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The Group-theoretic Description of 12-Fold and Microtonal Pitch Systems

Introduction

It is well-known that most of the musical instruments of the orchestra are constrained to produce 12 distinct pitch classes or categories per octave. This makes it rather easy to realize music based on a 12-fold octave division with such instruments but next to impossible to realize a piece of music based on any other pitch system. With the advent of the computer, the possibilities of exploring alternative *microtonal* systems of octave division broaden considerably. In the face of seemingly boundless freedom of choice, what is needed is a basis for selection that will tell us which systems offer the greatest resources and will thereby be the most likely to reward our exploration.

In fact, there is a deeper question than this, and that is the question of how one might appropriately describe the resources of a pitch system. To be sure, the ultimate resources of a pitch system are some function of its intervals, the primitive pairwise relations between pitches. So the question really boils down to one of how to conceive of intervals.

The commonly accepted answer is that the canonical definition of an interval is to be couched in terms of a frequency ratio, moreover a ratio of powers of small integers, a mathematical object of the form $2^p 3^q 5^r \ldots$, with p, q, r ranging over positive and negative integers. The resources of an equal-tempered n-fold pitch system of octave division are then a function of the "goodness-of-fit" between the equal log-frequency grid of the system and some set of ratios (Mandelbaum 1961; Stoney 1970; von Hoerner 1974; 1976). Certain ratios may be set aside as special in the sense that it is particularly important to approximate them closely, for example, $2^{-1}3^1$ (p5) or $2^{-2}5^1$ (M3).

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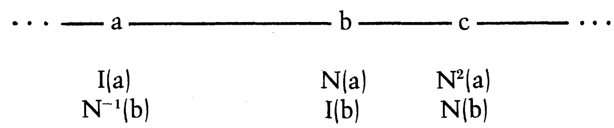
In this paper I shall argue for another way of assessing the resources of a pitch system, one that is independent of ratio concerns and that considers the individual intervals as transformations forming a mathematical *group*. Every equal-tempered system of n-fold octave division, as well as every system of n ratios that can be approximated by an equal-tempered system, possesses the structure of the so-called cyclic group of order n, C_n. We will examine the structure of C_{12} and see that it possesses rather special properties that make the sets we call diatonic scales possible. Our first major result will thus be that diatonic scales may be profitably represented in terms of C_{12}, without recourse to ratios. Inquiring into the specific nature of C_{12} that supports such pitch sets will suggest a method of generalization that yields a new family of microtonal systems. Unlike the goodness-of-fit approach, which leads to systems of size 9, 31, and 41 tones, our group-theoretic concerns will suggest systems of octave division based on 20, 30, and 42 tones. Further, it will be possible to specify diatonic scale "analogs" in each of these systems and to say in exactly what sense they are analogous. Our general method for presenting the groups by means of their isomorphisms in turn suggests a simple method of "simulating" diatonic melodies in C_{12} and their analogs in C_{20}, C_{30}, and C_{42}. By the end of the paper I will have presented not only specific systems and scales worthy of exploration, but in addition some strong arguments for considering the present *algebraic* description of pitch systems as a serious alternative to existing *acoustic* descriptions. To begin, I will briefly sketch the derivation of C_n for an n-fold system of octave division.[1]

Cyclic Group Structure and Pitch Systems

A group is one of the simplest of algebraic structures, consisting of a set G and a binary operation $*$ on elements of the set. Formally, the group is written as $\langle G, * \rangle$. If the set is closed under the operation, the operation is associative, and the elements include an identity element I and an inverse element e^{-1} for each element e ($e * e^{-1} = e^{-1} * e = I$) then the set is said to consitute a group under the given operation. Sometimes for ease of exposition we say the set is the group, the operation being implied only. When the operation is commutative, $a * b = b * a$ for all a and b, we have an *Abelian* group. For example, the set of positive integers is closed under the (associative) operation of addition, but does not constitute a group without both an identity element, 0, and the negative integers that are the inverse elements of the positive integers. The enlarged set is a group, the group of integers under addition, and it bears the name C_∞, the cyclic group

1. The octave is, of course, a 2/1 ratio. But any theory of music that recognizes some form of octave equivalence is ascribing to the octave a status that is qualitatively different from other ratios and not deducible from its 2/1 status per se. Mathematically, as I will show, the octave acts as an equivalence relation on pitches, a formal property shared by no other ratio. So either an acoustic or an algebraic theory must deal with the octave separately from the remaining intervals. The octave's equivalence-class-inducing properties are in fact better modeled by an algebraic formulation in any case.

Fig. 1. Fragment of a system consisting of three points (pitch places) on a line (log-frequency continuum).



of infinite order. All cyclic groups, including C_∞, are Abelian groups.

I have given the *static* interpretation of this group, where the elements are integers and the operation is addition. Every group admits also of a *dynamic* interpretation, where the elements are viewed as *transformations* and the operation as *succession*. In the first interpretation, two integers, say 3 and 4, are added to yield a third integer, 7. In the second interpretation, two transformations, "Add 3" and "Add 4" are applied in succession to yield a third transformation, "Add 7." This static/dynamic duality, which is always present (Whitehead 1948; Holland 1972), may be applied to our present concerns. The static elements correspond to pitch places in a system, the dynamic elements to the musical intervals as transformations generating sets of pitch places. We will adopt both ways of speaking, each one as it is appropriate to a particular context.

Both the static and the dynamic characterizations of the integers can be modeled by the set $\{\dots,-2,-1,0,1,2,\dots\}$ and the operation $+$. In fact, any realization of C_∞ can be modeled in this way. In particular, any pitch system as a set of points on a (log) frequency continuum is an example of C_∞. Consider the three points a, b, c in Fig. 1. We can describe their topological relationships via an adjacency predicate NEXT, whence b = NEXT(a), and c = NEXT(b). If we abbreviate NEXT with N, we also have $c = \mathrm{N}(b) = \mathrm{N}[\mathrm{N}(a)] = \mathrm{N}^2(a)$. N^{-1} also has a natural meaning as in $a = \mathrm{N}^{-1}(b)$, and for that matter $a = \mathrm{N}[\mathrm{N}^{-1}(a)] = \mathrm{N}^0(a) = \mathrm{I}(a)$. The indices or exponents combine like the integers—indeed, their domain is the integers—so perhaps it is not surprising that C_∞ obtains here. I will not pursue this *topological* application of C_∞ in this article; the main reason for mentioning it is to show that the group concept may be applied to pitch systems that are not equal-tempered systems, and is by no means restricted to the latter.[2] When the system is equal-tempered, however, NEXT has a strict interpretation in terms of (log) frequency, and this fact will make our exposition easier if we think in terms of equal-tempered systems. This move will also permit a more forceful interpretation of the group and will allow us to use the language of symmetry theory to describe the effects of group elements.

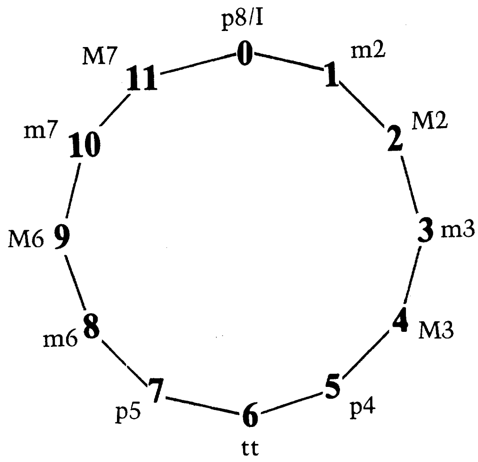
Every equal-tempered n-fold system of octave division, then, possesses a strong form of C_∞ symmetry. The various transformations of the group, N^i, may be considered as shifts of the entire system by a specific log-frequency amount. The N^i in particular constitute the set of transformations that leave the complete set of pitch places invariant, in the same way that rotations about the center of an equilateral triangle through angles of 120°, 240°, and 360° leave it invariant. The transformation N^1 is special in that it and its (positive and negative) powers are sufficient to generate the entire group; such an element is referred to as a *generator* of the group. N^{-1} is also a generator of C_∞, in fact the only other besides N, but it is not really a "new" generator. In general, the inverse of any element e of a group that is a generator (i.e., e^{-1}) is also a generator. The defining character of a cyclic group is that all of its elements may be described in terms of single generator. For any cyclic group with n elements, C_n, there is at least one element g such that $g^n = \mathrm{I}$. For example, the group of rotations of the equilateral triangle is C_3, since $g = 120°$ (or 240°) implies $g^3 = 360°$ (720°) = I.

