A GENERAL THEORY OF COMBINATORIALITY AND THE AGGREGATE

(Part 1)

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I

1.1 *Introduction*

The concept of combinatoriality originated in Schoenberg's row compositions and was subsequently generalized as a compositional technique by Babbitt and others during the 1950s.* It is a scheme for the contrapuntal combination of rows that is consistent with the more fundamental concept of pitch-class saturation which predated the initial canonization of row technique by at least a decade. Webern states:

> We felt that a pitch, frequently repeated either in direct succession or dispersed throughout a piece somehow 'got revenge', that the pitch established itself. That must have been satisfying—it was still possible then; but it demonstrated, for example, how disturbing it was if one pitch were repeated frequently throughout a passage which deliberately exhausted all twelve pitches. . . . In a word, it established itself as a procedural rule that before all twelve pitches had been used up, none of them could recur. The most important thing is that

* This paper was delivered by the authors in a special seminar held at Yale University in November 1974.

[1] For a synopsis of classical combinatorial techniques, see B. Fennelly, "Twelve-Tone Technique", in *Dictionary of Contemporary Music,* ed. John Vinton, New York: Dutton Books, 1974.

> the piece—the thought—the theme—through the working out of the twelve pitches had become a structural unit.[2]

Though the row was invented at first for this very purpose, the mere usage of rows is not sufficient to guarantee saturation when several rows at a time are used in counterpoint. Combinatoriality fills this gap in row technique, allowing rows to function as a means to realize Webern's ideal of tonal balance in contrapuntal frameworks.

In this paper we will generalize known row-combination techniques, present additional methods, and comment on some of the compositional implications of combinatoriality. We wish to make a point of the generality of the method and the freedom allowed within it. We hope that our discussion is self-contained, but the reader is referred where appropriate to earlier papers. Such writings on the subject have often given the impression that combinatoriality is possible only with the proviso that special rows are used, assembled from a small palette of "source chords", or that combinatoriality deals exclusively with the fragmentation of 12-note aggregates into 2, 3, 4, or 6 equal-sized parts, implying restrictions on the gestural make-up of compositions written with the system. While not all rows generate such "even" combinatorialities, the arbitrary row, possessing none of the classical hexachordal, tetrachordal, etc., properties, usually can be developed by various means to yield a variety of combinatorialities, among which hierarchically related groups of combinatorialities can often be found. Just as each row implies its own set of gestures and contours, it also implies a characteristic set of combinatorialities, which, depending on the row, may or may not include even ones. Combinatoriality can thus be viewed as general, pertaining to all rows, not just a sub-universe of them.

Since uneven combinatorialities seem to outnumber the even ones statistically,[3] we investigated operations which generated and related uneven combinatorialities, and searched for meaningful criteria by which different even and uneven combinatorialities could be evaluated for their differing gestural and motivic potentialities.

Most of the examples in this paper are based on two rows labelled throughout as X and Y:

$$X\colon\ 0\ 1\ 7\ 2\ 10\ 9\ 11\ 4\ 8\ 5\ 3\ 6$$
$$Y\colon\ 0\ 1\ 4\ 3\ 9\ 2\ 10\ 8\ 5\ 7\ 11\ 6$$

[2] Authors' own translation from Anton Webern, *Wege zur Neuen Musik,* Vienna: Universal Edition, 1960.

[3] Donald Martino, "The Source Set and its Aggregate Formations", *Journal of Music Theory,* 5/2(1961).

X is constructed from "all-combinatorial tetrachords", but *Y* is not. Neither has any of the classical hexachordal properties, and both rows are all-interval series, a property that is essentially independent of combinatorial potential,[4] which demonstrates how row material chosen for independent reasons might be dealt with in a combinatorial framework.

We include the cycle-of-fifths transform (multiplication of pitch-classes by 5 or 7 mod-12) as a basic row operation along with transposition, retrogression, and inversion. It enlarges the combinatorial possibilities of a row and can be handled with much the same conceptual framework of pitch-class mappings, invariances, and order inversions which we establish for the other operations.[5] The cycle-of-fifths transform is no more or less "audible" than the other pitch-class operations, primarily insofar as we are indeed regarding them here as *pitch-class* operations rather than *pitch* operations. While transposition and inversion can be used in registral space so as to preserve contour—as in the case of "mirror inversion"—and while the fifths transform might similarly be regarded as a "contour expansion", the contour corollaries of all the operations dissolve when we regard them purely in terms of pitch-class. In addition, when applied to unordered sets, the cycle-of-fifths transform can be very audible, since it invokes a specific change of interval content, which the other operations merely preserve.[6] One can point to its use in Jazz harmony, where chromatic bass lines are regularly substituted for sequences of fourths and fifths. Whatever one's reasons for using or not using the operation, the material presented below is in no way dependent on its inclusion as a row-operation—the point is simply that it fits well into the general framework of combinatoriality.

Section II establishes definitions for unordered sets, simple and composite operations, set-classes, and introduces the concepts of cliques and begin- and end-sets. Section III presents techniques by which combinatorialities can be constructed as a result of the structure of a row's segments. Cliques and cycles are used here both as generators of combinatorialities and as a basis for the substitution of groups of rows

[4] Morris & Starr, "The Structure of All-Interval Series", *Journal of Music Theory,* 18/2(1974).

[5] Discussed in Hubert S. Howe Jr., "Some Combinational Properties of Pitch Structures", *Perspectives of New Music,* 4/1(1965).

[6] John Clough, "Pitch-Set Equivalence and Inclusion", *Journal of Music Theory,* 9/1(1965).

in combinatorialities. Section IV * is devoted to operations performed on preexisting combinatorialities—the redistribution of PCs by swapping them between aggregates and the merging and complementation of entire combinatorialities—which not only yield structures unattainable with classical techniques, but can generate combinatorial structures of arbitrary complexity. Section V deals with various ways of comparing combinatorialities in qualitative, quantitative, and quasi-stochastic terms.

The discussion will be restricted to the partitioning of combinatorial structures into aggregates, and will not deal with any finer partitionings of aggregates into hexachords or other smaller units, except insofar as such partitionings pertain to aggregate structure. Thus, for instance, the discussion of all- and semi-combinatorial hexachords, the relations between them, and their trichordal generators will not be included in this study.

1.2 *Basics and Symbology*

Octave equivalence is assumed for the purposes of this discussion. Thus all 'C's belong to the pitch-class (henceforth 'PC') 0, all 'C♯'s belong to PC 1, all 'D's to PC 2, . . . , and all 'B♮'s belong to PC 11. The characters 'A' and 'B' are used to designate PCs 10 and 11 respectively, so that any PC can be written with a single character, and there is no confusion as to whether '10' is a single PC or a PC-pair consisting of '1' and '0'. Lower-case letters (e.g., a, b_i, x_i, etc.) are used for variables whose value is some PC.

Sets of PCs are written within braces and separated by commas, and constitute unordered combinations of from one to twelve of the 12 available PCs. Ex:

$$\{0, 1, 5\} \qquad \{7, B, A, 5, 3\} \qquad \{x_i, y_i, 6\}$$

The order in which the contents of a set are written is insignificant, so that

$$\{0, 1, 5\} = \{5, 0, 1\} = \{5, 1, 0\} = \text{etc.}$$

Upper-case letters (e.g., N, E, S, K_3, etc.) are used for variables denoting sets.

* Sections IV, V, and VI will appear in Vol. 16, No. 2—Ed.

Rows and *row-segments* are written as strings of PCs without commas. Ex:

0157 3B8 014295B38A76

A row is an ordered set of 12 PCs with no duplications. For this discussion, 'row' implies a specific ordering of PCs in contrast to 'set' which denotes an unordered collection.[7] Italic capital letters are used for row and row-segment variables (e.g., $E_{i.j}$, P, X, U_1, etc.). We refer to the PCs of a row with subscripted variables. Thus where $P =$ 0B1A29384756, $p_0 = 0$, $p_B = 6$, $p_4 = 2$, etc. The first PC of a row is subscripted with a 0 and the last with B.

