffel word $w = x \cdot y$ verifies:*

$$A_w = \begin{pmatrix} |x|_0 & |x|_1 \\ |y|_0 & |y|_1 \end{pmatrix}$$

Proof. The matrix $A_{(0,1)} = Id$ satisfies the formulation. We prove by induction that $A_{G(x,y)} = L \cdot A_{(x,y)}$ (the assertion $A_{D(x,y)} = R \cdot A_{(x,y)}$ can be proven in a similar way).

$$L \cdot A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} |x|_0 & |x|_1 \\ |y|_0 & |y|_1 \end{pmatrix} = \begin{pmatrix} |x|_0 & |x|_1 \\ |x|_0 + |y|_0 & |x|_1 + |y|_1 \end{pmatrix} =$$

$$= \begin{pmatrix} |x|_0 & |x|_1 \\ |xy|_0 & |xy|_1 \end{pmatrix} = A_{(x,xy)} = A_{G(x,y)} \qquad \blacksquare$$

We conclude by determining the relationship between the step pattern of dual scales. Let $\boxminus$ be the application

$$\begin{array}{ccc} SL(\mathbb{N},2) & \xrightarrow{\boxminus} & \mathbb{Q}^+ \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \frac{a+b}{c+d} \end{array}$$

This application (called the mediant ratio by T. Noll in Noll 2007) transforms the incidence matrix A_w of a Christoffel word $w = x \cdot y$ into $\frac{|x|}{|y|}$, that is, the slope of the dual word w^*. Denoting by A^*_w the matrix $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$(with the main diagonal flipped), then $\boxminus(A^*_w)=\frac{|w|_1}{|w|_0}$ is the slope of the original word w. In conclusion:

Proposition 9. *A^*_w is the matrix of the dual word of w and the transformation*

$$\begin{array}{ccc} SL(\mathbb{N},2) & \xrightarrow{*} & SL(\mathbb{N},2) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} d & b \\ c & a \end{pmatrix} \end{array}$$

is an anti-automorphism which induces the duality in Christoffel words, and therefore in the step patterns of well-formed scales modulo conjugation (Clough words).

Proposition 10. *The step patterns of dual scales are related to retrograde paths in the Christoffel tree. In other words, if $w = x \cdot y$ is the scale pattern of a scale and we have $W(0,1) = (0,1)$ for some word $W \in \{G, D\}^*$, then $w^* = y^* \cdot x^*$ is the pattern of the dual scale, where $(y^*, x^*) = \widetilde{W}(0,1)$.*

Proof. Let $A_w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(\mathbb{N},2)$ and let us suppose we have the decomposition $A_w = \Lambda_0 \cdot \ldots \Lambda_k$ with $\Lambda_i \in \{R, L\}$. One has that $A^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot A^t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and therefore:

$$A^*_w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot (\Lambda_0 \cdot \cdots \cdot \Lambda_k)^t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Lambda^t_k \cdot \cdots \cdot \Lambda^t_0 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observe that $L^t = R$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot L \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = R$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = Id$, thus:

$$\begin{aligned} A^*_w &= \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Lambda^t_k \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \cdot \cdots \cdot \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Lambda^t_0 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \\ &= \Lambda_k \cdot \cdots \cdot \Lambda_0 = A_{\widetilde{W}} \end{aligned}$$ ■

6 Final Remarks

1. Throughout the paper we have considered Christoffel words to be *positive*, that is, the path *over* the integer grid that lies under the semiline. If we

want to generate the *negative* Christoffel words (the path *under* the integer grid), we have just to change the root of the Christoffel tree and to consider $(1, 0)$ instead.

2. This paper should be understood as a point of departure in which we have identified the study of scale patterns with the algebraic theory of words. Therefore, one should continue research into related topics which potentially offer music-thoretical interpretations. For example, Sturmian words, which are obtained geometrically in the same way as are Christoffel words, but with rays of irrational slope, were discussed in Carey and Clampitt 1996, where they were referred to as *quasi-periodic sequences*.
3. The transformational theory for well-formed scales as proposed in (Noll 2007) is mainly covered by the theory of Sturmian morphisms. It is therefore challenging to review the music-theoretical interpretations in (Noll 2007) within the full algebraic context of Sturmian morphisms.

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