Since the character of C_∞ is common to all equal-tempered systems, it is not itself very helpful in describing the differences among systems. Representing the distinctive character of each system results from recognizing the redundancy of the system in every octave, N^n = octave, and exploiting this redundancy by a so-called homomorphic mapping of C_∞ to C_n. In essence, the mapping procedure corresponds to using the predicate "is the same pitch class as" or "is octave related to" as an equivalence relation, partitioning the N^i into n equivalence classes $\{..., \mathrm{N}^{-2n}, \mathrm{N}^{-n}, \mathrm{I}, \mathrm{N}^n, \mathrm{N}^{2n}, ...\}$, $\{..., \mathrm{N}^{-2n+1}, \mathrm{N}^{-n+1}, \mathrm{N}^1, \mathrm{N}^{n+1}, \mathrm{N}^{2n+1}, ...\}$, $\{..., \mathrm{N}^{-2+2}, \mathrm{N}^{-n+2}, \mathrm{N}2, \mathrm{N}^{2+2}, \mathrm{N}^{2n+2}, ...\}, \ldots, \{..., \mathrm{N}^{-2n+(n-1)}, \mathrm{N}^{-n+(n-1)}, \mathrm{N}^{n-1}, \mathrm{N}^{n+(n-1)}, \mathrm{N}^{2n+(n-1)}, ...\}$. In general, the homomorphic mapping may be written in the form $\mathrm{N}^{jn+k} \rightarrow k \pmod n$, with j ranging over the integers, and k taking on the values $0,1,...,n-1 \pmod n$.[3] The latter we will write as k_n whenever the embedding system is not clear from context. Thus the group C_{12} consists of the set $\{0_{12}, 1_{12}, ..., 11_{12}\}$ and the operation of modulo 12 addition. The identity element, 0_{12}, is the image of N^{-24}, N^{-12}, N^0, N^{12}, and so on, the set of transformations leaving pitch class invariant. As an active, dynamic transformation, 0 represents the set of pitch-class-preserving transformations; as a static element 0 is a potentially arbitrary origin, to which the addition rules apply no less and no more than to other elements. (Compare this to "zero" as a place versus "Add zero" as a transformation; the latter is highly individual in its effect, the former an equal member in a democracy (Holland 1972).)

2. In such cases, we must reverse the goodness-of-fit procedure to find an equal-tempered system that the given pitch system approximates. Such a fit may be determined by a least-squares procedure, for example. Another, less constraining possibility would take the given system and augment it with added pitch places as necessary to ensure that all of the representatives of N^i in the system are smaller in terms of log frequency than all of the N^{i+1}. A 12-fold Just Intonation system, for example, already satisfies this requirement in that all of the minor seconds are smaller than all of the major seconds, all of the major seconds smaller than all of the minor thirds, and so forth. This kind of constraint, which I call *coherence*, may be important, not only in linking a pitch system to the log-frequency continuum, but also in linking a scale or subset to the pitch system. The latter application of coherence is pursued briefly in the discussion of scales later in this article.

3. A *homomorphism* is a structure-preserving mapping in the sense that any true statement that links elements in the original group is also true of the mapped images of those elements: "the product of the images is equal to the image of the product." In the present case, whenever $\mathrm{N}^{j_1 n + k_1} * \mathrm{N}^{j_2 n + k_2} = \mathrm{N}^{j_3 n + k_3}$, $k_1 + k_2 = k_3 \pmod n$ in C_n. (In the language of group theory, C_n is a *quotient group* formed by "division" of C_n $\{\mathrm{N}^i \mid i \leftarrow \{..., -2, -1, 0, 1, 2, ...\}\}$ by its normal subgroup $\{\mathrm{N}^i \mid i \leftarrow \{..., -24, -12, 0, 12, 24, ...\}\}$. See, for example, Budden's book (1972). The product-preserving property is also true of *isomorphism*, a type of homomorphism that, unlike the present example, is a one-one mapping.

Fig. 2. *First representation of C_{12}: semitone space. "One-dimensional" structure generated by 1_{12} (m2).*



Please note that we are not claiming that notes related by octaves are "identical." It is just that the individual character of a system is most clearly revealed when the essential redundancy at the octave is incorporated into the formalism. Indeed, even though we may not say that a melody is the same if its notes are played in randomized octaves, we may say, for example, that the definitions of such constructs as C major scale or B-flat minor triad are octave-invariant. And, as we have said, the *rectilinear* aspect of any pitch system that is independent of octave equivalence is mirrored in C_∞, but our present interests dictate examination of the group C_n that is specific to each n-fold system of octave division. We now proceed to an analysis of C_{12}.

First Two Representations of C_{12}: Melodic and Key Relations

In this and the next section I will present three isomorphic representations of C_{12} based on different sets of generators. The representations are motivated almost purely from mathematical concerns, but we shall see that they directly address musical concerns as well. In particular, we shall see how diatonic scales and the pitch sets we call major and minor triads are woven, as it were, into the very fabric of C_{12}. I will also try to show how each of the three representations in turn captures something fundamental about melodic, harmonic, and key relations in C_{12}-based music.

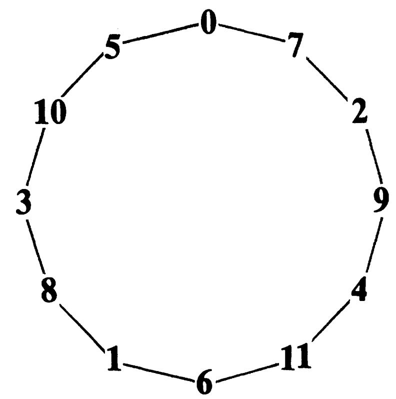
I have said that the three representations are isomorphic—if so, what is different about them? The answer is that structural relationships are unchanged in the isomorphisms, but proximity relations are changed. By structural relationships I mean simply the following: given an isomorphism between two groups $\langle G, *\rangle$ and $\langle H, \circ\rangle$, then for any three elements in G such that $g_1 * g_2 = g_3$, the corresponding elements in H show $h_1 \circ h_2 = h_3$ (see Footnote 3). By proximity relations I mean that elements transformationally "close" together in terms of the generator(s) of G are not necessarily transformationally close in terms of the generator(s) of H. To provide a very loose analogy: two squares on a chessboard that are close together from a knight's point of view will not be close together from a queen's point of view, although both knight and queen can individually "generate" an entire chessboard, that is, access any square in some number of moves.

The first and perhaps most obvious representation of C_{12} is given in Fig. 2. It is based on the (tempered) semitone or minor 2nd (1_{12}) as the generator of the space. Each adjacent pair of points is directly connected by a semitone transformation. Twelve iterations of a semitone result in an octave, or identity element: $(\text{m2})^{12} = I$, and no smaller number of iterations yields this result. We say the element 1 is of period 12 to summarize this fact. Note that $(\text{M7})^{12} = I$ is the outcome of running the cycle in reverse order. Indeed, $\text{M7} = (\text{m2})^{-1}$ (i.e., M7 and m2 are inverses) so they both generate the same space. Closeness of pitch points in this space is a function of the number of semitones separating them. When we refer to a melodic motion as "small" I believe it is closely related to the sense in which distances in this space may also be "small." For the sake of reference, call this configuration the *semitone group*, or *semitone space*.