1.3 *Rows, Matrices, and Saturation*

A row defines a succession of 12 distinct PCs. If we were to compose a passage consisting entirely of complete rows, we could guarantee that the frequency of all PCs would be the same—that is, the passage would exhibit overall PC-*saturation*. Let us further suppose that our passage is contrapuntally complex, involving simultaneously stated rows which overlap one another. Depending on which segments of which rows are involved in an isolated section of the passage, that section, taken out of context, might not be saturated at all, as it might, for instance, be the overlapping of the tail of one row and the head of another, which have several PCs in common, duplicating some PCs and perhaps skipping others. Thus while the usage of rows guarantees overall PC-saturation in a statistical sense, the interaction of immediately juxtaposed rows can be such as to neutralize saturation on a local basis.

This paper is essentially a study of how local saturation can be maintained in a context of contrapuntally stated rows. The problem is that the row is a one-dimensional ordering of PCs, while counterpoint is at least two-dimensional, with vertical relationships in addition to the horizontal relationships of the individual lines or voices being combined. We must then go beyond the linear saturation concept of the row to a two-dimensional saturation concept, insuring not only horizontal saturation in each line, but also vertical saturation in all overlaps of contrapuntally juxtaposed row-segments. We are thus motivated to think in terms of *combination matrices* as in Figures 1 and 2, in which

[7] Milton Babbitt uses the term 'set' in the same sense that 'row' is used here. He defines the twelve-tone row as a set of ordered pairs of order-numbers and PCs.

Y:			01	4	392A8	57	B6
$T_4(I(R(Y)))$:	A59	B8	627				1034
$T_3(Y)$:	347	6		0	5	1B	8A29
$T_7(I(R(Y)))$:	1082		B95A43	67			
$T_9(I(Y))$:				985	607B1	42A3	
$T_1(Y)$:		1254		A3B		9680	7
$T_5(I(R(Y)))$:	B6	A0973	8	21	4		5

Fig. 1

Y:		0143	92A8	57B6
$T_4(I(R(Y)))$:	A59B	8627		1034
$T_3(Y)$:	3476		051B	8A29
$T_7(I(R(Y)))$:	1802	B95A	4367	

Fig. 2

the matrix rows are transforms of each other under basic row operations, and the columns are of segments of those rows which sound simultaneously or in temporal proximity. Constant saturation is achieved, as in the examples, when each row and each column of such a matrix is an aggregate of 12 PCs represented once each (henceforth, simply 'aggregate'). Suffice to say that such matrices are not arrived at fortuitously.

A combination-matrix (henceforth, 'CM') of n rows will also have n columns, so that there will be n^2 matrix-positions, each containing a row-segment of 0 to 12 PCs.

II

2.1 *Row Operations*

We will restrict ourselves to the operations of inversion (I), transposition by n semitones (T_n), retrograde (R), and multiplication by 5 modulo-12 (M). We will regard these four operations as *basic*. The following are PC-wise descriptions of the basic operations applied to an arbitrary row $P = p_0, p_1 \ldots, p_B$ to produce a row $Q = q_0, q_1, \ldots, q_B$:

$$
\begin{array}{lll}
Q = \mathrm{T}_n(P): & q_i = p_i \oplus n, & \text{for i from 0 to B} \\
Q = \mathrm{I}(P): & q_i = \ominus p_i, & \text{for i from 0 to B} \\
Q = \mathrm{M}(P): & q_i = 5 \otimes p_i, & \text{for i from 0 to B} \\
Q = \mathrm{R}(P): & q_i = p_{B \ominus i}, & \text{for i from 0 to B}
\end{array}
$$

The encircled operators denote that the result of an arithmetic operation is to be taken modulo-12.[8] When we apply several of the basic operators in succession, we speak of the net result as a *composite operation,* which like the basic operations can also be described in PC-wise terms: For example:

$$Q = \mathrm{T}_n(\mathrm{M}(\mathrm{R}(P))): \quad q_i = (5 \otimes p_{B \ominus i}) \oplus n, \quad \text{for i from 0 to B}$$

When referring to this particular operation outside of the specific context of the rows P and Q, we would write '$\mathrm{T_nMR}$', omitting the parentheses. Note that in this composite operation, there is only one 'T_n', and it has been written as the left-most operator—that is, it is the basic operation that is applied *last* when the entire composite operation is applied to some row. Composite operations written in this way are in *canonical form.* Any row operation that has been written as a composite of basic operators that is not in canonical form may be renotated through algebraic manipulation of the PC-wise description of the operation to put it in canonical form. For example, the following operation:

$$
\begin{aligned}
Q = \mathrm{I}(\mathrm{M}(\mathrm{T}_7(P))): \quad q_i &= \ominus (5 \otimes (7 \oplus p_i)) \\
&= \ominus ((5 \otimes 7) \oplus (5 \otimes p_i)) \\
&= \ominus (5 \otimes p_i) \ominus (5 \otimes 7)
\end{aligned}
$$

can be rewritten in canonical form as $\mathrm{T_1IM}$. Canonical form avoids the confusion arising from the fact that T_n is commutable neither with I nor M.

There are ninety-six different row operations each of which has a unique representation in canonical form. Hereafter, we will use the term 'operation' to mean any one of these ninety-six possibilities, which might be a single basic operation or a composite of several. We will

[8] Birkoff & Bartee, *Modern Applied Algebra,* New York: McGraw-Hill Book Co., 1970, pp. 45ff.

use upper-case 'F' and 'G' to designate arbitrary operations in making generalizations.

Given an operation F such that $Q = \mathrm{F}(P)$, we often want to find its *inverse,* which is an operation G such that $P = \mathrm{G}(Q)$. For example, to find the inverse of $\mathrm{F}(P) = \mathrm{T}_n(\mathrm{M}(\mathrm{I}(P)))$, whose PC-wise description is

$$q_i = 5 \otimes (\ominus p_i) \oplus n$$

we solve for p_i:

$$\begin{aligned}
&q_i \ominus n = 5 \otimes (\ominus p_i) \\
&5 \otimes (q_i \ominus n) = 5 \otimes (5 \otimes (\ominus p_i)) \\
&5 \otimes (q_i \ominus n) = (5 \otimes 5) \otimes (\ominus p_i) \\
&(5 \otimes q_i) \ominus (5 \otimes n) = 1 \otimes (\ominus p_i) = \ominus p_i \\
&\ominus (5 \otimes q_i) \oplus (5 \otimes n) = p_i
\end{aligned}$$

and rewrite it as a composite operation in canonical form:

$$\mathrm{T}_{5 \otimes n}(\mathrm{I}(\mathrm{M}(Q))) = P$$

Calculating the inverse of an operation in canonical form always boils down to finding a new transposition operator, since the other components of an operation are always the same as those of its inverse. For example:

if $Q = \mathrm{T}_1(\mathrm{M}(P))$, then $P = \mathrm{T}_7(\mathrm{M}(Q))$
if $Q = \mathrm{T}_2(\mathrm{M}(\mathrm{I}(\mathrm{R}(P))))$, then $P = \mathrm{T}_A(\mathrm{M}(\mathrm{I}(\mathrm{R}(Q))))$
etc.

We can identify four classes of operations, for which the transpositional operators of an inverse can be calculated by formula:

(1) neither I nor M are included in the composite; T_n becomes $\mathrm{T}_{\ominus n}$
(2) I is included but not M; T_n is unchanged
(3) M is included, but not I; T_n becomes $\mathrm{T}_{\ominus (5 \otimes n)}$
(4) Both I and M are included; T_n becomes $\mathrm{T}_{(5 \otimes n)}$

Of special interest are reflexive operations, which are their own inverses—that is, whose inverses have the same transposition operator. $\mathrm{T}_n\mathrm{IM}$, for instance, is reflexive for values of n such that $n = 5 \otimes n$ where $n = 0$, 3, 6, or 9. Reflexive operations not involving R are starred in Table I (see sec. 3.7).

2.2 *Unordered PC-sets and set-classes*

There are 4096 subsets of the set {0, 1, 2, . . . B} of all 12 PCs. These sets, like rows, are subject to the three basic operators T_n, I, and M and their composites (R does not apply to unordered sets). We call such an operation a *set operation.* Table I presents the forty-eight possible set operations, identifying the inverse of each one, with reflexive operations starred.

