DOI: 10.1111/j.1468-2249.2007.00251.x

MICHAEL RUSS

Some Observations on $P_{M,N}$ Relations within Set Classes

Some of the recent writing on transformational and 'neo-Riemannian' theory and analysis has chosen to focus on the specific phenomenon of parsimonious voice-leading between triads and dominant and half-diminished sevenths.[1] Typical examples of this are where, in the case of the triad, two voices are held and one moves by a tone or semitone to produce another triad; or, in the case of the dominant and half-diminished seventh, two voices are held and two are displaced to produce another dominant or half-diminished seventh. This article, which complements work by Adrian Childs, Richard Cohn, Robert Gauldin, David Lewin and others, takes set class 4–27 [0, 2, 5, 8], the dominant or half-diminished seventh, as its starting point and then proceeds to establish the extent to which each tetrachordal set class has the capability of forming families of parsimonious relations.[2] Some tetrachordal set classes lack the capacity to do this, others limit it to pairs of sets and still others (such as 4–27) allow the formation of much larger families. As in Gauldin's article, the approach taken here is primarily set-theoretic. However, rather than exploring these families through the two- and three-dimensional graphic configurations that, following the lead given by Riemann's *Tonnetz*, many neo-Riemannian theorists have devised,[3] the present survey focuses upon interval strings and their rotations and the way in which these predict the capability of sets for forming families of various sizes. When dealing with sets of larger cardinality, it is also necessary to consider their capacity to hold subsets of n-1 and n-2 invariant under transposition and inversion (where n is the cardinality of the set class being examined). The primary concern here, therefore, is with defining how extensive the properties recognised in 4–27 and 3–11 [0, 3, 7] (the major and minor triad) are within the universe of pitch-class set classes, and with looking for examples of the exploitation of these properties in late nineteenth- and early twentieth-century music. Nonetheless, the final part of this article will proceed to examine some of the additional parsimonious relations that obtain between pentachords and hexachords.

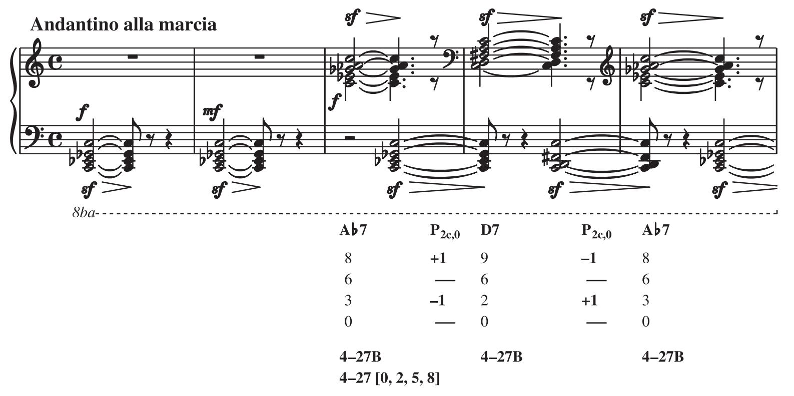
The approach adopted in the present instance is essentially empirical; no attempt is made to locate observations within a more generalised theory of efficient voice-leading. Recently, Clifford Callender, Ian Quinn and Dmitri Tymoczko have made significant advances in constructing a theory of generalised chord spaces, a theory which ultimately places the relationships considered here within a much larger context.[4] Yet as these authors have advised, theirs is still work in progress; moreover, the task of explaining the non-Euclidean geometry which supports such chord spaces requires a degree of mathematical abstraction and complexity that is beyond the scope of the current study. Nevertheless, the rigorous approach they have proposed obliges any related enquiry to define its assumptions with some precision. Specifically, their work proceeds from the careful definition of equivalence classes: octave (O), permutation (P), transposition (T), inversion (I) and cardinality (C).[5] In the present article, $P_{m,n}$ relations are assumed to hold regardless of the octaves within which individual pitch classes appear, or the ways in which sets are permuted in actual musical contexts. However, in defining $P_{m,n}$ relations within set classes, the changing T or T_n/I equivalences are significant. Pitch classes may also be freely duplicated, so the article proceeds by invoking O, P and C but not T or I. (Nor will it invoke two further equivalence classes defined by Callender, Quinn and Tymoczko: multiplication (M) and interval-class equivalence (Z).) Because relations hold between pitch classes, spelling is generally not of concern, and an assumption is therefore made in favour of equal-tempered tuning. Invoking all these equivalences allows the domain of the enquiry to be rendered more manageable. In particular musical contexts the voice-leading between $P_{m,n}$-related objects may be executed in different ways, as will be shown through the analysis of various examples from the repertoire.

The kind of parsimonious voice-leading described here is often a more efficient means than transposition of transforming one set into another of the same class. In the case of a tetrachord, for instance, transposition, in which a common integer is added to each pitch class, involves four actions. But in the case of a large number of tetrachords, the same mapping may be achieved more efficiently by the perturbation of one or two pitch classes by whole tones or semitones while the rest are retained. A simple and familiar way to conceptualise this is to think of the C major scale (7–35 {0, 2, 4, 5, 7, 9, 11}). In order to generate the dominant key (G), the entire set may be transposed by a fifth or, more efficiently, F♯ may be substituted for F♮. Similarly, a dominant seventh on A♭ may be transposed by six semitones to produce one on D. More expediently, however, as Musorgsky realised at the beginning of the Coronation Scene in *Boris Godunov* (1869), two pitch classes {0, 6} may be held and two perturbed by semitonal contrary motion, {3, 8} to {2, 9} (Ex. 1). Such parsimonious voice-leading between dominant sevenths is an example of what Lewin dubbed the 'DOUTH2' progression:

> we can say that pcsets X and Y are in the 'Douthett relation of degree 2' (DOUTH2) if Y can be obtained by discarding two member pitch-classes x1 and x2 of X, and then picking up two new pitch-classes y1 and y2 where y1 lies one semitone away from the discarded x1 and y2 lies one semitone away from the discarded x2.[6]

Although Lewin does not specify that X and Y should be of the same class in this definition, the context of the quotation makes it clear that this is the case. However, DOUTH2 is concerned only with the motion of two voices by semitone. On the grounds that such relations have an important place in the music of the

