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Scale Theory, Serial Theory and Voice Leading

Musical norms evolve over time. In the eleventh century parallel perfect fifths were tolerated, and perfect fourths were considered more consonant than thirds. In the eighteenth century parallel perfect fifths were not tolerated, and thirds were considered more consonant than perfect fourths. And in the twentieth century all manner of traditional prohibitions collapsed as composers revelled in the sense that everything was permitted. Despite such changes, however, certain musical principles remain relatively constant across styles. One of the most important of these dictates that harmonies should, in general, be connected by *efficient voice leading*. That is, notes should be distributed among individual musical voices so that no voice moves very far as harmonies change. Ex. 1 supplies a few representative passages. In each case, the music exploits the shortest possible path between successive chords, in a sense to be defined below.[1] Efficient voice leading is not simply a matter of performers' physical comfort, although that is certainly a factor. It also enables listeners to segregate the auditory stimulus into a series of independent musical lines, which is a prerequisite for understanding polyphony.[2]

Efficient voice leading is so ubiquitous that we tend to take it for granted. But upon reflection it is rather remarkable that composers manage so consistently to find the shortest route from chord to chord. There are 57,366,997,447 distinct voice leadings between two hexachords.[3] There are 288 bijective voice leadings from a half-diminished seventh chord to the twelve dominant seventh chords.[4] Yet almost any composer, theorist or undergraduate student of

Ex. 1 Efficient voice leading in several musical styles: (a) *Ad organum faciendum* (eleventh century); (b) Landini, *Sy dolce non sono*; (c) J.S. Bach, *Das wohltemperierte Klavier* I, Fugue No. 3 in C♯ major, BWV 869 (final cadence, upper four voices only); (d) Wagner, the 'Tarnhelm' motive (upper three voices only); and (e) a common jazz ii–V–I voicing (upper four voices only)

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harmony can quickly find the most efficient voice leading between two hexachords, or a maximally efficient bijective voice leading from a particular half-diminished seventh chord to any of the twelve dominant seventh chords. The puzzle lies in the mismatch between the large number of possibilities and our apparent ease in sorting through them.

We can complicate our puzzle by noting that students receive little explicit conceptual instruction in identifying efficient voice leadings. Music teachers tend to provide examples augmented with relatively uninformative exhortations: we enjoin our students to obey the 'law of the shortest way' – that is, to minimise the overall voice-leading distance between successive chords[5] – without telling them precisely what this means or how to do it. And students comply, easily and for the most part without complaint. How can this be? What is it that even the elementary harmony student knows which allows him or her to identify the smallest voice leading so readily from among such a large field of possibilities? Can we develop explicit algorithms or recipes that simulate the tacit knowledge regularly deployed by musicians?

This article attempts to answer this last question, offering tools for conceptualising voice leading and methods for identifying minimal voice leadings between arbitrary chords. Section I demonstrates that voice leadings, like chords, can be classified on the basis of transpositional and inversional equivalence. This allows us to group the large number of voice leadings into more manageable categories. Sections II–IV turn to voice leadings without *voice crossings.* Crossing-free voice leadings are important, first, because avoidance of voice crossings facilitates the perception of independent polyphonic voices,[6] and second, because for any 'reasonable' method of measuring voice-leading size (to be defined below), removing voice crossings never makes a voice leading larger. Thus we need only consider the crossing-free voice leadings in our search for a minimal voice leading between chords.

Sections II and III consider an even more restrictive, but musically very important class of voice leadings: those which are both crossing-free and bijective. This restriction permits an even more drastic act of cognitive simplification: there are 57,366,997,447 voice leadings between any two hexachords, 31,944 of which are crossing-free – but only *six* of these are both bijective and crossing-free. Bijective voice leadings are important because the number of voices in Western music typically remains constant, at least over small stretches of musical time. (In many cases, this is because of limits on the number of available instruments.) Furthermore, minimal voice leadings are often bijective, although they need not always be. Sections II and III present explicit methods for finding minimal bijective, crossing-free voice leadings. Section IV drops the requirement of bijectivity, showing how to identify a minimal voice leading (not necessarily bijective) between arbitrary chords. Together, these sections formalise and extend one aspect of the sophisticated intuitive knowledge that musicians possess.

It should be emphasised that this article formalises only one small portion of implicit musical knowledge. Musicians typically need to find efficient voice

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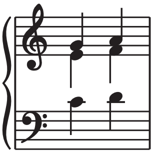
Ex. 2 A maximally efficient voice leading between C major and D minor triads which would be unacceptable to composers of the eighteenth century



leadings while also satisfying additional style-specific constraints. Thus the voice leading in Ex. 2, although minimal, would be unacceptable to an eighteenth-century composer, as it contains parallel perfect fifths. (It would be acceptable to a medieval or modern composer, however.) This paper will not attempt to model style-specific contrapuntal norms. Consequently, I will not discuss forbidden parallels, resolution of leading notes, completion of aggregates or the role of the bass voice in sounding chord roots. Instead, I will focus on a more general skill which I take to be important in a variety of different styles: finding efficient voice leadings between arbitrary chords. My methodological assumption is that finding efficient voice leadings in general is an isolable component of the skill involved in finding efficient voice leadings while also satisfying additional style-specific constraints. Furthermore, the ideas and algorithms in this article can be relatively easily adapted to account for these additional restrictions.

Although the main purpose of this article is to provide tools for thinking about voice leading, I will pause at several points to consider practical applications. Section I (c) shows how to extend 'neo-Riemannian' harmonic ideas, defining generalised 'dualistic chord progressions' and explaining why Riemann's thinking should have come to play such an important role in recent investigations of voice leading. Section II (b) investigates the principles underlying tritone substitution – a technique important in nineteenth-century chromaticism and central to modern jazz. Finally, Section III (b) outlines a new analytic approach to two celebrated nineteenth-century pieces: the Prelude to Wagner's *Tristan und Isolde* and Debussy's *Prélude à l'après-midi d'un faune*. My hope is that these investigations will motivate more analytically and historically minded readers to engage with the somewhat abstract theorising in the rest of the article.

I

Classifying Voice Leadings

(a) Definitions

I use scientific pitch notation, in which C4 is middle C, C3 is an octave below middle C, C5 is an octave above middle C, and so on. Spelling is unimportant:

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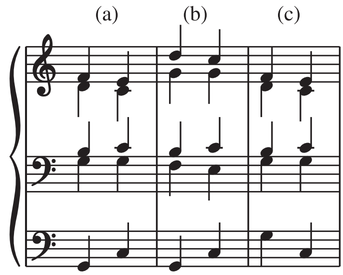
Ex. 3 Three voice leadings between G^7 and C



B♯3, C4 and D𝄫4 are all equivalent names for the same pitch. I also use real numbers to refer to pitches: here, C4 is 60, C♯4 is 61, D4 is 62, and so on. Real numbers in the range $0 \leq x < 12$ also refer to pitch classes: to find the pitch class corresponding to a pitch, divide by 12 and keep only the remainder. Curly brackets {} are used when order is not significant: $\{a, b, c\}$ is the same as $\{b, c, a\}$. Regular parentheses () are used when order is significant: (a, b, c) is distinct from (b, c, a). The term 'chord' refers either to multisets of pitches or pitch classes, as the context requires.[7]

Ex. 3a depicts a *pitch-space voice leading*, or *voice leading between pitch sets*: five melodic voices moving from the chord {G2, G3, B3, D4, F4} to {C3, G3, C4, C4, E4}. The voice leading can be represented using the notation (G2, G3, B3, D4, F4) → (C3, G3, C4, C4, E4). This indicates that the voice sounding G2 in the first chord moves to C3 in the second chord, that the voice sounding G3 in the first chord continues to sound G3 in the second chord, and so on. The order in which the voices are listed is not significant; what matters is the progression of each voice. Consequently, one could just as well represent Ex. 3a as (F4, D4, G2, B3, G3) → (E4, C4, C3, C4, G3). Formally, a voice leading between pitch sets A and B is a set of ordered pairs of pitches (a, b) such that a is a pitch in A and b is a pitch in B, and each pitch of each chord appears in at least one pair. A voice leading between pitch sets is *bijective* if each pitch of A appears as the first element in precisely one pair, and each pitch of chord B appears as the second element in precisely one pair.[8]

The voice leading in Ex. 3b is closely related to that in Ex. 3a; all that has changed is the octave in which some voices appear. We can represent what is common to the two voice leadings by writing $(G, G, B, D, F) \xrightarrow{5,0,1,-2,-1} (C, G, C, C, E)$. This notation indicates that one of the voices containing G, whatever octave it may be in, moves up five semitones to C; the other voice containing G is held over into the next chord; the B moves up by semitone to C; and so on. This sort of octave-free voice-leading schema can be described as a *pitch-class voice leading*, or *voice leading between pitch-class sets*.[9] The numbers above the arrow, here (5, 0, 1, –2, –1), determine a set of *paths in pitch-class*

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Ex. 4 Two instances of the same compositional schema

space.[10] The pitch-space voice leadings in Exs. 3a and b are *instances* of the pitch-class voice leading $(G, G, B, D, F) \xrightarrow{5,0,1,-2,-1} (C, G, C, C, E)$. Ex. 3c is not an instance of this pitch-class voice leading, since here G moves to C by seven descending semitones rather than five ascending semitones. (The specific path travelled by each voice matters!) If the number of semitones x moved by each voice lies in the range $-6 < x \leq 6$, then I will omit the numbers over the arrow. For example, I will write $(G, G, B, D, F) \rightarrow (C, G, C, C, E)$ for the pitch-class voice leading in Exs. 3a and b. This indicates that each voice moves by the shortest possible path to its destination, with the (arbitrary) convention being that tritones are assumed to ascend.[11] Formally, a voice leading between pitch-class sets A and B is a set of ordered pairs (a, p), where a is a pitch class in A, p is a real number which determines a path in pitch-class space from a to a pitch class b in B, and the voice leading contains a path associating each pitch class in one set with some pitch class in the other.[12]

Pitch-class voice leadings can be understood as convenient abstractions of the sort composers regularly deploy. Without such abstractions, musicians would need to conceptualise very similar voice leadings – such as those in Ex. 4 – as being completely distinct, with no relation to one another. It is far simpler, and far more practical, to understand the two voice leadings in Ex. 4 as instances of a single underlying compositional principle: *you can transform a C major triad into an F major triad by moving the E up by one semitone and the G up by two semitones.* Pitch-class voice leadings are simply tools for formalising such general principles and thus for modelling one aspect of the composer's craft. Of course, in actual compositional contexts, pitch-class voice leadings will necessarily be represented by specific pitches. But as we shall shortly see, composers often have a great deal of freedom over how to dispose pitch classes in register, and in such contexts pitch-class voice leadings are very useful. For this reason, the present article will be largely concerned with pitch-class voice leadings, rather than their more specific pitch-space counterparts.

A pitch-space voice leading is *crossing-free* if all pairs of voices satisfy the following criterion: if voice A is below voice B in the first chord, then voice A is not above voice B in the second.[13] A pitch-class voice leading is crossing-free if and only if all of its instances are crossing-free. The voice leadings in Exs. 3a and b are crossing-free when considered either as pitch-space voice leadings or as pitch-class voice leadings. The voice leading in Ex. 3c is crossing-free when

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considered as a pitch-space voice leading, but not when considered as a pitch-class voice leading. To see why, transpose the top voice down an octave: the transposed voice leading is an instance of the same pitch-class voice leading, but it has a crossing. Note that the term 'voice crossing', as applied to pitch-class voice leadings, is a technical term of art: ordinary musical discourse uses 'voice crossing' only to refer to crossings in pitch space.

Music theorists have proposed many ways to measure the size of a voice leading. Appendix A lists the main alternatives. Such proposals are at best approximations, attempts to make composers' intuitions, as embodied in Western musical practice, explicit. For this reason I will not adopt any single method. Instead, I will require that voice-leading metrics satisfy two intuitive constraints. First, the size of a voice leading should be a non-decreasing (monotonic) function of the distances moved by the voices.[14] Second, removing voice crossings should never make a voice leading larger: the pitch-space voice leading $(p_0, p_1, \ldots, p_{n-1}) \rightarrow (q_0, q_1, \ldots, q_{n-1})$ should be no larger than $(p_0, p_1, \ldots, p_{n-1}) \rightarrow (q_1, q_0, \ldots, q_{n-1})$ when $p_0 < p_1$ and $q_0 < q_1$.[15] Every existing music-theoretical method of measuring voice-leading size satisfies both requirements. I have elsewhere argued that any reasonable metric of voice-leading size *must* satisfy these constraints.[16] Rather than repeating that argument here, I will simply assume that one of the existing music-theoretical metrics – or some other metric satisfying the two constraints – is satisfactory. For such metrics, there is always a minimal voice leading between any two chords (both in pitch space and in pitch-class space) which is crossing-free.

This article's central question is: 'Given two pitch-class sets, how can we find the most efficient voice leading between them?' (Note: in what follows, I will often use the term 'voice leading' as shorthand for 'voice leading in pitch-class space'.) It should be emphasised that composers regularly confront this question. Suppose, for example, a composer has written the music shown in Ex. 5a: he or she has chosen to write a root-position V^7–I progression with five voices, and has determined the registral position of the notes in the first chord, as well as the registral position of the lowest note in the second chord. The question then becomes: 'How can the composer move the upper voices most efficiently so as to form a C major chord?' Since the notes of the second chord can be placed in any register, this question is equivalent to the question 'What is the most efficient (four-voice) pitch-class voice leading from G^7 to C?' A similar question might be asked by a composer who, having written the chord in Ex. 5b, would like to resolve it to a dominant seventh chord by maximally efficient voice leading. Which dominant seventh should the composer choose, and what is the optimal voice leading? Again, the freedom to dispose pitch classes in register makes it possible to use pitch-class voice leadings to answer this question.

To be sure, the question 'How do I find the minimal voice leading between two pitch-class sets?' is not the only one which must be answered when composing. In many tonal styles, for example, the bass voice often moves by

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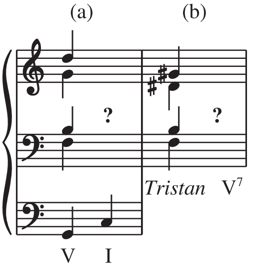
Ex. 5 Two practical compositional questions



leap; thus, efficient voice leading tends to occur only between the upper voices in the musical texture (Ex. 1c–e). Likewise, in many musical styles, some voice leadings, although efficient, will be forbidden – perhaps because they involve undesirable parallels (Ex. 2). Finally, composers often choose non-minimal voice leadings, if only for the sake of variety. Thus finding the most efficient voice leading between pitch-class sets is only one of the skills necessary to compose in a particular musical style. The goal of this article is to investigate this particular musical skill. This should be considered a prelude to, rather than a replacement for, more detailed study of the contrapuntal norms of specific styles.

