The Multiplicative Norm and Its Implications for Set-Class Theory

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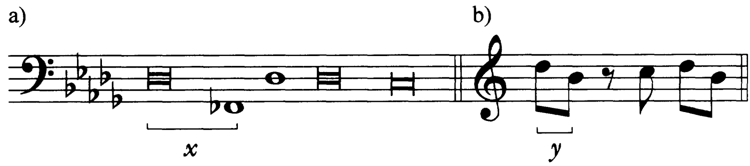
On pages 85–7 of his book *Generalized Musical Intervals and Transformations*, David Lewin makes a "methodological point," namely, that formal and intuitive truths may not always coincide. As an example he cites from the first movement of Chopin's B♭-minor Piano Sonata, op. 35, two ordered dyads (see Example 1). "The first melodic dyad of (b), marked *y* on the figure, belongs to the same interval class as *x*, the first melodic dyad of (a)," states Lewin, and immediately continues: "This relation . . . is formally 'true' but intuitively problematic" (86).

Two interpretations of Lewin's statement must be immediately discounted. First, the quotation marks enclosing "true" in "formally true"



EXAMPLE 1: TWO MELODIC DYADS FROM THE FIRST MOVEMENT OF CHOPIN'S "FUNERAL MARCH" SONATA

are clearly not meant to suggest that the assertion "*x* and *y* belong to the same interval class" is only ***apparently*** true, but in fact false. Second, the assertion "*x* and *y* belong to the same interval class" is not to be confused with the observation "*x* and *y*, as pitch-class intervals, sum up to the perfect prime, modulo the octave." There is nothing, I would think, intuitively problematic about the latter observation, although its analytic import is indeed unclear. What Lewin finds "intuitively problematic," in other words, is not that *x* and *y*—a descending major sixth and a descending minor third—are inversionally ***related*** as pitch-class intervals; rather, for Lewin it is formally true but intuitively problematic to assert that *x* and *y* are equivalent.[1]

The equivalence of inversionally related pitch-class intervals plays a central role in the theory of post-tonal music known as "set-class theory." Moreover, as Lewin's Chopin example demonstrates, a similar equivalence relation is often assumed to play a role in tonal music. After explaining that "intervals larger than 6 . . . are considered equivalent to their inversions with respect to the octave," Straus (1990, 8) adds in parentheses: "these equivalences are also observed in many aspects of tonal music and tonal theory."[2] It is not difficult to see that Lewin's "methodological" remark casts doubt not only on the notion that inversionally related pitch-class intervals are equivalent, but also on the more general notion that pitch-class sets of ***any*** given cardinality are equivalent, provided they are related transpositionally and/or inversionally. This latter generalization, to which I shall refer as "transpo/inversional equivalence" (henceforth, "*T/I* equivalence"), is currently the standard equivalence relation on pitch-class sets.

In the present study I paraphrase Lewin's remark and argue that it is valid but ***ad hoc*** to consider inversionally related pitch-class intervals as equivalent; similarly valid but ***ad hoc***, I shall argue, is the generally assumed equivalence of *T/I*-related pitch-class sets. By contrast, it is perfectly natural to view inversionally related *pitch* intervals as equivalent, and the same is true of *T/I*-related pitch sets. This fundamental difference between pitches and pitch classes, which thus far seems to have been largely overlooked in the theoretical and analytical literature, concerns a mathematical property generally known as "the multiplicative norm." To study this property in a proper musical context requires that we first consider two systems of pitches (or pitch classes) and their intervals. The "interval system," to begin with, is simply a Lewinian pitch or pitch-class GIS that allows for multiplying intervals by integer (or integer-class) operators. The "interval/distance system," in turn, is an interval system with an added distance on the set of pitches or pitch classes.

1. The Interval System

1.1 Definition: Let Z_1 be the infinite set of all integers, and let Z_k, $k \geq 2$ is any integer, be the finite set of integer classes, mod k. Let n be any natural number, and let G_n be the abelian group $(Z_n; +)$ under usual addition (if $n = 1$) or addition mod k (if $n = k \neq 1$). Let R_n be the ring $(Z_n; +, \cdot)$ under usual addition and multiplication (if $n = 1$), or addition and multiplication mod k (if $n = k \neq 1$), and let $M_n = (R_n, G_n, \cdot)$ be the module G_n over the ring R_n, again, under usual multiplication (if $n = 1$), or multiplication mod k (if $n = k \neq 1$).[3] Finally, let int be a function from $Z_n \times Z_n$ into G_n satisfying $\mathrm{int}(x, y) = (y - x)$ for any x, y in Z_n. We shall refer to any triple (Z_n, M_n, int) as an *interval system*.

We shall refer to the elements of the set Z_1 as *pitches*, and to the elements of any set Z_k, $k \geq 2$, as *(mod* k*) pitch classes.* We shall refer to the elements of the group G_1 as *pitch intervals*, and to the elements of any group G_k, $k \geq 2$, as *(mod* k*) pitch-class intervals.* Finally, we shall refer to the elements of the ring R_1 as *operators*, and to the elements of any ring R_k, $k \geq 2$, as *(mod* k*) operator classes.*

The following is stated without proof.

