A COMBINATORIAL PROBLEM IN MUSIC THEORY—BABBITT'S PARTITION PROBLEM (I)

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> Take a theme like that of Haydn's (St. Anthony Chorale), take the part of one of Brahm's variation corresponding to the first part of the theme, and set the task of constructing the second part of the variation in the style of its first part. That is a problem of the same kind as mathematical problems are. If the solution is found, say as Brahms gives it, then one has no doubt—that is the solution.
>
> —Ludwig Wittgenstein
> *Remarks on the Foundation of Mathematics*

INTRODUCTION

The mathematical questions dealt with in this paper are an extension of previous work involving Babbitt's Partition Problem [5], and form but a small part of the mathematical problems arising out of twelve-tone theory. It has been our feeling that these problems, far from being a specious excursion into numerology, form the beginnings of a new branch of applied mathematics. As such, they bear much the same relationship to twelve-tone theory proper and twelve-tone composition as do other branches of applied mathematics to their underlying cognitive disciplines.

From this vantage point it is not surprising that certain aspects of combinatorics, the theory of finite groups, ring theory, and graph theory should play a role in the analysis of the structure of the twelve-tone system. Music, which has always been regarded as an art, would not seem a very promising place to find meaningful mathematical problems. However the mathematical roots of music go deep, reaching as far back as Pythagoras, and it is likely that future generations will look upon the art/science dichotomy as more of a comment on our times rather than as a characteristic of the arts or sciences as such. As Boretz pointed out nearly a decade ago,* the

* "I mean to insist that the statements made about music by its practitioners are at least potentially capable of cognitive explication, whatever their evident deficiencies in rigor or coherence, and that this cognitive potential is independently supported by principles of thought developed with unique reference to the element—and relation—concepts particular to music.... Thus in school curricula, it would no longer be necessary or appropriate to distinguish the Arts from the Sciences, but only the Thoughts from the Acts" ([7], p. 4).

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division of cognitive activities into "art," on the one hand, and "science," on the other, has simply ceased to make sense in a world where art works are no longer offered up as entertainment but rather make unprecedented intellectual and perceptual demands upon anyone wishing to approach them at all. In this process mathematics becomes one tool among many, and its use can greatly facilitate our understanding of many aspects of modern music.

In our original paper we were concerned primarily with the music theoretic aspects of the "partition problem," and with establishing a framework within which it could be given a precise mathematical formulation. Here, we deal exclusively with the mathematical consequences of that formulation. Detailed background information can be found in two articles by Babbitt [2, 3], as well as by Kassler [10], and Halsey and Hewitt [8]. Our use of combinational concepts follows Hall [9] and Berge [6]. In particular, the recent book by Andrews [1] has also been extremely helpful. In order to fix terminology we shall review some essential concepts, together with relevant facts from these references.

Preliminaries

Throughout this paper, $A = Z_{12}$ will denote the ring of integers mod 12. In the theory of the twelve-tone system each element of A denotes an equivalence class of pitches, that is, the pitch of any note is identified with the equivalence class, or "pitch class number" to which it belongs. This is a consequence of the fact that the musical system initiated by Schoenberg, and his disciples, is based on the twelve tones of the equal-tempered chromatic scale. That is, the twelve tones are arranged in a specific order, so that each tone appears in a given sequence exactly once. This can be illustrated as follows [8].

Choose any initial note (on the pianoforte keyboard) and designate it by 0. Then number the succeeding notes, increasing in pitch by half tones, with the integers from 1 to 11. Begin this process again (one octave above the initial note) and then designate the first note that occurs one octave higher by 0, as well. Continue (increasing by half tones) and then by octaves, to the upper end of the keyboard. Next, reverse this process, descending in pitch (by half tones) from initial note 11, by labeling the succeeding notes with the integers from 10 to 0. Repeat, until you reach the lower end of the keyboard.

A series of tones, generated in the above fashion, is called a *chromatic scale* (Example 1 on the cassette). Octave translations are of no account in this theory, i.e., two pitches belong to the same equivalence class if they differ by an octave translation. A *twelve-tone row* is an ordered sequence of twelve pitch classes, that is, a specific ordering of the ring A. In the theory of the twelve-tone system we are concerned with the action of the following group G of all permutations on k-tuples ($\in A$) generated by the following transformations.

(1) $e = 1_A$ (identity on A)

(2) $\sigma_1(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$

(3) $\sigma_2^j(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\alpha_1 + j, \alpha_2 + j, \ldots, \alpha_k + j)$, where $0 \leq j < 12$ and integral multiples of 12 are set equal to zero.

(4) $\sigma_3(\alpha_1, \alpha_2, \ldots, \alpha_k) = [12 - \alpha_1 (\text{mod } 12), \ldots, 12 - \alpha_k (\text{mod } 12)]$, where $12 - 0 (\text{mod } 12)$ is of course 0.

The transformation σ_1 is called the *retrograde* of the k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and consists of reversing the order in which a sequence of pitch classes appear. For example, $\sigma_1(2, 3, 6, 7, 10, 11) = (11, 10, 7, 6, 3, 2)$. Metaphorically speaking, the retrograde operation "plays a particular passage backwards."

In an analogous fashion we define σ_2^j (translation by the integer j) to be the

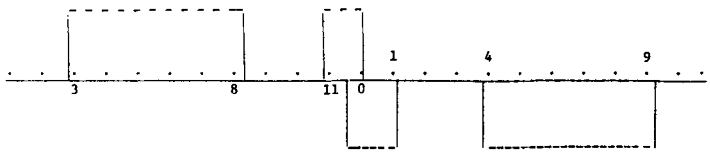


FIGURE 1.

transposition of the k-tupe $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. For instance, $\sigma_2^3(2, 3, 6, 7, 10, 11) = (5, 6, 9, 10, 1, 2)$. One of the most important characteristics of the transposition is that it "preserves intervalic distance," that is, k-tuples which differ by transposition have identical musical intervals. This suggests that we can define an equivalence relation on the collection of k-tuples obtained from A; two k-tuples being equivalent if they differ by transposition. For this reason it is desirable to establish a *standard form* for a k-tuple as a way of signifying the equivalence class to which it belongs; that is, we select the ordering for which the first element in the sequence is 0. Hence, (0, 2, 4, 6, 8, 10) is the standard form for the set of all transpositions of the 6-tuple (3, 5, 7, 9, 11, 1).

