Balzano and Zweifel: Another Look at Generalized Diatonic Scales

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I. Introduction

Nearly thirty years ago, an alternative basis for the derivation of the Western diatonic scale was proposed by Budden (1972), and developed more recently by Balzano (1980). This alternative basis is derived from Group Theory, rather than just-intonation ratios. The theory also predicts that other "diatonic" scales exist within microtonal scales.

This paper approaches these and other "diatonic" scales from the viewpoint of a composer seeking new materials for creative work, rather than trying for rigorous mathematical proof. The mathematical basis for the theory that follows can be found in any of the referenced texts. Instead, I have concentrated on the "topology" of the scales produced. In so doing

I discovered a number of other "diatonic" scales within other equal-tempered microtonal scales.

In addition to looking at these new tonal materials, I have also taken a look at ratio-based approaches to "diatonic" scale construction and related them to Balzano's work. This has provoked some interesting ideas on the creation of tonalities from basic musical intervals and perhaps also given an insight into the persistence of the 5-limit basis for Western tonality.

Evidently, the "diatonic" scales presented here (and the "pentatonic" scales that are not) all require detailed investigation. For reasons of space I have left out some topics of discussion;[1] inclusion of them all would have extended this paper to absurd length.

In the body of this paper, I use pitch-class number notation, which can be described simply as a means of using numbers to notate the different tones of a scale of any number of pitch classes to the "octave." These numbers begin with zero (0) and proceed by integer steps to a number that is one less than the number of pitch-classes in the scale, thus making the correct number of pitch classes in all.

The "C*n* scale" is a generic label applied to *all* equal-tempered scales of *n* pitch-classes. Specific instances of such scales are referred to by giving *n* a value. Thus, the equal-tempered scale of twelve pitch-classes is referred to as the C12 scale. Wherever the C12 scale itself is used, I use the common names for the intervals, but *not* letter names for the pitch-class numbers. For example the sequence of pitch-classes 0 4 8 0 represents a cycle of major thirds, beginning with pitch-class 0.

Throughout this paper, I use the terms "diatonic," "pentatonic" and "chromatic" in their generic senses, as follows:

1. A "diatonic" scale is a scale formed from two intervals of different sizes, such that groups of several adjacent instances of the larger interval are separated by single instances of the smaller interval.

2. A "pentatonic" scale is a scale formed from two intervals of different sizes, such that groups of several adjacent instances of the smaller interval are separated by single instances of the larger interval. Therefore a generic "pentatonic" can contain more than five tones.

3. "Chromatic" refers to the interval formed between adjacent pitch-classes of any equal-tempered scale.

Henceforth, these terms appear without qualifying quotation marks.

II. The Current Theory

In Balzano's paper, we are presented with a collection of diatonic scales generated by a specific collection of C*n* scales. Significantly, only those scales generated from values of *n* that are the product of two integers that have a difference of one are investigated. For example C12 is the product of C3 and C4, C20 is the product of C4 and C5, and so forth. The reason for the limitation to this sequence of scales is not discussed in the Balzano paper, but stems from theoretical considerations relating to group theory.[2]

The creation of these new diatonic scales is not easy to describe in words, but is relatively simple to show graphically. As an example, let us look at C12, and generate the group-theoretic diatonic for it. First, we draw a grid where major thirds proceed from left to right and minor thirds from bottom to top. Then a compact, connected structure is marked out on it as shown in Example 1 below. This structure consists of a number of steps alternating the smaller interval with the larger, until the starting pitch class recurs.[3]

1	5	9	1	5	9
10	2	6	10	**2**	6
7	11	3	**7**	**11**	3
4	8	**0**	**4**	8	0
1	**5**	**9**	1	5	9
10	**2**	6	10	2	6
7	11	3	7	11	3

EXAMPLE 1: C12, SEVEN-TONE DIATONIC SCALE

Notice that there are four minor thirds and three major thirds in the connected structure. Each of these cycles around an octave just once, so the starting pitch class reappears after six other pitch classes are presented, making seven pitch-classes in all. In Example 1, pitch-class 0 is the tonic. Formations of three adjacent connected pitch classes are the standard triads of the Western diatonic scale, consisting of three each of major and minor triads, and one diminished triad (the latter being the "wrap-around" point of the connected structure). It is these principles of grid construction and marking out of the diatonic scale which Balzano extends to C20, C30, and C42 scales and can also be extended to other C*n* scales of the type described in his paper.