1.1.1 Proposition: Let (Z_n, M_n, int), $M_n = (R_n, G_n, \cdot)$, $R_n = (Z_n; +, \cdot)$, $G_n = (Z_n; +)$, be any interval system. Then (Z_n, M_n, int) is a commutative Generalized Interval System (GIS) in the sense of Lewin (1983).

A Lewinian pitch or pitch-class GIS allows intervals to compose among themselves according to the laws of addition (e.g., M3 + m3 = P5). In the present study we are also interested in such relations as $2 \cdot \mathrm{P5} \equiv \mathrm{M2}$, i.e., the composition of intervals with *integer operators* according to the laws of multiplication. Hence, an "interval system" (Z_n, M_n, int) consists of a set, a *module* (rather than just a group), and the function int.

Note the use of integer subscripts to differentiate between the pitch-type interval system (Z_1, M_1, int), and interval systems of the pitch-*class* type (Z_k, M_k, int), $k \geq 2$. In the present section and the following one all definitions and propositions apply to interval systems (Z_n, M_n, int) of either type. In particular, the sign "=" may be read as either equality or congruence mod k.

1.2 Next we develop the notion of transpositional relation on pitch or pitch-class sets. Following Lewin (1987, 46–50), transposition in a commutative GIS is seen as an "interval-preserving" bijection from one set onto another. Note the following notational convention. If S is any non-empty set, and m is any natural number no larger than the number of elements in S, then $|S|^m$ denotes the set of all m-element subsets of S.

1.2.1 Definition: Let (Z_n, M_n, int) be any interval system. Fix any m, and let A and B be any two members of $|Z_n|^m$. Any bijection p from A onto B will be termed *interval-preserving* if $\text{int}(a, a') = \text{int}\,(p(a), p(a'))$ for any a, a' in A.

1.2.1.1 Proposition: Let (Z_n, M_n, int), $M_n = (R_n, G_n, \cdot)$, be any interval system. Fix any m, and let A and B be any two members of $|Z_n|^m$. Then p is an interval-preserving bijection from A onto B if, and only if, for some g in G_n, $p(a) = a + g$, for any a in A.

Proof. Assume that p is an interval-preserving bijection, and suppose that for some a in A, $p(a) = b$. Then there exists a unique g in G_n such that $b = a + g$. Let a' be any member of A. We have $\text{int}(a, a') = a' - a = \text{int}(p(a), p(a')) = p(a') - (a + g)$, from which we have $p(a') = a' + g$. Suppose next there exists a bijection p from A onto B such that, for some g in G_n, $p(a) = a + g$ for any a in A. Let a, a' be any two elements in A. Then $\text{int}(p(a), p(a')) = \text{int}(a + g, a' + g) = a' + g - (a + g) = \text{int}(a, a')$. This proves the proposition.

1.2.2 Definition: Let (Z_n, M_n, int) be any interval system. Fix any m, and let A and B be any two members of $|Z_n|^m$. We shall say that A is *transpositionally related* to B if there exists an interval-preserving bijection from A onto B. Clearly, transposition is an equivalence relation in $|Z_n|^m$.

Note that if A is ordered, B is transpositionally related to A, and the transposition is unique, then this induces an ordering on B.

1.3 We define the notion of inversion in analogy with transposition, using Lewin's idea of "interval-*reversing*" bijection (1987, 50–9). In an important sense, however, the analogy is misleading, as shall be pointed out subsequently.

1.3.1 DEFINITION: Let (Z_n, M_n, int) be any interval system. Fix any m, and let A and B be any two members of $|Z_n|^m$. Any bijection q from A onto B will be termed *interval-reversing* if $\text{int}(a, a') = -\text{int}(q(a), q(a'))$ for any a, a' in A.

1.3.1.1 PROPOSITION: Let (Z_n, M_n, int), $M_n = (R_n, G_n, \cdot)$, be any interval system. Fix any m, and let A and B be any two members of $|Z_n|^m$. Then q is an interval-reversing bijection from A onto B if, and only if, for some g in G_n, $q(a) = -a + g$, for any a in A.

Proof. Assume that q is an interval-reversing bijection, and suppose that for some a in Z_n, $q(a) = b$. Then there exists a unique g in G_n such that $b = -a + g$. Let a' be any member of A. We have $\text{int}(a, a') = a' - a = -\text{int}(q(a), q(a')) = (-a + g) - q(a')$, from which we have $q(a') = -a' + g$. Suppose there exists a bijection q from A onto B such that, for some g in G_n, $q(a) = -a + g$ for any a in A. Let a, a' be any two elements in A. Then $\text{int}(q(a), q(a')) = \text{int}(-a + g, - a' + g) = -a' + g - (-a + g) = -\text{int}(a, a')$. This proves the proposition.

The analogy between 1.2.1 and 1.2.1.1, and (respectively) 1.3.1 and 1.3.1.1, should not blur the crucial difference between transposition and inversion. *Transposition preserves intervals; inversion does not.* Moreover, there is only one interval-preserving bijection (namely transposition), compared to an indefinitely large number of bijections (of which inversion is one) that do not preserve intervals. Therefore, any equivalence relation on pitch or pitch-class sets other than transposition is necessarily *ad hoc*, at least in the context of the interval system. This is true, in particular, of *T/I* equivalence, defined below.