The transformation σ_3 is called an *inversion* and consists of subtracting each element of a given k-tuple from 12. Metaphorically speaking, k-tuples obtained from each other by inversion have the same "axis of symmetry." For example (FIGURE 1) $\sigma_3(0, 1, 4, 9) = (0, 11, 8, 3)$.

It must be emphasized that the concept of "interval," "intervalic distance," "pitch quantization," "axis of symmetry," and so on, are of music theoretic interest and will not concern us here. In the theory of the twelve-tone system they have achieved the status of quasitechnical terms and we mention them only to indicate that the mathematical objects we have defined have their roots in a musical system. The reader interested in a deeper understanding of this relationship is urged to consult the indicated references. If we denote the collection of k-tuples by $\mathscr{K}(A)$, then $\mathscr{K}(A)$ together with group G constitute structural objects out of which more complex structural entities are generated. In what follows we shall briefly describe some of these.

As mentioned earlier, a twelve-tone row is a specific ordering of the ring A. If we order A and put it in standard form (with 0 as the initial entry) there are 11! possible ways of ordering the remaining ring elements and each of these yields a distinct twelve-tone row. Using the group G of transformations we can form a 12×12 matrix from any given ordering of A by allowing A (in standard form) to be the first row and the inversion of A to be the first column. We obtain the remaining rows by translating elements of the first row by elements of the first column as follows:

$$\begin{bmatrix}
0 & 10 & 1 & 3 & 5 & 2 & 8 & 11 & 9 & 7 & 4 & 6 \\
2 & 0 & 3 & 5 & 7 & 4 & 10 & 1 & 11 & 9 & 6 & 8 \\
11 & 9 & 0 & 2 & 4 & 1 & 7 & 10 & 8 & 6 & 3 & 5 \\
9 & 7 & 10 & 0 & 2 & 11 & 5 & 8 & 6 & 4 & 1 & 3 \\
7 & 5 & 8 & 10 & 0 & 9 & 3 & 6 & 4 & 2 & 11 & 1 \\
10 & 8 & 11 & 1 & 3 & 0 & 6 & 9 & 7 & 5 & 2 & 4 \\
4 & 2 & 5 & 7 & 7 & 6 & 0 & 3 & 1 & 11 & 8 & 10 \\
1 & 11 & 2 & 4 & 6 & 3 & 9 & 0 & 10 & 8 & 5 & 7 \\
3 & 1 & 4 & 6 & 8 & 5 & 11 & 2 & 0 & 10 & 7 & 9 \\
5 & 3 & 6 & 8 & 10 & 7 & 1 & 4 & 2 & 0 & 9 & 11 \\
8 & 6 & 9 & 11 & 1 & 10 & 4 & 7 & 5 & 3 & 0 & 2 \\
6 & 4 & 7 & 9 & 11 & 8 & 2 & 5 & 3 & 1 & 10 & 0
\end{bmatrix}$$

For the purpose of selecting compositional material the matrix can be read both in the left-to-right direction, or in the right-to-left (retrograde) direction. Each column can be read either upward or downwards, hence there are a total of forty-eight ways to read from the matrix. Note the choice of the initial ordering of A is arbitrary. Once such an ordering is chosen, if distinct translates of A are then selected as new twelve-tone rows, new matrices can be constructed by means of the above procedure. From the standpoint of musical composition, all of these matrices are considered equivalent since the first row of each is some transposition of the same (initial) ordering of A.

Conversely, suppose we select a collection of $\mathscr{K}$ pitch class numbers from A (without regard to their order). Two such sets will be considered equivalent if they differ by a transposition $\sigma_2^j(0 \leq j \leq 12)$. Then a family of pitch class numbers chosen in the above fashion is called a chord. In the theory of the twelve-tone system, the selection of a chord is equivalent to the choice of a particular scale (Example 2 on the cassette consists of several chords).

Clearly if a k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ belongs to a particular chord then so do all of its translates $(\alpha_1 + j, \alpha_2 + j, \ldots, \alpha_k + j)$, where j is an integer between 0 and 12. Obviously, each k-tuple is contained in a *k-chord*, $\mathcal{O}_k$ (of which it represents one of the many possible orderings of the k-elements).

Given a chord $\mathcal{O}_k$ and any k-tuple contained in $\mathcal{O}_k$, we can consider the set of all pitch class numbers *not* belonging to that k-tuple. That is, if $\mathcal{O} \subset \mathcal{O}_k$, the set of elements *not* contained in $\mathcal{O}$ belongs to a chord called the *complement* of the original chord. Within the complementary chord there will exist a (12-k)-tuple consisting of those elements not contained in $\mathcal{O}$.

DEFINITION 1. Let $\mathcal{O} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a k-tuple. Then, by the *complement* $\tilde{\mathcal{O}}$ of $\mathcal{O}$ we mean the collection of all (12-k-)-tuples $\tilde{\mathcal{O}}$ (given in increasing order of elements) such that $\alpha_j \in A - \{\mathcal{O}\}$, where we are assuming that $\tilde{\mathcal{O}}$ is given in increasing order of elements. If an ordered k-tuple is equivalent to its complement $(\mathcal{O} \sim \tilde{\mathcal{O}})$ we say it is *self-complementary*.

For example, let $\mathcal{O} = (0, 1, 4, 5, 8, 9)$. Then $\tilde{\mathcal{O}} = (2, 3, 6, 7, 10, 11)$. Note that $\tilde{\mathcal{O}} = \sigma_2^2(\mathcal{O})$, hence $\mathcal{O} \sim \tilde{\mathcal{O}}$ as they differ by a transposition, that is $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are self-complementary.

An important collection of subsets of $\mathscr{K}(A)$ are the 6-tuples. In the theory of the twelve-tone system, 6-tuples are called *hexachords*. By definition, each hexacord is either self-complementary or belongs to a pair of mutually complementary hexachords.

Let $\mathscr{C}$ be a chord. For each choice of the elements of $\mathscr{C}$ form the collection of all sets of inverses $\{12 - c(\text{mod } 12)$, for $c \in \mathscr{C}\}$. Order them in increasing order. The collection of inverses so obtained belong to a chord called the inverse $\hat{\mathscr{C}}$ of $\mathscr{C}$. It can be shown that the inverse of a k-chord is a k-chord. If a chord is equivalent to its own inverse then it is called *self-invert*. Obviously, if $\mathscr{C}$ is self-invert, then any k-tuple contained in $\mathscr{C}$ is also self-invert.

