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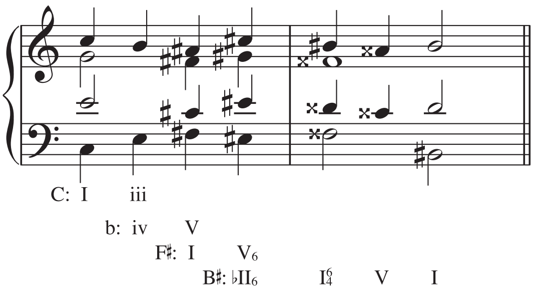
Paradoxes of Pitch Space

Music, like visual art, evokes images of space. The succession of pitches, harmonies and keys which make up a tonal composition may evoke movement through a space within which these musical 'objects' relate to one another as near or far, inside or outside, above or below. Given that we can measure distances between pitches according to their position on a frequency spectrum, it is tempting to imagine that we can map this space in geometric terms, representing pitches as points and keys as regions. Yet when we attempt to locate these musical objects precisely in relation to one another within this imaginary space, we quickly confront its contradictions. C1 and C8 are at opposite ends of the frequency spectrum, yet the ear, and the principle of octave equivalence, also tell us that they are the 'same'.[1] C major and A natural minor contain the same pitches, yet we perceive them as distinct tonal regions with different emotional qualities, C major sounding 'bright' or 'happy' and A minor 'dark' or 'sad'.

Composers at one time attributed different qualities to enharmonically equivalent keys as well: Beethoven, for example, described the key of C♯ major as 'hard' and D♭ major as 'soft'.[2] Evidence that the distinction between enharmonically equivalent keys continued to be taken seriously long after it was supposed to have been perceptually erased by equal temperament can be found in Max Reger's *Beiträge zur Modulationslehre* (1903; translated as *On the Theory of Modulation* (1948)). At the end of a series of exercises showing how to modulate from C major to all other keys, Reger presents his version of the shortest route for modulation from C major to B♯ major (Ex. 1). It is clear from the succession of keys which Reger advances that he intends this progression to be heard as ending a full twelve semitones higher along the circle of fifths than the point where it began. Yet given that the progression is only two bars long, we cannot help but notice that the sonority at the beginning is identical to the one at the end. In this passage, we are confronted with one of the most striking paradoxes of the twenty-four-key system: that keys identical in sound can nevertheless appear to be very far apart.[3]

Douglas R. Hofstadter (1999) notes that contradictions such as these are to be found not only in music, but also in visual art and mathematics. Illustrating his point with examples drawn from the music of J. S. Bach, the visual art of M. C. Escher and the mathematics of Kurt Gödel, Hofstadter proposes that these three modes of thought – musical, visual and mathematical – exhibit

Ex. 1 From Max Reger, *On the Theory of Modulation* (1948), p. 11



similar paradoxes, which he refers to as 'strange loops'. Hofstadter makes mathematics the central strand of his 'eternal braid'; the other two, music and visual art, provide colourful counterpoint to the subject of his 'metaphorical fugue'. This article will also draw together mathematics, music and visual art, but with music placed in the centre of the frame. It will show how mathematical models of musical and visual space on the one hand, and embodied models of musical and visual perception on the other, provide a framework which enables us to explain the paradoxes of pitch space by portraying them in visual form, thereby allowing us to 'see' how they work.[4]

Mathematical Models of Musical and Visual Space

Given that spatial relations in music, like those in visual art, can be described in geometric terms, we should not be surprised to discover that some of the same mathematical tools have been applied to these two art forms. The parallels between mathematical models of musical space and those of visual space have been brought to the fore by the recent emergence of neo-Riemannian and transformational theory, as can be seen by comparing Figs. 1 and 2.

Fig. 1 shows that now-familiar model of triadic pitch space, the *Tonnetz*, as it appears in an introductory article on neo-Riemannian theory by Richard Cohn (1998).[5] Strikingly similar, Fig. 2 is an example of what mathematicians refer to as *tiling of the plane*. To tile a plane means to cover it with polygons aligned edge to edge, with no overlap and no spaces left unfilled.[6] Tilings, part of the decorative arts since ancient times, have been the subject of mathematical study since the early seventeenth century, when Johannes Kepler (1619) discovered all eleven of the so-called Archimedean tilings of the plane.

Comparison of Figs. 1 and 2 reveals that the two are geometrically similar, each consisting of a gridlike arrangement of right-side-up and upside-down triangles. Yet further parallels exist between them, reflecting shared principles of organisation. First, each was created by replicating a single object – its basic shape. In the case of the *Tonnetz*, this basic shape is the major triad; in the case

Fig. 1 Neo-Riemannian *Tonnetz*

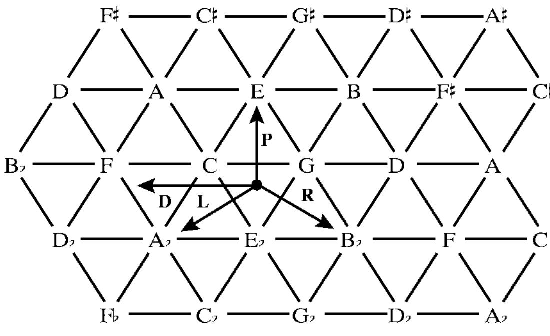
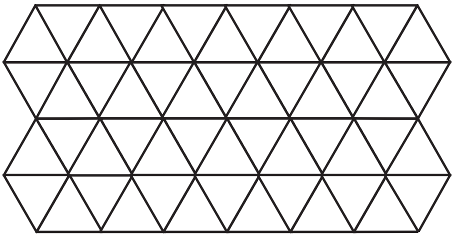


Fig. 2 Tiling of the plane



of the tiling, it is the equilateral triangle. Mathematicians refer to such an object as a *prototile*.

Second, the objects embedded within each grid can be manipulated in similar ways. In tiling the plane, replication of the prototile is carried out through what mathematicians call *congruence transformations*. As shown in Fig. 3, these consist of four basic types: *translation*, *rotation*, *reflection* and *glide reflection*. Translation shifts a tile (or group of tiles) a certain distance in one direction while preserving its overall orientation; rotation changes the orientation of a tile by rotating it around a point within the plane; reflection flips a tile from front to back across an axis lying within the plane; and glide reflection combines reflection with translation. As we can see by comparing Figs. 1 and 3, the neo-Riemannian operators D, P, R and L form a subset of this family of transformations. The D operator, by which a triad is transformed into its dominant or subdominant, corresponds to translation, while P, R and L operators, by which a triad is transformed into its parallel, relative or *Leittonwechsel*, correspond to reflection across the axis of its fifth, major third or minor third, respectively.

Fig. 3 Congruence transformations

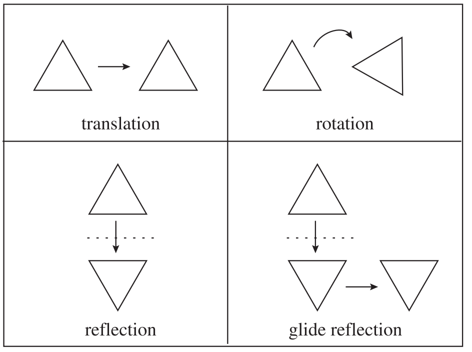
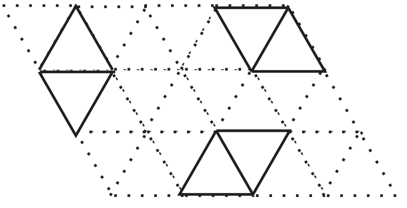


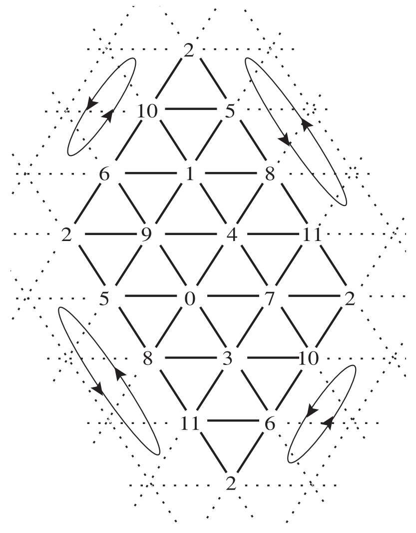
Fig. 4 Period parallelograms



Third, each of the patterns shown can be understood to repeat indefinitely in all directions. Mathematicians refer to such a pattern as a *periodic tiling*; it is constructed by replicating a *period parallelogram* (a parallelogram defined by the two basic translation vectors of the tiling). As Fig. 4 shows, the tiling of Fig. 2 can be constructed from any of three different period parallelograms, each formed by joining a right-side-up triangle to one of its three reflections. Repeated translation of any one of these objects in nonparallel directions will eventually cover the Euclidean plane, in the process forming a set of infinitely extending parallel axes which criss-cross in three directions. In the case of the *Tonnetz*, the pattern which is replicated is that of a major triad joined to a neighbouring minor triad, producing infinitely extending axes of perfect fifths, major thirds and minor thirds.

Finally, each grid can be made to take the shape of a torus. The tiling of Fig. 4 can be shaped into a torus by rolling the represented parallelogram up into a cylinder and then bending the cylinder around to join the ends.[7] The same can be done with the *Tonnetz* after first modifying it as shown in Fig. 5. In this modified *Tonnetz*, because equal temperament and enharmonic equivalence are presumed, pitch numbers take the place of pitch names, reducing

Fig. 5 Toroidal *Tonnetz*



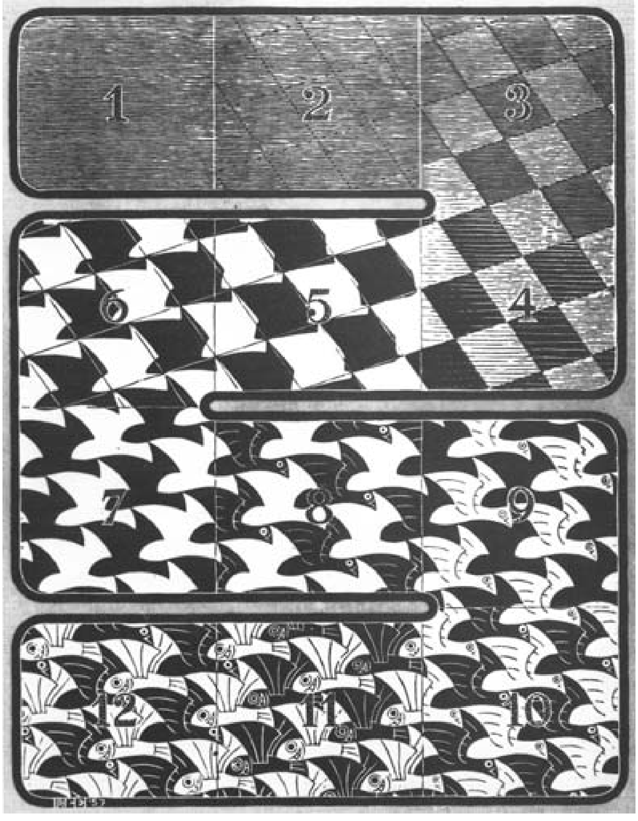
the number of distinct points within the grid from a potentially infinite number to twelve. These are arranged in the shape of a three-by-four-unit period parallelogram, which may be rolled up on itself in two directions, as shown by the arrows, to form a doughnut-shaped torus.[8]

Paradoxes of Visual Space in the Artwork of M. C. Escher

Just as music theorists have used the mathematics of neo-Riemannian theory to explore the geometry of musical space, visual artists have employed the mathematics of tiling to explore the geometry of visual space. Most notable among them is M. C. Escher.[9] Escher became fascinated with tilings of the plane after seeing the elaborate tile mosaics of the Alhambra Palace in Granada, Spain. Intrigued by these figures, he set out to explore systematically the possibilities for periodic tiling of the plane. The resulting notebook of sketches served as the basic material for much of his later artwork.[10]

However, Escher was not content simply to fill his artworks with abstract geometric figures. Instead, he carefully transformed each figure, bringing it to life by carving away portions of one side of a polygon to add to another until the original figure – triangle, square, rhombus or hexagon – took the shape of a bird, fish, lizard, horse or frog. We see this process of transformation in *Mouch* (Fig. 6), whose succession of frames, numbered 1 to 12, reveal the artist's working method. By transforming geometric figures into living creatures, Escher is able to highlight certain paradoxes of visual perception.

Fig. 6 M. C. Escher, *Mouch*



Their existence reveals how the human eye elaborates upon what is physically seen, suggesting that while visual space has properties in common with physical space, it also possesses a logic of its own.

In *Mouch* our attention is focused on the paradox of figure-ground reversal. When the eye is presented with a boundary dividing the visual field into two regions, it attempts to interpret each as lying either inside or outside – that is, as either figure or ground. In *Mouch*, figure-ground ambiguity results from the periodic division of the plane into black and white figures, which we perceive alternately as black objects against a white background or white objects against a black background. The paradoxical effect is heightened by Escher's use of another compositional device, that of *metamorphosis*. The black and white parallelograms which emerge in frames 1–4 are first transformed into black and white birds (frames 5–10), then into black and white fish (frame 11) and finally into white fish swimming to the left interspersed with black birds flying to the right (frame 12).

Escher was likewise fascinated by the paradox of impossible pathways for movement. In one of his best-known works, *Ascending and Descending*, the paradox of a staircase which doubles back on itself in a circle is endowed with human significance by the two lines of human figures, one apparently moving continuously upwards and the other continuously downwards.[11] In *Ascending and Descending*, Escher takes advantage of the eye's propensity to interpret two-dimensional images as three-dimensional while at the same time he provides contradictory cues for distance, height and depth to create a pathway which continuously rises while circling back upon itself to remain in place.

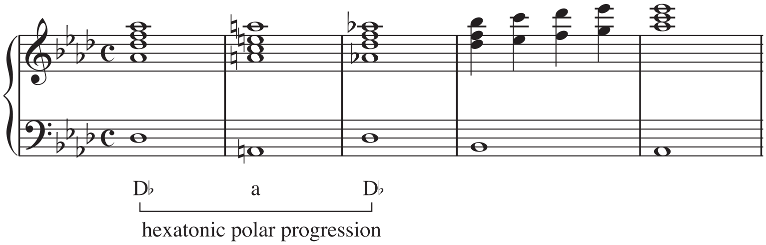
Escher's *Möbius Band II* confronts us with another 'impossible' pathway for movement: a one-sided yet two-sided surface, along which one can trace a continuous pathway from front to back. In this case the paradox is not just visual but physical: we can easily construct a Möbius strip by twisting a rectangular strip by 180° and joining its ends. Creating such an object allows us to discover firsthand its paradoxical properties. By placing a finger on one side of the strip and a thumb on the other, we can discover its *local* property of two-sidedness; by tracing our finger smoothly along its surface from front to back, we can discover its *global* property of one-sidedness.[12] In *Möbius Band II* the paradox of a one-sided yet two-sided surface is brought to our attention by the progression of a single, continuous line of ants travelling from 'front' to 'back'.

Paradoxes of Musical Space in Wagner's *Parsifal*

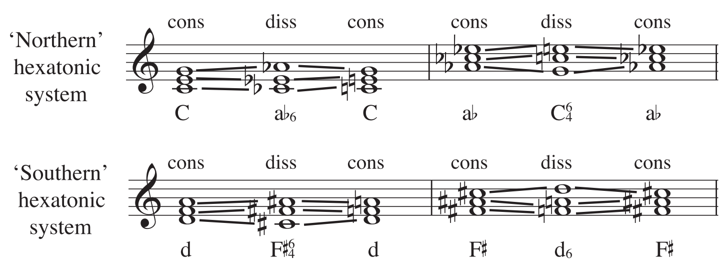
It was in part the presence of similar paradoxes in the chromatic music of Wagner and other nineteenth-century composers which inspired the development of neo-Riemannian theory. David Lewin (1984) suggests that Wagner uses a kind of sleight of hand to lead us from the everyday diatonicism of 'Stufen space' into the otherworldly enharmonicism of 'Riemann space'. To identify the differences between Stufen space and Riemann space, Lewin invokes the metaphor of a Möbius strip intersecting with a cylindrical loop, proposing that we pass from one to the other via a 'fault in the terrain', a 'splice' (1984, p. 347), created by an enharmonic C♭/B.