1.3.2 DEFINITION: Let (Z_n, M_n, int) be any interval system. Fix any m, and let A and B be any two members of $|Z_n|^m$. We shall say that A is *transpo/inversionally-related* to B if there exists either an interval-preserving bijection, or an interval-reversing bijection, from A onto B. Clearly, *T/I* relation is an equivalence relation in $|Z_n|^m$.

2. The Interval/Distance System

In general, given any two pitch or pitch classes x and y, int(x, y) $\neq$ int(y, x). In certain musical contexts, however, one intuits interval-like entities satisfying the relation "int"(x, y) = "int"(y, x). The existence of such entities motivates the following definition.

2.1 Definition: Let (Z_n, M_n, int)—n is any natural number—be any interval system. We shall refer to any quadruple Σ_n = (Z_n, M_n, int, dist) as an ***interval/distance system***, if dist is any distance on Z_n. In other words, dist is a function from $Z_n \times Z_n$ into the non-negative real numbers satisfying the following three conditions (A), (B), and (C) for any x, y, and z in Z_n.

(A): dist(x, y) = 0 iff (i.e., if and only if) $x = y$.
(B): dist(x, y) = dist(y, x).
(C): dist(x, z) $\leq$ dist(x, y) + dist(y, z).

2.1.1 Example: If dist is the trivial distance defined by dist(x, y) = 0 iff $x = y$, and dist(x, y) = c iff $x \neq y$, x and y are any two members of Z_n, c is any strictly positive real number, then Σ_n = (Z_n, M_n, int, dist) is an interval/distance system.

We shall assume henceforth that any distance dist is non-trivial.

2.2 While there is only one conceivable interval-preserving relation on pitch or pitch-class sets, namely transposition, when it comes to preserving ***distances*** there are two alternative relations to choose from. One relation (Definition 2.2.1 below) is analogous to transposition, as it involves a bijection from one set onto another. Another relation (Definition 2.2.2 below), a special case of which was originally proposed by Allen Forte (1964), involves a bijection from the set of all ***two-element subsets*** of one set onto the set of all two-element subsets of another set. The existence of two "distance-preserving" relations reflects the finding that unlike intervals, distances satisfy the relation dist(x, y) = dist(y, x).

2.2.1 Definition: Let Σ_n = (Z_n, M_n, int, dist) be any interval/distance system. Fix any m, and let A and B be any two members of $|Z_n|^m$. We shall say that A and B are ***delta related*** if there exists a bijection d from A onto B satisfying dist(a, a') = dist($d(a)$, $d(a')$) for any a, a' in A. Clearly, the delta relation is an equivalence relation in $|Z_n|^m$.

2.2.2 DEFINITION: Let $\sum_n$ be any interval/distance system. Fix any m, and let A and B be any two members of $|Z_n|^m$. We shall say that A and B are ***Forte related*** if either of the following conditions (A) or (B) is satisfied.

(A): $m = 1$.
(B): There exists a bijection f from $|A|^2$ onto $|B|^2$ satisfying $\text{dist}(a, a') = \text{dist}(f(a, a'))$ for any $\{a, a'\}$ in $|A|^2$.

Clearly, the Forte relation is an equivalence relation in $|Z_n|^m$.

Since we have not yet selected a particular distance among the many nontrivial distance functions satisfying Definition 2.1, it is premature to attempt to decide whether either of the two conceivable distance-preserving relations is preferable to the other.

In what follows, a well-known distance in $\sum_{12}$ is generalized. Note the following notational convention. If $n = k \geq 2$, we shall write $\pm\text{int}(x, y)$ to refer to the unique integer within the range $[0, k/2]$ that represents either $\text{int}(x, y)$ or $-\text{int}(x, y)$.

2.3 PROPOSITION: Let (Z_n, M_n, int) be any interval system, and let dist be any distance on Z_n satisfying $\text{dist}(x, y) = c \cdot \pm\text{int}(x, y)$ for any x, y in Z_n, (c is any fixed, strictly positive real number). Then $\sum_n = (Z_n, M_n, \text{int}, \text{dist})$ is an interval/distance system.

Proof. Clearly, it suffices to show that the proposition holds for $c = 1$.

If $n = 1$, $\text{dist}(x, y) = \pm\text{int}(x, y)$ is the usual distance on the reals, namely, the usual absolute value $|x - y|$ of $x - y$.

Suppose $n = k \geq 2$, and let x, y be any two elements in Z_k. Since (Z_k, M_k, int) is an interval system, we have $\text{dist}(x, y) = \pm\text{int}(x, y) = 0$ iff $x = y$, and condition 2.1A is satisfied. Since $\text{int}(x, y) = -\text{int}(y, x)$ and $-\text{int}(x, y) = \text{int}(y, x)$, we have $\text{dist}(x, y) = \pm\text{int}(x, y) = \mp\text{int}(y, x) = \text{dist}(y, x)$, and condition 2.1B is satisfied as well.