Balzano's presentation of his C20 diatonic scale is given below in Example 2. This grid is drawn up in the same way as the grid for the C12 scale, with a grid of two differently sized intervals and the marking out of a connected compact structure. (I give a generic method for creating these diatonic scales in section IV, below.) At first sight, this structure bears a strong resemblance to the C12 seven-tone diatonic scale: triads are formed from alternate tones from the (reordered) scale, and transposition upward by an interval of nine C20 chromatic steps results in the replacement of a tone (pitch-class 11) by one a chromatic step higher (pitch-class 12). However, on closer inspection there are curious anomalies: the raised tone does not act like a leading tone in the new "key," instead it operates as the second degree, and the scale structure itself is of a number of groups of smaller intervals separated by single larger intervals. The only scale contained within the C12 chromatic that looks like this is the anhemitonic pentatonic scale, with groups of whole tones separated by minor thirds.

6	11	16	1	6	**11**	16
2	7	12	17	2	**7**	12
18	3	8	13	**18**	**3**	8
14	19	4	**9**	**14**	19	4
10	15	**0**	**5**	10	15	0
6	**11**	**16**	1	6	11	16
2	**7**	12	17	2	7	12
18	3	8	13	18	3	8

EXAMPLE 2: C20 NINE-TONE SCALE (AFTER BALZANO)

These anomalies have also been noticed by Zweifel (1996). Zweifel also correctly deduces that the nine-tone C20 scale presented by Balzano is more like the C12 pentatonic scale in structure. Zweifel then proposes a new diatonic scale which more closely mirrors the properties of the C12 diatonic scale. Zweifel attempts to draw this diatonic scale on Balzano's original C4, C5 grid and makes some conclusions about its structure.[4] I also came to the same conclusions as Zweifel in 1993, but drew a different grid, which I present below. It was the drawing of this new grid that led to the developments presented here.

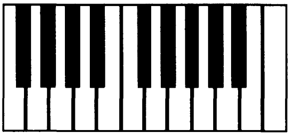
III. An Alternate View of the Eleven-Tone C20 Diatonic Scale

The following grid gives an alternate form for the new diatonic scale that Zweifel and I propose (Example 3).

13	19	5	11	17	3	**9**	15
8	14	0	6	12	18	**4**	10
3	9	15	1	7	**13**	**19**	5
18	4	0	16	**2**	**8**	14	0
13	19	5	**11**	**17**	3	9	15
8	14	**0**	**6**	12	18	4	10
3	**9**	**15**	1	7	13	19	5
18	**4**	10	16	2	8	14	0
13	19	5	11	17	3	9	15

EXAMPLE 3: C20, ELEVEN-TONE DIATONIC SCALE (AS PROPOSED)

However, my conclusions about where the tonic lies are different from those of Zweifel. As this new diatonic scale now contains C20 semitone scale steps, I thought it reasonable to assume that the base tone of the "diminished triad" (at the top of the connected shape in the above diagram) would act as a leading tone onto the tonic. Pitch-class 0 now becomes the tonic for the diatonic scale presented above. In modulating upward by an interval of 11 C20 semitone steps (the analogue of the fifth of the C12 diatonic scale), pitch-class 9 is replaced by pitch-class 10. This pitch class becomes the leading tone of the new key. This is the F → F♯ property that Balzano proposes as a fundamental feature of tonal systems. Importantly, the altered tone was eleven C20 semitone steps lower than the tonic (the analogue of the "subdominant" degree), and becomes the new leading tone, exactly analogous to the raising of the fourth degree of the C12 diatonic to become the leading tone of the "dominant" key. In terms of the "eleven-white, nine-black" keyboard that Zweifel proposes, the diatonic looks and "feels" like the C12 diatonic scale as represented on a standard keyboard, with adjacent white keys occurring in structurally similar places.[5]



EXAMPLE 4: THE C20 KEYBOARD
PITCH-CLASS 0 OCCUPIES THE KEY TO THE FAR LEFT
AND THE WHITE NOTES CORRESPOND TO THE DIATONIC
SCALE THAT IS PROPOSED IN THE TEXT

One interesting feature of this new C20 diatonic scale is that the grid does not wrap into a toroid as Balzano's diatonic scale does. I believe that the toroidal wrapping of the pitch space is a side effect of the scales that Balzano chooses to investigate rather than being a property of tonal systems in general. In addition, the triads of the new scale are not formed from alternate tones from the diatonic scale, but from every third tone. How this affects the tonal structure of music composed using this C20 diatonic scale has not been investigated.