Ex. 1 Musorgsky, *Boris Godunov*, Prelude, Scene 2, bars 1–5: $P_{2c,0}$ relation



nineteenth and early twentieth centuries, motions by either a single semitone or a single whole tone, as well as mappings to the same set class effected by voices moving by two whole tones, will also be considered here. In consequence, I shall employ the simple method formulated by Douthett and Steinbach to indicate degrees of parsimony. Their '$P_{m,n}$' relation indicates the degree of parsimony between two sets of the same size but not necessarily of the same class.[7] The first subscript indicates the number of pitches that move by half step and the second the number by whole step; thus $P_{2,0}$ indicates that two voices move by a half step, while $P_{0,1}$ indicates that one voice moves by a whole step. Douthett and Steinbach's nomenclature will nonetheless be subjected to a slight refinement in order to enable a distinction to be made between similar and contrary motion through the use of the subscripts 's' and 'c' (the absence of a letter indicates that the motion may be either similar or contrary). So, for example, '$P_{2c,0}$' indicates that voice-leading is contrary. For the purposes of this article, when the expression '$P_{m,n}$' is used, it should be assumed that the values of 'm' and 'n' lie between 0 and 2 and that, unless stated otherwise, the relation holds between sets of the same class.

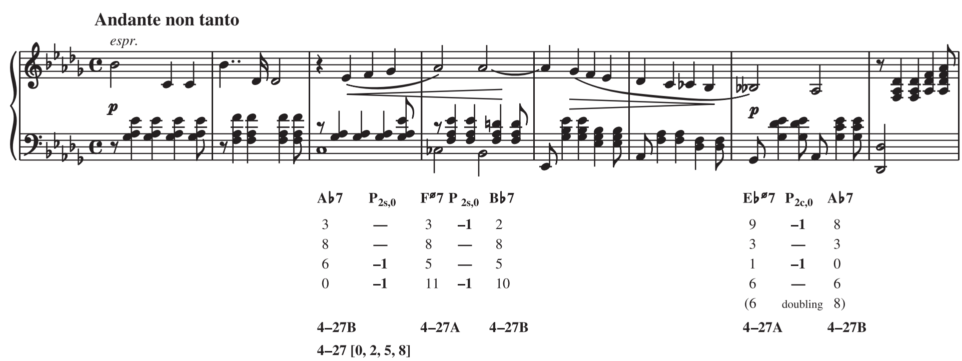
Nine of the 29 tetrachordal set classes are capable of forming at least one of the $P_{2c,0}$, $P_{2s,0}$, $P_{0,1}$, $P_{0,2c}$ and $P_{0,2s}$ relations without effecting a change of set class. Let us first of all examine the family of $P_{2,0}$ and $P_{0,1}$ relations within set class 4–27 using some examples drawn from the repertoire of the nineteenth century. This set class is uniquely profligate in that it is able to form nine $P_{2,0}$ relationships. Using $D^{ø7}$ [0258] as a starting point,[8] the six transformations to dominant sevenths and six to other half-diminished sevenths are listed in Table 1. As will be observed, both $P_{2s,0}$ and $P_{2c,0}$ relations are possible. Each $P_{2s,0}$ relation results in a transformation that inverts and transposes the original set, whereas each $P_{2c,0}$ relation results in a transformation that transposes the original set.[9] There are exactly twice as many of the former, because any similar-motion

Table 1 $P_{2,0}$ (DOUTH2) relations with $D^{ø}7$

Similar motion ($P_{2s,0}$)	G7	E7	C7	D7	F7	A♭7			
Contrary motion ($P_{2c,0}$)							$B^{ø}7$	$A♭^{ø}7$	$F^{ø}7$
I/T_n	I_7	I_4	I_1	I_2	I_5	I_8	T_9	T_6	T_3

* $D^{ø}7$ [0, 2, 5, 8] is the prime form of 4–27. The I/T_n numbers are those required to generate the transformation from the prime form.
$^{ø}7$ = half diminished seventh (4–27A).
7 = dominant seventh (4–27B).

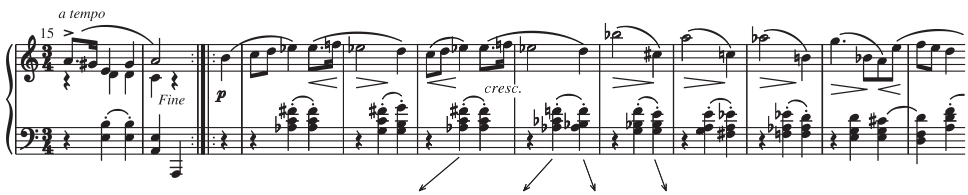
Ex. 2 Tchaikovsky, 'None but the Lonely Heart', Op. 6 No. 2, piano introduction



relation has a parallel which reverses the held and moving intervals; this is not the case with contrary-motion relations or with sets of larger cardinality, a phenomenon which will be explored further when interval strings are discussed below. Set class 4–27 is capable of a single $P_{0,1}$ relation (in the case of $D^{ø7}$, a transformation to $B♭^{7}$, where the seventh of the half-diminished seventh moves to the root of the dominant seventh). There is also one trivial $P_{0,2c}$ relation where the original chord maps onto itself (that is, in $D^{ø7}$, C moves to D and D moves to C).

Several analytical explorations of $P_{2,0}$ (DOUTH2) relations of the dominant seventh have centred on the more chromatic music of the nineteenth century. Such relations, while not common in Wagner, are not unusual either, as Childs, Lewin, Gauldin and others have noted.[10] Instances of $P_{2s,0}$ relations may also be found at the beginning of Tchaikovsky's song 'None but the Lonely Heart', Op. 6 No. 6 (1869–70) (Ex. 2). The chromatic progression from $A♭^{7}$ to $B♭^{7}$ via $F^{ø7}$ in bars 3 and 4 contains two overlapping $P_{2s,0}$ relations in the progression IV^{7}–$ii^{ø7}$–V^{7} ($ii^{ø7}$–V^{7} is the simplest and most familiar $P_{2,0}$ relation). The context is a brief tonicisation of ii, E♭ minor, which is then followed in bar 7 by a $ii^{ø7}$–V^{7} progression in the tonic, D♭ major. Tchaikovsky frequently highlights

Ex. 3 Chopin, Mazurka in A minor, Op. 7 No. 2, bars 15–24: $P_{2s,0}$ relations



A♭ Gm6	$P_{2s,0}$	F^{ø}7	$P_{2s,0}$	B♭7	$P_{2s,0}$	G^{ø}7
3	—	3	**–1**	2	**–1**	(1)
6	**–1**	5	—	5	—	5
0	**–1**	11	**–1**	10	—	10
8	—	8	—	8	**–1**	7
4–27B		**4–27A**		**4–27B**		**4–27A**
4–27 [0, 2, 5, 8]						

semitonal voice-leading in these progressions; several more instances of $P_{2,0}$ relations may be found throughout the song.