Richard Cohn (1996 and 2004) has likewise pointed out the paradoxical nature of hexatonic polar progressions, found not only in *Parsifal* but also in many other chromatic works of the nineteenth century. Noting that this progression has struck many commentators as 'uncanny', 'supernatural, magical, [and] weird' (Cohn 2004, p. 286),[13] Cohn describes the relationship between the triadic poles of a hexatonic system as 'a musical analogue of a Lewinian trope, the recursive gaze of Sieglinde and Siegmund' (Cohn 1996, p. 21). He notes that Wagner makes particularly effective use of this progression in the transformed version of the 'Grail' motive which appears at the very end of *Parsifal*, shown in Ex. 2. As Cohn observes, this progression conveys the mystery of the final event of the opera: Kundry's soul taking leave of her body.

Ex. 2 Wagner, *Parsifal*, Act III, bars 1123–1127 (Kundry's *Entseelung*)



Ex. 3 Hexatonic polar progressions



According to Cohn, the uncanniness of the hexatonic polar progression is due to two notable features: first, its maximally smooth voice leading, illustrated in Ex. 3, which gives rise to a powerful sense of resolution at the local level; and second, the extreme symmetry of the hexatonic collection, which makes it impossible to locate a triadic centre. As a result, each triadic pole can be heard to resolve the other, sounding alternately consonant or dissonant, stable or unstable.

The means by which the symmetry of hexatonic space contributes to the paradox of the hexatonic polar progression is revealed more clearly in Fig. 7. It shows how Northern and Southern hexatonic systems divide the neo-Riemannian *Tonnetz* symmetrically into halves, each of which is itself divided symmetrically into its polar complements.[14] The opposing perceptions of the consonance-dissonance relationship between hexatonic poles can be compared to what we experience when we look at a Necker cube (Fig. 8). The cube presents us with two equally possible but contradictory views of the same object, one consisting of the cube as viewed from above, the other from below. In hexatonic space such paradoxes appear not to be the exception, but the rule. It is a space within which two similar percepts can easily vie for our attention, neither asserting dominance over the other.

Fig. 7 Hexatonic polar progressions within the neo-Riemannian *Tonnetz*

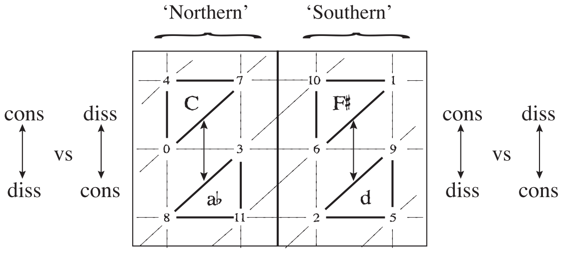
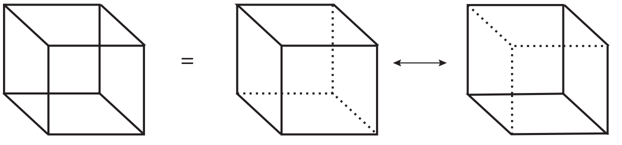


Fig. 8 Necker cube

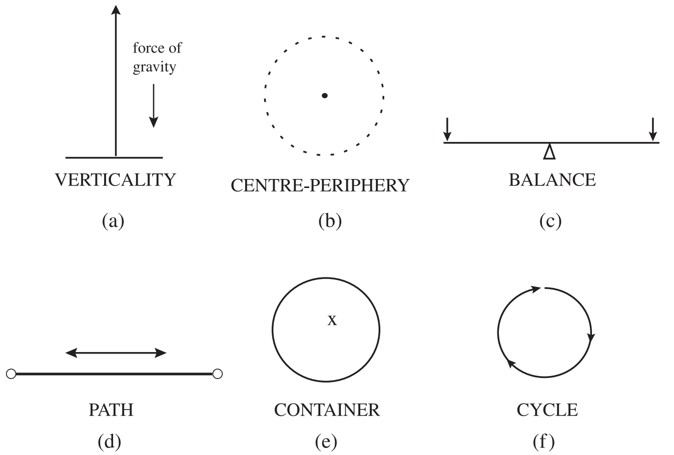


Neo-Riemannian theory has shed new light on the paradoxical nature of musical space by providing us with the means of visualising the geometry of triadic pitch space as a whole. Not only does it reveal symmetries which might otherwise go unnoticed, allowing us to see how spatial symmetry contributes to the paradox of the hexatonic polar progression, but it also neutralises the diatonic bias of everyday musical experience, thus allowing us to explore alternative modes of perception.

However, as this article will show, the mathematics of neo-Riemannian theory tells only part of the story. For just as we grasp the paradoxical nature of Escher's *Ascending and Descending* only by projecting ourselves into its world to experience vicariously the steps of the figures ever trudging both upwards and downwards, so we grasp the paradoxical nature of the polar progressions of Wagner's *Parsifal* only by projecting ourselves into its musical space, thereby experiencing from within that space the feelings of decentredness through which Wagner conveys the disembodiedness of Kundry's 'de-souling'.[15] To explain such paradoxes fully we must examine the perceptual processes which underlie our embodied experiences of both music and visual art.

What do the bodily sensations evoked by the music of Wagner's operas have in common with those evoked by Escher's 'impossible' representations? Recent theories of embodied cognition suggest that our most instinctive responses to both music and visual art originate in the body. It appears that musical sounds, like visual images, evoke bodily sensations due to our metaphorical projection of them into the domain of embodied experience. Furthermore, certain recurrent

Fig. 9 Embodied image schemas



patterns of embodied experience, known as *image schemas*, appear to play a central role in such metaphorical projections.[16]

The Embodied Nature of Musical and Visual Perception

Many cognitive scientists now believe that image schemas play a central role in human thought and perception, serving as patterns upon which we draw in order to construct meaning in the world around us. These schemas – sensorimotor images constituting the basic 'shapes' of bodily experience – are abstracted from such everyday experiences as balancing the body while walking, reaching out and grasping an object, moving along a pathway towards a particular goal, rising to a standing position and entering or exiting a closed space. It appears that we use these image schemas to make sense of patterns perceived in more abstract domains through metaphorical projections, or *cross-domain mappings*, which imbue those patterns with bodily meaning.

Among the image schemas that appear to organise our bodily experience are those shown in Fig. 9. Figs. 9a–c capture important features of our experience of the body. The VERTICALITY schema, represented as a vertical axis extending upwards from a horizontal plane, illustrates the way in which we experience the body in relation to the ground under the force of gravity. At a somatosensory level, we experience the ground as a maximally stable location, while associating ascending motion with overcoming the force of gravity (thus effortful or tensing) and descending motion as giving in to the force of gravity (thus relaxing). The CENTRE/PERIPHERY schema reflects our experience of the body as the centre and of the rest of the physical world as extending outwards from it in all directions. At a somatosensory level, we associate centredness with feelings of stability and rest, and departure from a centre with feelings of instability and an impulse to return. The balance schema reflects the ways in which we unconsciously balance the forces acting upon the body so as to remain stable, upright and at rest. Fig. 9c specifically reflects our somatosensory experience of the symmetry of the human body in terms of the weight of the left side balancing that of the right.

The schemas presented in Figs. 9d, e and f reflect ways in which we experience the body in motion. The path schema shows that we tend to move along pre-established pathways leading to specific destinations. Paths are either constructed or worn over time through repeated movement along the same trajectory; thus they become associated with such properties as continuity, regularity and predictability. The container schema reflects our experience of space as bounded – that is, as divided into insides and outsides. We experience the body itself as a container and its boundaries as creating the division between self and non-self. We also experience ourselves as moving within and between containers, such as rooms, buildings, cities and regions. We may associate containment either with stability and protection from external forces or, conversely, with a lack of freedom, with constraint and inhibition of movement. Finally, the cycle schema reflects the way in which we experience repeating patterns in space and time. We learn about cycles most directly through such repetitive actions of the body as breathing and walking, as well as from such higher-level cycles as waking and sleeping. The cycle schema regulates our experience of time, allowing us to anticipate future events, and is thus associated, like the path schema, with continuity, regularity and predictability.

Certain general features of the image schemas shown in Fig. 9 should be kept in mind when considering the roles they play in cross-domain mappings. First, although they are typically presented as visual images, image schemas combine input from all of the senses; each binds together visual, auditory, motor and somatosensory information into a single experiential gestalt. For example, the verticality schema includes not only the visual image of vertical and horizontal axes, but also the somatosensory experience of our feet resting on the ground and the kinaesthetic sensation of our muscles tensing to hold the body upright.

Second, many image schemas are *topological*; that is, they can be fitted to patterns of different shapes and sizes. Thus the container schema, although drawn as a circle in Fig. 9e, can be applied to any bounded region of space, regardless of its shape or the number of dimensions in which it exists. The container schema, when mapped onto a bounded region of visual space, causes us to perceive whatever falls on one side of the boundary as figure and whatever falls on the other as ground. Thus the container schema plays an important role in determining the way in which we divide visual space into objects, parts of objects and the empty space that surrounds them.

Finally, many of these image schemas have features or properties in common that allow them to be combined, as illustrated in Figs. 10a–c. For example,

Fig. 10 Combined image schemas

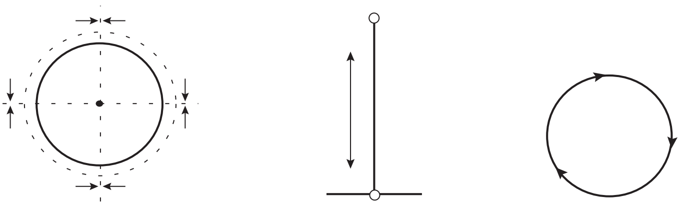
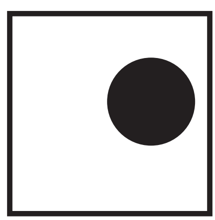


Fig. 11 From Rudolf Arnheim, *Art and Visual Perception: A Psychology of the Creative Eye*, published by the University of California Press, p. 12. © 1997 by the Regents of the University of California



CONTAINER, CENTRE-PERIPHERY and BALANCE schemas may be combined because they share the property of a centre, as shown in Fig. 10a. Likewise, VERTICALITY and PATH schemas may be combined because they share the property of linearity, yielding the schema for a VERTICAL PATH shown in Fig. 10b. Finally, CYCLE and CONTAINER schemas may be combined because they share the property of closure, yielding the schema for a CIRCULAR PATH shown in Fig. 10c.

The role these image schemas play in visual perception has been described by Mark Johnson (1987). Johnson shows how the BALANCE schema affects our experience of balance in visual art, illustrating his points with reference to experiments carried out by Rudolf Arnheim on the dynamic properties of visual 'objects' (Arnheim 1997). In one such investigation Arnheim presented viewers with the image of a disk within a square, such as that shown in Fig. 11, then asked them to describe their experience of the disk as either at rest or subject to forces of attraction or repulsion at different points within the square. Fig. 12 illustrates the ways in which viewers experienced the disk as acted upon by forces at different locations within the square, constituting what Arnheim refers to as its 'structural skeleton'. This figure suggests that the viewer's sense

Fig. 12 Arnheim's structural skeleton of the square. From Rudolf Arnheim, *Art and Visual Perception: A Psychology of the Creative Eye*, published by the University of California Press, p. 13. © 1997 by the Regents of the University of California

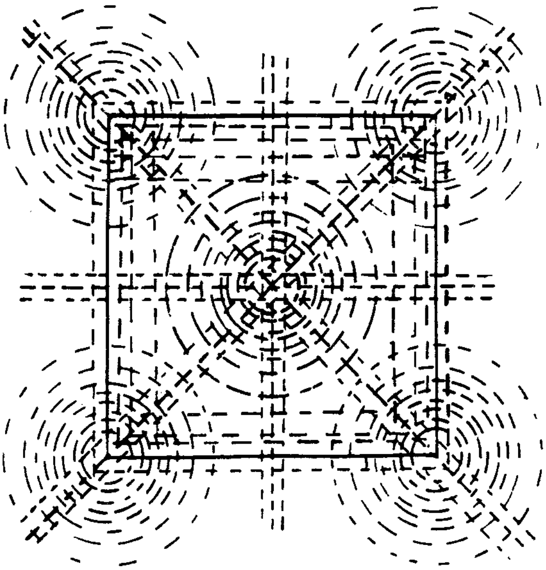
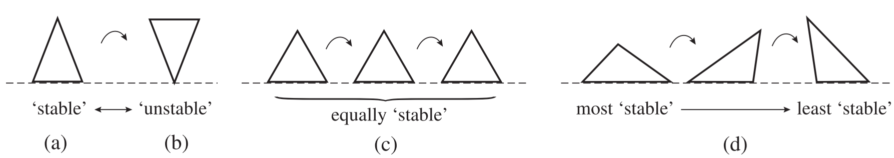


Fig. 13 Effect of shape and spatial orientation on visual stability. Adapted in part from Rudolf Arnheim, *Art and Visual Perception: A Psychology of the Creative Eye*, published by the University of California Press, p. 102. © 1997 by the Regents of the University of California



of balance is governed by the geometry of the square – its centre, edges, vertices and axes of symmetry. Further, it indicates that the viewer experiences the disk as most stable when it appears at the centre (its four axes of symmetry defining secondary areas of repose). The edges of the square also contribute to the viewer's perception of balance by exerting apparent forces, first of attraction as the disk approaches an edge, then of repulsion as it moves 'too close'. We can see in Arnheim's structural skeleton the combined influence of CENTRE-PERIPHERY, BALANCE AND CONTAINER schemas.

Arnheim further observes that our experience of visual 'objects' as stable or unstable depends upon their spatial orientation. He notes that a right-side-up triangle, such as that shown in Fig. 13a, appears to rest solidly on its base, while the upside-down version in 13b seems precarious, ready to topple to one side. Figs. 13c and d reveal even more clearly the combined effects of symmetry and spatial orientation. Although the equilateral triangle of Fig. 13c seems equally stable on each of its sides, the unequal triangle of Fig. 13d produces three different percepts, each more unstable than the last, reflecting the apparent rise in its centre of gravity and less evenly balanced distribution of weight. Figs. 13a–d clearly illustrate the contribution of VERTICALITY and BALANCE schemas to our embodied experience of visual 'objects', showing that we expect them to exhibit properties analogous to those exhibited by physical objects under the influence of the physical force of gravity.

That our embodied experience of visual 'objects' is determined in part by their geometry and spatial orientation suggests that the same may be true of the musical 'objects' represented in Fig. 1. At first glance, Fig. 1 might appear to capture our intuition about the relative stability of major and minor triads by representing the major triad as a right-side-up triangle resting on its stable base and the minor triad as an upside-down triangle perched on its apex. Yet, as Fig. 14 shows, this image presents a rather skewed picture of the triad's symmetry and spatial orientation. First, by representing the consonant triad as an equilateral triangle, the neo-Riemannian model suggests that the triad's tones are equidistant and thus imposes upon this type of chord a symmetry which it does not possess. Second, as the arrows in Fig. 14 indicate, the model portrays ascending intervals within the space of the triad as moving in three different directions: while the major third moves 'upwards', the minor third moves 'downwards', and the perfect fifth moves 'horizontally' across the page.[17]

Fig. 15 presents a modified version of Fig. 14 which preserves the original topology while changing its geometry and spatial orientation in order to reveal

Fig. 14 The neo-Riemannian 'basic shape'

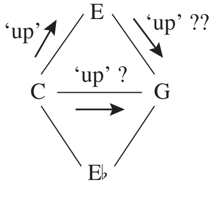
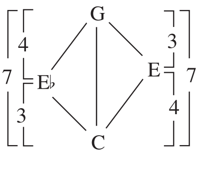


Fig. 15 Modified version of the neo-Riemannian 'basic shape'

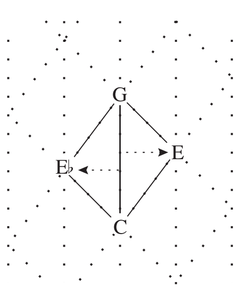


more clearly its image-schematic organisation. It first rotates the figure by 90° to place C at the bottom and G at the top, then flips it across the vertical axis to position E♭ on the left and E♮ on the right, locating these vertically so as to divide the seven semitones of the perfect fifth into 3 + 4 on the left and 4 + 3 on the right. The positioning of C at the bottom and G at the top allows every rising interval to ascend within the visual space of the page, thereby capturing our intuition that the root of a triad corresponds to its lowest, most stable note.