A system of letter names, key signatures, and seven-line staves can be constructed to write C20 diatonic music. Balzano proposes Roman numerals for marking the scale degrees, with uppercase for the analog of major triads and lowercase for minor and diminished triads.[6] I refrain from such markings and notation as the nature of the chords formed within this new diatonic scale may be totally different from those of the C12 diatonic scale. Certainly a thorough investigation of the acoustic properties of the components of this new C20 diatonic scale is in order, before notational and theoretic symbology is applied to it.

IV. Expansion to Other Scales

So far, the investigation has been limited to the C*n* scales as defined above. If, on the other hand, we look at *all* possible C*n* scales and see if diatonic scales exist within them, a rich seam of new musical possibilities opens up.

One empirical technique for detecting if there is a diatonic scale contained within a C*n* scale is described below.[7] This technique finds all of the diatonic and pentatonic scales to be found in the collection of scales that Balzano considers in his paper, but also finds many others in a wide range of C*n* scales. More recently, I have found it better to build a grid directly, and to search for diatonic scales within the grid. In all cases, I assume that the sequence of pitch classes ascending diagonally from left to right in the grid forms a cycle analogous to the cycle of fifths in C12.

i. Determine if there are interval cycles that step around all of the pitch classes in the C*n* scale under investigation, before returning to the starting pitch class. These scales are analogous to the cycle of fifths in the C12 diatonic system.
ii. Take a segment of this scale that has an odd number of pitch classes, that when reordered as an ascending scale, there are no points at which three adjacent pitch classes lie together. (To take C12 as an example, a seven-tone segment of the cycle of fifths complies with this rule, but a nine-tone segment does not.)
iii. Divide the segment into two parts, overlapping at the central tone. (For the seven-tone segment of C12: 5 0 7 2 9 4 11 → 5 0 7 2 and 2 9 4 11. Align these two subsegments, and alternate the tones from each segment, starting with the subsegment that begins with the repeated pitch class (2 9 4 11 in the example case). See Example 5.

  5   0   7   2
2   9   4   11

EXAMPLE 5

Drawing this so that the smaller interval steps vertically up the page and the larger interval steps horizontally across the page we arrive at the connected structure shown in Example 1, above. As in the example, the remaining tones from the grid can then easily be filled in.

The only additional rules I use to determine if the resulting formation is a diatonic scale are:

i. Check that the larger "horizontal" interval is less than twice the size of the smaller, "vertical" one. This ensures that the resulting diatonic scale is coherent in the sense given by Balzano in his article.

ii. Check that the scale conforms to the F→F♯ rule, as described above in the discussion on the C20 diatonic scale.

iii. Check that a pentatonic scale has not been created. All generated scales contain scale steps of two intervals, one larger than the other. If the number of instances of the smaller interval is greater than the larger interval then a pentatonic has been created, otherwise the scale generated is a diatonic.

Examples 6–13 contain a selection of grids and diatonic scales from other C*n* scales.

25	4	10	16	22	1	7	13
21	0	6	12	18	24	3	9
17	23	2	8	14	20	26	5
13	19	25	4	10	16	22	1
9	15	21	0	6	12	18	24
5	11	17	23	2	8	14	20
1	7	13	19	25	4	10	16
24	3	9	15	21	0	6	12
20	26	5	11	17	23	2	8

EXAMPLE 6: C27, ELEVEN-TONE DIATONIC SCALE

5	13	21	3	11	19
24	6	14	22	4	12
17	25	7	15	23	5
10	18	0	8	16	24
2	11	19	1	9	17
22	4	12	20	2	10
15	23	5	13	21	3

EXAMPLE 7: C26, SEVEN-TONE DIATONIC SCALE

22	0	9	18	27	5	14	23
14	23	1	10	19	28	**6**	15
6	15	24	2	11	**20**	**29**	7
29	7	16	25	**3**	**12**	21	30
21	30	8	**17**	**26**	4	13	22
13	22	**0**	**9**	18	27	5	14
5	**14**	**23**	1	10	19	28	6
28	**6**	15	24	2	11	20	29
20	29	7	16	25	3	12	21