In the example just given, $P_{2s,0}$ relations arise from mixing half-diminished and dominant sevenths; this is also the case in the passage from Chopin's Mazurka, Op. 7 No. 2 (1831) shown in Ex. 3.[11] This type of connective or prolongational passage featuring strong linear coherence is frequently the site for $P_{2,0}$ relations in Chopin. The chord in bar 19 is either a German sixth in the key of C minor or a misspelled dominant seventh in the key of D♭. The C minor orientation of the preceding phrase inclines us to the former interpretation, but in bars 19–20 the German sixth fails to behave in the anticipated way, progressing to B♭7 via F^{ø7} which, if a D♭ is implied, then moves to a G^{ø7} (although D would be an equally viable implied note here). As indicated in Ex. 3, if we imply the missing D♭ in the fourth chord, these harmonies form three interlocking pairs of $P_{2s,0}$ relations. It is evident that, in set-theoretic terms, the third chord (B♭^{7}) is a T_2 transposition of the first (A♭^{7}), whereby 4–27 {0, 3, 6, 8} becomes {2, 5, 8, 10}. However, the net voice-leading between these chords is such that two voices move by semitone, while one moves by a whole tone and the other not at all. This, a $P_{2,1}$ relation, is the most parsimonious route between the two chords, far more so than adding the constant 2 to each voice (the same relation holds between the second and [incomplete] fourth chords).

In Chopin's Mazurka, Op. 6 No. 1 (1830) (Ex. 4), a chromatically descending bass with root movement by fifth links the relative major (A in bar 4) with the dominant seventh (C♯^{7} in bar 8). The frequent alternation of half-diminished and dominant sevenths produces a large number of $P_{2,0}$ relations (seven out of the twelve pairs of chords) between forms of 4–27. The passage ends with a succession of three relations moving smoothly back to the dominant. Four pairs of chords involve motion between 4–27 and 4–28 (the diminished seventh [0, 3, 6, 9]). All pairs of successive dominant and diminished sevenths will either be in a $P_{1,0}$ or a $P_{3,0}$ relation; only the latter type is in evidence here. One other type of relation in this passage is $P_{2,1}$ between chords 2 and 3. The

Ex. 4 Chopin, Mazurka in A minor, Op. 6 No. 1, bars 1–9: $P_{2s,0}$ and $P_{3,0}$ relations

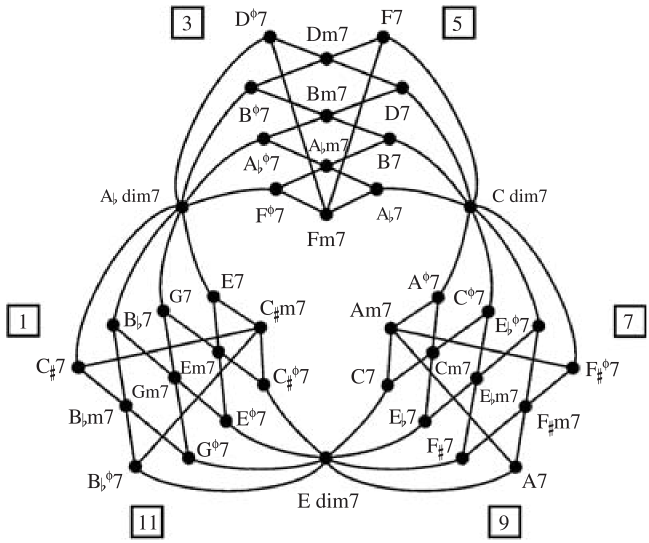
D♯ø7		G♯7		C♯7		C♯ø7		F♯7		F♯°7		Bø7		E7		E°7		Aø7		D7		G♯ø7		C♯7
	$P_{2s,0}$		$P_{2s,1}$		$P_{2s,0}$		$P_{2s,0}$		$P_{3s,0}$		$P_{3s,0}$		$P_{2s,0}$		$P_{3s,0}$		$P_{3s,0}$		$P_{2s,0}$		$P_{2s,0}$		$P_{2s,0}$	
1	–1	0	–1	11	—	11	–1	10	–1	9	—	9	–1	8	–1	7	—	7	–1	6	—	6	–1	5
6	—	6	–1	5	–1	4	—	4	–1	3	–1	2	—	2	–1	1	–1	0	—	0	–1	11	—	11
3	—	3	–2	1	—	1	—	1	–1	0	–1	11	—	11	–1	10	–1	9	—	9	–1	8	—	8
9	–1	8	—	8	–1	7	–1	6	—	6	–1	5	–1	4	—	4	–1	3	–1	2	—	2	–1	1
4–27A		4–27B		4–27B		4–27A		4–27B		4–28		4–27A		4–27B		4–28		4–27A		4–27B		4–27A		4–27B

4–27 [0, 2, 5, 8]
4–28 [0, 3, 6, 9]

amount of perturbation here is greater by a single semitone than that between the dominant and diminished sevenths.

