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Proving Musical Theorems I:  
The Middleground of  
Heinrich Schenker's Theory of Tonality

by

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PROVING MUSICAL THEOREMS I:  
THE MIDDLEGROUND OF HEINRICH SCHENKER'S  
THEORY OF TONALITY<sup>1</sup>

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Musical 'languages' share many attributes with natural languages such as English or French: their utterances (musical compositions or sentences) can be experienced as writing or as sound, and within the frame of sonic experiencing there are common nonisomorphic principal domains - acoustic, auditory, and (at least for vocal music) articulatory. Both kinds of languages have undergone diachronic change, have had regional or stylistic variant forms, and have proved sufficiently communicable to humans that, after appropriate experiencing and training, intersubjective judgements of the acceptability of utterances generally concur. The most significant distinction of natural from musical languages is the presence of a referential component in the former which is lacking in the latter - a component that connects linguistic utterances with the non-linguistic world outside.<sup>2</sup>

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1. This research has been supported by a grant from the John S. Guggenheim Memorial Foundation.
  2. Perhaps not unrelatedly, there is evidence that musical messages undergo fundamentally different neural processing than linguistic messages.

It is the presence of this component - or, more precisely, our present lack of understanding how to organise this component algorithmically - that has rendered infeasible, at least for the foreseeable future, the tasks of machine translation, machine information retrieval, automatic question-answering, and automatic speech recognition (excepting very small subtasks, such as document retrieval based upon descriptors provided by human indexers, whose solutions in no way generalize to the solutions of the tasks themselves). The investigation of the structure of musical languages - which have many of the important properties of natural language yet lack the one major component that has rendered intelligent language-processing machines beyond the limits of current feasibility - should attract artificial-intelligence researchers for its potential to clarify the position of these limits.<sup>3</sup>

To call a music-processing machine 'intelligent' we should want it to be able to carry out such 'central' processes as language identification (is a presented composition an instance of a particular tonality, the twelve-note system, or some other well-known musical language?), structural analysis (within an identified musical language, what syntactic relationships do the composition's notes, rests, chords, phrases, etc. possess?), and composition of coherent new utterances (within a particular musical language, and even within a particular musical 'style' that is a dialect

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3. There are, of course, other motivations: e.g., for the small number of professional musical theorists the investigation should be self-sufficiently interesting.

of such a language). Additionally, an 'intelligent' music-processing machine should be able to carry out several of the essentially different and totally separable 'I/O' processes - e.g., optical character-recognition of printed music, or electronic sound synthesis - that permit passage between compositions, on the one hand, and the written and sonic domains in which music actually is experienced, on the other. Orchestration, harmonizing a given melody, and correcting students' musical exercises, are simpler and largely subsidiary activities which an 'intelligent' machine could accomplish with comparatively little additional information. To endow a machine with such 'intelligence' it seems wise, as in other areas of artificial-intelligence research, first to give the machine a program embodying the best of hitherto existing theories of the subject matter and then to supplement or modify this with other available 'heuristic' concepts.

This report deals with tonality - the musical language that has governed most Western musical compositions from 1600 to 1900 and much subsequent popular music;<sup>4</sup> specifically, it deals with the theory of tonality formulated by Heinrich Schenker (1867-1935), particularly in his book Der Freie Satz (Vienna, 1935). This is not the place to provide a full or even an adequate statement of Schenker's theory, nor to recapitulate the music-analytical evidence

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4. Readers interested in the status of formalization of another musical language are referred to my article, 'Toward a theory that is the twelve-note-class system', Perspectives of New Music, vol.5, no.2 (1967) ppl-80.

provided by Schenker and his followers which has made the theory plausible.<sup>5</sup> Since the theory as published by Schenker is informal and in places unclear, ambiguous, or inconsistent - in part because Schenker changed the theory during the course of his publications - our initial task has been to explicate this theory, i.e., to replace Schenker's descriptions and examples of it by a structure which, though recognizably similar, has sufficient formalization that its constituent processes can be manipulated by computer and has sufficient consistency that it is sensible to test the structure against a corpus of suitable music for 'goodness of fit'. The import of our task derives both from tonality's prominent role in Western culture and from the hypothesis that Schenker's theory, when explicated, will embody much of the knowledge required for automatic identification and structural analysis of compositions instancing tonality.

The essence of Schenker's theory is a claim that every composition that is an instance of tonality - and no other composition - can be derived from one of three Ursätze - primitive musical compositions which Schenker regarded as axioms - by successive application of a small number of rules of inference called 'prolongation techniques'. Three stages are distinguished in the derivation of a composition (or of a movement in a multi-

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5. Schenker's theory, however, is not the only plausible theory of tonality. Another, radically different theory, proposed by A.F.C. Kollmann (1756-1829), is being explicated concurrently by the present writer, and will be discussed elsewhere in a subsequent report.

movement composition): the background, which comprises essentially just one Ursatz; the middleground, wherein the large-scale structure of the composition - e.g., its principal sections and their tonal 'centres' - is specified; and the foreground, wherein the harmonic, melodic, and rhythmic 'details' are specified that must be added to the large-scale structure to yield the composition itself. The middleground, accordingly, comprises a sequence of compositions that starts with an Ursatz and proceeds so that each subsequent composition in the sequence is derivable from its predecessor by application of one or another of the prolongation techniques that Schenker designated for the middleground. And the foreground thus comprises a sequence of compositions that starts with a final composition of some middleground sequence - i.e., a composition representing a possible large-scale structure for some instance of tonality - and proceeds so that each subsequent composition in the sequence is derivable from its predecessor by one or another of the prolongation techniques that Schenker designated for the foreground. The class of final compositions in one or another foreground sequence will coincide-if Schenker's theory is valid - with the class of compositions instancing tonality.

The nature of Schenker's theory should become clearer from the examples given later in this report. We would note here that this theory differs fundamentally from single-level Markovian theories, which have been embodied in most previous computer programs for composing music, and which, by implying that compositions can be accounted for by a single finite-state grammar whose productions are the chords present in the surface structure of the music, disallow the relevance of context having greater extent than the number of permitted states. Schenker's theory permits context as large as the composition itself, so the obvious empirical inadequacy of music previously composed by computer program does not bear upon the success or failure of our enterprise.