EXAMPLE 8: C31, ELEVEN-TONE DIATONIC SCALE

31	2	6	10	14	18	22	26	30	1	5	9
28	32	3	7	11	15	19	23	27	31	**2**	6
25	29	0	4	8	12	16	20	24	**28**	**32**	3
22	26	30	1	5	9	13	17	**21**	**25**	29	0
19	23	27	31	2	6	10	**14**	**18**	22	26	30
16	20	24	28	32	3	**7**	**11**	15	19	23	27
13	17	21	25	29	**0**	**4**	8	12	16	20	24
10	14	18	22	**26**	**30**	1	5	9	13	17	21
7	11	15	**19**	**23**	27	31	2	6	10	14	18
4	8	**12**	**16**	20	24	28	32	3	7	11	15
1	**5**	**9**	13	17	21	25	29	0	4	8	12
31	**2**	6	10	14	18	22	26	30	1	5	9
28	32	3	7	11	15	19	23	27	31	2	6

EXAMPLE 9: C33, NINETEEN-TONE DIATONIC SCALE

37	1	5	9	13	17	21	25	29	33	37	1	5	9
34	38	2	6	10	14	18	22	26	30	34	38	**2**	6
31	35	39	3	7	11	15	19	23	27	31	**35**	**39**	3
28	32	36	0	4	8	12	16	20	24	**28**	**32**	36	0
25	29	33	37	1	5	9	13	17	**21**	**25**	29	33	37
22	26	30	34	38	2	6	10	**14**	**18**	22	26	30	34
19	23	27	31	35	39	3	**7**	**11**	15	19	23	27	31
16	20	24	28	32	36	**0**	**4**	8	12	16	20	24	28
13	17	21	25	29	**33**	**37**	1	5	9	13	17	21	25
10	14	18	22	**26**	**30**	34	38	2	6	10	14	18	22
7	11	15	**19**	**23**	27	31	35	39	3	7	11	15	19
4	8	**12**	**16**	20	24	28	32	36	0	4	8	12	16
1	**5**	**9**	13	17	21	25	29	33	37	1	5	9	13
38	**2**	6	10	14	18	22	26	30	34	38	2	6	10
35	39	3	7	11	15	19	23	27	31	35	39	3	7

EXAMPLE 10: C40, TWENTY-THREE-TONE DIATONIC SCALE

4	12	20	28	36	3	11	19
38	5	13	21	29	37	**4**	12
31	39	6	14	22	**30**	**38**	5
24	32	40	7	**15**	**23**	31	39
17	25	33	**0**	**8**	16	24	32
10	18	**26**	**34**	1	9	17	25
3	**11**	**19**	27	35	2	10	18
37	**4**	12	20	28	36	3	11
30	38	5	13	21	29	37	4

EXAMPLE 11: C41, ELEVEN-TONE DIATONIC SCALE

3	9	15	2	8	14
17	4	10	16	**3**	9
12	18	5	**11**	**17**	4
7	13	**0**	**6**	12	18
2	**8**	**14**	1	7	13
16	**3**	9	15	2	8
11	17	4	10	16	3

EXAMPLE 12: C19, SEVEN-TONE DIATONIC SCALE

4	14	24	3	13	23
27	6	16	26	**5**	15
19	29	8	**18**	**28**	7
11	21	**0**	**10**	20	30
3	**13**	**23**	2	12	22
26	**5**	15	25	4	14
18	28	7	17	27	6

EXAMPLE 13: C31, SEVEN-TONE DIATONIC SCALE

Examples 12 and 13 are C19 and C31 seven-tone diatonic scales. These are included to show that the grid technique for the creation of new diatonic scales can also generate the same diatonic scale as best fit to just-intonation-ratio methods. I think that this is a good indication that these new diatonic scales created from other C*n* microtonal scales are also valid as tonal structures.

What was said above about the new C20 diatonic scale also applies to these new diatonic scales: notation and harmonic structure should only be proposed after investigation of the acoustic materials.

V. Another Look at Ratios

The C19 and C31 diatonic scales led me to the following conclusion: ***For all valid diatonic scales, the two intervals in the grid approximate to a given pair of just intonation ratios.***

For the seven-tone diatonic scale the ratios are 5/4 and 6/5. For other diatonic scales, the same principle holds, though the ratios involved may be more complex.[8] In order to test my conclusion I produced the following grid for the 5/4 and 6/5 ratios in cents (Example 14). A cent is merely the interval of 1/1200 of an octave, named after the fact that 100 cents make a C12 chromatic step.

