Coherent Tone-Systems: A Study in the Theory of Diatonicism
Author(s): Eytan Agmon
Source: *Journal of Music Theory*, Vol. 40, No. 1 (Spring, 1996), pp. 39-59
Published by: Duke University Press on behalf of the Yale University Department of Music
Stable URL: http://www.jstor.org/stable/843922
Accessed: 26/01/2009 15:29

COHERENT TONE-SYSTEMS: A STUDY IN THE THEORY OF DIATONICISM

Eytan Agmon

Looking back at the second half of the present century, a future historian of music theory might cite 1962 as the year modern diatonic theory came into being: the newly founded *Perspectives of New Music* published in 1962 an article by Milton Babbitt entitled "Twelve-Tone Rhythmic Structure and the Electronic Medium," including a brief yet thought-provoking allusion to the major/minor scale-system.[1]

When Babbitt introduced the integer model of pitch around the mid 1950's, his primary aim was to place the study of post-tonal music on a more rigorous, scientific basis; indeed, the model has since become a standard component in post-tonal theory, as testified by the work of theorists as Allen Forte, David Lewin, Robert Morris, George Perle, and John Rahn. Babbitt may have not foreseen that the integer model would also inaugurate an important line of research concerned with tonal, rather than post-tonal, music. In any event, today there exists a sizable body of literature on diatonic and related tone-systems considered in terms of Babbitt's integer model (or extensions thereof); representative works include Gamer (1967a), Balzano (1980; 1982), Clough and Myerson (1985), Brinkman (1986), Agmon (1989), Carey and Clampitt (1989), Clough and Douthett (1991), and Rahn (1991).[2]

One might assume that three decades or so of rigorous diatonic research would have produced some sort of consensus with regard to what, precisely, a "diatonic tone-system" is. Yet beyond some very general premises, no such consensus has thus far emerged. To be sure, certain ideas recur from work to work, for example, that a "diatonic set" is "fifth-generated." Even with such a seemingly basic property, however, its role as a *defining* property of diatonicism has become a matter of dispute.

This state of affairs is surely frustrating. What hope is there for a theory of Western "tonal" music, let alone a more comprehensive theory encompassing Western as well as non-Western music, if researchers are at odds with each other at such a preliminary level? (Imagine, for comparison, linguists who cannot agree on the definition of phonemes!)

Even more frustrating is the finding that a number of works, although presenting rather divergent views of diatonicism, nonetheless exhibit remarkable similarities; these works are Clough and Myerson (1985) and Clough and Douthett (1991), on the one hand, and Agmon (1989), on the other.[3] Indeed, the relationships among these three works are in certain respects so close—up to specific numerical results!—that one cannot but wonder whether between them a consensual definition of diatonicism has just narrowly been missed.

The present article is an attempt, if not to construct that elusive definition, at least to close the gap between opposing views, and to state as clearly as possible the nature of the remaining differences. The article is accordingly divided into three main sections. In the first section the studies of Clough and Douthett (1991) and Agmon (1989) are compared. (Since Clough and Douthett's (1991) study essentially absorbs Clough and Myerson's (1985) results, a comparison with the latter work is superfluous.) As it turns out, although the two works in question depart from rather different sets of preliminary assumptions, both exhibit interest in a certain property of diatonic intervals referred to as either "lack of contradiction" (Clough and Douthett) or "coherence" (Agmon). As the title of the present article suggests, the tone-systemic implications of this property are far-reaching. In the second section of the article these implications are examined formally; the formal examination leads in the third section to a definition of "diatonic tone-system" that considerably narrows the differences between Clough and Douthett's (1991) and Agmon's (1989) respective approaches.

I. Clough and Douthett (1991) and Agmon (1989) Compared

Table 1 summarizes the diatonically related assumptions and results in Clough and Douthett (1991) and Agmon (1989). A study of the table in conjunction with both works reveals that, for almost any assumption or result in Clough and Douthett (1991), there exists a corresponding set of assumptions and/or results in Agmon (1989). Thus, Clough and Douthett's R3 corresponds exactly to Agmon's A1, A2, and R4, combined; Clough and Douthett's A3 roughly corresponds to Agmon's A3b; Clough and Douthett's R1 ("lack of contradiction") corresponds to Agmon's A4 ("coherence"); Clough and Douthett's A1a corresponds to Agmon's R1; and finally, Clough and Douthett's R2 nearly exactly corresponds to Agmon's A1, R2b, R3, and R5. Note that the framed R2a is Clough and Douthett's preferred definition of "diatonic tone-system"; note also the emphasized (c,d)=1 in R2b and R3a, indicating that R2a entails R3b.

In the course of the present study it shall become apparent why in some cases the correspondences between the two works are only approximate. More important at the moment is to note the following: (1) even though the two works depart from rather different sets of assumptions, they arrive at such remarkably similar results as R3b and R2c (Clough and Douthett), and R4 and R5 (Agmon), respectively; (2) in particular, *there exists no corresponding assumption in Agmon's study to Clough and Douthett's central assumption A2: maximal evenness.* How is this possible? The answer, as we shall see, is quite simple: where Clough and Douthett's and Agmon's studies overlap, the properties of maximal evenness and coherence are equivalent. The work of Section 2 is aimed at reaching this conclusion, formally expressed as a theorem (Theorem 3.1). Here we shall consider informally the following notions: "scalar system" (Clough and Douthett's point of departure), "cyclic system" (Agmon's point of departure), and "coherence."

Scalar system. We shall refer to Clough and Douthett's preliminary assumptions as "scalar system." Clough and Douthett adopt the combined "specific" and "generic" perspectives as put forth by Clough and Myerson (1985). Thus, generalizing the familiar chromatic (mod 12) and diatonic (mod 7) universes, they begin by selecting a partial set $D_{c,d}=D_0, D_1, D_2, \ldots, D_{d-1}$ of d different elements out of $U_c=0, 1, 2, \ldots, c-1, d \leq c$ (Definitions 1.1 and 1.2). Clough and Douthett also state that they "generally assume $D_0 < D_1 < \ldots < D_{d-1}$" (Definition 1.2). As shall be seen, this second assumption is crucial to their "lack of contradiction" result (R1).

Cyclic system. We shall refer to Agmon's preliminary assumptions (slightly revised in the discussion below) as "cyclic system." The essential idea here is that a set of "scale steps" is generated *cyclically,* by repeatedly adding some constant value (the cyclic generator). For example, in the familiar diatonic system the set of scale steps {C, D, . . . , B} may be generated cyclically by fifths: taking F as the point of origin, we have C=F+one fifth, G=F+two fifths, etc., all modulo the octave.

