GENERALIZED DIATONIC AND PENTATONIC SCALES: A GROUP-THEORETIC APPROACH

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I. INTRODUCTION

THIS IS THE SECOND in a planned series of papers in which musical scales, tuning, and temperament are studied from a mathematical point of view.[1] In the present paper, group-theoretic methods are used to study N-tone scales for various N. The methods used there were described initially by Budden (1972) and elaborated by Balzano (1980). To assist the reader who may not be familiar with these references, a brief review is given below, but for a complete understanding the references should be studied, particularly the paper of Balzano. A number of authors have studied N-tone scales with $N > 12$. Typically, they have attempted to find good approximations to the canonical tuning ratios (Rossing 1990, 178). Examples are papers by Hall (1985) and Fuller (1991). A somewhat different approach has been followed by Clough and Myerson (1986) and Clough and Douthett (1991), as well as references cited therein. The ideas in these studies is to use number-theoretic arguments to choose N's containing acceptable diatonic subsets.[2] My approach is *structural*—I seek N's and diatonic subsets which have musical structure similar to the familiar twelve-tone set with its seven-note diatonic scale. To anticipate my conclusion, I suggest that the only viable alternative to $N = 12$ is $N = 20$. Interestingly enough, when Douglas Keislar (1991) interviewed six American composers on their microtonal choices,[3] none of them used $N = 20$; one or more have used 11, 13, 17, 19, 22, 24, 31, and 144.

A division of the octave into N logarithmically equal frequency intervals is called an *N-tone scale*. The ratio between two successive notes of the scale is seen to be $2^{1/N}$ or, in cents, $1200/N$¢.[4] Pitch classes, whose elements are the same scale step in different octaves, are viewed as equivalence classes which form the elements of the cyclic group C_N. If the classes are numbered sequentially from 0 to $N-1$, then the group operation is addition, mod N, of the numbers of the sequence.[5] For the most familiar case, $N = 12$ and the intervals are called (100¢) semitones, denoted S. The sequential ordering begins with C = 0, C♯ = D♭ = 1, and so forth, up through $B = 11$.

The groups C_N are characterized by primitive elements, or generators: n is a generator of C_N if $nN = 0$, mod N and $nK \neq 0$, mod N, $K < N$. Putting it another way, in the sequence $n, 2n, 3n, \ldots (N-1)n$ (mod n), each element of the group occurs exactly once. The group C_{12} has four generators: 1 and its inverse 11; and 5 and *its* inverse 7.[6] The sequence generated by 1 is simply the ascending chromatic scale, that by 11 the descending chromatic scale, while 5 and 7 generate the circle of fourths and circle of fifths, respectively (or, alternatively, the descending and ascending circle of fifths). When we deal with the groups C_N for $N > 12$, we shall identify certain of the primitive elements as *generalized* fifths and fourths. The various ordering of group elements generated by different primitive elements are called automorphisms of the group.[7]

C_{12} has *subgroups* C_2, C_3, C_4, and C_6.[8] In general, C_K is a subgroup of C_N if and only if (iff) K divides N, i.e. if $N = 0$, mod K. A *coset* of C_K in C_N is obtained by adding, mod N, to each element of C_K the same element of C_N. It turns out there are N/K distinct cosets (cosets differing only in the order of elements are considered the same coset, expressing the musical fact that inversions of chords are identified as the same chord). As an example, the subgroup C_4 of C_{12} is (0, 3, 6, 9). The subgroup itself is one coset (obtained by adding the group element 0 to each element of the subgroup). The other cosets are (1, 4, 7, 10), obtained by adding 1 (or 4, or 7 or 10, mod 12) to each element; and (2, 5, 8, 11).

These three cosets form the so-called ***quotient*** group C_{12}/C_4. They are the three diminished-seventh chords of the twelve-tone scale.

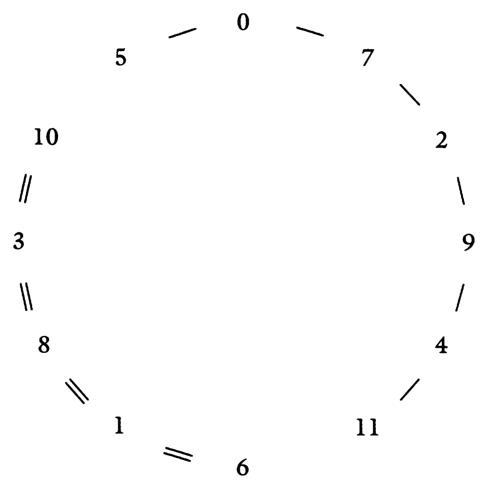
The other quotient groups are C_{12}/C_2, representing the six tritones; C_{12}/C_3, representing the four augmented triads; and C_{12}/C_6 representing the two whole-tone scales. The fact that the quotient group C_N/C_K has N/K elements makes it easy to figure out how many chords or scales of a given type exist. This is not too important for $N = 12$, because everybody already knows the answer anyway, but it can be useful for pitch systems with $N > 12$.