Douthett and Steinbach's 'power towers' provide a useful way of interpreting this passage.[12] Each tower (Fig. 1) is octatonic and is erected from T_3 cycles of dominant, minor seventh and half-diminished seventh chords. Each dominant and each half-diminished chord is capable of forming $P_{1,0}$ relations with two of the minor seventh chords at the centre of the tower and, via the minor sevenths, three $P_{2,0}$ relations with the chords on the opposite side of the tower. Diminished sevenths couple the towers together, enabling movement between them. In addition, these chords indicate the means by which each diminished and each dominant seventh can form four more $P_{2,0}$ relations with members of the next tower, at the same time providing a route back to the three remaining half-diminished or dominant sevenths on the side of the tower where the expedition began. The nine $P_{2,0}$ relations within set class 4–27 are thus accounted for graphically as the compounds of two $P_{1,0}$ relations via 4–28, a topic I shall return to later. In the Chopin excerpt reproduced in Ex. 4, no chord is repeated; moreover, if a line is traced from E♭^{ø7}, the enharmonic equivalent of the opening chord of the passage, it will be observed that Chopin always moves anticlockwise around the towers, alighting three times on two of them and four times on the other. This analysis may seem a little outlandish in a passage which essentially features a familiar stepwise sequence in the melodic parts combined with an underlying root progression through the cycle of fifths. However, it is not the

Fig. 1 Douthett and Steinbach's 'power towers'

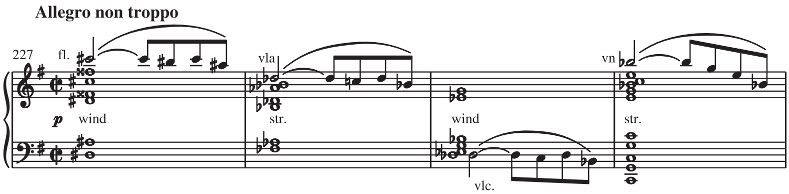


root progression but the chromatic voice-leading that commands our attention, and such passages prepare the ground for later ones where $P_{m,n}$ relations are an even more significant means of generating harmonic coherence.

The retransition of the first movement of Brahms's Fourth Symphony (Ex. 5) provides a more extended example of $P_{2,0}$ relations between dominant and half-diminished sevenths. As with the two Chopin examples, these relations form part of a clear linear structure within a connective passage. In this case it links a pizzicato variant of the main theme in G♯ minor, a major third above the tonic (bar 219), with the return to the home key of E minor and an augmented version of the principal theme (bar 246). At bar 227, the harmonic sequence reaches V^7 of G♯ minor (enharmonically re-notated as E♭^{7} for reasons of clarity in Ex. 6a). As indicated by the boxes beneath Ex. 5 and the brackets beneath Ex. 6a, the passage may be grouped into two four-bar, four-chord groups, followed by two two-bar, two-chord groups that preserve the relationship between the outer chords in the four-chord groups but remove the middle two. The passage ends with a ii^7–V^7 progression in E minor, itself a $P_{2,0}$ relation.

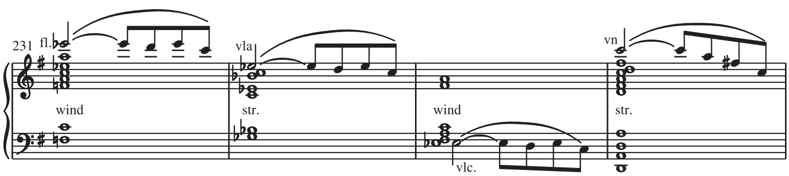
If viewed as a passage through the power towers, this music traverses a rather different journey to the Chopin example examined above. Starting in the same place, it moves clockwise to the other side of the next tower before returning to the original one. The T_2 transposition between the groups of chords then effects an anticlockwise rotation of this excursion. As is apparent from the arrows within the boxes in Ex. 5, shifts in register result in voice-leading that is much more fractured than in the examples considered up until

Ex. 5 Brahms, Symphony No. 4 in E minor, Op. 98, first movement, bars 227–46



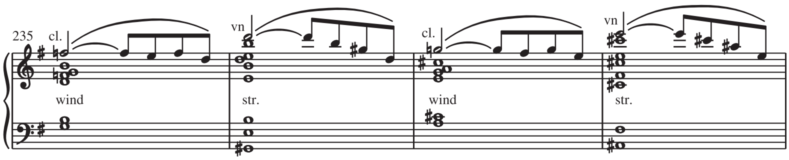
D♯7 (E♭7)	$P_{2s,0}$	B♭ø7	$P_{2s,0}$	E♭7	$P_{2c,0}$	C7
1	—	1	—	10	—	10
7	+1	10	—	7	—	4
10	—	8	–1	3	+1	7
3	+1	4	–1	1	–1	0
4–27B		4–27A		4–27B		4–27B

T_2



F7	$P_{2s,0}$	Cø7	$P_{2s,0}$	F7	$P_{2c,0}$	D7
3	—	3	—	9	—	0
9	+1	0	—	0	—	6
0	—	10	–1	5	+1	9
5	+1	6	–1	3	–1	2
4–27B		4–27A		4–27B		4–27A

T_2

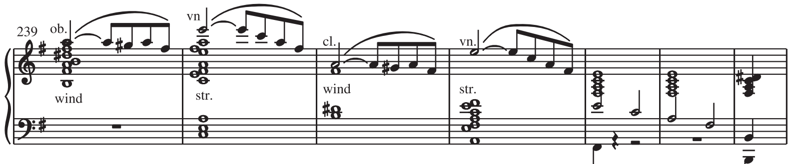


G7	$P_{2c,0}$	E7
5	–1	2
2	—	4
11	—	11
7	+1	8
4–27B		4–27B

T_2

A7	$P_{2c,0}$	F♯7
7	–1	4
4	—	1
1	—	6
9	+1	10
4–27B		4–27B

T_2

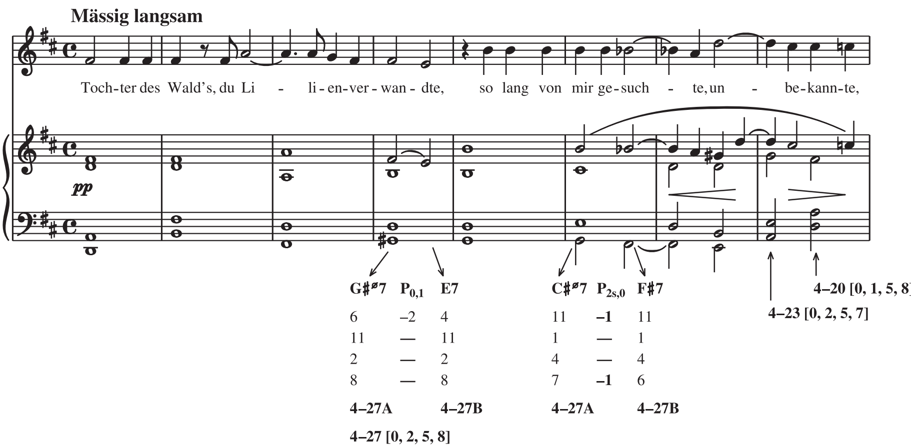


B7	$P_{2s,0}$	F♯ø7	$P_{2s,0}$	B7	$P_{2s,0}$	F♯ø7
9	—	4	–1	9	—	4
3	+1	9	—	6	—	6
6	—	6	—	3	+1	0
11	+1	0	–1	11	+1	9
4–27B		4–27A		4–27B		4–27B