(b) Classifying Voice Leadings

Pitch-class set theory categorises equal-tempered chords on the basis of transpositional and inversional equivalence. This section will extend these classifications to voice leadings. The result can be described, doing only moderate violence to the English language, as the *set theory of voice leadings*. As we will see, there is a crucial difference between traditional set theory and the 'set theory of voice leadings'. This is because there are two ways in which transposition and inversion can act upon a voice leading: *uniformly*, where the same transposition or inversion applies to both chords; and *individually*, where different transpositions or inversions apply to the chords. This distinction between individual and uniform relatedness does not appear in traditional set theory, which is concerned only with the classification of isolated pitch-class sets.[17]

Intuitively, the voice leadings (C, E, G) → (C, F, A) and (G, B, D) → (G, C, E) are very similar: they exhibit the same musical pattern at different transpositional levels, holding the root of a major triad fixed, moving the third up by semitone and moving the fifth up by two semitones (Ex. 6a). Such voice leadings can be said to be *uniformly transpositionally related*, or *uniformly T-related*.[18] We can transform any instance of a voice leading into an instance of a uniformly T-related voice leading simply by transposing all notes in pitch space.

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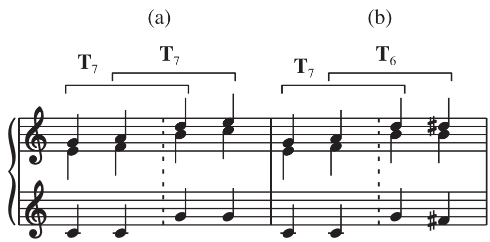
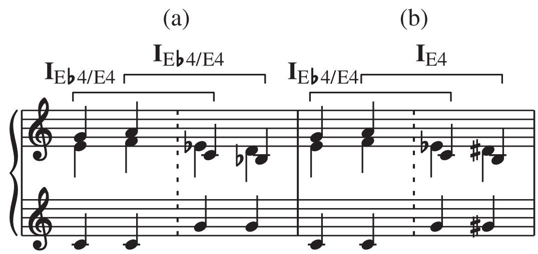
Ex. 6 Individually and uniformly T-related voice leadings



Ex. 7 Individually and uniformly I-related voice leadings



The voice leadings (C, E, G) → (C, F, A) and (G, B, D) → (F♯, B, D♯) are also similar, albeit slightly less so; each maps the root of the first chord onto the fifth of the second, the third of the first chord to the root of the second and the fifth of the first chord to the third of the second. Ex. 6b shows that we can transform one voice leading into the other by applying a *different* transposition to each chord: we transpose (C, E, G) by *seven* semitones to produce (G, B, D); but we transpose (C, F, A) by *six* semitones to produce (F♯, B, D♯). For this reason, such voice leadings can be said to be *individually transpositionally related*, or *individually T-related*. The possibility of individual transpositional relatedness marks the main difference between the 'set theory of voice leadings' and traditional set theory.

The distinction between individual and uniform relatedness extends naturally to inversion. Ex. 7a shows that an instance of the voice leading (C, E, G) → (C, F, A) can be inverted to produce an instance of (G, E♭, C) → (G, D, B♭): here, inversion around E♭4/E4 sends C4 to G4, E4 to E♭4, F4 to D4, G4 to C4 and A4 to B♭3. More generally, a single inversion will transform any instance of the voice leading (C, E, G) → (C, F, A) into an instance of the voice leading (G, E♭, C) → (G, D, B♭). These voice leadings are therefore *uniformly inversionally related*, or *uniformly I-related*. Similarly, the voice leadings (C, E, G) → (C, F, A) and (G, E♭, C) → (G♯, D♯, B) are *individually inversionally related*, or *individually I-related*, since it takes *two* inversions to transform an

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instance of the first voice leading into an instance of the second. Ex. 7b demonstrates: here, we invert the first chord around E♭4/E4 but we invert the second chord around E4. (Note: individual inversional relatedness still requires that each chord in the first voice leading be related by some inversion to the corresponding chord in the second voice leading; it is not permissible to invert only one of the two chords.[19])

The musical significance of the uniform relationships is clear: uniformly T-related voice leadings exhibit the underlying musical pattern at different transpositional levels, while uniformly I-related voice leadings are 'mirror images' of one another. Uniformly T- or I-related voice leadings always move their voices by the same distances, although possibly in opposite directions.[20] The musical significance of *individual* T and I relationships is perhaps less clear: the voice leading $(\mathrm{C}, \mathrm{E}, \mathrm{G}) \xrightarrow{0,1,2} (\mathrm{C}, \mathrm{F}, \mathrm{A})$suggests a standard I–IV chord progression, whereas the voice leading $(\mathrm{C}, \mathrm{E}, \mathrm{G}) \xrightarrow{-1,0,1} (\mathrm{B}, \mathrm{E}, \mathrm{G}\sharp)$ sounds like an instance of Schubertian chromaticism. Nevertheless, there is a clear sense in which two individually T-related voice leadings are similar: as we will shortly see, each voice leading relates structurally analogous notes, and their voices move by the same distance up to an additive constant.[21] Analogous points can be made about individually I-related voice leadings.[22] I will suggest that it can be quite profitable to focus on these individual voice leading relationships, as they allow us to sort the overwhelming multitude of voice leading possibilities into a much smaller set of categories. This will be useful when we are searching for minimal voice leadings between chords. It can also alert us to new relationships within and between musical works.

(c) Excursus I: Dualism and the Theory of Voice Leading

We can describe our new 'set theory of voice leadings' using the inversionally symmetrical ('dualistic') language of Oettingen and Riemann. Following these theorists, let us label minor triads *from top to bottom*, so that G is the 'root', E♭ the 'third' and C the 'fifth' of the minor triad {C, E♭, G}. Let us also define the 'principal direction' of the triad as the direction in which the root moves by seven semitones to reach the fifth. Using this labelling, we can say that (C, E, G) → (C, F, A) and (G, E♭, C) → (G, D, B♭) each hold the root of the first chord constant so that it becomes the fifth of the second chord, move the third of the first chord by semitone in the principal direction so that it becomes the root of the second chord, and move the fifth of the first chord by two semitones in the principal direction so that it becomes the third of the second chord. The lesson here is general. Uniformly T- or I-related voice leadings can always be described using precisely the same terms, as long as we conceive of chords dualistically: if we label the elements of set class A by arranging them in the ordering $(a_0, a_1, \ldots, a_{n-1})$, then we must label the elements of set-class $\mathbf{I}_x(A)$ based on their position in the *inversion* of this ordering, $(\mathbf{I}_x(a_0), \mathbf{I}_x(a_1), \ldots, \mathbf{I}_x(a_{n-1}))$.[23]

Thus the language of Riemannian dualism arises naturally when we attempt to classify voice leadings on the basis of transpositional and inversional

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equivalence. This is, on one level, unsurprising: Oettingen and Riemann were among the first theorists to emphasise inversion, and they explicitly used transposition and inversion to categorise chord relationships. Nevertheless, there is something somewhat unexpected about the appearance of dualistic concepts in our current enquiry. For Riemannian dualism is fundamentally a *harmonic* theory – a theory whose principal concern is chords ('*Klangs*') and their relations. By contrast, voice leading is a *contrapuntal* matter and involves lines, melodies and voices. Why, exactly, should dualistic harmonic concepts be so useful to an investigation that is essentially contrapuntal?

It is worth pausing to consider this question. Let us define a *chord progression* as a succession of unordered chords.[24] For instance, the chord progression {C, E, G, B♭} ⇒ {E, G♯, B}, or C^7 ⇒ E, associates the unordered collection of pitch classes {C, E, G, B♭} with the unordered collection of pitch classes {E, G♯, B}.[25] Unlike voice leadings, chord progressions do *not* associate the elements of their respective chords: whereas the voice leading (C, E, G) → (B, E, G♯) maps C onto B, E onto E and G onto G♯, the chord progression {C, E, G} ⇒ {B, E, G♯} does not specify any particular mapping between its notes. A chord progression is simply a sequence of chords or *Klangs*, each considered as an indivisible harmonic unit. Consequently, while any voice leading can be uniquely associated with a particular chord progression, the converse is not true.

Clearly, chord progressions, just like voice leadings, can be said to be individually or uniformly T- and I-related, depending on whether it takes one or two such operations to transform one into another. I will say that two chord progressions are *dualistically equivalent* if they are uniformly T- or I-related. Thus, a chord progression $A \Rightarrow B$ is dualistically equivalent to $\phi(A) \Rightarrow \phi(B)$ for any transposition or inversion ϕ. For example, the chord progression C^7 ⇒ E is dualistically equivalent to a♭^{ø7} ⇒ g, since $\mathbf{I}_6(C^7)$ = a♭^{ø7} and $\mathbf{I}_6$(E) = g.[26] Similarly, both progressions are dualistically equivalent to D^7 ⇒ F♯, a^{ø7} ⇒ g♯, and so on. It is easily seen that two progressions between major and minor triads are dualistically equivalent if and only if they are instances of the same Riemannian *Schritt* or *Wechsel*.[27] Our definition simply extends the *Schritt*s and *Wechsel*s, allowing us to say, for any two chord progressions, whether they are dualistically equivalent or not.[28]

What does dualistic equivalence, so defined, have to do with voice leading? We can now answer this question clearly. Transposition and inversion are the only *distance-preserving* operations defined over all of pitch and pitch-class space.[29] Therefore uniformly T- and I-related voice leadings move their voices by precisely the same distances, and are (for any reasonable metric) the same size. For instance, the C dominant seventh chord can be linked to the E major triad by the semitonal voice leading (C, E, G, B♭) → (B, E, G♯, B). Here two voices move up by semitone, a third moves down by semitone and the fourth does not move. Since a♭^{ø7} ⇒ g is dualistically equivalent to C^7 ⇒ E, we can construct a precisely analogous voice leading between the A♭ half-diminished

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seventh chord and the G minor chord: (A♭, C♭, D, F♯) → (G, B♭, D, G). Here two voices move *down* by semitone, a third moves up by semitone and the fourth does not move. From the standpoint of the theory of voice leading, then, dualistic equivalence is important because it identifies pairs of chords which can be linked by structurally analogous voice leadings that are of exactly the same size.

Riemannian dualism re-entered contemporary music theory in the work of David Lewin and Brian Hyer.[30] This early 'neo-Riemannian theory' was largely harmonic in character, treating 'neo-Riemannian transformations' as functions between chords. Richard Cohn was the first to note that dualistic harmonic ideas have a natural application to questions about voice leading: Cohn pointed out, for example, that two consonant triads can be linked by single-semitone voice leading only if they are related by the neo-Riemannian L and P transformations.[31] This simple but profound observation led to an explosion of interest in voice leading, much of it conducted in dualistic, neo-Riemannian terms.[32] Surprisingly, however, there have been few attempts to explain why dualistic harmonic terminology should play such a central role in fundamentally contrapuntal investigations. Our 'set theory of voice leadings', now extended to chord progressions, provides a clear account of why this is so.

II

Crossing-Free Voice Leading and Scalar Transposition

(a) The Scalar Interval Matrix

We will now use ideas from scale theory to investigate crossing-free voice leadings. Sections II and III consider voice leadings which are both crossing-free and bijective – mapping every element of the source chord onto precisely one element of the target and vice versa.[33] The musical justifications for this restriction have already been discussed. Section IV broadens the focus to include non-bijective crossing-free voice leadings.

The first task is to describe how to assign scale-degree numbers to an arbitrary multiset. Define the *ascending distance* from pitch class a to b as the length of the shortest non-descending path in pitch-class space from a to b.[34] Let A be a multiset of pitch classes, and let $(a_0, a_1, \ldots, a_{n-1})$ order the objects of A based on increasing ascending distance from an arbitrarily chosen pitch class a_0.[35] We can identify each object's position in the list with its *scale degree*. By convention, we apply the same scale-degree numbers to all the transpositions of A: thus, $a_0 + x$ is the first scale degree of $\mathbf{T}_x(A)$, $a_1 + x$ is the second, and so on.[36] (Note that any chord can be considered a scale; the term carries no particular implications or ontological import beyond signifying that the chord's notes have been assigned scale-degree numbers.) In the special case where the scale contains multiple instances of some pitch class, we need to define a

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variant of the ascending distance as follows: the *ascending scalar distance* from a_i to a_j is equal to the ascending distance from a_i to a_j unless the two pitch classes are equal and $j < i$, in which case it is equal to 12. Thus the ascending scalar distance from the third to the second scale degree of (a_0, a_1, a_2) = (C, G, G) is 12 rather than 0.[37]

A *scalar transposition* is a crossing-free voice leading from a chord to itself which moves every note by the same number of scale degrees.[38] (Mathematically, it maps each scale degree a_i onto the scale degree $a_{i+c \pmod n}$, c scale degrees away from a_i.) It can be shown that any bijective crossing-free voice leading from A to any of its transpositions can be written as a combination of two crossing-free voice leadings:

$$(a_0, a_1, \ldots, a_{n-1}) \xrightarrow{d_0, d_1, \ldots, d_n} (a_{0+c \pmod n}, a_{1+c \pmod n}, \ldots, a_{n-1+c \pmod n}) \xrightarrow{x, x, \ldots, x} (a_{0+c \pmod n} + x, a_{1+c \pmod n} + x, \ldots, a_{n-1+c \pmod n} + x)$$

The first voice leading is a scalar transposition which moves each note upwards by c scale degrees; the second is a chromatic transposition which moves each note by x semitones. The distances d_i are the ascending scalar distances from a_i to $a_{i+c \pmod n}$. Appendix B provides a proof.

This is a simple but powerful formula. It tells us that bijective crossing-free voice leadings between transpositionally related chords are individually T-related to scalar transpositions. Furthermore, we know from Section I (a) that, for any reasonable method of measuring voice-leading size, there is always a minimal voice leading between any two chords which is crossing-free. This voice leading will combine a scalar transposition (by c scale steps) with a chromatic transposition (by x semitones) which comes as close as possible to neutralising it.

Fig. 1 illustrates. Fig. 1a presents a 'transpositional' voice leading which sends each member of the C major triad up four semitones to the corresponding member of the E major triad. Fig. 1b precedes this chromatic transposition by scalar transposition down one scale step – moving G to E, E to C and C to G. Since addition is commutative, it does not matter which voice leading occurs first, as Fig. 1c shows. The two voice leadings combine to produce the minimal bijective voice leading between the C and E major chords.[39] For the major triad, scalar transposition downwards by step comes closest to negating chromatic transposition upwards by four semitones. Equivalently, scalar transposition down by scale step provides the best approximation to $\mathbf{T}_{-4}$.[40] Fig. 1d shows the minimal voice leading between C and E♭ dominant seventh chords. This combines scalar transposition by one descending step with chromatic transposition by three ascending semitones.

The investigation of bijective crossing-free voice leadings is greatly facilitated by what I will call the *scalar interval matrix*. The rows of this matrix identify the size of a multiset's ascending scalar intervals. Equivalently, its rows record the effect of scalar transposition upon each of the multiset's objects.