Our explication proceeds according to the logistic method: we replace Schenker's prose and illustrative musical examples by a sequence of formalized languages whose assertions - i.e., theorems under principal interpretation - are musical compositions. Generally theorems of one formalized language become axioms of the next language in sequence. Some aspects of our music/logic analogy have been intimated already: axioms of the first formalized language in the sequence replace Ursätze, and rules of inference usually replace prolongation techniques. Decision procedures not only determine whether or not a presented musical composition is an assertion but collectively will produce for every given assertable composition of the final formalized language

a step-by-step derivation, of the composition from an Ursatz, that constitutes a structural analysis (literally, a structural synthesis) of the composition.<sup>6</sup>

The result here reported is that the middleground of Schenker's theory has been explicated, and that decision procedures for the two formalized languages which constitute this explication have been programmed.<sup>7</sup> Preliminary testing indicates that the explication accounts satisfactorily for the musical structures it is intended to explain. At present the explication is limited to compositions in major mode; however, its extension to include compositions in minor mode is not problematic.

The underlying logistic systems of these two formalized languages - i.e., the languages without their principal (musical) interpretations - are named  $S_1$  and  $S_2$ . The structure of these logistic systems can be presented rigorously - e.g., the primitive symbols and the formation rule of  $S_1$  can be specified as follows:<sup>8</sup>

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6. Because of the finitary nature of the musical systems treated there is no question of their decidability; the problems, rather, are those of combinational complexity. See also, footnote 18 below.
  7. In the APL programming language, and implemented both on an IBM 360/50 computer at the University of New South Wales and on a CDC Cyber 72 computer at the University of Sydney.
  8. For a rigorous presentation of a system substantially the same as the present system  $S_1$ , see my Ph.D. dissertation, A Trinity of Essays (Princeton University 1967, University Microfilms order number 68-2490). There too the principal interpretations of  $S_1$  (which hold also for  $S_2$ ) are sketched in greater detail than possible here: these <sup>2</sup>'map' well-formed formulas into corresponding musical compositions. For a general discussion of the logistic method, of logistic systems, and formalized languages, see Alonzo Church, Introduction to Mathematical Logic, Vol.1, Princeton 1956.



PRIMITIVE SYMBOLS: 0 1 2 3 4 5 6 7 8 9

Primitive unit - a formula having four symbols.

Note-code - a primitive unit whose leftmost symbol is 0, whose next symbol is not 0, and whose rightmost two symbols denote (according to the decimal system) a number from zero to eleven, inclusive.

FORMATION RULE:  $\Gamma$  is well-formed if  $\Gamma$  is  $\Gamma_1 8000 \Gamma_2$  and (1)

$\Gamma_1$  and  $\Gamma_2$  have the same number of symbols, (2) the number of symbols in  $\Gamma$  is a multiple of four, (3) the only primitive units into which  $\Gamma_1$  and  $\Gamma_2$  partition are note-codes or 8100, (4) 8100 is not the leftmost primitive unit of either  $\Gamma_1$  or  $\Gamma_2$ , and (5) if 8100 is the  $i$ th primitive unit of  $\Gamma_1$  then 8100 is not the  $i$ th primitive unit of  $\Gamma_2$  (for all  $i$  less or equal to the number of primitive units of  $\Gamma_1$ ).

In the present report we proceed with less rigour. We use an alternative two-row matrix representation of well-formed formulas of  $S_1$ , made possible by the placement of the primitive unit '8000' in and only in the middle of every well-formed formula: the elements of the first row are the primitive units preceding '8000' (in that order); the elements of the second row are the primitive units following '8000' (in that order). Each element of the matrix is either a note-code or '8100'. A note-code stands for a musical note in the following way: its second digit is the number of the note's octave<sup>9</sup> and its rightmost two digits specify the note's note-class.<sup>10</sup>

9. Following convention (R.W. Young, 'Terminology for Logarithmic Frequency Units', Journal of the Acoustical Society of America, vol.11, no.1 (1939), pp.134ff.), Middle C is fourth-octave C, and octaves include the span from a C to the next higher B. The formation rule permits octave numbers from 1 through 9, inclusive.

10. I.e., the class of all notes  $n$  octaves away from the note, where  $n$  is a non-negative integer. The explication presupposes enharmonic equivalence - e.g., of D# and E-flat in the same octave - which implies a partitioning of each octave into exactly twelve notes but does not necessarily imply equal temperament. Those who would question this presupposition can be reminded of its inherence in the design of keyboard and other musical instruments for which very many compositions instancing tonality have been written.

Thus the note-code 0406 represents the F# (or G-flat or E-double-sharp) six semitones above Middle C, and the note-code 0306 represents the F# six semitones below Middle C. '8100' indicates that the immediately preceding note is tied, i.e., still sounds. Primitive units in the same column are to be regarded as simultaneous.<sup>11</sup> In the background and middleground stages of Schenker's theory there is no durational differentiation - the rhythm is only that implied by note succession, ties and rests (represented in  $S_2$ , and subsequently, by '8200'). Also in these stages there is no indication of intensity or dynamic marks, of timbre or instrumentation, of text in vocal music, or of tempo: all these properties belong in Schenker's theory - to the extent that they belong at all - to the foreground stage.

The musical entity corresponding to each row of a matrix is a part performable on a monophonic instrument, i.e., an instrument that at any one time can perform at most one note. Such a part of a composition we call a lyne.<sup>12</sup> The compositions which are assertions of  $S_1$  (according to a principal interpretation) can be characterized as dilynear compositions since they consist of exactly two lynes.

We have implemented an INPUT routine which permits the notes, rests, and ties to be entered from a computer terminal, one

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11. The formation rule ensures that no row starts with a tie ('8100') and that no column consists entirely of ties.

12. So spelt to distinguish from, e.g., the lines that form the staff.

lyne at a time, using a simple alphanumeric notation: the routine then computes the matrix representation which is utilized by the APL software. Complementarily, we have implemented a PRINT routine which permits the content of matrices (representing well-formed formulas) to be printed on a computer terminal in a notation that should be easier to read than the wholly numeric notation. A note-code is output by the PRINT routine as 'C', 'CS', 'D', 'DS', 'E', 'F', 'FS', 'G', 'GS', 'A', 'AS', or 'B', according as its rightmost two digits are 00 through 11, respectively, followed immediately by the octave number of the note. ('S' stands for 'sharp'.) '8100' and '8200' are output as 'TIE' and 'RST', respectively. Should any other primitive unit be encountered it is output as 'HELP'.

The three axioms of  $S_1$  are now shown in the matrix representation, in the output format of the PRINT routine, and in ordinary musical notation. They are called '3-to-1 Axiom', '5-to-1 Axiom', and '8-to-1 Axiom', since the top lyne - which Schenker termed the Urlinie - descends a major third, a perfect fifth, or a perfect octave.