By focusing attention on the vertical arrangement of notes within the triad, Fig. 15 also highlights an important difference between the geometry of visual 'objects' and that of musical 'objects'; that is, visual 'objects' have both height and width, but musical 'objects' have only height. This can be seen clearly in Fig. 15 in the way that the 'sides' of the C major 'triangle' fail to add up. To form a triangle with sides three, four and seven units in length, we would have to set the internal angles at 180° and 0°, which would cause the shorter sides to collapse onto the longer one. This lack of a horizontal dimension also applies to the space within which we imagine musical 'objects' to be in motion. Although we often describe notes and triads as moving 'up' or 'down', it seems nonsensical to describe them as moving to the 'left' or to the 'right'.

What, then, is the meaning of the horizontal dimension in Fig. 15? What determines the placement of pitches along the horizontal axis? These questions are answered visually in Fig. 16. As the arrows in Fig. 16 show, we can conceive of the triangles used to model major and minor triads as having been formed by displacing the third of each triad away from the vertical fifth axis, the third of the minor triad to the left and the third of the major triad to the right. This allows the geometry of the space as a whole to be seen more clearly, much as we might open out the gathered folds of a net in order to reveal the geometric pattern of its construction. Not only does the horizontal displacement of the thirds allow major and minor triads to be more easily differentiated by eye, but it also allows the interval of a semitone (that is, E♭ to E♮) to be read as ascending from left to right. Thus, Fig. 16 makes clear that the horizontal dimension is nothing more than an aid to visualisation. Although the horizontal

Fig. 16 Reinterpretation of the neo-Riemannian 'basic shape'



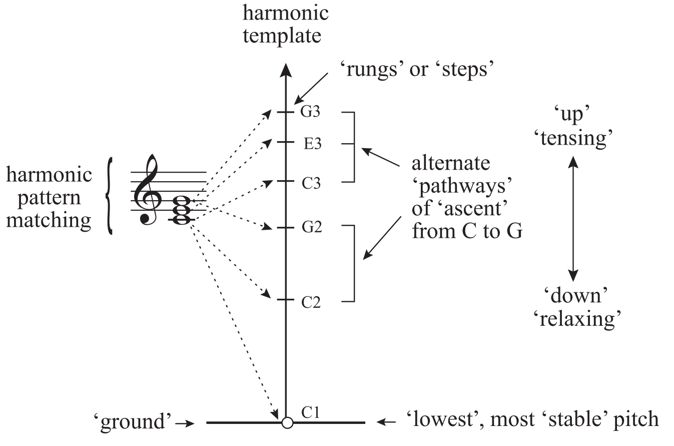
displacements of E♭ and E♮ make them look as if they are far apart, measurement along the vertical axis reveals that they are in fact quite close.

Although Fig. 16 allows us to 'see' the vertical arrangement of tones within major and minor triads, it offers no clue as to their relative stability. This reflects the fact that whereas visual 'objects' map more or less directly onto physical objects through their shared projection on the retina, the mapping of visual 'objects' onto musical 'objects' involves not only different domains but also different sense modalities – sight versus sound. Thus we should not expect the visual model to portray directly the embodied qualities of musical 'objects', but only those features of their geometry which contribute to their image-schematic organisation.[18]

It is to psychoacoustics that we must turn for clues concerning our embodied experience of musical sounds. Research in this field suggests that our perception of tone combinations as stable or unstable is governed in part by the way in which their intervallic patterns correlate with that of the overtone series. It has been found that the overtone series, having been stored in memory through repeated exposure to the complex sounds of music and speech, serves as a template for pattern matching carried out at an unconscious level.[19] It is here that we find the most likely explanation for our perception of the major triad as more stable than the minor: its intervallic pattern forms a better match with the intervallic pattern of the overtone series.

We can gain further insight into the embodied nature of our experience of musical sounds by examining the image-schematic organisation of the overtone series itself. As Fig. 17 shows, the overtone series takes on an image-schematic structure of its own when mapped onto a VERTICAL PATH schema. Fig. 17 portrays the fundamental of a pitch as its 'ground' – the lowest, most stable location within the space of a complex pitch – and the upper partials as the rungs

Fig. 17 Mapping of the overtone series onto a VERTICAL PATH schema



of a vertical pathway or 'ladder' which ascends indefinitely towards a vanishing point on the perceptual horizon. This image suggests that our experience of the root of a triad as the lowest, most stable pitch can be attributed to its mapping onto the fundamental – the 'ground' element of the overtone series.

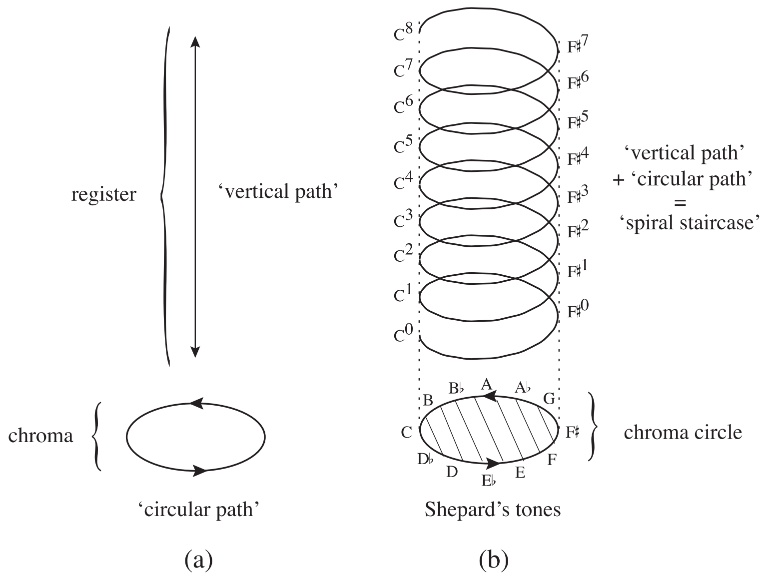
Fig. 17 also provides an image-schematic explanation for our experience of fifths and thirds as pathways for harmonic motion. As Fig. 17 shows, the first six notes of the series provide two distinct pathways for ascent from C to G, the first proceeding directly by fifth, and the second indirectly by major and then minor third. Fig. 16 represents these alternative pathways, the direct route from C to G vertically and the more indirect route from C to E to G as zigzagging first to the right and then to the left.

The fact that pitches are replicated within the overtone series helps to explain not only why we interpret C–G and C–E–G as alternative pathways for harmonic motion, but also why we reduce the pitches making up the major triad to three.[20] C2 and C3 are perceived to be the 'same' as C1; thus they take on similar feelings of 'groundedness', reinforcing our impression that the root of a triad, regardless of the register in which it sounds, corresponds to its 'lowest', 'most stable' pitch. Likewise, we perceive G3 to be the 'same' as G2, reinforcing our impression of the fifth of a triad as its 'highest' note and of the third as the 'middle'. This helps to explain why movement from root to fifth, or from root to third to fifth, tends to be associated with tensing and its opposite with relaxing, as shown by the arrows on the right.[21] These feelings of tension and relaxation may be understood as a reflection of the bodily sensation of moving up and down with respect to the ground.

In its replication of pitches from one octave to another, we find another way in which pitch space differs from both visual and physical space, yielding a dimension not easily captured in the visual domain. Psychologists account for this dimension by describing pitch as having not one but two components, *register* and *chroma*. Change in register is modelled as movement along a vertical path and change in chroma as movement along a circular path, as shown in Fig. 18a.[22] The two components are sometimes modelled together in the form of a 'spiral staircase', as shown in Fig. 18b, which portrays our experience of increasing frequency as continuously ascending motion accompanied by a repeating cycle of changing timbral colouration within each octave.[23] This model of a spiral staircase has been used by the psychologist Roger Shepard to create so-called Shepard's tones, a chromatic scale within which each component's distribution of partials gradually shifts downwards as the pitch ascends, so that by the time the pitch has risen a full octave, the distribution of partials is the same as was the case at the outset.[24] The result is a musical version of Escher's *Ascending and Descending* – a scalar series of pitches which can be heard to rise continuously while also remaining in place.[25]

This pattern of repeating tone colours lacks any perceptual correlate in visual space. Although Fig. 16 does not portray this cyclic dimension of perceived pitch directly, the dotted vertical lines in the background hint at its

Fig. 18 (a) Register and chroma; (b) Shepard's tones



existence. We can conceive of each vertical line as having been formed by a metaphorical tracing wheel which was rolled from the bottom to the top of the page; the serrations along its outer rim represent the twelve pitches of the chroma circle. Thus, each dotted vertical line is the imprint of this repeating series of twelve chromatic pitches upon the page.

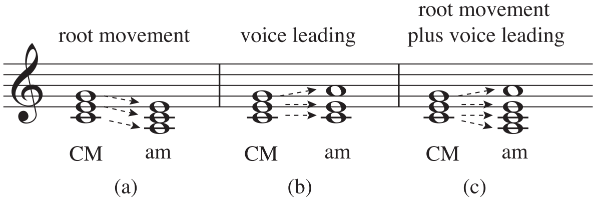
We can use the image-schematic model of the C major triad shown in Fig. 16 to construct an embodied model of triadic pitch space which incorporates these geometric and image-schematic features, thus enabling us to model our embodied experience of motion within triadic pitch space as a whole. Using principles of tiling, we can form additional triads from the C major triad which fit together to form the larger musical space within which we experience movement among pitches, triads and keys. We may begin with a purely tuned C major triad, making adjustments of tuning along the way as needed in order to permit the musical space to close in on itself and thus to form both diatonic and chromatic collections.

Rather than positing the space all at once, as was done with the neo-Riemannian *Tonnetz*, we will construct it in separate stages by applying two principles in alternation.[26] First, during each stage we will add only those triads needed to achieve some new form of symmetry and/or closure. Second, we will continue to add triads until musical space closes in on itself in all directions to form an embodied version of the neo-Riemannian *Tonnetz*. In the course of constructing this space, we will witness the formation of a set of nested spaces, each exhibiting properties of closure and/or symmetry which allows the space to be mapped onto a CIRCULAR PATH and/or a CONTAINER/CENTRE-PERIPHERY/BALANCE schema, thus giving rise to the embodied experience of containment, centredness and balance. We will also witness the emergence of spatial paradoxes analogous to those found in the artwork of M. C. Escher: figure-ground reversal, circular staircases and the Möbius strip.

Because the present model is constructed using neo-Riemannian operators (D, P, L and R) and its completed structure resembles that of the neo-Riemannian *Tonnetz*, it is important to note the ways in which it differs from the neo-Riemannian model. First, although neo-Riemannians disregard the acoustic origins of the consonant triad, according equal weight to root, third and fifth, the present model privileges the root, reflecting the contribution of these origins to our sense of bodily orientation within triadic pitch space. Moreover, it views the root as standing in some sense for the triad as a whole, much as the fundamental does for a complex pitch. Second, although neo-Riemannian theory defines harmonic distance in terms of voice leading, thus privileging L, P and R transformations as maximally smooth, the present theory defines harmonic distance in terms of root movement. This is because voice-leading motion is presumed to take place within another dimension of musical space, one which I have referred to elsewhere as *melodic space* (Brower 2000). In consequence, whereas neo-Riemannians treat D as equivalent to L + R because these operations yield identical voice leadings, the present model views the same operations as distinct: D produces the harmonic motion of a single step along a pathway of fifths, while L + R produces the motion of two steps along a pathway of alternating major and minor thirds. Finally, although neo-Riemannians treat P, L and R as a family of operators because each produces stepwise motion within a single voice, the present model treats P as distinct from L and R because it produces not root movement, but a change in the position of the third of the triad with respect to its root and fifth.

The difference between these two interpretations of harmonic motion is illustrated in Ex. 4. Ex. 4a depicts the progression C major–A minor as root movement; thus, it maps root onto root while preserving internal relations among pitches. By contrast, Ex. 4b represents the same progression as voice leading; thus it preserves the common tones while mapping the fifth of the first

Ex. 4 Root movement and voice leading



triad onto the root of the second so as to create movement along the shortest melodic path. These two representations are combined in Ex. 4c, which shows both the harmonic pathway followed by root movement in the bass and the melodic pathways followed by voice leading in the upper voices.

The present model's treatment of root movement and voice leading as distinct components of harmonic motion helps to highlight the paradoxical nature of the relationship between them. Harmonic and melodic distances tend to be inversely related, so that what sounds melodically close often sounds harmonically distant and vice versa.[27] Furthermore, root movement and voice leading tend to proceed in opposite directions, with root movement down by fifth or third producing voice leading upwards by step and vice versa. As a result, whether the perceived direction of harmonic motion will be governed by root movement or voice leading cannot be determined independently of the musical context. In passages predominantly made up of root-position triads, our sense of direction is likely to be governed by root movement; in passages formed primarily from inverted triads, it is likely to be governed by voice leading.

The model also shares certain features with the cognitively derived model of pitch space advanced by Fred Lerdahl (2001). Using empirical data provided by Diana Deutsch and John Feroe (1981), Carol Krumhansl (1990 and 1998) and others, Lerdahl constructs three different models of tonal pitch space, which he refers to as *basic pitch space*, *chordal space* and *regional space*. His basic pitch space corresponds to what I am calling *melodic space*, while his chordal and regional spaces combine in the present model to form what I am calling *triadic space*. Lerdahl criticises Riemann and his followers for attempting to represent distances within musical space using a single map, namely that of the *Tonnetz*, in which, owing to a conflation of descriptive levels, points are sometimes taken to represent pitch and on other occasions triad or key. The present model avoids this problem by depicting pitches as points and triads and keys as groups of points, each of which forms a closed region of musical space. Because Lerdahl's model represents triads and keys, like pitches, as points rather than as regions, it cannot accommodate overlaps among these regions. This is particularly problematic in the case of relative major and minor keys. I will show here that the latter do not constitute different regions of musical space, but rather alternative ways of perceiving the same region.[28]

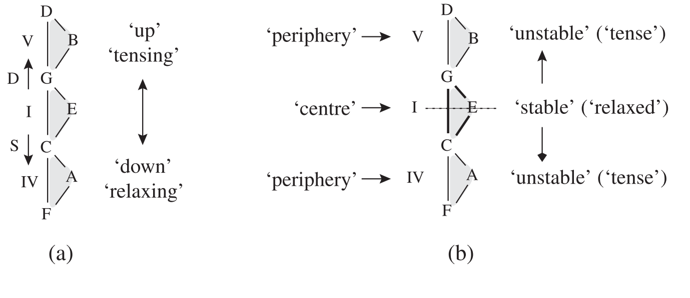
In emphasising the embodied character of our experience of movement through triadic pitch space, I do not claim that all listeners will hear changes of harmony or key as rising or falling, or tensing or relaxing, movement. Nor do I suggest that we can extrapolate from the experiences of Western subjects to those of non-Western listeners. Because such metaphorical mappings require active engagement of the bodily imagination, as well as familiarity with the conventions of the major-minor system, individual listeners may differ markedly in the degree to which they experience such sensations.[29] Nevertheless, we find evidence of significant intersubjective agreement among trained listeners concerning such experiences in the form of a strong consensus among eighteenth- and nineteenth-century composers and theorists as to how these properties correlate with specific changes of harmony and key (Steblin 2002).