A ***diatonic scale*** in C_{12} is defined to be seven connected elements in the circle of fifths (or fourths). Balzano (1980) observes that it enjoys the F → F♯ property, which means that moving from one such connected set to the next adjacent one leaves six notes the same and changes the other by a minimal amount, i.e. up or down a semitone, depending upon which way one moves. This is illustrated in Example 1, which is Figure 4 of Balzano's paper. The connected set, joined by lines, represents the C major scale or one of its modes, depending on the (cyclic) ordering. In fact, these seven modes form the elements of the cyclic group C_7 (which is ***not*** a subgroup of C_{12}). The group elements are cyclic permutations which act on the symbolic sequence $T\ T\ S\ T\ T\ T\ S$, representing sequences of tones (T) and semitones (S).[9] The modes will be studied in more detail in a forthcoming paper.

Balzano demonstrated the important result that the various properties considered desirable in constructing scales (the F → F♯ property and the isomorphism to a Cartesian product) occur only for N-tone systems with $N = k(k+1)$ for some integer k. For this reason, I shall, like Balzano, consider only such values of N (see Appendix).

The pentatonic scale is defined to be five connected elements in the circle of fifths; it also has the F → F♯ property, a fact which was not noted by Balzano. The existence of two scales with this property will be important when the cases $N > 12$ are considered. In terms of the pentatonic sequence (M2)(m3)(M2)(M2)(m3), the pentatonic scale, and its five modes, can be considered elements of the group C_5.

Balzano also discusses, in great detail, how the isomorphism $C_{12} \approx C_3 \times C_4$ (proved, more generally, in the Appendix) can be used in the construction of chords. This isomorphism means, arithmetically, that every integer 0, 1, ... 11 can be written in the form $3n + 4m$, mod 12, for integers $0 \leq n \leq 4$ and $0 \leq m \leq 3$. Musically, it says that every interval can be decomposed into major thirds and minor thirds, e.g. P5 = M3 + m3. Musicians usually do not think of expressing arbitrary intervals in terms of thirds—e.g. a minor second equals three minor thirds plus a major third—perhaps because, unlike mathematicians, they do not consider the second and the ninth the same interval!

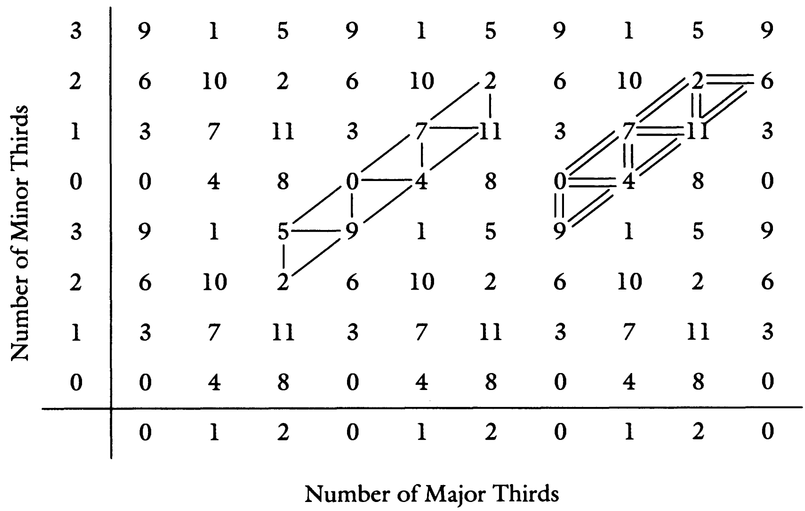


The circle of fifths. A single line connects the notes of the diatonic scale. A double line connects the notes of the pentatonic scale. Observe that with respect to clockwise rotation, the diatonic scale has the F → F♯ property, while the pentatonic scale has the F♯ → F property. The reason for this reversal is that the pentatonic scale is based on the circle of fourths, and ascending keys should rotate counterclockwise in the above diagram.

EXAMPLE 1

Example 2 here is similar to Balzano's Figure 5; each point of the array consists of p major thirds (the x-coordinate) plus q minor thirds (the y-coordinate). Points in the array labelled with the same numbers are to be considered the same point, i.e. the array represents an unrolled torus. By joining the seven notes of the diatonic scale, a chordal structure is produced in the figure. Right triangles with vertices pointing to the right represent major chords; those with vertices pointing to the left represent minor chords. A straight line three units long parallel to the y-axis represents a diminished chord. A similar line parallel to the x-axis would represent an augmented triad, but there is none in the figure. (They will enter the study of permuted scales, to be presented in a forthcoming paper.)

The diminished chord is not immediately visible, because of the toroidal geometry, but it is there (11–2–5, B–D–F, for the C-Major case; 6–9–0, F♯–A–C, for the G-Major).



Balzano diagram for the diatonic scale. For different key signatures, the diagram is of identical shape, but translated: — 0 mode; = 7 mode.

