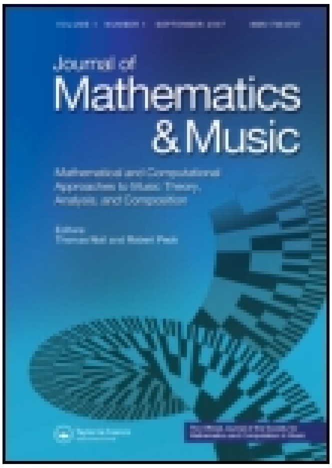

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Motivic analysis according to Rudolph Réti: formalization by a topological model

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RESEARCH ARTICLE

Motivic analysis according to Rudolph Réti: formalization by a topological model

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This paper proposes a formalization of the neutral niveau of Rudolph Réti's approach to motivic analysis within our mathematical model based on topological spaces of motives. Réti developed a substantial approach favouring melodic relationships below the musical surface. However, his approach has been much criticized for reasons such as an evident lack of methodology. This paper suggests that, when Réti's terminology is redefined in a precise mathematical setup, his approach can fit a computer-aided motivic analysis, and a topological solution to his problematic identity concept and limitation of transformations can be proposed. Our mathematical model, based on motif, contour, gestalt, and motif similarity, involves neighbourhoods of a motif that include similar motives of different cardinalities. It yields a topological space on the set of all motives of a composition, and in which Réti's concepts of shape, imitation, variation, and transformation are naturally formalized. The '*germinal motif*' corresponds to the 'most dense' motif in the space.

Keywords: Réti's approach; motivic structure formalization; motivic space of a score; topological model

1. Introduction

'Why is it that we cannot produce a convincing musical composition by taking a group or a section from one work and linking it to that of another?' This question, posed by Rudolph Réti [1, p. 348], still remains a challenge at the beginning of the 21st century. Long-established analytical theories, such as Schenkerian analysis, do not provide thorough answers to the question, neither do more recent cognitive analytical approaches, e.g. the Generative Theory [2,3]. But this question incontestably addresses a central issue for our understanding of music [4, p. 89]. For Rudolph Réti, who as a student had once asked this provocative question in his composition class, it remained a whole career focus, though not without controversy.

Réti proposed that thematic ideas in a composition are realized through developments stemming from a single germinal motive, which then contributes to unity within the composition [1,5]. First critics of his approach most often expressed skepticism, emphasizing the problem of hearing Réti's

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complex connections in the analysed music and lack of clear methodology, but acknowledged the importance of Réti's questions – e.g. [6,7]. Unavoidably, some rejected Réti's approach outright, such as Bauman [8], who labelled it mysticism.

Later, eminent analysts Meyer [9, pp. 59–70] and Rosen [10, p. 41] did not support Réti's approach, though then criticized the limitations of Schenkerian analysis. While Epstein emphasizes the weak methodology in Réti's approach [11, p. 10], Cook critiques that Réti was most concerned above all '*to prove his theory right, regardless of the particular qualities of the music he's talking about*' [12, p. 3]. Devoting some 26 pages of his book to Réti, Cook's main critique is that Réti's method does not focus on the '*music as it is heard, but about the compositional process that gave rise to it*' [12, p. 111]. On the other hand, Dunsby and Whittall recognize Réti's contribution to music analysis: '*It is inevitable that Réti's historical diagnoses are undervalued, in view of the particular bias of nineteenth-century studies*' [4, p. 91], but also stress some issues, e.g. '*Réti's terminology is ill-defined and inconsistent... and the two main studies, of the* Pathétique *and* Appassionata *sonatas, do not offer clear models for analysis*' [4, p. 92]. Furthermore, Dunsby describes Réti's approach to motivic and thematic analysis as favouring '*immanent relationships below the musical surface ... [but his approach is] certainly not scientifically reproducible*' [13].

Walker addresses three main arguments for the rejection of Réti's approach [14]. The first is '*(1) The complex connections that Réti observes as unifying contrasts cannot be heard*' [14, p. 33]. Walker replies that complexity is no argument for rejection, and he reinforces this by mentioning his experimental result on the perception of such complex connections. The second argument, '*(2) Thematic connections may be unconscious to the composer, therefore they cannot be meaningful*' [14, p. 33], is answered by referring to composers (e.g. Schoenberg and Mahler) who themselves made piercing observations of their own work. Finally, Walker replies to '*(3) Taken to its logical limits Réti's method produces bizarre results. You can establish the relatedness of any two works*' [14, p. 33] by clarifying that a relation between two pieces does not abolish their respective internal unity. This third issue was also stressed by Maw [15], who writes that there is '*unlimited possibility for similarity as well as for variety*' [15, p. 492], but concludes, '*so long as the technique does not get out of hand much can be learnt. Readers should bear this in mind while grasping at Dr. Réti's revelations*' [15, p. 493].

In view of these confusing judgments of Réti's theory, we propose in this paper a topological approach to motivic structure and analysis of compositions.[1] We propose a discourse on the neutral niveau [19] in order to attempt to realize a precise formalization of Réti's principles. By taking this perspective, the two first rejection arguments discussed by Walker are directly discarded. Indeed, cognitive studies, such as [20], show that we do not 'hear' the retrograde of a short melody, although it is unanimously acknowledged as being part of Bach's fugues not only from the poietic perspective but also on the neutral level [19]. The third rejection argument is addressed within our topological model by giving a detailed account of the dynamics of motivic connections as a function of variable similarity thresholds (see Section 4). And contrary to Dunsby's belief [13] we want to suggest that, when Réti's terminology is redefined in a precise mathematical setup, his approach can become 'scientifically reproducible' and fit in a computer-aided motivic analysis.

Our topological model is based on the concepts of motif, contour, gestalt, motif similarity, and neighbourhood of a motif, including (similar) motives of different cardinalities, and also formalizing Réti's concepts of shape, imitation, variation and transformation. Our mathematical construct yields a topological space on the set of all motives of a composition in which Réti's germinal motif corresponds to the 'most dense' motif in the space. It is true that any model of germinal motives stemming from the consideration of similarities among motives of different cardinalities must be a complex theory. However, our model has been entirely implemented in a computer program: see [21] for our stand-alone and extended version, called *Melos*, of the software modules MeloTopRUBETTE® [22] and MeloRUBETTE® in RUBATO® (see [23–25]), and for which a program in the software *OpenMusic* [26] was designed for the visualization of its multiple output results. The complexity of calculations could be managed and considerably reduced by use of specific mathematical results [16,22]. In fact, the computational complexity of the model is such that it requires an implementation for any explicit meaningful application to a music piece. A number of investigations, e.g. into Schumann's *Träumerei* [27,28] and *Von Fremden Ländern und Menschen* [29], to Webern's *Variation für Klavier*, op. 27, no. 1 [23], and to Bach's *Kunst der Fuge* [30,31], support the validity of our model. Thus, in view of our proposed formalization of Réti's approach within our topological model, this may suggest to revisit the analytical power of Réti's ideas once they have been made precise.