Where one set can be transformed into another by a set operation, we say that both sets are in the same *set-class.* Conversely, for any pair of sets in the same set-class (henceforth, 'SC'), there is at least one operation (often several) allowing us to transform one set into the other.[9] The following sets, for instance, are all in the same SC:

$$\{0,\ 1,\ 5,\ 8\} = S_0$$
$$\{3,\ 4,\ 8,\ B\} = T_3(S_0)$$
$$\{B,\ 4,\ 0,\ 3\} = T_B(M(S_0))$$
$$\{7,\ 6,\ 2,\ B\} = T_7(I(S_0))$$

We can define the operation C which gives us the complement of an unordered set—that is, the set of all PCs that are not in the original set. Ex:

$$C(\{0,\ 1,\ 5,\ 8\}) = \{2,\ 3,\ 4,\ 6,\ 7,\ 9,\ A,\ B\}$$

A set is not generally in the same SC as its complement.

The *set-union* operator '$\cup$' allows us to combine two sets into a set containing all the PCs in either initial set represented once. We define the set-union operator as follows: a PC p is in the set D $\cup$ E if and only if p is in D, in E or in both D and E. For example:

$$\{0,\ 1,\ 2,\ 6\} \cup \{5,\ 6,\ 9,\ B\} = \{0,\ 1,\ 2,\ 5,\ 6,\ 9,\ B\}$$
$$\{4,\ 6,\ 7,\ 9,\ 5\} \cup \{9\} = \{4,\ 6,\ 7,\ 9,\ 5\}$$
$$\{0,\ 1\} \cup \{3,\ 5\} = \{0,\ 1,\ 3,\ 5\}$$

The set-intersection operator '$\cap$' lets us extract a set of PCs that are

[9] The set-class concept is developed in a variety of ways for different compositional and analytic purposes in Allen Forte, *The Structure of Atonal Music,* New Haven: The Yale University Press, 1973, Howe *op. cit.,* Richard Chrisman, "Identification and Correlation of Pitch-sets", *JMT,* 15/1 & 2(1971), George Perle, *Serial Composition and Atonality,* Berkeley: Universty of Calfornia Press, 1963, Herman Van Sann, "Sundry Notes Introductory to the Theoretical Mechanics of Mathematical Music", *Interface,* 2/1(1973), and Martino *op. cit.*

common to two other sets: A PC p is in the set $D \cap E$ if and only if p is in D and in E. Ex:

$$\{0, 1, 2, 6\} \cap \{5, 6, 9, B\} = \{6\}$$
$$\{4, 6, 7, 9, 5\} \cap \{2, 9, 8, B\} = \{9\}$$
$$\{0, 1\} \cap \{0, 1, 3, 5\} = \{0, 1\}$$

2.3 *Invariant Sets*

In certain cases, a set remains unchanged when subjected to a certain operation. For example, {0, 1, 5, 6} yields itself again when subjected to T_6I, because that operation "sends" the PCs of the set to each other rather than to PCs outside of the original set. We say then that the set is *invariant under* T_6I, or that the set is T_6I-*invariant.* The following are examples of invariance under different types of operations:

{0, 1, 4, 5, 8} is M-invariant
{0, 1, 2, 6, 7, 8} is T_6-invariant and T_8I-invariant
{1, 2, 7, 8} is IM-invariant, T_6-invariant, and T_6IM-invariant

Note that the latter two examples are invariant under more than one operation, as are the sets belonging to many SCs. It can be proven that if a set S is invariant under some operation F, then all members of the SC to which S belongs will be invariant under T_nF for some n—that is, returning to the four classes of operation identified in section 2.1, if one member of a SC is invariant under some operation, then each member of the SC will be invariant under an operation belonging to the same class. Note that all sets are invariant under the *identity operation,* which might be written T_0, which maps any set to itself. As this is a trivial invariance, not particular to any set or class of sets, when we speak hereafter of a set as being invariant, it will be implicit that we are speaking of some operation other than the identity operation.

2.4 *Cliques*

An *n-clique* is a set containing a cycle of PCs j, $j \oplus n$, $j \oplus (2 \otimes n)$, $j \oplus (3 \otimes n), \ldots, j \oplus (k \otimes n) = j$. Every n-clique is also a $\ominus$n-clique, as the same cycle of PCs can be generated by adding the interval n in one direction, or alternately by adding its complement, $\ominus$n, in the other direction around the cycle (i.e., j, $j \ominus n$, $j \ominus (2 \otimes n), \ldots$

$j \ominus (k \otimes n) = j$). Thus a "whole-tone hexachord" is a 2-clique ($n = 2$), a "diminished seventh tetrachord" is a 3-clique, an "augmented triad" is a 4-clique and a tritone is a 6-clique, leaving the 1-clique and the 5-clique which are both 12-PC sets.[10]

All the PCs of an n-clique taken modulo-n are equal. Ex:

{0, 4, 8} is a 4-clique: (0)mod-4 = (4)mod-4 = (8)mod-4 = 0

{1, 4, 7, A} is a 3-clique: (1)mod-3 = (4)mod-3 = (7)mod-3 = (A)mod-3 = 1

etc.

so that the PCs of an arbitrary set or row can be classified as members of different n-cliques by examining them modulo-n. Ex:

Y: 0 1 4 3 9 2 A 8 5 7 B 6

0 1 0 1 1 0 0 0 1 1 1 0 mod-2

same 2-clique

same 2-clique

0 1 1 0 0 2 1 2 2 1 2 0 mod-3

same 3-clique

same 3-clique

same 3-clique

In all, there are two distinct 2-cliques, three distinct 3-cliques, four 4-cliques, and six 6-cliques (only one 5-, 7-, 11-, and 1-clique, however).

All n-cliques are invariant under T_n and $T_{k \times n}$, for any positive integer k, but moreover, the following invariances should be noted:

(1) Both 2-cliques are I-invariant
(2) All three 3-cliques are IM-invariant
(3) All four 4-cliques are M-invariant
(4) All six 6-cliques are IM-invariant

[10] The clique concept is discussed in Milton Babbitt, "Set Structure as a Compositional Determinant", in *Perspectives on Contemporary Music Theory,* eds. Boretz & Cone, New York: W. W. Norton & Co., 1972, Howe *op. cit.,* and Iannis Xenakis, *Formalized Music,* Bloomington, Indiana: The Indiana University Press, 1971, pp. 94–97. In addition, 6-cliques are discussed in Morris & Starr *op. cit.* as a possible basis for the structure of all-interval series.

Furthermore, any union of two or more n-cliques will be invariant under an operation above, under which both cliques are individually invariant. For example, the set {0, 1, 4, 5, 8, 9} which is the union of the two 4-cliques {0, 4, 8} and {1, 5, 9} will be T_nM-invariant according to (3) above. The four rules can be extended as follows: a 2-clique can be regarded as the union of two 4-cliques or alternately of three 6-cliques, from which it follows that 2-cliques are both M-invariant and IM-invariant. Similarly, it might have been inferred that a 3-clique is IM-invariant specifically because it can be regarded as the union of two 6-cliques. The four rules and their extensions can be incorporated along with the transpositional invariances of cliques noted at first to form a large number of composite operations. Thus 2-cliques are invariant under T_nI, T_nM, and T_nIM where n is even, 3-cliques are T_nIM-invariant where n = 0, 3, 6, or 9, and so on.

2.5 *Begin-sets and End-sets of Rows*

The first n PCs of a row considered as an unordered set constitute that row's 'begin-set' of size n. Similarly, the last n notes of a row make up that row's 'end-set' of size n. Example:

Y: 0 1 4 3 9 2 A 8 5 7 B 6

n	begin-set of size n	end-set of size n
3	{0, 1, 4}	{6, 7, B}
4	{0, 1, 3, 4}	{5, 6, 7, B}
5	{0, 1, 3, 4, 9}	{5, 6, 7, 8, B}
6	{0, 1, 2, 3, 4, 9}	{5, 6, 7, 8, A, B}

Fig. 3

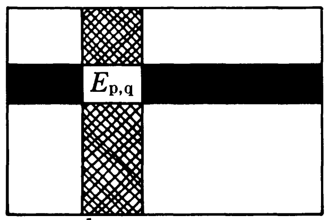
Note that the operation R "interchanges" a row's begin-sets with its end-sets; a begin-set of some row *P* is an end-set of R(*P*) and vice versa.