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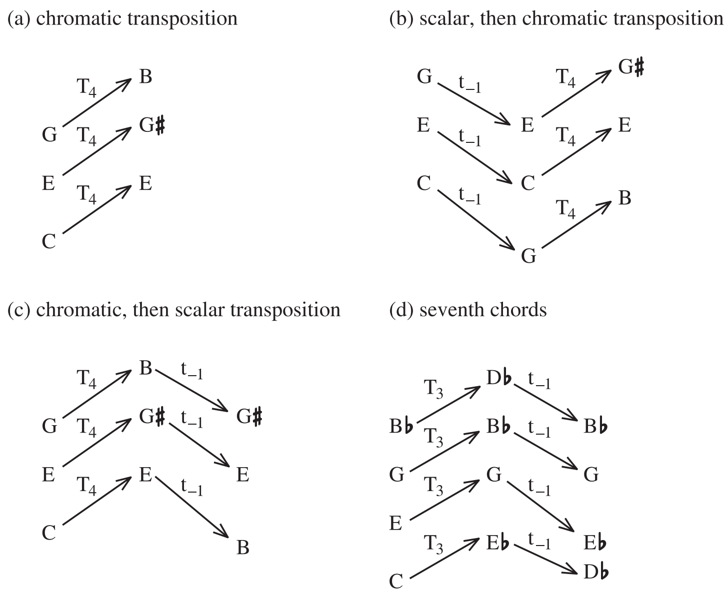
Fig. 1 Scalar and chromatic transposition



t_{-1} indicates scalar transposition downward by step

Definition 1. Let A be a multiset with n objects, labelled $(a_0, a_1, \ldots, a_{n-1})$ and ordered by increasing ascending distance from pitch class a_0. The *scalar interval matrix* associated with A has elements $M_{i,j}$ equal to the ascending scalar distance from a_j to $a_{i+j \text{ (mod } n)}$.[41]

The element $M_{0,0}$ records the ascending scalar distance from a_0 to a_0, $M_{1,2}$ records the ascending scalar distance from a_2 to a_3, $M_{2,1}$ records the ascending scalar distance from a_1 to a_3, and so on.[42] Musically, the first row of the interscalar interval matrix records the size of the zero-step scalar intervals, the second row records the size of the one-step scalar intervals, and so on. Students of serial theory will immediately recognise that the scalar interval matrix is closely related to a 'rotational array'.[43] This means that the questions we shall be asking about voice leading have close analogues in the realm of serial composition, as will be explored in Section IV (b).[44]

Fig. 2 lists the scalar interval matrix for the diatonic collection. Here I have arbitrarily chosen the traditional major-scale tonic as the first scale degree. The first row shows, trivially, that all of the zero-step intervals in the scale have chromatic size 0. The second row shows that the one-step intervals in the scale, starting with the first scale degree, are of successive size (2, 2, 1, 2, 2, 2, 1); that is, transposing a major scale up one scale step shifts the first scale degree upwards by two semitones, the second scale degree by two semitones, the third by one semitone and so forth. Likewise, the third row shows that the two-step

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Fig. 2 The scalar interval matrix for the diatonic collection

0 steps	0	0	0	0	0	0	0
1 step	2	2	1	2	2	2	1
2 steps	4	3	3	4	4	3	3
3 steps	5	5	5	6	5	5	5
4 steps	7	7	7	7	7	7	6
5 steps	9	9	8	9	9	8	8
6 steps	11	10	10	11	10	10	10

(The traditional tonic of the major scale has been arbitrarily selected as the first degree.)

intervals in the scale, starting with the first scale degree, are of size (4, 3, 3, 4, 4, 3, 3). The rest of the rows identify the size of the scale's three-, four-, five- and six-step intervals.

Scalar interval matrices make it easy to calculate the combined effects of scalar and chromatic transposition: the rows of the matrix represent scalar transpositions; we represent chromatic transposition by adding a constant to these values. Thus we can use Fig. 2 to calculate the effect of transposing the major scale upwards by three steps and downwards by five semitones:

5	5	5	6	5	5	5	diatonic transposition by 3 ascending steps
–5	–5	–5	–5	–5	–5	–5	chromatic transposition by 5 descending semitones
0	0	0	1	0	0	0	

The result signifies that the fourth scale degree ascends by one semitone and all other scale degrees remain fixed: in other words, it describes the voice leading (C, D, E, F, G, A, B) → (C, D, E, F♯, G, A, B). Mathematically, one can add the vector (0, 0, 0, 1, 0, 0, 0) to (0, 2, 4, 5, 7, 9, 11) (which lists the elements of the C major scale in scale-degree order) to produce (0, 2, 4, 6, 7, 9, 11). The result contains the seven pitch classes of the G major collection, starting with the fourth scale degree. To perform further calculations with this vector we would need to rotate it so that it is once again in scale-degree order.

Note that we perform this calculation with *real numbers*, representing paths in pitch-class space, rather than mod-12 pitch-class intervals. This permits us to distinguish transposition by ascending perfect fifth (+7) from transposition by descending perfect fourth (–5). Indeed, each of the numbers {… , –17, –5, +7, +19, …} represents a distinct path in pitch-class space mapping C onto G. The advantage of using real numbers, rather than mod-12 intervals, is that real numbers allow us to distinguish ascending from descending motion: in the previous paragraph, the vector (0, 0, 0, 1, 0, 0, 0) indicates that one pitch class moves *up* by semitone, and not that it moves down by eleven semitones. Traditionally minded readers, who may feel uncomfortable with paths in pitch-class space, can convert to pitch-class intervals simply by performing all calculations mod 12.

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Let us now use Fig. 2 to answer some simple questions about voice leading. The point of these questions is not yet to produce novel musical insights; it is just to develop a feel for working with the scalar interval matrix. Keep in mind that a minimal *crossing-free* voice leading is also a minimal voice leading, since no reasonable metric of voice-leading size can favour a crossed voice leading over all of the uncrossed alternatives.

1. What is the minimal bijective crossing-free voice leading between two diatonic collections a minor third apart? To answer this question we search for that row of the scalar interval matrix whose values will best neutralise the chromatic transposition: if we transpose downwards by three semitones, we look for the row whose values come closest to the value 3, since $3 + -3 = 0$. (The relevant sense of 'closeness' here is given by a specific metric of voice-leading size; see Appendix A.) According to any reasonable metric, the third row is closest to 3. Adding (4, 3, 3, 4, 4, 3, 3) to (–3, –3, –3, –3, –3, –3, –3), we obtain (1, 0, 0, 1, 1, 0, 0), indicating that the first, fourth and fifth scale degrees move upwards by semitone. This is the voice leading (C, D, E, F, G, A, B) → (C♯, D, E, F♯, G♯, A, B).

2. What is the minimal bijective crossing-free voice leading which transforms the Ionian mode into the Locrian? The minimal voice leading between two transpositions of a multiset does not depend on our identification of scale degree 1. Therefore, any minimal crossing-free voice leading between the Ionian and Locrian modes will equally be a minimal crossing-free voice leading between Dorian and Ionian, between Phrygian and Dorian, and so on. This allows us to rephrase the question in a more general way: *Which chromatic transposition best neutralises scalar transposition downwards by scale step?* Here we begin with the last row of Fig. 2: (11, 10, 10, 11, 10, 10, 10). This represents scalar transposition upwards by six steps; to model scalar transposition downwards by step, subtract 12 from each entry, producing (–1, –2, –2, –1, –2, –2, –2). The constant transposition which best neutralises this diatonic transposition is (2, 2, 2, 2, 2, 2, 2), corresponding to transposition upwards by two semitones.[45] Adding the vectors produces (1, 0, 0, 1, 0, 0, 0), indicating that the first and fourth scale degrees move up by semitone. This voice leading is the maximally efficient voice leading which transforms C Ionian into C♯ Locrian – and equally, D Ionian into D♯ Locrian, E Ionian into E♯ Locrian, and so forth.

3. Which diatonic collections are linked by the smallest bijective voice leading? Here we look for those rows of Fig. 2 whose elements most closely approximate a constant value. These are the fourth and fifth rows: each changes a single pitch class by a single semitone when combined with the appropriate chromatic transposition. These voice leadings underlie the familiar practice of modulation from tonic to dominant keys.[46]

4. What is a minimal bijective voice leading which maps the C of a C diatonic collection onto the F of a D♭ diatonic collection? This question is not as easy as it seems: by specifying the mapping C → F, we have lost the ability to make use

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of the 'no crossings' principle. This is because the minimal bijective voice leading which sends note x of collection A into note y of collection B may involve a crossing: for example, the minimal bijective voice leading which sends note C of the C diatonic collection into note B of the G diatonic collection is (C, D, E, F, G, A, B) → (B, D, E, F♯, G, A, C). To answer our question we must therefore ask: What is a minimal bijective voice leading which sends the C diatonic collection, minus the note C, into the D♭ diatonic collection, minus the note F? We then combine this voice leading with the voice leading C → F to obtain the desired voice leading. Since this procedure will in general require us to identify minimal voice leadings between arbitrary multisets, we do not yet have the tools to pursue it. However, the techniques discussed in Section III will allow us to answer such questions.

5. *Why is it that some chords have smaller voice leadings to their transpositions than others do?* The minimal voice leading between the C major and E major triads, (C, E, G) → (B, E, G♯), moves two notes by semitone and holds one note fixed. By contrast, the minimal bijective voice leading from {C, C♯, D} to its $\mathbf{T}_4$ form, {E, F, F♯}, involves at least *twelve* total semitones of motion among the three voices. What accounts for this difference? Why is the major triad capable of participating in such efficient voice leadings to its transpositions?

Although a rigorous answer to this question is beyond the scope of this article, the basic principles are readily explained.[47] Recall that a minimal bijective crossing-free voice leading combines a chromatic transposition with the scalar transposition which comes closest to neutralising it. Thus, if a scalar transposition is almost equal to transposition by x semitones, then a chord will have an efficient voice leading to its $\mathbf{T}_{-x}$ form. In general, the more evenly the chord divides the octave, the more closely its scalar transpositions will resemble chromatic transpositions.[48] For perfectly even chords, such as the tritone, augmented triad, diminished seventh chord and whole-tone scale, scalar transpositions are always exactly equal to chromatic transpositions; hence they can be used to offset chromatic transposition precisely. For nearly even chords, such as the perfect fifth, major triad, dominant seventh chord and major scale, scalar transposition is *almost* equal to chromatic transposition.[49] Here, scalar transposition almost offsets chromatic transposition, producing efficient voice leading.[50]

(b) Excursus II: Tritone Substitution

Let us now use scalar interval matrices to investigate an important but poorly understood musical practice: 'tritone substitution', in which a dominant seventh chord is replaced by its tritone transposition. We will see that tritone substitution is as much a contrapuntal phenomenon as a harmonic one: the substitution works not simply because tritone-related dominant seventh chords share common tones, but also because their remaining notes can be linked by efficient voice leading. Consequently, a dominant seventh chord can be replaced by its tritone transposition without seriously disrupting a piece's harmonic or contrapuntal fabric. Scalar interval matrices will show us not only

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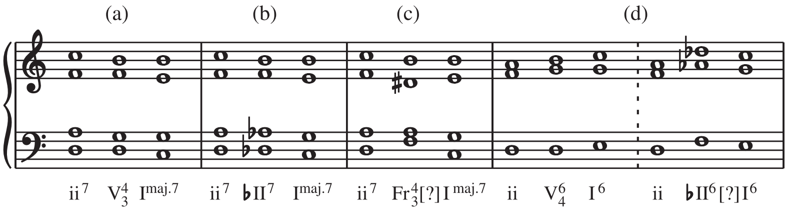
Ex. 8 Tritone substitution as both a contrapuntal and a harmonic phenomenon



why this is so, but also that a surprisingly wide range of chords can participate in tritone substitutions.

Ex. 8a presents an elementary voice-leading schema common to classical music and jazz: a descending-fifth progression in which root-position and second-inversion seventh chords alternate. Ex. 8b substitutes a ♭II^7 chord for the V$^4_3$. Tritone substitution preserves two important features of the original: the tritone F–B, containing the most active notes of the V^7 harmony, and the stepwise descending voice leading. Many discussions of tritone substitution focus only on the first feature. But the second is also important: Ex. 8c replaces the V^7 with a French sixth containing the tritone F–B. The result does not convey convincing dominant functionality. Similarly, Ex. 8d substitutes a ♭II triad for a V triad; the substitution does not have as convincing an effect as that of Exs. 8a and b. These examples suggest that tritone substitution is not simply a matter of 'preserving the tritone' or learning to hear chords on ♭II as having dominant function. It is also important that tritone substitution preserve a phrase's contrapuntal structure.

How does tritone substitution work? And what other chords might allow for it? Scalar interval matrices can be used to answer these questions. Figs. 3a and b contain the scalar interval matrices for the tritone and perfect fifth. The matrices show that there is a trivial voice leading between any tritone and its tritone transposition,[51] since (6, 6) + (–6, –6) = (0, 0). We also see that there is a voice leading between any perfect fifth and its tritone transposition which moves the two notes by semitonal contrary motion, since (7, 5) + (–6, –6) = (1, –1). Now consider any chord which can be partitioned into perfect fourths, tritones or perfect fifths (a 'generalised fourth chord'). Any such chord can be connected to its tritone transposition by efficient voice leading: this is because tritones can be held fixed, while perfect fourths can be transformed into tritone-related perfect fifths (and vice versa) by contrary semitonal motion.[52] Therefore, one can always replace any generalised fourth chord in any sequence of chords by its tritone transposition: tritones will be preserved, and the progression's overall voice leading will remain recognisable.