In Section 2, we briefly recall our topological model of motivic structure and analysis. It is followed in Section 3 by a detailed discussion of our formalization of Réti's approach. In Section 4, we propose a topological solution to Réti's problematic identity concept and limitation of transformations. Finally, in Section 5, we briefly discuss the generic character of our approach, which provides us with a number of variable analytical perspectives stemming from varying the topological parameters: this type of variable perspective is a typical application of the Yoneda lemma [18].

2. Modelling motivic structure through topological spaces

In this section we briefly recall the main concepts of our mathematical model needed for our discussion in Section 3. We refer the reader to, for example, [30] for a description of our model presented with motivational comments for each introduced concept. We restrict our attention to a minimal setup in order to make the essential clear; for details see [16,18,32].

We introduce the **tone parametrization space** $\mathbb{R}^{\{O,P,D,L,C,G\}} \cong \mathbb{R}^6$ for which the parameters are, respectively, *onset* O, *pitch* P, *duration* D, *loudness* L, *crescendo* C, and *glissando* G. We consider the **space of notes** $\mathbb{R}^{\{O,P,\ldots\}}$ parametrized by a subset $\{O, P, \ldots\} \subset \{O, P, D, L, C, G\}$ containing at least onset and pitch parameters. A **motif** M is a non-empty finite set of notes

$$M = \{m_1, \ldots, m_n\} \subset \mathbb{R}^{\{O,P,\ldots\}}$$

such that the canonical projection of M to the onset axis $\mathbb{R}^{\{O\}}$ is a bijection. Therefore, in a motif only one note is heard at a given onset. We set $card(M) = n$ and call it the **cardinality of the motif** M. This is the number of notes in the motif M. A **submotif** N **of a motif** M is a motif such that $N \subset M$. The set of all possible motives with cardinality n is denoted MOT_n, and the **space of motives**, denoted $MOT := \coprod_n MOT_n$, is the set of all motives. Given a **composition** S, i.e. a subset $S \subset \mathbb{R}^{\{O,P,\ldots\}}$, we denote by $MOT(S)$ an arbitrarily selected finite collection of motives $M \subset S$. We further impose that $MOT(S)$ satisfies the *Submotif Existence Axiom* (*SEA*) below.

SEA: Let $n_{\min}$ denote the cardinality of the smallest motif in $MOT(S)$. If M is a motif in $MOT(S)$ and N is a submotif of M with $card(N) \geq n_{\min}$, then N is also a motif in $MOT(S)$.

We illustrate these concepts in Schumann's Träumerei. *We decide to use the rhythmic/melodic segmentation of the soprano voice proposed by Bruno Repp* [33] : *see Figure* 1. *In this example we consider the space of notes* $\mathbb{R}^{\{O,P\}}$. *We take value 60 for* C_4, *value 1 for a semi-tone, value* 3/4 *for the onset of the first score note* C, *and value 1 for a bar duration. Then the E-D-C-F motif 2, which we denote* M, *can be represented by*

$$M = \{(2.75, 76), (2.875, 74), (3, 72), (3.125, 77)\}.$$

For $MOT(S)$, *we select all 28 motives from Figure* 1, *together with all their submotives down to cardinality 2, yielding, by use of our implementation* Melos, *a total of 1438 motives defining the set* $MOT(S)$. *Note that if instead the motive collection was containing the 28 motives only, it*

Figure 1. The rhythmic/melodic segmentation of the soprano voice of Schumann's *Träumerei* as proposed by Repp [33]. Its space of motives $MOT(S)$, constructed with our implementation *Melos*, of all 28 motives together with their submotives down to 2 notes contains a total of 1438 motives (not necessarily made of consecutive notes). If we do not consider this segmentation but instead consider all motives in the soprano voice containing between 2 to 10 notes with a maximum span of 2 bars, the space of motives increases to 237 736 motives.

would not satisfy the SEA (e.g. the 2-note submotives of motif 2 would not be in $MOT(S)$*). Another example for* $MOT(S)$ *could be all motives from the soprano voice that contain between 2 to 10 notes and for which the first and last notes are at most 2 bars apart from one another. In this case,* $MOT(S)$ *would have a total of 237 736 motives. It is important to note that our model* is not restricted to monophonic music. *As such, another example for* $MOT(S)$ *is the set of all motives in the* Träumerei *(all voices together) that contain between 2 to 10 notes for which the first and last notes are at most 1 bar apart from one another; this would lead to a total of 711 198 motives in* $MOT(S)$*. This illustrates the need for an implementation of explicit constructs for our model.*

The **contour**[2] **of a motif** M is the value of M under a set map $t\colon MOT \to \Gamma_t$ for a certain set Γ_t. Usually we define such a map t as a disjoint union of maps $t_n\colon MOT_n \to \Gamma_{t,n}$, so that $\Gamma_t = \coprod_n \Gamma_{t,n}$. There are several **contour types** t we will use in this paper. The **Rigid** contour map

$$Rg\colon \coprod_n MOT_n \longrightarrow \coprod_n \mathbb{R}^{2n}$$

projects a motif to the onset-pitch plane and orders the notes according to their onset values. The **COM-Matrix** contour map

$$Com\colon \coprod_n MOT_n \longrightarrow \coprod_n \{-1, 0, 1\}^{n\times n}$$

assigns to an n-note motif M an $n \times n$ matrix with entries in $\{-1, 0, 1\}$. If $(q_1, \ldots, q_n)$ are the onset-ordered pitch values from $Rg(M)$, the COM-matrix $Com(M)$ is defined (see, for example, [34]) as

$$(Com(M))_{ij} = \begin{cases} 1 & \text{if } q_j - q_i > 0 \\ 0 & \text{if } q_j = q_i \\ -1 & \text{if } q_j - q_i < 0. \end{cases}$$

The **Diastematic** contour map

$$Dia\colon \coprod_n MOT_n \longrightarrow \coprod_n \mathbb{R}^{n-1}$$

assigns to a motif M the vector of its consecutive pitch differences (i.e. of consecutive intervals). Finally, the **Diastematic Index** contour map

$$India\colon \coprod_n MOT_n \longrightarrow \coprod_n \{-1, 0, 1\}^{n-1}$$

assigns to a motif M the vector of signs (represented by -1, 0, and 1) of consecutive pitch differences.

For example, if we consider again motif $M = motif\ 2$ from Figure 1, we have $Rg(M) = ((2.75, 76), (2.875, 74), (3, 72), (3.125, 77))$,

$$Com(M) = \begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix},$$

$Dia(M) = (-2, -2, 5)$, *and* $India(M) = (-1, -1, 1)$.