2.6 *Combination Matrix Notation and Concatenation*

Combination matrices will be notated as doubly-subscripted arrays of row-segments, and will be denoted by boldface capitals (e.g., **D**, **E**). An individual row-segment in the i^{th} row and j^{th} column of a CM **E** will be written $E_{i,j}$ and the number of PCs in such a segment will be given by $\#E_{i,j}$. Row segments can be concatenated by the operator '|' giving longer segments. We can join the first two segments of the i^{th} row of $\mathbf{E}$ by writing $E_{i,1}|E_{i,2}$ whose length is $(\#E_{i,1}) + (\#E_{i,2})$. To consider some $E_{i,j}$ as an unordered set, we use the notation $[E_{i,j}]$. It follows then then $[E_{i,1}] \cup [E_{i,2}] = [E_{i,1}|E_{i,2}]$.

2.7 *The Row-Column Theorem*

Consider the set $[E_{p,q}]$ which is in the p^{th} row and the q^{th} column of some CM $\boldsymbol{E}$:



Since each CM-column contains an aggregate, the other sets in column q together comprise the complement of $[E_{p,q}]$ (crosshatched area above). Similarly, since each CM-row also contains an aggregate, the other sets in row p together form the complement of $[E_{p,q}]$, too (shown by solid shading). Formally then, the *row-column theorem* states:[11]

$$\bigvee_{i \neq p} [E_{i,q}] = \bigvee_{j \neq q} [E_{p,j}] = C([E_{p,q}])$$

Thus the row-complement and column-complement of any $[E_{p,q}]$ are the same set, so that in a 3-row CM, for example:

$E_{1,1}$	$E_{1,2}$	$E_{1,3}$
$E_{2,1}$	$E_{2,2}$	$E_{2,3}$
$E_{3,1}$	$E_{3,2}$	$E_{3,3}$

the following statements could be made:

$$[E_{1,1}] \cup [E_{2,1}] = [E_{3,2}] \cup [E_{3,3}],$$
$$[E_{1,1}] \cup [E_{3,1}] = [E_{2,2}] \cup [E_{2,3}],$$
$$[E_{2,1}] \cup [E_{3,1}] = [E_{1,2}] \cup [E_{1,3}], \quad \text{etc.}$$

[11] The expression $\bigvee_{i \neq p} [A_{i,q}]$ is to be read, 'the set-union of all sets $[A_{i,q}]$ where i does not equal p.'

In Fig. 1, for example, we can extract the set {B, 9, 5, A, 4, 3} from the fourth row and third column of the CM. The union of the remaining positions of the fourth row comprise the set [1082] ∪ [67] = {0, 1, 2, 6, 7, 8}, which is the same set as the union of the sets in the remaining positions of the third column, [01] ∪ [627] ∪ [8].

III

3.1 *2-row Combination Matrices*

A 2-row CM E can be diagrammed as follows:

P:	$E_{1,1}$	$E_{1,2}$
$\mathrm{F}(P)$:	$E_{2,1}$	$E_{2,2}$

where $\mathrm{F}(P) = E_{2,1} \mid E_{2,2}$ is a transform of $P = E_{1,1} \mid E_{1,2}$. It follows from the row-column theorem that $[E_{1,1}]$, a begin-set of P, is the same as $[E_{2,2}]$ which is an end-set of $\mathrm{F}(P)$, since both are complements of $[E_{1,2}]$. Similarly, $[E_{1,2}] = [E_{2,1}]$. One can thus construct a 2-row CM when some begin-set of a row is the same as the end-set of one of its transforms.

A class of 2-row CMs results when any row is combined with its retrograde, since R exchanges all of a row's begin-sets with its end-sets (see sec. 2.5). Ex:

X:	01439	2A857B6
$\mathrm{R}(X)$:	6B758A2	93410

Fig. 4

Such CMs exist for any row, for begin-set-end-set pairs of any length.

There are two general rules for forming 2-row CMs:

(1) When some begin-set of a row P maps to one of its end-sets under some operation F—that is, *when the row possesses a begin-set and an end-set both of which are in the same set-class*—a 2-row CM exists with P and $\mathrm{F}(P)$. Example:

$T_6(M(\{0, 1, 2, 7, 9, A\})) = \{3, 4, 5, 6, 8, B\}$

X:	0172A9	B48536
$T_6(M(X))$:	6B5483	12A790

Fig. 5

(2) Where some begin-set or end-set is invariant under some operation G, a 2-row CM exists with the rows P and $G(R(P))$. Example:

$\{0, 1, 2, 7\}$ is T_2I-invariant

X:	0172	A9B48536
$T_2(I(R(X)))$:	8B96A354	0712

Fig. 6

3.2 *3-row CMs*

Unlike 2-row CMs, the 3-row CMs based on an arbitrary row cannot be exhaustively listed from a perusal of the row's begin- and end-sets. A certain amount of trial and error is necessary. We present here a means to generate 3-row CMs which ultimately would be exhaustive, were one to pursue it at length, which would most likely be the domain of a computer program.

For any 3-row CM $\boldsymbol{E}$:

P:	$E_{1,1}$	$E_{1,2}$	$E_{1,3}$
$F(P)$:	$E_{2,1}$	$E_{2,2}$	$E_{2,3}$
$G(P)$:	$E_{3,1}$	$E_{3,2}$	$E_{3,3}$

it follows from the row-column theorem that $[E_{1,2} \mid E_{1,3}] = [E_{2,1}] \cup [E_{3,1}]$. A necessary but not sufficient condition, then, for the existence of such a CM is that there must be some end-set of P equal to the union of some pair of begin-sets of the rows $F(P)$ and $G(P)$.

Consider the row Y, whose begin- and end-sets are given in Fig. 3 (sec. 2.5, above). This row's end-sets of lengths 6 and 10 can be broken down as follows:

(a) $\{5, 6, 7, 8, A, B\} = \{5, 6\} \cup \{7, 8, A, B\}$
(b) $\{2, 3, 4, 5, 6, 7, 8, 9, A, B\} = \{4, 7, 8, A, B\} \cup \{2, 3, 5, 6, 9\}$

into component sets which are transforms of begin- and end-sets of the row:

(a) $\{5, 6\} = T_5(\{0, 1\})$
$\{7, 8, A, B\} = T_B(I(\{0, 1, 3, 4\}))$

(b) $\{4, 7, 8, A, B\} = T_7(\{0, 1, 3, 4, 9\})$
$\{2, 3, 5, 6, 9\} = T_6(I(\{0, 1, 3, 4, 9\}))$

This suggests the possibility of two CMs, though it turns out that partition (b) fails to generate a CM, as it is impossible to separate the right-hand part of the resultant matrix into columnar aggregates:

(a)

Y:	014392	A857B	6
$T_5(Y)$:	56		982731A04B
$T_B(I(Y))$:	BA78	2913640	5

(b)

Y:	01	4392A857B6
$T_7(Y)$:	78BA4	9530261
$T_6(I(Y))$:	65239	7B142A3

Fig. 7

This method might be extended to generate CMs of arbitrary size, by always splitting an end-set into the requisite number of (possibly null) begin- or end-set transforms. Some further restrictions or heuristic might be found to limit the amount of trial and error to make a similar method practical for larger CMs.

3.3 *Cliques of Rows*

An *n-clique of rows* is a collection of transpositionally related rows at transpositional levels corresponding to the PCs of an n-clique.[12] Thus the following is a 4-clique of rows:

[12] Such aspects of cliques and cycles of rows are discussed in Babbitt *op. cit.* (pp. 84, 90) as related to the rows of Schoenberg's String Trio and Fourth Quartet.