It can be seen from Fig. 2 that C_{12} contains elements of many different periods, a fact directly attributable to the compositeness of 12. Thus the elements 2 and 10—M2 and m7—are both of period 6, $(\text{M2})^6 = (\text{m7})^6 = I$. Either one generates not the full group but only the elements $\{0,2,4,6,8,10\}$. This set corresponds to a whole-tone scale, to be

Fig. 3. Second representation of C_{12}: fifths space. "One-dimensional" structure generated by 7_{12} (p5).

Automorphism of first representation.



sure, but for our purposes what is more important is that these elements constitute the group C_6, a subgroup of C_{12}. It is easily verified that the set $\{0,2,4,6,8,10\}$ is a group under mod 12 addition. It is, in a sense, a system within a system, and only our labeling of the elements as $\{0,2,4,6,8,10\}$ instead of $\{0,1,2,3,4,5\}$ (with mod 6 addition) informs us that there are other elements to be considered that do not fall in this smaller group. Similarly, 3 and 9 are both of period 4 and generate the subgroup C_4—$\{0,3,6,9\}$—also known as a diminished-seventh chord. In general, given any group element of order k, such an element generates a subgroup C_k of the group under consideration. The other subgroups of C_{12} are C_3—$\{0,4,8\}$—generated by M3 or m6 and corresponding to an augmented triad, and C_2—$\{0,6\}$—the infamous tritone that generates only itself and the identity due to the fact that the tritone is its own inverse, $tt = (tt)^{-1}$, the only group element with this property. A moment's consideration should reveal that the subgroups of any C_n are simply all the C_k such that k divides n evenly.

The lone elements that have not come into play yet are 5 and 7, the p4 and p5. Neither one belongs to any of C_{12}'s subgroups because each one generates C_{12} in its own right. They are the generators of our second representation of C_{12}, shown in Fig. 3. The mapping from Fig. 2 to the arrangement of points here is structure-preserving, for example, the mapping $0 \rightarrow 0$, $1 \rightarrow 7$, and so on is an isomorphism. To illustrate briefly, take three elements from the semitone group, m2 * M2 = m3, whose images in Fig. 3 are p5, M2, and M6 respectively. The isomorphism ensures that p5 * M2 is equal to M6, which is readily verified.

This isomorphic mapping between a set $\{0,1,...,11\}$ and itself is called an *automorphism*. Every Abelian group possesses a trivial automorphism obtained by swapping inverse elements (in our case, "reflecting" through the unison-tritone axis in Fig. 2). In addition, variants of any given (nontrivial) automorphism may be obtained by inverse-swapping in either of the arrangements involved in the mapping (i.e., reflecting either Fig. 2 or 3), so we could have mapped $1 \rightarrow 5$ instead of $1 \rightarrow 7$. But other than these variations, the arrangement in Fig. 3 depicts the only structure-preserving mapping between the set $\{0,1,...,11\}$ and itself.

Those readers who recognize in Fig. 3 the familiar *circle (dodecagon, cycle) of fifths* may see the latter in a rather new light. It is a structure that is isomorphic to the even more familiar and perceptually immediate cycle of semitones, and it is moreover the only such structure. Note that the cycle of fifths and its special status are a function of p5-as-7_{12}, and not p5-as-3/2. We have been taught that the cycle (circle) of fifths is a *mnemonic*, which promotes a rather superficial conception of it. But from a group-theoretic point of view the cycle of fifths is anything but superficial; it is deeply ingrained in the structure of our pitch system.

Yet, of course, the cycle of fifths is a mnemonic, a way of remembering key signatures and key relations. Why should this be? The short answer, not always made clear in courses of musical instruction, is that diatonic scales of the form $\{0,2,4,5,7,9,11\}$ show an overlap when transposed that is a direct function of distance on the cycle of fifths; and key signatures were developed, it seems, with diatonic scales specifically in mind.

Let us, however, not overlook the following simple fact: a major scale, say C, when transposed up a p5 to G, does lead to a scale with all but one element the same, the changed element altered by a semitone: F→F#. Our "answer" thus leads to another question: Why should this be? The short answer is that what we are calling diatonic scales are very special subsets of the cycle of fifths. As Fig. 4 reveals, a diatonic scale is a connected region

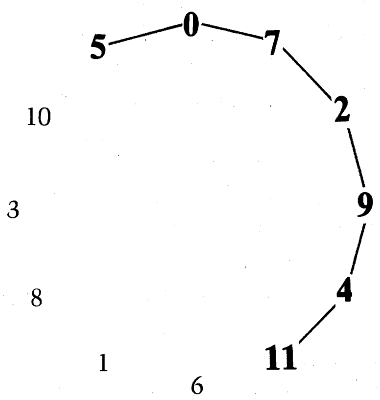


Fig. 4. Diatonic scale as a connected region of fifths space.

in the space of fifths, a set of points maximally "close" to one another in the sense implied by our second representation. It should be evident that any fully connected subset of size $m = 7$ in this space is a diatonic scale; there are 12 different subsets of this kind corresponding to the 12 transpositions of a diatonic scale, 1 for each group element. Because of its connectedness, a diatonic scale necessarily changes only one element when transformed by a p5. It is not difficult to show that only connected sets behave in this way.

In everyday musical parlance, we implicitly invoke the cycle of fifths to talk about "close" and "distant" keys. The point I wish to make here is that it is the diatonic scale that realizes this spatial potential of C_{12} by marking off a region of the space and by exhibiting transpositional overlap that is directly related to distance in the space. No other 7-element scale behaves in this way.

There are, of course, connected regions of size 5, 6, 8, and the like, and they do also exhibit the overlap-transformational distance relation. One of these, the $m = 5$ set, is none other than the pentatonic scale. The set of size 6 is Guido's hexachord from the 10th century. Besides m, a number of other features differentiate these sets. One feature is the set of intervals of the group that is represented in the scale. Smaller scales do not possess a sufficient number of interrelations to contain representatives of all intervals; in fact, the $m = 7$ scale is the smallest one that does contain at least one token of every interval. (The pentatonic scale has no tt or m2.) For scales containing more than 7 elements, all intervals are represented but the overlap-distance relation deteriorates rapidly. Every pair of size-10 scales save adjacent pairs share 8 of 10 elements; the adjacent pairs share 9. Even for size-8 scales, five transpositions share exactly 4 of 8 elements and are therefore not distinct in terms of overlap.

Perhaps more important are distinguishing features of the scales taken with reference to the semitone space and C_∞. It turns out that many of these connected scales are not "coherent" with respect to semitone space and the embedding C_∞ space in the sense that their scalestep sizes are not a monotonically increasing function of semitones (see Footnote 2). Consider the set $\{0,2,4,5,7,9\}$, Guido's hexachord: at some places in this set three semitones correspond to two scalesteps, in other places one scalestep; five semitones corresponds to three scalesteps in some places, to two in others. Diatonic scales fail to exhibit coherence only with the tritone interval, but it can be shown that any scale with an odd number of elements and one or more tritones must map this interval into two different-sized scalesteps. In the diatonic case, the tritone is an augmented fourth in one place and a diminished fifth in another.

Our final distinguishing feature, and perhaps the most important, selects connected scales of sizes 5 and 7 due to the fact that $(\mathrm{p5})^5 = (\mathrm{m2})^{-1}$ and $(\mathrm{p5})^7 = \mathrm{m2}$. That is to say, transposing one of these scales by a perfect fifth not only leads to a scale with $m - 1$ members in common, but the changed element has undergone a minimal change in the sense given in semitone and C_∞ space. It is this property that underlies F $\rightarrow$ F$^\#$, and the very possibility of key signatures.