To show that condition 2.1C ("triangle inequality") is satisfied as well, we shall fix any two elements x, z in Z_k. Then either (1) k is even, and $\text{dist}(x, z) = \pm\text{int}(x, z) = k/2$; or (2) $\text{dist}(x, z) = \pm\text{int}(x, z) < k/2$.

If k is even, and $\text{dist}(x, z) = \pm\text{int}(x, z) = k/2$, we have $\text{dist}(x, y) + \text{dist}(y, z) = \pm(\text{int}(x, y) + \text{int}(y, z)) = \pm\text{int}(x, z) = k/2 = \text{dist}(x, z)$, for any third element y in Z_k.

Suppose $\text{dist}(x, z) = \pm\text{int}(x, z) < k/2$.

If y is any third element in Z_k such that $\text{dist}(x, y) = \text{int}(x, y)$ and $\text{dist}(y, z) = \text{int}(y, z)$, we have $\text{dist}(x, y) + \text{dist}(y, z) = \text{int}(x, y) + \text{int}(y, z) =$

$\mathrm{int}(x, z)$. Then, if $\mathrm{dist}(x, z) = \mathrm{int}(x, z)$, we have $\mathrm{dist}(x, y) + \mathrm{dist}(y, z) = \mathrm{dist}(x, z)$, and if $\mathrm{dist}(x, z) = -\mathrm{int}(x, z)$, we have $\mathrm{dist}(x, y) + \mathrm{dist}(y, z) > \mathrm{dist}(x, z)$.

If y is any third element in Z_k such that $\mathrm{dist}(x, y) = -\mathrm{int}(x, y)$ and $\mathrm{dist}(y, z) = -\mathrm{int}(y, z)$, we have $\mathrm{dist}(x, y) + \mathrm{dist}(y, z) = -\mathrm{int}(x, y) - \mathrm{int}(y, z) = -\mathrm{int}(x, z)$. Then, if $\mathrm{dist}(x, z) = \mathrm{int}(x, z)$, we have $\mathrm{dist}(x, y) + \mathrm{dist}(y, z) > \mathrm{dist}(x, z)$, and if $\mathrm{dist}(x, z) = -\mathrm{int}(x, z)$, we have $\mathrm{dist}(x, y) + \mathrm{dist}(y, z) = \mathrm{dist}(x, z)$.

Finally, if y is any third element in Z_k such that either (1) $\mathrm{dist}(x, y) = \mathrm{int}(x, y)$ and $\mathrm{dist}(y, z) = -\mathrm{int}(y, z)$; or (2), $\mathrm{dist}(x, y) = -\mathrm{int}(x, y)$ and $\mathrm{dist}(y, z) = \mathrm{int}(y, z)$, we have $\mathrm{dist}(x, y) + \mathrm{dist}(y, z) = \mathrm{dist}(x, y) + \mathrm{dist}(y, x) + \mathrm{dist}(x, z) = 2 \cdot \mathrm{dist}(x, y) + \mathrm{dist}(x, z) > \mathrm{dist}(x, z)$.

Since in all cases condition 2.1C is satisfied, $\mathrm{dist}(x, y) = \pm\mathrm{int}(x, y)$ is a distance on Z_k. This proves the proposition.

If $n = k \geq 2$ (and assuming $c = 1$), it is not difficult to see that the distance $c \cdot \pm\mathrm{int}(x, y)$ generalizes Forte's familiar "interval-class" function. Moreover, using $\pm\mathrm{int}(x, y)$ as a distance, the Forte relation (Definition 2.2.2) generalizes Forte's familiar "interval-vector" relation. However, while Forte's distance is valid, as we have just seen, as a distance on Z_k, in the absence of additional assumptions the distance is *ad hoc*, since clearly, $c \cdot \pm\mathrm{int}(x, y)$ is not the only distance satisfying the conditions set forth in Definition 2.1.[4]

As we shall see in the following section, in Σ_1 there exists a set of assumptions that renders $c \cdot \pm\mathrm{int}(x, y)$ a rather privileged distance.

3. The Normed Interval/Distance System Σ_1

3.1 Definition: Let $\Sigma_1 = (Z_1, M_1, \mathrm{int}, \mathrm{dist})$ be any interval/distance system, $M_1 = (R_1, G_1, \cdot)$, $R_1 = (Z_1; +, \cdot)$, $G_1 = (Z_1; +)$. We shall refer to the system as *normed* if there exists a function norm from G_1 into the non-negative real numbers satisfying the following two conditions (A) and (B).

(A): $\mathrm{norm}(g) = \mathrm{dist}(x, y)$ for any g in G_1 and any x, y in Z_1 satisfying $\mathrm{int}(x, y) = g$.

(B): $\mathrm{norm}(\alpha \cdot g) = |\alpha| \cdot \mathrm{norm}(g)$, for any operator α in R_1 and any pitch-interval g in G_1.

3.1.1 Theorem: Let $\Sigma_1 = (Z_1, M_1, \mathrm{int}, \mathrm{dist})$ be any interval/distance system. Then the following two conditions (A) and (B) are equivalent.

(A): Σ_1 is normed.

(B): $\text{norm}(g) = c \cdot |g| = \text{dist}(x, y)$ for any g in G_1 and any x, y in Z_1 satisfying $\text{int}(x, y) = g$, c is any fixed, strictly positive real number.