504 502 500 E5 D5 C5

300 307 300 C3 G3 C3

(3-to-1 Axiom)



507 505 504 502 500 G5 F5 E5 D5 C5

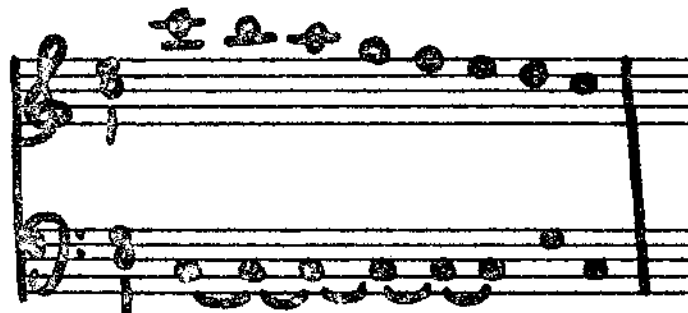
300 8100 8100 307 300 C3 TIE TIE G3 C3

(5-to-1 Axiom)

600 511 509 507 505 504 502 500 C6 B5 A5 G5 F5 E5 D5 C5

300 8100 8100 8100 8100 8100 307 300 C3 TIE TIE TIE TIE TIE G3 C3

(8-to-1 Axiom)



These axioms correspond to Schenker's three (major-mode) Ursätze, in C-major and at the 'original' registral position (i.e., top lyne ending in the fifth octave and bottom lyne in the third octave). Other (major-mode) keys and registral positions are achievable by applying the Rules of Transposition and Rules of Octave Adjustment, respectively.

The compositions that are assertions of  $S_2$  (under a principal interpretation) all are trilynear - besides their two outer lyns, they possess a middle lyne (Mittelstimme). A well-formed formula of  $S_2$  accordingly is represented by a three-row matrix. Each middleground prolongation technique - at least in our explication - either transforms dilynear compositions into dilynear compositions, in which case it is assigned to  $S_1$ , or transforms trilynear compositions into trilynear compositions, in which case it is assigned to  $S_2$ ; moreover, no  $S_1$  technique need be applied after any of the  $S_2$  techniques has been applied. Accordingly, our explication of the middleground takes, for any given composition, the general form of a derivation (i.e., proof) within  $S_1$  followed by a derivation within  $S_2$ . The systems  $S_1$  and  $S_2$  are formally connected by the axiom schema of  $S_2$ , which asserts that a three-row matrix represents an axiom of  $S_2$  if the two-row matrix formed by disregarding the middle row constitutes a theorem of  $S_1$  and every primitive unit of the middle row is 8200. Thus the middle lyne of a composition corresponding to an axiom of  $S_2$  consists entirely of rests.

The following list relates Schenker's middleground prolongation techniques to the rules of inference of  $S_1$  and  $S_2$ . It indicates the comparatively small number of rules of inference Schenker supposed were sufficient. (Transposition and Octave Adjustment are each explicated as classes of rules of inference, but this is a detail.) Remarkably, Schenker's specification of the foreground stage does not call for a substantially larger number of rules.

Schenker's Prolongation Technique	Corresponding Rule of Inference in $S_1$
Aufwärts-Bassbrechung	Rule of Bass Arpeggiation and Rule of Bass Ascent
----13	Rule of Bass G Transfer
----13	Rule of Bass Descent
Gliederung (of $\hat{3}$ & $\hat{5}$ )	Rule of Articulation
Mischung (in major mode) <sup>14</sup>	Rule of Mixture
Nebennote	Rule of Neighbour-Note Prolongation
Züge - rising (Anstieg)	Rule of Preliminary Ascent
- falling (in $S_2$ )	
Brechung	Rule of Preliminary Arpeggiation
(implicit)	Rules of Transposition
(implicit)	Rules of Octave Adjustment

13. The explication here deviates from Der Freie Satz by allowing the bass lyne to descend from third-octave C to second-octave G as well as to ascend from third-octave C to third-octave G. Although Schenker explicitly permits such descent only in the foreground, no good reason was found to disallow the symmetry in the middleground. Should a good reason be found subsequently, it will be easy to alter the explication by removing these two rules.

14. As compositions in minor mode are not yet included in the explication, the Rule of Mixture explicates only that portion of Schenker's Mischung prolongation technique that proceeds: major→minor→major. For the same reason, Schenker's 'Phrygian 2' prolongation technique, which applies only to compositions in minor mode, has not yet been explicated.

Schenker's Prolongation Technique	Corresponding Rule of Inference in $S_2$
Züge - falling	Rule of First-Order Descending Progression
Übergreifen	Rule of Overlapping
Untergreifen	Rule of Middle-Lyne Ascent
Ausfaltung <sup>15</sup>	Rule of Left Unfolding and Rule of Right Unfolding

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15. The Rules of Left Unfolding and Right Unfolding also embrace some, but not all, aspects of Schenker's Höherlegung, Tieferlegung and Koppelung prolongation techniques. In particular, explication of shifts of top-lyne register has been deferred to the foreground. Schenker's Vertretung prolongation technique, because of its admittedly limited applicability, is not yet included in our explication.