As an example, if $\mathcal{O} = (0, 1, 2, 3, 4, 5) \subset \mathscr{C}$, then $(0, 7, 8, 9, 10, 11) \subset \hat{\mathscr{C}}$. Since $(0, 7, 8, 9, 10, 11) = \sigma_2^6(0, 1, 2, 3, 4, 5)$, $\mathcal{O}$ is self-invert.

Let $\mathscr{H}(A) \subset \mathscr{K}(A)$ denote the subcollection of 6-tuples. As stated earlier, each element $\mathscr{H} \subset \mathscr{H}(A)$ is called a hexacord. The inversion of a hexacord may or may not be equivalent to its complement. If it is, it is called *invert-complementary*. For example, the hexachord $\mathscr{H} = (0, 1, 4, 5, 8, 9)$ is equivalent to $\mathscr{H} = (0, 3, 4, 7, 8, 11)$. Here it is invert-complementary. Note $\mathscr{H}$ is also self-invert. If a hexachord is self-invert and self-complementary, it is called *self-invert-complementary*; or, in the terminology of Babbitt, *all combinatorial*. As noted in [8], "the importance of these categories in Babbitt's analysis rests upon the observation that twelve-tone compositional hexachords are almost always invert-complementary and very often self-invert complementary" (Example 3 on the cassette).

Denote by $\mathscr{A}(\mathscr{H})$ the collection of all combinatorial hexachords contained in $\mathscr{H}(\mathscr{A})$. It can easily be shown that there are a total of six (up to equivalence) elements in $\mathscr{A}(\mathscr{H})$, which can be listed (in standard form) as follows:

(1) (0, 1, 2, 3, 4, 5)	(4) (0, 1, 2, 6, 7, 8)
(2) (0, 2, 3, 4, 5, 7)	(5) (0, 1, 4, 5, 8, 9)
(3) (0, 2, 4, 5, 7, 9)	(6) (0, 2, 4, 6, 8, 10)

Note, by inspection of each element of the list, that for any all combinatorial hexachord $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ we have that $\alpha_1 + \alpha_6 = \alpha_5 + \alpha_2 = \alpha_3 + \alpha_4$. This is an important property which we shall later exploit.

It must be emphasized that the above notions are not vacuous. It can easily be shown that (0, 1, 2, 3, 6, 9) is self-invert, but not self-complementary. Furthermore, (0, 1, 2, 4, 5, 9) is self-invert, but not invert-complementary. Finally, (0, 1, 2, 3, 4, 7) is neither self-invert, self-complementary, nor invert-complementary.

Let n be a positive integer. By a *partition* of the integer n into j parts we mean a finite sequence of positive integers $\alpha_1, \alpha_2, \ldots, \alpha_j$ such that $\sum \alpha_i = n$. The integers α_i are called the *parts* of the partition. Suppose $\alpha = (\alpha_1, \ldots, \alpha_j)$ is a partition of n into j parts. The partition $\alpha' = (\alpha'_1, \ldots, \alpha'_j)$ obtained from alpha by choosing α'_1 as the number of parts of α that are $\geq i$ is called the *conjugate partition* associated with α.

It is a standard result [6, 1] that α' can be graphically obtained from α by a rotation of the Ferrer's diagram of α about the diagonal. Suppose $\alpha = (\alpha_1, \ldots, \alpha_j)$ is given. We have introduced a special class of tableaux associated with the partition α called *blank tableaux* [5].

DEFINITION 2. By a blank tableau associated with the partition α we mean an array $[(b^m_m)]$ of boxes, such that the mth row $(b^m_1, b^m_1, \ldots, b^m_{\alpha_m})$ of $[(b^m_n)]$ contains exactly α_m boxes, where we are assuming that the boxes are empty until explicity filled. A box in a blank tableau is called a *place*.

Clearly, the above notion generalizes the usual definition (cf. [6]) of the tableau associated with a partition. Hence the class of blank tableaux associated with a partition α includes as a subclass the collection of Ferrer's diagrams corresponding to α, as well as a subcollection of the conjugates obtained from those diagrams.

Suppose we are given $\alpha = (\alpha_1, \ldots, \alpha_j)$. Let $K = [(k^m_n)]$ be a tableau associated with α. Suppose we consider the mth row of K. Note the mth row $(k^m_1, k^m_2, \ldots, k^m_{\alpha_m})$ contains exactly α_m integers. Following Berge [6], we call K *normal* if:

(1) k^m_n's are all distinct;
(2) $m > r \rightarrow k^m_s > k^r_s$, for all r;
(3) $m > r \rightarrow k^t_m > k^t_r$, for all t.

If normal tableau K associated with the partition α uses only the integers 1, 2, …, n; K is called a *standard tableau* associated the partition α. The class of standard tableaux will ultimately play an important role in the partition problem.

THE BASIC PROBLEM

Let $A = Z_n$, and $M_m(A)$ denote the ring of $m \times m$ matrices over A, where λ, m and n are positive integers. Its most general setting, what we have called *Babbitt's*

Partition Problem, deals with the decomposition of matrices into block designs generated by partitions of the integer n into m parts. In the following we restrict ourselves to the case where $m = 4$ and $n = 12$. In particular, we prove the existence (and give a procedure for the construction) of a specific class of matrices which decompose into block designs generated by the above partitions. These designs were first discovered by Babbitt in his musical composition "partitions" for $n = 12$, $m = 4$. However it must be emphasized that everything we do carries over to the general case. In a subsequent paper we shall present these results (as well as estimates and a series of formulas for computing the various numbers arrived at by means of our basic construction procedure), however we felt it best to present this problem within the setting that first inspired it. The problem will be stated in three parts. The first part asks to:

(i) Find an algorithm to compute, and systematically list, the number k of all matrices belonging to $M_4(A)$ having the property that each of their rows and each of their columns add up to 12 such that the element 0 is never used.

(1) Repetitions among ring elements are allowed.

(2) Addition is taken in the usual sense.