Ex. 6 Brahms, Symphony No. 4 in E minor, Op. 98, first movement, bars 227–46, voice-leading graph

this point. Nevertheless, there is a strong underlying linear framework as Ex. 6a illustrates. The transposition of the chordal groups creates a structure in which the final dominant seventh of each spans four whole tones, C^7–D^7–E^7–$F\sharp^7$. The series of notes attached to the beam in the upper stave of the example are the continuous pitches within each group; these proceed to rise by step to reach E in bar 237. Following this, E remains the highest active note with only two temporary interruptions (bars 239 and 241), resolving (as the seventh in $F\sharp^{ø7}$) to D♯ in bar 246 in the context of a dominant ninth preparing for the return to E minor. The upper line is covered at the beginning of each four-bar or two-bar unit, but the progression suggested by these cover tones (spanning D♭–G) does not carry lasting implications for the long-term structure. In addition to the series of maximally parsimonious relations at the foreground, the middleground may be analysed as unfolding the interval A♯–E within the chord of F♯ (V of V), but with an initial $P_{2,0}$ related substitution of C^7 for $F\sharp^7$ (Ex. 6b). This analysis, demonstrating the impact of $P_{2,0}$ relations on two structural levels, perhaps accounts better for the harmonic coherence of this passage than one based simply on harmonic function.

Ex. 7 Wolf, 'Auf eine Christblume', *Mörike-Lieder*, bars 1–8

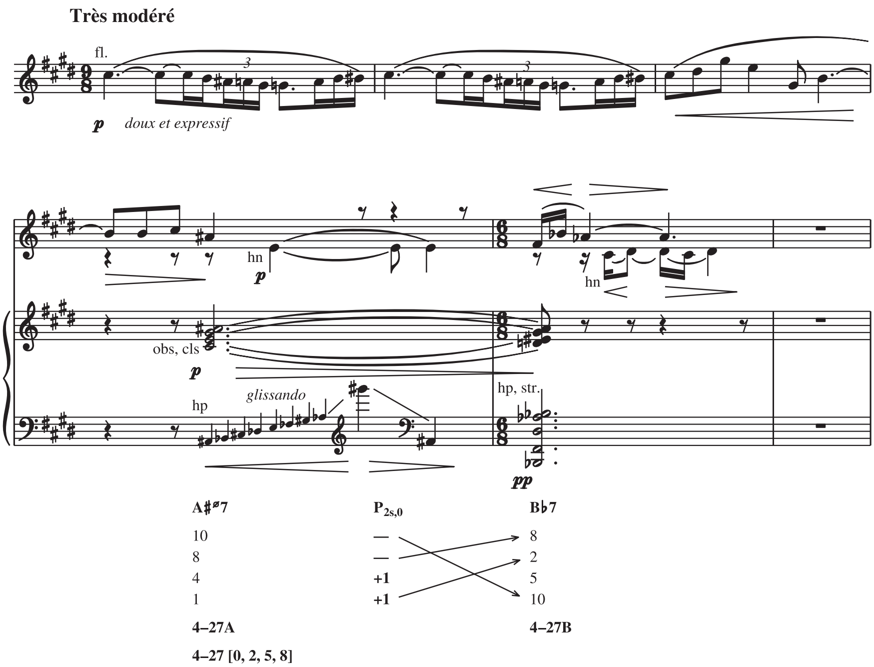


To complete this section, an example of transformation through the perturbation of a single tone by a whole tone is provided. Ex. 7, from a song by Wolf, shows a $P_{0,1}$ relation in bar 4, where the whole-step fall of F♯ to E (a 7–6 suspension) transforms G♯^{ø7} into E^7. Douthett and Steinbach's power towers show this motion to be not a single $P_{0,1}$ relation, but the compound of two $P_{1,0}$ relations connecting the two chords via a diminished seventh. Ex. 7 also illustrates a $P_{2s,0}$ relation in the sixth bar of the same passage.

The progressions identified in Brahms, Chopin, Tchaikovsky and Wolf occur primarily in connective roles. In Musorgsky and Debussy, $P_{2,0}$ relations are used more colouristically. Successions of half-diminished chords are rare in the tonal repertoire.[13] Dominant sevenths occur more frequently in succession, as in the multiple repetitions of A♭^{7} and D^7 at the beginning of the Coronation Scene in Musorgsky's *Boris Godunov* referred to above (see again Ex. 1). This T_6 progression is sometimes regarded as an example of harmonic progressiveness in that it moves between two harmonies which are remote in terms of the cycle of fifths and which cannot belong to a single diatonic scale (but which can nonetheless belong to a single octatonic one). The tonic here is C; its importance is asserted partly by its role as one of the two common tones in a $P_{2c,0}$ relation and partly by emphasis and registral positioning.[14] The fact that one chord can be elegantly transformed into the other through the simple semitonal perturbation of two notes reveals their closeness from a parsimonious perspective, a relationship quite different to those routinely encountered in the context of functional harmony. Clearly Musorgsky heard a strong affinity between these harmonies, perhaps in the same way that some jazz musicians hear an affinity between a chord and its tritonal substitution.