EXAMPLE 2

A similar diagram exists for the pentatonic scale, but it has little structure and will not be drawn. The graph in Example 2 will be called a "Balzano diagram." It will be particularly useful in analyzing the twenty-tone system. It is also helpful in the study of permuted scales in that the chordal structure of the subdominant-tonic-dominant progression can be read off immediately for any mode. Take the 0-mode. The progression is seen to be 5–9–0, 0–4–7, 7–11–2, three right triangles and hence three major chords (M M M). Now look at the 5 mode. The progression is seen as 11–2–5, 5–9–0, 0–4–7, i.e. d M M (d means "diminished"). Or the 9 mode: m m m. All this is well-known in the twelve-tone system, but in N-fold systems with higher N the Balzano diagram is very useful in figuring out the chordal progressions which tell us, for example, which mode should be considered major and which natural minor.

Before continuing with a group-theoretical analysis of microtonal pitch systems, it would be well to review some of the microtonal systems already in existence, and to compare them with that proposed here.

Microtonality, of course, has a long musical history, much of which has been reviewed and analyzed in two earlier issues (vol. 29, nos. 1 and 2) of this journal. As is explained there, the term "microtonality" means different things to different composers. For example, Ben Johnston (Fonville 1991) begins with a justly-tuned twenty-four note enharmonic scale (identical to that proposed in my own article on just tuning (Zweifel 1994)) and then proceeds to modify it in an attempt to tune the higher partials. The result is a seven-note diatonic scale which is supposed to improve consonance in string ensembles (Elster 1991). No attempt is made to deal with scales with more than seven notes.

Similarly, Easley Blackwood (1991) divides the octave into N equal intervals, $N > 12$, as is done in this paper, but then constructs seven-note diatonic scales consisting of five wide and two narrow intervals (analogous to the usual tones and semitones of standard practice, but with ¢ ratios different from 2:1). He then goes on to study the consonance of various chords in such schemes concentrating on the (analogues of) the subdominant, dominant, and tonic chords of the standard system. (The use of microtonal scales such as Blackwood's and microtonal-capable instruments such as the saxophone and, of course, the strings, to implement them goes back in this century to the earliest days of jazz.)

The work described originally by Balzano (1980) and amplified here is essentially different, in that the analogues of diatonic and pentatonic scales with more than seven and five notes respectively are constructed (by suitably generalizing, in group-theoretical terms, the definitions of "diatonic" and "pentatonic"). Chordal structure is stressed, as in Blackwood's work, but there are more than three basic triads. (In the seven-note scales there are of course seven triads, three of which are considered "primary"; in the eleven-note scale espoused in this article, five of the eleven triads are considered primary, as is explained in the next section.) The approach to scale construction begun by Balzano and continued here is in accordance with Blackwood's advice: "When investigating a tuning for which there is little or no repertoire or tradition, it will be helpful to look for similarities between the new tunings and the familiar twelve-note equal tuning" (Blackwood 1991, 167).

This is exactly Balzano's approach, but the similarities are perceived as a congruent mathematical (group-theoretical) structure rather than melodic or chordal ideals. That means that one seeks scales which enjoy the various group-theoretical properties of the $N = 12$ system (especially the F $\rightarrow$ F♯ property). This requires a faith that musical expression has an underlying mathematical structure, in particular the group-theoretical structure of Balzano, and that preservation of this structure in generalizing the present system will lead to music which stands to present practice in evolution rather than revolution.

It is obvious that the development of computers and computer-driven instruments is what makes musically feasible the generalization of the $N = 12$ system to higher values of N. The systems of Johnston and Blackwood, while adaptable to the computer, can also be realized on conventional instruments as already noted.

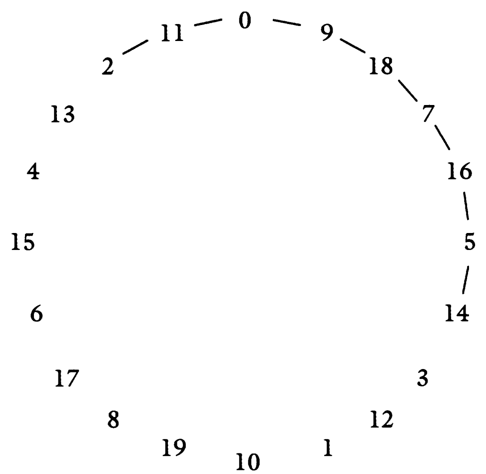
II. The Case of $N = 20$

It is necessary to study the group C_{20}. It has the primitive elements 1, 3, 7, 9 and their inverses 19, 17, 13, and 11. It has subgroups C_{10}, C_5, C_4, and C_2. If by abuse of nomenclature the 60¢ interval of the twenty-tone scale is referred to also as a semitone, then the quotient group C_{20}/C_{10} gives the two whole-tone scales. The group C_{20}/C_4 represents five diminished chords if five semitones, 300¢, are considered to be a minor "third." Since this is the precise width of the (tempered) minor third in the twelve-tone system, this seems reasonable.[10] In anticipation of the scale which will be constructed, the elements of C_{20}/C_4 will be called "ninth chords." A typical element is (0, 5, 10, 15). It is actually the analogue, in the twenty-tone system, of the diminished seventh chord in $N = 12$. In fact, in cents it is identical.