In the previous example, the rigid contour of the motif is almost the same as the motif itself. When other parameters are taken into account for representing notes, such as duration or loudness, the rigid contour is a very useful contour type. Note too that COM matrices are anti-symmetric and have a main diagonal of zeros, making it sufficient to represent the COM-matrix contour of a motif strictly by the upper-triangular values of the matrix, as implemented and formally defined in our model. However, for the purpose of this paper, we keep the usual whole representation of COM matrices.

We consider the **counterpoint group** CP generated by translations in time, transpositions, inversions and retrogrades. More precisely, CP is the affine Klein group, a subgroup of the general affine group $\overrightarrow{GL}\,(\mathbb{R}^{\{O,P,\ldots\}})$. For example, given a note $m = (o, p, \ldots) \in \mathbb{R}^{\{O,P,\ldots\}}$, the retrograde k is defined as $k \cdot m := (-o, p, \ldots)$ and the translation in time t_x by x is defined as $t_x \cdot m := (o + x, p, \ldots)$.

Thus, CP acts pointwise on *MOT*, i.e. for any $M = \{m_1, \ldots, m_n\} \in MOT$ and any $g \in CP$, then $g \cdot M = \{g(m_1), \ldots, g(m_n)\}$. We suppose that this action induces an action of CP on Γ_t, i.e. for any $g \in CP$ and any $M \in MOT$, then $t(g \cdot M) = g \cdot t(M)$. We consider a subgroup P of CP that we call a **paradigmatic group**[3] (and where possibly $P = CP$). The **(t,P)-gestalt of a motif** M, denoted $Ges_t^P(M)$, is the set of all motives with contours in the same P-orbit as its contour $t(M)$, that is $Ges_t^P(M) := t^{-1}(P \cdot t(M))$. Gestalts[4] conceptualize the identification of motives with their *imitations*.

For example, if we consider the group $P = Tr$ generated by the transpositions and translations in time and the diastematic contour type, motif $M = motif\ 2$ and $M_1 = motif\ 19$ in Figure 1 clearly have the same gestalt, since $Dia(M_1) = Dia(M)$, and therefore $M_1 \in Ges^{Tr}_{Dia}(M)$. With *the same group but with the COM-matrix contour type, the G-F-E-A submotif N_1 composed of the 1st, 2nd, 3rd and 5th notes from motif 7, is also identified with motif 2: $N_1 \in Ges^{Tr}_{Com}(M)$. By using Melos with Schumann's* Träumerei*, the 1438 motives in $MOT(S)$, as constructed previously, are regrouped in 382 (Dia, Tr)-gestalts and in 322 (Dia, CP)-gestalts, whereas they are regrouped in 164 (Com, Tr)-gestalts and in 123 (Com, CP)-gestalts. This shows that the choice of contour type and paradigmatic group will greatly affect the landscape of the constructed topological space.*

We introduce pseudo-metrics[5] d_n on $\Gamma_{t,n}$ that we retract to motives: the **t-distance between two motives M and N with the same cardinality n** is $d_t(M, N) := d_n(t(M), t(N))$, and their **gestalt distance** is the minimum distance between the two gestalts, i.e. between the two motives when considering all their imitations:

$$gd_t^P(M, N) := \inf_{p,q \in P} d_n(p \cdot t(M), q \cdot t(N)).$$

If P is a group of isometries[6] with respect to d_t, then gd_t is a pseudo-metric on MOT_n. For example, we can use the Euclidean distance (denoted Ed) and Relative Euclidean[7] distance (denoted REd) on $\Gamma_t \subset \mathbb{R}^m$, for any contour type t, or, for example, the CSIM or C^+SIM values [37] for the COM-matrix contour type as, for example, was used by Quinn [38] in his fuzzy logic approach to identifying similar contours. The distance value between motives with the same cardinality corresponds to a *contour similarity* measure.[8]

For example, if we take the E♭-D-E-C motif 8 in Figure 1*, which we denote N_3, i.e. $N_3 = \{(7.875, 75), (8.0, 74), (8.25, 76), (8.5, 72)\}$ with COM-matrix contour*

$$Com(N_3) = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

with counterpoint paradigmatic group $P = CP$ and relative Euclidean distance function $d_n = REd$, then $d_{Com}(M, N_3) = \sqrt{20}/4 \approx 1.118$, and $gd_{Com}(M, N_3) = \sqrt{4}/4 = 0.5$ since the inverse of N_3, denoted $inverse(N_3)$, minimizes the distance between the two gestalts, i.e. $gd_{Com}(M, N_3) = d_{Com}(M, inverse(N_3))$.

A crucial step in our model in order to formalize the contour similarity for *different motif cardinalities*, i.e. to formalize variations and transformations of motives, is the introduction of the **(t, P, d)-ϵ-neighbourhood of a motif** M for a given similarity threshold $\epsilon > 0$, denoted $V_\epsilon^{t,P,d}(M)$: it consists of all motives N that contain a submotif N^* with gestalt distance less than ϵ to M (or, speaking directly, N^* needs to be ϵ-similar to M). More precisely, given $\epsilon > 0$,

$$V_\epsilon^{t,P,d}(M) := \{N \in MOT | \exists N^* \subset N \text{ s.t. } gd_t^P(N^*, M) < \epsilon\},$$

or simply denoted $V_\epsilon(M)$, and we introduce the **(t, P, d)-ϵ-variations of a motif** M, as the set

$$\{N \in MOT | N \in V_\epsilon^{t,P,d}(M) \text{ or } M \in V_\epsilon^{t,P,d}(N)\}.$$

Note that ϵ-neighbourhoods are not symmetric, in the sense that it is possible that M is in the ϵ-neighbourhood of N while at the same time N is *not* in the ϵ-neighbourhood of M.

Why do we introduce these two concepts since it is clear that each pseudo-metric $gd_{t,n}$ naturally defines a topological space on MOT_n? These spaces on MOT_n actually yield a motif similarity structure on motives *of the same cardinality n* only. But our concept of neighbourhood links similar motives of different cardinalities in such a way, as will be seen later in this paper, that a topological structure is formed and corresponds to the similarity structure of a music piece. Whereas it is the concept of ϵ-neighbourhood that is needed in our construction of the topological space, the ϵ-variation concept will be used for the formalization of the germinal motives (in the definition of presence, content and weight functions).