Construction of the triadic pitch-space model in stages also provides the basis for a more ambitious claim, namely, that the logic governing this construction is similar to that which governed the evolution of the tonal system, from the emergence of triadic harmony in the fifteenth century, with its emphasis on diatonicism and the major mode, to the extreme chromaticism of the late nineteenth century and the emergence of hexatonic and octatonic collections. One could view this evolutionary pathway as reflecting the combined workings of two opposing compositional impulses: the impulse continually to push the boundaries of triadic pitch space outwards to expand the possibilities for harmonic motion, and the contrary inclination to form spaces which are symmetrical, centred and/or contained. In the final section of this article, support for this claim will be offered in the form of analyses of works from the fifteenth to the nineteenth centuries which hint at this pathway. These analyses also provide evidence that the paradoxes of triadic pitch space are more than purely mathematical abstractions, which suggests that at least some composers have intentionally exploited these paradoxes in order to evoke movement of the body through a space topologically very different from our own.

Constructing an Embodied Model of Triadic Pitch Space

In the first stage of construction we replicate our C major 'prototile' above and below via translation to form the symmetrical space shown in Fig. 19a. This expands the total number of pitches from three to seven – from those of the C major triad to those of the key of C major – while extending the vertical pathways for movement from the single fifth C–G to a chain of fifths, F–C–G–D, and from the succession of thirds C–E–G to a chain of alternating major and minor thirds, F–A–C–E–G–B–D. Within this space we may conceive of a succession of pitches as the movement of a single note along any of the pathways illustrated and a succession of triads as the movement of a single triad along the pathway of fifths shown on the left.[30] From a mathematical perspective, the

Fig. 19 Construction of triadic pitch space, stage 1



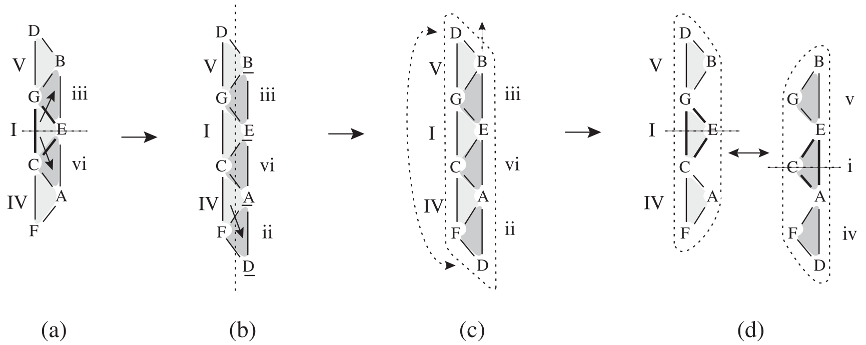
latter involves using the D operator to transform a triad into its dominant and the S operator to transform a triad into its subdominant.[31] From an embodied perspective, these correspond to motion upwards or downwards, tensing or relaxing, respectively, as shown by the arrows on the right.

Fig. 19b presents an alternative mapping of this constructed space onto a CENTRE-PERIPHERY/BALANCE schema. It portrays the key of C major as a musical 'object' within which are nested three smaller objects, IV, I and V, with I at the centre balanced by V above and IV below. This mapping reveals another dimension of our embodied experience of motion within diatonic space, that of the tonic as centre and thus a maximally stable location. This in turn supports our experience of the motion from tonic to either dominant or subdominant as tensing and the motion of return as relaxing. On an embodied level, we can understand this as reflecting the tensing and relaxing of our muscles as we move with respect to our own centre of gravity.

The two mappings of Figs. 19a and b yield somewhat conflicting image-schematic interpretations, with motion from tonic to subdominant connoting relaxation in the mapping of the VERTICALITY schema and tension in the mapping of the CENTRE-PERIPHERY/BALANCE schema. This suggests that alternative image-schematic mappings may at times yield contradictory meanings, and that our embodied experience of musical sound may at times reflect those contradictions.

In the second stage of expansion, shown in Fig. 20a, we begin by filling in the spaces left open in Fig. 19 with E minor and A minor triads formed through the operation of reflection, thus creating closure while preserving symmetry. It is at this stage of construction that the minor triad first appears as a musical 'object'. The relationship between major and minor triads can be viewed as analogous to that which emerges between the black and white areas in Escher's *Mouch*: areas initially perceived as ground take on the characteristics of figures because of their similarity to the objects that they surround.

Fig. 20 Construction of triadic pitch space, stage 2



Transformation of a major triad into a minor triad produces a lowering of the third which we experience on an embodied level as a decrease in its stability, reflecting its imperfect correspondence to the overtone series. The bodily sensation of the lowering of the third can be understood in relation to the VERTICALITY schema as akin to the weakening of the muscles which hold the body upright. This contrast in the affective quality of major versus minor has been expressed in such terms as 'strong' versus 'weak', 'hard' versus 'soft', 'happy' versus 'sad' or 'bright' versus 'dark'.[32] Because these properties are not directly represented by the visual model, Fig. 20 reminds us of them by shading major triads light and minor triads dark.

The addition of two minor triads expands the possibilities for harmonic motion by accommodating root movement by fifth along the newly formed pathway of fifths on the right, as well as root movement by third along the zigzag chain of thirds. We can trace a descent along this pathway by applying L and R operators in alternation, transforming major triads into minor and vice versa.[33]

Fig. 20b extends the chain of fifths on the right by reflecting F major across its major-third axis to form D minor. The addition of one more triad produces a new degree of symmetry which balances the three major triads on the left with three minor triads on the right. The underlining of the pitches D, A, E and B on the right reminds us that the triads of this space were constructed from purely tuned fifths and thirds; thus, $\underline{\text{D}}$ differs from D by a syntonic comma. As a result, if we were to continue the process of adding pure triads above and below, each of the interlocking chains of fifths and thirds would extend into infinity without reaching closure. However, we can easily impose closure on this musical space through a process known as *shearing*. By holding the pitches on the left in place while shifting those on the right upwards by a syntonic comma, we can align D at the bottom with D at the top, as shown in Fig. 20c.[34] By transforming purely tuned thirds into Pythagorean ones in this manner, we can form a 'circular staircase' from the vertical chain of thirds, along which motion can ascend or descend to return to its starting position.

This modification yields another spatial paradox comparable to one found in Escher's drawings, that of figure-ground reversal. As Fig. 20d shows, nestled within this closed diatonic space are two smaller spaces, each filling in the interstices of the other. One consists of the primary triads of C major and the other the primary triads of A minor. Yet because each of these nested spaces has its own centre-periphery organisation and maps independently onto a CONTAINER/CENTRE-PERIPHERY/BALANCE schema, we may flip back and forth between the two opposing perceptions, just as we do when we look at a Necker cube or the interlocking pattern of black birds and white birds in *Mouch*. If we imagine D to be at the 'top', C major will appear at the centre, straddled by its dominant and subdominant; if we imagine D to be at the 'bottom', A minor will appear at the centre, straddled by its dominant and subdominant.[35]

Fig. 21 Construction of triadic pitch space, stage 3

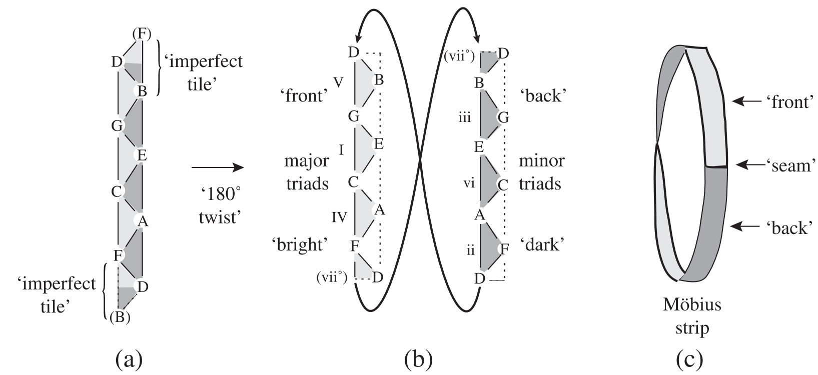


Fig. 22 Circle of fifths progression along the diatonic Möbius strip

'seam' 'seam' 'seam' 'seam'

'front' ↓ 'back' ↓ 'front' ↓ 'back' ↓ 'front'

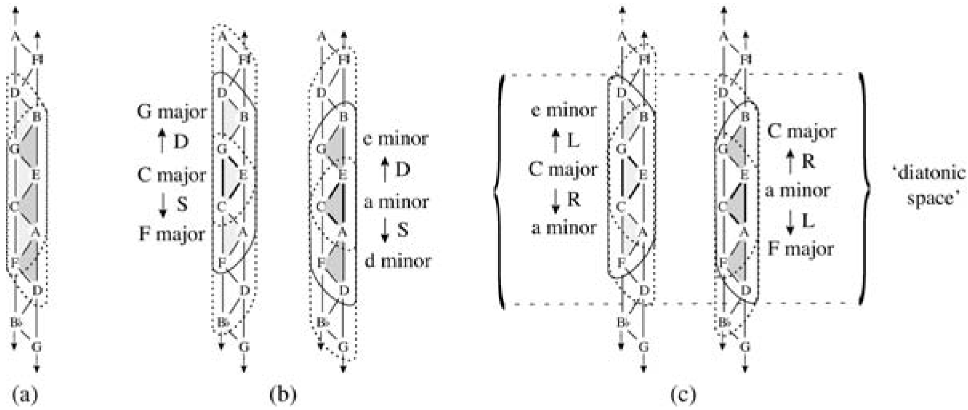
I – IV – vii° – iii – vi – ii – V – I – IV – vii° – iii – vi – ii – V – I

'bright' 'dark' 'bright' 'dark' 'bright'

Having formed a circular pathway from the zigzag chain of thirds, we can form another from the two chains of fifths. As shown in Fig. 21a, this involves twisting the double chain of fifths and thirds by 180° while bringing the ends together to align D at the top with D at the bottom, then inserting an 'imperfect tile', the B diminished triad, into the space remaining in order to link B to F via a diminished fifth. The result of this operation, as shown in Figs. 21b and c, is that one-sided yet two-sided geometric object the Möbius strip.[36]

With the insertion of the B diminished triad, we can trace a circular route for harmonic motion along the diatonic chain of fifths which passes through the three major triads on the front of the Möbius strip to the three minor triads on the back via the diminished triad straddling the seam, as shown in Fig. 22. Whereas the 'two sides' of the Möbius strip portrayed in Escher's *Möbius Band II* are indistinguishable, the two sides of the diatonic Möbius strip are easily differentiated owing to the appearance of major triads on the 'front' and minor triads on the 'back'. Furthermore, whereas in *Möbius Band II* the dividing line between 'front' and 'back' has been erased by the artist, yielding an apparently seamless one-sided surface, the seam in the diatonic Möbius strip remains marked by the presence of the diminished fifth, its imperfection serving to orientate us within the circle of fifths.[37]

Fig. 23 (a) Construction of triadic pitch space, stage 4, part 1; (b) and (c) modulation using D, S, L and R operators

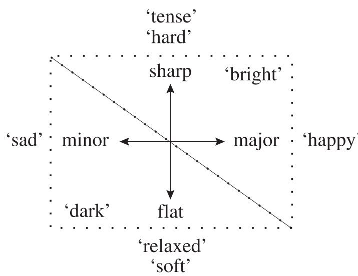


We may pursue the process of expansion begun in Figs. 19 and 20 by continuing to add major and minor triads above and below. As shown in Fig. 23a, the first few steps of this expansion involve adding D major/B minor above and B♭ major/G minor below. Within this expanded space, the key space of C major/A minor appears as a bounded object dividing triadic pitch space into two regions, diatonic and chromatic.

Just as we were previously able to transform one triad into another, so we can now apply D, S, L and R operators to this musical object in order to transform one key into another, yielding once again the metaphor of an object in motion. As Figs. 23b and c show, D and S operators produce movement upwards or downwards by fifth – to the sharp side or to the flat side – while L and R operators produce movement either upwards or downwards by third and from 'right' to 'left' or 'left' to 'right', transforming major into minor and vice versa. That each of these operations produces upwards or downwards movement helps to explain why modulation to the sharp side has been associated with tensing or hardening and that to the flat side, conversely, with relaxing or softening, these being the feelings we associate with upwards and downwards movements of the body. Likewise, the association of modulation to the sharp side with brightening and to the flat side with darkening may be accounted for in terms of our experience of the brightness of the sky versus the darkness of the ground when turning our eyes upwards or downwards.[38] Finally, the experience of modulation from major to minor as a shift from 'bright' to 'dark' or from 'happy' to 'sad' may be explained as the result of a lowering of the third of each primary triad.

Fig. 24 portrays the embodied sensations associated with modulation in these four directions as the four points of an 'embodied compass'. The dotted

Fig. 24 The four points of the embodied compass



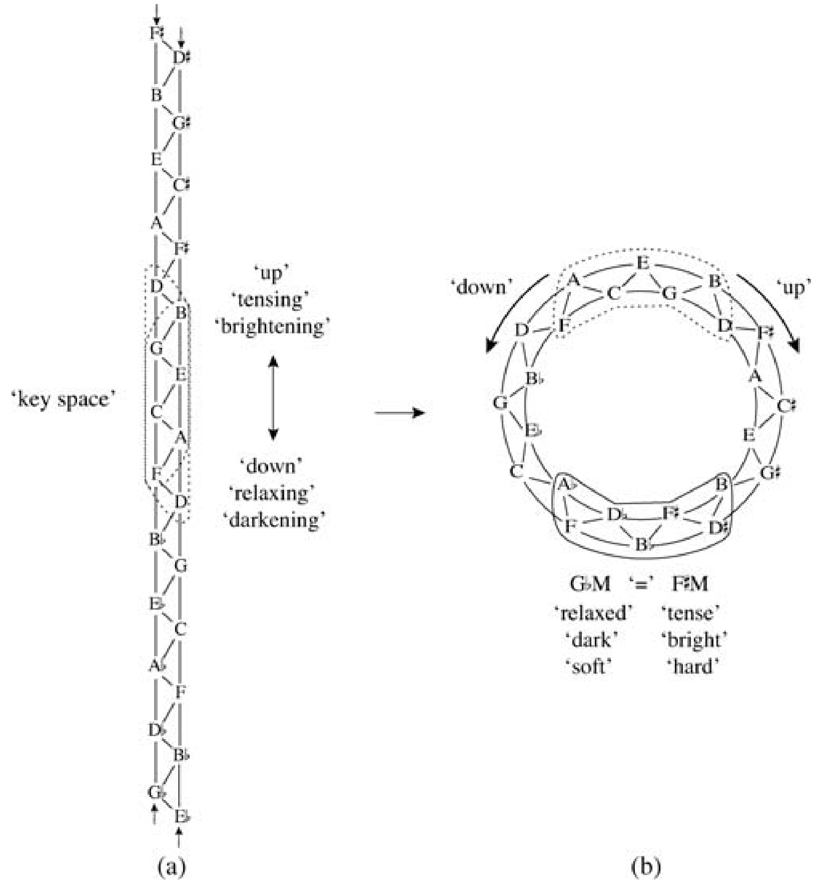
diagonal line draws attention to the fact that modulation to the sharp side or to a major key tends to be experienced somewhat similarly as 'brightening', and that to the flat side or to a minor key likewise as 'darkening.' This effect is attributable to the raising or lowering of the key container in the former case and of the thirds of the primary triads in the latter.[39]

Fig. 23c also reveals the special nature of the transformation of a major key into its relative minor and vice versa. Whereas application of D, S and L operators changes the pitch content of the key, application of the R operator changes only the mapping of the key container onto the CENTRE-PERIPHERY schema. Using the R operator to transform the key of C major into that of A minor causes the pitch D, which formerly appeared at the 'top' of the key space, to now appear at the 'bottom'. When D occupies the upper position, C major appears in the centre of the key container, straddled by its major dominant and subdominant; when D is in the lower position, A minor appears in the centre, straddled by its minor dominant and subdominant.