Proof. Consider any interval/distance system $\Sigma_1 = (Z_1, M_1, \text{int}, \text{dist})$, and suppose condition 3.1.1A holds. From condition 3.1A, in conjunction with condition 2.1A, we have $\text{norm}(g) = 0$ iff $g = 0$. Fix $\text{norm}(1) = c$, c is any strictly positive real number. Since we can write any pitch-interval g as the product $\alpha \cdot 1$, $\alpha = g \in R_1$, from condition 3.1B we have, for any g in G_1, $\text{norm}(g) = c \cdot |g|$. Together with condition 3.1A, condition 3.1.1B holds.

Suppose condition 3.1.1B holds. Then condition 3.1.1A immediately follows. This proves the theorem.

We shall assume henceforth that $c = 1$, and thus the norm and the usual absolute value are one and the same.

3.1.2 Example: Let $\alpha = -2$, and let $\text{int}(x, y) =$ "ascending perfect fifth." An ascending perfect fifth times -2 is a descending major ninth. The ***distance*** represented by any two pitches spanning the interval of a descending major ninth is the absolute value of this interval, namely a major ninth. This distance (major ninth) is equal to an ascending fifth times 2 (the absolute value of -2).

One may well wonder why Σ_n in Definition 3.1 is assumed ***a priori*** to satisfy $n = 1$. The answer is straightforward. In the multiplicative condition 3.1B the usual absolute value on R_1 is assumed. Suppose $n = k \geq 2$. Then, if k is not a prime, an absolute value on R_k cannot exist. On the other hand, if k is a prime, the only absolute value on R_k that exists is the trivial absolute value defined by $|x| = 0$ iff $x = 0$, and $|x| = 1$ iff $x \neq 0$.[5] This implies that the distance dist is also trivial, contrary to our assumption.

The following final theorem establishes the equivalence of the delta relation (Definition 2.2.1) and the T/I relation (Definition 1.3.2), in the context of the normed interval/distance system Σ_1.

3.2 Theorem: Consider the normed interval/distance system $\Sigma_1 = (Z_1, M_1, \text{int}, \text{dist})$. Fix any natural number m, and let A and B be any two members of $|Z_1|^m$. Then A is delta related to B if, and only if, A is T/I related to B.

To prove the theorem we shall first prove the following

LEMMA. Consider the normed interval/distance system $\Sigma_1 = (Z_1, M_1,$ int, dist), and let A and B be any two members of $|Z_1|^3$ such that A is delta related to B. Then A is T/I related to B.

Proof. Set $A = \{u, v, w\}$ and $B = \{x, y, z\}$, $u < v < w$ and $x < y < z$. Clearly, A is delta related to B under either (1) the bijection $d = \{(u, x), (v, y), (w, z)\}$; or (2), the bijection $d' = \{(u, z), (v, y), (w, x)\}$.

If $\text{int}(u, w) = \text{int}(x, z)$, we have

$$\begin{aligned}\text{int}(u, w) &= \text{int}(u, v) + \text{int}(v, w) = \text{int}(x, z) = \text{int}(x, y) + \text{int}(y, z)\\ &= -\text{int}(z, x) = -(\text{int}(z, y) + \text{int}(y, x)),\end{aligned}$$

and A is either (1) transpositionally related to B under the bijection d; or (2), inversionally related to B under the bijection d'. If $\text{int}(u, w) = -\text{int}(x, z)$, we have

$$\begin{aligned}\text{int}(u, w) &= \text{int}(u, v) + \text{int}(v, w) = -\text{int}(x, z) = -(\text{int}(x, y) + \text{int}(y, z))\\ &= \text{int}(z, x) = \text{int}(z, y) + \text{int}(y, x),\end{aligned}$$

and A is either (1) inversionally related to B under the bijection d; or (2) transpositionally related to B under the bijection d'. This proves the lemma.

Proof of the theorem. Let A and B be any two members of $|Z_1|^m$, and suppose that A is T/I related to B. Since $\text{dist}(x, y) = \pm\text{int}(x, y)$ for any x, y in Z_1, it follows that A is delta related to B.

Let A and B be any two members of $|Z_1|^m$, and suppose A is delta related to B. If $m = 1$ or $m = 2$ the theorem is satisfied trivially. If $m = 3$ the theorem is satisfied by the lemma.

Suppose $m \geq 4$. By Definition 2.2.1 there exists a bijection d from A onto B satisfying $\text{dist}(a, a') = \text{dist}(d(a), d(a'))$ for any a, a' in A. By the lemma, given any three-element subset A′ of A, $A' = \{a_1, a_2, a_3\}$, A′ is T/I related to the three-element subset B′ of B, $B' = \{d(a_1), d(a_2), d(a_3)\}$. The theorem follows from the observation that A can be written as a union of exactly $m - 2$ distinct three-element subsets, ordered in such a way that any two consecutive subsets have exactly two elements in common. In particular, since $\text{int}(x, y) \neq \text{int}(y, x)$ holds for any two distinct pitches x and y, any two three-element subsets (A′, A″) of A having two elements in common must be both either transpositionally or inversionally related to their corresponding images under d, the three-element subsets (B', B'') of B.