Before moving to our third pitch representation, I should point out that the isomorphic semitone space is just as structurally adequate a basis for connected scales as the fifths space. The diatonic scale analog in semitone space is simply the set $\{0,1,2,\ldots,6\}$. Of course, to use this space as a space of scales and scale relations does nothing to bring out the perceptually less immediate fifths space, and therefore constitutes a less "balanced" use of the resources of C_{12}. More than this, however, the connected scales in semitone space are all hopelessly lacking in what we've called coherence. The size-7 scale given previously, for example, contains distances of one and two scalesteps that are actually smaller, in semitones, than distances of three, four, and five scalesteps.

To review briefly, we have seen that C_{12} can be generated independently by two different elements, and the two isomorphic spaces reflect different senses of "closeness" that are operative in our music. One is a kind of melodically based proximity of single notes; the other a scale-based proximity of keys. Between the single note and the scale is an intermediate level of structuring present in most diatonic music from the 17th century on, namely the *triad*, or chord. Our third C_{12} representation, as it turns out, addresses this level of structure.

Third Representation of C_{12}: Harmonic Relations

The semitone group and fifths group are both one-dimensional in the sense that their elements are all described in terms of a single generator, a 1-tuple. *Direct product groups*, like Cartesian products, are sets of n-tuples, vectors each component of which is a 1-tuple from some smaller group. Thus the product group $C_3 \times C_2$ consists of 2-tuple elements of the form (member of C_3, member of C_2). Elements combine componentwise; using integers mod 3 and mod 2, the following is a true statement about $C_3 \times C_2$: $(1_3, 1_2) * (2_3, 1_2) = (0_3, 0_2)$. There are evidently six elements in this group: (0,0), (1,0), (2,0), (0,1), (1,1), (2,1). From our example the last two of these are shown to be inverses.

It turns out that C_{12} is isomorphic to the direct product of two of its subgroups, C_3 and C_4. We write $C_{12} \cong C_4 \times C_3$. Unlike the 1-tuples of the first two groups, (a), $a \leftarrow \{0,1,...,11\}$, an element in $C_3 \times C_4$ is described as (a,b), $a \leftarrow \{0,1,2\}$, $b \leftarrow \{0,1,2,3\}$. The rule of combination is

$$(a,b) * (a',b') = ([a + a'] \bmod 3, [b + b'] \bmod 4), \qquad (1)$$

a simple "vector addition." What we are saying, then, is that the elements (0,0), (0,1), (0,2), (0,3), (1,0), (1,1), . . . , (2,2), (2,3), subject to the operation in Eq. 1, give rise to a structure that is isomorphic to the C_{12} structure shared by our first two representations. The isomorphism itself is rather simple:

$$(a,b) \longleftrightarrow (4a + 3b)_{12}. \qquad (2)$$

The interpretation is also quite simple. In terms of intervals, there is one axis generated by major thirds, (4_{12} generates C_3) the other by minor thirds (3_{12} generates C_4), and each interval is a point corresponding to the number of major and/or minor thirds "contained" in that interval. A p5, for example, may be broken down into one M3 and one m3. Thus p5 is just the point (1,1). Every interval can be described in terms of zero to two major thirds and zero to three minor thirds, and every such combination in turn corresponds to one and only one interval. To facilitate direct comparison with the previous representations, Fig. 5 displays the product group representation with the points labeled as numbers mod 12. The space should properly be represented on a torus, but we have unwrapped it and arranged several duplications together to show all the adjacency relationships. Keep in mind that every representative of, say, 2, is really the same point.

The first thing to note about this space is that the maximally compact, connected structures are none other than the four basic triads, with the major and minor triads represented as congruent right triangles related by 180° rotation and the diminished and augmented triads as "degenerate" triangles spanning only one of the two dimensions of the space. Perhaps even more significant is the higher-order, compact structure that is the diatonic scale. Just as this scale could be built up in our second representation by adjoining fifths, and just as triads were built up in the present space by adjoining thirds, so the diatonic scale is here built up by adjoining triads in a manner that results in a figure that is convex, compact, and spans a maximum amount of space along both axes. The spanning property can be seen from the double occurrence of 2 in the scale region; since both 2's are really the same point, the set spans a region from the lower left to the upper right corner of the space, passing through all possible values on both axes. None of

Fig. 5. Third representation of C_{12}: thirds space. Unwrapped "two-dimensional" structure generated by 4_{12} (M3) and 3_{12} (m3). Product group $C_4 \times C_3$ isomorphic to first and second representations. Italicized points are a single replication of the full space, with "seams." Right triangles correspond to major and minor triads, boldfaced parallelogram spanning region to a diatonic scale.

0	4	8	0	4	8	0	4	8	0	4	8	0	4	8
9	1	5	9	1	5	9	1	5	9	1	5	9	1	5
6	10	2	6	10	2	6	10	2	6	10	2	6	10	2
3	7	11	3	7	11	3	7	11	3	7	11	3	7	11
0	4	8	0	4	8	0	4	8	0	4	8	0	4	8
9	1	5	9	1	5	9	1	5	9	1	5	9	1	5
6	10	2	6	10	**2**	**6**	**10**	*2*	6	10	2	6	10	2
3	7	11	3	7	**11**	**3**	*7*	*11*	3	7	11	3	7	11
0	4	8	0	4	**8**	*0*	*4*	**8**	0	4	8	0	4	8
9	1	5	9	1	*5*	*9*	**1**	**5**	9	1	5	9	1	5
6	10	2	6	10	*2*	**6**	**10**	**2**	6	10	2	6	10	2
3	7	11	3	7	11	3	7	11	3	7	11	3	7	11
0	4	8	0	4	8	0	4	8	0	4	8	0	4	8
9	1	5	9	1	5	9	1	5	9	1	5	9	1	5
6	10	2	6	10	2	6	10	2	6	10	2	6	10	2
3	7	11	3	7	11	3	7	11	3	7	11	3	7	11

the other connected-fifth scales ($m \neq 7$) have this property; none are compact, "space-filling," connected-third/triad scales.

The connected-semitone scale with $m = 7$ is contained in this space as well, a "reflected" image of a diatonic scale. No "rigid motion" in this space will map one figure onto the other. Perhaps here we have another case of "handedness in nature" (Gardner 1979), where two nonsuperposable mirror-image forms exist in principle but only one is found in nature. Interestingly, the triad-analogs in the reflected scale do not correspond to triads in the sense of "every other note" in the scale. Clearly, the diatonic scale has emerged as a unique pitch set.

There are no other isomorphisms of C_{12} in terms of n-tuples. $C_2 \times C_6$ is a group of order 12, but it is not isomorphic to C_{12}. It is easily shown, for example, that the group $C_2 \times C_6$ contains no elements of order 12 and therefore no elements that behave structurally like semitones or fifths. It is thus impossible to map the intervals onto the elements of this group. No other product group with 12 elements (e.g., $C_2 \times C_2 \times C_3$) is isomorphic with C_{12} either, so the set of representations is complete.

Of course, each representation in a sense "contains" the other two. Because we have depicted our $C_3 \times C_4$ representation as an infinite lattice on a plane, it is possible to see how this group contains the others. The family of lines analogous to $y = x + c$ are unwrapped cycles of fifths separated by thirds. The family of lines $y = -x + c$ of opposite "handedness" are all cycles of semitones. These long cycles must pass "outside" the more compact thirds space in Fig. 5 in order to close. In a toroidal representation of thirds space, this would mean that the full cycle of fifths (or semitones) does not close until after several revolutions around the torus of thirds.