The process of adding major and minor triads above and below the diatonic space of C major/A minor is continued in Fig. 25 to produce a series of twelve major triads on the left balanced by twelve minor triads on the right. At this stage, closure can once again be achieved by aligning pitches at the bottom with those at the top. Whereas in diatonic space we employed shearing to bring D at the bottom into alignment with D at the top, in chromatic space we use *scaling* to bring G♭ into alignment with F♯, and likewise E♭ with D♯, thus transforming Pythagorean fifths and thirds into equal-tempered ones.[40]

As Fig. 25b shows, this chromatic chain of fifths and thirds can be doubled back on itself to form not a Möbius strip, but a cylindrical loop. Within this cylindrical loop, the imperfection of the diminished fifth and the 'twist' of the diatonic Möbius strip have disappeared. Thus, repeated application of the D or S operator will produce movement along either one chain of fifths or the other. If we follow the pathway on the left we get a series of twelve major triads; if we follow the pathway on the right, a series of twelve minor triads. Or, by applying L and R operators in alternation, we encounter all twenty-four major

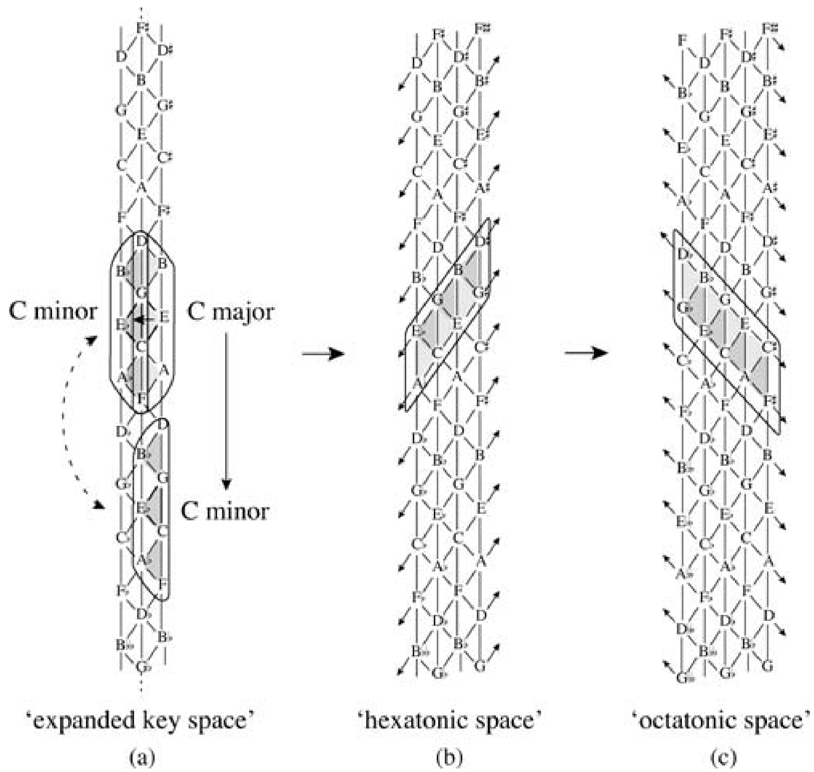
Fig. 25 Construction of triadic pitch space, stage 4, part 2



and minor triads in succession, yielding the equivalent of what Cohn (1997) has dubbed the <LR> cycle.[41]

Within this space we once again confront the paradox of a pathway which is both vertical, extending infinitely upwards and downwards, and circular, closing in on itself to return to where it began. As a result, we may ascend from C major to F♯ major or descend from C major to G♭ major to reach the 'same' location. Yet we do not experience modulation to these two keys in the same way. Rather, we experience movement along the sharp side of the circle of fifths as 'tensing', 'hardening' or 'brightening' and movement along the flat side as 'relaxing', 'softening' or 'darkening'. Thus F♯ major and G♭ major take on different meanings in the embodied imagination – 'tense' versus 'relaxed', 'hard' versus 'soft', 'bright' versus 'dark' – reflecting their positions at opposite ends of a vertically oriented space centred around C major.

Fig. 26 Construction of triadic pitch space, stages 5–7



Having filled the musical plane completely in the vertical direction, we can continue to add triads on both sides of our double chain of fifths, thus expanding musical space in the 'horizontal' direction. We begin by reflecting our prototile to the left, using the P operator to produce its parallel, C minor. From this inverted prototile we can construct a series of interlocking major and minor triads on the left which parallels the one on the right, thus producing a space which is symmetrical around the fifth axis containing our starting pitch, C, as shown in Fig. 26a.

This expansion increases the possibilities for modulation by adding the P operator to those already in place. Using D, S, L, R and P operators we can transform any major or minor key into any one of its five most closely related keys: dominant, subdominant, mediant, submediant and parallel.[42] Within this expanded key space we may modulate from any major key to its parallel minor (or vice versa) by following one of two different routes. As shown in Fig. 26a, we can move from C major to C minor by following a pathway of descent along the double chain of fifths and thirds, or we can move directly from C major to

C minor by using the P operator to reflect the key container across the fifth axis.[43] As in the case of enharmonically equivalent keys, these two modulations are not the same, for although they both end in the key of C minor, each takes us on a different journey through triadic pitch space. The appearance of these two regions at different stages of mathematical construction is reflected in the history of compositional practice as well: composers of the late Baroque favoured descent along the circle of fifths, while Classical and Romantic composers increasingly favoured the immediate contrast of mode made possible by direct modulation.[44]

As a consequence of the previous expansion, two sets of diagonal pathways have opened up, one made up of successive major thirds and the other of successive minor thirds. We may once again expand musical space to a point of closure by simply extending each diagonal chain of major thirds by one step, as shown in Fig. 26b. We can then align the pitch at the top of each chain of major thirds with the one at the bottom to form the set of circular pathways shown by the arrows, marking off that closed region of triadic pitch space constituting the hexatonic collection. Within hexatonic space we can move from one major or minor triad to the next via translation, or we can apply P and L operators in alternation to produce Cohn's '<LP> cycle'. Or we can move directly from one hexatonic pole to the other via glide reflection, producing the maximally smooth voice leading shown in Ex. 3.

By adding one more link to the chain of consecutive minor thirds which unfolds along the other diagonal, we achieve closure by yet another means, as shown in Fig. 26c. This forms another set of circular pathways marking off that region of triadic pitch space which constitutes the octatonic collection. Within octatonic space we can again move from one major or minor triad to the next via translation, or we can alternate between P and R operations to realise the <PR> cycle.

Because hexatonic and octatonic collections constitute closed regions contained within a larger chromatic space, we may conceive of them as we do the diatonic collection – as containers or objects which may themselves be set in motion. As Fig. 27 shows, it is within the space of the neo-Riemannian *Tonnetz* that hexatonic and octatonic modulation takes place, the hexatonic container moving through four possible locations within neo-Riemannian space (H1–H4) and the octatonic container through three (O1–O3). Each of these containers can be visualised as a cylindrical band which wraps around the hexatonic or octatonic torus, respectively, sliding around it three or four times before returning to its starting position.

Over the course of the expansions of Figs. 26a–c, a fourth, 'hidden' axis has emerged consisting of a series of chromatic semitones … C♭–C–C♯ … . We can fill the musical plane completely by extending this chromatic axis by a full twelve semitones in order to form the embodied *Tonnetz* of Fig. 28. Like the three-by-four *Tonnetz* of Fig. 27, this twelve-by-twelve *Tonnetz* can be rolled up in two directions, as shown by the arrows, to form a torus.[45] This final

Fig. 27 Hexatonic and octatonic modulation

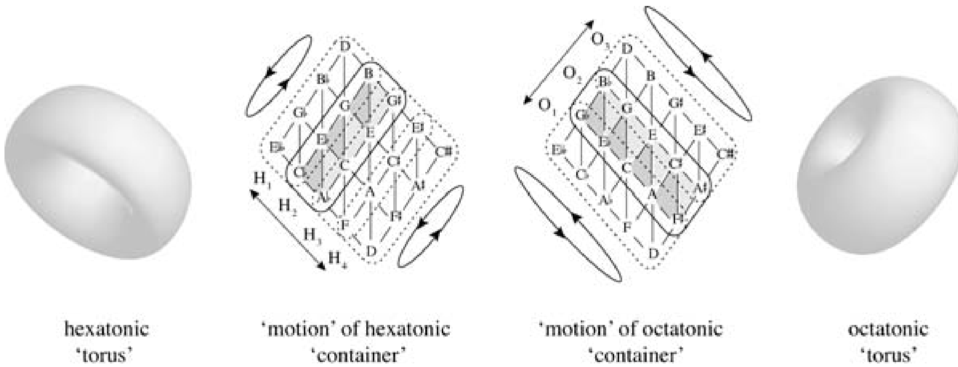


Fig. 28 Embodied *Tonnetz*

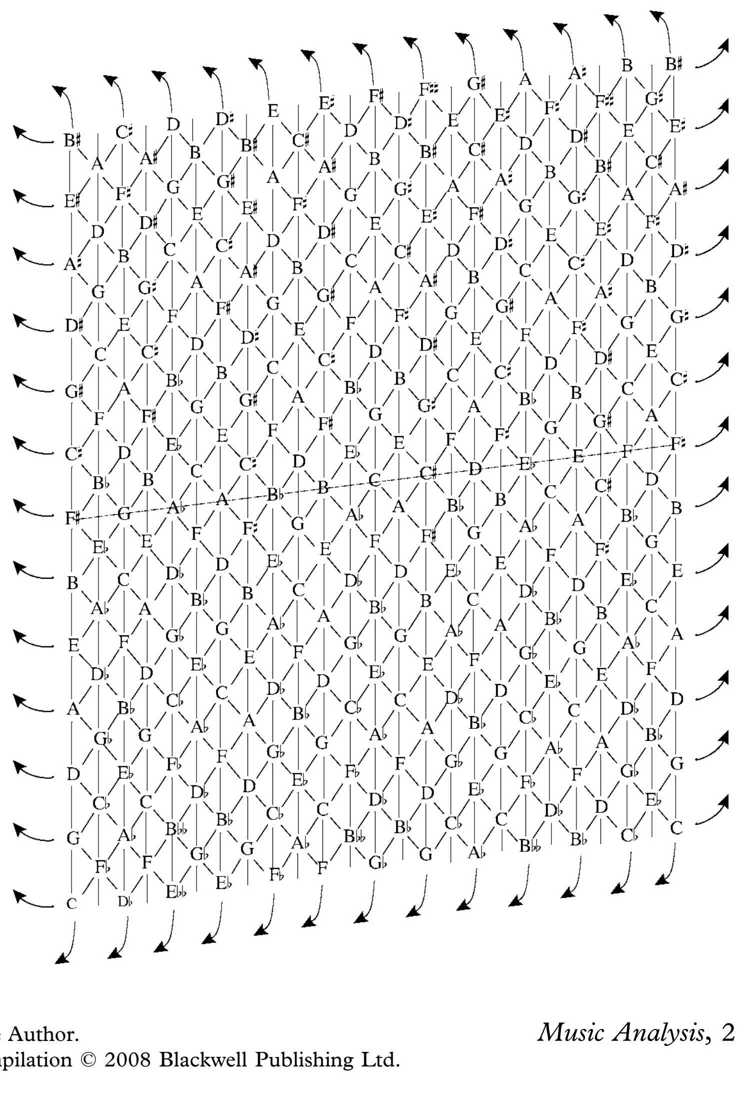
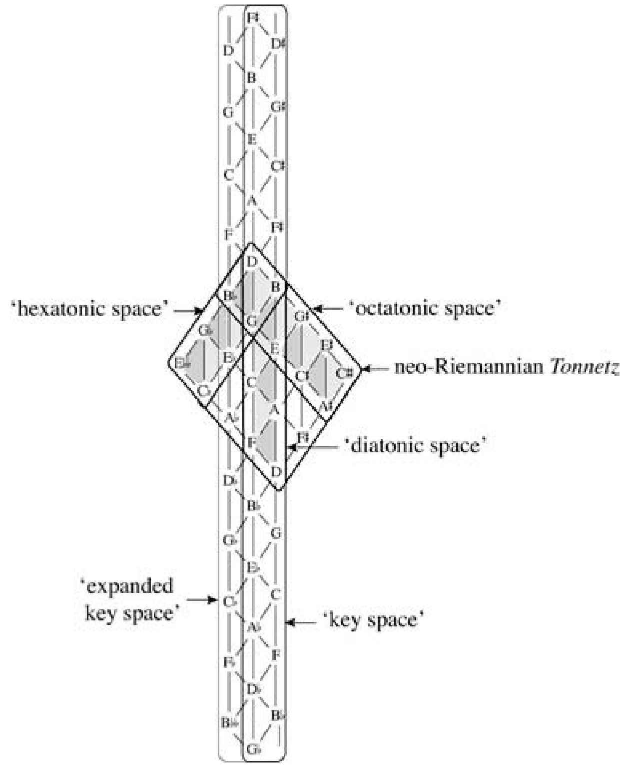


Fig. 29 Nested and intersecting spaces within the embodied *Tonnetz*



transformation produces a space which has neither a fixed centre nor fixed boundaries, our starting pitch, C, having become the geometric equivalent of every other. Yet the nested regions of musical space formed over the course of its construction have not vanished, although their boundaries have become invisible to the eye. Rather, their existence is made apparent through our embodied experience of movement within and between these regions. Just as the kinaesthetic and somatosensory sensations of the body serve to orientate us within physical space, the bodily sensations associated with changes of harmony and key allow us to intuit our location within the harmonic space modelled by the embodied *Tonnetz*.

Fig. 29 shows that the nested and overlapping spaces which emerged in the construction of the embodied *Tonnetz* combine to form a coherent geometric pattern. At the centre of the figure appears that rhomboidal-shaped region of triadic pitch space which constitutes the neo-Riemannian *Tonnetz*. Contained within it are diatonic, hexatonic and octatonic spaces, each forming one of the *Tonnetz*'s main constituent axes. Diatonic space unfolds vertically from D to D, top to bottom, while hexatonic and octatonic spaces unfold diagonally along each of its sides, along the major-third and minor-third axes, respectively. The neo-Riemannian *Tonnetz* – the natural home of hexatonic and octatonic collections – overlaps with those regions of the embodied *Tonnetz* which I have designated as *key space* and *expanded key space*. Together they make up the natural home of the diatonic collection.

By representing diatonic, hexatonic and octatonic collections as intersecting regions of a single space, Fig. 29 highlights how differently we experience movement within them. Diatonic space offers a strong sense of vertical orientation, giving rise to feelings of tension and relaxation as we move up and down within its space by fifth or by alternating major and minor thirds. Its dual centre-periphery organisation also makes it possible for us to orient ourselves within its boundaries with respect to a triadic centre, either C major or A minor, while its diminished fifth likewise provides orientation by locating us at the 'boundaries' of diatonic space – the 'seam' in the Möbius strip. Finally, when the key container itself is set in motion within key space and expanded key space, we can attune to feelings of tensing and relaxing, brightening and darkening, to sense whether we are moving from sharps to flats, or from major to minor, or vice versa.