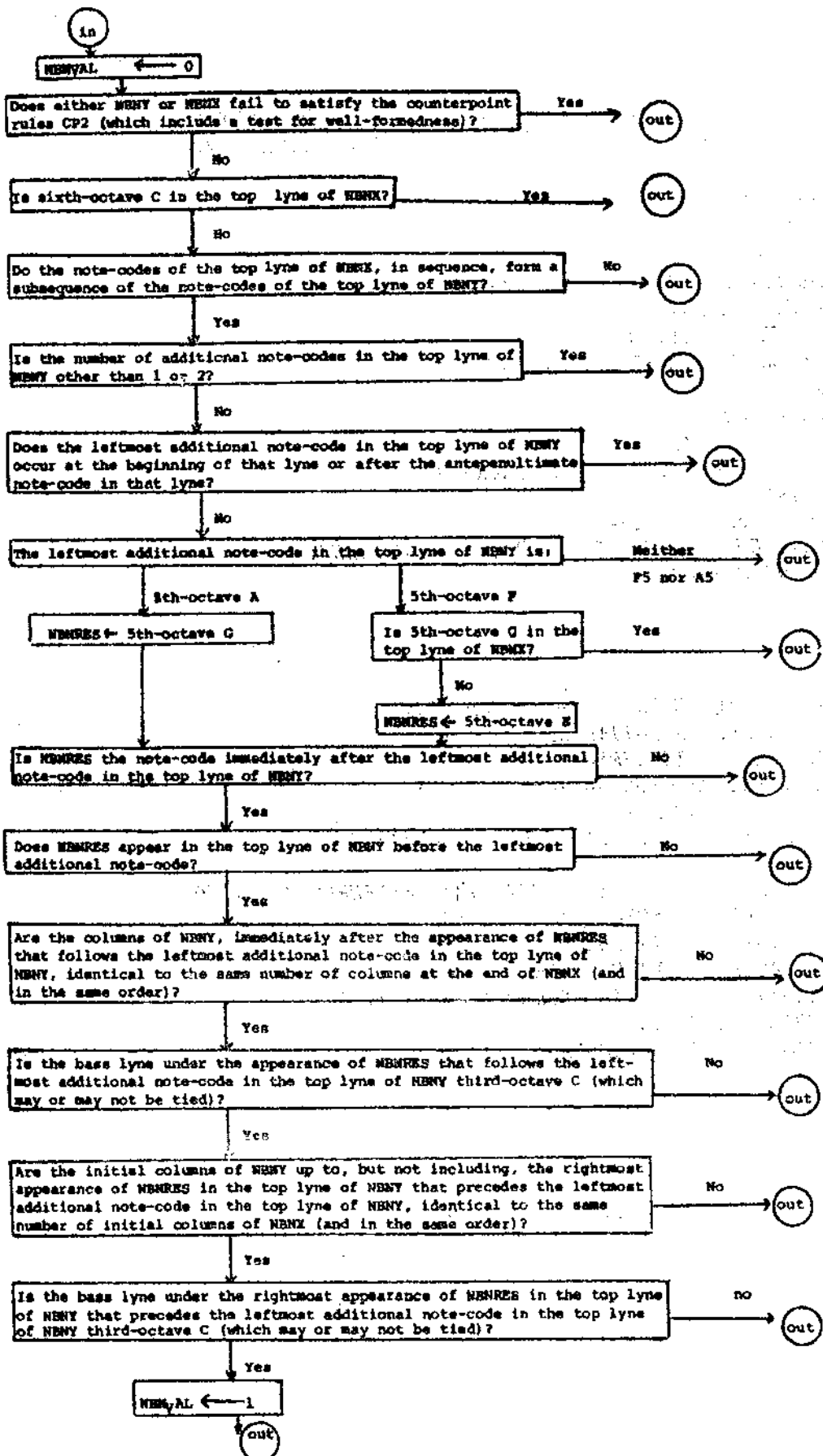


To exemplify a rule of inference, we present the APL function NBN which serves to explicate Schenker's middleground Nebennote prolongation technique: the function NBN has two arguments, NBNX and NBNY (in that order), and  $S_1$ 's Rule of Neighbour-Note Prolongation permits, or does not permit, the composition corresponding to NBNY to be immediately inferred from the composition corresponding to NBNX according as the value NBNVAL of the function for those arguments is 1 or 0. A flowchart following the function should make its logic clear.

$\nabla$  NBN[ ] $\nabla$   
 $\nabla$  NBNVAL $\leftarrow$ NBNX NBN NBNY;NBNJNCX1;NBNJNCY1;NBN1OR2;NBNJNCXYIX;  
 NBNLOCJNCY;NBNRES;NBNYIX;NBN1;NBNWA;NBNVAL

[1] NBNVAL $\leftarrow$ 0  
 [2]  $\rightarrow(\sim(\text{CP2 NBNY}))/0$   
 [3]  $\rightarrow(\sim(\text{CP2 NBNX}))/0$   
 [4]  $\rightarrow(600\epsilon\text{NBNX}[1;])/0$   
 [5]  $\odot$  NO NEIGHBOUR-NOTE TO OCTAVE DESCENT.  
 [6] NBNJNCX1 $\leftarrow$ JUSTNC NBNX[1;]  
 [7] NBNJNCY1 $\leftarrow$ JUSTNC NBNY[1;]  
 [8] NBNJNCXYIX $\leftarrow$ NBNJNCX1 SUBVIN NBNJNCY1  
 [9]  $\rightarrow(\text{NBNJNCXYIX}=0)/0$   
 [10] NBN1OR2 $\leftarrow$ ( $\rho$ NBNJNCY1)-( $\rho$ NBNJNCX1)  
 [11]  $\rightarrow(\sim(\text{NBN1OR2}\epsilon(1,2)))/0$   
 [12] NBNLOCJNCY $\leftarrow$ ( $\times$ (NBNJNCXYIX-( $\backslash$ ( $\rho$ NBNJNCX1)))) $\backslash$ 1  
 [13]  $\rightarrow(\text{NBNLOCJNCY}<2)/0$   
 [14]  $\rightarrow(\text{NBNLOCJNCY}>((\rho\text{NBNJNCY1})-3))/0$   
 [15]  $\rightarrow(((509,505)\epsilon\text{NBNJNCY1}[\text{NBNLOCJNCY}]),1)/(\text{NBN5},\text{NBN3},0)$   
 [16] NBN3 $\leftarrow$ ( $507\epsilon\text{NBNJNCX1}))/0$   
 [17]  $\odot$  NO G.  
 [18] NBNRES $\leftarrow$ 504  
 [19]  $\rightarrow$ NBNTRES  
 [20] NBN5:NBNRES+507  
 [21] NBNTRES $\leftarrow$ (NBNJNCY1[(NBNLOCJNCY+1)] $\neq$ NBNRES)/0  
 [22]  $\rightarrow(\sim(\text{NBNRES}\epsilon((\text{NBNLOCJNCY}-1)+\text{NBNJNCY1}))/0$   
 [23] NBNYIX $\leftarrow$ NBNJNCY1 SUBVIN NBNY[1;]  
 [24] NBN1 $\leftarrow$ (NBNIX[(NBNLOCJNCY+2)]-1)-( $\neg$ 1 $\rightarrow$ ( $\rho$ NBNY))  
 [25]  $\odot$  POST-RESOLUTION ELEMENTS  
 [26]  $\rightarrow(\sim((,((2,\text{NBN1})+\text{NBNX}))\times=(,((2,\text{NBN1})+\text{NBNY}))))/0$   
 [27]  $\odot$  ENSURE EQUAL ENDS  
 [28]  $\rightarrow(\sim((\text{NBNY SIMUL NBNYIX}[(\text{NBNLOCJNCY}+1)])\times=(\text{NBNRES},300)))/0$   
 [29] NBNWA $\leftarrow$ (NBNYIX[(NBNLOCJNCY-1)]+NBNY[1;]  
 [30] NBN1 $\leftarrow$ (( $\rho$ NBNWA)-( $\phi$ NBNWA) $\backslash$ NBNRES))+1  
 [31]  $\odot$  FIND IMMEDIATELY PREVIOUS NBNRES.  
 [32]  $\rightarrow(\sim((,((2,\text{NBN1})+\text{NBNX}))\times=(,((2,\text{NBN1})+\text{NBNY}))))/0$   
 [33]  $\odot$  ENSURE EQUAL BEGINNINGS.  
 [34]  $\rightarrow(\sim((\text{NBNY SIMUL NBN1})\times=(\text{NBNRES},300)))/0$   
 [35] NBNVAL $\leftarrow$ 1

$\nabla$



The function NBN permits the well-formed formula corresponding to the following composition (2) to be inferred from the 3-to-1 Axiom, which corresponds to composition (1), by a single application of the Rule of Neighbour-Note Prolongation. Similarly, (4) can be so inferred from (3). However, despite superficial appearances, (6) cannot be inferred from (5) by this rule, as the counterpoint rules embodied in the APL function CP2 do not allow the bass lyne to leap to a dissonance such as the F over the G in the second simultaneity of (6).