We have not yet had anything to say about the concept of a *tonic*, partially because without thirds space it would have been difficult to provide an intelligible account. Prior to the 17th century there is only scant evidence that certain members of the diatonic set were to be preferred as tonics, but with the growth of harmony and the major/minor triadic basis of music, the diatonic set came more and more to represent only two scales: the so-called major and natural minor scales. As Wilding-White (1961) has pointed out, the central notion here was not so much one of a tonic as of a tonic triad, and this idea fits in rather nicely with thirds space. It can be seen in Fig. 5 that the diatonic set consists of three "major" and three "minor" triangles. The centrally located major triangle in the set $\{0,2,4,7,9,11\}$ is $\{0,4,7\}$ and the centrally located minor triangle is $\{9,0,4\}$. These are none other than the tonic triads of the surviving major and natural minor modes of the diatonic set. Neither of these triads nor their roots 0 or 9 possesses any privileged location in the fifths-space representation of a diatonic scale. I am tempted to speculate that only when music came to draw more fully upon the resources of the thirds representation did the related concepts of tonic and triad really come into full force.

Third-relatedness of triads in a given diatonic scale, like fifth-relatedness of scales and semitone-relatedness of the individual notes of the scale, is but another sense of "closeness" that is potentially operative in music. It is sometimes said, for example, that a ii chord may be substituted for a IV chord in many contexts; these two triads in Fig. 5 are the edge-sharing sets $\{2,5,9\}$ and $\{5,9,0\}$. A diatonic scale, in fact, may be seen as a set where each element plays each of the three triadic roles exactly once, a property that inevitably leads to a set of adjoining triads alternately sharing m3 and M3 edges. Stated in this way, the structure of the diatonic scale seems to possess a family resemblance to certain finite geometries with m lines passing through each of n points as in the theorems of Pappas and Desargues (Budden 1972). I will not pursue this connection here.

We have sketched out the basic form of the three isomorphisms of C_{12} and showed something of their bearing on harmonic, melodic, and key relations in music. We have also seen the remarkable properties of diatonic scales with reference to the different spaces, and how this scale instantiates the more abstract relations in fifths and thirds spaces while remaining coherent with respect to semitones. What remains now is to demonstrate the existence of other C_n's that share most of these important form-bearing properties with C_{12}, and to describe the analogous structures to the triads, cycle-of-keys, and diatonic scales in these other groups.

Generalization to n-Fold Systems

Every system possesses an analog to a semitone group where the generator is simply the smallest or *unit* element. Our question is, Which of these systems have two other groups isomorphic to the semitone group, one of which is a single-generator cycle of "keys," the other a product group of "triads," such that a "diatonic" connected subset on the cycle of keys is also a convex, connected structure in the product space and is coherent with respect to "semitones"?

There are not many groups having these structural resources, but the ones that do are all of the form $C_n \cong C_k \times C_{k+1}$ (i.e., $n = k(k+1)$ for integer

k). It is easy to show this to be a sufficient condition for satisfying the above constraints, but to demonstrate the necessity of $n = k(k + 1)$ invokes technicalities too extensive to be treated here and depends somewhat on exactly how the constraints are stated. I will be content to convince the reader of the sufficiency by way of example and to make some prefatory remarks on necessity of a quasi-intuitive nature.

First of all, for a set to be a *spanning region* built up by alternating fundamental triangles of a $C_k \times C_{k'}$ product space, the orders of the constituent groups must differ by at most 1, $|k - k'| \leq 1$. If the orders differ by more than 1, the generated parallelogram will necessarily span the smaller cycle before the larger cycle is completed. Four a's and three b's can all be used in an alternating cycle if the cycle is *abababa*, but there is no way five a's and three b's can all be deployed in this way because we run out of b's too soon. Four a's and four b's would work, of course, but in no case does the isomorphism $C_k \times C_k = C_{k^2}$ hold true. It can be shown, with some labor, that a nonspanning region in a product space cannot be a connected sequence of powers of a group generator. So in order for a connected cycle-of-fifths scale to be convexly representable in an isomorphic product space, that product space must be of the form $C_k \times C_{k+1}$.

It turns out that for $k < 3$, the resulting C_n (C_6) is not sufficiently rich to possess more than one generator, so no cycle of fifths is possible. The case $k = 3$ is just C_{12} and, as we have seen, 7_{12} (or 5_{12}) is the only nonunit generator available as a basis for the space of keys. When $k > 3$, there are seemingly many more generators to choose from, but in each case only one will give rise to a scale whose triadic structure is mirrored in the product space. That one generator is the element $(2k + 1)_n$, and the diatonic scale built up from this element contains $2k + 1$ notes, is a spanning region in $C_k \times C_{k+1}$, and has the F → F# property that adjacent scales differ in a single note changed by one "semitone." The $(2k + 1)_n$ element behaves structurally like a perfect fifth, and the elements k_n and $(k + 1)_n$ behave like minor and major thirds respectively.

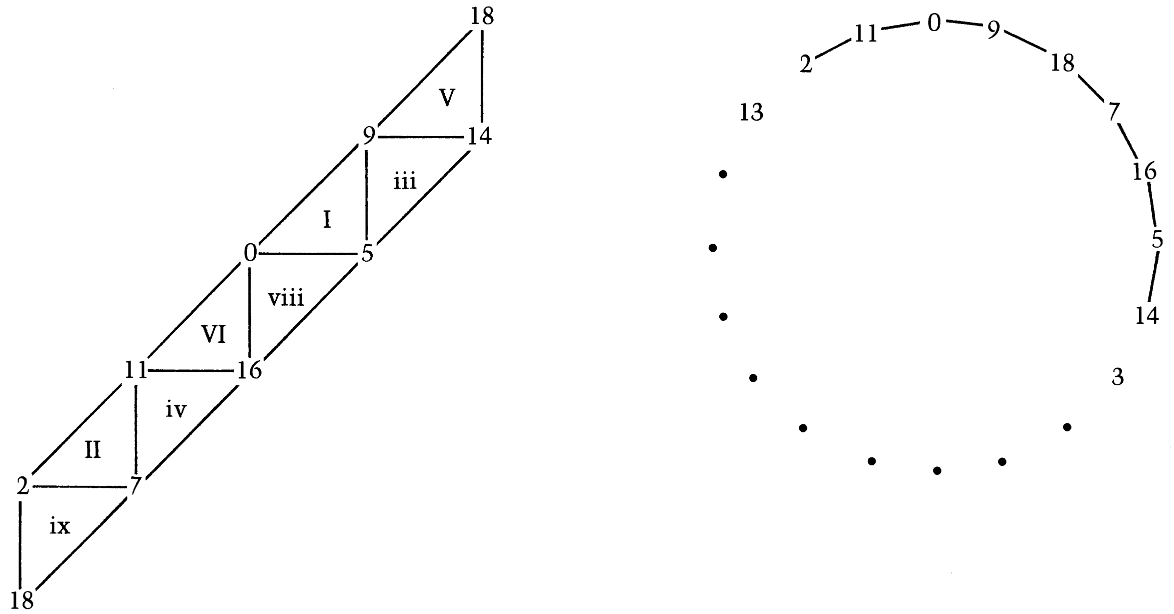
Many of these properties are demonstrated easily, but because I wish to move quickly to the specific systems themselves, I will sketch out only one here. For a scale generated by an element $(2k + 1)_n$, we wish to show that if this scale contains $2k + 1$ elements it possesses the F → F# property. This boils down to showing that $[(2k + 1)(2k + 1)] \bmod [k(k + 1)] = 1$. We can rewrite the expansion $4k^2 + 4k + 1$ as $4k(k + 1) + 1$, and it is readily seen that, since the first term in the latter is 0 mod $k(k + 1)$, the sum is always $0 + 1 = 1$. The $2k + 1$ scale size also ensures its representability as a spanning region covering the $k + 1$ elements of C_{k+1} and the k elements of C_k in an *abab . . . ba* arrangement, as the reader will see from the figures to follow.