X:	0	1	7	2	A	9	B	4	8	5	3	6
$T_4(X)$:	4	5	B	6	2	1	3	8	0	9	7	A
$T_8(X)$:	8	9	3	A	6	5	7	0	4	1	B	2

Fig. 8

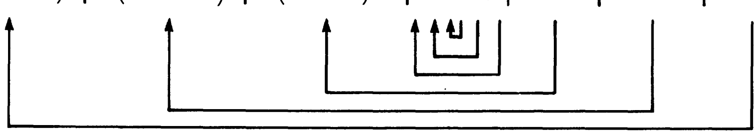
Each column comprises a 4-clique, and since all four 4-cliques are M-invariant (sec. 2.4), any column of such a 4-clique of rows, taken as an unordered set, would contain the same PCs if M were applied to all three rows:

$M(X)$:	0	5	B	A	2	9	7	8	4	1	3	6	
$T_8(M(X))$:	8	1	7	6	A	5	3	4	0	9	B	2	$=M(T_4(X))$
$T_4(M(X))$:	4	9	3	2	6	1	B	0	8	5	7	A	$=M(T_8(X))$

Similarly, the columns of any 2-, 3-, or 6-clique of rows are invariant under the operations specified in sec. 2.4.

In addition, the columns of cliques of rows with certain properties are invariant under composite operations involving R. Like the PCs of a row, we will number the columns of a clique of rows from 0 through 11. If each $(11-i)^{th}$ column of a clique of rows transforms to the i^{th} column for values of i from 0 to 5 under the same operation F, then the columns of the clique of rows are FR-invariant:

col. 0 =	col. 1 =	col. 2 =	. . . .	col. 9	col. 10	col. 11
F(col. 11)	F(col. 10)	F(col. 9)	. . . .			



We can test for this kind of columnar invariance in an n-clique of rows by examining the row which generated it modulo-n. Where

$$(p_i)\text{mod-}n = (F(x_{11-i}))\text{mod-}n, \text{ for i from 0 to 5,}$$

an invariance exists under FR. Example:

X:	0	1	7	2	A	9	B	4	8	5	3	6
(X)mod-2:	0	1	1	0	0	1	1	0	0	1	1	0

Here $(x_i)\text{mod-}2 = (x_{11-i})\text{mod-}2$, so that the columns of a 2-clique of rows based on this row are invariant under T_nR, where $(n)\text{mod-}2 = 0$. The columns of all 2-cliques of rows are in addition I-invariant, so that there is a further invariance under T_nIR which is the composite of T_nR and I.

Taking the same row modulo-4,

$$(X)\text{mod-4}: \quad 0\ 1\ 3\ 2\ 2\ 1\ 3\ 0\ 0\ 1\ 3\ 2$$

Fig. 9

we note that a symmetrical nest results in which paired elements differ by 2; that is, $(x_i \oplus 2)\text{mod-4} = (x_{11-i})\text{mod-4}$, so that the columns of 4-cliques of this row are invariant under T_nR, where $(n)\text{mod-4} = 2$, and also under $T_{\ominus(5 \otimes n)}MR$, the composite of T_nR and M, since all four 4-cliques are invariant under M.

As a final example, we will construct several rows, the columns of whose 3-cliques of rows are T_1MIR-invariant. T_1MI maps PCs 0 through 2 as follows, taken modulo-3:

x	$T_1(M(I(x)))$	$(T_1(M(I(x))))\text{mod-3}$
0	1	1
1	8	2
2	3	0

on the basis of which we might construct a symmetrical nest:

$$2\ 0\ 2\ 1\ 0\ 1\ 0\ 2\ 0\ 1\ 2\ 1$$

The following rows, then are realizations of this modular structure:

2	3	5	1	0	A	9	B	6	7	8	4
5	6	8	4	0	7	3	B	9	A	2	1
8	3	B	A	6	7	9	5	0	1	2	4

etc. . . .

3.4 *Cliques of Rows as CMs*

Depending on the structure of a row, one can often use an n-clique of rows directly as a CM by adding the requisite number of vertical partitions. The following CM, for example, a 4-clique of rows:

X:	0172	A9B4	8536
$T_4(X)$:	45B6	2138	097A
$T_8(X)$:	893A	6570	41B2

Fig. 10

results from the structure of the row taken modulo-4:

$$(0\ 1\ 3\ 2)(2\ 1\ 3\ 0)(0\ 1\ 3\ 2)$$

which is such that in each 4-PC segment, the four distinct 4-cliques are represented once each, making successive aggregates. Similarly, a 2-clique of rows results in a 6-row CM for the same row, because of the modulo-2 structure of the row:

$$(01)(10)(01)(10)(01)(10)$$

X:	01	72	A9	B4	85	36
$T_2(X)$:	23	94	0B	16	A7	58
$T_4(X)$:	45	B6	21	38	09	7A
$T_6(X)$:	67	18	43	5A	2B	90
$T_8(X)$:	89	3A	65	70	41	B2
$T_A(X)$:	AB	50	87	92	63	14

Fig. 11

An alternate way to generate this class of CMs is to inspect the interval content of the various equal-lengthed segments of the row.[13] If the successive 2-PC row-segments contain no instances of even interval-classes, then a 6-row CM like Fig. 11 results. Similarly, if interval-classes 3 and 6 are absent from the four successive non-overlapping trichords, a 4-row CM results which is a 3-clique of rows, and if interval-class 4 is missing from the successive tetrachords, a 3-row CM like that of Fig. 10 results. In addition, one can use a

[13] Martino *op. cit.*

6-clique of rows—two rows related transpositionally at the tritone—as a 2-row CM if neither "hexachord" contains instances of interval-class 6.

3.5 *Transposition CMs and their Derivatives*

A 12-row CM can always be constructed from the twelve transpositions of any row. Example:

X:	0	1	7	2	A	9	B	4	8	5	3	6
$T_1(X)$:	1	2	8	3	B	A	0	5	9	6	4	7
$T_2(X)$:	2	3	9	4	0	B	1	6	A	7	5	8
$T_3(X)$:	3	4	A	5	1	0	2	7	B	8	6	9
$T_4(X)$:	4	5	B	6	2	1	3	8	0	9	7	A
$T_5(X)$:	5	6	0	7	3	2	4	9	1	A	8	B
$T_6(X)$:	6	7	1	8	4	3	5	A	2	B	9	0
$T_7(X)$:	7	8	2	9	5	4	6	B	3	0	A	1
$T_8(X)$:	8	9	3	A	6	5	7	0	4	1	B	2
$T_9(X)$:	9	A	4	B	7	6	8	1	5	2	0	3
$T_A(X)$:	A	B	5	0	8	7	9	2	6	3	1	4
$T_B(X)$:	B	0	6	1	9	8	A	3	7	4	2	5

Fig. 12

One can always transform a clique of rows in such a CM with an operation under which the clique is invariant, yielding a more complex CM.[14] For example, one can substitute six inverted rows for a 2-clique of rows in the above CM:

X:	0	1	7	2	A	9	B	4	8	5	3	6	
$T_B(I(X))$:	B	A	4	9	1	2	0	7	3	6	8	5	$= I(T_1(X))$
$T_2(X)$:	2	3	9	4	0	B	1	6	A	7	5	8	
$T_9(I(X))$:	9	8	2	7	B	0	A	5	1	4	6	3	$= I(T_3(X))$
$T_4(X)$:	4	5	B	6	2	1	3	8	0	9	7	A	
$T_7(I(X))$:	7	6	0	5	9	A	8	3	B	2	4	1	$= I(T_5(X))$
$T_6(X)$:	6	7	1	8	4	3	5	A	2	B	9	0	
$T_5(I(X))$:	5	4	A	3	7	8	6	1	9	0	2	B	$= I(T_7(X))$
$T_8(X)$:	8	9	3	A	6	5	7	0	4	1	B	2	
$T_3(I(X))$:	3	2	8	1	5	6	4	B	7	A	0	9	$= I(T_9(X))$
$T_A(X)$:	A	B	5	0	8	7	9	2	6	3	1	4	
$T_1(I(X))$:	1	0	6	B	3	4	2	9	5	8	A	7	$= I(T_B(X))$

Fig. 13

[14] This was thought to be true only for rows with certain properties in Peter Westergaard, "Toward a Twelve-Tone Polyphony", in *Perspectives on Contemporary Music Theory*, eds. Boretz & Cone, New York: W. W. Norton & Co., 1972.