We have established the conceptual validity of T/I equivalence as a distance-preserving relation in the context of the normed interval/distance system $\sum_1$. However, T/I equivalence is not the only conceivable distance-preserving relation. Another conceivable distance-preserving relation, we have seen, is the Forte relation (Definition 2.2.2).

Abstractly considered, the Forte and the delta (or T/I) relations are equally valid distance-preserving relations in the context of the normed interval/distance system $\sum_1$; nonetheless, from a *perceptual* viewpoint the Forte relation is problematic. Whether or not two given pitch sets of equal cardinalities are T/I equivalent is fairly easy to establish. Clearly, the interval-preserving bijection from one set onto the other, if it exists, maps lowest pitch onto lowest pitch, second-lowest pitch onto second-lowest pitch, and so forth (similarly, the interval-*reversing* bijection maps lowest pitch onto highest pitch, second-lowest pitch onto second-highest pitch, and so forth). By contrast, to establish whether or not two given pitch sets of equal cardinalities are Forte related, one must generally go through the tedious process of tallying all distances. I shall revisit the question of formal versus perceptual music-theoretic considerations in my closing remarks.

4. Analytic Application

In his book *Introduction to Post-Tonal Theory*, Joseph Straus (1990, 40, example 2–9) cites the opening section of Schoenberg's *Klavierstück*, op. 11, no. 1. "The passage is shown . . . with a number of pitch-class sets circled or joined by a beam," Straus comments. "All of these pitch-class sets are members of the same set class." Straus is using, of course, the currently standard equivalence relationship on pitch-class sets, namely T/I equivalence. If, however, as argued above, T/I equivalence is *ad hoc* in the context of any interval system other than the normed interval/distance system $\sum_1$, Straus's statement must be modified. To facilitate discussion, Straus's example 2–9 is reproduced (with minor modifications and with added alphabetic labels to Straus's sets) in Example 2.[6]

Sets *a*, *b*, and *h*, consisting each of the three pitch classes G, G♯, and B, are members of the same set class by virtue of *set equivalence*, the axiomatic sense by which any (unordered) pitch or pitch-class set is equivalent to itself. In fact, sets *a* and *b* are set-equivalent as either pitch or pitch-class sets.

EXAMPLE 2: *T/I*-RELATED PITCH-CLASS SETS FROM SCHOENBERG'S *KLAVIERSTÜCK*, OP. 11, NO. 1. AFTER JOSEPH STRAUS, *INTRODUCTION TO POST-TONAL THEORY*, EXAMPLE 2–9 ON PAGE 40

Sets *a*, *d*, *f*, and *i* are all transpositionally equivalent, given the bijections $a \rightarrow d \rightarrow f \rightarrow i$ tabulated in Example 3.

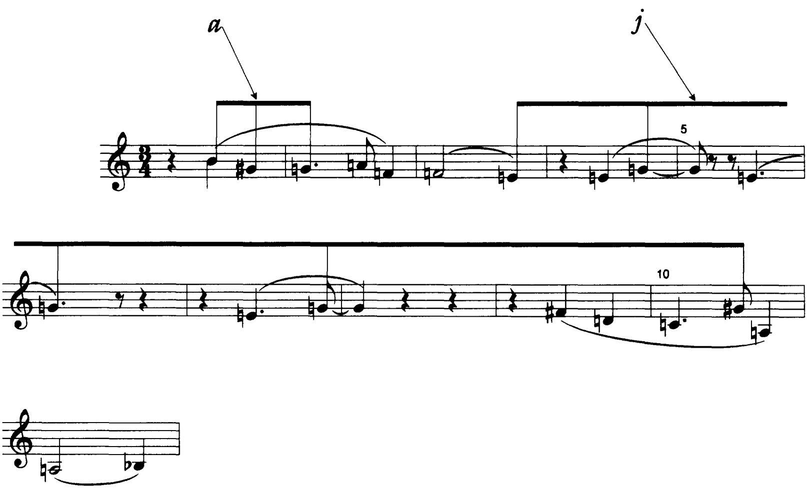
Sets *a* and *e* are inversionally related, given the bijection $\{a \rightarrow e$: (B, G), (G♯, B♭), (G, B)$\}$. However, unlike Straus, I shall refrain from asserting equivalence in this case, for reasons previously discussed.

Finally, consider the sets *a*, *c*, and *g*. Again, we have sets that are non-equivalent, though inversionally related. However, unlike sets *a* and *e*, sets *a*, *c*, and *g*, *regarded as pitch rather than pitch-class sets*, are delta equivalent. Indeed, in Example 4 another compositionally significant pitch set belonging to the same "set class" is identified by the letter *j*.