For example, the (Com, Tr, REd) *- 0.001-neighbourhood*[9] *of motif 2 in Figure 1 includes motif 19, and its 0.4-neighbourhood contains additionally motives 7, 11, 15 and 24. At* $\epsilon = 0.001$*, all the motives in the neighbourhood are linked with a strict submotif relation with same gestalt as motif 2; in this case motives 19 and 2 simply have the same gestalt. For example, motif 2 is in the 0.001-neighbourhood of motif 0 (and of motives 5, 9, 13, 17 and 22) since it contains a submotif, for example, E-F or C-F, that has the same COM-matrix contour as motif 0. At* $\epsilon = 0.4$*, small variations are taken into account. When considering the rigid contour type, motif 2 is in the* (Rg, Tr, REd)*-0.001-neighbourhood of motives 9 and 13, but not in the neighbourhood of motives 0, 5, 17 and 22 since the onset difference in the latter motives and in submotif C-F of motif 2 is not the same, and the smallest non-zero distance value is 0.089.*

If our setup (defined by fixing a contour type t, the action of a paradigmatic group P, and pseudo-metrics d_n on $\Gamma_{t,n}$) fulfils the inheritance property[10] [18,32], the collection of all these neighbourhoods $V_\epsilon^{t,P,d}$ forms a basis for a topology $\mathcal{T}_{t,P,d}$ on the set *MOT* of all motives. In contrast to all other contour types, we observe that the $India$ contour type does not satisfy the inheritance property since it has lost the 'global' motif structure information: see [18,32]. In other words, the assumption of inheritance in our model does not exclude any standard examples that have been shown, except for $t = India$. The topological space for the set $MOT(S)$ of selected motives in a composition S, called the **motivic space of the composition** S, is the relativization of $\mathcal{T}_{t,P,d}$ to $MOT(S)$.

The resulting topology, of type T_0 (i.e. not necessarily Hausdorff), represents the motivic structure of a composition, and the germinal motives are formally represented by the '*most dense*' motives in the space for a given similarity threshold radius $\epsilon > 0$ [16], *as if they were* in a Hausdorff space: a most dense motif M in the space would be a motif with the largest number of motives in its ϵ-neighbourhood. More precisely, with the aim of quantifying ϵ-variations in the topological space, we first record the presence of a motif M in a motif N at radius $\epsilon > 0$ by counting the number of submotives of N with same cardinality as M and that are less than ϵ-distant to M

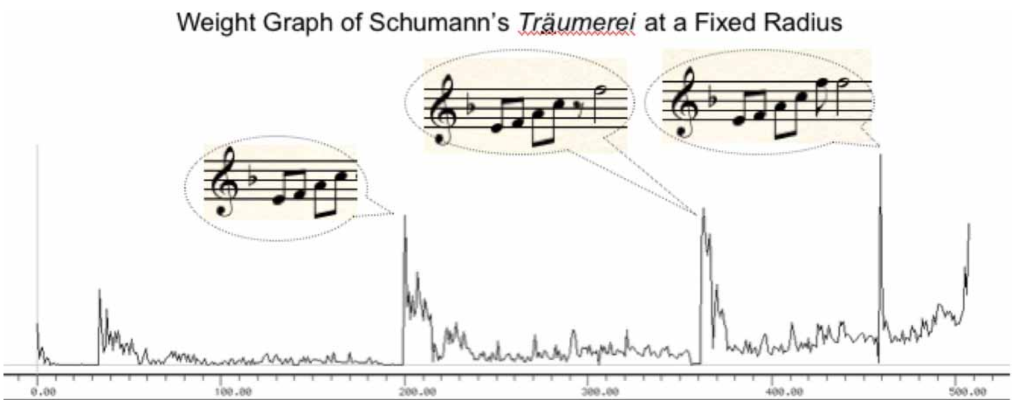

Figure 2. The weight function graph, produced by our visualization program *OM-Melos* [21] in *OpenMusic*, of Schumann's *Träumerei* with motivic topology $\mathcal{T}_{El,Tr,REd}$ (note that $t = El$ is the elastic contour type [16,18]) at fixed similarity radius $\epsilon = 0.001$. In this case, we used the following generalized weight function [16]: $weight(M,\epsilon) := pres(M,\epsilon) \cdot cont^2(M,\epsilon) \cdot card(M)$. The initial set of motives $MOT(S)$ contains 1438 motives of the soprano voice, i.e. the 28 motives shown in Figure 1 together with their submotives down to cardinality 2. This set of motives reduces to 507 (El, Tr)-gestalts. In this graph, the gestalts are along the horizontal axis (lexicographically ordered) and the weights are along the vertical axis. It reads as follows: the higher, the more significant. The three most significant gestalts in this motivic space, at this fixed similarity threshold, are highlighted.

(i.e. using the '$N \in V_\epsilon(M)$'- part of ϵ-variations of M). If $N \in V_\epsilon(M)$, the count is at least one. Since the larger the cardinality difference between M and N the more probable submotives of N are in the ϵ-disc of M, we multiply this count by a factor involving the cardinality difference. And, by letting the motif N vary through the whole space $MOT(S)$ of motives in S, we define the **presence** [16,18] **of a motif M at radius** ϵ as

$$pres_\epsilon(M) := \sum_{N \in MOT(S)} \frac{1}{2^{n-m}} \cdot \#\{N^* \subset N | gd_t(N^*, M) < \epsilon\},$$