This passage offers a straightforward demonstrations of the means by which neo-Riemannian analysis serves to answer Cohn's by now familiar question: 'If

Ex. 8 Debussy, *Prélude à l'après-midi d'un faune*, bars 1–5: $P_{2s,0}$ relation



this music is not fully coherent according to the principles of diatonic tonality, by what principles does it cohere?'[15] In a similar way, identification of a $P_{2s,0}$ relation is enlightening with regard to the first two chords in Debussy's *Prélude à l'après-midi d'un faune* (1894) (Ex. 8). The chord in bar 4 is an added sixth on C♯, synonymous with a half-diminished seventh on A♯; it is followed by B♭^{7}, thereby creating an ambiguous, 'nonfunctional' progression. However, a $P_{2s,0}$ relation exists between these two chords, a correspondence that plainly compensates for the apparent lack of 'functional' coherence. This interpretation balances approaches which, taking a lead from the orchestration, might describe the passage as 'colouristic', emphasising the distinctiveness of the chords rather than the logic of the voice-leading operation between them. As with the Musorgsky excerpt reproduced in Ex. 1, this $P_{2s,0}$ relation involves chords within a single power tower; both harmonies are therefore subsets of the same octatonic scale. The use of such a progression at this point in Debussy's creative development links him with Wagner, from whom Lewin's own examples of DOUTH2 progressions are drawn.[16] As Anthony Pople has pointed out,

> Debussy was thoroughly familiar with one of the classic harmonic gambits of *Tristan und Isolde* – involving chromatic voice-leading between half-diminished and dominant-quality chords and the occasional use of enharmonic changes to send such motion in unexpected directions. A little later in Debussy's career, in

Ex. 9 PLR transformations (with aggregate sets)

> his work from *Pelléas et Mélisande* (1902) onward, his reliance on this type of chromaticism decreased, and he began to emphasise new modes such as the whole-tone, acoustic and octatonic collections, bound together with the characteristic dominant-quality sonorities that are easily associated with these collections, and also with diatonic melodic fragments that give a veneer of traditional tonality to much of the music. Parallel motion between chords also offered something quite distinct from Wagnerian precedent.[17]

Two questions arise from the exploration of $P_{2,0}$ and $P_{0,1}$ relations between forms of 4–27 exemplified above. Why is this set class able to form these relations, and how common is this property among tetrachords and set classes of other sizes? It is feasible to suggest some relatively simple answers. In the first instance, the $P_{m,n}$ relations within set class 4–27 relate closely to the PLR group of transformations that may be effected on triads as described by Riemann (Ex. 9). He noticed that in each of these progressions two voices can be held while the remaining one is moved by a whole tone or a semitone. In the process the triad inverts: major becomes minor, and the voice-leading is maximally parsimonious.[18] (The aggregate sets formed by the union of the original and transformed triads shown in Ex. 9 will be of interest later in this article.) Furthermore, as Cohn and others have observed, the triad is unique among trichords in its ability to transform in this way:

> (1) Among mod-12 trichords, the consonant triad alone is susceptible to parsimonious voice-leading under the three PLR-family operations; (2) This circumstance is a function of the trichord's step-interval sizes which are an aspect of its internal structure; (3) the optimal voice-leading properties of triads therefore stand in incidental relation to their optimal acoustic properties.[19]

In other words, the PLR transformations of the triad encapsulate the fundamental properties of the tonal system in that the latter favours interval classes 3, 4 and 5 as intervals of simultaneity (reflecting the structure of the harmonic series), but favours 1 and 2 as intervals of voice-leading (reflecting the natural desire of the voice for conjunct motion).[20]

The P and L operations are a subgroup of what Lewin dubs 'Cohn flips'. In a Cohn flip we may discard one pitch class in a set and replace it with an adjacent one, in this case a semitone away; in the process an inverted form of the original set is produced. The P and L operations fulfil this condition. The ability to transform a set into one of the same class through the semitonal motion of a single voice is limited to sets of odd cardinality, and the numbers of set classes with this capability are tiny. Within this group there is an even smaller family of sets which, like the triad, have the ability to transform in two different ways. Lewin calls these 'Cohn sets'.[21] In addition to singletons, this group comprises the principal tonal sets: 3–11 (the major/minor triad), 5–35 (the pentatonic scale), 7–35 (the major scale) and 9–11 (the complement of the major/minor triad). As a straightforward example of this double capacity for transformation, we can extend the example of the major scale referred to above. Not only can the C major scale transform into G major by replacing F with F♯, but it can also transform into the F major scale if B is replaced with B♭. In both cases the substitution of one pitch class with an adjacent one a semitone away takes place without a change in set class.[22]

In the above definition, 'adjacency' is taken to mean 'move to a note a semitone away', and the relations therefore take place within a modulo 12 context. Within a modulo 7 context such as the major scale, adjacency may allow motion both by a tone (as in the Wolf example above) and by a semitone. The R operation exists in just such a modulo 7 context. If the context is a diatonic scale, adjacency may be defined as motion by tone or semitone according to the starting scale step. For example, C minor may flip to E♭ major (C descends to B♭ by a whole tone) or to A♭ major (G ascends by semitone) within the modulo 7 context of C minor/E♭ major.

A simple, intuitive way of understanding the arithmetic of PLR family operations will now be given, for it provides a model that can be extended to encompass tetrachords. Self-evidently, if the particularities of the operative gamut are disregarded, parsimonious voice-leading adds 1 or 2 to, or subtracts 1 or 2 from, the pitch classes that constitute a set. The PLR family is a special group comprising the three interval class 1 or 2 perturbations of 3–11 that leave the set class unchanged (the remaining nine will produce a trichord other than 3–11). Set class 3–11 is unique in its capability to 'flip' in three ways: set classes 3–2, 3–5 and 3–7 have the capacity to make one semitonal flip; and classes 3–3 and 3–8 the capacity to make a single whole-tone flip. As can be seen from the left-hand side of Table 2, the operations P, L and R when

Table 2 PLR operations applied to C minor triad {0, 3, 7}

Pitch classes		Interval strings	
3–11:	{0, 3, 7}	<3 4 5>	C minor
	0, +1, 0	+1, –1, 0	
=3–11:	{0, 4, 7}	<4 3 5>	C major (P)
3–11	{0, 3, 7}	<3 4 5>	C minor
	0, 0, +1	0, +1, –1	
=3–11	{0, 3, 8}	<3 5 4>	A♭ major (L)
3–11	{0, 3, 7}	<3 4 5>	C minor
	–2, 0, 0	+2, 0, –2	
=3–11	{10, 3, 7}	<5 4 3>	E♭ major (R)