The cosets of C_5 in C_{20} comprise four elements of the form (0, 4, 8, 12, 16). Since the intervals are 240¢, they are closely related to the 200¢-wide whole-tone scale of the twelve-tone system, although they have one less note. They might also be viewed as a variant of the pentatonic scale, and might well be investigated performance-wise.

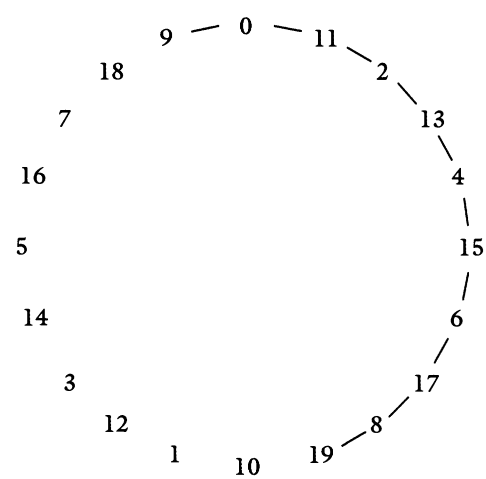
Finally, there is the group C_{20}/C_{10}, two 600¢ tritones, as always.

If one considers a scale based on 0, then 10 is the tritonic element. It lies between the generators 11 and 9. By analogy with the twelve-tone system, 11 is the "generalized fifth" and 9 the "generalized fourth." From this point of view, the diatonic scale in the twenty-tone system is based on the automorphism generated by the primitive element 11, the "circle of sevenths," it turns out, and it has eleven elements. The scale constructed in Balzano 1980 is based on the primitive element 9, and has nine notes. It is the analogue in C_{20} of the pentatonic scale. Note (Examples 3a and 3b) that both scales have the F → F♯ property, but, as we go on to show, the eleven-note scale has many other properties which make it a better candidate for a scale in the twenty-tone system than the nine-note scale.



The nine-note scale of Balzano. It is based on the circle of sixths (generalized circle of fourths). Under clockwise rotation, 2 → 3, other notes remain the same.

EXAMPLE 3a



The eleven-note scale espoused in the present article. It is based on the circle of sevenths (generalized circle of fifths).

EXAMPLE 3b

I have already noted one of these properties, the 300¢ minor fourth which takes the place of the minor third of the twelve-tone system. The major fourth of 360¢ which takes the place of the twelve-tone major third is reasonably close to the 400¢ tempered twelve-tone major third and even closer to the just 386¢ major third. In the nine-note scale the corresponding intervals are 240¢ for the minor interval and 300¢ for the major, which seems much less desirable.

I should now justify the terms "ninth chord"; "major\minor fourth" and "circle of sevenths" which have already been introduced. I shall show presently that the major scale has the interval structure $T\,T\,T\,T\,S\,T\,T\,T\,T\,S\,T$ with, we recall, $S = 60$¢ and $T = 120$¢. Note that this scale contains both the "generalized fourth," $9S = 540$¢, which now is a sixth; and the generalized fifth, $11S = 660$¢, which is a seventh (hence "circle of sevenths"). Note also that 15 is a ninth of the scale, hence my calling (0, 5, 10, 15) the ninth chord.[11] Furthermore, $6S = 360$¢ is indeed a major fourth and so $5S = 300$¢ is a minor fourth.

By now, every reader should be convinced that the eleven-note scale is the correct analogue of the seven-note diatonic scale in the twelve-tone system, but one more piece of evidence is still available. In going from a twelve-tone to a twenty-tone pitch system, the number of notes is increased by a factor of 5/3. It is reasonable that the notes of the diatonic scale should increase by roughly the same factor; $5/3 \times 7 = 11\ 2/3 \approx 11$.[12]

The chordal structure still must be studied, and this means considering the isomorphism between C_{20} and $C_4 \times C_5$ (Appendix). In Example 4 the Balzano graphs of the eleven-note and nine-note scales are shown. They are drawn exactly like Figure 6 in Balzano 1980, in complete analogy with the corresponding graph (Example 2) for $N = 12$. It is clear that the eleven-note scale has a perfectly acceptable connected structure, only with acute angles taking the place of right triangles. There are five major chords (left pointing acute angles, e.g. 9–15–0 et cetera), five minor chords (right pointing acute angles, e.g. 4–9–15 et cetera), and a diminished chord, 19–4–9. (The fact that the eleven-note scale has a central chord is also an argument in its favor, since the nine-note scale does not have such a chord. See Balzano 1980.)