Clearly (i) resembles the problem of computing magic squares, however, here we do not require that the sum of the diagonal elements be 12. A typical matrix satisfying (i) is:

$$\begin{bmatrix} 5 & 4 & 2 & 1 \\ 5 & 3 & 2 & 2 \\ 1 & 2 & 4 & 5 \\ 1 & 3 & 4 & 4 \end{bmatrix}$$

Given a matrix M which satisfies (i), each row or column of M represents a partition of the integer 12 into four parts. There are fifteen such partitions (listed in nonincreasing order):

$$(9\ 1^3), (8\ 2\ 1^2), (7\ 3\ 1^2), (7\ 2^2\ 1), (6\ 4\ 1^2), (6\ 3\ 2\ 1), (6\ 2^3),$$
$$(5^2\ 1^2), (5\ 4\ 2\ 1), (5\ 3^2\ 1), (5\ 3\ 2^2), (4^2\ 3\ 1), (4^2\ 2^2), (4\ 3^2\ 2),$$
$$(3^4)$$

Applying a well-known formula [9], there are a total of $\binom{12-4+4-1}{4-1} = \binom{11}{3} = 165$ possible ways in which the above partitions can be ordered and each one of these ways represents a possible row or column of a matrix M satisfying (i).

We shall now apply the formalism we have developed up to this point to formulate the second part of the partition problem. In particular this will allow us to obtain a lower bound on k.

From any such matrix M we can take a row or column and form a blank tableau where, as noted earlier, each position is considered empty until explicitly filled. For example, the blank tableaux $T_1 - T_4$ in FIGURE 2 (whose places are the individual boxes) are derived from the columns of the matrix,

$$\begin{bmatrix} 4 & 1 & 4 & 3 \\ 4 & 1 & 4 & 3 \\ 2 & 5 & 2 & 3 \\ 2 & 5 & 2 & 3 \end{bmatrix}$$

In forming blank tableaux from a matrix M we proceed (as in the previous example) from left to right along the columns of M. Note that we do not assume

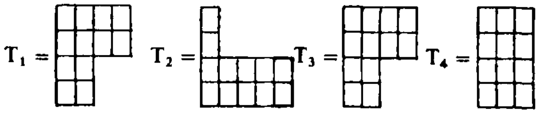


FIGURE 2.

anything about the justification of the blank tableaux; that is, they can be left justified, right justified, or neither.

Suppose M is a solution of (i) and T_k $(k = 1, \ldots, 4)$ is a blank tableau generated by the kth column of M. Note there are a total of i $(i = 1, 2, \ldots, 12)$ places; hence let t_i^k denote the position corresponding to the ith place of T_k (where we are assuming that the sequence of places $t_1^k, \ldots, t_{12}^k$ are assigned by starting in the upper left-hand corner of T_k, and proceeding horizontally across a row until one runs out of positions, returning to the next row, and so on, until all positions have been exhausted). For example,

$T =$

t_1^1	t_2^1	t_3^1	t_4^1
t_5^1	t_6^1	t_7^1	t_8^1
t_9^1	t_{10}^1		
t_{11}^1	t_{12}^1		

Two blank tableaux T_k and $T_{k'}$ are said to be equal if they have the same *shape*. For example, $T_1 = T_3$ in FIGURE 3.

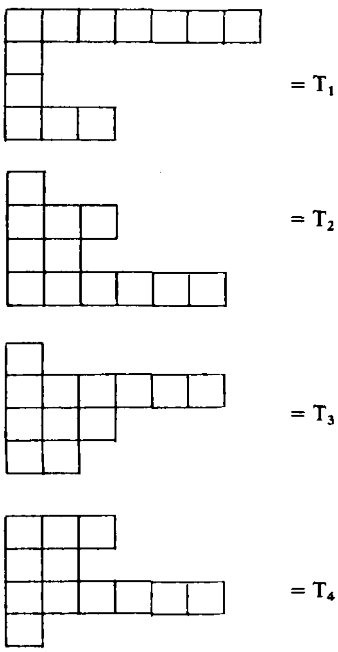


FIGURE 3.

Suppose T_k is a blank tableau and let T^k be the tableau obtained from T_k by a function $f: A \to T_k$, which assigns to each place t_m^k an element $f(a)$ of the ring A.

DEFINITION 3. We say that $T^k = T^{k'}$, if $T_k = T_{k'}$ and if they are filled in the way by the same elements. Hence in the previous example, $T_1 = T_3$, but $T^1 \neq T^3$, below.

$T^1 =$

0	10	1	2
9	11	8	7
3	5		
6	6		

$T^3 =$

2	1	10	0
7	8	11	9
5	3		
4	6		

Suppose $\tilde{A} \subset \mathscr{K}(A)$ is some ordering of the ring A, and $\{\sigma_i\}$ $(i = 1, \ldots, 4)$ is a sequence of any four transformations of $\tilde{A}$ by elements of the group B. Note that this sequence generates a 4×12 matrix $(a_{i,j})$:

$$(a_{i,j}) = \begin{bmatrix} a_{1,1} \cdots\cdots\cdots a_{1,12} \\ a_{2,1} \cdots\cdots\cdots a_{2,12} \\ a_{3,1} \cdots\cdots\cdots a_{3,12} \\ a_{4,1} \cdots\cdots\cdots a_{4,12} \end{bmatrix}$$

For instance,

$$(a_{i,j}) = \begin{bmatrix} 0 & 10 & 1 & 2 & 5 & 3 & 9 & 11 & 8 & 7 & 4 & 6 \\ 9 & 11 & 8 & 7 & 4 & 6 & 0 & 10 & 1 & 2 & 5 & 3 \\ 3 & 5 & 2 & 1 & 10 & 0 & 6 & 4 & 7 & 8 & 11 & 9 \\ 6 & 4 & 7 & 8 & 11 & 9 & 3 & 5 & 2 & 1 & 10 & 0 \end{bmatrix}$$

is generated by the transformation $1_{\tilde{A}}, \sigma_2^9\sigma_3, \sigma_1\sigma_2^9$, and σ_1, where A has been given the ordering (0, 10, 1, 2, 5, 3, 9, 11, 8, 7, 4, 6).

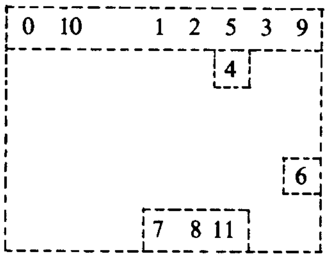
Let M be a solution to (i), T_k the blank tableau generated by the kth column of M, and $(a_{i,j})$ be as above. Then if R_i $(i = 1, \ldots, 4)$ is the ith row of $(a_{i,j})$, let a_α^i, a_β^i denote the elements of R_i belonging to the αth and βth column of $(a_{i,j})$, where we assume $(1 \leq \alpha \leq \beta \leq 12)$.