Table 3 $P_{2,0}$ relations among tetrachords

Set class	Prime	Interval string	$P_{2s,0}$ relations	$P_{2s,0}$ voice-leading operations	$P_{2c,0}$ relations	$P_{2c,0}$ voice-leading operations
4–2	[0, 1, 2, 4]	<1128>	2	(+1, 0, –1, 0)		
4–11	[0, 1, 3, 5]	<1227>	2	(+1, 0, –1, 0)		
4–14	[0, 2, 3, 7]	<2145>			1	(–1, +1, +1, –1)
4–Z15	[0, 1, 4, 6]	<1326>	2	(+1, 0, –1, 0)		
4–16	[0, 1, 5, 7]	<1425>	2	(+1, 0, –1, 0)		
			2	(0, +1, 0, –1)		
4–19	[0, 1, 4, 8]	<1344>	2	(0, +1, 0, –1)		
4–22	[0, 2, 4, 7]	<2235>	2	(+1, 0, –1, 0)		
4–26	[0, 3, 5, 8]	<3234>			1	(–1, +1, +1, –1)
4–27	[0, 2, 5, 8]	<2334>	2	(+1, 0, –1, 0)		
			2	(0, +1, 0, –1)		
			2	(+1, –1, +1, –1)		
4–27	[0, 2, 5, 8]	<2334>			1	(+1, +1, –1, –1)
					1	(+2, –1, 0, –1)
					1	(+1, 0, +1, –2)
4–Z29	[0, 1, 3, 7]	<1245>	2	(+1, –1, +1, –1)		

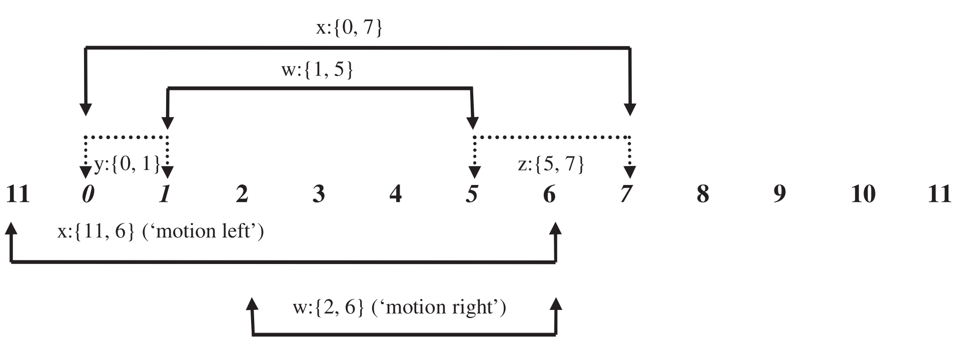
applied to the prime form of 3–11 result from the addition of 1 to 3 or 7, or from the subtraction of 2 from 0. Although PLR operations on triads may be understood as an addition to or subtraction from one of the integers that represent pitch classes, they may also be understood as operations carried out on interval strings (as shown on the right-hand side of Table 2). In the latter case, perturbation will necessarily affect two intervals: one interval will increase in size and, in compensation (because the string must sum to 12), another will shrink. For this to work, the string must contain pairs of adjacent entries that differ by one or two, a property unique to set class 3–11. It should also be noticed that each operation results in a retrograde rotation of the original string and hence an inverted form of the set.

Returning to tetrachords and $P_{2,0}$ relations, it is evident that a small number of sets rich in interval class 1 will transform in a trivial way such that their constituent pitch classes swap around and the set maps onto a replica of itself (as in the case of 4–1, where the ordered set {0, 1, 2, 3} can be 'transformed' to {0, 2, 1, 3} through the voice-leading operation (0, +1, –1, 0).[23] Once these trivial relations are eliminated, only nine of the 29 tetrachordal set classes exhibit the capacity to generate $P_{2,0}$ relations. Table 3 lists these set classes and also indicates the number of $P_{2s,0}$ and $P_{2c,0}$ relations for each. Clearly 4–27 takes the lead, but it is not alone in forming more than two such relations: 4–16 has the capacity to form four.

The potential a set class has to form a family of $P_{2s,0}$ relations may be assessed by inspecting its interval string. $P_{2s,0}$ relations are possible only when the interval string contains one pair of non-adjacent integers or two pairs of adjacent integers that differ by one. The flipping of these integers must result in a rotation of the original interval string. So, for example, through the addition of one to the first integer and the subtraction of one from the third, the interval string for 4–11 <1227> is subjected to a retrograde rotation <2217>, which will result in a $P_{2s,0}$ relation with the prime form of the set ([0, 1, 3, 5] transforming to {0, 2, 4, 5}). This can be described as a (+1, 0, –1, 0) voice-leading operation. The forms these voice-leading operations take are listed in the fifth and seventh columns of Table 3. In contrast to the transformation of 4–1 described above, the voice-leading vector is now applied to the members of the interval string and not to the constituent pitch classes (and is consequently shown in roman rather than italic type). The vectors are given with reference to the interval strings of the prime forms. If the interval strings are retrograded or rotated (as happens in T/I forms), the voice-leading operations will also be rotated. The interval strings of 4–27 <2334> and 4–Z29 <1245> each contain two pairs of adjacent integers that differ by one. $P_{2s,0}$ relations result from the flipping of both pairs simultaneously. Thus, in the case of 4–Z29, <1245> becomes <2154> and [0, 1, 3, 7] transforms to {0, 2, 3, 8}. In examining Ex. 7, it will be observed that $P_{2s,0}$ relations between tetrachords always result in inversion because the interval string always appears in retrograde. This phenomenon will be explored further later in this article.