I now use the Balzano diagram, as described at the end of Section I, to identify the major and natural minor modes of our diatonic scale. I shall call the five chords appropriate to each mode the sub-sub-dominant, the sub-dominant, the tonic, the dominant and the super-dominant. They are, respectively, the XI (or xi); VI (or vi); I(i); VII(vii) and II (or ii) chords. From Example 4, it is seen that the eleven-mode[13] contains all five major chords, and has every right to be called the major scale:

11 13 15 17 19 0 2 4 6 8 9

with *TS* sequence

T T T T S T T T T S T.

Number of 240¢ intervals												
3	12	17	2	7	12	17	2	7	12	17	2	7
2	8	13	18	3	8	13	8	3	8	13	18	3
1	4	9	14	19	4	9	14	19	4	9	14	19
0	0	5	10	15	0	5	10	15	0	5	10	15
4	16	1	6	11	16	1	6	11	16	1	6	11
3	12	17	2	7	12	17	2	7	12	17	2	7
2	8	13	18	3	8	13	18	3	8	13	18	3
1	4	9	14	19	4	9	14	19	4	9	14	19
0	0	5	10	15	0	5	10	15	0	5	10	15
4	16	1	6	11	16	1	6	11	16	1	6	11
3	12	17	2	7	12	17	2	7	12	17	2	7
2	8	13	18	3	8	13	18	3	8	13	18	3
1	4	9	14	19	4	9	14	19	4	9	14	19
0	0	5	10	15	0	5	10	15	0	5	10	15
	0	1	2	3	0	1	2	3	0	1	2	3

Number of 300¢ intervals

EXAMPLE 4: BALZANO DIAGRAMS FOR THE ELEVEN-NOTE (—) AND NINE-NOTE (=) SCALES IN THE TWENTY-TONE SYSTEM

It does lack a leading tone—this should not be terribly important when the whole tone is only 120¢ wide. If it is, the 0-mode would be a possible alternative, as it has a leading tone plus four major chords, and a diminished sub-sub-dominant chord, as is easily read from Example 4. The 9-mode is the other leading-tone mode, but its chordal sequence is m d M M M, which is bad. It also contains the tritone (just like the F, or Lydian, mode of the twelve-tone system, of which it is the analogue). The natural minor is evidently the 6-mode as it has five minor chords. Its *TS* sequence is

$$TSTTTTTSTTT.$$

If one accepts the 11-mode as the major scale, then the major intervals can be defined as in Example 5.

Interval	*Semitones*
$M2$	2
$M3$	4
$M4$	6
$M5$	8
$M8$	13
$M9$	15
$M10$	16
$M11$	18

EXAMPLE 5: THE MAJOR INTERVALS OF THE ELEVEN-NOTE SCALE IN THE TWENTY-TONE PITCH SYSTEM

The natural minor 6-mode scale has the perfect sixth and seventh, as it should. It has, starting from the tonic, six minor intervals and two major intervals, the second and the eighth. This is in good accord with the twelve-tone case, where the natural minor scale—the 9-mode in that case—has as its first interval a major second.

Evidently a piano built to play the twenty-tone system should have eleven white keys and nine black keys,[14] the black keys being placed between every pair of white keys except the fifth and sixth and the ninth and tenth. The note I have called "11" previously might be called C, with the white notes continuing

C D E F G H I J K A B.

Then C major and K minor are played on the white keys (as are the other nine modes, the analogue of the ecclesiastical modes) based on D E F G and so on. The key of I major has one sharp—B♯—while H major has one flat, G♭.

The whole array of key signatures can be constructed easily by adding a perfect seventh (11*S*) for sharps and a perfect sixth (9*S*) for flats in complete analogy with the use of the circle of fifths for sharps and circle of fourths for flats in the twelve-tone system. In Example 6, the key signatures are presented for twenty-three different keys.

The relative minor is of course lower than the major by a minor fourth which, as has already been seen, is the analogue of the twelve-tone minor third.

The same analysis of chordal structure keys, and so on, could be carried out for the nine-note scale which, in analogy with the pentatonic scale in the twelve-tone system, could easily be played on the piano we have described as appropriate for the eleven-note scale. It has a similar structure to the ordinary pentatonic scale in that it has two wide intervals (180¢) and six narrow intervals (120¢). The modulation structure of the ordinary pentatonic scale is not usually studied in music theory courses, but one easily visualizes that in the twenty-tone system the nine-note scale could be more important. For this reason, an analysis is presented here of the nine-note scale in the context of the keyboard described above.

The Balzano diagram for the nine-note scale appears in Example 4 and indicates that the "0" mode is the "major," since it contains four "major chords," and the 16 mode is the "minor." Expressing this in terms of the note names as introduced above, the major scale, as read from the circle of sixths, Example 3a, is

H I J♯ K♯ B C D♯ E♯ F♯.

Still using the circle of sixths, this scale, transposed to the key of E, has no sharps or flats:

E F H I J K B C D.

The key signatures containing sharps are obtained simply by moving clockwise around the circle of sixths. For example, one step gives the scale

J K B C D E F♯ H I.