174	561	947	133	520	906
1059	245	631	1018	**204**	590
743	1129	316	**702** —	**1088**	275
427	814	**0** —	**386**	773	1159
112	**498** —	**884**	71	457	843
996	**182**	569	955	141	528
680	1067	253	639	1026	212

EXAMPLE 14: THE 5/4–6/5 DIATONIC SCALE EXPRESSED IN CENTS

Many observations could be made about the contents of this grid, but importantly, notice that the ends of the marked-out diatonic scale (C = 0 cents) give one D at 204 cents and another D at 182 cents. This scale clearly marks out the difference between the 10/9 and 9/8 ratios, or the "syntonic" comma. All equal-tempered approximations to this 5/4, 6/5 grid, with the exception of the fifty-three-tone system (not exampled here), confound these two ratios (to use Partch's terminology), i.e. assume that they are the same size.

I believe that the diatonic scales derived from the different C*n* scales conform to background formations existing independently of the number of tones in the base C*n* scale. These scales merely "sample" the pitch space giving rise to diatonic and pentatonic scales that approximate to a particular background formation. Certainly the seven-tone diatonic scales existing within the C12, C17, C19, C26, C31, C41, and C53 scales indicate that they are all approximations to a seven-tone diatonic consisting of pure 5/4 and 6/5 intervals (major and minor thirds), as marked out in Example 14. Another conclusion that can also be drawn from Example 14 and the C53 diatonic scale is that diatonic scales can also exist without the requirement to have the same pitch class at the top and bottom of the connected structure. This has profound implications for the creation of new diatonic scales, in that a diatonic scale can be seen not as a reordered segment of a "cycle of fifths" or its microtonal equivalents, but a compact structure in a grid space of two alternating intervals.

Example 15, below, shows a grid of traditional letter names laid out in the same way as the cent grid for the 5/4, 6/5 ratios. I have found that many insights into Western diatonic music can be obtained from the study of this grid.[9] I should point out that in Example 15 there are no enharmonic tones; every tone is unique.

...	...	D𝄫++	F♭++	A♭++	C+	E+
...	G𝄫++	B𝄫++	D♭+	F+	A+	C♯
C𝄫++	E𝄫++	G♭+	B♭+	**D**	F♯	A♯
A𝄫++	C♭+	E♭+	**G**	**B**	D♯ –	F𝄪 –
F♭+	A♭+	**C**	**E**	G♯ –	B♯ –	D𝄪 – –
D♭	**F**	**A**	C♯ –	E♯ –	G𝄪 – –	B𝄪 – –
B♭	**D**–	F♯ –	A♯ –	C𝄪 – –	E𝄪 – –	...
G–	B–	D♯ – –	F𝄪 – –	A𝄪 – –	...	...

EXAMPLE 15: THE 5/4–6/5 GRID IN LETTER NAMES (EXCERPT). THIS GRID CAN BE EXTENDED INDEFINITELY. AN ELLIPSIS MARKS WHERE TRIPLE SHARPS OR FLATS WOULD BE REQUIRED. PLUS AND MINUS SIGNS INDICATE DIFFERENCES OF A SYNTONIC COMMA

VI. Manifold Diatonic Scales

So far, all of the grids produced have been formed from two intervals. While working with these grids, I also decided that a third interval could be added, transforming the two-dimensional grid into a three-dimensional lattice. Consequently, every pitch or pitch class in this lattice is at the center of a cube of pitches or pitch classes. The first application of this extension of Balzano's idea was to form a lattice of the ratios 5/4, 6/5, and 7/6. This lattice corresponds to the extension of pitch relations to the 7-limit.

Immediately the question arises: what form do diatonic scales take in such a lattice? At the time of writing, I have determined that there are two distinct possibilities:[10]

i. A pitch or pitch-class in the lattice is chosen, and the intervals in the lattice are selected one after the other in rotation until the same pitch or pitch class recurs (or a pitch or pitch class close to the start appears). This produces a helix within the lattice.

ii. Add together three orthogonal diatonic scales, one scale from each of the three lattice planes or grids. The intersection point should be the tonic of each the three different diatonic scales.

Of the two methods of building these "multidimensional" diatonic scales, I prefer the helical basis, as then the harmonic components of the scale are connected tetrahedra or "tetrads." These tetrads can be of several types, depending upon the combination of intervals in the chord.

The alternative method, of creating three intersecting tonalities, may also be valid from the point of view that any three-dimensional tonal system may have tonal functions in different planes, rather than retaining the linear basis that helical tonalities imply. In addition, diatonic scales with tonics in different places may intersect at common pitch classes to produce "poly-diatonic" structures existing in the three-dimensional space of the lattice. In recognition of the fact that these structures exist in multidimensional space, I have borrowed a term from topology and refer to them collectively as "manifold" diatonic scales.