Since Agmon (1989) also uses a combined modular system in the

Table 1: Comparison between Clough and Douthett (1991) and Agmon (1989)

Clough and Douthett	*Agmon*
ASSUMPTIONS	
(A1) (a) $D_{c,d}=\{D_0, D_1, \ldots, D_{d-1}\}$ is a set of d distinct elements out of $U_c=\{0, 1, \ldots, c-1\}$, $d \leq c$; (b) "generally," $D_{c,d}$ satisfies $D_0<D_1< \ldots < D_{d-1}$ (Def. 1.2)	(A1) (a,b)=1 (Def. 1) (A2) Existence of a cyclic generator (q,r) (Def. 2)
(A2) "Maximal evenness" (Def. 1.7; but see also Theorems 1.2 and 1.5) (A3) The spectrum of $M_{c,d}$ is the smallest spectrum including all clens while containing at least one ambiguity (pp. 134–35; see also Def. 2.6)	(A3) "Efficiency" (p. 12): (a) Any set of scale steps {(s,t)} is the smallest set of its type that contains any t=0, 1, . . . , b-1 at least once; (b) The set of diatonic intervals {(u,v)} is the smallest set of its type that contains any u=0, 1, . . . , a-1 at least once (A4) "Coherence" (p. 14)
RESULTS	
(R1) $M_{c,d}$ contains no contradiction (Lemma 2.1)	(R1) Number of scale steps is b (Prop. 1)
(R2) Equivalence of: (a) Mc,d has precisely one ambiguity; (b) c=2d-2 and **(c,d)=1**; (c) c=2d–2 and $c \equiv 0 \pmod 4$, etc. (Theorem 2.2)	(R2) (a) a=2b–1 or (b) a=2b–2 (Prop. 1) (R3) If a is even, existence of exactly one pair of enharmonically equivalent diatonic intervals (Prop. 2)
(R3) Equivalence of: (a) **(c,d)=1**; (b) existence of a cyclic generator g satisfying $g = \frac{cI \pm 1}{d}$, etc. (Theorem 3.1A)	(R4) $q = \frac{ar \pm 1}{b}$ (from Prop. 1 and the theorem on p. 14) (R5) If a is even, a=8+4n, n=0, 1, . . . (p. 17)

manner of Clough and Myerson (1985), his cyclic generator is an *integer pair* (q,r), satisfying $1 \leq q \leq a-1$, $1 \leq r \leq b-1$ ($a>b>1$). Thus, a set of scale steps is a set of N+1 distinct integer pairs $\{(s,t)\}$, $0 \leq s \leq a-1$, $0 \leq t \leq b-1$, namely, the set $\{(0,0), (q,r), 2(q,r), \ldots, N(q,r)\}$, all mod (a,b).

Coherence. The property of coherence ("lack of contradiction," in Clough and Douthett's terminology) is best introduced by way of example. In the familiar diatonic tone-system, any two different intervals have the following property: measured in semitones, on the one hand, and diatonic steps, on the other, the larger interval (in semitones) is never smaller (in diatonic steps). For example, an interval of 7 semitones is larger than an interval of 6 semitones; correspondingly, a fifth ("perfect") is equal to a fifth ("diminished"), and larger than a fourth ("augmented"). The cognitive content of coherence is not difficult to see, particularly with respect to the property's role in facilitating the processing of semitonal information diatonically.[4]

While coherence plays a central role in Agmon's study, it is somewhat less prominent in Clough and Douthett's, where "maximal evenness" is highlighted instead. Nonetheless, Clough and Douthett's R1 suggests that focusing on coherence in both the cyclic and scalar contexts is a promising avenue of research in terms of understanding the correspondences between their study and Agmon's. Note that Agmon's (1989) study, while presupposing a cyclic system, also presupposes that the system is efficient in considering the necessary and sufficient conditions for coherence (p. 14). The present study, by contrast, considers coherence apart from efficiency in both the scalar and cyclic contexts.

II. Coherent Tone-Systems: A Formal Study

This section consists of three subsections. The first subsection studies the scalar system; the second subsection studies the cyclic system; and the third subsection studies the "cyclic/scalar" system, a system that qualifies as cyclic as well as scalar.

Since Clough and Douthett (1991) and Agmon (1989) adopt somewhat different notational conventions and terminology, a necessary preliminary step is to form a common notational and terminological ground. In the present study the notational conventions and terminology of Agmon shall be used. Thus, for example, Clough and Douthett's variables c and d are notated throughout as a and b, respectively. Moreover, the use of subscripts to express correspondences between elements $s=0, 1, \ldots, a-1$ and $t=0, 1, \ldots, b-1$ is replaced by parenthesized integer pairs; for example, $(s,t)=(a-1, b-1)$ is the present notational equivalent to Clough and Douthett's $D_{d-1}=c-1$. As to terminology, "scale steps," "diatonic intervals," "coherence," and so forth, shall be used.

It is important to emphasize that, apart from such purely notational and terminological matters (and except for one or two insignificant or irrelevant details, duly noted), Clough and Douthett's work is faithfully transcribed in what follows.

1. The Scalar System

We begin by developing a formal definition of "scalar system," which serves as Clough and Douthett's point of departure. After then defining "coherence," a theorem concerning coherent scalar systems is stated and proved.

Definition 1.1: Octave Equivalence. Let a and b be two integers, $a>b>1$.[5] In what follows we shall be interested in pairs of integers (x,y) divided into classes in the following way: each class consists of pairs of integers of the form (s+ia,t+jb), where s=0, 1, . . . , a-1; t=0, 1, . . . , b−1; and i and j run over all integers. We shall refer to any two integer pairs (s+ma,t+nb) and (s+m′a,t+n′b) belonging to the same class as congruent, mod (a,b), or "octave equivalent," and shall write

$$(s+ma,t+nb)\equiv(s+m'a,t+n'b) \bmod (a,b).$$

To represent an arbitrary integer-pair class (s+ia,t+jb) we shall use the notation (s,t) mod (a,b). In particular, we shall use the notation

$$(x,y)=(s,t) \bmod (a,b)$$

to mean "(s,t) is the *representative* mod (a,b) of (x,y), $0\leq s\leq a-1$, $0\leq t\leq b-1$."