C	–											
I	B♯											
D	B♯	H♯										
J	B♯	H♯	C♯									
E	B♯	H♯	C♯	I♯								
K	B♯	H♯	C♯	I♯	D♯							
F	B♯	H♯	C♯	I♯	D♯	J♯						
A	B♯	H♯	C♯	I♯	D♯	J♯	E♯					
G	B♯	H♯	C♯	I♯	D♯	J♯	E♯	K♯				
B♯	B♯	H♯	C♯	I♯	D♯	J♯	E♯	K♯	F♯			
H♯	B♯	H♯	C♯	I♯	D♯	J♯	E♯	K♯	F♯	A♯		
C♯	B♯	H♯	C♯	I♯	D♯	J♯	E♯	K♯	F♯	A♯	G♯	
H	G♭											
B	G♭	A♭										
G♭	G♭	A♭	F♭									
A♭	G♭	A♭	F♭	K♭								
F♭	G♭	A♭	F♭	K♭	E♭							
K♭	G♭	A♭	F♭	K♭	E♭	J♭						
E♭	G♭	A♭	F♭	K♭	E♭	J♭	D♭					
J♭	G♭	A♭	F♭	K♭	E♭	J♭	D♭	I♭				
D♭	G♭	A♭	F♭	K♭	E♭	J♭	D♭	I♭	C♭			
I♭	G♭	A♭	F♭	K♭	E♭	J♭	D♭	I♭	C♭	H♭		
C♭	G♭	A♭	F♭	K♭	E♭	J♭	D♭	I♭	C♭	H♭	B♭	

EXAMPLE 6: KEY SIGNATURES FOR THE ELEVEN-NOTE MAJOR MODES, $N = 20$

This corresponds to moving forward a perfect sixth of the eleven-note scale. The scales with flats are obtained from moving in the opposite direction. The first is

K B♭ C D E F H I J.

These scales correspond to moving forward a perfect *seventh* of the eleven-note scale. The key signatures are given in Example 7. It should be noted that the note names G and A do not enter any of the nine-note scales. Again, it should be noted that this scale contains nine modes, which can be created by playing ten successive white-key notes E F H I J K B C D in the nine possible cyclic orders. The E mode is, as I have noted, the "major"; the I mode is the "minor"—it lies four semitones below the major mode, a "minor third" of the nine-note scale, as expected.

This completes the discussion of the twenty-tone pitch system. In a forthcoming paper, permutations of the diatonic and pentatonic (or "generalized pentatonic"—i.e. "nonotonic") scales will be studied. The permutations, which are taken over the "wide" and "narrow" intervals of the scales, lead to new scales which are disconnected sets of the circle of fifths (or of generalized fifths). One disconnected scale has already been encountered: the whole-tone scale is a maximally disconnected set—every other note—in the (generalized) circle of fifths. The analyses of these permuted scales is considerably simplified through the use of the group-theoretical methods used here, particularly the "generalized fifths" automorphism and the Balzano diagram.

III. The Case $N = 30$

This section will be brief, since the thirty-tone system seems to have little to offer. First, it has subgroups C_{15}, C_{10}, C_6, C_5, C_3, and C_2. Recalling that if two different group C_N and C_M have the same subgroup C_K, the subgroups are identical *in cents*,[15] we see that C_{15} is the only subgroup which has not already been encountered. It represents a fifteen-note equal-interval scale, the intervals being 80¢ wide. The thirty-tone system also lacks the subgroup C_4 present in C_{12} and C_{20}, which represents a deficiency.

But the major problem with the thirty-tone system is that the two F → F♯ scales are generated by the primitive elements 19 (the generalized fifth) and 11 (the generalized fourth). A simple calculation indicates that a nineteen-note scale must have eleven "tones" (intervals 80¢ wide)

Key	Signatures								
J	F♯								
D	F♯	K♯							
I	F♯	K♯	E♯						
C	F♯	K♯	E♯	J♯					
H	F♯	K♯	E♯	J♯	D♯				
B	F♯	K♯	E♯	J♯	D♯	I♯			
F♯	F♯	K♯	E♯	J♯	D♯	I♯	C♯		
K♯	F♯	K♯	E♯	J♯	D♯	I♯	C♯	H♯	
E♯	F♯	K♯	E♯	J♯	D♯	I♯	C♯	H♯	B♯
K	B♭								
F	B♭	H♭							
B♭	B♭	H♭	C♭						
H♭	B♭	H♭	C♭	I♭					
C♭	B♭	H♭	C♭	I♭	D♭				
I♭	B♭	H♭	C♭	I♭	D♭	J♭			
D♭	B♭	H♭	C♭	I♭	D♭	J♭	E♭		
J♭	B♭	H♭	C♭	I♭	D♭	J♭	E♭	K♭	
E♭	B♭	H♭	C♭	I♭	E♭	J♭	E♭	K♭	F♭

EXAMPLE 7: KEY SIGNATURES FOR THE NINE-NOTE MAJOR MODES, $N = 20$

and eight "semitones" (40¢ intervals). Use of the circle of "generalized fifths," i.e. the automorphism generated by 19 yields a rather nasty sequence of *T*'s and *S*'s:

$$T\,T\,S\,T\,S\,T\,T\,S\,T\,S\,T\,S\,T\,T\,S\,T\,S\,T\,S$$

and, of course, its modal cyclic permutations. In fact, if it is given that two S's may not occur together (the analogue, in the twelve-tone system, of forbidding the interval of the diminished third anywhere in the scale) the only possibility is for either one or two T's to be sandwiched between S's, so there is not much hope of improving on this scale. So it is discarded, forthwith!