a:	B	G♯	G
d:	D♭	B♭	A
f:	D	B	B♭
i:	C	A	G♯

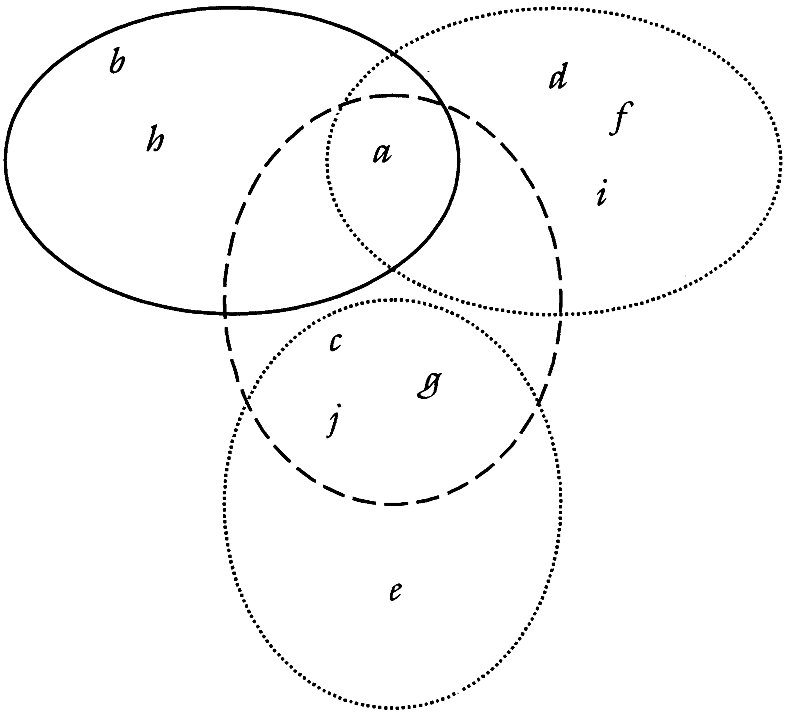
EXAMPLE 3: THE INTERVAL-PRESERVING BIJECTIONS $a \rightarrow d \rightarrow f \rightarrow i$ IN SCHOENBERG'S *KLAVIERSTÜCK*, OP. 11, NO. 1, MEASURES 1–11 (SEE EXAMPLE 2)

(Note that the temporal ordering of sets *a* and *j* enhances their inversional relationship. Moreover, these two sets have two pitches in common.)



EXAMPLE 4: TWO DELTA-EQUIVALENT (OR *T/I*-EQUIVALENT) PITCH SETS FROM THE MELODIC LINE OF SCHOENBERG'S *KLAVIERSTÜCK*, OP. 11, NO. 1, MEASURES 1–11

Example 5 provides an overview of the relations discussed. Although the intersection of all four set classes portrayed is empty, one can see that the intersection of three of the four set classes contains exactly one element, namely set *a*. "Set *a*," however, is not the same object in all three cases. In the context of the set-equivalence class {*a*, *b*, *h*} and the transpositional-equivalence class {*a*, *d*, *f*, *i*}, set *a* is a pitch-class set; in the context of the delta (or *T/I*) equivalence class {*a*, *c*, *g*, *j*}, set *a* is a pitch set. Note that transitivity (a defining property of equivalence relations in general) does *not* generally hold where different objects or different equivalence relations on these objects are involved. For example, set *a* and (say) set *c* are delta equivalent as pitch sets, whereas sets *c* and *e* are transpositionally equivalent as pitch-*class* sets. Thus, one cannot assume that sets *a* and *e* are equivalent.



EXAMPLE 5: FOUR SET CLASSES FROM SCHOENBERG'S *KLAVIERSTÜCK*, OP. 11, NO. 1, MEASURES 1–11 (SEE EXAMPLES 2 AND 4). *SOLID OVAL*: SET EQUIVALENCE; *DOTTED OVALS*: TRANSPOSITIONAL EQUIVALENCE; *DASHED OVAL*: DELTA (OR *T/I*) EQUIVALENCE

5. Conclusion

In the present study a sharp distinction was drawn between pitch and pitch-class sets *vis-à-vis* the status of T/I equivalence. In the context of pitch sets, we have seen, T/I equivalence is a privileged equivalence relation that preserves distances; by contrast, in the context of pitch-*class* sets T/I equivalence is *ad hoc*, a valid equivalence relation among indefinitely many others.

Although algebraic arguments have dominated the discussion, arguments of an entirely different type have also surfaced at one point. Towards the end of Section 3 the Forte relation (Definition 2.2.2) was seen as *perceptually* problematic, compared to the alternative distance-preserving relation, namely the delta relation (Definition 2.2.1). Let me conclude this study by reflecting briefly on the distinction between formal and perceptual music-theoretic considerations.

Consider again the familiar relation of transposition (Definition 1.2.2). So long as the *existence* of an interval-preserving bijection between two given sets can be established, from the formal point of view the two sets are transpositionally related. Indeed, from the formal point of view it is entirely irrelevant whether, to determine that two given sets are transpositionally related, one applies one's intuitions, consults a list of some sort, utilizes a rule-of-thumb, or resorts to paper-and-pencil calculations. By contrast, from the perceptual point of view such "operational" considerations are crucial. Thus, to the extent that paper-and-pencil calculations are needed to establish the existence of a Forte relation in a particular case, the relation is problematic from the perceptual point of view.

Interestingly, and despite appearances to the contrary, the perceptual validity of the "simple" relation of transposition is not self-evident either. Even in the realm of pitches, where the existence of a transpositional relationship between two given sets can be easily verified by mapping lowest pitch onto lowest pitch, second-lowest pitch onto second-lowest pitch, and so forth, the most familiar case of transposition involves *ordered* sets (that is, melodies), where one can also map first pitch onto first pitch, second pitch onto second pitch, etc.