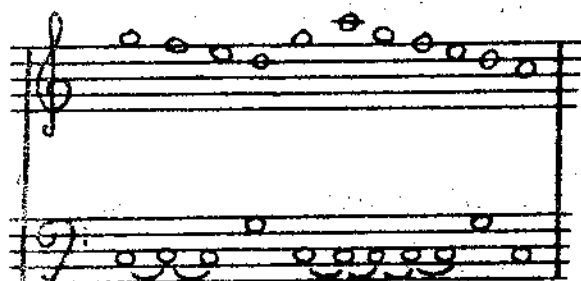
(1)



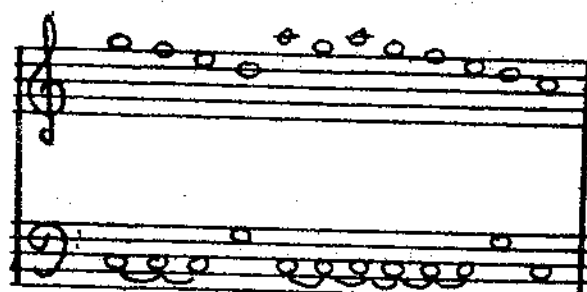
(2)



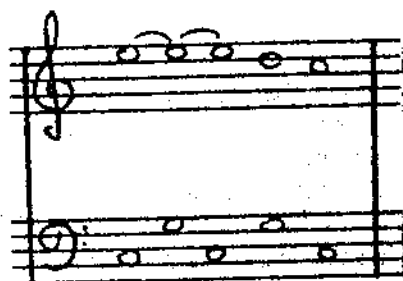
(3)



(4)



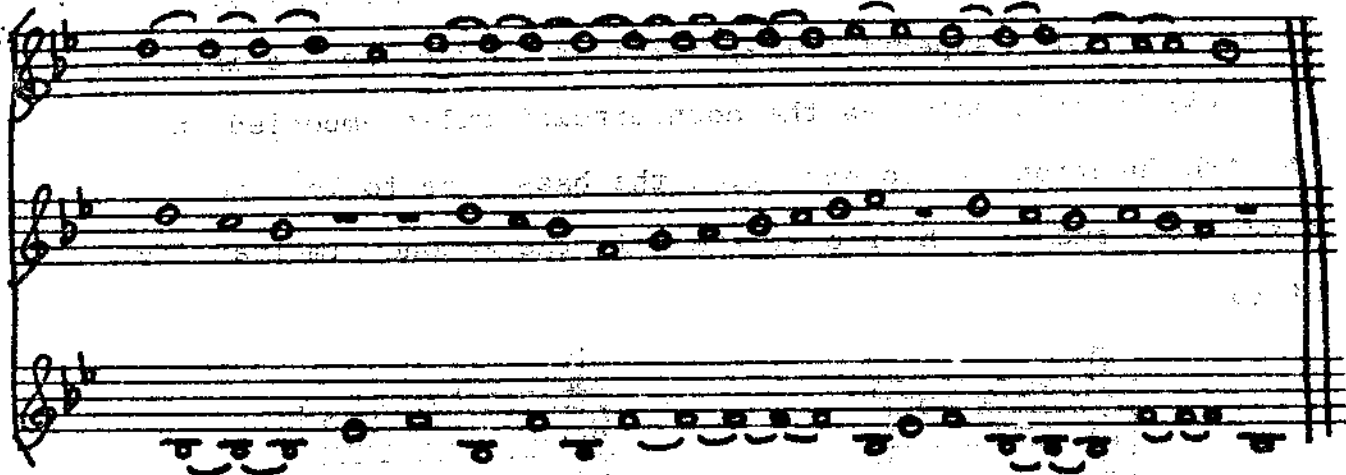
(5)



(6)



We turn next to the decision procedures for the systems  $S_1$  and  $S_2$  and exhibit a computer-produced proof of the three-row matrix (call it  $M$ ) corresponding to the following trilynear musical composition:



The user of the system, after having obtained  $M$  from the above composition by utilizing the INPUT routine, enters from his terminal:

PROVES1S2 M

This causes the computer to proceed, essentially without further human intervention,<sup>16</sup> either to produce a proof of  $M$  or to produce a message such as:

16. The present versions of the program have the decision procedures for  $S_1$  and  $S_2$  located in different APL workspace. This requires<sup>2</sup> the user, at appropriate times, to store one on disk and load another. In a computer system permitting larger workspaces, or with more compactly written software, this human intervention clearly could be eliminated.

NOT A THEOREM. COMPOSITION LACKS ACCEPTABLE HEADNOTE.  
which provides a reason why the matrix entered cannot be proved  
(and at the same time can notify the user what stage in the  
decision procedure the computer had reached when the determin-  
ation that the matrix is not a theorem was made).

M is, in fact, a theorem of  $S_2$ , and the proof of M  
obtained by the program as follows. Observe that the computer  
output consists of a proof in the system  $S_1$  followed by a proof  
in the system  $S_2$ : whilst the latter is a proof of M itself  
(in the system  $S_2$ ), the former is a proof (in the system  $S_1$ )  
of a matrix consisting of the outer rows of the first well-  
formed formula in the proof of M, and together with the all-  
rest middle lyne establishes that this first well-formed formula  
is an axiom of  $S_2$ .

P R O O F

(IN THE SYSTEM S1)

E5 D5 C5  
C3 G3 C3

(3-TO-1 AXIOM.)

E5 TIE TIE D5 C5  
C3 G3 C3 G3 C3

(INFERRED FROM LAST BY RULE OF BASS ARPEGGIATION.)

E5 D5 E5 D5 C5  
C3 G3 C3 G3 C3

(INFERRED FROM LAST BY RULE OF ARTICULATION.)

E5 D5 E5 F5 E5 D5 C5  
C3 G3 C3 TIE TIE G3 C3

(INFERRED FROM LAST BY RULE OF NEIGHBOUR-NOTE PROLONGATION.)

E5 D5 E5 TIE F5 E5 D5 C5  
C3 G3 C3 G3 TIE C3 G3 C3

(INFERRED FROM LAST BY RULE OF BASS ARPEGGIATION.)