The only other possibility is the generalized pentatonic scale based on the generator 11, the generalized fourth. This scale has, in fact, been written down by Balzano (1980) as

$$0\ 3\ 6\ 8\ 11\ 14\ 17\ 19\ 22\ 25\ 28\ 30.$$

That this is the "major" mode was concluded from the Balzano diagram, not reproduced here, where it is seen that the mode has five major chords. We note, as usual, that this generalized pentatonic scale consists of narrow (80¢) and wide (120¢) intervals. Converting to cents, the scale becomes:

$$0\ 120\ 240\ 320\ 440\ 560\ 680\ 760\ 880\ 1000\ 1120\ 1200.$$

This is to be compared with the eleven-note scale presented in the previous section which is, in cents:

$$0\ 120\ 240\ 360\ 480\ 540\ 660\ 780\ 900\ 1020\ 1140\ 1200.$$

These two scales seem similar enough that there is little reason to develop both. Further, the one based on the twenty-tone system has a major chord, in cents, of:

0–300¢–660¢

while that generated by the thirty-tone system has a rather dismal major chord of:

0–240¢–440¢

which follows from the isomorphism (Appendix):

$$C_{30} \approx C_6 \times C_5 .$$

The forty-two-tone system discussed by Balzano will not be considered here.

IV. Disclaimer

It is important that the reader should understand the essential contributions of Budden and Balzano. Budden pioneered the use of group theory to analyze musical structure. Balzano proved the crucial theorem (that only N-tone scales with $N = k(k+1)$ need be considered) and furthermore introduced the diagrammatic technique which makes the analysis so easy. Although in the present paper it has been suggested that some of Balzano's recommendations for specific scales might be improved upon, this in no way detracts from the seminal nature of his work.

V. Acknowledgment

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NOTES

1. The first of the series, "Just Tuning and the Unavoidable Discrepancies," was published in *Indiana Theory Review* 15, no. 2 (Fall 1994): 89–120.

2. In particular, Clough and Douthett, footnote 8, stress the corelationship of diatonic and pentatonic scales.

3. This article is part of the two-part "Forum: Microtonality Today," Part 1, 29, no. 1 (1991): 172–257; Part 2, 29, no. 2 (1991): 72–211. The use of microtonal scales was suggested originally by Ferruccio Busoni in 1911 according to Harold C. Schonberg in *The Lives of the Great Composers*, Revised Edition (New York: W. W. Norton and Co., 1981), 554.

4. This indicates that only equal temperament will be considered here. The question of tuning justification will be addressed in a forthcoming paper.

5. We recall the definition of addition, mod N: Let p, q, and r be integers. Then $p+q = r$, mod N, if upon division of $p+q$ by N, the remainder is r. Thus $5+7 = 0$, mod 12. Musically, this states that a perfect fifth ($7S$) plus a perfect fourth ($5S$) yields an octave ($12S$) equivalent to a unison ($0S$).

6. The element P is the inverse of q (written q^{-1}) if $p+q = 0$, mod N. Every element of a group has a unique inverse. Further, if n is a generator, so is n^{-1}.

7. The set of operations by which one automorphism of a group G is mapped into another itself forms a group, the automorphism group of G, written *autG*. For example, $autC_{12} = C_2 \times C_2$ (Budden 1972, 423 and 139). It is hoped that such groups will prove to be useful in the mathematical theory of tuning. For example, we shall observe that C_2 represents the tritone. The group $C_2 \times C_2$ represents the *two* tritones, augmented fourth and diminished fifth and, in a sense, explains mathematically why they are distinct. This is connected with the concept of *coherence*, introduced by Balzano (1980).

8. A subgroup S of a group G is defined as a subset of G which is a group in its own right, i.e. if p and q are elements of S then so is $p+q$. Also for every p in S, the element p^{-1} is in S and the identity element 0 must be an element of S.

9. Hopefully, it is not confusing to the reader to see some groups, like C_{12}, whose elements are integers 0, 1, . . . , 11 which combine by addition, mod 12, while other groups, like C_7, consist of operations on symbolic sequences. In Balzano 1980 the dual—static and dynamic—interpretation of groups is explained in detail. In our use of C_7 to represent the modes, it is important to remember that it is the cyclic permutations—the operations—not the various sequence of T's and S's which are the group elements. These permutations are *isomorphic* to addition of the integers 0, 1, . . . , 6, mod 7.