Figure 6 shows the "thirds space" for C_{20}, the $k = 4$ case representing the next system after C_{12}. The analogs to the cycle of fifths and cycle of semitones can be read from the positive- and negative-going diagonal lines respectively, just as in C_{12} thirds space. The "diatonic scale" is generated by the element 9_{20} ($= 2k + 1$) and consists of nine connected elements on this cycle, that is, $\{2,11,0,9,18,7,16,5,14\} = \{0,2,5,7,9,11,14,16,18\}$. (To be sure, C_{20} has other generators: 3, 7, and their inverses. But no subsets based on these generators fill out triadic regions in the product space.) Since the underlying system is a 20-fold octave division, the "semitone" is 60 cents, and thus the 9 element, for example, corresponds to an interval 540 cents wide. When the new diatonic scale is transposed up a p5 by the addition of 9 to each element, we get all the original elements back except that 2 is replaced by 3, as advertised: $\{9,11,14,16,18,0,3,5,7\}$. The "major triad," $\{0,5,9\}$, gives the same appearance as its C_{12} counterpart, as does the "minor triad" $\{0,4,9\}$, differing from the major by a "lowered third." It can be seen from Fig. 6 that the scale is both a connected series of "fifths" and an overlapping set of nine triads, eight of which are major or minor (four each) and one of which is "diminished" ($\{14,18,2\}$). A closer look at this scale is provided in Fig. 7. It is in fact the case that all of these scales of size $2k + 1$ have k major, k minor, and one diminished triad. Note also that, as expected, the triads do indeed consist of "every other note" in the scale. The matter of which triads are most suitable for tonic triads is left somewhat open in C_{20}, as for all even k, since there are two centrally located major

Fig. 6. Thirds space for C_{20}, demonstrating $C_5 \times C_4 \cong C_{20}$. Triads and diatonic scale as in Fig. 5.

0	5	10	15	0	5	10	15	0	5	10	15	0	5	10
16	1	6	11	16	1	6	11	16	1	6	11	16	1	6
12	17	2	7	12	17	2	7	12	17	2	7	12	17	2
8	13	18	3	8	13	18	3	8	13	18	3	8	13	18
4	9	14	19	4	9	14	19	4	9	14	19	4	9	14
0	5	10	15	0	5	10	15	0	5	10	15	0	5	10
16	1	6	11	16	1	6	11	16	1	6	11	16	1	6
12	17	2	7	12	17	2	7	12	17	2	7	12	17	2
8	13	18	3	8	13	**18**	**3**	**8**	**13**	*18*	3	8	13	18
4	9	14	19	4	9	**14**	**19**	**4**	*9*	*14*	19	4	9	14
0	5	10	15	0	5	**10**	**15**	*0*	*5*	**10**	15	0	5	10
16	1	6	11	16	1	**6**	*11*	*16*	**1**	**6**	11	16	1	6
12	17	2	7	12	17	*2*	*7*	**12**	**17**	**2**	7	12	17	2
8	13	18	3	8	13	*18*	**3**	**8**	**13**	**18**	3	8	13	18
4	9	14	19	4	9	14	19	4	9	14	19	4	9	14
0	5	10	15	0	5	10	15	0	5	10	15	0	5	10
16	1	6	11	16	1	6	11	16	1	6	11	16	1	6
12	17	2	7	12	17	2	7	12	17	2	7	12	17	2
8	13	18	3	8	13	18	3	8	13	18	3	8	13	18
4	9	14	19	4	9	14	19	4	9	14	19	4	9	14

Fig. 7. The diatonic scale for C_{20}. This family of 20 transpositionally related subsets is the only one that is a connected region on both a single-generator cycle and an isomorphic, dual-generator product space. Every pair of these transpositionally related subsets shows an overlap that is a direct function of distance in the single-generator space, and the overlap relation is such that all the scales can be coded with nine letter names and the symbols # and b. The scale consists of four "major" triads {0,5,9}: I, II, V, and VI; four "minor" triads {0,4,9}: iii, iv, viii, and ix; and one "diminished" triad {0,4,8}: vii_0.



diatonic scale = {0,2,5,7,9,11,14,16,18}

scale with one sharp = {9,11,14,16,18,0,3,5,7}

scale with one flat = {11,*13*,16,18,0,2,5,7,9}

Triads									
fifth	9	11	14	16	18	0	2	5	7
third	5	7	9	11	14	16	18	0	2
root	0	2	5	7	9	11	14	16	18
	I	II	iii	iv	V	VI	vii_0	viii	ix

and minor triads to choose from. The option taken in Fig. 7 for the major-scale analog entails that I, II, V, and VI are the major triads of the scale, iii, iv, viii, and ix the minor triads, and vii_0 the lone diminished triad. The implication of the latter, in the spirit of triadic substitutability determined by thirds space, is that vii_0 in this scale may well end up being absorbed by V as a variant of a V^7 chord. This is similar to what obtains in our present major scale with the vii_0 and V^7 chords.

It would appear that a completely revised set of letter names and system of staff notation is necessary to describe this scale and system in traditional music-theoretic terms. Instead of seven letter names for notes—appropriate to a seven-note diatonic scale—we would need nine names (H and I?)

in order to write the two preceding scales in a key-signature-minded way, such as {A,B,C,D,E,F,G,H,I}, {E,F,G,H,I,A,B#,C,D}. To grow apace with the pitch materials and to prevent confusion in interpretation, a six-line staff might also be necessary. The label "octave" becomes a mild misnomer, based as it is on seven-note diatonic thinking. "Eighths" and "ninths" are here not compound but simple intervals, the "minor eighth" (m8) being 15_{20} "semitone" units wide, the M8 interval 16 units, and the m9 and M9 at 17 and 18 units respectively. The "octave" in this system is better called a "dectave" or just a p10.

Rather than dwell on C_{20}, I should like to illustrate the principles of derivation by moving on to C_{30} and C_{42}. Of course, other systems with higher k (and n) can be described, containing 56, 72, and 90 divisions per octave, but if the ideas presented in this article have some validity, the smaller systems of 20, 30, and 42 contain enough resources to keep us busy for some time.

The $C_5 \times C_6$ product space isomorphic with C_{30} is shown in Fig. 8. The canonical major and minor triads are {0,6,11} and {0,5,11} respectively, the scale-generating interval is 11_{30}, and the 11-tone diatonic scale is given by
{8,19,0,11,22,3,14,25,6,17,28} =
{0,3,6,8,11,14,17,19,22,25,28}. This set, given in Fig. 8, contains {0,6,11} as the centrally located major triad and {25,0,6} as the centrally located "relative minor" triad. Again, the F → F# property is easily verified; this time, transposing the scale one location on the cycle of keys ($+11_{30}$) sharpens the 4th element of the scale, 8, mapping the raised pitch into the 11th element of the transposed scale. This is similar to the C major to G major transition in C_{12} where the 4th element of the C scale is sharpened to yield the "raised 7th" of the G scale.

The inquisitive reader may note that C_{30} (like C_{20}) has several generators: 1, 7, 11, 13 and their inverses. In addition, there are two other cyclic product groups of order 30 that are isomorphic to C_{30}: $C_2 \times C_{15}$ and $C_3 \times C_{10}$. The lattices are easy to construct, as are the single-generator cycles. Working with these directly is perhaps the best way to convince ourselves of the impossibility of using any other generator to construct a connected scale that is also a symmetrical region in any of the product spaces.[4]

Our final example is $k = 6$, $n = 42$. The $C_6 \times C_7$ product space is shown in Fig. 9, along with a diatonic region. Here the space of key relations is generated by 13_{42}, and the diatonic scale is a connected series of 13 places in the space of keys. Once again, this corresponds to a spanning region consisting of six {0,7,13} major triads and six {0,6,13} minor triads (and one diminished {28,3,8} triad). For the interested reader, isomorphic product spaces that will not embed a diatonic scale are $C_2 \times C_{21}$ and $C_3 \times C_{14}$; single generators for C_{42} are naturally all the k_n that are relatively prime to 42.