DEFINITION 4. By a *segment* $\{a_{\alpha,\beta}^i\}$ of R_i with *initial point* a_α^i and *terminal point* a_β^i we mean the *subrow* of R_i of all elements of the form $\{a_\lambda^i\}\alpha \leq \lambda \leq \beta$. By the *length* of a segment we mean the number of elements in $\{a_{\alpha,\beta}^i\}$. A segment consisting of a single element has only one subscript. For instance, $\{a_{1,7}^1\} = [0\ 10\ 1\ 2\ 5\ 3\ 9]$ is a segment of length 7 consisting of the first row and first seven columns of the matrix $(a_{i,j})$ from the previous example.

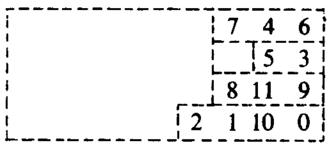
Two segments will be said to be *equal* if they have the same length, contain the same elements, and their elements occupy identical order positions. If two segments are unequal they are said to be distinct.

DEFINITION 5. By a *segmented block* B corresponding to $(a_{i,j})$ we mean a sequence $\{a_{\alpha,\beta}^i\}_k$ of segments, where the indices $\{k\}$ indicate the order in which segments of the block are to be filled.

For example, the sequence of segments $\{a_{\alpha,\beta}^i\}_k = \{\{a_{1,7}^1\}_1, \{a_5{}^2\}_2, \{a_7{}^3\}_3$ $\{a_{3,5}^4\}_4\}$ yield the segmented block,



while $\{a^i_{\alpha,\beta}\}_k = \{\{a^4_{9,12}\}_1, \{a^3_{10,12}\}_2, \{a^2_{11,12}\}_3, \{a^1_{9,12}\}_4\}$ yields the segmented block,

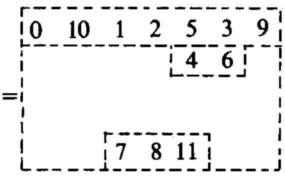


DEFINITION 6. Suppose B is some segmented block of a matrix $(a_{i,j})$. Then B is said to be *properly filled* by elements of $(a_{i,j})$ if each element of A occurs *exactly once* in B.

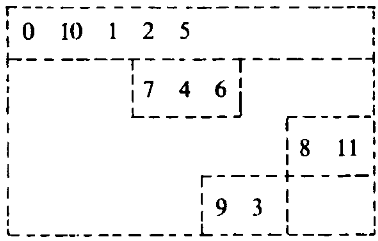
For example, both of the above segmented blocks are properly filled by elements from the 4×12 matrix $(a_{i,j})$, above. Note here we do *not* allow repetitions among ring elements, hence a properly filled segmented block decomposes into segments which contain a given ring element only once. Henceforth, properly filled segmented blocks will be called *proper blocks.*

DEFINITION 7. A blank tableau T_k is said to be *properly filled* by elements of a fixed matrix $(a_{i,j})$ if there is a proper block B and a bijection $f: B \to T_k$ which maps distinct segments of B onto distinct rows of T_k. That is, a blank tableau can be properly filled if a 1-1 correspondence can be established between segments of some proper block and the rows of the blank tableau. As the elements of B are distinct the definition says that each $a \in \{a^i_{\alpha,\beta}\}$ is mapped onto a distinct place of a fixed row of T_k. Formally, we write $f: (a^i_{\alpha,\beta})_t \to t_m{}^k$, where $(a^i_{\alpha,\beta})_t$ is the element of $(a_{i,j})$ belonging to the segment $\{a^i_{\alpha,\beta}\}$ of B.

Before continuing, we give some examples of mappings which are illegal. Clearly, although the block,

$B =$ 

is proper, the segments of B cannot be bijectively mapped onto any blank tableau derived from a column of some M which is a solution to (i) since each such blank tableau must have four rows, while B has only three. On the other hand, the block,



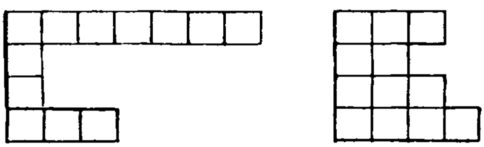
derived from the matrix $(a_{i,j})$ satisfies all of the requirements of the definition, however the mapping from the segments of B into the places of the blank tableau corresponding to the partition $(5\ 3\ 2^2)$ defined by

0	10	1	2	5
7	4	8		
11	6			
9	3			

is clearly illegal since it splits the second and third segments of B. That is, distinct segments do not map onto distinct rows, even though the above filling does set up a 1-1 correspondence between elements of distinct segments and places of the blank tableau.

Another way of phrasing the definition would be to say that a blank tableau can be properly filled by elements of a fixed matrix $(a_{i,j})$ if there is a proper block B (with the aforementioned characteristics) such that for each segment of B we can find a row of the blank tableau (which is a permutation of some element of a four-part partition of 12) corresponding *to the number of elements contained in that segment.*

Hence the blank tableaux in FIGURES 3 and 4 can be bijectively filled by the segmented blocks of our initial example to form the indicated tableaux. These



0	10	1	2	5	3	9
4						
6						
7	8	11				

7	4	6	
5	3		
8	11	9	
2	1	10	0

FIGURE 4.

tableaux correspond to permutations of $(7\ 3\ 1^2)$ and $(4\ 3^2\ 2)$. Note that,

2	1	10	0
7	4	6	
8	11	9	
5	3		

is also a tableau that can be formed from the aforementioned blocks.

To give some indication of the utility of these concepts, consider the proper blocks (B_1, B_2, B_3, B_4) in FIGURE 5. Note there are a total of forty-eight entries in the blocks B_1–B_4. Indeed, they exhaust all of the elements of the matrix $(a_{i,j})$ from which they were derived. Note also that B_1–B_4 can, respectively, fill the blank tableau T_1–T_4 of FIGURE 3.

The tableau T_1 corresponds to a permutation of the order of the partition $(7\ 3\ 1^2)$, while T_2, T_3, and T_4 all correspond to permutations of the order of the partition

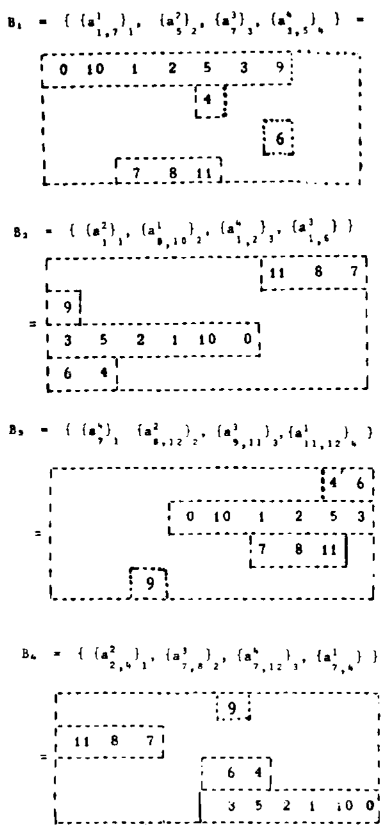


FIGURE 5.