E5 D5 E5 TIE TIE TIE F5 E5 D5 C5  
C3 G3 C3 G3 C3 G3 TIE C3 G3 C3

(INFERRED FROM LAST BY RULE OF BASS ARPEGGIATION.)

E5 D5 E5 TIE TIE TIE TIE TIE F5 E5 D5 C5  
C3 G3 C3 G3 C3 G3 C3 G3 TIE C3 G3 C3

(INFERRED FROM LAST BY RULE OF BASS ARPEGGIATION.)

E5 D5 E5 TIE TIE TIE TIE F5 TIE E5 D5 C5  
C3 G3 C3 G3 C3 C3 F3 G3 C3 G3 C3

(INFERRED FROM LAST BY RULE OF BASS ASCENT.)

E5 TIE D5 E5 TIE TIE TIE TIE F5 TIE E5 D5 C5  
C3 F3 G3 C3 G3 C3 G3 C3 F3 G3 C3 G3 C3

(INFERRED FROM LAST BY RULE OF BASS ASCENT.)

D5 TIE C5 D5 TIE TIE TIE TIE DS5 TIE D5 C5 AS4  
AS2 DS3 F3 AS2 F3 AS2 F3 AS2 DS3 F3 AS2 F3 AS2

(INFERRED FROM LAST BY RULE OF TRANSPOSITION TO AS4.)

D5 TIE C5 D5 TIE TIE TIE TIE DS5 TIE D5 C5 AS4  
AS3 DS4 F4 AS3 F4 AS3 F4 AS3 DS4 F4 AS3 F4 AS3

(INFERRED FROM LAST BY RULE OF OCTAVE ADJUSTMENT OF LYNE 2.  
UP 1 OCTAVE.)

(Q.E.D.)

P R O O F  
(IN THE SYSTEM S2)

D5 TIE C5 D5 TIE TIE TIE TIE DS5 TIE D5 C1 AS4  
RST RST RST RST RST RST RST RST RST RST RST RST RST  
AS3 DS4 P4 AS3 P4 AS3 P4 AS3 DS4 P4 AS3 P4 AS3  
(AXIOM.)

D5 TIE C5 D5 TIE TIE TIE TIE DS5 TIE D5 C5 TIE TIE AS4  
RST RST RST RST RST RST RST RST RST RST RST RST RST  
AS3 DS4 P4 AS3 P4 AS3 P4 AS3 DS4 P4 AS3 P4 TIE TIE AS3  
(INFERRED FROM LAST BY RULE OF FIRST-ORDER DESCENDING PROGRESSION.)

D5 TIE C5 D5 TIE TIE TIE TIE DS5 TIE D5 TIE TIE C5 TIE TIE AS4  
RST RST RST RST RST RST RST RST RST RST RST RST RST  
AS3 DS4 P4 AS3 P4 AS3 P4 AS3 DS4 P4 AS3 TIE TIE P4 TIE TIE AS3  
(INFERRED FROM LAST BY RULE OF FIRST-ORDER DESCENDING PROGRESSION.)

D5 TIE C5 D5 TIE TIE TIE TIE TIE TIE TIE TIE DS5 TIE D5 TIE TIE C5 TIE TIE AS4  
RST RST RST RST RST RST RST P4 G4 A4 AS4 C5 D5 DS5 RST D5 C5 AS4 C5 AS4 A4 RST  
AS3 DS4 P4 AS3 P4 AS3 P4 TIE TIE TIE TIE AS3 DS4 P4 AS3 TIE TIE P4 TIE TIE AS3  
(INFERRED FROM LAST BY RULE OF MIDDLE-LINE ASCENT.)

D5 TIE C5 D5 TIE TIE TIE TIE TIE TIE TIE TIE DS5 TIE D5 TIE TIE C5 TIE TIE AS4  
RST RST RST D5 C5 AS4 P4 G4 A4 AS4 C5 D5 DS5 RST D5 C5 AS4 C5 AS4 A4 RST  
AS3 DS4 P4 AS3 P4 AS3 P4 TIE TIE TIE TIE AS3 DS4 P4 AS3 TIE TIE P4 TIE TIE AS3  
(INFERRED FROM LAST BY RULE OF FIRST-ORDER DESCENDING PROGRESSION.)

D5 TIE TIE TIE C5 D5 TIE TIE TIE TIE TIE TIE TIE TIE DS5 TIE D5 TIE TIE C5 TIE TIE AS4  
D5 C5 AS4 RST RST D5 C5 AS4 P4 G4 A4 AS4 C5 D5 DS5 RST D5 C5 AS4 C5 AS4 A4 RST  
AS3 TIE TIE DS4 P4 AS3 P4 AS3 P4 TIE TIE TIE TIE AS3 DS4 P4 AS3 TIE TIE P4 TIE TIE AS3  
(INFERRED FROM LAST BY RULE OF FIRST-ORDER DESCENDING PROGRESSION.)

(Q.E.D.)

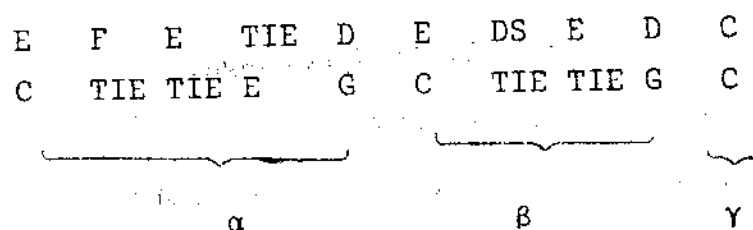


A decision procedure, of course, provides one proof for any given theorem, rather than all possible proofs. In the case of the middleground of Schenker's theory of tonality, this is no disadvantage so far as the musical interpretation is concerned, for different minimal<sup>17</sup> proofs of the same theorem differ essentially only in the order in which the rules of inferences are applied rather than in a more fundamental way - e.g., involving different rules of inference. For instance, because of the commutativity of musical transposition with some other operations constituting rules of inference, there are minimal proofs of M in which transposition from C to B-flat occurs elsewhere than in the computer-produced proof, not to mention absurd minimal proofs of M in which transposition is effected from C to some other key before a final transposition to B-flat. But such alternative minimal proofs of M in no way demonstrate a structural ambiguity of M in any semantically important sense. The situation is likely to change significantly when the foreground stage is explicated, for - so far as is now known - the same foreground composition may well be derivable from fundamentally different middleground entities.