The entire development of these pitch systems has been effected without reference to ratios. The special nature of the diatonic scale in C_{12} is usually thought to be a function of the simple ratios it contains (Schenker 1906/1973; Helmholtz 1885/1954), but we have shown—and others such as Babbitt (1972) have hinted—that the diatonic scale can be defended as a case of "survival of the fittest" independently of its ratios. In light of how scarce diatonic-scale-supporting pitch systems are, it is truly remarkable that we have come up with these musical materials without group-theoretic thinking. By representing musical intervals and their interrelations algebraically instead of acoustically, we have found our present diatonic scale and its embedding C_{12} as the simplest member of a family of microtonal systems including 20-fold, 30-fold and 42-fold systems of octave division. In fact, when we line up the two families of candidates for microtonal systems, one based on algebraic and the other on acoustic concerns, it appears that our 12-fold system is the only system appearing in both families.

It perhaps goes without saying that anyone seeking a fit to whole-number ratios will enjoy increased chances of success as the partitioning of

4. It is not that these alternate product spaces are inherently uninterpretable, but that all our interpretations have been based on analogy with C_{12}, where there are no alternate product spaces. I do not by any means wish to rule out the possibility that these extra spaces may come to represent some as-yet-undiscovered potential musical functions not available in C_{12}.

Fig. 8. Thirds space for C_{30} demonstrating $C_6 \times C_5 \cong C_{30}$. Triads and diatonic scale as in Fig. 5.

0	6	12	18	24	0	6	12	18	24	0	6	12	18	24
25	1	7	13	19	25	1	7	13	19	25	1	7	13	19
20	26	2	8	14	20	26	2	8	14	20	26	2	8	14
15	21	27	**3**	**9**	**15**	**21**	**27**	*3*	9	15	21	27	3	9
10	16	22	**28**	**4**	**10**	**16**	*22*	*28*	4	10	16	22	28	4
5	11	17	**23**	**29**	**5**	*11*	*17*	**23**	29	5	11	17	23	29
0	6	12	**18**	**24**	*0*	*6*	**12**	**18**	24	0	6	12	18	24
25	1	7	**13**	*19*	*25*	**1**	**7**	**13**	19	25	1	7	13	19
20	26	2	*8*	*14*	**20**	**26**	**2**	**8**	14	20	26	2	8	14
15	21	27	*3*	**9**	**15**	**21**	**27**	**3**	9	15	21	27	3	9
10	16	22	28	4	10	16	22	28	4	10	16	22	28	4
5	11	17	23	29	5	11	17	23	29	5	11	17	23	29
0	6	12	18	24	0	6	12	18	24	0	6	12	18	24
25	1	7	13	19	25	1	7	13	19	25	1	7	13	19
20	26	2	8	14	20	26	2	8	14	20	26	2	8	14
15	21	27	3	9	15	21	27	3	9	15	21	27	3	9
10	16	22	28	4	10	16	22	28	4	10	16	22	28	4
5	11	17	23	29	5	11	17	23	29	5	11	17	23	29

Fig. 9. Thirds space for C_{42} *demonstrating* $C_7 \times C_6 \cong C_{42}$*. Triads and diatonic scale as in Fig. 5.*

0	7	14	21	28	35	0	7	14	21	28	35	0	7	14
36	1	8	15	22	29	36	1	8	15	22	29	36	1	8
30	37	2	9	16	23	30	37	2	9	16	23	30	37	2
24	31	38	3	**10**	**17**	**24**	**31**	**38**	**3**	*10*	17	24	31	38
18	25	32	39	**4**	**11**	**18**	**25**	**32**	*39*	*4*	11	18	25	32
12	19	26	33	**40**	**5**	**12**	**19**	*26*	*33*	**40**	5	12	19	26
6	13	20	27	**34**	**41**	**6**	*13*	*20*	**27**	**34**	41	6	13	20
0	7	14	21	**28**	**35**	*0*	*7*	**14**	**21**	**28**	35	0	7	14
36	1	8	15	**22**	*29*	*36*	**1**	**8**	**15**	**22**	29	36	1	8
30	37	2	9	*16*	*23*	**30**	**37**	**2**	**9**	**16**	23	30	37	2
24	31	38	3	*10*	**17**	**24**	**31**	**38**	**3**	**10**	17	24	31	38
18	25	32	39	4	11	18	25	32	39	4	11	18	25	32
12	19	26	33	40	5	12	19	26	33	40	5	12	19	26
6	13	20	27	34	41	6	13	20	27	34	41	6	13	20
0	7	14	21	28	35	0	7	14	21	28	35	0	7	14
36	1	8	15	22	29	36	1	8	15	22	29	36	1	8
30	37	2	9	16	23	30	37	2	9	16	23	30	37	2
24	31	38	3	10	17	24	31	38	3	10	17	24	31	38
18	25	32	39	4	11	18	25	32	39	4	11	18	25	32

Table 1. A possible set of whole-number ratios for C_{20} intervals

Group Element	*Ratio*	$2^p \cdot 3^q \cdot 5^r \cdot 7^s$	*Cents*	*"Ideal" Cents*
0	1/1	2^0	0.00	0
1	28/27	$2^2 \cdot 3^{-3} \cdot 7$	62.96	60
2	15/14	$2^{-1} \cdot 3 \cdot 5 \cdot 7^{-1}$	119.44	120
3	10/9	$2 \cdot 3^{-2} \cdot 5$	182.40	180
4	8/7	$2^3 \cdot 7^{-1}$	231.17	240
5	32/27	$2^5 \cdot 3^{-3}$	294.13	300
6	49/40	$2^{-3} \cdot 5^{-1} \cdot 7^2$	351.34	360
7	32/25	$2^5 \cdot 5^{-2}$	427.37	420
8	21/16	$2^{-4} \cdot 3 \cdot 7$	470.78	480
9	48/35	$2^4 \cdot 3 \cdot 5^{-1} \cdot 7^{-1}$	546.82	540
10	45/32	$2^{-5} \cdot 3^2 \cdot 5$	590.22	600

the octave grows finer. By way of example, Table 1 shows one possible array of simple ratios $2^p 3^q 5^r 7^s$ that matches equal-tempered C_{20} reasonably well. Certain highly valued ratios like 3/2 are indeed missing, but there are some familiar intervals to be found in this system. The tritone is here ($10_{20} \longleftrightarrow$ 600 cents), and the $2^5 3^{-3}$ "Pythagorean minor third" (294.13 cents) is used as a reasonable match to the interval 5_{20} (300 cents), although $3^{-1}5^2 7^{-1}$ would provide an ever closer match (301.85 cents). Whenever C_4 is contained as a subgroup in C_n, as it is in C_{20}, the system will have intervals 300 and 900 cents wide, "minor thirds" and "major sixths," but we should recognize that these names are appropriate only to their diatonic C_{12} functions. Similarly, the presence of a C_3 subgroup implies major thirds and minor sixths, a C_6 subgroup implies major seconds and minor sevenths, and a C_{12} subgroup—present in $C_{72} \cong C_8 \times C_9$—implies the presence of ("real") perfect fifths/fourths and semitones. Last but not least, a C_2 subgroup, and all of the new systems presented here have one of these, implies the presence of a tritone. Ironically, the tritone, the last interval of our current set of 12 to be accepted into the fold, appears to have the least specific allegiance to C_{12} and is a general characteristic of all even-numbered systems of octave division.

Computer Realization of C_n Pitch-Set Constraints

The microtonal systems advocated here necessitate a rather thorough break with traditional thinking about music and its putative mathematical basis. Even the existing staff and letter-name notation, I have argued, is subtly tied to C_{12} and its diatonic subset; to "add" pitch places named "F double-sharp" or "E prime" and notate them within existing media is to embellish but remain essentially within C_{12} constraints. Intervals we call minor thirds can actually behave more like major thirds in $C_{20} \cong C_4 \times C_5$, or even not like thirds at all (in C_{56}).