(6 3 2 1). Hence T_1, T_2, T_3, and T_4 are all derivable from columns of the matrix,

$$\begin{bmatrix} 7 & 1 & 1 & 3 \\ 1 & 3 & 6 & 2 \\ 1 & 2 & 3 & 6 \\ 3 & 6 & 2 & 1 \end{bmatrix}$$

which is obviously a solution to (i).

This seems to suggest that we should search for matrices $(a_{i,j})$ that can be decomposed into blocks that "fit together," i.e., which properly fill blank tableau derivable from matrices of the above form. This is the motivation behind the notion of *uniform decomposition.*

DEFINITION 8. Let $(a_{i,j})$ be a 4×12 matrix obtained by a sequence of four transformations of some 12-tuple contained in $\mathscr{K}(A)$ by elements of the group G. Then $(a_{i,j})$ is *uniformly decomposable* into proper blocks, if there exists a decomposition of $(a_{i,j})$ into *nonoverlapping* proper blocks B_1–B_4. In other words, a matrix is uniformly decomposable if it can be partitioned by *some* of its proper blocks into distinct segments, no two of which intersect, and the totality of which yield the original matrix. The blocks B_1–B_4 form a *uniform decomposition* of $(a_{i,j})$. For example, the matrix given by,

$$\begin{bmatrix} 3 & 4 & 11 & 0 & 7 & 8 & 2 & 1 & 6 & 5 & 10 & 9 \\ 2 & 1 & 6 & 5 & 10 & 9 & 3 & 4 & 11 & 0 & 7 & 8 \\ 8 & 7 & 0 & 11 & 4 & 3 & 9 & 10 & 5 & 6 & 1 & 2 \\ 9 & 10 & 5 & 6 & 6 & 2 & 8 & 7 & 0 & 11 & 4 & 3 \end{bmatrix}$$

can be decomposed into the proper blocks given in FIGURE 6.

The blocks B_1–B_4 partition the matrix into distinct segments, no two of which overlap, and the union of which yield the given matrix, hence $(a_{i,j})$ is uniformly decomposed by the blocks B_1–B_4.

On the other hand, the matrix $(a_{i,j})$ is not uniformly decomposed by the proper blocks given in FIGURE 5 since the segments $\{a^1_{1,7}\}_1 \in B_1$ and $\{a_7{}^1\}_4 \in B_4$ overlap (they both contain the matrix element 9).

As noted earlier, the segments of the matrix $(a_{i,j})$ can be bijectively mapped onto blank tableaux derivable from,

$$\begin{bmatrix} 7 & 1 & 1 & 3 \\ 1 & 3 & 6 & 2 \\ 1 & 2 & 3 & 6 \\ 3 & 6 & 2 & 1 \end{bmatrix} \tag{1}$$

while the segments of the matrix of our above example bijectively fill blank tableaux corresponding to,

$$\begin{bmatrix} 5 & 4 & 2 & 1 \\ 4 & 2 & 1 & 5 \\ 2 & 1 & 5 & 4 \\ 1 & 5 & 4 & 2 \end{bmatrix} \tag{2}$$

Note that the matrix (2) has the property that each column can be paired into 2-tuples, the sum of which is six, while matrix (1) cannot. This fact will play a significant role in the following discussion.

If a matrix $(a_{i,j})$ has a uniform decomposition, then we say that it *decomposes*

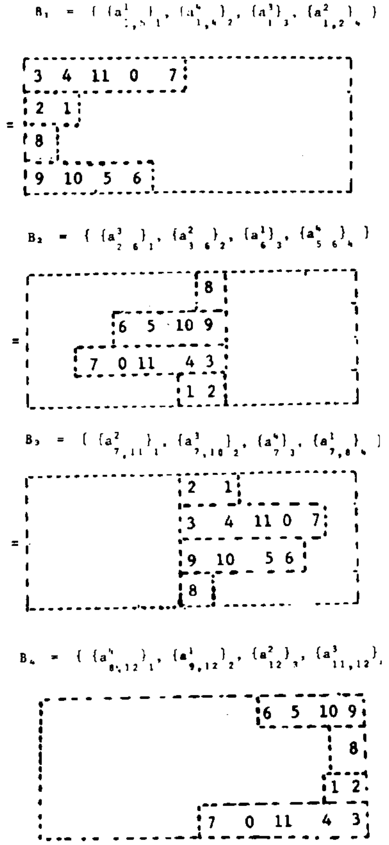


FIGURE 6.

into a *block design* generated by that decomposition. Hence, every uniform decomposition of a matrix generates a block design of a particular configuration. In every such configuration a total of forty-eight elements will be distributed over sixteen segments. Each element will occur in exactly four different segments and the totality of segments will piece together to form the original matrix.

Suppose $\mathscr{M}(M)$ denotes the collection of all matrices belonging to $M_4(A)$ which satisfy (i), note that $|\mathscr{M}(M)| = k$, where $|\ |$ denotes cardinality. Let $\mathscr{T}(T_k)$ be the collection of all blank tableau $\{T_k\}$ generated by the *columns* of elements of $\mathscr{M}(M)$. For a given sequence $\{\sigma_i\}$ of transformations with fixed matrix $(a_{i,j})$ let $\mathscr{U}(T_k)$ be the subcollection of $\mathscr{T}(T_k)$, consisting of all blank tableau T_k which uniformly decompose the matrix $(a_{i,j})$. That is, each element of $\mathscr{U}(T_k)$ can be properly filled by some proper block belonging to a uniform decomposition of $(a_{i,j})$. Let $\beta(T_k)$ be the number of *distinct* elements of $\mathscr{U}(T_k)$. Then the second and third parts of the Partition Problem ask to:

(ii) Find an algorithm for computing $\beta(T_k)$.

(iii) Find an algorithm to compute and systematically list the number of ways $(a_{i,j})$ can be uniformly decomposed into block designs generated by the $\beta(T_k)$ elements of $\mathscr{U}(T_k)$.