The operation M can then be applied to both a 4-clique of uninverted rows—$T_2(X)$, $T_6(X)$, and $T_A(X)$—and a 4-clique of inverted rows —$T_9(I(X))$, $T_5(I(X))$, and $T_1(I(X))$—yielding a CM consisting of four 4-cliques of rows. Ex. cont'd:

X:	0	1	7	2	A	9	B	4	8	5	3	6	
$T_B(I(X))$:	B	A	4	9	1	2	0	7	3	6	8	5	
$T_A(M(X))$:	A	3	9	8	0	7	5	6	2	B	1	4	$= M(T_2(X))$
$T_9(M(I(X)))$:	9	4	A	B	7	0	2	1	5	8	6	3	$= M(I(T_3(X)))$
$T_4(X)$:	4	5	B	6	2	1	3	8	0	9	7	A	
$T_7(I(X))$:	3	2	8	1	5	6	4	B	7	A	0	9	
$T_6(M(X))$:	2	7	1	0	4	B	9	A	6	3	5	8	$= M(T_6(X))$
$T_1(M(I(X)))$:	1	8	2	3	B	4	6	5	9	0	A	7	$= M(I(T_7(X)))$
$T_8(X)$:	8	9	3	A	6	5	7	0	4	1	B	2	
$T_3(I(X))$:	7	6	0	5	9	A	8	3	B	2	4	1	
$T_2(M(X))$:	6	B	5	4	8	3	1	2	A	7	9	0	$= M(T_A(X))$
$T_5(M(I(X)))$:	5	0	6	7	3	8	A	9	1	4	2	B	$= M(I(T_B(X)))$

Fig. 14

Instead, one could apply the operation T_2R to any of the 4-cliques of Fig. 13, since 4-cliques of this particular row are also invariant under that operation (see Fig. 9 in sec. 3.3):

X:	0	1	7	2	A	9	B	4	8	5	3	6	
$T_B(I(X))$:	B	A	4	9	1	2	0	7	3	6	8	5	
$T_4(R(X))$:	A	7	9	0	8	3	1	2	6	B	5	4	$= T_2(R(T_2(X)))$
$T_B(I(R(X)))$:	5	8	6	3	7	0	2	1	9	4	A	B	$= T_2(R(I(T_3(X))))$
$T_4(X)$:	4	5	B	6	2	1	3	8	0	9	7	A	
$T_7(I(X))$:	3	2	8	1	5	6	4	B	7	A	0	9	
$T_8(R(X))$:	2	B	1	4	0	7	5	6	A	3	9	8	$= T_2(R(T_6(X)))$
$T_7(I(R(X)))$:	9	0	A	7	B	4	6	5	1	8	2	3	$= T_2(R(I(T_7(X))))$
$T_8(X)$:	8	9	3	A	6	5	7	0	4	1	B	2	
$T_3(I(X))$:	7	6	0	5	9	A	8	3	B	2	4	1	
$R(X)$:	6	3	5	8	4	B	9	A	2	7	1	0	$= T_2(R(T_A(X)))$
$T_3(I(R(X)))$:	1	4	2	B	3	8	A	9	5	0	6	7	$= T_2(R(I(T_B(X))))$

Fig. 15

3.6 *Row substitution in clique-generated CMs*

Just as cliques of rows can be substituted in transposition CMs, one can operate on smaller CMs in the same way to make them more complex. Though this is feasible in any CM, it is particularly useful in dealing with CMs resulting from cliques of rows, in which all the rows are transpositionally related, as in the transposition CM.

Consider the 6-row CM of Fig. 11 which results from a 2-clique of rows. It can also be seen as two 4-cliques of rows, which, as we have seen (sec. 3.3, Fig. 9), exhibit columnar invariance under T_2R, because of the modulo-4 structure of the particular row used (X), in which the ith pair of PCs taken mod-4 maps under T_2R to the $(5 - i)$th pair, where the pairs are numbered 0 to 5:

$$(01)(32)(21)(30)(01)(32)$$

Thus one 4-clique of rows in the original 6-row CM can be subjected to T_2R to yield a new CM:

X:	01	72	A9	B4	85	36
$T_4(R(X))$:	A7	90	83	12	6B	54
$T_4(X)$:	45	B6	21	38	09	7A
$T_8(R(X))$:	2B	14	07	56	A3	98
$T_8(X)$:	89	3A	65	70	41	B2
$T_0(R(X))$:	63	58	4B	9A	27	10

Fig. 16

In the 3-row CM of Fig. 10, we can substitute a single row whose successive tetrachords are invariant under T_AM:

$T_4(X)$: (45B6) (2138) (097A)
$T_6(M(X))$: (6B54) (8312) (A790)

X:	0172	A9B4	8536
$T_6(M(X))$:	6B54	8312	A790
$T_8(X)$:	893A	6570	41B2

Fig. 17

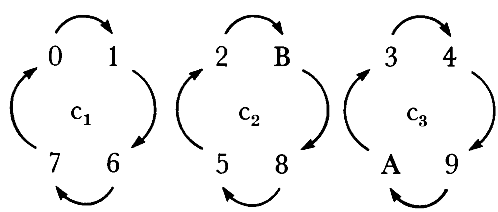
This is a different sort of invariance from the invariance of cliques of rows, and cannot be inspected for by taking a row with respect to any modulus. Put differently, a single row might be seen as a zero-clique of rows, and as such it has *already* been taken with respect to the desired modulus—namely, 12, which is an alias for 0 in the modulo-12 arithmetic system—since it is a structure of PCs (i.e., mod-12 numbers) to begin with. Furthermore, there is no operation under which "all zero-cliques of the row" (or of any row, for that matter) are invariant, as that would in effect be an operation under which any transposition of the row would be segmentally invariant. Specifically, the successive tetrachords of X are T_2M-invariant, those of $T_4(X)$ are T_AM-invariant, and those of $T_8(X)$ are T_6M-invariant, so that X, $T_4(X)$, and $T_8(X)$ are tetrachordally equivalent to $T_2(M(X))$, $T_6(M(X))$, and $T_A(M(X))$, respectively.

3.7 *Cycles*

Given a starting PC, successive applications of the operation T_n generate an n-clique. Suppose, however, we used some other operation instead. For example, T_1M maps individual PCs as follows:

S:	0	1	2	3	4	5	6	7	8	9	A	B
$T_1(M(S))$:	1	6	B	4	9	2	7	0	5	A	3	8

resulting in the following cycles:



Note that while the cycles 0-1-6-7 and 3-4-9-A are in the same SC, 2-5-8-B is not.[15] Such cycles can be used like cliques to generate CMs:

	c_1 c_2 c_3	c_2 c_1 c_3	c_2 c_1 c_3	c_3 c_2 c_1	
U:	0 2 9	5 6 4	B 7 A	3 1 8	
$T_1(M(U))$:	1 B A	2 7 9	8 0 3	4 6 5	
$T_6(U)$:	6 8 3	B 0 A	5 1 4	9 7 2	$=T_1(M(T_1(M(U))))$
$T_7(M(U))$:	7 5 4	8 1 3	2 6 9	A 0 B	$=T_1(M(T_6(U)))$

Fig. 18

in which each CM-position contains 1 PC from each cycle.

Table I[16] gives the cyclical properties of all composites of T_n, I, and M. Where some operation generates a group of cycles, the inverse of that operation generates the same cycles in reverse order, and hence

[15] Use is made here, in the following section, and in Table I, of a notation for cycles that consists of PCs separated by hyphens. 0-1-6-7 is the same cycle as 1-6-7-0 and 7-0-1-6 which are different notations for the same thing, since they are all cyclic permutations of each other.