Ex. 9 shows that such substitutions can be found in both nineteenth-century music and jazz. In Ex. 9a, the tritone substitution affects the upper four voices

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Fig. 3 Some dyadic scalar interval matrices

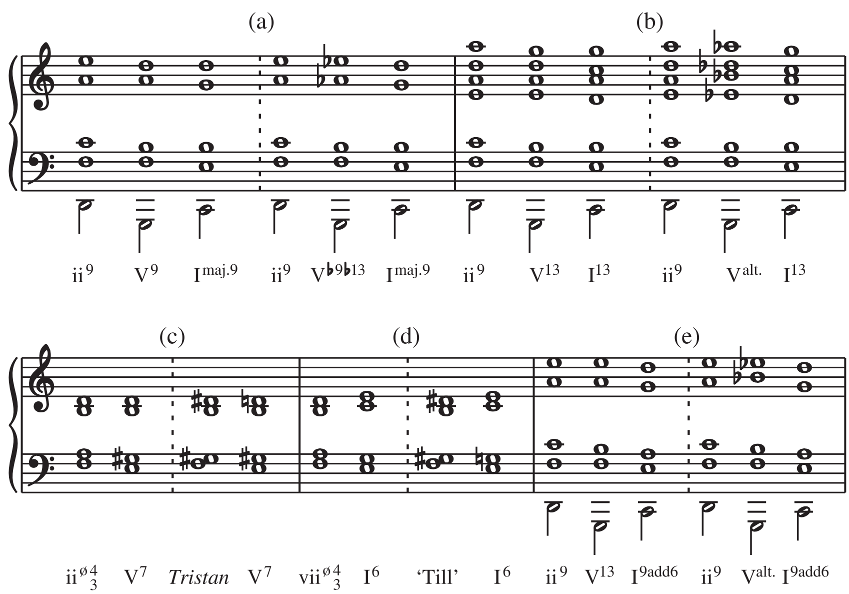
(a) the six-semitone interval in chromatic space

0 steps	0	0
1 step	6	6

(b) the seven-semitone interval in chromatic space

0 steps	0	0
1 step	7	5

Ex. 9 Tritone substitution in jazz and classical music



of a ii^9–V^9–I^9 sequence; the result is a ii^9–V^9–I^9 with an altered dominant chord. (The bass does not participate in the substitution.) Ex. 9b applies the same process to the six upper voices of a seven-voice ii–V–I progression. In Ex. 9c, tritone substitution replaces the predominant chord in a ii$^{ø}{}^{4}_{3}$–V^7 progression; the result, as Finn Hansen has observed, is the opening of *Tristan*.[53] (One could also apply tritone substitution to the dominant chord, producing a transposed version of the *Tristan* progression.) Ex. 9d, the *Till Eulenspiegel* progression, relates by tritone substitution to the familiar vii$^{ø}{}^{4}_{3}$–I progression. Finally, Ex. 9e presents a canonical modern-jazz ii–V–I voice leading: tritone substitution

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Ex. 10 Individually T-related voice leadings between dominant seventh chords

creates voice leading that, although not strictly stepwise, is still quite close to the original.[54]

Ultimately, tritone substitution exploits the very same intervallic properties which permit efficient voice leadings between fifth-related seventh chords. Ex. 10a depicts an efficient voice leading between a dominant seventh chord and its transposition by descending fifth. Ex. 10b depicts an efficient voice leading between a dominant seventh chord and its tritone transposition. The two voice leadings are individually T-related, and are very similar in size. (Both voice leadings are efficient because the chords can be partitioned into fifths and tritones, which bisect the octave evenly or almost evenly.) Traditional tonal syntax exploits the voice leading in Ex. 10a to connect sequential fifth-related dominant seventh chords. Tritone substitution exploits the voice leading in Ex. 10b to *replace* a dominant seventh chord with its tritone transposition. In this sense, the possibility of tritone substitution is latent in the basic voice-leading routines of traditional tonality. Over the course of its history, tonal harmony exploits this latent possibility with increasing frequency – beginning with the introduction of augmented sixths in the eighteenth century, progressing through the occasional use of tritone substitutions in the early nineteenth century and culminating in their universal acceptance in modern jazz.

III

Interscalar Transpositions

(a) The Interscalar Interval Matrix

Section II developed tools for understanding bijective, crossing-free voice leadings between transpositionally related chords. However, such voice leadings make up only a small portion of Western contrapuntal practice; even the simplest student exercise will feature voice leadings between major and minor triads, minor sevenths and dominant sevenths, and so on. Accordingly, Section III generalises the tools we have already developed. We will see that *interscalar interval matrices* describe bijective crossing-free voice leadings between arbitrary chords. These matrices are powerful analytical tools which allow us to

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consider real musical works in all their complexity. Section III (b) demonstrates by sketching a new approach to two famous works of nineteenth-century chromaticism.

Let A and B be n-object multisets. As before, we provide each multiset with scale-degree numbers, ordering their notes on the basis of increasing ascending distance from arbitrarily chosen pitch classes a_0 and b_0. Any bijective crossing-free voice leading from A to any transposition of B can be written as a combination of three voice leadings:[55]

$$(a_0, a_1, \ldots, a_{n-1}) \xrightarrow{\alpha_0, \alpha_1, \ldots, \alpha_n} (b_0, b_1, \ldots, b_{n-1}) \xrightarrow{\beta_0, \beta_1, \ldots, \beta_n}$$
$$(b_{0+c \pmod n}, b_{1+c \pmod n}, \ldots, b_{n-1+c \pmod n}) \xrightarrow{x, x, \ldots, x}$$
$$(b_{0+c \pmod n} + x, b_{1+c \pmod n} + x, \ldots, b_{n-1+c \pmod n} + x)$$

The first voice leading is a crossing-free voice leading which maps scale degree i of A onto scale degree i of B. The second is a scalar transposition from B to B. The third is a chromatic transposition which adds x to each note of B. We can combine the first two voice leadings

$$(a_0, a_1, \ldots, a_{n-1}) \xrightarrow{d_0, d_1, \ldots, d_n} (b_{0+c \pmod n}, b_{1+c \pmod n}, \ldots, b_{n-1+c \pmod n})$$

where $d_i = \alpha_i + \beta_i$. This is a crossing-free voice leading which sends scale degree i in chord A to scale degree $i + c$ in chord B. I will call this an *interscalar transposition by* c *steps*, since it adds c to each scale degree while changing the underlying scale in the process.[56] (Note that the number of steps in an interscalar transposition is relative to the arbitrary choice of scale degree 1 in the two collections.) As before, a minimal bijective crossing-free voice leading from A to B combines a chromatic transposition x with the interscalar transposition which comes as close as possible to neutralising it.

The *interscalar interval matrix* is constructed as follows:

> **Definition 2.** Let A and B be multisets each with n objects, labelled $(a_0, a_1, \ldots, a_{n-1})$ and $(b_0, b_1, \ldots, b_{n-1})$ and ordered by increasing ascending distance from pitch classes a_0 and b_0. Let $B_{i,j}$ be the scalar interval matrix associated with B, and let
>
> $$(a_0, a_1, \ldots, a_{n-1}) \xrightarrow{\alpha_0, \alpha_1, \ldots, \alpha_n} (b_0, b_1, \ldots, b_{n-1})$$
>
> be a crossing-free voice leading from A to B. The interscalar interval matrix from A to B has entries $M_{i,j}$ equal to $B_{i,j} + \alpha_j$.[57]

Each row i of the interscalar interval matrix contains a crossing-free voice leading which sends note a_j to $b_{j+i \pmod n}$: the top row contains a voice leading which sends a_i to b_i, the next row contains a voice leading which sends a_i to $b_{i+1 \pmod n}$, and so on. As before, the n rows of the matrix combine with

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chromatic transpositions to yield the complete set of bijective crossing-free voice leadings from any transposition of the first chord to any transposition of the second. Students of serial theory will again recognise that interscalar interval matrices are closely related to rotational arrays, as we shall note further in Section IV (b).[58]

Fig. 4 shows how to construct the interscalar interval matrix which takes {C, E♭, G♭, B♭}, the half-diminished (or *Tristan*) chord, to {C, E, G, B♭},

Fig. 4 Constructing an interscalar interval matrix

(a) Step 1. Arbitrarily choose a pitch class to serve as scale degree 1 for each of the two chords, labelling the remaining notes on the basis of increasing ascending distance from this pitch class

$$
\begin{array}{ll}
a_0 = \mathrm{C} & b_0 = \mathrm{C} \\
a_1 = \mathrm{E}\flat & b_1 = \mathrm{E} \\
a_2 = \mathrm{G}\flat & b_2 = \mathrm{G} \\
a_3 = \mathrm{B}\flat & b_3 = \mathrm{B}\flat
\end{array}
$$

(b) Step 2. Construct a crossing-free voice leading in which both chords are listed in scale-degree order

$$(\mathrm{C}, \mathrm{E}\flat, \mathrm{G}\flat, \mathrm{B}\flat) \xrightarrow{0,1,1,0} (\mathrm{C}, \mathrm{E}, \mathrm{G}, \mathrm{B}\flat)$$

(c) Step 3. Construct the scalar interval matrix for chord *B*

0 steps	0	0	0	0
1 step	4	3	3	2
2 steps	7	6	5	6
3 steps	10	8	9	9

(d) Step 4. Add the numbers above the arrow in Step 2 to each row of the matrix in Step 3

0 steps	0	0	0	0		0	1	1	0		0	1	1	0
1 step	4	3	3	2	+	0	1	1	0	=	4	4	4	2
2 steps	7	6	5	6		0	1	1	0		7	7	6	6
3 steps	10	8	9	9		0	1	1	0		10	9	10	9

(e) Result. The interscalar interval matrix for (C, E♭, G♭, B♭) and (C, E, G, B♭)

0 steps	0	1	1	0
1 step	4	4	4	2
2 steps	7	7	6	6
3 steps	10	9	10	9

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the dominant seventh chord. We begin by assigning scale-degree numbers to each chord. We then identify a crossing-free voice leading between them, (C, E♭, G♭, B♭) $\xrightarrow{0,1,1,0}$ (C, E, G, B♭), which lists each chord in scale-degree order. We add the vector of numbers (0, 1, 1, 0) to each row of the scalar interval matrix belonging to {C, E, G, B♭}. The result is the interscalar interval matrix for the *Tristan* and dominant seventh chords. The first row sends the root of the half-diminished seventh to the root of the dominant seventh. It corresponds to the voice leading (C, E♭, G♭, B♭) → (C, E, G, B♭). According to our labelling of scale degrees, this is 'interscalar transposition by zero steps'. The second row sends the root of the half-diminished seventh to the third of the dominant seventh, producing (C, E♭, G♭, B♭) → (E, G, B♭, C). According to our labelling, this is 'interscalar transposition by one step'. The third row sends the root to the fifth and the fourth row sends the root to the seventh, corresponding to (C, E♭, G♭, B♭) $\xrightarrow{7,7,6,6}$ (G, B♭, C, E) and (C, E♭, G♭, B♭) $\xrightarrow{10,9,10,9}$ (B♭, C, E, G), respectively. Note that the numbers above the arrows are just the numbers in the appropriate rows of the scalar interval matrix.

We can calculate voice leadings with this matrix exactly as before. To show how this is done, I will pose five questions which echo our earlier investigation of the diatonic scale.

1. What is the minimal bijective crossing-free voice leading between the C half-diminished seventh and the F♯ dominant seventh? We look for that row of Fig. 4e whose values come closest to six. This is the third row. Adding –6 to each entry we obtain (C, E♭, G♭, B♭) → (C♯, E, F♯, A♯).

2. What is the minimal bijective crossing-free voice leading which sends the root of a half-diminished seventh into the third of some dominant seventh? We cannot answer this question without specifying which metric of voice-leading size we are using. For example, the 'smoothness' metric (see Appendix A) measures the total number of semitones moved by all the voices. For this metric, chromatic transposition by four descending semitones best neutralises interscalar transposition by ascending step, since (C, E♭, G♭, B♭) → (C, E♭, G♭, A♭) moves the four voices by a total of two semitones. Other metrics, such as 'parsimony' and the 'L^∞ norm', depend on the *largest* distance moved by any single voice. For these metrics, chromatic transposition by three descending semitones minimises interscalar transposition by ascending step.[59] This is because the voice leading (C, E♭, G♭, B♭) → (C♯, E, G, A) moves none of its four voices by more than a semitone. Smoothness and parsimony are equally legitimate conceptions of voice-leading size. Their divergence underscores the need to remain flexible about how we measure voice leading.

3. What is the minimal bijective crossing-free voice leading between a half-diminished seventh and a dominant seventh? Here we find an embarrassment of riches. All reasonable metrics agree that the half-diminished seventh is connected by minimal bijective voice leading to no fewer than *six* distinct dominant seventh chords; in each case, two voices each move by single semitone. In addition, every half-diminished chord is connected to two more dominant seventh chords by

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Fig. 5 Minimal crossing-free voice leadings between {C, E♭, G♭, B♭} and several transpositions of {C, E, G, B♭}

	T_{11}	T_0	T_2	T_3	T_5	T_6	T_8*	T_9†
0 steps	C → B E♭ → D♯ G♭ → F♯ B♭ → A	C → C E♭ → E G♭ → G B♭ → B♭						
1 step							C → C E♭ → E♭ G♭ → G♭ B♭ → A♭	C → C♯ E♭ → E G♭ → G B♭ → A
2 steps					C → C E♭ → E♭ G♭ → F B♭ → A	C → C♯ E♭ → E G♭ → F♯ B♭ → A♯		
3 steps			C → C E♭ → D G♭ → F♯ B♭ → A	C → D♭ E♭ → E♭ G♭ → G B♭ → B♭				

* maximally smooth, but not maximally parsimonious (see Appendix A)
† maximally parsimonious, but not maximally smooth

the voice leadings described in the previous paragraph. Fig. 5 lists these possibilities in the form of a table, where the vertical axis corresponds to interscalar transposition and the horizontal axis corresponds to chromatic transposition.[60] As we will see, the large number of voice leadings is due to the fact that both the half-diminished and dominant seventh chords divide the octave almost evenly.

4. *What minimal bijective voice leading sends the root of a C half-diminished seventh chord to the seventh of an F♯ dominant seventh chord?* This is a question of the type we left previously unanswered (see again Section II (a), question 4). We can answer it now that we have the tools to find minimal bijective voice leadings between arbitrary chords.

Since we have specified that the C must move to E, we do not know that the overall voice leading will be crossing-free. However, we can require that the voice leading between the remaining notes be crossing-free. Thus we can reformulate the question as follows: What minimal bijective crossing-free voice leading sends the C half-diminished seventh, minus the C, to the F♯ dominant seventh, minus the E? In other words, what is the minimal bijective voice leading between {E♭, G♭, B♭} and {F♯, A♯, C♯}? Fig. 6 presents the relevant interscalar interval matrix. We look for the row whose values are closest to 0 or 12, since we do not need to transpose either chord. This is the third row, representing the voice leading (E♭, G♭, B♭) → (C♯, F♯, A♯). We combine this with C → E to produce the voice leading (C, E♭, G♭, B♭) → (E, C♯, F♯, A♯). This voice leading, although not crossing-free, is the minimal bijective voice leading satisfying our constraint.

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Fig. 6 The interscalar interval matrix for (E♭, G♭, B♭) and (G♭, B♭, D♭)

0 steps	3	4	3
1 step	7	7	8
2 steps	10	12	12

5. *Why is it that some pairs of (transpositional) set classes can be linked by efficient voice leading, while others cannot?* For example, members of the half-diminished and dominant seventh set classes can be linked by very efficient voice leading, whereas no half-diminished chord can be linked to any chromatic tetrachord by very efficient voice leading.[61] Why is this? It can be shown that the size of the minimal voice leading between the members of two transpositional set classes depends on the similarity between their respective series of one-step intervals.[62] It follows that members of closely 'clustered' set classes, such as {C, C♯, D, E♭}, can be linked by efficient voice leading to members of other clustered set classes, such as {C, C♯, D♯, E}, while members of extremely 'even' set classes, such as {C, E♭, G♭, B♭}, can be linked by efficient voice leading to members of other, extremely 'even', set classes such as {C, E, G, B♭}. But there can be no particularly efficient voice leading between 'clustered' and 'even' chords. Voice leading thus provides a quantitative representation of set-class similarity, allowing us to measure the 'distance' (or 'difference') between set classes using the size of the smallest voice leading between their members.[63]

Section IV (b) will show that the rows of the scalar interval matrix for the chord $\mathbf{I}_x(A)$ are the rotated retrogrades of the corresponding rows of the scalar interval matrix of A. It follows from the previous paragraph that a chord can be linked by smooth voice leading to its inversion if and only if the second row of its scalar interval matrix is approximately retrograde-symmetrical. For example, the second row of the half-diminished seventh chord's scalar interval matrix is (3, 3, 4, 2), which is approximately equal to its retrograde, (2, 4, 3, 3); this is why the voice leading (C, E♭, G♭, B♭) → (C, D, F♯, A) is efficient.[64] Why are there so many other efficient voice leadings between half-diminished and dominant seventh chords? As we saw in Section II, there are always efficient voice leadings between distinct transpositions of extremely 'even' set classes such as {C, E♭, G♭, B♭}: if A is extremely even, then there will be some voice leading $A \rightarrow \mathbf{T}_x(A)$ which is small. Consequently, if the voice leading $B \rightarrow A$ is small, then the combined voice leadings $B \rightarrow A \rightarrow \mathbf{T}_x(A)$ will also be relatively small. There will therefore be efficient, bijective voice leadings from any even chord B to multiple transpositions of another even set class A. This is why there are so many voice leadings evident in Fig. 5.