A problem with any microtonal system, but one that is especially acute with the proposed systems, lies in describing how we shall compose music with it. In light of the vast differences in detail between even C_{12} and C_{20}, many of our intuitions about pitch may simply be inappropriate to composing with C_{20}. One possibility is for the composer to start from scratch like a beginning theory student: to learn to spell out the 20 diatonic scales, to learn the hierarchy of sharps and flats and how they relate to the scale degrees, to understand the nature of relations among triads and their constituent notes in and out of each scale, and so on. It would appear to be a massive undertaking—not that we should expect it to be easy—and there are furthermore no musical examples to guide the way.

An alternate procedure, one we are currently pursuing, is to use the computer as a tool for exploration, making use of the orderly constraints imposed by the manifestations of C_n and drawing heavily on those analogies that do exist between the systems. We start with C_{12} and consider the qualitative character of its representations. Given the proposed conceptions of melodic, harmonic, and key spaces, the structure of C_{12} provides a clear suggestion about how to proceed.

We conceive of a rudimentary melody as subject to pitch-set constraints at several nested levels. At the most local level, single notes may change while the underlying triad does not change. Most of these single-note pitch motions will be small in the sense of semitone space. At the next level, triads themselves may change more slowly over time but leave the underlying diatonic scale invariant, that is, they may consist of motions desirable as I-ii-IV-V-I (or, in C_{20}, perhaps I-VI-ix-V-I). Here we would describe closeness in terms of distance in thirds space. At an even more global temporal level the scale itself may change by modulation, but the underlying system C_n is left invariant. A "small" change at this level is reckoned in terms of fifths-space distance. The system itself is a kind of global invariant, never changing within a piece of music (although system-to-system modulation $C_n \rightarrow C_{n'}$ is a provocative notion).

Thus we put each of the three spaces in control of three different levels of pitch variation, in a way not unlike received descriptions of traditional tonal music. We propose to explore admittedly stochastic music, but nonetheless music of discriminable degrees of "musicality." For example, an otherwise random series of notes constrained to the pitch set $\{0,2,4,5,7,9,11\}$ sounds more "musical"[5] than a series constrained only to the elements of C_{12}—the chromatic scale. It is possible to hear the added constraint, so to speak. This effect may be observed even more clearly if the active pitch set corresponds to a pentatonic scale; even a random series of notes selected from this small pitch set sounds constrained. The space of fifths is the source of these most global weightings on pitch selection. On a more local time scale, the root, third, and fifth of the currently active triad are additionally weighted. Most locally, notes near (in semitones) to the currently active note are weighted more heavily. The quantitative form of these weights can be varied empirically with a view to optimizing the perceived musical sense of the result. Constraint changes that are undetectable can be deleted or set at will. We seek to impose a minimal amount of structure to get a musical result, but we are not averse to adding stronger, more "artificial" constraints to achieve this if necessary. For example, we can provide a piece with a more individual style by supplying lower-level harmonic and melodic *motives* (e.g., I-IV-V; $\hat{1}$-$\hat{2}$-$\hat{3}$-$\hat{2}$-$\hat{1}$), and even rhythmic motives, the frequency of recurrence of which can also be weighted.

Given a reasonably satisfactory but minimal set of serial constraints for C_{12} music, the next step is to leave the constraints essentially the same but change the system, to C_{20} for example. Some modification of the constraints of a parametric sort would be necessary to incorporate the larger arrays of the larger system. But other than this our best guess as to what constitutes intelligible music in the new system almost has to be in terms of constraints couched in the representational language that led us to the system in the first place. Should

5. Those who are uncomfortable with connotations of musical value may think of what we have called "musicality" in terms of the ability of the sonic events to hold a listener's attention, or the extent to which the events sound nonrandom, or make sense to a listener. Voss and Clarke employed a similar methodology quite effectively, and determined that music modeled on white noise—$1/f^0$ music—sounded "too random," $1/f^2$ music sounded "too correlated," and $1/f^{(1)}$ music was "close to being 'just right'" (1978, p. 263).

the audible result prove to have detectable structure, it then becomes a task for the psychology-minded among us to test the hypothesis of common perceptible constraints between systems.

It will almost surely be the case that music from these alternate C_n's will sound like nothing we have ever heard before. I cannot believe that it will sound like nothing at all, however. The recurring triadic and diatonic set structures in changing environments will almost surely be distinguishable both from random pitch changes and from C_{12}-based wanderings. If our hearing facility cannot stretch beyond our present C_{12} categories, then that is a problem common to all microtonal systems, one that all but dooms them to ultimate failure. For those of us who have listened to and even begun to hear the audible sense in pitch systems other than our own, the prospects appear hopeful. What I have tried to provide is a new set of microtonal pitch systems that is endowed not only with ample structural resources, but with resources affording similar possibilities to those so thoroughly mined from C_{12} in the 18th and 19th centuries. I have attempted to describe these systems in a way that permits systematic exploration, but ultimately I think we should have to educate our own compositional intuitions in order to produce genuinely musical results from these systems.

Conclusion

This has been a whirlwind tour of a rather vast terrain. Let me sum up what has been shown and provide some perspective.

The acoustic description of musical intervals leads to an evaluation of equal-tempered pitch systems in terms of goodness-of-fit to small, whole-number ratios. In this article an alternative is outlined in which the intervals are conceived as elements of a group of transformations generating a pitch system C_n. Three isomorphic representations of C_{12} (our current system) bear striking parallels to the melodic, harmonic, and key relations as exhibited in tonal music, and the diatonic scale is revealed as a special subset of C_{12} that is in fact the simplest embodiment of the abstract relations given in two of the three representations. A logical extension of the analytic method used for C_{12} leads to the construction of other microtonal systems of n-fold octave division, $n = k(k + 1)$; it was shown that these systems and only these systems offer structures maximally analogous to our present cycle of keys, diatonic scale, and major and minor triads. Within the present framework, we are invited to think of "tonality" as corresponding to a perceptible region in the spaces generated by the group representations.[6]

The three groups are conceived of as controlling pitch-set structure at nested time levels, with the space of "semitones" the most local, the space of "thirds" next, and the space of "fifths" the most global. This in turn suggests a method for simulating C_n music on a computer by weighting pitches, pitch sets, and the transformations among them on the basis of spatial relations in the group representations. This method may well prove useful for exploratory purposes and initial familiarization with some of the musical possibilities of each system, as well as for empirical study of pitch-set constraint perceptibility, but we cannot expect to find the key of any microtonal system without making as much as we can know of its structure part of our intuitive working knowledge. When we do, the computer will still be our primary means of realizing music based on any of these microtonal systems.

In retrospect, when we look at the Just, Pythagorean, and equal-tempered tunings of our fa-

6. There has been a growing recognition of the usefulness of concepts related to group theory for describing human perception and internal representation (Cassirer 1944; Gibson 1966; Shaw and Pittenger 1977; Balzano 1978; Shepard 1979). Typically, in recent discussions of such matters, the spaces through which transformations occur and in which invariants are preserved are akin to the "visual space" of our surrounding worldly environment. The groups that characterize the potentially audible spaces of pitch relations are quite different, but even here a group-theoretic means of description promises a viable framework for thinking about problems of music perception. Speculatively, perhaps the uniqueness of the musical experience is due in part to the uniqueness of the group structures music makes available to a perceiver.

miliar major scale, it is evident that there is an important sense in which they all work in substantially the same way: we don't have to learn to compose or hear separately for each one. The ratios, which are different in all three tuning schemes, do not really address this fundamental commonality. This fact alone should have told us that ratios per se are not the basic descriptors of what goes on in a piece of diatonic C_{12} music; that ratios, in terms of which we must call the three tunings "different," may even be an inappropriate level of description for pitch systems. Now there is no question that without ratios we would have never discovered and refined the splendid 12-fold system. But let us not confuse historical importance with perceptual importance. It may well be that the group-theoretic properties, as it became possible for them to come into play, were the more perceptually important all along.

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