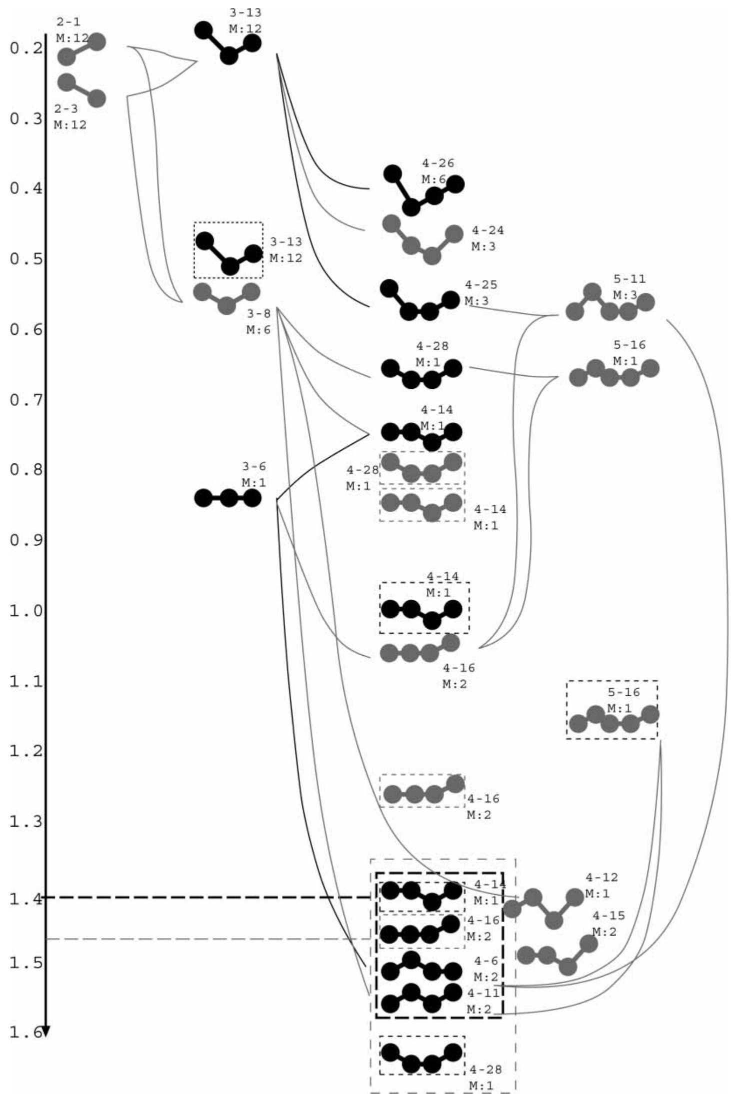

Figure 3. The motivic evolution tree of Bach's *Kunst der Fuge* 8-note main theme, as detailed in [30]: it shows the germinal motif representatives (in black) and the second most omnipresent motives (in grey), for different similarity thresholds (vertical axis, growing from top to bottom). Short connecting lines correspond to submotif relations. Top labels correspond to motif lexicographic number, bottom ones to *multiplicity* (number of motives within the gestalt). This graphic representation of the 8-note main theme's motivic space was calculated for $t = Com$, $P = Tr$ and $d = REd$.

where $m = card(M)$ and $n = card(N)$. Similarly, for quantifying the '$M \in V_\epsilon(N)$'-part of ϵ-variations of M, its **content** [16,18] is defined as

$$cont_\epsilon(M) := \sum_{N \in MOT(S)} \frac{1}{2^{m-n}} \cdot \#\{M^* \subset M | gd_t(M^*, N) < \epsilon\}.$$

Finally, the **weight**[11] **of a motif** M **at radius** ϵ [16,18] is defined as $weight_\epsilon(M) := pres_\epsilon(M) \cdot cont_\epsilon(M)$. A **germinal motif**, given a similarity threshold ϵ, is formalized as a motif with largest weight in the composition. For example, see the *Träumerei*'s weight function graph for a fixed radius in Figure 2. We also graphically represent a germinal motif as a function of the similarity threshold ϵ in what we call a *Motivic Evolution Tree*. Figure 3 shows an example of a motivic evolution tree: Bach's *Kunst der Fuge* 8-note main theme – see [16,30] for details.

3. Réti's approach and its realization within motivic spaces

In this section, we summarize Réti's approach referring to [1], and we redefine it within motivic spaces. As indicated in the introduction, our formalization is achieved by taking a neutral niveau perspective.

3.1. *Definition of a germinal motif according to Réti*

Despite his accurate overview of the evolution of a motif through a composition, Réti uses vague terminology (the same word for different concepts or different words for the same concept), which unfortunately produces hazy deductions, not to say contradictions. Nevertheless, behind his words, Réti's conception of a first sketch for a motivic theory is substantial.

In fact, Réti is not even concerned with a reliable terminology:

> *In general, the author does not believe in the possibility or even desirability of enforcing strict musical definitions.* [1, p. 12]

Therefore, before any attempt to reproduce or implement Réti's motivic analysis, we must take a closer look at his terminology in order to bring consistency to his approach. In trying to interpret the definition of a germinal motif,[12] the very wording of concepts may serve as a starting point.

For Réti, a germinal motif is:

> *... any musical element, be it a melodic phrase or fragment or even only a rhythmical or dynamical feature which, by being constantly repeated and varied throughout a work or a section, assumes a role in the compositional design somewhat similar to that of a motif in the fine arts.* [1, pp. 2–3]

Réti points out a difference between the two concepts of germinal motif and theme by defining a theme as

> *... a fuller (compared to a motif) group or 'period' which acquires a 'motivic' function in a composition's course.* [1, p. 3]

According to him, both a 'germinal motif' and a 'theme' have the same function and therefore have been reduced to almost the same concept. The length distinction between the two concepts has consequences for how and where germinal motives and themes can occur in the composition. A motif being short, it can be easily repeated throughout the composition. It can also be modified and inserted in parts having different moods. These frequent occurrences create a certain unity within the piece. A theme has a similar function but cannot be modified or/and inserted in the composition as easily as the germinal motif.

A crucial idea in Réti's approach is that abstractions have to be made in order to compare themes and germinal motives:

> *Similar liberties were taken in our analysis on several other occasions...by rearranging the design..., by exchanging the octaves in which some phrases were notated in the score.* [1, p. 243].

In a more general way, when talking about classical symphonies, Réti claims that

> *He (the classical composer) strives toward homogeneity in the inner essence... toward variety in the outer appearance. Therefore he changes the surface but maintains the substance of his shapes... Tempo, rhythm, melodic detail, in fact the whole character and mood are altered and adjusted to the form in which the composer conceived them fitting to the new movement.* [1, pp. 13–14]

Réti clearly affirms that we have to perform abstractions when comparing sequences of notes, be it merely the cancellation of loudness of the notes or be it more complex procedures such as the exchange of octave positions.

Réti never uses the word 'gestalt'; instead, he is concerned with comparisons of 'shapes'. The importance of the shape concept is that it carries the idea of an abstraction. Nevertheless, Réti does not have a well-defined use of this concept. In similar situations, such as in examples 328 [1, p. 210] and 334 [1, p. 213], both words 'motif' and 'shape' are used with the same meaning. The reason for this lack of rigour is simply that when comparing motives, natural abstractions like the abolition of loudness of the motif are somehow unconsciously operated. For Réti, starting with shapes for comparison seems trivial.

According to Réti, the shape of a germinal motif is endlessly heard, literally or modified, through the piece. The shape is imitated, varied, transformed or has indirect affinity:

> (1) imitation*, that is literal repetition of shapes, either directly or by inversion, reversion, and so forth;*
> (2) varying*, that is, changing of shapes in a slight, well traceable manner;*
> (3) transformation*, that is, creating essentially new shapes, though preserving the original substance;*
> (4) indirect affinity*, that is, creating an affinity between independent shapes through contributory features.*
>
> *Between imitation and nonrelationship lies a whole complex of features comprising all degrees of structural relationship. Varying, that is, altered repetition, is gradually intensified until it becomes transformation, which forms the central, the most concentrated expression of the thematic phenomenon.* [1, p. 240]

In other words, a germinal motif has a 'smooth shape' meaning that its shape can be imitated or smoothly modified and remains the same, or as Réti asserts, it remains 'identical'.[13] However, the accepted transformations of shapes can reach a limit:

> *...his (Beethoven's) most impressive thematic constructions lie on the border line between being matchless master strokes of transformation or utterances wherein the thematic bond has almost dissolved in that very transformating process by which they were created.* [1, p. 355].