(b) Excursus III: Tristan *and the* Prélude à l'après-midi d'un faune

The interscalar interval matrix can deepen our understanding of individual works by showing how composers exploited the voice-leading possibilities

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Ex. 11 Half-diminished/dominant seventh progressions in the *Tristan* Prelude

available to them. To illustrate this, we will explore Wagner's *Tristan* and Debussy's *Prélude,* focusing on the similarities between the two composers' treatment of the half-diminished and dominant seventh chords. As we will see, interscalar interval matrices provide useful tools for analysing these pieces, allowing us to make sense of their otherwise elusive contrapuntal structures.

Ex. 11 presents a series of progressions drawn from the Prelude to *Tristan und Isolde*. In each case, a *Tristan* (or half-diminished) chord resolves to a dominant seventh. The actual voice leading here is *not* always crossing-free: indeed, it is a central motivic feature of the opera that ascending chromatic motion produces voice crossings in pitch-class space. However, there is a well-established analytical tradition that views these crossings as 'surface' events – 'voice exchanges' embellishing a more fundamental stepwise structure.[65] We shall find that it is analytically advantageous to adopt this view. For minimal voice-leadings, although not present on the musical surface, play an important role in determining the piece's deeper voice-leading structure.

The bottom stave of Ex. 11 identifies the crossing-free voice leading most closely corresponding to each progression. Below that stave, I describe each voice leading using the notation $\mathbf{T}_x\mathbf{t}_y$. Here $\mathbf{T}_x$ refers to chromatic transposition, while $\mathbf{t}_y$ refers to the voice leading shown in row y of the matrix in Fig. 4e, numbering rows from 0. (Thus, $\mathbf{t}_y$ indicates interscalar transposition by y steps, with the root of each chord considered to be scale degree 1.) The voice leading

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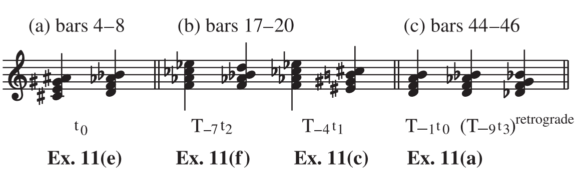
in Ex. 11a is labelled $\mathbf{T}_{-1}\mathbf{t}_0$, which indicates that we apply the top row of Fig. 4e to the chord (F, G♯, B, D♯), and then transpose the resulting voice leading down by one semitone. This produces the voice leading (F, G♯, B, D♯) $\rightarrow$ (E, G♯, B, D) as shown on the bottom stave of Ex. 11a.

Ex. 11 shows that Wagner comprehensively explored the full range of interscalar transpositions between the two chord types, making use of all four rows of Fig. 4e. Furthermore, Ex. 11f, from bars 82 to 84, presents a single *Tristan* chord acting as a 'pivot' between two different minimal voice leadings. The first of these corresponds to a traditional $\mathrm{ii}^{\varnothing 7} \rightarrow \mathrm{V}^4_3$ progression, while the second corresponds to the prototypical *Tristan* resolution (Ex. 11a). This passage is emblematic of the Prelude as a whole, suggesting that alternative resolutions of the *Tristan* chord are indeed one of its central preoccupations.

Most of the voice leadings shown in Ex. 11 are minimal voice leadings, according to some standard measure of voice-leading size; indeed, all but two of the voice leadings are uniformly T-related to a voice leading shown in Fig. 5. This suggests a hypothesis about the deep structure of the *Tristan* Prelude: although the surface of Wagner's music involves voice exchanges and non-minimal voice leadings, it embellishes a deeper structure that is fundamentally determined by efficient voice leading. (Here it should be noted that the very possibility of Wagner's ubiquitous minor-third voice exchanges require 'background' voice leadings in which the relevant voices stay approximately fixed; only then will the exchanging voices travel by approximately a minor third. Hence the voice exchanges themselves subtly exploit the fact that the background voice leadings are minimal.) Indeed, Wagner's piece seems to be propelled by the same facts about musical structure we used to generate Fig. 5. We can imagine (re)composing the *Tristan* Prelude by first identifying minimal voice leadings between *Tristan* and dominant seventh chords, then decorating these voice leadings by adding motivic voice exchanges. In this sense, the voice leadings in Fig. 5 may provide the underlying voice-leading patterns from which the music's surface is constructed.

Ex. 12 shows some progressions from an equally famous late nineteenth-century work, Debussy's *Prélude à l'après-midi d'un faune*. Here again we see the composer making comprehensive use of the four interscalar transpositions between a *Tristan* chord and its inversion. Ex. 12a, from the opening of the

Ex. 12 Half-diminished/dominant seventh progressions in the *Prélude à l'après-midi d'un faune*



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Prélude, recalls Ex. 11e. Ex. 12b uses a single *Tristan* chord to pivot between two separate maximally efficient resolutions – just as Wagner does in Ex. 11f, although the second resolution in Ex. 12b is different from Wagner's.[66] (Note that the first three chords in the two progressions are identical.) Ex. 12c shows another three-chord progression: chords 1–2 involve the prototypical *Tristan* resolution of Ex. 11a; chords 2–3 utilise the retrograde of the interscalar transposition in the fourth row of Fig. 4e. As in *Tristan,* all of the different interscalar transpositions are explored. And, like Wagner, Debussy appears to be particularly interested in minimal voice leadings – indeed, *all* of the voice leadings shown in Ex. 12 are minimal voice leadings for some common metric.

Despite the many differences between Debussy's and Wagner's musical languages, the resemblance between Exs. 11 and 12 is striking. The examples suggest that the two works are animated by a similar compositional strategy of exploring the efficient voice-leading possibilities between *Tristan* and dominant seventh chords. The ideas and concepts in Sections I–III have given us techniques for describing this process, allowing us to look in detail at the various ways in which the two composers exploited the possibilities available to them. What results, perhaps, is a more rigorous perspective on these two famously unruly masterpieces of nineteenth-century chromaticism, one in which voice leading plays a vital constructive role.[67]

IV

The T-Matrix and Non-Bijective Voice Leadings

(a) Doublings and the T-Matrix

Section III provides tools for investigating bijective crossing-free voice leadings between arbitrary chords. But it did not show us how to find non-bijective voice leadings, which of course play an important role in Western music: for instance, elementary harmony exercises frequently involve minimal voice leadings between four-note dominant seventh chords and three-note major triads. Obviously, in such voice leadings, at least one note of the major triad must be doubled. Interscalar interval matrices do not provide an efficient means for determining which note to double: the best we can do is to generate three separate interscalar interval matrices, corresponding to the three ways of doubling one note of the major triad, and then compare the twelve voice leadings in these three matrices.

Moreover, it occasionally happens that the minimal voice leading between two chords is non-bijective. Ex. 13a presents one of the two minimal bijective voice leadings between T_6-related 3–8B {0, 4, 6} trichords. Ex. 13b presents the minimal four-voice voice leading between these same trichords. Two pitch classes, E and F♯, in the source chord are related to a single pitch class, F♯, in the target. Likewise, two pitch classes, C and A♯, in the target chord are related

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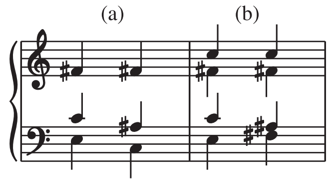
Ex. 13 Doubling and minimal voice leading



to a single pitch class, C, in the source. According to any of the standard metrics of voice-leading size, Ex. 13b is smaller than Ex. 13a. Somewhat surprisingly, increasing the number of voices decreases the overall voice-leading size.[68]

The fact that minimal voice leadings are sometimes non-bijective makes it all the more remarkable that musicians manage to identify minimal voice leadings so easily. Somehow, even the elementary harmony student can readily answer questions such as 'Given two chords A and B, what is the minimal voice leading between them if we allow notes to be freely doubled?' and 'How do we find the minimal voice leading from chord A to any transposition of chord B if doublings are freely allowed?' These simple-sounding questions are surprisingly difficult – indeed, as we will see, it is no easy task to come up with an efficient algorithm for answering them. Fortunately, however, the T-matrices of serial theory provide useful tools for investigating non-bijective voice leadings.[69] In many practical cases, simple inspection of the T-matrix suffices to identify the minimal voice leading (not necessarily bijective) between arbitrary chords.

We define the *T-matrix* associated with two multisets as follows:

> **Definition 3.** Let A and B be multisets labelled $(a_0, a_1, \ldots, a_{n-1})$ and $(b_0, b_1, \ldots, b_{m-1})$ and ordered by increasing ascending distance from pitch classes a_0 and b_0. The *T-matrix* which takes A to B has elements $\mathrm{M}_{i,j}$ equal to the traditional pitch-class interval from a_j to b_i.

(Note that we use traditional pitch-class intervals rather than paths in pitch-class space: we consider intervals 11 and 1 to have size 1.[70]) Since the interval from a_j to b_i is given by $(b_i - a_j)_{\text{mod } 12}$, each element of the T-matrix contains the difference (mod 12) between its row label and its column label. It is, in short, a simple subtraction table. Its elements represent what Morris (1998) calls the 'total voice leading' between A and B – that is, every possible pitch-class interval which can be formed between a note in the first chord and one in the second.

Fig. 7 presents the T-matrix linking {G, B, D, F} to {C, E, G}. Each square of the matrix contains the traditional pitch-class interval between its column

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Fig. 7 The T-matrix linking {G, B, D, F} to {C, E, G}

	7	**11**	**2**	**5**
0	5	1	10	7
4	9	5	2	11
7	0	8	5	2

label and its row label. The crossing-free voice leadings form a special sort of closed loop through the matrix: a crossing-free voice leading is defined by a series of one-unit right, downwards or diagonally right-and-downwards moves in the matrix, wrapping around the matrix as a 2-torus, beginning at any point in the matrix, ending at the same point and touching each column and row of the matrix at least once. These restrictions ensure that an ascending scale step in one collection is mapped onto either a unison or an ascending scale step in the other.

Let us return now to the first question we asked above – 'Given two multisets *A* and *B*, what is the minimal crossing-free voice leading between them, if we allow notes to be freely doubled?' To answer this question we can construct the T-matrix linking the multisets. We then look for the closed loop containing values closest to 0, which only moves downwards, to the right or diagonally downwards and to the right and which passes through every row and every column. (The sense of 'closest' is, as usual, given by a metric of voice leading size.) To answer the second question – 'How do we find the minimal voice leading from chord *A* to any transposition of chord *B* if doublings are freely allowed?' – we look for the closed loops which always move down, to the right or diagonally down and to the right, passing through every row and every column, and whose values come closest to a constant value.

Unfortunately, a good deal of computation is required to search these paths. There are quite a large number of them, and for sizeable multisets with many doublings, it is necessary to use a computer to investigate the possibilities. Indeed, the problem of finding an efficient algorithm to perform the search is non-trivial: even with midsize chords, there are enough possibilities (31,644 for two hexachords) to make a brute-force solution unappealing, particularly in real-time applications such as interactive computer music. Fortunately, there is an alternative: we can use the ubiquitous computer-science technique of 'dynamic programming' to solve the problem in polynomial time.[71] The technical details of this algorithm are spelled out elsewhere.[72]

However, in practice, we can answer many musical questions by simple inspection of the T-matrix. This is because adding voices generally tends to increase the size of voice leadings. Finding the minimal voice leading between multisets is therefore a matter of balancing the need to minimise the number of voices with the need to make each voice move by minimal distances. Luckily for the music theorist, it is often clear how to do this.

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Fig. 8 The T-matrix linking {C, E, F♯} to {F♯, A♯, C}

	0	**4**	**6**
6	6	2	0
10	10	6	4
0	0	8	6

We will illustrate by asking two questions which address loose threads left untied in the course of the previous discussion.

1. We saw above that one can reduce the voice leading between T_6-related {0, 4, 6} trichords by doubling one note. Is there any way to reduce the voice leading further by doubling more than one note? Fig. 8 provides the T-matrix for the voice leading between {C, E, F♯} and {F♯, A♯, C}. The voice leading shown in Ex. 13a begins with the first element of the second row, moving down-and-right, down-and-right and down-and-right. This yields the collection of values {10, 8, 0}, corresponding to the intervals moved by the three musical voices. The voice leading shown in Ex. 13b begins at the same place, but moves down, down-and-right, right, and down-and-right producing {10, 0, 2, 0}. This allows the path to avoid the 8, instead touching upon the values 0 and 2. The sum of these values, 2, is smaller than 4, the interval class to which 8 belongs. This is why the doubling decreases the size of the voice leading for all standard metrics.

Have we found the minimal voice leading between these chords? The answer is yes. The four intervals in our voice leading, {10, 0, 2, 0}, are the four entries in the matrix closest to 0; therefore any smaller voice leading must have fewer voices. Looking at the down-and-right (NW/SE) diagonals of Fig. 8, we see that all of the three-voice voice leadings are larger than our four-voice voice leading. The four-voice voice leading in Ex. 13b is the minimal voice leading between the two multisets – even if arbitrary doublings are allowed.

2. Earlier, we investigated the minimal voice leading between the Tristan *chord and the dominant seventh. Does doubling ever allow us to decrease the size of the minimal voice leading between chords belonging to these two set classes?* Fig. 5 demonstrates that the *Tristan* chord can be connected by two-semitone voice leadings to seven different dominant seventh chords, and by four-semitone voice leading to an eighth dominant seventh chord. For any acceptable metric, these voice leadings are minimal, because each pitch class in the first chord moves to the closest possible pitch class in the second. Since each pitch class of the first chord moves to only one pitch class in the second, it is not possible to decrease the size of these voice leadings further.