We shall use the following operation involving integer-pair classes mod (a,b):

$$(s,t)-(s',t') \bmod (a,b)\equiv(s-s' \bmod a,t-t' \bmod b).$$

Definition 1.2: Set of Scale Steps. Let {(s,t)}, $0\leq s\leq a-1$, $0\leq t\leq b-1$, be a set of b distinct integer pairs. We shall refer to {(s,t)} as a set of scale steps S if the set {s} of all first members in {(s,t)} contains exactly b distinct integer classes (mod a), and similarly, the set {t} of all second members in {(s,t)} contains exactly b distinct integer classes (mod b).

We shall use the notation S(a,b) to refer to any set of scale steps {(s,t)} satisfying $0\leq s\leq a-1$ and $0\leq t\leq b-1$.

Definition 1.3: Diatonic Interval. Let (s,t) and (s′,t′) be two members (not necessarily distinct) of a given set of scale steps S(a,b). We shall refer to the difference $(u,v)=(s',t')-(s,t) \bmod (a,b)$, $0 \le u \le a-1$, $0 \le v \le b-1$, as the diatonic interval from (s,t) to (s',t').

Definition 1.4: Set of Diatonic Intervals. Given a set of scale steps $S(a,b)$, we shall refer to the set $\{(u,v)\}$ of all distinct diatonic intervals (u,v) from any member (s,t) to any other member (s',t') of $S(a,b)$ as the set of diatonic intervals I, affiliated to $S(a,b)$. We shall use the notation $I(S(a,b))$ to refer to the set $\{(u,v)\}$ of diatonic intervals affiliated to $S(a,b)$.

Definition 1.5: Scalar System. We shall refer to a given set of scale steps $S(a,b)$ and the affiliated set $I(S(a,b))$ of diatonic intervals as a scalar system $SS(a,b)$.[6]

Definition 1.6: Coherence. Given a set of integer pairs $\{(u,v)\}$, $0 \le u \le a-1$, $0 \le v \le b-1$, we shall say that the set is coherent if for any pair of integer pairs (u,v) and (u',v') in the given set such that $u>u'$, the relation $v \ge v'$ holds.

Corollary: Given a coherent set of integer pairs $\{(u,v)\}$, for any pair of integer pairs (u,v) and (u',v') in the coherent set such that $v>v'$, the relation $u \ge u'$ holds.

Definition 1.7: Coherent Scalar System. We shall say that a scalar system $SS(a,b)$ is coherent if $\{(u,v)\}=I(S(a,b))$ is coherent.

We are now ready to state a theorem concerning coherent scalar systems; the theorem—which essentially says that coherent scalar systems exist—is indebted to Clough and Douthett's work.

THEOREM 1.1. *Any scalar system SS(a,b) satisfying the relation*

$$S(a,b)=\{(s,t)\}=\{(\left\lceil \frac{na}{b} \right\rceil, n),\ n=0,1,\ldots,b-1\}, \tag{1.1}$$

is coherent.[7]

Proof. See Lemma 2.1 in Clough and Douthett (1991).

A careful study of Clough and Douthett's proof reveals that it ultimately relies on their Lemma 1.2 (p. 103). This latter lemma, however, does not hold unless $D_{c,d}$ satisfies $D_0 < D_1 < \ldots < D_{d-1}$ in Clough and Douthett's Definition 1.2.[8] It follows that at least as far as the property of coherence (or "lack of contradiction") is concerned, Clough and Douthett's "maximally even set" and the present relation (1.1) are equivalent.[9]

2. *The Cyclic System*

We begin with a formal definition of the "cyclic system" serving as Agmon's point of departure. Agmon's original 1989 definitions, however, are slightly revised here, as shall be subsequently discussed.

Definition 2.1: Octave Equivalence. (Same as Definition 1.1.)

Definition 2.2: Quintic Class. We shall refer to a class (q,r) mod (a,b) as quintic if q and r satisfy $1 \leq q \leq a-1$, $1 \leq r \leq b-1$.

Definition 2.3: Set of Scale Steps. Let N be a positive integer. We shall refer to a set {(s,t)} of N+1 distinct integer pairs as a set of scale steps S if {(s,t)}={n(q,r) mod (a,b), n=0, 1, . . . , N}, where the product n(q,r) mod (a,b) is defined as (nq mod a,nr mod b). We shall use the notation S(a,b;q,r;N) to refer to the set S of scale steps defined by parameters a, b, q, r, and N.

Definition 2.4: Diatonic Interval. Let (s,t) and (s′,t′) be two members (not necessarily distinct) of a given set of scale steps S(a,b;q,r;N). We shall refer to the difference (u,v)=(s′,t′)–(s,t) mod (a,b), $0 \leq u \leq a-1$, $0 \leq v \leq b-1$, as the diatonic interval from (s,t) to (s′,t′).

Definition 2.5: Set of Diatonic Intervals. Given a set of scale steps S(a,b;q,r;N), we shall refer to the set {(u,v)} of all distinct diatonic intervals (u,v) from any member (s,t) to any other member (s′,t′) of S(a,b;q,r;N) as the set of diatonic intervals I, affiliated to S(a,b;q,r;N). We shall use the notation I(S(a,b;q,r;N)) to refer to the set {(u,v)} of diatonic intervals affiliated to S(a,b;q,r;N).

Note that any diatonic interval (u,v) in I(S(a,b;q,r;N)) is a member of the set Ψ={n(q,r) mod (a,b), n=0, ±1, . . . , ±N}. However, since Ψ may contain some repetition, it is not necessarily the same as I(S(a,b;q,r;N)).

Definition 2.6: Coherent Cyclic System. We shall refer to a set of scale steps S(a,b;q,r;N) and the affiliated set I(S(a,b;q,r;N)) of diatonic intervals as the cyclic system CS(a,b;q,r;N). We shall say that a cyclic system CS(a,b;q,r;N) is coherent if {(u,v)}= I(S(a,b;q,r;N)) is coherent (see Definition 1.6).