Hexatonic and octatonic spaces differ in that they provide few cues to spatial orientation. Each unfolds diagonally along either the major-third or the minor-third axis, thus prohibiting root movement upwards or downwards by fifth. Each has an even rather than an odd number of triads and thus lacks the centre-periphery organisation of the diatonic collection. And each confronts us with the paradox of enharmonic equivalence more quickly than movement of the diatonic container through key space, their smaller circles of major and minor thirds bringing us from sharp to flat over the space of just an octave. Furthermore, the maximally smooth voice leading and frequent changes of inversion associated with movement within hexatonic and octatonic spaces greatly undermines the effect of being grounded in the root of each triad.[46] As a result, movement within these spaces can seem uncanny, mysterious and unsettling, as if the laws of nature have been transcended, causing us to lose our sense of tonal centredness, stability, groundedness and balance.

Like the patterns of tilings which filled Escher's notebooks, Figs. 19–29 constitute no more than an inventory of possible spaces; it is left to composers to realise these in the context of particular musical works. The remainder of this article will examine passages from four pieces in which the paradoxical features of these spaces are brought to light in musically significant ways: the fifteenth-century motet 'Absalon fili mi';[47] the finale of Haydn's String Quartet in G major, Op. 76 No. 1; Brahms's Intermezzo in B minor, Op. 119 No. 1; and Wagner's *Parsifal*. These analyses suggest that at least some composers have been aware of the paradoxes of triadic pitch space and have intentionally exploited them, allowing listeners to experience sensations like those associated with Escher's *Mouch*, *Ascending and Descending* and *Möbius Band II*.

The Paradox of the Circular Staircase in 'Absalon fili mi'

We find what may be the earliest example of a circular staircase in the famed 'Absalon fili mi' (Ex. 5). Cited by Edward Lowinsky (1946) and others as an example of what was to become known as *musica reservata*, this work stands out as among the most harmonically daring of its time.[48] Noted especially for its remarkable text painting, the closing reference to an infernal descent is set musically by a displacement into newly chromatic regions of triadic pitch space, forming a progression which, according to Howard Mayer Brown, 'not only demonstrates . . . the outer limits of musical space, but also symbolizes the idea of a descent into hell in an almost physically palpable way' (1976, p. 131). This departure culminates in a shift from major to minor mode, capturing the meaning of the closing line of text, 'Let me live no longer, but descend into hell weeping'. The following analysis realises that descent in visual form, showing how the journey ends, paradoxically, in a region parallel to the one in which it began.

Although clearly modal in its overall conception, 'Absalon fili mi' exhibits many features which mark it as tonally progressive, including key signatures of two to four flats, notated accidentals and numerous V–I cadences.[49] From a compositional perspective, we may view its remarkable harmonic excursion as a stroke of compositional genius, one which makes the most of the expressive resources of the modal system. From a historical perspective, we can view its partial key signature and its modulation far to the flat side of the circle of fifths in bars 77–83 as a preliminary stage in the formation of the key space shown in Fig. 25. In this work, triadic harmony appears to have been released from its diatonic moorings, possibly for the first time in the history of Western music, carving out through its remarkable trajectory precisely those regions of triadic pitch space within which movement from one key to another becomes possible.[50]

As indicated by its key signature of two to four flats, the motet begins in a region of triadic space already remarkably low for its time. Yet in its closing section, heard for the second time in bars 69–85, we are taken on a journey which extends to new depths of key space. Our final destination is among the darkest of its regions, the key of B♭ minor. The precise nature of the journey is revealed in Ex. 6 and Fig. 30, which show how three distinct musical 'objects' – single pitch, triad and key – follow the same descending pathway.

The journey begins in bar 78, immediately following a V–I cadence in B♭ major which confirms the tonality of the motet as a whole. Over the next six bars, all four voices join together in a double canon, with the tenor imitated by the bassus at the fifth and echoed an octave higher by the superius and altus, thereby outlining a series of descending thirds from F to G♭. Throughout this canonic passage, melodic motion cascades from one voice to another along a chain of alternating major and minor thirds. Ex. 6 illustrates the way in which the melodic descent is carried out by the tenor and bassus in tandem.

Ex. 5 ‘Absalon fili mi’, bars 69–85

Ex. 6 'Absalon fili mi', bars 77–83, tenor and bassus

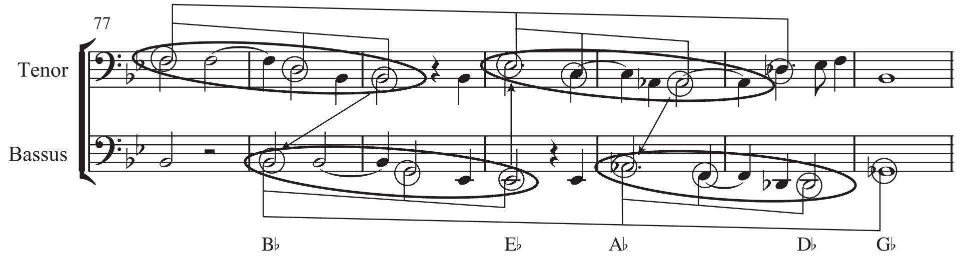
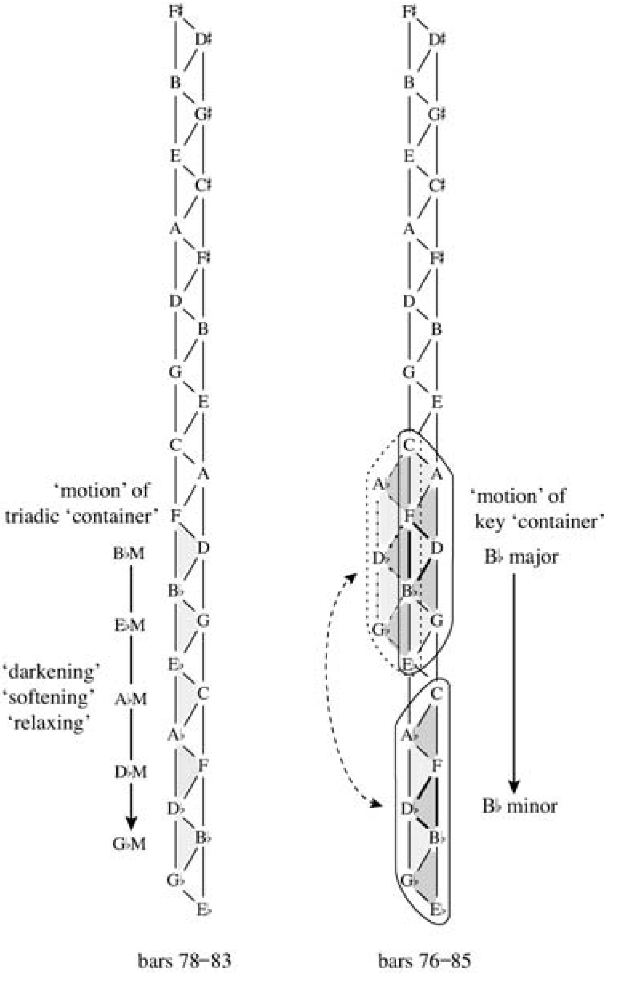


Fig. 30 'Absalon fili mi', bars 76–85: harmonic and tonal journeys



As the beaming indicates, the tenor voice initiates the descent by falling through the space of a B♭ major triad, a movement continued by the bassus in bar 79. Movement by falling thirds passes back and forth between the two voices to outline a succession of major triads, B♭–E♭–A♭–D♭–G♭. Adding to the overall feeling of descent is a melodic sequence, marked by the higher-level beams, which forms the pattern F–E♭–D♭ in the tenor and B♭–A♭–G♭ in the bassus.

The succession of major triads ending on G♭ in bar 83 might lead us to expect a cadence in D♭ major. But when the voices break out of their sequence, the G♭, rather than ascending to A♭, descends to F, leading to a V–i cadence in B♭ minor. The lack of a third in the final sonority lends a certain ambiguity of mode, suggesting that the work may have ended after all in the key of B♭ major. But as Fig. 30 makes clear, the harmonic and tonal motion of the intervening bars leads to a much lower region of triadic pitch space than the one in which it began. The feeling of falling, softening and darkening as we descend along the circle of fifths and shift from major to minor powerfully evokes a grieving father's descent into hell. This suggests that the paradoxical nature of triadic pitch space – that it is open yet closed, vertical yet circular, one-sided yet two-sided – can be intuited directly via its effect on both body and mind.

The Paradox of Enharmonic Modulation in Haydn's String Quartet in G Major, Op. 76 No. 1

It is the uncanny effect of enharmonic modulation that is exploited in the last movement of Haydn's String Quartet in G major, Op. 76 No. 1 (Ex. 7). Haydn is like a master magician showing off his compositional tricks, impressing and enchanting us with his enharmonic sleight of hand. The modulation in question appears at the height of the development section, ushered in by a new motive, marked 'x' in the score. This motive serves as a 'vehicle' for the modulation, its identity as an object in motion confirmed by sheer repetition and preservation of shape.[51]

Haydn prepares us for his show of enharmonic ingenuity in a number of ways. Bar 94 brings about an abrupt change of texture from imitative counterpoint to sustained chords, followed over the next two bars by a *diminuendo* and a fall in register in all voices. In bar 98 harmonic motion comes almost to a standstill: D♭ minor is sustained for four bars while the *subito pianissimo* in bar 100 encourages us to listen ever more closely. It is on the downbeat of bar 102 that the composer performs his distinctive magic. Having introduced the enharmonic pivot chord D♭ minor as iv in A♭ minor, he causes us to reinterpret it as C♯ minor, or iii in the key of A major, by transforming D♭/C♯ minor into A major via semitonal ascent from G♯ (notated as A♭) to A♮ in the first violin. With this slender melodic thread Haydn leads us out of the gloomy depths of the key of A♭ minor into the brilliant heights of A major, a change of key confirmed by the ensuing V–I cadence.[52] The tonal distances traversed by

Ex. 7 Haydn, String Quartet in G major, Op. 76 No. 1/iv, bars 94–106

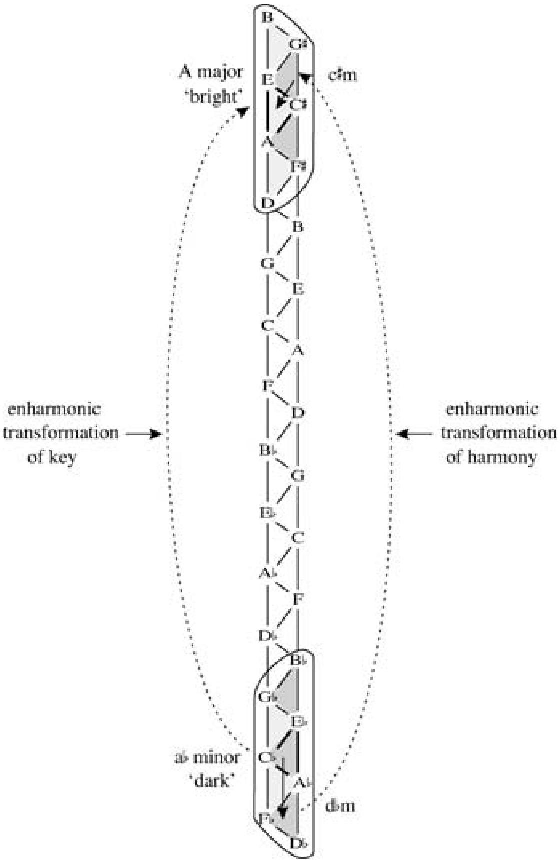
harmony and key at this instant – from seven flats to three sharps and from minor to major – are represented in Fig. 31.[53]

By contrasting the minuteness of the melodic motion required to carry out the transformation with the vastness of the key space traversed, Haydn draws attention to the paradoxical nature of the relationship between melodic and harmonic motion: that what we experience as near in one dimension may appear to be quite distant in another. In the nineteenth century composers increasingly exploited this inverse relation, using chromatic voice leading to move easily yet mysteriously to seemingly distant tonal regions.

Figure-Ground Reversal and the Möbius Strip in Brahms's Intermezzo in B minor, Op. 119 No. 1

That the odd geometry of diatonic pitch space should have escaped theoretical notice for so long is perhaps not surprising given that one does not normally

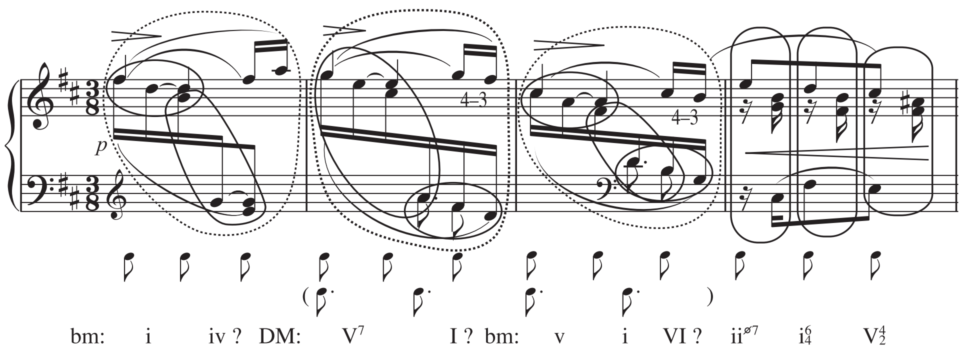
Fig. 31 Enharmonic transformation in Haydn, String Quartet in G major, Op. 76 No. 1/iv, bars 101–102



look for strangeness in what seems most familiar. Yet Brahms's unusual treatment of the diatonic collection in the opening bars of his Intermezzo in B minor, Op. 119 No. 1 (Ex. 8), suggests that he was aware of its paradoxical features and meant to bring them to light. The steady semiquaver rhythm and continuously falling thirds of the opening present a nearly seamless musical surface reminiscent of Escher's *Möbius Band II*. Its continuous cycling through the diatonic collection locates us firmly within the key space of D major/B minor. Yet figure-ground ambiguity causes the ear to flip back and forth between these two keys, which are counterposed in almost perfect balance.[54]

In the first three bars, the ambiguity of harmony and metre reinforces that of key. Bar-length units are marked off by the repeating pattern of the right-hand melody and the left-hand arpeggiation, as shown by the dotted lines in Ex. 8. Yet these units resist segmentation into clearly defined beats and

Ex. 8 Brahms, Intermezzo in B minor, Op. 119 No. 1, bars 1–4

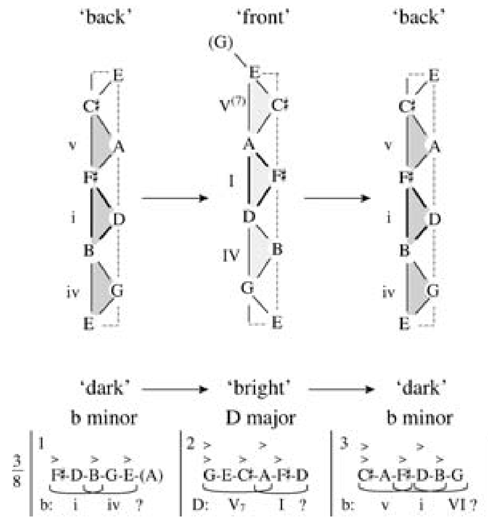


harmonies. The division into beats is obscured by the uniform surface rhythm and the 3/8 metre, which lends itself to alternative groupings of 2 + 2 + 2 or 3 + 3. Similarly, the unbroken succession of diatonic thirds and the sustaining of every note to the end of the bar serves to blur triadic boundaries such that each pitch may be heard as the root, third or fifth of the sounding harmony. Some pitches can be heard as members of more than one harmony, as shown by the overlapping circles in Ex. 8, a phenomenon which produces a smoothly shifting, rather than a discretely changing, harmonic landscape. Thus, we are encouraged by the seamlessness of the musical surface to process bars as wholes rather than as sums of their constituent parts.