Réti states that a shape cannot be endlessly transformed and still remain identical with its initial shape: the identity has a limit. This brings two issues: where is the limit? And is the identity relation not fully transitive? We discuss these issues in Section 4.

In summary, the **germinal motif of a composition according to Réti** is the first appearance of a sequence of notes whose shape is 'everywhere in the composition' in the sense of imitation, variation, transformation or indirect affinity (of the first shape). Although Réti did not include the word 'gestalt' in his work, the concept is nevertheless omnipresent. Réti's implicit definition of a gestalt is first the choice of an abstraction perspective on a motif, and second all the immalleable transformations (the imitations) of the motif's shape in the chosen perspective.

3.2. *Réti's germinal motif in the motivic space of a composition*

Using the above 'definition' of germinal motif according to Réti, we make a concept correspond to each of its parts within our model of motivic spaces.

3.2.1. '... *sequence of notes*...'

Réti's implicit first step in his motivic analysis of a composition S is the choice of a determined set $MOT(S) \subset MOT(\mathbb{R}^{\{O,P,D,L\}})$ of motives in S, i.e. 'sequences of notes' that he wants to compare with each other in order to find imitations, variations and transformations of the shape of a certain sequence of notes which he will then call a 'germinal motif'. Most motives in $MOT(S)$ are in no way qualified for becoming contributors to Réti's germinal motives. They are just a kind of motivic 'raw material'. Furthermore, Réti naturally allows himself to compare parts (subsequences) of the chosen motives (sequences). He normally does not regress to the single notes, thus he gives himself a lower bound for motif cardinality. This all means that, in the mathematical model, the SEA must be satisfied by $MOT(S)$. Most of the time, as applied by Réti, the limit is the interval, i.e. $n_{\min} = 2$.

3.2.2. '... *whose shape(s)*...'

As seen in Section 3.1, the choice of an abstraction of the motives is the starting point for comparison. Réti's abstraction procedure corresponds in our model to selecting a contour type t, and we model the shape abstraction of a motif $M \in MOT(S)$ by its associated contour $t(M)$.

For example, when discussing the shape of Beethoven's *Ode an die Freude* (example 20 in [1, p. 21]), Réti claims:

> ... *we recognize the second example as literally identical to the first, merely with changed rhythm and transposed to B-flat.* [1, p. 21]

Indeed, their diastematic (and therefore also their COM-matrix) contours are the same, see Figure 4: (0, 1, 2, 0, −2, −1, −2, −2, 0, 2, 2, 0, −2). Due to the rhythmical change and transposition, however, their rigid contours are different. The choice of abstraction (the choice of the contour type) strongly influences our view on a composition, and this is a significant characteristic of this approach and of our model since it takes into consideration different qualities and characters of the composition.

3.2.3. '... *in the sense of imitation*...'

First, we emphasize that declaring that a shape is an 'imitation of another' suggests the choice of a collection of permissible imitation transformations. In our mathematical model this corresponds to the choice of a paradigmatic group P. For example, most of the time strict repetitions (translations in time) are permissible, and it is clear that the choice of P depends on the selected shape abstractions. The implicitly conceived gestalt of a motif in Réti's approach perfectly corresponds to the gestalt of a motif in the motivic space of a composition.

For example, in the first scene of Schumann's *Kinderszenen*, motif I (B-G-F#-E-D) in example 34 [1, p. 32], is imitated (translated in time with respect to any contour type *Com*, *Rg*, *Dia* or

Figure 4. According to Réti, the two versions of Beethoven's *Ode an die Freude* are 'literally identical'. In our topological model, they have same diastematic contour.

Figure 5. The motif B-G-F#-E-D in the first two bars of Schumann's *Kinderszenen* first scene is imitated in the 3rd-4th bars.

India) in measures 3–4 after hearing it for the first time in measures 1–2: see Figure 5. Indeed Réti claims:

> *The piece starts with motif I, which is first repeated literally* ... [1, p. 32–33]

3.2.4. *'...variation, transformation or indirect affinity (of the first shape)'*

The gestalt distance gd_t^P between two motives with the same cardinality in our model gives a measure for their similarity. For Réti, two motives or contours are identical (imitated), closely alike (varied or transformed) or different (not related). In a motivic space, these relations are more subtle. The contours of two motives 'being identical' means having distance 0 in the pseudo-metric d_n on $\Gamma_{t,n}$, 'being closely alike' requires one to know 'how close' they are from each other, and being different requires one to know 'how far' they are from each other. There are no longer only three rough categories, but instead we are given subtly nuanced distance relations (i.e. similarity measures) between motives.

According to Réti, there is a limit to the transformation of a contour such that it still relates to the original contour, i.e. such that it remains 'closely alike'. In the mathematical model, given a distance function d_n 'measuring' the transformation (i.e. the similarity) between two contours with the same cardinality, there is a maximal admissible distance $\epsilon_{\max}$ between the motives. This similarity threshold corresponds to a fixed radius $\epsilon_{\max}$. The latter is a bound such that if two contours with the same cardinality are less than $\epsilon_{\max}$ distant, then they remain 'closely alike'. However, variations, transformations and indirect affinities do not necessarily maintain the number of elements. They correspond in the motivic spaces to three different situations (*Cases I, II and III* in the following) in which a transformation is differentiated from a variation by the need for a larger radius. We comment on indirect affinities at the end of this section. Given two shapes α and β of respective motives M and N, i.e. in our topological model M and N are motives in $MOT(S)$ with $t(M) = \alpha$ and $t(N) = \beta$, there are various possibilities for their degree of nearness.

Case I. If α and β have the same cardinality and are 'closely alike', then in our topological model their motives M and N are in the other's $\epsilon_{\max}$-neighbourhood, that is $M \in V_{\epsilon_{\max}}(N)$ and $N \in V_{\epsilon_{\max}}(M)$. For example, in Palestrina's *Missa Papae Marcelli*, Réti compares the beginning of the two first sections [1, Examples 85 and 86, p. 62], the *Kyrie Eleison* and the Christe Eleison (see Figure 6):

> *... And it is easily seen that these bars [motives N_1 and N_2] are a reiteration of the second half [motif M] of the opening theme...* [1, p. 62]

By taking their diastematic contours and the Euclidean distance, we have $gd_{Dia}^{CP}(M, N_1) = 4$ and $gd_{Dia}^{CP}(M, N_2) = \sqrt{3}$.

Figure 6. In Palestrina's *Missa Papae Marcelli* the two motives N_1 and N_2 are similar to motif M: $gd^{CP}_{Dia}(M, N_1) = 4$ and $gd^{CP}_{Dia}(M, N_2) = \sqrt{3}$ by taking the Euclidean distance function $d = Ed$.

Figure 7. The motif b in the first theme of Brahms's Second Symphony contains in any of its ϵ-neighbourhoods the motif b in the new theme when considering the toroid contour type.