When comparing the above definitions with those of Agmon (1989), one notes a number of differences. For example, the "coprimeness conditions" (a,b)=1, (a,q)=1, and (b,r)=1 in Agmon's 1989 definitions of "octave equivalence" and "quintic class" have been dropped.[10] In Agmon's earlier study the three coprimeness conditions were introduced to ensure *a priori* that no two "diatonic intervals" are identical. However, while these conditions, together with the restriction $N < \frac{ab}{2}$, also dropped in the present study, are indeed sufficient "distinctness" conditions, they are by no means *necessary.* Thus, rather than making these unnecessary assumptions, the present definitions simply stipulate that any two diatonic intervals (or scale steps) in a *set* of diatonic intervals (or scale steps) must be distinct.[11]

In what follows, three theorems concerning coherent cyclic systems are stated and proved. The first theorem states a sufficient condition for coherence; the second theorem states a necessary condition for coherence; and the third theorem states a necessary as well as sufficient condition for coherent cyclic systems. The proof of the third, culminating theorem draws upon the theory of Farey series of elementary number theory.[12] See Agmon (1995) for further discussion of Farey series and diatonicism.

THEOREM 2.1. *Consider a cyclic system CS(a,b;q,r;N). Set* $\Delta = ar-bq$, *and assume* $\Delta \neq 0$. *If the relation*

$$N < \frac{b}{|\Delta|} \tag{2.1}$$

holds, then CS(a,b;q,r;N) is coherent.[13]

Proof. Since any diatonic interval (u,v) in I(S(a,b;q,r;N)) is a member of the set $\Psi=\{n(q,r) \bmod (a,b),\ n=0, \pm 1, \ldots, \pm N\}$, we shall prove that Ψ is coherent; we shall then show that $\{(s,t)\}=\{n(q,r) \bmod (a,b),\ n=0, 1, \ldots, N\}$ consists of N+1 distinct members, and is therefore a set of scale steps S(a,b;q,r;N) satisfying Definition 2.3.

Let n be any integer such that $-N \leq n \leq N$. Set

$$m = \left[\frac{nq}{a}\right],\ m' = \left[\frac{nr}{b}\right].$$

We propose to show that $m=m'$. We have:

$$\frac{nq}{a} = m + \frac{i}{a},\ \frac{nr}{b} = m' + \frac{j}{b}, \tag{2.2}$$

where i, j are integers u, v, $0 \leq i \leq a-1$, $0 \leq j \leq b-1$. Subtracting the first equation in (2.2) from the second we find that

$$m' - m = \frac{nr}{b} - \frac{nq}{a} + \frac{i}{a} - \frac{j}{b} = \frac{n\Delta}{ab} + \frac{i}{a} - \frac{j}{b}.$$

If $m' \geq m$ we find that

$$(2.3) \qquad m' - m \leq \frac{n\Delta}{ab} + \frac{i}{a} \leq \frac{|n||\Delta|}{ab} + \frac{i}{a}.$$

If $m' \leq m$ we find that

$$(2.4) \qquad m - m' \leq -\frac{n\Delta}{ab} + \frac{j}{b} \leq \frac{|n||\Delta|}{ab} + \frac{j}{b}.$$

From (2.1) we have $|n||\Delta| \leq N|\Delta| < b$, and therefore if $m' \geq m$

$$0 \leq m' - m < \frac{b}{ab} + \frac{i}{a} = \frac{i+1}{a} \leq 1,$$

which implies that $m'-m=0$. Similarly, if $m' \leq m$ using (2.4) we find that

$$m - m' < \frac{a}{ab} + \frac{j}{b} = \frac{j+1}{b} \leq 1,$$

which again implies that $m=m'$.

We shall now use the result $m=m'$ to prove that $\Psi=\{n(q,r) \bmod (a,b), n=0, \pm 1, \ldots, \pm N\}$ is coherent.

Let n_1, n_2 be two distinct integers such that $-N \leq n_k \leq N$, $k=1,2$. Using the result $m=m'$ we have

$$(2.5a) \qquad n_1 q = m_1 a + i_1,\ n_1 r = m_1 b + j_1,$$

$$(2.5b) \qquad n_2 q = m_2 a + i_2,\ n_2 r = m_2 b + j_2,$$

where

$$m_1 = \left[\frac{n_1 q}{a}\right] = \left[\frac{n_1 r}{b}\right],$$

$$m_2 = \left[\frac{n_2 q}{a}\right] = \left[\frac{n_2 r}{b}\right],$$

$$0 \leq i_k < a,\ 0 \leq j_k < b,\ k = 1, 2.$$

To prove that the coherence condition holds we may assume that $i_1 \neq i_2$ and $j_1 \neq j_2$ (if for instance $i_1 = i_2$ there is nothing to prove).

Multiplying the first equation in (2.5a) by b and the second by a, and subtracting we get:

$$(2.6a) \qquad n_1 (ar - bq) = aj_1 - bi_1.$$

Similarly, from the two equations in (2.5b) we find

(2.6b) $$n_2(ar - bq) = aj_2 - bi_2.$$

Subtracting (2.6a) from (2.6b) we find:

(2.7) $$(n_2 - n_1)\Delta = a(j_2 - j_1) - b(i_2 - i_1).$$

It follows from (2.7) and our assumption (2.1) (since $|n_2 - n_1| \leq 2N$) that

(2.8) $$|a(j_2 - j_1) - b(i_2 - i_1)| \leq 2N|\Delta| < 2b.$$

This implies that if $j_2>j_1$ then we must have $i_2>i_1$. Otherwise, if $i_2<i_1$, then $j_2-j_1\geq 1$ and $i_1-i_2\geq 1$, so that we have:

(2.9) $$a + b \leq a(j_2 - j_1) - b(i_2 - i_1) < 2b,$$

leading to the contradiction that $a<b$. Similarly it follows from (2.8) that if $j_2<j_1$ then we must have $i_2<i_1$. This proves that $\Psi=\{n(q,r) \bmod (a,b), n=0, \pm 1, \ldots, \pm N\}$ is coherent.

To conclude the proof of Theorem 2.1 it remains to show that $\{(s,t)\} =\{n(q,r) \bmod (a,b), n=0, 1, \ldots, N\}$ consists of N+1 distinct members. This, however, follows from (2.7), since $i_1=i_2$ and $j_1=j_2$ ($n_2\neq n_1$) imply that $\Delta=0$, contrary to our assumption.

COROLLARY 2.1. *I(S(a,b;q,r;N))*= $\{n(q,r) \bmod (a,b), n = 0, \pm 1, \ldots, \pm N\} = \Psi$.

Proof. The corollary also follows from (2.7), since $n_k=0, \pm 1, \ldots, \pm N$.