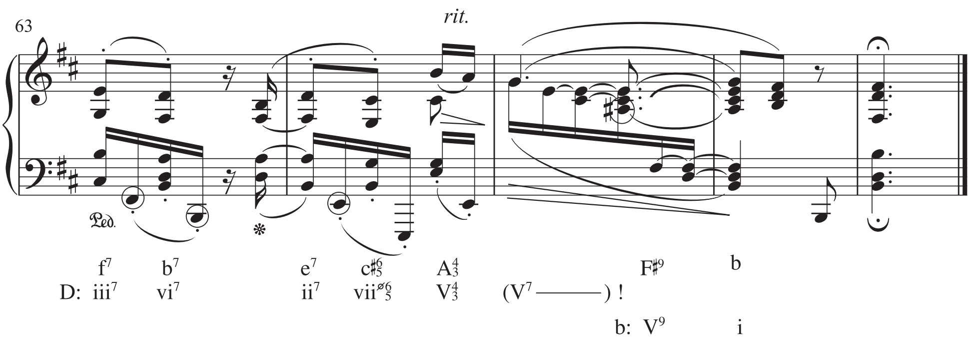
A seamless surface coupled with pattern repetition at the level of the individual bar creates ideal conditions for the kind of figure-ground reversal illustrated in Fig. 20d. In bar 1, emphasis on the pitches F♯, B and E tilts the ear towards hearing first i and then iv of B minor. This gives way in bar 2 to an impression of V^7 and then I of D major, an impression reinforced by the 4–3 suspension on the last beat of the bar. This is superseded in bar 3 by emphasis on the pitches C♯, F♯ and B, suggesting v and then i of B minor. Thus, as shown in Fig. 32, each of these first three bars presents us with a different view of the diatonic Möbius strip: first its ‘dark’ side in bar 1, followed by a turn to its ‘bright’ side in bar 2, then another to its ‘dark’ side in bar 3. Only in bar 4 do we experience a momentary lifting of the tonal haze as we hear the clear outlines of a $\hat{4}$–$\hat{3}$–$\hat{2}$ descent in the melody supported by $ii^{ø7}$–i^6_4–V^4_2 in B minor.

That Brahms intends the tonal ambiguity of the opening bars to come to the listener’s attention is further suggested by the way it plays out over the course of the movement as a whole. Its ABA′ form is set in the keys of B minor and D major, thus reflecting on a formal level the figure-ground reversal of the first three bars. Yet neither of the first two sections ends with a perfect cadence which might serve to confirm its tonic.[55] In fact, the identity of the tonic key, whether

Fig. 32 Brahms, Intermezzo in B minor, Op. 119 No. 1, bars 1–3



Ex. 9 Brahms, Intermezzo in B minor, Op. 119 No. 1, bars 63–67



D major or B minor, remains open to question until the final cadence. Bars 63 and 64 present the beginnings of a circle-of-fifths progression which seems to prepare for a V^7–I ending in D major, an expectation confirmed by the recall of material from bar 2 (Ex. 9). Yet at the last moment Brahms undermines our expectations of this resolution by substituting A♯ for A♮ (bar 65) while continuing the pattern of falling thirds across the bar line. Above this we hear the G once again function as a suspension, but in place of the expected 4–3, a 6–5 motion is introduced, all four upper voices sustaining the constituent pitches of the diminished seventh chord across the bar line before resolving simultaneously as part of a V^9–i cadence in B minor. In retrospect, the surrounding of D major by its 'alter ego', B minor, in the first three bars foreshadows its eventual 'downfall', expressed in the penultimate bar through the movement of the falling third, D to B.

An Enharmonic Seam between Diatonic and Hexatonic Space in Wagner's *Parsifal*

The last excerpt to be analysed is taken from Act III of Wagner's *Parsifal*, the same passage discussed at length in Lewin (1984). It was this work perhaps more than any other which motivated Lewin to investigate the transformational nature of Riemann's harmonic theories and to extend them in group-theoretical ways, thus laying the groundwork for the later development of neo-Riemannian and transformational theory. In his analysis of Act III, Lewin contrasts the otherworldly chromaticism and enharmonicism of 'Riemann space', which he associates with the 'Arabian … world of magic and miracle', with the everyday diatonicism of 'Stufen space', which he associates with the Grail brotherhood of Act I (1984, p. 347). According to Lewin, passage between Stufen space and Riemann space is effected via the 'hidden seam' (p. 347) of the enharmonic equivalence between C♭ and B. The transformation of the former into the latter is marked both dramatically and musically as being among the opera's most significant events. Most notably, it accompanies both the kiss which Kundry gives Parsifal in Act II and Parsifal's unveiling of the Holy Grail near the end of the opera, at the point when the chalice begins to shine.

It is in reference to the latter passage (Act III, bars 1097–1099) that we find Lewin's intriguing yet elusive reference to a musical Möbius strip. He presents us with two different versions of the 'Grail' motive, first as it appears in its diatonic version at the end of Act I, where it accompanies the unveiling of the Holy Grail by Amfortas (Ex. 10a), then as it appears transformed in the parallel passage near the end of Act III, to accompany the unveiling of the Grail by Parsifal, armed now with the reclaimed Spear (Ex. 10b). Lewin re-notates this motive as in shown Ex. 10c to show how its tonal journey might be represented if carried out in Stufen space, thereby leading to an extraordinarily distant E♭♭♭ at the point of termination.

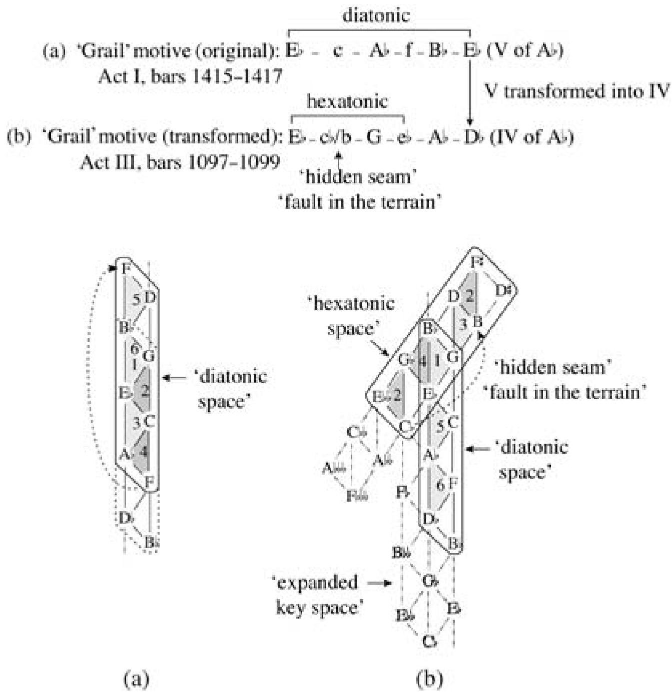
The analysis shown in Ex. 10c is ultimately deemed 'untenable' (p. 346) on the grounds that it suggests a much greater distance than that which we seem to have covered over the final three diatonically related harmonies of the passage. Yet Lewin rejects as equally untenable the two analyses shown in Ex. 10b since they require us to interpret either the E♭–C♭/B major third or the C♭/B–G major third as a diminished fourth.[56] It is with the intention of resolving this conflict that he invokes the metaphor of a Möbius strip intersecting with a cylindrical loop, noting that 'in some contexts the two spaces [Riemann space and Stufen space] may coexist locally without apparent conflict; in this way the surface of a Möbius strip would *locally* resemble the surface of a cylinder to an ant who had not fully explored the global logic of the space' (p. 345; italics as original). Later he goes on to add that

Ex. 10 Lewin's analysis of the 'Grail' motive, original and transformed

> if we imagine pressing together between our fingers a section of a Möbius strip and a section of a cylindrical loop, then the E♭♭♭ chord can be imagined metaphorically on some part of the loop that diverges from the strip, and the subdominant function can be imagined on some part of the strip that diverges from the loop. (p. 346)

We can make sense of Lewin's foray into the world of Escherian logic by reinterpreting the passage in light of the model presented here. If we take 'Stufen space' to correspond to diatonic space and 'Riemann space' to correspond to hexatonic space, we may proceed to rationalise the passage in question by mapping it onto the model of triadic pitch space shown in Fig. 29.[57] Fig. 33a shows that the original version of the 'Grail' motive fits neatly within the key

Fig. 33 Pathways of harmonic motion in Wagner, *Parsifal*, Act I, bars 1415–1417, and Act III, bars 1097–1099



space of E♭ major, outlining a purely diatonic progression along the Möbius strip. It initially descends by third and then by fifth, eventually bringing us full circle to where we began, and hence prolongs the dominant of A♭ major. Fig. 33b, on the other hand, reveals the more circuitous route taken by the transformed version of the 'Grail' motive. Its descent begins in the same place – E♭ major functioning as V of A♭ major – but subsequently diverges, using the 'hidden seam' of C♭/B as its point of exit to move along a diagonal axis, first to C♭ minor, its hexatonic polar complement, then back around to the top of the loop via the enharmonic relation between C♭ and B. It then continues by way of an L transform to G major, which is in turn transformed into its own polar complement, E♭ minor. Having brought us to a point within expanded key space parallel to that at which we began, the phrase exits the hexatonic system to complete its journey in diatonic space, following the pattern formed by the original diatonic version of the motive and descending along the fifth axis from E♭ to A♭ to D♭, to end not on E♭, but on D♭, thus accomplishing the large-scale modulation from I to V to IV.

In his interpretation of the passage, Lewin suggests that this progression through Riemann space forms an important part of the listener's experience of the opera's magical effects. As we pass through hexatonic space, our sense of gravitational pull, vertical orientation and centredness is disrupted, giving rise to a momentary sense of uncanniness, only to be restored, along with our sense of tonal groundedness, at the final cadence. At the same time, chromatic transformation of the 'Grail' motive, accomplished via the enharmonic relationship between C♭ and B, serves to change what was once heard as a dominant into a subdominant. This in turn prepares us for the large-scale plagal cadence which closes the entire opera. Lewin's analysis reveals that the enharmony of C♭/B, which might otherwise appear to be nothing more than a notational inconvenience – a flaw in the logic of the diatonic system – takes on powerful symbolic and affective meaning with respect to the drama as a whole.

Hofstadter points out that it was the presence of paradoxes such as these in the field of mathematics which inspired Bertrand Russell and Alfred North Whitehead to attempt to banish them from their great monument to the tenets of logical empiricism, *Principia Mathematica*. This treatise, inspired by Russell's 'desperate quest for a way to circumvent paradoxes of self-reference' (Hofstadter 1999, p. iv), was an attempt to construct a mathematical system which would be both complete and free of internal contradiction. The futility of their quest was revealed only thirty years later by Kurt Gödel's incompleteness theorem, which revealed the inevitability of the contradictions that appear in even the most rigorously derived mathematical systems.

New light has been shed on the paradoxes of mathematics in a recent book by George Lakoff and Rafael E. Núñez on the bodily basis of mathematical thought (Lakoff and Núñez 2000). The authors show that mathematical reasoning is largely spatial in nature, and that many of its basic concepts – those which form its axiomatic foundations – have image-schematic origins. Concepts such as equality, line, circle and set appear to originate in cross-domain mappings of image schemas such as BALANCE, PATH, CYCLE and CONTAINMENT. Lakoff and Núñez propose that mathematical systems such as algebra, analytical geometry, number theory and set theory begin with axioms which are image-schematic in nature, then build through successive layerings of image-schematic mappings towards increasingly higher levels of complexity and abstraction. As one metaphorical mapping is overlaid upon another, contradictions arise within and between these systems, reflecting their different image-schematic origins. For example, the axiom of Euclidean geometry which asserts that nonparallel lines must intersect at a single point is contradicted by axioms governing the geometry of elliptical and hyperbolic spaces, in which lines are both 'straight' and 'curved'. Thus mathematics, like music, reflects the capacity of the embodied imagination to produce internally coherent systems which are logical yet also paradoxical.

In the case of music, these paradoxes appear to be woven into the very fabric of musical space itself – a residue of its cyclic nature. We may remain oblivious to the fact that we are moving in circles; we may fail to notice when traversing the chromatic circle of fifths that we have returned to the point where we began; and we may be unaware of the twist in the diatonic Möbius strip as we move from front to back. Because these paradoxes are not only pervasive but inescapable, a composer must make special efforts to bring them to our attention. The analyses presented here suggest that at least some composers have shared with Escher the urge to create works of art which force us to confront these contradictions.

Music transports us to a world in which the logic of everyday space, time and movement gives way to impossible possibilities, leading us into contradiction no matter which way we turn – pathways which are vertical yet circular, locations which are the same yet different, near yet far, above yet below. Far from constituting flaws in the logic of an otherwise perfect system, such paradoxes help to make music meaningful for us, a source of never-ending mystery and delight.

NOTES

1. Octave equivalence appears to be one of the few truly universal principles of pitch perception, applying not only to humans, but to other primates as well. See Wright, Rivera, Hulse, Shyan and Neiworth (2000).
2. As reported by Schindler (1966, p. 368).
3. In his late essay 'Ideas for a "Study on the Imagination of Tone"', Hugo Riemann (1914–15) identifies enharmonic equivalence as among the most important theoretical puzzles remaining to be solved, predicting that '*the study of enharmonic identification … ultimately will solve and explain the contradictions between the results of tone-psychological investigations and the practical experiences of musicians*' (Wason and Marvin 1992, p. 110; italics as original).
4. As noted by Lakoff and Johnson (1980), the verb 'to see' is often used metaphorically to mean 'to understand', reflecting the workings of the conceptual metaphor SEEING IS UNDERSTANDING. I use quotation marks here and elsewhere to draw attention to the metaphorical connotations of a term wherever they might otherwise be overlooked.
5. This article describes the origins of the *Tonnetz*, or 'table of tonal relations', in the work of Hugo Riemann and other nineteenth-century German theorists and explains how the *Tonnetz* came to be resurrected and recast in mathematical terms largely through the efforts of Lewin (1982 and 1987) and Hyer (1989 and 1995).
6. For a useful introduction to the mathematics of tiling, see Grünbaum and Shephard (1989).
7. Thurston (1997) notes that this property is shared by all periodic tilings.
8. It is precisely this property which distinguishes the neo-Riemannian *Tonnetz* from its nineteenth-century predecessors. The numerical version of the *Tonnetz* was first proposed by Hyer (1995).

9. Other artists who have incorporated tilings into their artwork include Koloman Moser (1868–1918) and Victor Vasarely (1908–97).

10. The contents of Escher's notebooks, unpublished during his lifetime, appear in Schattschneider (1990).

11. This and some of Escher's other artworks can be viewed online at http://www.mcescher.net/ (accessed 6 June 2008).

12. For a simple explanation of local versus global properties, see Weeks (2002).

13. In this article Cohn offers examples of hexatonic polar progressions spanning more than 300 years, from Carlo Gesualdo's 'Moro lasso' to Schoenberg's String Trio, Op. 45.

14. The alternative layout of fifths along diagonal axes and major and minor thirds along vertical and horizontal axes shown in Fig. 7 was introduced by Balzano (1980) and is favoured by Cohn (1997). The geographic labelling of the four hexatonic systems as Northern, Southern, Eastern and Western was introduced by Cohn (1996).

15. The term 'de-souling' is Cohn's (2004, p. 297) rendering of the term *Entseelung*, which translates literally as 'removal of the soul'.

16. The concept of an image schema, first introduced by the linguist George Lakoff and the philosopher Mark Johnson (Lakoff 1987; Johnson 1987; and Lakoff and Johnson 1999), has since given birth to a thriving sub-discipline of cognitive science which has engaged scholars in such fields as linguistics (Langacker 1987 and Talmy 2000), psychology (Gibbs and Colston 1995), literature (Turner 1991 and 1996), anthropology (Strauss and Quinn 1997), mathematics (Lakoff and Núñez 2000), visual art (Esrock 2002) and music (Saslaw 1996 and 1997–8; Mead 1997–8; Brower 1997–8 and 2000; Zbikowski 1997 and 2002; Larson 1997–8 and 2004; Cox 1999; Snyder 2000; Kemler 2001; Butterfield 2002; Iyer 2002; Eitan and Granot 2004; and Straus 2006). The following discussion is particularly indebted to Johnson (1987).