It may be of interest to record that the decision procedure for  $S_1$  makes use of the 'juncture metatheorem' which allows an articulated structure such as:

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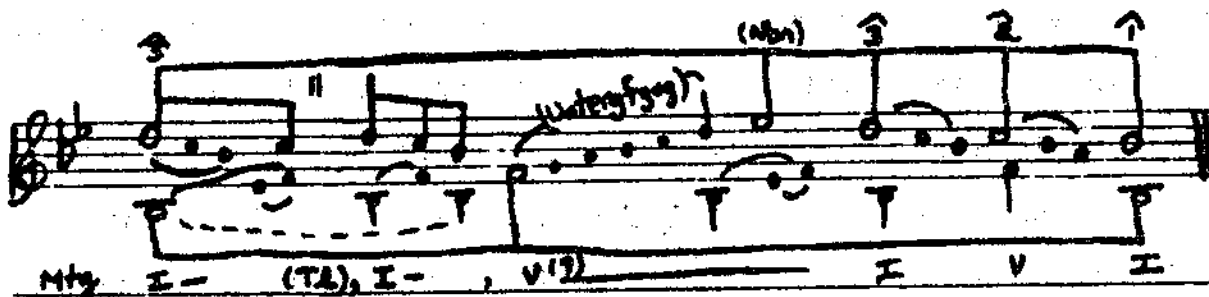
17. A proof in  $S_1$  or  $S_2$  is minimal if each non-first well-formed formula in the proof is inferred from its immediate predecessor, and if no well-formed formula appears more than once in the proof. The decision procedures implemented for  $S_1$  and  $S_2$  produce minimal proofs.



to be split into two smaller structures. The metatheorem provides that if  $\alpha$  followed by  $\beta$ , and  $\alpha$  followed by  $\gamma$ , are theorems satisfying a number of conditions (e.g., C-major, original registrational position, no sixth-octave C in the top lyne, no preliminary arpeggiation or preliminary ascent to  $\beta$ ), then so is  $\alpha$  followed by  $\beta$  followed by  $\gamma$ . The computer program examines well-formed formulas to determine whether splitting is potentially feasible; if so,  $\beta$  is stored while a proof of  $\alpha$  followed by  $\gamma$  is attempted; if such a proof is found,  $\alpha$  is then stored while a proof of  $\beta$  followed by  $\gamma$  is attempted; this endeavour may involve a further splitting of  $\beta$ ; eventually, if the subsidiary proofs have been obtained, a proof of the whole well-formed formula is constructed from them. The decision procedure for  $S_1$  also employs a limited amount of back-tracking - e.g., when it tries to simplify a well-formed formula by undoing the effect of applying a particular rule of inference and cannot do so because it recognizes that some other rule of inference must have its effect undone first, the decision procedure will note to retry the simplification at a later time. The decision procedure

for  $S_2$  is comparatively straightforward.<sup>18</sup>

Schenker, in his publications, generally did not display each constituent well-formed formula of a proof individually, as the computer has done in the proof of M presented above, but rather epitomized the entire proof in one or more musical 'sketches' written in a neologistic notation frequently supplemented or clarified by textual commentary. For instance, the entire proof of M might have been summarized in a sketch such as this:



18. A comparison of the structure of  $S_1$  and  $S_2$  with well-known algebraic-linguistic models<sup>1</sup> has not been undertaken. That successive well-formed formulas in a proof in these systems need not have a greater number of columns may impinge upon this comparison - sometimes a structure such as:

tie note  
note tie

will be replaced by:  
note  
note

However, no rule of inference of  $S_1$  or  $S_2$  allows as conclusion a well-formed formula that contains fewer note-codes than the premiss.

Here the greatest-numbered circumflected numeral designates which one of the three Ursätze is involved, differences in stem notation separate the upper from the middle lyne, '||' indicates articulation, '(Nbn)' indicates neighbour-note prolongation, '(Untergfzug)' indicates the prolongation technique of middle-lyne ascent, greater-valued notes generally imply that note's presence in the proof prior to the first appearance of lesser-valued notes, phrase-marks delimit progressions, etc.

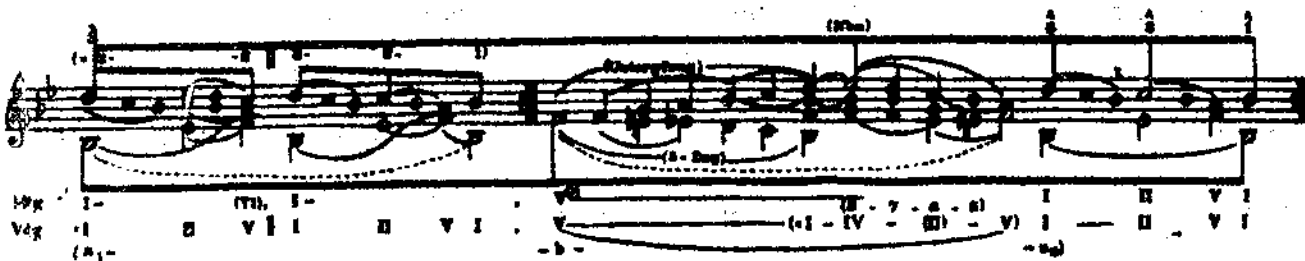
The foreground prolongation techniques collectively must account for many more notes and rests than the middle-ground prolongation techniques. However, a remarkable facet of Schenker's theory is that nearly every foreground technique closely resembles a corresponding middleground technique which Schenker called by the same name. Thus the research reported here should extend readily to an explication of Schenker's entire theory.

Schenker published a sketch that can be regarded as an extension of the proof of M into the initial strata of the foreground. A comparison of this sketch (reproduced from Der Freie Satz<sup>19</sup>) with the preceding one shows which notes have been added as a result of foreground prolongation techniques: the passage E-flat D C, starting at the fourth simultaneity in the following sketch, is a foreground descending progression rather

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19. Anhang, 2nd ed., (Vienna, 1956), p.16 ex.42 no.2.

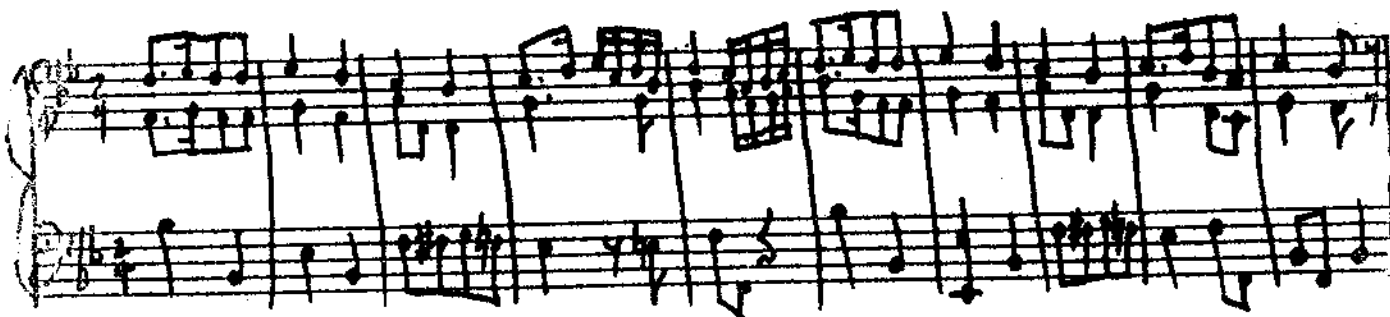
than a middleground descending progression because, amongst other reasons, it leads to a top-lyne note (fifth-octave C) rather than from one; the descending progression C B-flat A starting at the tenth simultaneity belongs to the foreground because it descends from a middle-lyne note rather than from a top-lyne note; the E-flat in the bottom lyne at the fourth simultaneity from the end belongs to the foreground rather than the middleground because, at least in the Der Freie Satz exposition of the theory, a middle-ground first-order descending progression must have its first and last notes supported in the bottom lyne by the same note; etc.