In the realm of (unordered) pitch-class sets, where registral distinctions are meaningless, the existence of a transpositional relationship between two given sets can be far from obvious. I believe that the relative facility by which one identifies transpositional relationships among the triads and seventh chords of tonal music is highly misleading. After all, triads and seventh chords are very special types of "pitch-class sets." In particular, in the absence of chromatic alterations any two given triads (or any two seventh chords) are transpositionally equivalent, if at all, under a unique bijection that maps root onto root, third onto third, etc.

Perceptual considerations, in short, add an important dimension to the formal theory of "set equivalence" that has been the focus of the present study. In a strange way, therefore, we have come back a full circle to Lewin's methodological claim from which our investigation departed:

> One should not ask of a theory that every formally true statement it can make about musical events be a perception-statement. One can only demand that a preponderance of its true statements be *potentially* meaningful in sufficiently developed and extended perceptual contexts (1987, 87).

Notes

An early version of this paper was presented at the Nineteenth Annual Meeting of the Society for Music Theory, Baton Rouge, 1996. An advanced version was read at the Third Symposium on Neo-Riemannian Theory, Buffalo, 2001. I am grateful to David Lewin and Ehud De-Shalit for valuable comments and suggestions.

1. "At least," states Lewin by way of qualification, "the [equivalence] relation of *x* and *y* dyads is hard to hear when we first hear the first theme, the first time through the exposition. But, I claim, the asserted [equivalence] relation has the *potential* for becoming audible, and in fact it does become audible, even highly significant, the second time through the exposition." While Lewin proceeds with a motivic analysis of the movement to prove his point, the analysis, insightful as it surely is, is irrelevant even to the weakened claim that *x* and *y* are *potentially* equivalent. To be sure, a potential relation of one type or another exists between any two intervals, and, as Lewin convincingly demonstrates, Chopin composes the movement in such a way so as to bring this generic potential to fruition in the case of *x* and *y*. Moreover, the choice of *x* and *y* was probably not arbitrary, given that they both descend, and (as Lewin notes) they both begin with the same pitch class, namely D♭. However, unless *x* and *y* are potentially *equivalent* pitch-class intervals in the first place, a notion that is seriously questioned in the present article, the assertion "*x* and *y* belong to the same interval class" remains intuitively problematic, even in the context of Chopin's ingenious motivic network.

2. The notion that inversionally related pitch-class intervals are equivalent is sometimes misconstrued as a logical consequence of octave equivalence. For example, Castine (1994) writes as follows: "Since pitch classes embody octave equivalence, there is no way to determine which of two pitch classes is 'higher' than the other." [*Footnote*: "The reader familiar with Shepard scales will recognize a physical illustration of this idea."] "To make the point using traditional notation, consider the notes D♭ and A♭. If we ignore register, there is no way to determine whether the interval is that of a perfect fourth or a perfect fifth. So, when dealing with pitch class, we must consider interval inversions to be equivalent" (29). But clearly, although a conceptually infinite number of distinct intervals can be spanned from D♭ (*any* D♭) to A♭ (*any* A♭), the class of *all* intervals from D♭ to

A♭ is distinct from the class of all intervals from A♭ to D♭. For example, the former class contains the ascending perfect fifth and the descending perfect fourth, whereas the latter class contains the *descending* perfect fifth and the *ascending* perfect fourth.

3. A ring is an algebraic structure consisting of a set and two operations, "addition" and "multiplication," where the latter is distributive over the former. A module is a composite algebraic structure consisting of a ring R, a group G, and a "multiplication" function from R × G into G.

4. Note that strengthening Definition 2.1 by assuming a distance invariant under transposition will not yield a unique distance either.

5. "Absolute value" is a real-valued function satisfying four conditions: (1) $|x| \geq 0$; (2) $|x| = 0$ iff $x = 0$; (3) $|x \cdot y| = |x| \cdot |y|$; and (4) $|x + y| \leq |x| + |y|$. Suppose that x, y are elements of Z_k, $k \geq 2$. From condition 2 (interpreted as $|x| = 0$ iff $x \equiv 0$) we have $|k| = |0| = 0$. Suppose k is not a prime. Then k can be factored into powers of primes. However, since $|k| = 0$ it follows from condition 3 that the absolute value of at least one of these primes equals 0, which contradicts condition 2. Suppose k is a prime. From condition 3 we have $|1| = |1|^2$, implying that $|1| = 1$. By Fermat's "little" theorem (see LeVeque 1990, 43–4) we have, for any $x \neq 0$, $x^{k-1} \equiv 1 \pmod{k}$. By condition 3 we have $|x^{k-1}| = |1| = |x|^{k-1} = 1$, implying that $|x| = 1$ for any $x \neq 0$.

6. Other analyses of Schoenberg's well-known piano piece include Wittlich (1972), Forte (1981), and Perle (1991). See also Haimo (1996). It is interesting to compare Straus's (1990) example 2–9 with Forte's (1981) example 5c (140).

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