THEOREM 2.2. *Consider a cyclic system CS(a,b;q,r;N); suppose the system is coherent. Then for any n=0, 1, . . . , N, the following holds:*

either

(2.10) $$\frac{nq}{a} = \frac{nr}{b} \text{ is an integer;}$$

or

(2.11) *there is no integral multiple of* $\frac{1}{2}$ *that lies strictly between* $\frac{nq}{a}$ *and* $\frac{nr}{b}$.

To prove the theorem we shall first prove two lemmas.

LEMMA 2.1. *Consider a cyclic system CS(a,b;q,r;N); suppose the system is coherent. Then for any n=2, 3, . . . , N, if* $\frac{nq}{a}$ *is an integer,* $\frac{nr}{b}$ *is also an integer, and vice versa.*

Proof. Suppose by contradiction that $\frac{nq}{a}$ is an integer, while $\frac{nr}{b}$ is a non-integer. Set nr=v and (n–1)r=v′ (mod b); we have n(q,r)=(0,v) and (n–1)(q,r)=(a-q,v′) mod (a,b), where v≠0, v≠v′. If v′=0, coherence is violated, contradicting our assumption; thus v′≠0. To prove that $\frac{nr}{b}$ is an integer we shall examine three cases: v= $\frac{b}{2}$; v′= $\frac{b}{2}$; v,v′≠ $\frac{b}{2}$.

Case a: v= $\frac{b}{2}$. Since v≠v′, v′≠ $\frac{b}{2}$. It follows that –(n–1)(q,r)=(q,b–v′) is distinct mod (a,b) from (n–1)(q,r)=(a-q,v′). Consider the three diatonic intervals (0, $\frac{b}{2}$), (a–q,v′), and (q,b–v′). These intervals violate coherence, since on the one hand both q and a-q are larger than 0, yet on the other, either v′ or b-v′ must be smaller than $\frac{b}{2}$.

Case b: v′= $\frac{b}{2}$. Here coherence is violated among the three diatonic intervals (0,v), –n(q,r)=(0,b–v), and (a–q, $\frac{b}{2}$).

Case c: neither v nor v′ equal $\frac{b}{2}$. Consider the four diatonic intervals (0,v), (0,b–v), (a–q,v′), and (q,b–v′). If v′<v, then necessarily b–v′>b–v; conversely, if v′>v, then necessarily b–v′<b–v. It is not difficult to see that, in either case, coherence is violated, whether by (0,v) and (a–q,v′), or by (0,b–v) and (q,b–v′).

Since coherence is violated in all three cases contrary to our assumption, $\frac{nr}{b}$ must be an integer. If {(u,v)} is coherent, then for any pair of diatonic intervals (u,v) and (u′,v′) such that v>v′, the relation u≥u′ holds (corollary to Definition 1.6). Thus, one can use a similar argument to prove that if $\frac{nr}{b}$ is an integer, $\frac{nq}{a}$ is also an integer. This proves the lemma.

LEMMA 2.2. *A necessary condition for a cyclic system CS(a,b;q,r;N) to be coherent is*

(2.12) $$\left[\frac{nq}{a}\right] = \left[\frac{nr}{b}\right],\ n = 1, 2, \ldots, N.$$

Proof. Suppose the system is coherent. We shall prove (2.12) by induction.

Clearly (2.12) holds for n=1. Suppose it holds for some n≤N–1. Set

$$m = \left[\frac{(n+1)q}{a}\right] = \left[\frac{nq}{a} + \frac{q}{a}\right], \text{ and}$$

$$m' = \left[\frac{(n+1)r}{b}\right] = \left[\frac{nr}{b} + \frac{r}{b}\right].$$

Since

$$\left[\frac{nq}{a}\right] = \left[\frac{nr}{b}\right]$$

by assumption, and since

$$\left[\frac{q}{a}\right] = \left[\frac{r}{b}\right] = 0,$$

m and m′ can differ by 1 at the most. It is not difficult to see that unless m=m′ coherence is violated within the pair of diatonic intervals (u,v)=n(q,r) mod (a,b) and (u′,v′)=(n+1)(q,r) mod (a,b). For example, if m–m′=1 we have

$$u > u+q-a = u',$$

whereas

$$v < v+r = v',$$

contradicting our coherence assumption. A similar contradiction of coherence results if m–m′=–1. This proves the lemma.

COROLLARY 2.2. *A necessary condition for a cyclic system CS(a,b; q,r;N) to be coherent is*

$$\left[\frac{nq}{a}\right] = \left[\frac{nr}{b}\right],\ n = \pm 1, \pm 2, \ldots, \pm N. \tag{2.13}$$

Proof. The corollary follows from Lemmas 2.1 and 2.2, and the observation that given positive numbers x and y that are either both non-integers or both integers, if the integral part of x equals the integral part of y, then the integral part of –x equals the integral part of –y.

Proof of the theorem. We are now in a position to prove Theorem 2.2. Set

$$m = \left[\frac{nq}{a}\right], \text{ and}$$

$$m' = \left[\frac{nr}{b}\right],\ 1 \le n \le N.$$

According to the coherence condition, if the representative (mod a) of nq is larger than the representative (mod a) of –nq, n=1, 2, . . . , N, then the representative (mod b) of nr is larger than or equal to the representative (mod b) of –nr. In other words, if

$$nq - ma > a - (nq - ma),$$

then

$$nr - m'b \geq b - (nr - m'b).$$

By (2.12) m=m′, and therefore we may state the condition as follows:

$$\text{if } \frac{nq}{a} > m + \frac{1}{2}, \text{ then } \frac{nr}{b} \geq m + \frac{1}{2}. \tag{2.14a}$$

Clearly, the following related condition must also be true:

$$\text{if } \frac{nq}{a} < m + \frac{1}{2}, \text{ then } \frac{nr}{b} \leq m + \frac{1}{2}. \tag{2.14b}$$

Since v>v′ implies u≥u′ in a coherent set {(u,v)} (corollary to Definition 1.6), we also have:

$$\text{if } \frac{nr}{b} > m + \frac{1}{2}, \text{ then } \frac{nq}{a} \geq m + \frac{1}{2}, \text{ and} \tag{2.15a}$$

$$\text{if } \frac{nr}{b} < m + \frac{1}{2}, \text{ then } \frac{nq}{a} \leq m + \frac{1}{2}. \tag{2.15b}$$

The theorem follows from these relations, together with Lemmas 2.1 and 2.2.