17. Hyer (1995) notes the contradiction implied by the visual layout of Riemann's original *Tonnetz*: that whereas we *descend* in register when moving downwards along the major-third axis, we *ascend* in register when moving downwards along the minor-third axis.

18. As Butterfield (2002) observes, the concept of a musical object is itself metaphorical. He suggests that we identify as musical 'objects' those features of a musical work – including the work itself – which share properties with objects in the physical world, such as replicability, durability, containment and preservation of shape and size.

19. See especially Ernst Terhardt's work on the perception of virtual pitch, a term used by Terhardt to refer to our perception of the fundamental frequency of a complex tone as its sounding pitch even when it is not actually sounded (Terhardt 1974 and 1979). The implications of this work have been extended to the perception of combinations of tones (Terhardt 1984; and Parncutt 1988 and 1989). According to Terhardt (1984, p. 293), 'the nature of the fundamental note (root) of musical chords is identical with that of the virtual pitch of individual complex tones'.

20. The pitches shown in Figs. 15 and 16 constitute a type of pitch class because they subsume those pitches that share the same letter name, regardless of register. However, they differ from the pitch classes of atonal set theory in that they

exclude their enharmonic equivalents. Throughout this article, the use of a letter name alone should be taken to imply this more limited notion of what we might call letter-name pitch class.

21. The metaphor of root as ground also helps to explain why major and minor triads sound more stable in root position than in first or second inversion. When the root appears in the bass, it functions as the fundamental – as a stable foundation for the sonority as a whole.

22. Although Western musicians tend to take for granted the correlation between increasing frequency and ascending movement, cross-cultural studies suggest that this mapping may not be universal (Zbikowski 2002 and Ashley 2004). For example, Balinese and Javanese musicians refer to 'high' and 'low' pitches as 'small' and 'large', while the Suya of the Amazon basin refer to them as 'young' and 'old' (Zbikowski 2002). That changes of pitch tend to evoke feelings of rising and falling movement in Western listeners may reflect the pervasively triadic nature of Western tonal music, which supports the mapping of root (or tonic) as ground and of melodic motion as rising and falling in relation to this ground.

23. Révész (1954) and Shepard (1964).

24. These are described in Shepard (1964).

25. For an auditory demonstration of Shepard's tones, see http://www.cs.ubc.ca/nest/imager/contributions/flinn/Illusions/ST/st.html (accessed 6 June 2008). A similar effect can be experienced when one listens to certain organ works played on mixture stops. For example, the bass line of Bach's Fantasia in G minor, BWV 542, includes a sequential series of stepwise descents (bars 31–34) which seem to create the illusion of a continuous descent in this respect.

26. Construction of the model in stages makes it possible to address certain problematic features of the neo-Riemannian model noted by Lerdahl (2001): first, that it appears to privilege root relations by third over those by fifth despite the predominance of the latter even in the highly chromatic music which inspired development of the model; and second, that it appears to posit the existence of the chromatic collection before the diatonic even though the latter is both syntactically central and historically prior to the former.

27. A particularly clear example of this inverse relation is to be found in Ex. 3.

28. Although the embodied features of the present model have yet to be empirically tested, its representation of relative distances among chords and keys, similar to those offered by Lerdahl (2001), fits well with the results of probe-tone studies (Krumhansl 1990 and 1998). They also correlate effectively with the results of a recent study showing that key distances are represented in the form of a dynamic topography in the brain (Janata, Birk, Van Horn, Leman, Tillmann and Bharucha 2002).

29. Riemann (1914–15) also stresses the role played by the imagination in transforming musical sounds. He notes that 'the "Alpha and Omega" of musical artistry … exists in the mental image of musical relationships that occurs in the creative artist's *imagination* – a mental image that lives before it is transformed into notation and re-emerges in the imagination of the hearer' (Wason and Marvin 1992, p. 82; italics as original). Later he observes that such images are at least partly somatosensory in nature: 'As a result of the valuation of pitch-level motion as

alternating rising and falling … a psychic experience emerges comparable to an ascent into the lighter regions and a descent into the darker, like the flight of the bird in the air or the swimming of the fish in water – not as something viewed, but as something actually experienced' (p. 96).

30. That a succession of triads can be understood metaphorically as a single object in motion reflects a key feature of our experience of the movement of physical objects, namely, preservation of shape. Psychologists have shown that shape preservation plays an important role in our perception of the apparent motion of visual 'objects'. Ternus (1938) found that observers who were shown a group of dots in which one dot moved to a different location while the other two stayed in place saw all of the dots as moving together so long as the shape of the whole was preserved.

31. Neo-Riemannian theorists generally treat S as simply the inverse of D (D – 1). Here, I treat S as its own operator to reflect the difference in our embodied experience of upwards and downwards movement. I also restore Riemann's original connotations for D and S; thus, the D operator transforms any triad into its dominant, while the S operator transforms the same triad into its subdominant.

32. As Steblin (2002) notes, such descriptors, applied to keys as well as triads, have a history going back at least to Zarlino. It appears that the association of major keys with happiness and minor keys with sadness may be a response not only to the lowering of the third of the primary triads, but also to our tendency to map melodic motion onto bodily movements carried out under the influence of tonal and gravitational forces, which appear to pull more strongly downwards in minor than in major keys (Brower 2000 and Arnheim 1984). That the association of major with happy and minor with sad has a bodily explanation and is not merely the product of enculturation is suggested by empirical studies showing that these associations are made even by very young children (Kastner and Crowder 1990) and brain-damaged adults (Peretz, Gagnon and Bouchard 1998).

33. In applying L and R operators to the triangular shapes of Fig. 20, we are once again reminded that musical objects differ geometrically from the visual 'objects' used to portray them. When the operation of reflection is used to transform the triangles of Fig. 20 into their adjacent neighbours, they require not one but two reflections, the second perpendicular to the first, owing to the extra horizontal dimension of the visual model. Because pitch space lacks this dimension, the axis of reflection actually constitutes a point, not a line, in pitch space – namely, that point which lies halfway between the pitches to be preserved as common notes.

34. Shearing is a term used by physicists and graphic artists alike to refer to the distortion which results from sliding two parts of an object in a direction parallel to their plane of contact.

35. These opposing perceptions are not equally likely, however, because the greater stability of the major triad biases us towards perception of the major mode. The Necker cube induces a similar bias: it is more frequently seen as if viewed from the top than from the bottom. The former appears more stable than the latter because it conforms more closely to our everyday view of cube-shaped objects. It seems likely that the raising of the leading note in the minor mode came about in part to compensate for this perceptual bias.

36. That diatonic space takes the shape of a Möbius strip when arranged as a chain of interlocking fifths and thirds was first noted by Mazzola (1990) and Mazzola,

Göller and Müller (2002). Interestingly, the Möbius strip was discovered by a procedure similar to that shown in Fig. 21. In experimenting with different ways of forming closed loops from rectangular strips created by gluing triangles together, A. F. Möbius discovered that any strip containing an even number of triangles forms a cylindrical loop, while any strip containing an odd number of triangles must be twisted by 180° to make the ends meet, forming what is now known as the Möbius strip (Biggs 1993).

37. A precursor of the model shown in Figs. 19–21 appears in Hauptmann ([1893] 1991; first published in 1853 as *Die Natur der Harmonik und der Metrik*). Hauptmann portrays the diatonic pitches of C major as a chain of thirds, F–a–C–e–G–b–D (where the capital and lowercase letters refer to the roots of major and of minor triads respectively), made up of a 'triad of triads' (p. 9), I, IV and V, which he describes as closing in on itself to form a circle. Because Hauptmann's model is one-dimensional, however, it does not show the 'twist' in the chain of interlocking fifths and thirds. Furthermore, because Hauptmann, like many of his contemporaries, favoured just intonation, the third D–F is a syntonic comma smaller than pure, thus producing not one but two 'diminished' triads. It appears that, like many other German theorists of the nineteenth century, Hauptmann viewed triadic pitch space as extending indefinitely towards increasingly chromatic regions from the centrally located key of C major (Engebretsen 2002).

38. Although one finds varying descriptions of the qualities of individual keys in the eighteenth and nineteenth centuries, there appears to have been a strong consensus concerning the association of modulation to the sharp side of the circle of fifths with tensing, hardening and brightening, and to the flat side with relaxing, softening and darkening (Steblin 2002). For example, according to Rameau (1754), 'the side of the dominant, that of the rising fifth, is rightly the side of strength, so that the more fifths there are in going up, the more this strength increases; the same reasoning holds conversely for softness, on the side of the subdominant' (quoted in Steblin 2002, p. 97). And according to Vogler (1778), 'if we go up by fifths through G, D, A, and E, there is always an increase of strength, effect, cutting quality and penetration. If we go down by fifths through F, B♭, E♭, and A♭, all strength is reduced and the impression becomes duller and darker' (quoted in Steblin 2002, p. 121).

39. Steblin (2002) notes that some theoretical confusion arose in the eighteenth century concerning the distinction between major mode and sharp keys on the one hand and minor mode and flat keys on the other because movement from sharp to flat keys and modulation from major to minor mode both require the addition of flats and, likewise, movement from flat to sharp keys and modulation from minor to major mode both require the addition of sharps. It seems likely that this confusion was caused in part by the similar experiential qualities of these changes in key and mode.

40. Scaling, a term used by graphic artists to refer to a uniform change in the size of an object, here refers to a reduction in the size of each perfect fifth by one-twelfth of a Pythagorean comma.

41. An early representation of the <LR> cycle appears in Vogler (1778). Vogler portrays the cycle in circular score notation, the falling thirds in the bass being filled in with passing notes in order to allow each triad to be approached by its dominant. See Wason (1985), p. 17.

42. The results of a probe-tone study suggest that the modulations produced by applying these five operators do in fact correspond to those keys heard as most closely related (Krumhansl 1998).

43. Within this space, we can also chart a circular route around our starting pitch of C by applying P, L and R operators in succession to produce what Cohn has termed the <PLR> cycle. The resultant hexagon-shaped musical object, which binds together all of the triads containing the pitch C, appears to be what Riemann meant by *Klangvertretung*, translated by Wason and Marvin as 'representation of a tonal complex' (1992, p. 86). According to Wason and Marvin, Riemann used this term to refer to the psychological processing of a note in terms of its six possible harmonic functions as root, third or fifth of a major or minor triad.

44. We find evidence of such preferences in the writings of composers and theorists of the time. Georg Andreas Sorge, for example, notes the possibility of direct modulation in his *Vorgemach der musicalischen Composition* (1745–7), only to reject it as too abrupt, comparing the sounding of C major and C minor together to 'a knock on the head' (p. 30).

45. Technically speaking, every horizontal row of pitches should consist of successive sharpings and flattings of the same pitch, C♮ becoming C♭♭♭♭♭♭ on the left and C♯♯♯♯♯♯ on the right. This permits every pitch in the grid to have its own letter designation. For practical reasons, I have substituted the pitches of the chromatic scale, thus yielding enharmonic spellings of triads which become more obvious as we move away from the centre of the grid.

46. Cohn (1996) proposes treating the pitches of a triad within a hexatonic system as equally weighted, regardless of their status as root, third or fifth. His proposal underlines the degree to which the maximally smooth voice leading associated with hexatonic progressions undermines the privileged position of the root within the triad.

47. We find evidence of such intentions in the score of Bach's 'Canon per tonos' from *Das musikalische Opfer*, a work which sequences upwards by major second six times in order to conclude in the key in which it began. That Bach meant the paradox of a circular staircase to come to the attention of its royal dedicatee, Frederick the Great, can be inferred from his inscription: 'As the modulation rises, so may the King's glory'. But although Hofstadter (1999, p. 10) describes the work as an 'Endlessly Rising Canon' and indicates by this that Bach intended the canon to portray the king's glory as ascending infinitely, Chafe (1984) argues that Bach most likely intended just the opposite: not infinite ascent but merely the illusion of ascent. Thus the dedication in fact reveals the earthly limitations of the king's glory.

48. See Lowinsky (1946). Although this motet was long attributed to Josquin Desprez, its authorship is now questioned, with many scholars arguing, largely on stylistic grounds, that it was composed by Pierre de la Rue (Benthem 1989; Rifkin 1991; and Meconi 1998). However, Davison (1996) maintains that the work cannot be definitively assigned to either composer and that its authorship must remain uncertain.

49. Randel (1971) notes the ubiquity of the V–I cadence in fifteenth-century music and refers to it as an example of 'emergent triadic tonality'.

50. It is perhaps not surprising that the expansion of triadic space should have proceeded far to the flat side before moving in the opposite direction. The medieval gamut offered the 'soft' (flat) B as an alternative to the 'hard' (natural)

B, providing an extra link in the chain of perfect fifths. It seems only natural that composers would have viewed the B♭ at the bottom of the chain as the more logical point for expansion. The closing bars of 'Absalon fili mi' reveal an intermediate stage in the formation of key space, one which extends the chain of major and minor thirds by seven steps, thus permitting a return to the initial tonic but not to the initial mode.

51. That Haydn conceived of this motive as a 'vehicle' for the modulation is further suggested by its only other appearance in the movement: in bar 143 of the recapitulation, where it forms a prominent feature of the new theme which ushers in an equally striking change of key from G minor to G major.

52. The remarkable effect of enharmonic modulation is noted by Anton Reicha in his *Traité de mélodie* (1814); he observes that 'when the key of F♯ is suddenly changed into G♭, and C♯ into D♭ (and vice versa), we fall from a very brilliant key into a very sombre key, or from a very sombre into a very piercing key' (quoted in Steblin 2002, p. 126).

53. One could hear this modulation as simply a continuation of the descent along the circle of fifths to the key of the Neapolitan, B𝄫 major. However, assuming that one has kept track of one's position within the key space, this only delays the arrival of the enharmonic modulation needed to return to the home key of G minor. Given Haydn's unusual treatment of this passage, his notation of the modulation at precisely this point, and the fact that the key of B𝄫 major falls far outside the range of keys that in the late eighteenth century were judged acceptable, it seems likely that the composer intended the modulation to be heard as it is notated. Further support for this hearing is provided by the upwards resolution of the A♭, strongly suggesting its reinterpretation as G♯.

54. Many other ambiguous features of this piece have been pointed out by Dunsby (1981).

55. Jordan and Kafalenos (1989) explore the tonal ambiguity of this composition in terms of a 'double trajectory' of B minor and D major, showing how Brahms maintains the ambiguity throughout.

56. The B♮ shown in parentheses in Ex. 10b reflects Wagner's own notation, while the C♭ reflects our initial hearing of this triad as a flattened submediant in the key of E♭ major.

57. From the passages cited above, it would appear that Lewin conceived of Riemann space as the Möbius strip and Stufen space as the cylindrical loop, precisely the opposite of the interpretation offered here. This suggests that he had not fully worked out the implications of his topological metaphor but was attempting rather to capture the idea of two musical spaces that intersect, yet which remain topologically distinct.

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ABSTRACT

Parallels between the mathematics of tiling, which describes geometries of visual space, and neo-Riemannian theory, which describes geometries of musical space, make it possible to show that certain paradoxes featured in the visual artworks of M. C. Escher also appear in the pitch space modelled by the neo-Riemannian *Tonnetz*. This article makes these paradoxes visually apparent by constructing an embodied model of triadic pitch space in accordance with principles drawn from the mathematics of tiling, on the one hand, and from cognitive science, on the other – specifically, the notion that our experience of pitch relationships is governed in part by the metaphorical projection of patterns abstracted from embodied experience known as image schemas. These paradoxes are illustrated with reference to passages drawn from four compositions to whose expressive character such paradoxes contribute: the fifteenth-century motet 'Absalon fili mi'; the finale of Haydn's String Quartet in G major, Op. 76 No. 1; Brahms's Intermezzo in B minor, Op. 119 No. 1; and Wagner's *Parsifal*.