Fig. 9 shows the T-matrix connecting {C, E♭, G♭, B♭} to {C, E, G, B♭}. We will use it to examine the voice leading between the *Tristan* chord and the four remaining dominant seventh chords. These dominant sevenths are T_1, T_4, T_7 and T_{10} of {C, E, G, B♭}. Since we are looking for voice leadings that come

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Fig. 9 The T-matrix linking {C, E♭, G♭, B♭} to {C, E, G, B♭}

	0	**3**	**6**	**10**
0	0	9	6	2
4	4	1	10	6
7	7	4	1	9
10	10	7	4	0

closest to negating chromatic transposition by one, four, seven and ten semitones, we will be searching for the paths through Fig. 9 whose elements come closest to eleven, eight, five and two.

We can find two cases in which doubling decreases the size of the voice leading. The first begins in the first square of the second row, moving down-and-right, down-and-right, down, down-and-right and right. This produces the vector (4, 4, 4, 6, 6), which combines with the chromatic transposition (7, 7, 7, 7, 7) to yield the voice leading (C, E♭, G♭, G♭, B♭) → (B, D, F, G, B). (This voice leading is closely related to the *Till Eulenspiegel* resolution of the *Tristan* chord to the major triad, shown in Ex. 9d.) For all standard metrics, this voice leading is smaller than the smallest bijective voice leading between the two collections, (C, E♭, G♭, B♭) → (B, D, F, G). Since the values {4, 4, 4, 6, 6} are the five matrix elements closest to 5, it follows that further doublings will not decrease the size of the voice leading.

The second voice leading begins on the first square of the third row; its values (7, 7, 9, 10, 9) combine with the transposition (4, 4, 4, 4, 4), producing (C, E♭, E♭, G♭, B♭) → (B, D, E, G♯, B). For some metrics, this voice leading is smaller than the minimal four-voice alternative, (C, E♭, G♭, B♭) → (B, D, E, G♯). Close inspection of the matrix again reveals that further doublings will not produce a smaller voice leading.

In the remaining cases, doubling does not reduce the size of the minimal four-voice voice leading – at least not for any of the standard metrics. We conclude that the minimal voice leading between a *Tristan* chord and a dominant seventh will have four voices, except in the two cases mentioned above. We have now found a minimal voice leading between the *Tristan* chord on C and each of the twelve distinct transpositions of the dominant seventh, as shown in Fig. 10. This table illustrates the possibilities available to a composer interested in exploring minimal voice leadings between half-diminished and dominant seventh chords. We have already seen that the table can be useful in investigating the music of Wagner and Debussy. Further exploration would show that it can help to elucidate the music of a range of other nineteenth- and early twentieth-century composers as well.

(b) Scale Theory and Serial Theory

The scalar and interscalar interval matrices are closely related to serial theory's rotational arrays, which are in turn closely related to T-matrices. Related

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Fig. 10 The minimal voice leadings between {C, E♭, G♭, B♭} and all transpositions of {C, E, G, B♭}

	T_0	T_1	T_2	T_3	T_4*	T_5	T_6	T_7	T_8	T_9	T_{10}†	T_{11}
0 steps	C → C E♭ → E G♭ → G B♭ → B♭	C → D♭ E♭ → F G♭ → A♭ B♭ → C♭									C → B♭ E♭ → D G♭ → F B♭ → A♭	C → B E♭ → D♯ G♭ → F♯ B♭ → A
1 step									C → C E♭ → E♭ G♭ → G♭ B♭ → A♭	C → C♯ E♭ → E G♭ → G B♭ → A	C → D E♭ → F G♭ → A♭ B♭ → B♭	
2 steps					C → B E♭ → D G♭ → E B♭ → G♯	C → C E♭ → E♭ G♭ → F B♭ → A	C → C♯ E♭ → E G♭ → F♯ B♭ → A♯					
3 steps		C → C♭ E♭ → D♭ G♭ → F B♭ → A♭	C → C E♭ → D G♭ → F♯ B♭ → A	C → D♭ E♭ → E♭ G♭ → G B♭ → B♭	C → D E♭ → E G♭ → G♯ B♭ → B							
non-bijective					C → B E♭ → D E♭ → E G♭ → G♯ B♭ → B			C → B E♭ → D G♭ → F G♭ → G B♭ → B				
smoothness	**2**	**6**	**2**	**2**	**6**	**2**	**2**	**5**	**2**	**4**	**6**	**2**

* The bijective voice leadings are maximally smooth but not maximally parsimonious.

† There are four additional non-bijective voice leadings that are maximally smooth but not maximally parsimonious: (0, 3, 6, 10, 10) → (10, 2, 5, 8, 10), (0, 3, 6, 6, 10) → (10, 2, 5, 8, 10), (0, 3, 6, 10, 10) → (2, 2, 5, 8, 10) and (0, 3, 6, 6, 10) → (2, 2, 5, 8, 10).

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matrices also appear in tonal theory, as they represent both scalar transpositions and key signatures. In this section I will explore these connections in greater depth. Some of the ideas discussed (see for example subsections 1–2, below) will be familiar to students of serial theory. Others (for example 5–6) generalise observations familiar from tonal theory. Still others (for example 3–4) are indigenous to the subject of voice leading. The discussion will therefore suggest some interesting correspondences between fairly disparate fields of music theory.

The major difference between the matrices I use here and those of traditional serial theory is that I use paths in pitch-class space rather than traditional pitch-class intervals. This is because I want to have my cake and eat it too: I want the generality which comes from speaking about pitch classes, but I do not want to eradicate the difference between ascending and descending motion. The combination of these two desires, which might at first blush seem contradictory, leads me to consider C4 and C5 to be instances of the same object (the pitch class C), while still distinguishing the motion C4 → C5 (an ascending octave, represented by a clockwise circuit around the pitch-class circle) from the motion C5 → C4 (a descending octave, represented by an anticlockwise circuit around the circle).[73]

1. Source and target chords in the matrix. We can consider the numbers in a scalar or interscalar interval matrix (mod 12) to represent pitch classes rather than paths or intervals: thus –1 represents B, –2 represents B♭, and so on. If we do this, we find that the columns of an interscalar interval matrix contain the 'modes' of the target sonority, while the matrix's up-and-right (SW/NE) diagonals contain the inversions of the modes of the source sonority. Thus the columns of Fig. 2 contain the modes of the diatonic scale. The columns of Fig. 4e contain the modes of the dominant seventh chord, as do the columns of Fig. 11a. The first column of each matrix is in scale-degree order. In Fig. 11a, the SW/NE diagonal running from the lower left corner of the matrix to the upper right inverts the source chord in scale-degree order.[74] (This is also true of Figs. 2 and 4e, although in both cases the inversion of the target chord is the same as the source chord.[75]) The diagonals parallel to the SW/NE diagonal contain the inversions of the remaining modes of the source sonority.[76]

2. Interscalar interval matrices and T-matrices. As Babbitt, Morris and others have noted, an interscalar interval matrix can be transformed into a T-matrix by rotating each column so that the first row instead lies along its major diagonal. (The major diagonal of a matrix descends from the upper left corner to the lower right.) This amounts to rotating the *i*th column of the matrix downwards by *i* places. (Numbering columns starting from 0.) Thus, appropriate rotation of the matrix shown in Fig. 11a produces Fig. 11b, whose four columns (again considered as pitch classes) represent four transpositions of a dominant seventh chord, and whose four rows (considered as pitch classes) represent four inversions of the source chord. This would be a T-matrix were it not for the value –1 in the lower right corner. This number appears because

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Fig. 11 Various matrices linking (C, E♭, G, B) and (C, E, G, B♭)

(a) the interscalar interval matrix from (C, E♭, G, B) to (C, E, G, B♭)

0 steps	0	1	0	–1
1 step	4	4	3	1
2 steps	7	7	5	5
3 steps	10	9	9	8

(b) rotating the columns of Fig. 11(a) produces a T-matrix containing paths rather than intervals

	0	**3**	**7**	**11**
0	0	9	5	1
4	4	1	9	5
7	7	4	0	8
10	10	7	3	–1

(c) the interscalar interval matrix from (C, E, G, B♭) to (C, E♭, G, B)

0 steps	0	–1	0	1
1 step	3	3	4	2
2 steps	7	7	5	5
3 steps	11	8	8	9

(d) rotating the columns of Fig. 11(c) produces a T-matrix containing paths rather than intervals

	0	**4**	**7**	**10**
0	0	8	5	2
3	3	–1	8	5
7	7	3	0	9
11	11	7	4	1

interscalar interval matrices use real numbers representing paths in pitch-class space rather than traditional mod-12 pitch-class intervals.

3. Matrix rows and inverse voice leadings. Fig. 11c presents an interscalar interval matrix which takes {C, E, G, B♭} to {C, E♭, G, B}. Each row of the matrix in Fig. 11a can be rotated and added to a row of Fig. 11c to produce a constant vector ($12i$, $12i$, $12i$, $12i$), where i is an integer. For example, adding the first row of one matrix to the first row of the other produces the values (0, 0, 0, 0). Adding the second row of one matrix to the fourth row of the other, rotated by one place, yields the values (12, 12, 12, 12). The third rows also add to (12, 12, 12, 12) when one is rotated by two places. This is because the

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interscalar transpositions in one matrix counteract those in the other: for every bijective crossing-free voice leading from {C, E♭, G, B} to {C, E, G, B♭} there is a bijective crossing-free voice leading that sends {C, E, G, B♭} *back* to {C, E♭, G, B}, returning every voice to the pitch class from which it began. (Typically, the combination of the two voice leadings produces an additional octave transposition; this is because both matrices represent *ascending* scalar transpositions.) Similarly, the second row of Fig. 2 is the rotated complement (mod 12) of the seventh row, the third row is the rotated complement (mod 12) of the sixth row, and so on. This reflects the fact that scalar transpositions counteract each other: transposition upwards by one diatonic step, followed by transposition upwards by six diatonic steps, is equal to transposition upwards by an octave, and so on.

4. Matrices and inversion. Let $M_{A \to B}$ be the interscalar interval matrix from chord A to chord B. The rows of the interscalar interval matrix from $\mathbf{I}_x(B)$ to $\mathbf{I}_x(A)$ will be the rotated retrogrades of the corresponding rows of $M_{A \to B}$.[77] It follows that the rows of the matrix from A to $\mathbf{I}_x(A)$ must be invariant under a combination of rotation and retrograde, as Figs. 2 and 4e show. Similarly, the rows of $M_{I_x(A)I_x(B)}$ will be the rotated retrogrades of the complements (mod 12) of the rows of $M_{A \to B}$.[78] When A and B are both inversionally symmetrical, then each row i of $M_{A \to B}$, when rotated and retrograded, can be added to a row j of $M_{A \to B}$ (with i possibly equal to j) to produce a constant vector $(c, c, \dots, c)$. The constant c depends on the chords A and B.

Musically, this means that for any crossing-free voice leading between two inversionally symmetrical chords, there will be a corresponding 'inverted' crossing-free voice leading between the first chord and a chord belonging to the same set class as the second: for example, the voice leading (C, D, E, F, G, A, B) → (C, D, E♭, F, G, A, B) transforms the C major scale into an acoustic (melodic minor ascending) scale by lowering one note by a semitone. Since both chords are inversionally symmetrical, we know there must be a uniformly I-related voice leading (C, D, E, F, G, A, B) → (C♯, D, E, F, G, A, B) which sends the C major scale to an acoustic scale by raising one note by a semitone. Thus, the crossing-free voice leadings between inversionally symmetrical chords can be grouped into uniformly I-related pairs. Such voice leadings are an important component of Debussy's modulatory practice.[79]

5. Sum of matrix rows. The (absolute) sum of the elements in each row i of a scalar interval matrix will be equal to $12i$. This is because scalar transposition by n ascending steps spans a total distance of exactly n ascending octaves. For example: the voice leading (C, E, G) → (E, G, C), which transposes the C major triad up by one step, moves its three voices C → E, E → G and G → C by a total of one octave.[80] The rows of an interscalar interval matrix sum to $12i + c$, where c is the sum of the paths α_i in the crossing-free voice leading $(a_0, a_1, \dots, a_{n-1}) \xrightarrow{\alpha_0, \alpha_1, \dots, \alpha_n} (b_0, b_1, \dots, b_{n-1})$ which is used to generate the matrix (Definition 2). Thus the rows of Fig. 4e sum to $12i + 2$, while the rows of Fig. 11a sum to $12i$. This feature of interscalar interval matrices links them

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more closely to scale theory, which is often concerned with ascending scalar intervals, than to serial theory, which is more typically concerned with traditional pitch-class intervals.

6. *Key signatures.* As Julian Hook first observed, the rows of the scalar interval matrix in Fig. 2 can be understood as *key signatures* for the enharmonically equivalent keys C major, B♯ major, A𝄪 major, G𝄪♯ major, etc.[81] Here the elements in each row indicate the number of sharps applied to each letter name (C, D, E, F, G, A, B). Any of the standard major-scale key signatures can be formed by adding a constant to any row of the matrix.[82] Thus we can use scalar interval matrices (and interscalar interval matrices) to explore key signatures in precisely the same way that we use these matrices to explore voice leadings. I have argued that there is a precise analogy between voice leadings and key signatures: key signatures simply are voice leadings from a 'basic scale', which has been assigned the unadorned letter-names A, B, C, … , to a target scale, which is the scale represented by the key signature. Many of the concerns in the present article, such as the avoidance of voice crossings and minimising voice-leading size, arise naturally in the context of key signatures. Thus there is a close connection between serial theory's rotational arrays and tonal theory's key signatures, mediated by the topic of voice leading.

V

Conclusion

How, then, do musicians manage to find the minimal voice leading between arbitrary chords? Have we internalised the procedures described in the preceding sections, so that while composing we subconsciously construct scalar and interscalar interval matrices? Do we use some other algorithm for determining minimal voice leadings? Can we simply determine minimal voice leadings 'by sight', in an intuitive manner which it is difficult for algorithms to simulate? Or do we use some simpler but non-optimal heuristic which manages to identify minimal voice leadings in the majority of interesting cases? (For example, do we simply choose the smallest bijective voice leading and hope for the best?) These important questions remain problems for future research. This article is not a piece of empirical psychology, and it does not attempt to describe the conscious or subconscious mental mechanisms involved in musical composition. Instead, it provides explicit recipes which allow us to extend and systematise the implicit knowledge we seem regularly to deploy. This opens up a range of new music-theoretical questions, including questions about how the processes used by actual composers relate to the idealised processes described here.

While our investigation does not yet bear directly on empirical psychology, it does deepen our understanding of voice leading, a central but poorly understood music-theoretical topic. As the mathematical music theorist Guerino

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Mazzola has written, 'Although the theory of categories has been around since the early 1940s and is even recognised by computer scientists, no attempt is visible in American Set Theory to deal with the morphisms between pcsets'.[83] This article, in essence, proposes a method for dealing with 'morphisms between pcsets', understood as voice leadings between chords. We have seen that a systematic understanding of voice leading allows us to shed new light on topics as diverse as neo-Riemannian theory, tritone substitution, the relation between serial theory and scale theory, key signatures, the *Tristan* Prelude, similarities between Debussian impressionism and Wagnerian chromaticism and the role of doublings in decreasing voice-leading size. It remains for future work to extend this list. In particular, it can be shown that many of the ideas in this article have a straightforward geometrical interpretation, and that the mathematical concepts of 'quotient space' and 'orbifold' provide powerful and natural tools for understanding voice leading.[84] Even without this extension, however, the techniques described in this article provide practical tools for understanding, analysing and perhaps even composing music.