COROLLARY 2.3. *Consider a cyclic system CS(a,b;q,r;N). Set Δ= ar–bq. A necessary condition for the system to be coherent is*

$$|\Delta| < \frac{ab}{2N}. \tag{2.16}$$

Proof. From 2.10 and 2.11 we have

$$-\frac{1}{2} < \left(\frac{nq}{a} - \frac{nr}{b}\right) < \frac{1}{2}, \; 1 \leq n \leq N.$$

Multiplying by $\frac{ab}{n}$ we get

$$-\frac{ab}{2n} < (ar - bq) < \frac{ab}{2n}.$$

Now let n=N.

THEOREM 2.3. *Consider a cyclic system CS(a,b;q,r;N) satisfying (a,q)=1 and (b,r)=1. Set* $\Delta=ar-bq$. *If the relation* $b > N \geq \frac{a-1}{2}$ *holds, then* $\Delta=\pm 1$ *is both a necessary and sufficient condition for the system to be coherent.*

Proof. The sufficiency follows directly from Theorem 2.1. We shall now prove that $\Delta=\pm 1$ is also necessary.

Assume the system is coherent. Since $\frac{q}{a}$ and $\frac{r}{b}$ are reduced fractions with a>b, we have $\frac{q}{a} \neq \frac{r}{b}$. By definition, the reduced fractions $\frac{q}{a}$ and $\frac{r}{b}$ are two distinct terms (whether adjacent or not) in the Farey series F_a. From the necessary condition (2.11) it follows that no integral multiple of $\frac{1}{2n}$, n=1, 2, . . . , N, lies strictly between $\frac{q}{a}$ and $\frac{r}{b}$ (clearly, the necessary condition 2.10 cannot be satisfied for any $n \neq 0$). By assumption $2N \geq a-1$, and therefore a reduced fraction $\frac{x}{y}$ satisfying $2 \leq y \leq a-1$ does not lie strictly between $\frac{q}{a}$ and $\frac{r}{b}$. Suppose $\frac{r}{b}$ is not adjacent to $\frac{q}{a}$ in F_a. Then $\frac{q+1}{a}$ or $\frac{q-1}{a}$ must be reduced and adjacent to $\frac{q}{a}$ in F_a. This contradicts a Farey-series theorem which states that no two adjacent terms of F_n (n>1) have the same denominator.[14] Thus $\frac{q}{a}$ and $\frac{r}{b}$ are adjacent terms in F_a; by a famous Farey-series theorem, $ar-bq=\pm 1$.

3. The "Cyclic/Scalar" System

A cyclic/scalar system is a system that satisfies both sets of definitions 1.1-1.4 and 2.1-2.5. Since these two sets of definitions are compatible, a cyclic/scalar system is simply a cyclic system CS(a,b;q,r;N) with a set of scale steps {(s,t)} (see Definition 2.3) that satisfies the scalar-systemic requirements of Definition 1.2, namely, {s} is a set of exactly b distinct integer classes (mod a), and {t} is a set of exactly b distinct integer classes (mod b). The following propositions concerning cyclic/scalar systems are easily proven:

PROPOSITION 3.1. *For any cyclic/scalar system, the relations (b,r)=1 and N=b–1 hold.*

PROPOSITION 3.2. *Any maximally even cyclic system (i.e., any cyclic system satisfying relation 1.1), is a cyclic/scalar system.*

PROPOSITION 3.3. *Any coherent cyclic system satisfying (b,r)=1 and N=b–1 is a cyclic/scalar system.*[15]

As it happens, a coherent cyclic/scalar system satisfying $a<2b$ and $(a,q)=1$ is exactly the intersection between Clough and Douthett's and Agmon's studies; hence the significance of the following:

THEOREM 3.1. *Let CS(a,b;q,r;N) be a cyclic system satisfying (b,r)=1 and N=b–1. If the relations a<2b and (a,q)=1 hold, then maximal evenness (in the sense of relation 1.1) is both a necessary and sufficient condition for the system to be coherent.*

Proof. If CS(a,b;q,r;N) is maximally even, it is a cyclic/scalar system (Proposition 3.2); thus the sufficiency is given by Theorem 1.1. We shall now prove that maximal evenness is also necessary.

Suppose CS(a,b;q,r;N) is coherent. Since the relations $(a,q)=1$, $(b,r)=1$, and $b>N\geq\frac{a-1}{2}$ are satisfied, by Theorem 2.3 $ar-bq=\pm1$. By Corollary 2.1 I(S(a,b;q,r;N)) consists of $2N+1=2b-1$ distinct elements. Excluding the diatonic interval (0,0), we can write the remaining $2b-2$ diatonic intervals $\{(u,v)\}=\{n(q,r) \bmod (a,b),\ n=\pm1,\ \pm2,\ldots,\pm(b\text{-}1)\}$ as a set of $b-1$ pairs: $|n|(q,r)$ and $(-b+|n|)(q,r) \bmod (a,b)$. Since $-br\equiv0 \pmod b$, each pair is of the form $\{(u,v),(u',v)\}$, where clearly, different pairs have a different v; moreover, since $ar-bq=\pm1$ (Theorem 2.3), $-bq\equiv\pm1 \pmod a$, so that we have $|u-u'|=1$ (neither u nor u' equals 0). As Clough and Douthett prove in their Theorem 1.5, this property implies maximal evenness in the sense of (1.1) here (see also note 8).

Theorem 3.1 states that given a cyclic/scalar system satisfying $a<2b$ and $(a,q)=1$, *the properties of coherence and maximal evenness are equivalent*: if the system is maximally even it is coherent, and if it is coherent it is maximally even. (Note that the condition $(a,q)=1$ follows from Clough and Douthett's A3 and Agmon's A3b.) This equivalence relationship explains the correspondences initially noted between Clough and Douthett's and Agmon's results. For example, the correspondence between Clough and Douthett's R2c and Agmon's R5 is explained by observing that in both cases the relation $a=2b-2$ (equivalently, $c=2d-2$) is assumed.