Schenker asserted that the above sketch is a sketch of Haydn's Chorale St. Antoni.<sup>20</sup> We interpret this claim

20. This chorale, used by Brahms as the theme for his Variations on a Theme by Haydn, is no longer attributed to Haydn, but that is another matter. Although music-theoretical propositions (i.e., those whose truth-value is computable just from the notes, rests, and other 'primitive symbols' of compositions) can express membership of a composition in one or another 'style' of composition, ascription of authorship requires other 'external' evidence, e.g., of a composer's handwriting, which, though musicological, transcends the music-theoretical level of explanation.

to mean that, when the foreground stage has been explicated, the proof of M can be extended to account not merely for all the notes in the above sketch but for all the notes and rests in the entire theme, whose first phrase follows:



The fifth-octave E-flats in bars one and two, which are not included in the previous sketch (Schenker implies thereby that they should be regarded as being nearer to the surface structure of the composition than any note included in the sketch), result from two applications of the foreground neighbour-note prolongation technique. Two other foreground prolongation techniques permit the fourth simultaneity in the previous sketch, particularly the E-flat over the C, both in the fifth octave, to be expanded first into the passage C D E-flat and then into the arpeggiation E-flat C which appear in the fourth bar of the chorale. Other foreground prolongation techniques allow introduction of additional notes from the same harmony, chromatic passages, and at least some rudimentary durational differentiation.

Even before explication of the foreground, now under way, is completed, the capability of the middleground explication to reproduce Schenker's middleground structures can be tested. For instance, the next computer-produced proof of a middleground structure of a Bach chorale includes, as the final assertion in the proof of  $S_1$ , the same composition as the second in the following series of five progressively more surface-structure sketches Schenker published<sup>21</sup> of this chorale:

[The following text is extremely faint and largely illegible, appearing to be a series of musical sketches or a very faded transcription of a musical passage.]

21. Recently reprinted as Five Graphic Music Analyses (New York, 1969), pp. 32-33. An obvious misprint in the bottom lyne of the second sketch has been corrected.

P R O O F  
(IN THE SYSTEM S1)

E5 D5 C5  
C3 G3 C3

(3-TO-1 AXIOM.)

E5 TIE TIE D5 C5  
C3 G3 C3 G3 C3

(INFERRED FROM LAST BY RULE OF BASS ARPEGGIATION.)

E5 D5 E5 D5 C5  
C3 G3 C3 G3 C3

(INFERRED FROM LAST BY RULE OF ARTICULATION.)

E5 D5 E5 F5 E5 D5 C5  
C3 G3 C3 TIE TIE G3 C3

(INFERRED FROM LAST BY RULE OF NEIGHBOUR-NOTE  
PROLONGATION.)

E5 D5 E5 F5 E5 D5 C5  
C3 G3 C3 F3 G3 TIE C3

(INFERRED FROM LAST BY RULE OF BASS ASCENT.)

C5 AS4 C5 CS5 C5 AS4 GS4  
GS2 DS3 GS2 CS3 DS3 TIE GS2

(INFERRED FROM LAST BY RULE OF TRANSPOSITION TO GS4.)

(Q.E.D.)

P R O O F  
(IN THE SYSTEM S2)

C5 AS4 C5 CS5 C5 AS4 GS4  
RST RST RST RST RST RST RST  
GS2 DS3 GS2 CS3 DS3 TIE GS2  
(AXIOM.)

C5 AS4 C5 TIE TIE CS5 C5 AS4 GS4  
RST RST C5 AS4 GS4 RST RST RST RST  
GS2 DS3 GS2 TIE TIE CS3 DS3 TIE GS2

(INFERRED FROM LAST BY RULE OF FIRST-ORDER DESCENDING  
PROGRESSION.)

C5 TIE TIE AS4 C5 TIE TIE CS5 C5 AS4 GS4  
C5 AS4 GS4 RST C5 AS4 GS4 RST RST RST RST  
GS2 TIE TIE DS3 GS2 TIE TIE CS3 DS3 TIE GS2

(INFERRED FROM LAST BY RULE OF FIRST-ORDER DESCENDING  
PROGRESSION.)

(Q.E.D.)



J. S. BACH CHORALE: "ICH BIN'S, ICH SOLLTE BÜSSEN"

Ursatz

1. Schicht

2. Schicht

3. Schicht

Takter 1  
(Fermaten:

4. Tafel

Extensive testing of the suitability of our middleground explication - involving, amongst other things, extensive comparison of Schenker's writings and examples of the middleground with theorems of  $S_2$  and analysis of the discrepancies - remains to be done. We have chosen to defer this testing until the foreground is explicated, in part because Schenker's writings do not place the middleground/foreground boundary always in the same place, in part because initial evidence (including, but not limited to, the proofs presented in this report) indicates that what has been done so far is satisfactory, and in large part because our ultimate goal is not to explicate the somewhat artificial middleground structures that Schenker composed but the class of all musical compositions instancing tonality.

As a significant by-product of our research, a computer-based 'library' has been implemented of the music-theoretical concepts that we have explicated as algorithms. This facility enables one to interact comparatively rapidly with the computer, which automatically can determine the applicability of such a concept, or of a given function suitably composed from such concepts, to particular pieces of music.<sup>22</sup> The freeing of the

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22. Technically feasible hardware and software engineering can produce faster response times than presently experienced and an APL compiler for debugged programs that executes faster than the present interpreters. Only time and budgetary constraints preclude implementation of a machine aid to musical theorists in which computations having the complexity of the theorem-proving reported here are performed so fast as to appear instantaneous.

user from having to carry out important but non-creative tasks allows him to think more about those components of the processes of theory formation and theory validation that have not been mechanised. The experience of other fields suggests that the intensive man/machine interaction and increased opportunity for creative thought which such a facility makes possible will result in important intellectual advances.