It should not surprise the reader that everything that has been said about three-dimensional lattices formed from three orthogonal intervals can also apply to four and more orthogonal intervals. Such formations exist in four- and higher-dimensional lattice spaces.

VII. At the 7-limit and Beyond

One question remains: why has Western music largely stopped at the 5-limit? After all, if the other tonal systems that bear likeness to the C12 diatonic scale are also valid, why are they not used? I suspect that the reason is complexity. The inclusion of a third ratio, the 7-limit, implies that a two-dimensional grid has been transformed into a three-dimensional lattice. In effect, the concept of tonality is transformed from a two-dimensional shape into a three-dimensional one. Incorporating yet higher limit ratios implies lattices of higher dimensions.[11]

Ultimately the discrimination of the human ear puts limits on what is possible. From my own experience, the ear is very elastic and can hear finer pitch gradations than I at first thought possible. Whether this extends to the aural comprehension of functional relationships between these new pitch materials will only come with use, and that requires composers to write them, and performers to articulate them.

References

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Zweifel, Paul. 1996. "Generalized Diatonic and Pentatonic Scales: A Group Theoretic Approach." *Perspectives of New Music* 34, no. 1 (Winter 1996): 140–61.

Note: The reader is also directed to the two *Microtonality Forum* special editions of *Perspectives of New Music* 29, no. 1 and 29, no. 2 (1991) for a more general overview of compositional approaches to microtonality.

Notes

1. Topics include: scale formation, harmonic structure, progression structure, coherence, and the "left-handed" variant of the diatonic scale also discussed by Balzano.

2. Balzano, 75: "It is easy to show $[k(k+1)]$ to be a sufficient condition for satisfying the above constraints, but to demonstrate the necessity of $n = k(k+1)$ invokes technicalities too extensive to be treated here and depends somewhat on exactly how the constraints are stated." It is for this reason that I relaxed the above restraint in my own researches, and consequently made some very interesting discoveries.

3. In all diatonic and pentatonic scales formed from alternating an interval m with a larger interval n, the diatonic can be generalized to $+(m+n)+m$ which means: any positive nonzero number of instances of intervals m and n in alternation, plus an additional interval m.

4. The assumption of a central tonic major and relative-minor triad pair is based on Balzano rather than the structure of the generated scales themselves. My assumption is that the connecting diminished triad provides the focus for determining the tonic in that its base tone (I hesitate to call it the "root") should lead onto the tonic. For diatonic scales this is the case; the opposite is true for pentatonic scales in that the base tone of the diminished triad is the second degree of the pentatonic scale. I find it significant that the diminished triad (which contains the "incoherent" interval of the diatonic scale, to use Balzano's expression) should contain a tone maximally close to the tonic. The inherent sense of "resolution" from the incoherent interval to the coherent "major third" or its equivalent is something that seems therefore to be independent of the number of tones in the base scale, i.e. it is a group-theoretic concept.

5. I include this diagram to reinforce my proposed view of the eleven-tone C20 diatonic scale. It clearly shows the "leading note" relationship, and it also reinforces the idea that the raised generalized fourth should become the leading tone of the key a generalized fifth above.

6. As the triads of the diatonic scale are constructed of "fourths," then different voice-leading rules must come into play. I suspect that there are more types of harmony within this diatonic scale than traditional seven-tone diatonic music has terminology for. In any case the current diatonic terminology is based on concepts other than group

theory, and I feel that though topological similarities can be used to determine basic properties of the new diatonic scales generated, the full palette of harmonic practice and theory can only be determined from actual compositional use.

7. This technique was discovered by accident. I used it continuously until 1997 when I came across the principles that are discussed in the following section on ratios. I now use a computer spreadsheet program to generate the grids for any value of C*n* and for any values of the two grid intervals, but the tests my original technique used are still valid for determining the validity of scales extracted from these grids.
8. For the C20 eleven-tone diatonic I arrive at 6/5 for the smaller interval and 16/13 for the larger.
9. Some of the insights include usage and derivation of all augmented sixth chords, plus the use of chromatic intervals and the use of diminished intervals. It should be noted that the augmented sixth is an interval approximating to the ratio interval of 7/4.
10. There are a large number of three-dimensional connected structures in a lattice, many of which have no analogue on a two-dimensional grid. The two possibilities I propose are analogues of connected grid structures, but this does not imply that the other manifold connected structures are not valid musical entities. The study of these alternate tonal structures would in itself be at least a lifetime's work.
11. Partch's 11-limit system exists in a four-dimensional lattice space, for example.