Since $(a,b)=1$ and $(a,q)=1$ are not assumed in the present study, it is possible to have a cyclic/scalar system satisfying $\frac{q}{a}=\frac{r}{b}$. Indeed, we have the following:

THEOREM 3.2. *Let CS(a,b;q,r;N) be a cyclic system satisfying (b,r)=1 and N=b–1. If* $\frac{q}{a}=\frac{r}{b}$, *then the system is both coherent and maximally even.*

Proof. If $\frac{q}{a} = \frac{r}{b}$ it follows easily that $\Psi=\{n(q,r) \bmod (a,b), n=0, \pm 1, \ldots, \pm N\}$ is coherent for any N. Since $\frac{r}{b}$ is a reduced fraction, we have $q=ir$ and $a=ib$ for some integer $i>1$. Thus $\{(s,t)\}=\{n(q,r) \bmod (a,b), n=0, 1, \ldots, b-1\}=\{(ni,n)\}=\{(\frac{na}{b},n)\}=\{(a-\frac{na}{b},b-n)\}$, and $\{(s,t)\}$ is a maximally even set of scale steps S(a,b) by definition (see relation 1.1). Note that if $r=1$ we have $q=i=\frac{a}{b}$; if $r=b-1$, $q=(b-1)i=a-\frac{a}{b}$.

It is not difficult to see that Theorem 3.2 corresponds to Clough and Douthett's Theorem 3.1B (not represented in Table 1).

Theorem 3.2 completes the formal study of coherent tone-systems. In the following section we turn to a special type of coherent tone-system known as "diatonic."

III. Towards a Consensual Definition of "Diatonic Tone-System"

As initially noted, despite the striking correspondences between Clough and Douthett's (1991) and Agmon's (1989) studies analyzed in the previous section, these studies offer rather divergent views of diatonicism. Clough and Douthett define "diatonic tone-system" as a maximally even scalar system that has precisely one ambiguity, meaning that the set of diatonic intervals contains exactly one enharmonically equivalent pair. Agmon, on the other hand, defines the same notion as an efficient and coherent cyclic system, stipulating further that the ordering of the set of scale steps by the quintic class must be distinct from its ordering by the major or minor second (1989, 16–17). Although the two definitions lead to similar numerical results, the difference between them must not be underestimated. For example, the existence of a cyclic generator is for Agmon a defining property of diatonicism, but not so for Clough and Douthett.

Clough and Douthett's definition of "diatonic tone-system" certainly has the advantage of being more concise. Is it, however, intuitively satisfying? Clough and Douthett claim that it is (1991, 141), but fail to support this claim with a convincing argument.[16] Indeed, it is difficult to see why "having a single ambiguity"—or having any ambiguity at all—must count as a tone-systemic advantage. The existence of enharmonically equivalent pairs of diatonic intervals is clearly a disadvantage from the cognitive point of view, for in processing semitonal information the listener is forced to draw upon contextual information in order to arrive at a unique "solution" (for example, to decide whether six semitones represent an augmented fourth or a diminished fifth). To define diatonicism on the basis of ambiguity is therefore counterintuitive rather than intuitive; indeed, the definition of diatonicism must be such that the single (!)

ambiguity is seen as a *necessary sacrifice*, given other, more important systemic properties.

The remaining nine equivalent "diatonic" conditions in Clough and Douthett's Theorem 2.2 (1991, 138–39) are of no apparent intuitive appeal either. Condition 5, however, contains in addition to $c=2(d-1)$ the condition $(c,d)=1$; this latter condition is equivalent (as Clough and Douthett prove in their Theorem 3.1) to the existence of a cyclic generator, a condition whose intuitive appeal is well supported by tradition. Thus, although generation by fifth is not part of Clough and Douthett's definition of "diatonic tone-system," indirectly they acknowledge the property's significance; indeed, it seems doubtful that an intuitively satisfying definition of "diatonic tone-system" is possible which excludes this basic property.

Once ambiguity is discarded as a defining property of diatonicism, and "cyclic generation" is invoked instead, the difference between Agmon's and Clough and Douthett's definitions of "diatonic tone-system" is essentially reduced to the difference between the properties of coherence and maximal evenness. By Proposition 3.1, any cyclic/scalar system satisfies the relations $(b,r)=1$ and $N=b-1$; these relations follow by Agmon's approach from A3a (see again Table 1). Moreover, stripped of the ambiguity requirement, Clough and Douthett's A3 becomes identical to Agmon's A3b; the assumption thus yields in addition to the relations $(a,q)=1$ and $b=\frac{a}{2}+1$, a is even, the relations $(a,q)=1$ and $b=\frac{a+1}{2}$, a is odd (cf. Agmon's R2). It follows that the terms of Theorem 3.1, under which coherence and maximal evenness are seen as equivalent, are fully met. Since Clough and Douthett seem to accept Agmon's argument concerning the quintic and secundic scale-step orderings, an agreement on the basis by which an odd a may be ruled out seems to exist as well.[17] Indeed, having distinct quintic and secundic scale-step orderings seems, intuitively, a sufficiently important property to outweigh the attendant systemic disadvantage of ambiguity.

In defining "diatonic tone-system" as a cyclic system that satisfies either (1a) coherence or (1b) maximal evenness, and in either case, satisfies in addition (2) "efficiency" and (3) the requirement that the quintic and secundic scale-step orderings be distinct,[18] we have come a long way towards establishing a consensual definition of diatonicism. If such a definition shall eventually lead to a consensus regarding the definition of "tonality" as well, the goals of the present study will have been more than fulfilled.

NOTES

In the course of preparing the present article the author has held many helpful discussions with Shmuel Agmon.

1. Babbitt (1962), 76–77; see also Babbitt (1965), 54.
2. Additional studies are Gamer (1967b); Clough (1979); Browne (1981); Gauldin (1983); and Clough, Douthett, Ramanathan, and Rowell (1993). Special mention must be made of Regener's (1973) idiosyncratic yet fascinating study of "pitch notation."
3. Clough and Douthett's work is based to a considerable extent on Clough and Myerson's. Correspondences between Agmon's work and Clough and Myerson's or Clough and Douthett's works are noted in Agmon (1989, 23–24), Clough and Douthett (1991, 123 and 135), and Block and Douthett (1994, 35–38).
4. The property of coherence seems to have been first defined in Balzano (1982, 327–28). The term "coherence" was coined by Balzano; "[lack of] contradiction" is a term Clough and Douthett borrow from Rahn (1991).
5. Clough and Douthett assume $a \geq b \geq 1$. However, no significant result is lost if the cases $a=b$ and $b=1$ are *a priori* ruled out.
6. The notion of a directed measurement from (s,t) to (s′,t′) (see Definition 1.3) may bring to mind Lewin's (1987) "Generalized Interval System" (GIS). A Scalar System SS(a,b), however (and similarly, a Cyclic System—see Definition 2.6), is not a GIS, for it fails to satisfy an important defining property of GIS structure; see Condition (B) in Lewin's Definition 2.3.1 (1987, 26).
7. The square bracket notation "[x]" (where x is not necessarily an integer) means: "the greatest integer less than or equal to x."
8. More accurately, $D_{c,d}$ must satisfy $i+D_j < i+D_{j+1} < \ldots < i+D_{j+d-1}$ for some fixed i and j, $0 \leq i \leq c-1$, $0 \leq j \leq d-1$ (the terms and their subscripts are reduced mod c and mod d, respectively, to bring them if necessary within the ranges 0, 1, . . . , c-1, or 0, 1, . . . , d–1).