DEFINITION 9. A blank tableau T_k will be called a *solution to the partition problem* if it is an element of $\mathscr{U}(T_k)$.

In the theory of the twelve-tone system proper blocks are called *aggregates.* Hence an aggregate is a collection of the full twelve chromatic pitch classes. In many respects aggregates can be regarded as the fundamental building blocks of many contemporary musical compositions. That is, the elements of $\mathscr{U}(T_k)$ are precisely the "four part aggregate-forming partitions [5]," while the number $\beta(T_k)$ represents the total number of four-part aggregate-forming partitions obtainable from a fixed matrix $(a_{i,j})$. In the remainder of this paper we outline a procedure that allow us to construct blank tableau which are solutions to the partition problem in the above sense.

THE BASIC CONSTRUCTION

We noted in the introduction that $\mathscr{A}(\mathscr{H})$ consists of the set of all 6-tuples that are self-invert-complementary, i.e., all combinational. As noted before, given any $(a_1, a_2, \ldots, a_6) \in \mathscr{A}(\mathscr{H})$ we have that $a_1 + a_6 = a_2 + a_5 = a_3 + a_4 = a'$ for some $a' \in A$. Suppose $\mathscr{S}_1 = (a_1, a_2, a_3)$ and $\mathscr{S}_2 = (a_4, a_5, a_6)$. LEMMA 1 is easily proven.

LEMMA 1. $\mathscr{S}_2 = \sigma_1[2^{a'}[\sigma_3(\mathscr{S}_1)]]$.

Let $\mathcal{O} \subset \mathscr{A}(\mathscr{H})$. As $\mathcal{O}$ is self-complementary there is an integer j such that $\tilde{\mathcal{O}} = \sigma_2{}^j(\mathcal{O})$. Suppose we take $\sigma_1(\tilde{\mathcal{O}}) = \sigma_1[\sigma_2{}^j(\mathcal{O})]$. In this fashion we generate two 6-tuples; namely, $\mathcal{O}_0 = (a_1, a_2, \ldots, a_6) = \mathcal{O}$ and $\mathcal{O}_6 = \sigma_1(\tilde{\mathcal{O}}) = \sigma_1[\sigma_2{}^j(\mathcal{O})] = (a_7, a_8, \ldots, a_{12})$.

Hence given any element of $\mathscr{A}(\mathscr{H})$ we can induce an ordering of A by forming the union of $\mathcal{O}_0$ with $\mathcal{O}_6$. For example, if we take $\mathcal{O}_0 = (0, 2, 3, 4, 5, 7)$, we have $\tilde{\mathcal{O}} = \sigma_2{}^6(\mathcal{O}) = (6, 8, 9, 10, 11, 1)$. Since $\mathcal{O}_6 = \sigma_1(\tilde{\mathcal{O}}) = (1, 11, 10, 9, 8, 6)$ we have that $A_1 = (0, 2, 3, 4, 5, 7, 1, 11, 10, 9, 8, 6)$ is the ordering of A induced by $\mathcal{O}_0$ and $\mathcal{O}_6$.

Note that by the property of translation invariance of the elements of $\mathscr{A}(\mathscr{H})$, any translate of $\mathcal{O}_0$ can be used to construct an ordering of A in the following way. If $\mathcal{O}^* = \sigma_2{}^j(\mathcal{O}_0)$ for some integer j, decompose $\mathcal{O}^*$ into 3-tuples (a_1, a_2, a_3) and (a_4, a_5, a_6) having the property that $a_1 + a_6 = a_2 + a_5 = a_3 + a_4 = a'$. Then place the elements of $\mathcal{O}_0{}^*$ in the order $(a_1, a_2, \ldots, a_6) = \mathcal{O}_0{}^*$. As the complement of $\mathcal{O}_0{}^*$ will be some translate of $(a_1, a_2, \ldots, a_6)$ by computing σ_1 of the complement we generate $\mathcal{O}_6{}^*$ and its union with $\mathcal{O}_0{}^*$ yields the aforementioned ordering of A.

For example, let $\mathcal{O}^* = \sigma_2{}^{10}(0, 2, 3, 4, 5, 7) = (10, 0, 1, 2, 3, 5)$. Note the pairing into 3-tuples by (0, 10, 1) and (2, 5, 3) induces an ordering of $\mathcal{O}^*$ into $\mathcal{O}_0{}^* = (0, 10, 1, 2, 5, 3)$. Then $\sigma_2{}^6(\mathcal{O}_0{}^*) = (6, 4, 7, 8, 11, 9)$ and $\sigma_1[\sigma_2{}^6(\mathcal{O}_0{}^*)] = (9, 11, 8, 7, 4, 6) = \mathcal{O}_0{}^*$. The union of $\mathcal{O}_0{}^*$ with $\mathcal{O}_6{}^*$ induces the ordering of A given by $A_1 = (0, 10, 1, 2, 5, 3, 9, 11, 8, 7, 4, 6)$ which, incidently, is the first row of the matrix $(a_{i,j})$.

Suppose $\mathcal{O}_0$ and $\mathcal{O}_6$ are as defined and A has been given the ordering induced by $\mathcal{O}_0$ and $\mathcal{O}_6$. Consider the action of the sequence of transformations $\{1_{A_1}(A_1) = A_1, \sigma_2{}^7[\sigma_3(A_1)] = A_2, \sigma_3(A_1) = \sigma_1(A_2) = A_3, \sigma_1(A_1) = A_4\}$. These generate a 4×12 matrix $(a_{i,j})$,

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_{11} & a_9 & a_{10} & a_{11} & a_{12} \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_{12} & a_{11} & a_{10} & a_9 & a_8 & a_7 \\ a_{12} & a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{bmatrix}$$

where the noted properties of the set construction are summarized in FIGURE 7.

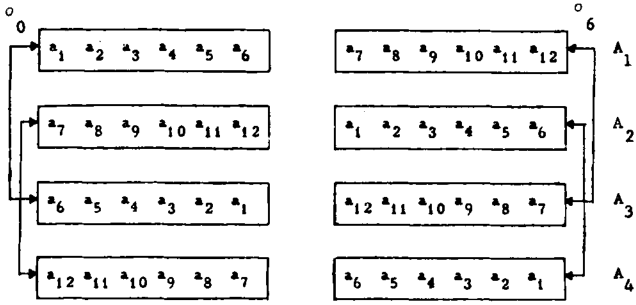


FIGURE 7.