Appendix A

Measuring Voice Leading

Let $(x_0, x_1, \ldots, x_{n-1}) \to (y_0, y_1, \ldots, y_{n-1})$ be a voice leading between two multisets of pitches. Its displacement multiset is the multiset of distances $\{|y_i - x_i|, 0 \leq i < n\}$. Every existing music-theoretical method of measuring voice-leading size depends only on the displacement multiset. We can therefore model methods of comparing voice leadings as methods of comparing multisets of nonnegative real numbers.

A. Smoothness. The size of a voice leading is the sum of the objects in the displacement multiset.[85] According to the smoothness metric, $\{2, 2\} > \{3.999\} > \{1, 1, 1\}$. Smoothness is sometimes called the 'taxicab norm'. It reflects aggregate physical distance on keyboard instruments.

B. L^p norms. Smoothness is analogous to the L^1 vector norm, although vectors are ordered whereas multisets are not. The analogues to the L^p vector norms can also be used to measure voice leadings when $p \geq 1$.[86] Callender uses the L^2 vector norm.[87]

C. L^∞. According to the L^∞ vector norm, the size of a displacement multiset is its largest element. The musical terms 'semitonal voice leading' and 'stepwise voice leading' refer to this measure of voice-leading size. Semitonal voice leadings have an L^∞ norm of 1; stepwise voice leadings have an L^∞ norm less than or equal to 2. The L^∞ norm measures the largest physical distance moved by any single voice on a keyboard instrument.

D. Parsimony. Parsimony is related to lexicographic ordering. It generalises a notion introduced by Richard Cohn and developed by Jack Douthett and Peter Steinbach.[88] Given two voice leadings α and β, α is smaller (or 'more parsimonious') than β if and only if there exists some real number j such that

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1. for all real numbers $i > j$, i appears the same number of times in the displacement multisets of α and β; and
2. j appears fewer times in the displacement multiset of α than β.

Thus, according to the metric of parsimony, $\{3 + \varepsilon\} > \{3, 3\} > \{3\}$, where ε is any positive nonzero quantity, no matter how small.

Appendix B

A Bijective Crossing-Free Voice Leading from a Chord to One of Its Transpositions Can Be Written as a Combination of Scalar and Chromatic Transpositions

Consider a bijective crossing-free voice leading from a chord $A = (a_0, a_1, \ldots, a_{n-1})$ to one of its transpositions $\mathbf{T}_x(a_0, a_1, \ldots, a_{n-1})$. Assume A has no pitch-class duplications. Suppose pitch class a_0 moves to pitch class $(a_c + x)_{\text{mod } 12}$ by a path of d_0 semitones. We can write $d_0 = \delta^{\uparrow}(a_0, a_c) + y$, $0 \le c < n$, where $\delta^{\uparrow}(a, b)$ is the ascending distance from a to b. Here $\delta^{\uparrow}(a_0, a_c)$ represents a scalar transposition within chord A, while y represents a chromatic motion which will be the same for all the notes in the chord.

Now consider pitch class a_1, immediately above a_0 in the source chord. Since the voice leading is crossing-free, a_1 must move to the pitch class $(a_{c+1 \text{ (mod } n)} + x)_{\text{mod } 12}$, which is the pitch class in the destination chord immediately above that to which a_0 moves. (Henceforth all subscript additions will be modulo n unless otherwise noted: a_{c+1} stands for $a_{c+1(\text{mod } n)}$.) To see why, assume the contrary. We could then construct an instance containing the four pitches p_0, q_0, p_1 and q_1, where p_0 belongs to pitch class a_0, q_0 is the pitch to which p_0 moves $(p_0 + d_0)$, p_1 is the source-chord pitch immediately above $p_0[p_0 + \delta^{\uparrow}(a_0, a_1)]$ and q_1 is the target-chord pitch immediately above $q_0[q_0 + \delta^{\uparrow}(a_c, a_{c+1})]$. If the voice originating at p_1 does not cross the voice originating at p_0, then it is mapped onto a note above q_1. (By hypothesis, it is not mapped onto q_1, which belongs to the pitch class $(a_{c+1} + x)_{\text{mod } 12}$.) But then the voice terminating at q_1 originates either below p_0 or above p_1 and hence creates a crossing. This contradicts the hypothesis. By induction on the notes in chord A, we conclude that each pitch class a_i in the source chord moves to pitch class $(a_{i+c} + x)_{\text{mod } 12}$.

Now consider an instance containing p_0, q_0 and p_1 as in the above paragraph. We have shown that p_1 must be mapped onto $q_1 + 12k$ for some particular integer k. If $k < 0$, then the voice originating at p_1 crosses $p_0 \rightarrow q_0$, since $q_1 - 12 = q_0 + \delta^{\uparrow}(a_c, a_{c+1}) - 12$ is below q_0. (This follows from the fact that A has no pitch-class duplications, and hence the ascending distance from a_c to a_{c+1} is greater than 0 and less than 12.) Similarly, if $k > 0$, we can construct an instance containing pitches p_1, $p_0 + 12$, and $q_0 + 12$. (That is, we simply move the voice $p_0 \rightarrow q_0$ up by an octave.) The voice originating at p_1 crosses $(p_0 + 12) \rightarrow (q_0 + 12)$ because $p_1 = p_0 + \delta^{\uparrow}(a_0, a_1) < p_0 + 12$ and $q_1 + 12 > q_0 + 12$. (Again, A has no pitch-class duplications, so $0 < \delta^{\uparrow}(a_0, a_1) < 12$.) It follows that $k = 0$ and p_1 is mapped onto q_1. But

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$$\begin{aligned} q_1 - p_1 &= (q_0 + \delta^{\uparrow}(a_c, a_{c+1})) - (p_0 + \delta^{\uparrow}(a_0, a_1)) \\ &= p_0 + d_0 + \delta^{\uparrow}(a_c, a_{c+1}) - p_0 - \delta^{\uparrow}(a_0, a_1) \\ &= \delta^{\uparrow}(a_0, a_c) + y + \delta^{\uparrow}(a_c, a_{c+1}) - \delta^{\uparrow}(a_0, a_1) \\ &= \delta^{\uparrow}(a_1, a_{c+1}) + y \end{aligned}$$

Using induction over the chord A, we see that each pitch class a_i moves by a distance of $\delta^{\uparrow}(a_i, a_{i+c}) + y$, where the first term represents scalar transposition by c steps, and the second term represents chromatic motion by y semitones.

When A contains multiple copies of a single pitch class, the above proof will not work since the distance $\delta^{\uparrow}(a_i, a_j)$ may be equal to 0 or 12. However, because we are working in continuous pitch-class space, we can take any crossing-free voice leading $A \rightarrow \mathbf{T}_x(A)$ and add small quantities ε_i to any duplicate pitch classes in A, producing a new voice leading $A^\star \rightarrow \mathbf{T}_x(A^\star)$ which is crossing free, has no pitch-class duplications, and is equal to $A \rightarrow \mathbf{T}_x(A)$ in the limit where the ε_i all go to 0. (Because the voice leading is bijective, this process is straightforward: construct a crossing-free instance $(p_0, \ldots, p_{n-1}) \rightarrow (q_0, \ldots, q_{n-1})$, so that both chords are in nondescending order and each spans an octave or less; if either chord spans precisely an octave, then lower all copies of its top pitch by a small amount δ so that it spans less than an octave; then add $i\varepsilon$ to each note p_i and q_i, choosing ε so that $n\varepsilon < \delta$.) The above proof then gives us the voice leading $A^\star \rightarrow \mathbf{T}_x(A^\star)$. In the limit where δ and ε go to 0, the distance $\delta^{\uparrow}(a_i, a_{i+c})$ between any duplicate pitch classes a_i and a_{i+c} goes to 0 for $i + c < n$, and to 12 for $i + c \geq n$. This is the ascending scalar distance between the notes.

NOTES

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1. The term ‘chord’ is somewhat anachronistic here, because early composers were likely to think of harmonies as products of musical lines rather than as entities in themselves. Nevertheless, they shared the sense that some verticalities (for example, C–G) were acceptable and others (for instance, C–C♯–D) were not, and that voices in general were to move efficiently so as to form acceptable verticalities.

2. See Huron (2001).

3. The actual count depends somewhat on how one chooses to individuate voice leadings. Here I am permitting ‘non-bijective’ voice leadings, in which a single note in one chord is associated with multiple notes in the other; I am also assuming that each pitch class moves to its destination by the shortest possible path, as explained in Section I (a). A voice leading in this sense can be represented as a set of ordered pairs (x, y) in which x is a pitch class in the first chord, y is a pitch class in the second, and every element of each chord appears in some pair.

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4. A mapping between two collections $A \rightarrow B$ is bijective if and only if it maps every object in A onto a unique object in B and if a unique object in A is mapped onto every object in B.

5. I use the phrase 'law of the shortest way' to refer to the general principle that one should try to minimise the overall distance moved by the voices in any voice leading. See Masson ([1694] 1967), Hostinsky (1879) and Schoenberg ([1911] 1983). Sources in the Western theoretical tradition do not fully specify what it means to 'minimise overall distance': contemporary theory's various metrics of voice-leading size represent different ways of interpreting this prescription (as indicated in Appendix A). All metrics agree, however, that the minimal voice leading between two chords A and B need not map each note in A onto the nearest note in B: for example, (C, C♯, D) → (B♭, C♯, D) is minimal (for all standard metrics) even though C is closer to C♯ than to B♭.

6. See Huron (2001).

7. A multiset is an unordered collection in which cardinality is significant. Thus the multiset of pitches {C4, C4, G4} is distinct from {C4, G4}. When analysing voice leading it is often simpler to use multisets rather than sets. This is because the number of 'doublings' in a chord affects its voice-leading capabilities: for instance, the chord {C4, C4, C4, F5, F♯5, G5} can be linked to one of its pitch-space inversions by very efficient bijective voice leading, while the chord {C4, F5, F♯5, G5} cannot be; see Tymoczko (2006) and Callender, Quinn and Tymoczko (2008). Consequently, Sections I–III of this article will model chords using multisets. Section IV considers non-bijective voice leadings between *sets* of pitch classes, thereby making closer contact with standard music-theoretical terminology.

8. When A and B are multisets which potentially contain multiple copies of a given pitch, we need to require that there be at least as many pairs whose first element is a as there are copies of a in A, and at least as many pairs whose second element is b as there are copies of b in B.

9. Or perhaps, less euphoniously, a 'pitch-class-space voice leading'. This definition corresponds to what I have elsewhere called a 'path-specific' voice leading between pitch-class sets because it specifies the path along which each pitch class moves to its destination. Other theorists, such as Morris (1998), Lewin (1998) and Straus (2003), favour 'path-neutral' voice leadings which do not specify particular paths. See Tymoczko (2005) for further discussion.

10. A path in pitch-class space is a real number indicating how far and in what direction a pitch class moves: thus motion by eleven ascending semitones (+11) is distinct from motion by one descending semitone (–1), even though these two paths link the same pair of pitch classes. This is precisely analogous to the fact that an ant could travel between 3 and 2 on an ordinary clock face either by moving $^{1}/_{12}$ of a circumference anticlockwise or $^{11}/_{12}$ of a circumference clockwise (see Tymoczko 2005).

11. Note that in passing from pitch to pitch-class space we lose the ability to say whether a note in one voice is above or below a note in another. However, we can still speak about the direction of the motion between notes in the same voice: given the pitch-class voice leading (G, G, B, D, F) → (C, G, C, C, E) we can

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say that the F moves down by semitone to the E, but we cannot say whether the E is above or below the B.

12. Once again, when A and B are multisets, we need to add the proviso that there are as many paths beginning with pitch class a as there are copies of a in A, and as many paths ending with pitch class b as there are copies of b in B.

13. If two voices sound the same pitch, then neither is above or below the other.

14. For example, the size of the voice leading (C4, E4) → (B♭3, G4) should depend only on the fact that one voice moves by two semitones, while the other voice moves by three semitones. Reducing the distance moved by any voice should not increase the voice leading's size.

15. One piece of evidence in favour of this requirement is that classical composers typically choose to avoid voice crossings rather than preserve common tones: given the melody D5 → C5 and the chord progression V–I in C major, classical composers generally prefer the upper-voice pattern (G4, B4, D5) → (E4, G4, C5), which avoids voice crossings, to (G4, B4, D5) → (G4, E4, C5), which preserves the common tone G4. My proposal is that we measure voice leadings such that this preference is consistent with the 'law of the shortest way'.

16. See Tymoczko (2006).

17. Tymoczko (2004b) explicitly distinguished between individual and uniform equivalence. At about the same time, Callender (2004) implicitly utilised this notion in developing geometrical models where individual equivalences were naturally represented. After a long and intensive period of collaboration, the connection between these two investigations was described in Callender, Quinn and Tymoczko (2008).

18. Note that although the definitions are stated here using pitch-class voice leadings, they have analogues in pitch space.

19. The terminology employed in the current article diverges somewhat from that of Callender, Quinn and Tymoczko (2008): what is here called individual I-relatedness is there described as uniform I and individual T. The difference ultimately derives from two different ways of conceiving of the TI group: Callender, Quinn and Tymoczko analyse it as combining transpositions with a single 'fixed' inversion $\mathbf{I}_0$, whereas I imagine it in this context as containing an equal number of transpositions and inversions, with none of the latter being privileged. It is hoped that the simplicity of the current terminology will outweigh any potential confusion.

20. If a voice leading moves a_i to b_i by d_i semitones, then a uniformly T-related voice leading moves $\mathbf{T}_x(a_i)$ to $\mathbf{T}_x(b_i)$ by d_i semitones and a uniformly I-related voice leading moves $\mathbf{I}_x(a_i)$ to $\mathbf{I}_x(b_i)$ by $-d_i$ semitones. For example, the voice leadings (C, E, G) → (C, F, A), (G, B, D) → (G, C, E) and (G, E♭, C) → (G, D, B♭) each move one voice by zero semitones, one voice by one semitone and one voice by two semitones.

21. More formally: if a voice leading moves pitch class a_i to b_i by d_i semitones, then any individually T-related voice leading will move $\mathbf{T}_x(a_i)$ to $\mathbf{T}_y(b_i)$ by $d_i + c$ semitones.

22. If a voice leading moves pitch class a_i to b_i by d_i semitones, then any individually I-related voice leading will move $\mathbf{I}_x(a_i)$ to $\mathbf{I}_y(b_i)$ by $c - d_i$ semitones.