Case II. If α and β are 'closely alike' in β, then the motif N with contour β is in the $\epsilon_{\max}$-neighbourhood of the motif M with contour α, that is $N \in V_{\epsilon_{\max}}(M)$. For example, in the first movement of Brahms's Second Symphony (see Figure 7), Réti compares what he calls the first theme with the new theme [1, Example 118, p. 80]:

> ... *while motifs b and c appear at original pitch; motif b in the new theme merely expanded through a figuration.* [1, p. 80]

The contour $t(b)$ of the first theme is closely alike in the contour $t(b)$ of the new theme, with respect to toroid contour type (identification of pitches with same name: see [16,18]), i.e. the motif b of the first theme contains in its $\epsilon_{\max}$-neighbourhood the motif b of the new theme.

Case III. If α and β are 'closely alike' with a part α^* of α in β, then there is a submotif M^* of motif M with contour α^* such that the motif N with contour β is in the $\epsilon_{\max}$-neighbourhood of M^*, that is $N \in V_{\epsilon_{\max}}(M^*)$. For example, in Brahms's First Rhapsody, Réti compares [1, Example 218, p. 143] a part of the first theme with the second theme; see Figure 8.

Figure 8. By fixing $t = Dia$, $P = CP$ and $d = Ed$, the motif of the second theme in Brahms's First Rhapsody is in the 1.5-neighbourhood of the first theme's submotif F#-E-D-C#. As such, the two themes are 'closely alike'.

Indeed, with regard to *Case III*, we add that Réti affirms

> *... by singling out certain notes,... two shapes which as such have nothing in common can nonetheless become organic parts of an architectural whole through a mediator, a third shape related to both...* [1, p. 240–241]

Note that *Case II* corresponds in our model to the '$N \in V_\epsilon^{t,P,d}(M)$'-part of ϵ-variations, *Case III* corresponds to the '$M \in V_\epsilon^{t,P,d}(N)$'-part, whereas *Case I* is included in both parts. As for indirect affinities, Réti gives an example within Beethoven's Quartet, op. 130, in B-flat major [1, Example 372, pp. 238–239]. He brings together voices in the theme of the Presto movement in order to show the similarity with the Adagio theme; see Figure 9. In our model, this corresponds first to consider in $MOT(S)$ motives with notes from different voices. This increases considerably the collection $MOT(S)$ of motives, but both the model and its implementation *Melos* can deal with such a large collection. Secondly, it involves most likely Cases *II* or *III* or possibly Case *I*, i.e. variations and transformations. In the example from Figure 9, it is an instance of *Case III* with toroid contour type.

Finally, it is important to observe that the gd_t^P distance is a priori defined on *MOT* rather than its subset $MOT(S)$. This means that the calculation of gd_t^P for two motives *within* a given composition is determined by first looking at their contours (following a given abstraction), and then, given a paradigmatic group, by comparing not only all their imitations (in a 'finite' gestalt) within the composition but also all the *a priori* imaginable imitations (in a gestalt). According to the rigid contour type, we easily see that a similar distance '$gd_{Rg}^{P\star}$' defined *only* on $MOT(S)$ would yield absurd results. And, as we saw in the previous examples, these 'imaginary' contours are naturally, but probably also unconsciously, used by Réti.

3.2.5. *'...is everywhere in the composition...'*

The germinal motif's attribute '...is everywhere in the composition...' encompasses other attributes previously formalized for any motives, such as *imitation, variation, transformation or indirect affinity (of the first shape)*. But it also involves them in the global environment, i.e. in the whole composition. More precisely, these imitations, variations and transformations of shapes, whenever they are '*the most dense with regard to any other contours in the composition*', do characterize the germinal motif.

We recall that variations, transformations and indirect affinities of motives relate to ϵ-neighbourhoods, whereas imitations relate to gestalts. But also, in a motivic composition space,

Figure 9. Example of an indirect affinity within Beethoven's Quartet, op. 130: the Adagio theme has an indirect affinity to the Presto theme when bringing the voices together.

ϵ-neighbourhoods of motives are unions of motif gestalts[14] [16], therefore, the characteristic '*the most dense*' in motivic composition spaces has to be understood with ϵ-neighbourhoods corresponding to variations and transformations: *Case II* corresponds to the presence at a radius $\epsilon_{\max}$ of a motif *M in the motivic space of the composition*; *Case III* corresponds to the content at radius $\epsilon_{\max}$ of *M in the motivic space of the composition*; and *Case I* corresponds to both the presence and the content at radius $\epsilon_{\max}$ of M. These three points together correspond to the weight at radius $\epsilon_{\max}$ of *M in the motivic composition space of the composition*, where $\epsilon_{\max}$ is a fixed neighbourhood radius corresponding to a similarity threshold. Therefore Réti's concept '... is everywhere in the composition...' defining the germinal motives of the composition is formalized, in our mathematical model, by the motives with highest weights, at a certain similarity threshold $\epsilon_{\max}$, with respect to all other motives in the motivic space of the composition.

By use of weight functions on motivic spaces in our mathematical model, we obtain a fine description of (relative) motivic significance of all motives in a composition. This enriches the traditional motivic analysis of music, such as Réti's approach, which normally determines the germinal motives, describes them in the composition, but does not deal with other less significant motives. In other words, traditional motivic analysis normally corresponds in our model to considering '*the heaviest, the most significant*', and our model also proposes '*the heavier, the more significant*': see, for example, Figure 2. For a simplified graphical representation of a complete motivic hierarchy, i.e. with an evolving similarity threshold, we refer the reader to the motivic evolution trees as illustrated in Figure 3 or detailed in [16,30].

In summary: Réti's '*sequences of notes*' corresponds to motives, '*whose shape(s)*' correspond(s) to contours of motives with respect to a contour type t, '*imitation*' corresponds to gestalts of a motif (with respect to a paradigmatic group P), '*variation, transformation, or indirect affinity*' correspond to containing or being contained in the $\epsilon_{\max}$-neighbourhood of a motif (with respect to a distance function d for t), i.e. as defined by ϵ-variations, for a certain similarity threshold $\epsilon_{\max}$, and '*is everywhere in the composition*' corresponds to the largest weight value at $\epsilon_{\max}$.

4. Revision of incoherence in Réti's work

The following shows why we suggest that a revision of Réti's theory should be undertaken by use of the topological concepts of our model.

4.1. *Réti's problematic identity relation*

Réti states that a shape cannot be endlessly transformed and still remain identical with its initial shape (see the last quotation in Section 3.1). In other words, the identity relation is not fully transitive for him. This is by itself a contradiction. The commonly accepted concept of 'identity' is by no means non-transitive. However, there are subtle identification problems which, when the analyst is not precise enough (i.e. does more identification than the situation requires) cause contradictions while the essential points drop out. Unfortunately, Réti did not realize the implications of his abuse of the identity concept.