Note that, since we begin with a specific 6-tuple, this procedure is entirely constructive. As noted earlier, the class of 4×12 matrices which can be constructed in this fashion was suggested to the second author while analyzing Babbitt's composition Partitions; they have been termed *B-constructible* (Example 4 on the cassette realizes such a configuration).

Suppose $(a_{i,j})$ is a fixed matrix. Consider the subcollection $\mathscr{B}(T_k) \subset \mathscr{T}(T_k)$ consisting of those blank tableau which have the property that their entries can be paired into 2-tuples, the sum of which is 6.

DEFINITION 10. We say that a blank tableau T_k is B-constructible if it belongs to $\mathscr{B}(T_k)$.

Our main theorem, which we stated without proof in [5], can now be proved.

THEOREM 1. A sufficient condition for a blank tableau T_k to be a solution to the partition problem is that $(a_{i,j})$ be B-constructible.

Proof: What we show is that any B-constructible blank tableau generates a uniform decomposition of $(a_{i,j})$. Suppose $(a_{i,j})$ is B-constructible and that $T_k \subset \mathscr{B}(T_k)$. Then, by definition, T_k is obtainable from some partition $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of the integer 12 into four parts, which has the property that it can be paired into 2-tuples, the sum of which is 6. For simplicity we assume that the pairing is given by $\{(\alpha_1, \alpha_4), (\alpha_2, \alpha_3)\}$, where $\alpha_1 \geq \alpha_4$ and $\alpha_3 \leq \alpha_2$. By means of cyclic permutations we obtain the matrix:

$$M = \begin{bmatrix} \alpha_1 & \alpha_4 & \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_1 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_4 \end{bmatrix}$$

Note that blank tableau derivable from columns of M are again B-constructible. We now show that these blank tableau yield a uniform decomposition of $(a_{i,j})$.

Beginning with a_1 we can form a segment consisting of elements from a_1 to a_{α_1}, out of the *first row* of $(a_{i,j})$. Since $\alpha_1 \leq 5$, we can form a *second segment* from the *third row* of $(a_{i,j})$ by selecting α_4 elements from a_{α_1+1} to a_6. We obtain a *third* segment by choosing the α_2 elements from the *second* row of $(a_{i,j})$ starting with the element $a_{7+(\alpha_2-1)}$ and ending with a_7. Finally we obtain a *fourth* segment from the last row of $(a_{i,j})$ consisting of those elements from $a_{7+\alpha_2}$ to a_{12}. In other words, we have formed segments consisting of α_1 elements from the first row, α_4 from the third, α_2 from the second, and α_3 from the fourth. These segments do not overlap,

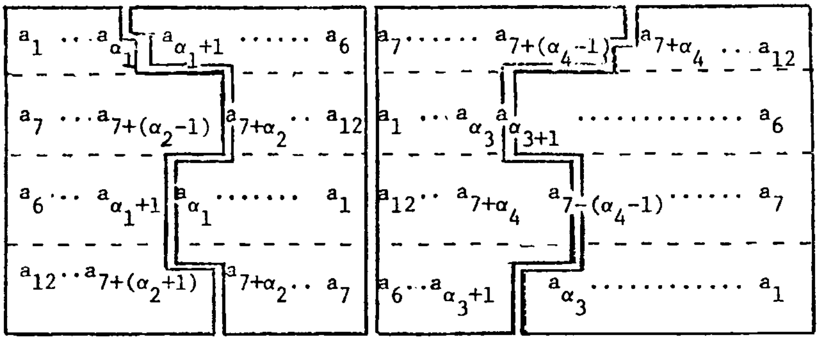


FIGURE 8.

contain all of the ring elements, and properly fill the blank tableau corresponding to the first column of M, hence they decompose into a proper block B_1.

In order to obtain a second proper block B_2 we use the fact that $\alpha_1 + \alpha_4 = 6$. Since the first segment of B_1 consists of the α_1 elements from a_1 to a_{α_1}, we have that there are α_4 elements remaining between a_{α_1} and a_6. Hence we form the *first* segment of B_2 by selecting those elements from a_{α_1+1} to a_1. A *third segment* is obtained from the *fourth* row of $(a_{i,j})$ by selecting the α_3 elements from a_6 to $a_{7+(\alpha_2-1)}$; finally, a *fourth segment* is obtained from the second row by choosing the α_1 elements from a_{12} to $a_{7+\alpha_2}$. By now the sequence of steps in the procedure should be obvious. If we proceed to form the proper blocks B_3 and B_4 we obtain a uniform decomposition of the matrix $(a_{i,j})$ given in FIGURE 8.

Since T_k corresponds to the above uniform decomposition of $(a_{i,j})$ we have $T_k \subset \mathscr{U}(T_k)$, hence T_k is a solution to the partition problem and the theorem is proved. □

If T_k is not B-constructible but simply obtained from a solution to (ii) it is easily shown that $(a_{i,j})$ can still be uniformly decomposed by the columns of T_k, however the corresponding block design will have an entirely different shape. *Note*: it is precisely the fact that T_k can be paired into 2-tuples (the sum of which is 6) which yields the "shape" of the above design, i.e., the segments divide clearly along the center of the matrix (cf. FIGURE 7). When the T_k's are not B-constructible, the resulting matrix decomposition will not exhibit such a symmetry (i.e., segment will cross hexachordal boundaries). The fact that $(a_{i,j})$ is B-constructible is only a sufficient condition can be seen from the following example. It is easily checked that,

$$\begin{bmatrix} 0 & 10 & 1 & 3 & 5 & 2 & 8 & 11 & 9 & 7 & 4 & 6 \\ 9 & 11 & 8 & 6 & 4 & 7 & 1 & 10 & 0 & 2 & 5 & 3 \\ 3 & 5 & 2 & 0 & 10 & 1 & 7 & 4 & 6 & 8 & 11 & 9 \\ 6 & 4 & 7 & 9 & 11 & 8 & 2 & 5 & 3 & 1 & 10 & 0 \end{bmatrix}$$

is not B-constructible, but can be uniformly decomposed by the columns of,

$$\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

On the other hand, we conclude with the following *conjecture*: The matrix,

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix}$$

cannot be uniformly decomposed by the columns of,

$$\begin{bmatrix} 7 & 3 & 1 & 1 \\ 3 & 7 & 1 & 1 \\ 1 & 1 & 7 & 3 \\ 1 & 1 & 3 & 7 \end{bmatrix}$$

Acknowledgments

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