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23. For example: suppose we order the elements of the C major triad so that C is first, E is second and G is third. We can invert each element in the sequence (C, E, G) around the semitone D–E♭ to produce the sequence (G, E♭, C). We then use this ordering to label the elements of the C minor triad so that G is first, E♭ is second and C is third. This is precisely what Riemann did, although he used the labels 'root', 'third' and 'fifth' rather than 'first', 'second' and 'third'.

24. Strictly speaking, this is a 'pitch-class chord progression'. One could also define progressions between chords of pitches.

25. Note that I use curly brackets {} and a double arrow ⇒ for chord progressions, reserving regular parentheses () and the single arrow → for voice leadings.

26. In this paragraph capital letters refer to major triads, lower-case letters to minor triads.

27. For a discussion of *Schritt*s and *Wechsel*s, see Klumpenhouwer (1994).

28. In the special case where chord B is invariant under every transposition and inversion which leaves A invariant, we can model dualistically equivalent progressions $\phi(A) \Rightarrow \phi(B)$ using a function F which commutes with transposition and inversion: $F(\phi(A)) = \phi(F(A)) = \phi(B)$, for all transpositions and inversions ϕ. Thus for example the neo-Riemannian 'P' transformation can be modelled as a function which takes a major or minor triad as input and outputs the other major or minor triad sharing its perfect fifth. See Lewin (1987) and Satyendra and Fiore (2005). However, we cannot do this when A has symmetries which B does not, because $F(\phi_1(A)) = F(A) = B \neq \phi_1(B)$, for some ϕ_1. This is why I prefer to model chord progressions as *objects* related by functions (transposition and inversion), rather than as functions themselves.

29. See Lewin (1987) for an abstract, group-theoretical discussion. It is a fact of elementary geometry that translations and reflections are the only isometries of the line and circle. Musically, this means that transposition (translation) and inversion (reflection) are the only distance-preserving functions over pitch space (the line) and pitch-class space (the circle).

30. See Lewin (1982) and Hyer (1995).

31. See Cohn (1996, 1997 and 1998).

32. See the various articles published in the particular issue of the *Journal of Music Theory* dedicated to research in this area (1998), in particular those by Callender, Childs and Douthett and Steinbach. The present article can be considered to be part of the 'explosion of interest' occasioned by Cohn's work.

33. There is a subtle point here with the potential to cause much confusion. Because we permit chords to have duplicate pitch classes, bijective voice leadings may have doublings. For instance, the voice leading (G, G, B, D, F) → (C, G, C, C, E), shown in Ex. 3a, is a bijective voice leading from {G, G, B, D, F} (a chord with two instances of the pitch class G) to {C, C, C, E, G} (a chord with three instances of pitch class C). This is because every item of each chord (including each of the individual 'duplicates') is related to one and only one item in the other. Section IV relaxes this restriction, making it possible to treat (G, G, B, D, F) → (C, G, C, C, E) as a *non-bijective* voice leading from {G, B, D, F} to {C, E, G}.

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34. Mathematically, this is the smallest non-negative real number x such that $a + x$ is congruent to b modulo 12. 'Ascending distance' might just seem to be a new name for the familiar pitch-class interval. However, theorists traditionally consider the pitch-class interval 11 to be the same size as the interval 1 – the two intervals belong to the same interval class and represent equal-sized steps taken in opposite directions. By contrast, the ascending distance of eleven is eleven times as large as the ascending distance of one (Tymoczko 2005). Ascending distance plays an important role in scale theory. See, for example, Clough and Myerson (1985).

35. The multiset {0, 4, 4, 7} has three *elements* (0, 4 and 7) and four *objects* (0, the first 4, the second 4 and 7). When putting the multiset in scale-degree order we keep track of the multiple instances of each element by assigning each object a different scale-degree number.

36. Note that scale degrees are typically numbered from 1, whereas modular arithmetic requires subscripts starting with 0. Thus, in the chord $(a_0, a_1, \ldots, a_{n-1})$, a_0 is the first scale degree, a_1 is the second scale degree, and so on. This is an unfortunate but unavoidable inconsistency between standard musical terminology and standard mathematical notation.

37. The reasons for this are given in n. 80 and Appendix B.

38. Theorists often consider scalar transpositions in which a smaller chord is transposed along a larger scale – for instance, when a C major triad is transposed along the C major scale. Here I am concerned with the special case in which the chord and scale are the same, such as when we shift the entire C major scale up by scale step.

39. When I refer to a voice leading as 'minimal' without specifying a particular metric, I mean it is minimal with respect to all metrics satisfying the two constraints in Section I (a).

40. In other words, scalar transposition down by step is a not-very-fuzzy 'fuzzy transposition'. The term 'fuzzy transposition' was introduced by Quinn (1996); see also Lewin (1998) and Straus (2003).

41. Here as elsewhere the elements of the matrix are labelled starting with 0, so that the upper left corner is $M_{0,0}$ and the lower right is $M_{n-1,\,n-1}$. The scalar interval matrix is defined relative to the arbitrary choice of a first scale degree. (Choosing a different first scale degree rotates the columns of the matrix.) This choice does not affect in any important way the information contained in the matrix or the way we calculate with it. I will therefore speak as if there were a unique scalar interval matrix for each chord. See Rothenberg (1978).

42. An important terminological caution: the scalar interval matrix contains 'ascending scalar distances', as defined in the text. These are paths in pitch-class space, rather than (traditional) pitch-class intervals.

43. A serial 'rotational array' has as its first row a series of pitch classes, usually starting on zero. Its second row rotates the first to the left by one place, transposing it so that its first pitch is again zero. Subsequent rows continue this process through all the rotations of the original row. Rotational arrays were used by Stravinsky and have been studied by Babbitt (1964 and 1997), Rogers (1968),

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Morris (1988) and others. There are two differences between scalar interval matrices and traditional arrays. First, scalar interval matrices contain paths in pitch-class space rather than traditional pitch-class intervals. Thus these matrices can contain the value 12 as well as 0. (Interscalar interval matrices, as we will see, can also contain negative values.) Second, scales are ordered in pitch-class space, whereas traditional twelve-note rows can be arbitrarily ordered.

44. Suppose a composer uses a row to present the main theme of a composition, while in the recapitulation he or she wishes to evoke the main theme without repeating it exactly. The composer might therefore look for that rotated and transposed row which minimises the interval class between each order position in the original theme and the corresponding order position in the recapitulation. The relation between these two rows is analogous to the relation between two chords linked by minimal crossing-free voice leading. Roeder (1984 and 1987) explores the parallels between the temporal ordering in serial composition and the registral ordering needed to analyse voice leading. Roeder's concerns and methods are important precursors to those of the present article. Lewin (1977) proposes – but does not explore – an approach similar to Roeder's. Regener (1974) and Chrisman (1977) articulate points of intersection between scale theory and set theory.

45. Again, when I do not specify a metric all reasonable metrics are in agreement.

46. See Cohn (1996) and Tymoczko (2005).

47. See Tymoczko (2006).

48. Or if the chord can be represented as the union of equal-sized subsets which themselves divide the octave evenly. Agmon (1991) and Cohn (1996) make closely related observations. Block and Douthett (1994) describe a continuous measure of evenness.

49. For example, the scalar transposition (C, E, G) → (E, G, C) moves its voices by four, three and five semitones. These values are approximately equal to 4.

50. Alert readers will note that for very 'clustered' chords, such as {C, C♯, D}, the values in the scalar interval matrix, when considered as traditional pitch-class intervals, are all approximately equal to 0. The scalar transpositions do not generate minimal voice leadings, because they only enable us to offset chromatic transposition by zero semitones. (Furthermore, these voice leadings involve crossings when each pitch class moves to its destination by the shortest possible path.) However, they can yield efficient voice leadings from the chord to itself. See Tymoczko (2006).

51. A trivial voice leading moves each voice by 0 semitones.

52. For example, {C, E, G, B♭} is a generalised fourth chord, because it can be partitioned into {C, G} and {E, B♭}. Efficient voice leading connects it to its tritone transposition {C♯, E, F♯, A♯}, moving the perfect fifth {C, G} by semitonal contrary motion to {C♯, F♯} and holding the tritone {E, B♭} fixed.

53. See Hansen (1996).

54. Here, tritone substitution introduces a three-semitone ♭7 → 5 leap in the alto voice. Such three-semitone leaps will appear when the original progression contains a perfect fourth or perfect fifth which descends in parallel by major second.

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55. The proof follows the same basic outlines as that given in Appendix B: we derive an inequality $0 < X + 12k < 12$ and show $0 < X < 12$, which implies $k = 0$.

56. Santa (1999) discusses scale-to-scale operations related to interscalar transposition, and Hook (2007) discusses interscalar transpositions in pitch space.

57. The form of the interscalar interval matrix will depend on our identification of the first scale degree and our choice of the crossing-free voice leading from A to B. Since the same essential information is conveyed by all of these variant matrices, I will speak somewhat loosely of *the* interscalar interval matrix.

58. See in particular Morris (1988).

59. Here we are attempting to find that value x such that the maximum of $|4 + x|$ and $|x + 2|$ is as small as possible. Hence $x = -3$.

60. There is a striking similarity between this table and the voice leadings surveyed in Childs (1998), which approaches some of the topics treated in this article from a neo-Riemannian perspective.

61. Such voice leadings can involve no fewer than seven semitones of total motion, as in (C, E♭, G♭, B♭) → (C♯, D, E♭, C).

62. See Roeder (1984 and 1987). The point here is simply that the crossing-free voice leading $(x_0, x_1, \ldots, x_{n-1}) \rightarrow (y_0, y_1, \ldots, y_{n-1})$ is small only when $x_i \approx y_i$, for all i. Hence, the ascending scalar distance from x_0 to x_1 must be approximately equal to the ascending scalar distance from y_0 to y_1, the ascending scalar distance from x_1 to x_2 must be approximately equal to the ascending scalar distance from x_1 to x_2, and so on. The vector of ascending scalar distances between adjacent x_i is simply the second row of the chord's scalar interval matrix.

63. It is interesting to compare this notion of 'set class similarity' to other measures which have been proposed. See Quinn (2001), Straus (2003) and Callender, Quinn and Tymoczko (2008).

64. Interested readers are encouraged to derive formulas relating the second row of two chords' scalar interval matrices to the crossing-free voice leadings between them.

65. Compare Chapter 36 of Gauldin (1997).

66. Debussy uses similar progressions in the passage between rehearsal numbers 10 and 14 of 'Fêtes', the second of the orchestral *Nocturnes.*

67. In the course of future research I intend to apply these analytical techniques more comprehensively with a view to determining the ways in which Debussy's voice-leading practices depart from those of Wagner. For example, in Tymoczko (2004a), I argue that Debussy's chords often give rise to non-diatonic scales at the surface level, whereas Wagner's typically do not.

68. Roughly speaking, this is because the three notes of the chord {C, E, F♯} cluster in two approximately antipodal regions of the pitch-class circle. In this sense, {C, E, F♯} is 'close' to {C, F♯} and consequently inherits some of its voice-leading properties. {C, F♯} has a *trivial* voice leading to its tritone transposition, and chords close to {C, F♯}, such as {C, E, F♯} and {C, F}, have *efficient* voice leadings to their tritone transpositions. In the same way, the four notes of the

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chord {E, G♯, B, D♯} cluster in three more or less evenly spaced regions of pitch-class space; consequently, {E, G♯, B, D♯} inherits some of the voice-leading properties of {E, G♯, C} and can be linked to its major-third transposition by efficient voice leading.

69. T-matrices are a long-standing fixture of serial and atonal theory and have been studied under various names by Alphonce (1974), Gamer and Lansky (1976) and Morris (1987, 1988 and 1998). (The term 'T-matrix' is due to Morris.) Some of this work addresses voice leading, either obliquely (as in Gamer and Lansky 1976) or explicitly (as in Morris 1998). The present article continues this tradition.

70. The reason for this is that when using T-matrices, we assume each note will move to its destination by the shortest possible path. We could create T-matrices using paths in the range $-6 < x \leq 6$, but these are virtually equivalent to pitch-class intervals.

71. A problem can be solved in polynomial time if the number of calculations it requires is determined by some polynomial of the input variables.

72. See Section 8 of the supplementary online materials to Tymoczko (2006). I have written a simple computer program, which runs as a Java applet in Max, to find the minimal voice leading between arbitrary chords using dynamic programming. See http://music.princeton.edu/~dmitri/.

73. See Tymoczko (2005) and Callender, Quinn and Tymoczko (2008) for more discussion of paths in pitch-class space.

74. Reading from the first entry in the fourth row of Fig. 11a upwards along the diagonal to the fourth entry in the first row, one finds the pitch classes (B♭, G, E♭, C♭). Inverting this sequence around B, one obtains (C, E♭, G, B), which is the source chord of Fig. 11a, listed in scale-degree order.

75. This is because Fig. 2 connects an inversionally symmetrical set class to itself, while Fig. 4e connects a set class to its own inversion.

76. These diagonals wrap around, moving off the right edge of the matrix to reappear on the left.

77. This is because pitch-class inversion reverses a multiset's step-interval vector. See Regener (1974) and Chrisman (1977) for discussion.

78. The first row of $M_{\mathbf{I}_x(A)\to\mathbf{I}_x(B)}$ will be the rotated retrograde of the complement of the first row of $M_{A\to B}$, the second row of $M_{\mathbf{I}_x(A)\to\mathbf{I}_x(B)}$ will be the rotated retrograde of the complement of the last row of $M_{A\to B}$, and so on.

79. See Tymoczko (2004a).

80. See Clough and Myerson (1985). There is a slight complication here in the case of chords with multiple copies of a single pitch class: given the scale $S = (a_0, a_1, a_2) = (C, C, C)$, if we want scalar transposition by one step to span an octave, then we should take a_0 to move to a_1 by 0 semitones, a_1 to move to a_0 by 0 semitones and a_2 to move to a_0 by 12 semitones. This is what motivates the definition of 'ascending scalar distance' in Section II (a).

81. See Hook (2003 and 2004).

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82. See Tymoczko (2005).

83. See Mazzola, Göller and Müller (2002), p. 257.

84. See Callender (2004), Tymoczko (2006) and Callender, Quinn and Tymoczko (2008).

85. See Roeder (1984), Lewin (1998) and Straus (2003).

86. The L^p norm of the multiset $\{x_0, x_1, \ldots, x_{n-1}\}$ is the quantity $(\Sigma|x_n|^p)^{1/p}$.

87. See Callender (2004).

88. See Cohn (1997) and Douthett and Steinbach (1998).

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ABSTRACT

Efficient voice leading, in which melodic lines move by short distances from chord to chord, is a hallmark of many different Western musical styles. Although musicians can often find maximally efficient voice leadings with relative ease, theorists have not adequately described general principles or procedures for doing so. This article formalises the notion of voice leading, shows how to classify voice leadings according to transpositional and inversional equivalence and supplies algorithms for identifying maximally efficient voice leadings between arbitrarily chosen chords. The article also includes analytical and theoretical discussions of neo-Riemannian theory, the 'tritone substitution' in contemporary jazz, the music of Wagner and Debussy, the relation between harmony and counterpoint and the connections between scale theory and serial theory.

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