From another point of view, Réti avoided this transitivity problem by constant reference to the first appearance of a motif in the score. Réti usually affirms that a shape is a transformation, imitation or variation of a motif by, most of the time, comparing the shape of the first motif appearance with the considered transformation. In a sense, this 'identity' is an order relation, the first appearance A being 'greater' than all the later appearances, $B, C, \ldots : A > B, A > C, \ldots$ But transitivity is not relevant here because, besides $A > B, A > C, \ldots$, no relation $B > \ldots, C > \ldots$, etc. is considered. So the transitivity question $A > B$ and $B > C \overset{?}{\Rightarrow} A > C$ is of no interest.

However, again, we are confronted with a non-sense definition of the identity: it is not symmetric. In any case, the common concept of identity is violated. Although Réti was imprecise he was not wrong. He abuses the term 'identity', meanwhile he somewhat clarifies the concept of germinal motif, unfortunately hidden by a fuzzy terminology.

We propose to revisit Réti's concept of identity within our topological model by considering identity through the gestalt relation: two motives M and N are considered 'identical' whenever $Ges_t^P(M) = Ges_t^P(N)$. In fact, since $Ges_t^P(M) = Ges_t^P(N)$ if and only if $M \in V_\epsilon(N)$ and $N \in V_\epsilon(M)$ for all $\epsilon > 0$ [16], we can understand the gestalt identity concept from a purely topological perspective. Thus we can see that Réti's violation of the strict concept of identity is related in our model to a violation of one or several parts of the above neighbourhood condition for gestalt identity. It is then undeniable that the identity of motives should be limited to imitations, or in other words, the identification of motives with their gestalts brings consistent statements. The identification with transformations creates a system where every motif can be identified with another motif, and this was indeed one of Réti's concerns all along in his work [1]: the *limitation* of shape identification through transformations.

4.2. *Réti's problem of transformational limitation*

It is clear that eventually every motif can become a transformation (in Réti's sense) of any other motif, but where does one stop? In fact, a recurrent argument for the rejection of Réti's approach was that he had pushed the transformations to unacceptable limits. For example, Meyer states: '*Methodologically … then almost any melody can be related to any other whether within or between works*' [9, p. 62]. Réti himself was not clear about how far to go with this issue, all the more since his poietic perspective cannot rely on sufficient information about the composers' intentions. However, by taking a neutral perspective with our mathematical model, Réti's problem of transformational limitation can be addressed by considering all similarity thresholds, and observing how the germinal motif evolves as a function of the threshold: see the motivic evolution trees in [16,30] and illustrated in Figure 3. In other words, the topological character of our neutral model circumvents the transformational limitation problem: ϵ-neighbourhoods are constructed by use of the similarity threshold ϵ yielding a motivic structure evolving with ϵ and in which motivic significance is formalized by 'the more neighbours, the more significant'.

5. Yoneda lemma in Réti's approach

A strong point in Réti's analysis is certainly that it considers different perspectives on one and the same composition. From the examples presented in this paper, it turns out that Réti relates motives by using different abstractions. In view of the mathematical principles backing our model, Réti's method follows the spirit of Yoneda's lemma [18, Chapter 9]. Intuitively speaking, this lemma states that the knowledge of an object is equivalent to looking at it from all possible points of view. This insight is realized with our approach and its implementation, *Melos*, in that a motivic analysis of a composition generates a family of motivic spaces, each of which correspond to a specific perspective.

However, despite his vicinity to the Yoneda approach to motivic perspectives, Réti did not use the philosophy at a crucial point. He tried to explain a composition uniquely by motivic analysis but he neglected the perspectives of harmonic and rhythmic analyses, as stressed for example by Epstein [11, p. 10]. In this sense, we are fully aware that our model must be complemented by models of harmonic and rhythmic analyses. Such models have been implemented in RUBATO software as MetroRUBETTE (metric analysis) [18,39] and HarmoRUBETTE (harmonic analysis) [18,40].

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Notes

1. This paper summarizes the formalization that has been presented in Buteau's theses [16,17] and in Mazzola's comprehensive book [18].
2. Observe that in previous publications we also used the terminology 'shape' and 'abstract motif' for contours.
3. This terminology refers to Nattiez's use of *paradigm* [35].
4. In Morris's approach [34,36], the sets of equivalent contours, called *csegs*, correspond in our approach to Ges^{Id}_{Com}, where $P = Id$ is the identity group, and the set of transformationally equivalent contours or csegs correspond to Ges^{CP}_{Com}. Note however that the sets in Morris's approach contain contours or csegs, whereas in our formalization the sets (i.e. the gestalts) contain motives which are regrouped by a relation on their contours. Note too that our model includes Morris's generalized contours and the strictly cseg I-relation.
5. Pseudo-metrics are similar to metrics except that the distance between two distinct objects is not required to be positive. In our context, for example, we want two different motives with the exact same COM-matrix to have distance 0.
6. This means that the distance between two motives will stay the same if we, for example, equally translate the two motives. More precisely, this means that for any motives M and N of the same cardinality and for any $p \in P$: $d_t(M, N) = d_t(p \cdot M, p \cdot N)$.
7. The Relative Euclidean distance is the Euclidean distance times a factor in order to take into account the number of notes. For example, if M and N are motives with cardinality n, then $REd_{\mathrm{Com}} := Ed_{\mathrm{Com}}(M, N)/n$. These factors are introduced for a better comparison of similarity values between motif pairs of different cardinalities.
8. Note that our model is of generic type, and that, for example, more perception-meaningful similarity contour measures can be used instead of the Euclidean distance functions.
9. Note that the smallest non-zero distance value between motives of the same cardinality is, in this case, 0.167.
10. This property ensures that if two motives are similar, so are their respective submotives: given a motif M, a submotif M^* of M and $\epsilon > 0$, then there exists a $\delta > 0$ such that for all motives N, if $d_t(M, N) < \delta$, then N has a submotif N^* such that $d_t(M^*, N^*) < \epsilon$.
11. These presence, content and weight functions can all be extended to gestalts [16], i.e. on the quotient space of gestalts, which significantly reduces computations in *Melos*. Also, these functions can be generalized to include other quantifications based on different heuristics as we do in Figure 2; see [16].
12. Réti uses only the word 'motif'. But in order to clarify our discussion about his approach and our model we add the adjective 'germinal' to motif because in our mathematical model any sequence of notes is a motif. The topological construction then determines which of these motives is 'most omnipresent', i.e. the germinal motif, and this is what Réti is dealing with.
13. The words 'identity' and 'identical' are used over and over in the book, and their meaning is not the one that one is accustomed to in mathematical logic. For example, see pages 13, 21, 38, 167, 243, 102, ...
14. By definition, if M and N are imitations of one another, they belong to the same gestalt, and therefore their gestalt distance is 0.

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