10. We shall see that this interval is a minor fourth in the twenty-tone system and six semitones = 360¢, a major fourth. Interestingly enough, the subgroup C_4 of C_{12} is identical to the subgroup C_4 of C_{20} in cents—they are both (0, 300¢, 600¢, 900¢)—although they differ when represented as integers. This is a general rule: C_K as a subgroup of C_N is the same, *in cents*, as C_K as a subgroup of C_M for every N and M. The quotient groups, C_M/C_K and C_N/C_K of course are different; in particular they contain different numbers of elements.

11. Not to be confused with the ninth-chord of the conventional twelve-tone system.

12. This was the original reason that an alternative to Balzano's nine-note scale was sought.

13. The mode is named by the number of the chromatic scale note on which it begins.

14. The number of white and black keys always are equal, respectively, to the number of semitones in the generalized fifth and generalized fourth.

15. See Note 10.

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Appendix

Here a number of results are proved which are relevant to the group-theoretical analyses of musical tone systems. For reasons explained by Balzano (1980), only N-tone systems with $N = j(j+1)$ for integer j are considered.

Proposition 1. *The cyclic group* $C_{j(j+1)}$ *is isomorphic to* $C_j \times C_{j+1}$.

Proof.

Clearly, $C_{j(j+1)}$ and $C_j \times C_{j+1}$ both have $j(j+1)$ elements. To prove isomorphism, it is sufficient to demonstrate that $C_j \times C_{j+1}$ has an element of period $j(j+1)$. (See Budden 1972.) Clearly 1 is an element of period j in C_j and of period $j+1$ in C_{j+1}. Therefore $(1, 1)$ is an element of period equal to the lowest common multiple of j, $j+1$. The lowest common multiple is clearly $j(j+1)$, completing the proof.

Remark.

The proof makes it clear why although C_{12} is isomorphic to $C_3 \times C_4$, it is *not* isomorphic to $C_6 \times C_2$, since the lowest common multiple of 6 and 2 is 6. This proposition is an important part of the rationale for restricting N to $j(j+1)$.

The proposition does not explicitly demonstrate the isomorphism. It may be taken as follows. If $p \in C_j$, $q \in C_{j+1}$, and $r \in C_{j(j+1)}$, then $r = (j+1)p + jq \mod j(j+1)$. As we have already seen in the text for $j = 3$, $N = 12$, this isomorphism expresses any note of the twelve-tone scale as a sum of major thirds (C_3) and minor thirds (C_4).

Proposition 2. *For* $C_{j(j+1)}$, $2j+1$ *(and its inverse* $j^2 - j - 1$*) are generators.*

Proof.

Clearly $(1, 1)$ is a generator for $C_j \times C_{j+1}$. Using the isomorphism described above $(1, 1) \approx (j+1) \cdot 1 + j \cdot 1 = 2j+1$. It is a standard fact (Budden 1972) that if k is a generator, so is its inverse.

Proposition 3. *The elements* $2j+1$ *and its inverse* $j^2 - j - 1$ *generate* F $\rightarrow$ F♯ *scales in the* $j(j = 1)$*-tone system.*

Proof.

Consider first $2j+1$. The scale consists of $2j+1$ connected elements in the $(2j+1)$-automorphism. The proposition states that starting with a given note and progressing $2j+1$ intervals of the automorphism leads to a note which differs from the starting note by one semitone ("F → F♯" moving up or "B → B♭" moving down).

Moving up we see that the result follows if $(2j+1)^2 = 1 \bmod j(j+1)$. But $(2j+1)^2 = 4j(j+1)+1 = 1 \bmod j(j+1)$. On the other hand, moving down, the intervals are of length j^2-j-1 so we need also $(2j+1)(j^2-j-1) = -1 \bmod j(j+1)$. But

$$(2j+1)(j^2-j-1) = (2j+1)(j(j+1)-2j-1)$$

$$= (2j+1)j(j+1) - (2j+1)^2 = -1 \bmod j(j+1)$$

by the "up" calculation above.

For the j^2-j-1 scale one needs to show that $(j^2-j-1)^2 = (j(j+1)-(2j+1))^2 = 1, \bmod j(j+1)$ and $(j^2-j-1)(2j+1) = -1 \bmod j(j+1)$. The first of these is obvious and the second has already been shown in the first part of the proposition. This concludes the proof.

It might be noted that for $j>3$, i.e. for all $N>12$, $2j+1<j^2-j-1$. For example in the twenty-tone system, $2j+1$ corresponds to the nine-note scale emphasized by Balzano (1980) and j^2-j-1 to the eleven-note scale recommended in the article. For the twelve-tone system, the situation is reversed, i.e. $2j+1 = 7$ (diatonic) and $j^2-j-1 = 5$ (pentatonic). In all other tone systems considered, $2j+1$ generates the analogue of the pentatonic scale and j^2-j-1 the diatonic analogue.