[16] After Rothgeb, "Some Ordering Relationships in the Twelve-Tone System", *JMT*, 10/2(1966).

the same CMs. This redundancy is noted in Table I. (See pp. 29–30.) The table can, however, be reduced even further, since the collections of cycles generated by certain operations are transpositionally related to other such collections. Example:

T_2MI-cycles: 0-2-4-6-8-A, 1-9-5, 3-B-7
T_8MI-cycles: 1-3-5-7-9-B, 2-A-6, 4-0-8

Operations generating transpositionally related cycles are connected with brackets in Table I, giving fifteen brackets or distinct cyclic patterns in all. Reflexive operations, which are their own inverses, generate cycles consisting of no more than 2 PCs, and include all operations of the form T_nI.

Note that the cycles resulting from the operation T_2MI, for example, are not all of the same length:

c_1: 0-2-4-6-8-A, c_2: 1-9-5, c_3: 3-B-7

but a useful 3-row CM can still be derived from them by splitting the larger cycle in two at any point:

0-2-4, 6-8-A or 8-A-0, 2-4-6 etc.

and using the cycle halves in separate columns:

	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_1	c_3	c_2	c_1	c_1
W:	8	1	7	2	9	B	6	0	3	5	A	4
T_2(M(I(*W*))):	A	9	3	4	5	7	8	2	B	1	0	6
T_4(*W*):	0	5	B	6	1	3	A	4	7	9	2	8

Fig. 19

Note that in this cyclic CM, unlike the CM of Fig. 18, the operation under which the cycles are generated, when applied to the bottom-most row, does not yield the top-most original row *W*, because of the way in which the longer cycle was divided. We actually are using only half of a cycle of rows here.

TABLE I: Cycles generated by set operations

Starred operations are reflexive. Brackets connect cyclic patterns which are transpositionally related.

T_0	all singleton sets
T_1	0-1-2-3-4-5-6-7-8-9-A-B
T_2	0-2-4-6-8-A, 1-3-5-7-9-B
T_3	0-3-6-9, 1-4-7-A, 2-5-8-B
T_4	0-4-8, 1-5-9, 2-6-A, 3-7-B
T_5	0-5-A-3-8-1-6-B-4-9-2-7
* T_6	0-6, 1-7, 2-8, 3-9, 4-A, 5-B
T_7	(inverse of T_5)
T_8	(inverse of T_4)
T_9	(inverse of T_3)
T_A	(inverse of T_2)
T_B	(inverse of T_1)

* I	0, 1-B, 2-A, 3-9, 4-8, 5-7, 6
* T_1I	0-1, 2-B, 3-A, 4-9, 5-8, 6-7
* T_2I	0-2, 1, 3-B, 4-A, 5-9, 6-8, 7
* T_3I	0-3, 1-2, 4-B, 5-A, 6-9, 7-8
* T_4I	0-4, 1-3, 2, 5-B, 6-A, 7-9, 8
* T_5I	0-5, 1-4, 2-3, 6-B, 7-A, 8-9
* T_6I	0-6, 1-5, 2-4, 3, 7-B, 8-A, 9
* T_7I	0-7, 1-6, 2-5, 3-4, 8-B, 9-A
* T_8I	0-8, 1-7, 2-6, 3-5, 4, 9-B, A
* T_9I	0-9, 1-8, 2-7, 3-6, 4-5, A-B
* T_AI	0-A, 1-9, 2-8, 3-7, 4-6, 5, B
* T_BI	0-B, 1-A, 2-9, 3-8, 4-7, 5-6

TABLE I (*cont.*)

Starred operations are reflexive. Brackets connect cyclic patterns which are transpositionally related.

* M	0, 1-5, 2-A, 3, 4-8, 6, 7-B, 9
T_1M	0-1-6-7, 2-B-8-5, 3-4-9-A
* T_2M	0-2, 1-7, 3-5, 4-A, 6-8, 9-B
T_3M	0-3-6-9, 1-8-7-2, 4-B-A-5
* T_4M	0-4, 1-9, 2, 3-7, 5, 6-A, 8, B
T_5M	0-5-6-B, 1-A-7-4, 2-3-8-9
* T_6M	0-6, 1-B, 2-4, 3-9, 5-7, 8-A
T_7M	(inverse of T_1M)
* T_8M	0-8, 1, 2-6, 3-B, 4, 5-9, A, 7
T_9M	(inverse of T_3M)
* T_AM	0-A, 1-3, 2-8, 4-6, 5-B, 7-9
T_BM	(inverse of T_5M)

* MI	0, 1-7, 2, 3-9, 4, 5-B, 6, 8, A
T_1MI	0-1-8-9-4-5, 2-3-A-B-6-7
T_2MI	0-2-4-6-8-A, 1-9-5, 3-B-7
* T_3MI	0-3, 1-A, 2-5, 4-7, 6-9, 8-B
T_4MI	0-4-8, 1-B-9-7-5-3, 2-6-A
T_5MI	(inverse of T_1MI)
* T_6MI	0-6, 1, 2-8, 3, 4-A, 5, 7, 9, B
T_7MI	0-7-8-3-4-B, 1-2-9-A-5-6
T_8MI	(inverse of T_4MI)
* T_9MI	0-9, 1-4, 2-B, 3-6, 5-8, 7-A
T_AMI	(inverse of T_2MI)
T_BMI	(inverse of T_7MI)

3.8 *Compound cycles*

Any operation—even if written with a single symbol—can be decomposed into two operations whose composite is the original operation. That is, if $Q = F(P)$ there is always an F_1 and an F_2 such that $Q = F(P) = F_1(F_2(P))$. Ex:

(a) $F(P) = T_4(P) = T_6(M(T_2(M(P))))$; $F_1 = T_6M$ & $F_2 = T_2M$
(b) $F(P) = T_9(P) = T_6(M(T_3(M(P))))$; $F_1 = T_6M$ & $F_2 = T_3M$
(c) $F(P) = T_1(M(I(P))) = T_2(M(T_7(I(P))))$;
$F_1 = T_2M$ & $F_2 = T_7I$

Since each component operation can be further decomposed, it follows that operations can be decomposed into arbitrary numbers of components. Ex:

(d) $F(P) = T_4(P) = T_9(I(T_6(I(T_1(P))))) = F_1(F_2(F_3(P)))$;
$F_1 = T_9I$ & $F_2 = T_6I$ & $F_3 = T_1$

Where F generates cycles or cliques, a decomposition of F into two or more components generates *compound cycles*. For example, (a) above, a decomposed transposition operator, maps PCs as follows through an "intermediate value":

(a) S:	0	1	2	3	4	5	6	7	8	9	A	B
$T_2(M(S))$:	2	7	0	5	A	3	8	1	6	B	4	9
$T_4(S)$:	4	5	6	7	8	9	A	B	0	1	2	3

yielding compound cycles:

c_1: 0-2-4-A-8-6 c_2: 1-7-5-3-9-B
c_3: 2-0-6-8-A-4 c_4: 3-5-7-1-B-9

Note that these cycles contain the same PCs as 2-cliques, but not in the same order—cycles c_1 and c_3 are reverses of each other as are c_2 and c_4. As unordered sets, c_1 and c_3 are complements of c_2 and c_4, so that a 6-row CM can be generated, whose columns are paired complementary cycles:[17]

[17] This CM can alternatively be derived by taking the clique-generated CM of Fig. 11 and subjecting one 4-clique of rows in that CM to the operation M, under which all 4-cliques are invariant.

	c_1	c_2	c_4	c_3	c_3	c_2	c_4	c_1	c_1	c_2	c_4	c_3	
X:	0	1	7	2	A	9	B	4	8	5	3	6	
$T_2(M(X))$:	2	7	1	0	4	B	9	A	6	3	8	5	$>T_2M$
$T_4(X)$:	4	5	B	6	2	1	3	8	0	9	7	A	$>T_6M$
$T_A(M(X))$:	A	3	9	8	0	7	5	6	2	B	1	4	$>T_2M$
$T_8(X)$:	8	9	3	A	6	5	7	0	4	1	B	2	$>T_6M$
$T_6(M(X))$:	6	B	5	4	8	3	1	2	A	7	9	0	$>T_2M$
													T_6M

Fig. 20

Example (b) above generates three cycles:

S:	0	1	2	3	4	5	6	7	8	9	A	B	c_1: 0-3-9-0-6-9-3-6
$T_3(M(S))$:	3	8	1	6	B	4	9	2	7	0	5	A	c_2: 1-8-A-5-7-2-4-B
$T_9(S)$:	9	A	B	0	1	2	3	4	5	6	7	8	c_3: 8-7-5-4-2-1-B-A

Cycles c_2 and c_3 contain the same 8-PC set ordered differently, while c_1 contains the complementary 4-PC set with each PC duplicated. Together, $c_{1\text{-}3}$ form two aggregates, each of which share half of the PCs in c_1. Thus the following CM is possible:[18]

	c_1		c_1		c_1		c_1		
	c_2	c_3	c_2	c_3	c_2	c_3	c_2	c_3	
W:	10	8	A3	5	76	2	49	B	
$T_3(M(W))$:	8	37	5	64	2	91	B0	A	$>T_3M$
$T_9(W)$:	A9	5	7	02	43	B	16	8	$>T_6M$
$M(W)$:	5	04	2	31	B	6A	8	97	$>T_3M$
$T_6(W)$:	7	62	4	9B	10	8	A	35	$>T_6M$
$T_9(M(W))$:	2	91	B0	A	8	37	5	64	$>T_3M$
$T_3(W)$:	43	B	16	8	A9	5	7	02	$>T_6M$
$T_6(M(W))$:	B6	A	89	7	5	04	23	1	$>T_3M$

Fig. 21

[18] This CM possesses the maximal fragmentation (sec. 5.1) for an 8-row CM.

Example (c) above results in two 12-PC cycles which are reverses of each other:

S:	0	1	2	3	4	5	6	7	8	9	A	B
$T_7(I(S))$:	7	6	5	4	3	2	1	0	B	A	9	8
$T_1(M(I(S)))$:	1	8	3	A	5	0	7	2	9	4	B	6

c_1: 0-7-1-6-8-B-9-A-4-3-5-2
c_2: 2-5-3-4-A-9-B-8-6-1-7-0

resulting in a 12-row CM [19] for any row:

	c_1	c_1	c_2	c_1	c_1	c_2	c_2	c_1	c_1	c_2	c_2	c_2	
Y:	0	1	4	3	9	2	A	8	5	7	B	6	
$T_7(I(Y))$:	7	6	3	4	A	5	9	B	2	0	8	1	T_7I
$T_1(M(I(Y)))$:	1	8	5	A	4	3	B	9	0	2	6	7	T_2M
$T_6(M(Y))$:	6	B	2	9	3	4	8	A	7	5	1	0	T_7I
$T_8(Y)$:	8	9	0	B	5	A	6	4	1	3	7	2	T_2M
$T_B(I(Y))$:	B	A	7	8	2	9	1	3	6	4	0	5	T_7I
$T_9(M(I(Y)))$:	9	4	1	6	0	B	7	5	8	A	2	3	T_2M
$T_A(M(Y))$:	A	3	6	1	7	8	0	2	B	9	5	4	T_7I
$T_4(Y)$:	4	5	8	7	1	6	2	0	9	B	3	A	T_2M
$T_3(I(Y))$:	3	2	B	0	6	1	5	7	A	8	4	9	T_7I
$T_5(M(I(Y)))$:	5	0	9	2	8	7	3	1	4	6	A	B	T_2M
$T_2(M(Y))$:	2	7	A	5	B	0	4	6	3	1	9	8	T_7I

Fig. 22

[19] This CM is analogous to the one in Fig. 14 and can be generated by the same row-substitutions used there, applied to a transposition CM based on Y.

Example (d) gives a function decomposed into three components, mapping PCs through two intermediate values:

S:	0	1	2	3	4	5	6	7	8	9	A	B
$T_1(S)$:	1	2	3	4	5	6	7	8	9	A	B	0
$T_5(I(S))$:	5	4	3	2	1	0	B	A	9	8	7	6
$T_4(S)$:	4	5	6	7	8	9	A	B	0	1	2	3

resulting in four 9-PC cycles.

c_1: 0-1-5-4-5-1-8-9-9 $= H \cup I \cup I$

c_2: 1-2-4-5-6-0-9-A-8 $= H \cup I \cup J$

c_3: 2-3-3-6-7-B-A-B-7 $= J \cup K \cup K$

c_4: 3-4-2-7-8-A-B-0-6 $= H \cup J \cup K$

with each cycle containing exactly three intact 4-cliques whose PCs are connected with brackets above. It is possible to form aggregates by gathering together the four 4-cliques:

$$H = \{0, 4, 8\} \qquad I = \{1, 5, 9\}$$
$$J = \{2, 6, A\} \qquad K = \{3, 7, B\}$$

in each column of a resultant 9-row CM:

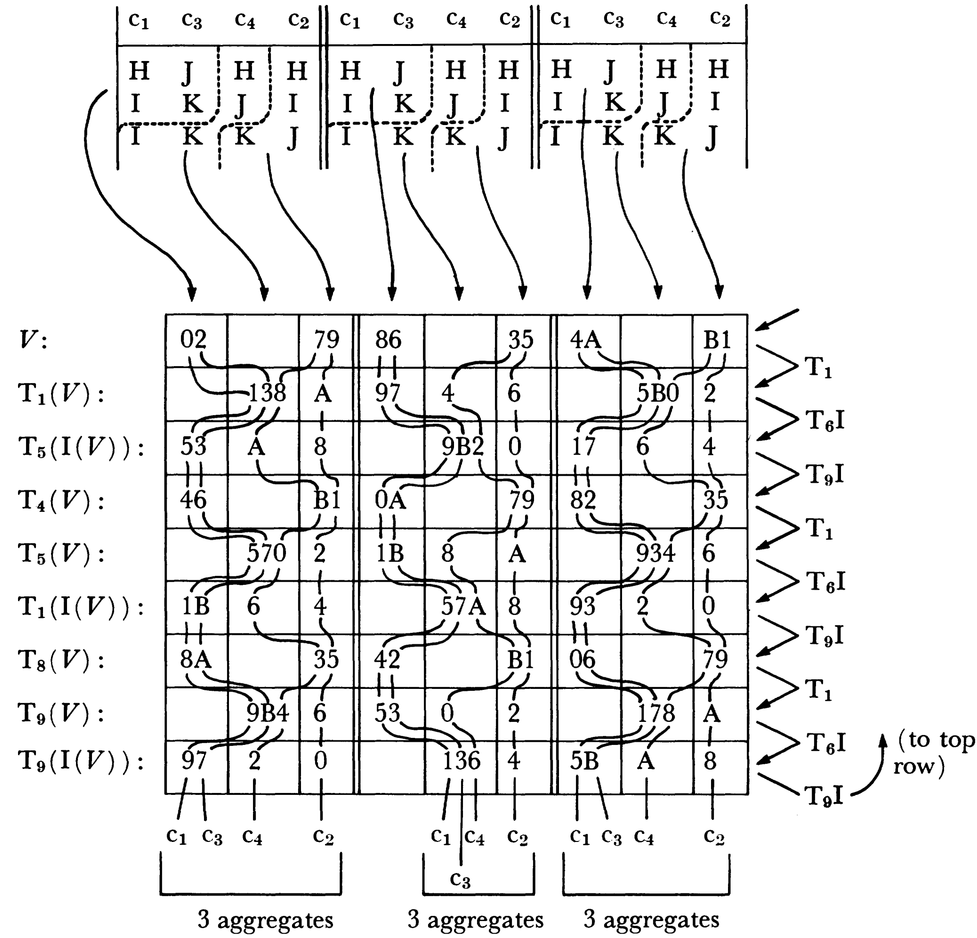


Fig. 23

Compound cycles thus present many interesting and varied CM possibilities, as demonstrated by the examples. It is usually difficult to match an arbitrary row to a pattern of compound cycles which generates a CM based on it, though the row *X* which recurs throughout the examples in this paper was susceptible to the compound cycles of ex. (a), and those of ex. (c) would generate a 12-row CM for any row. Some *ad hoc* reasoning must usually be applied when working with compound cycles because of the repetition of PCs within cycles and since they involve partitions of two or more aggregates together rather than single aggregates as is the case with simple cycles and cliques.