   Given a set $D_{c,d}$ satisfying $D_0 < D_1 < \ldots < D_{d-1}$, it is not difficult to see that $i+D_j$, $i+D_{j+1}, \ldots, i+D_{j+d-1}$ is some "chromatic" and "diatonic" "transposition" of $D_{c,d}$. For example, the "C-major" set $D_{12,7}=0_0, 2_1, 4_2, 5_3, 7_4, 9_5, 11_6$ has 12 distinct chromatic transpositions ("keys"), for each of which 7 distinct diatonic transpositions ("modes") are possible. The existence of such transpositions is irrelevant to the present argument, and is therefore conveniently ignored (for Clough and Douthett's treatment of chromatic and diatonic transposition, see especially their Theorems 1.9 and 1.10). Note in particular that in the absence of chromatic and diatonic transposition, and assuming $D_0 < D_1 < \ldots < D_{d-1}$, Clough and Douthett's "J-set" (Definition 1.9) and the present relation (1.1) are equivalent.
9. In Theorems 1.2 and 1.5 Clough and Douthett establish that "being a J-set" (Definition 1.9) is equivalent to "having the property that the spectrum of each dlen is either a single integer or two consecutive integers" (Definition 1.7). Thus Clough and Douthett's "maximal evenness" (assuming $D_0 < D_1 < \ldots < D_{d-1}$) and the present relation (1.1) are the same (see in this connection also note 8).
10. The notation "(x,y) = z" (where x, y, and z are integers) means: "z is the greatest common divisor of x and y."
11. Another improvement on Agmon's (1989) study here is the definition of "diatonic intervals" on the basis of "scale steps," rather than vice versa.

12. See for example Hardy and Wright (1938), 23–37.
13. The reader is encouraged to prove as an exercise that if $\Delta=0$ (i.e., $\frac{q}{a}=\frac{r}{b}$), $\Psi=\{n(q,r) \bmod (a,b), n=0, \pm 1, \ldots, \pm N\}$ is coherent for any N.

    The vertical-bar notation " $|x|$ " means "the absolute value of x" (i.e., if x is positive $|x| = x$, and if x is negative $|x| = -x$).
14. See Hardy and Wright (1938, 24).
15. The proposition may be proved using Lemma 2.1.
16. Clough and Douthett's appeal to "the power of the single ambiguity" is circular. Although assuming a single ambiguity yields the desired result, one must provide an independent argument for making that assumption.
17. Clough and Douthett (1991, 123): ". . . Agmon quite reasonably rules out the sets of family A [i.e., c=2d–1] on the ground that they are inherently less interesting than those of family B [c=2d–2], because the former are generated by the whole-step (clen 2) or its complement, while (for c>4) the latter are generated by a 'skip' of more than one diatonic step (e.g., in the usual diatonic set, clen 5 or 7—the perfect 4th or perfect 5th)." Note that Agmon's requirement rules out not only the case a is odd, but also the case a=4.
18. Alternatively, the third condition may simply stipulate that a "quintic class" and a "second" (or an inversion of a second) are distinct diatonic intervals.

LIST OF WORKS CITED

Agmon, E. 1989. "A mathematical model of the diatonic system." *Journal of Music Theory* 33: 1–25.

———. 1995. "Diatonicism and Farey series." *Muzica* 6/1: 68–74.

Babbitt, M. 1962. "Twelve-tone rhythmic structure and the electronic medium." *Perspectives of New Music* 1: 49–79.

———. 1965. "The structure and function of music theory: I." *College Music Symposium* 5: 49–60.

Balzano, G. 1980. "The group-theoretic description of 12-fold and microtonal pitch systems." *Computer Music Journal* 4: 66–84.

———. 1982. "The pitch set as a level of description for studying musical pitch perception." In Manfred Clynes (ed.), *Music, Mind, and Brain: The Neuropsychology of Music*. New York: Plenum.

Block, S., and Douthett, J. 1994. "Vector products and intervallic weighting." *Journal of Music Theory* 38: 21–41.

Brinkman, A. 1986. "A binomial representation of pitch for computer processing of musical data." *Music Theory Spectrum* 8: 44–57.

Browne, R. 1981. "Tonal implications of the diatonic set." *In Theory Only* 5: 3–21.

Carey, N., and Clampitt, D. 1989. "Aspects of well-formed scales." *Music Theory Spectrum* 11: 187–206.

Clough, J. 1979. "Aspects of diatonic sets." *Journal of Music Theory* 23: 45–61.

Clough, J., and Myerson, G. 1985. "Variety and multiplicity in diatonic systems." *Journal of Music Theory* 29: 249–70.

———, and Douthett, J. 1991. "Maximally even sets." *Journal of Music Theory* 35: 93–173.

———, Douthett, J., Ramanathan, N., and Rowell, L. 1993. "Early Indian heptatonic scales and recent diatonic theory." *Music Theory Spectrum* 15: 36–58.

Gamer, C. 1967a. "Some combinatorial resources of equal-tempered systems." *Journal of Music Theory* 11: 32–59.

———. 1967b. "Deep scales and difference sets in equal-tempered systems." *American Society of University Composers: Proceedings of the Second Annual Conference.*

Gauldin, R. 1983. "The cycle-7 complex: Relations of diatonic set theory to the evolution of ancient tonal systems." *Music Theory Spectrum* 5: 39–55.

Hardy, G., and Wright, E. 1938. *An Introduction to the Theory of Numbers.* Oxford: Clarendon.

Lewin, D. 1987. *Generalized Musical Intervals and Transformations.* New Haven: Yale University Press.

Rahn, J. 1991. "Coordination of interval sizes in seven-tone collections." *Journal of Music Theory* 35: 33–60.

Regener, E. 1973. *Pitch Notation and Equal Temperament: A Formal Study.* Berkeley: University of California Press.