As the fourth column of Table 3 shows, each similar-motion voice-leading operation has the potential to generate two $P_{2s,0}$ relations with the original set. Using prime forms as an example, the leftmost element in the voice-leading operation may enlarge the first interval, either by adding to the second pitch class, or by subtracting from the first. For example, the prime form of 4–Z15 [0, 1, 4, 6] may be transformed by the voice-leading operation (+1, 0, –1, 0) on its interval string <1326> to either {0, 2, 5, 6} or {11, 1, 4, 5}. Both are $P_{2s,0}$-related to the original set. In the first case, the first and third pitch classes are held and the other pair move; in the second, the situation is reversed. To help understand this property, Fig. 2 shows set class 4–16 [0, 1, 5, 7] arranged

Fig. 2 4–16 [0, 1, 5, 7] $P_{2s,0}$ relations arranged in one-dimensional pitch-class space



in a one-dimensional space comprising the twelve pitch classes. The pitch classes [0, 1, 5, 7] may be paired such that a smaller-interval dyad {1, 5} = w is contained within a larger-interval one {0, 7} = x. Imagine also two further dyads {0, 1} = y and {5, 7} = z that combine, respectively, the first and second pitch classes of the first two dyads. Assuming dyads y and z have interval classes that differ by one, then it is evident that if (a) dyad x is subjected to motion left by one semitone to {11, 6}, or (b) dyad w is subjected to motion right by one semitone to {2, 6}, then dyads y and z will be expanded or contracted and exchange interval classes. In the case of the prime form of 4–16, the first three integers of the interval string (that is, the intervals created by dyads y, w and z) are <142>; these will reverse in both cases to <241>. The one remaining integer required to complete the string must be 5 = 12 – (1 + 2 + 4). In every four-integer string, reversing three elements and leaving one fixed simply results in a rotated retrograde of the original: <1425> becomes <2415>. Thus each $P_{2s,0}$ relation will not only exist in two forms representing the motion right or left of the constituent dyads, but will also result in an inversion of the original set since the original interval string is subjected to retrogression and rotation. This quality is limited to tetrachords and their complements.

In contrast to the way in which $P_{2s,0}$ relations come in pairs, $P_{2c,0}$ relations exist in only one form on account of the fact that each is a compound of two similar-motion operations. We can take as an example set class 4–14. Its <2145> interval string is very similar to that of 4–Z29 in that it contains pairs of adjacent integers that differ by one. However, because the order of these integers is no longer ascending (in the first pair <21> the larger interval precedes the smaller), each member of this set class is capable of forming only one $P_{2c,0}$ relation. It will be noted from Table 3 that the voice-leading operation which produces the $P_{2c,0}$ relation is made up of two interlocking rotations of the similar-motion voice-leading operation (+1, 0, –1, 0). To arrive at the $P_{2c,0}$ relation, the pair of voice-leading motions can be separated: 4–14 [0, 2, 3, 7] <2145> subjected to the voice-leading operation (–1, 0, +1, 0) results in <1155> (= 4–6 [0, 1, 2, 7]), which, subjected to the voice-leading operation (0, +1, 0, –1), results in <1254> (= 4–14 {0, 1, 3, 8}). Set class 4–26 is similar in structure to 4–14; it is not able to form $P_{2s,0}$ relations, but is able to form a $P_{2c,0}$ relation since a rotation of its interval string may be engendered by interlocking two rotations of the voice-leading operation (+1, 0, –1, 0) to form (+1, +1, –1, –1). This $P_{2c,0}$ relation is equivalent to the transposition of 4–26 (0) to 4–26 (9). As is particularly apparent when considering hexachordal set classes, contrary motion is nearly always equivalent to transposition; 4–14 is a unique exception. The $P_{2s,0}$ relation with the prime form [0, 2, 3, 7] is 4–14 (I_3) {0, 1, 3, 8}. It will be apparent that the original and transformed sets can be aligned into a series of sum 3 dyads (necessary for inversion) since the two moving voices 2 → 1 and 7 → 8 both create sum 3 dyads, as do the two fixed pitch classes {0, 3}.

Table 4 4–27 [0, 2, 5, 8] $P_{2s,0}$ and $P_{2c,0}$ relations

	Prime	String	$P_{2s,0}$ Voice-leading	$P_{2c,0}$ Voice-leading	New string	Result
4–27	[0, 2, 5, 8]	<2334>	(+1, 0, −1, 0)		<3324>	{0, 3, 6, 8}
			(+1, 0, −1, 0)		<3324>	{11, 2, 5, 7}
			(+1, −1, +1, −1)		<3243>	{0, 3, 5, 9}
			(+1, −1, +1, −1)		<3243>	{11, 2, 4, 8}
			(0, +1, 0, −1)		<2433>	{0, 2, 6, 9}
			(0, +1, 0, −1)		<2433>	{11, 1, 5, 8}
			(+1, 0, −1, 0)		<3324>	{11, 2, 5, 7}
			plus (+1, −1, +1, −1) =	(+2, −1, 0, −1)	<4233>	{11, 3, 5, 8}
			(+1, −1, +1, −1)		<3243>	{11, 2, 4, 8}
			plus (0, +1, 0, −1) =	(+1, 0, +1, −2)	<3342>	{11, 2, 5, 9}
			(0, +1, 0, −1)		<2433>	{11, 1, 5, 8}
			plus (+1, 0, −1, 0) =	(+1, +1, −1, −1)	<3423>	{11, 2, 6, 8}

Table 5 Transformation of 4–27 {0, 2, 5, 8} into {2, 6, 8, 11} via 4–27 {11, 2, 5, 7}

4–27 {0, 2, 5, 8}		0	2	5			8
4–27 {11, 2, 5, 7} ($P_{2s,0}$ {0, 2, 5, 8})	11		2	5		7	
4–27 {11, 2, 6, 8} ($P_{2c,0}$ {0, 2, 5, 8})	11		2		6		8
4–28 {11, 2, 5, 8}	11		2	5			8

Only set class 4–27 is able to form $P_{2,0}$ relations by both similar and contrary motion; the complete family is shown in Table 4. Like 4–14 and 4–26, it must form $P_{2c,0}$ relations though compounds of $P_{2s,0}$ (although, unlike 4–14, the intermediate set is a form of 4–27 itself and the $P_{2c,0}$ relation results in transposition, not inversion). This set class has two pairs of adjacent and two pairs of non-adjacent integers that differ by one. This allows contrary motion to result from the following compounds: (+1, 0, −1, 0) and (0, +1, 0, −1) result in (+1, +1, −1, −1); (+1, 0, −1, 0) and (+1, −1, +1, −1) result in (+2, −1, 0, −1); and (0, +1, 0, −1) and (+1, −1, +1, −1) result in (+1, 0, +1, −2). Table 5 illustrates one of these compound relations whereby 4–27 [0, 2, 5, 8] transforms to {11, 2, 6, 8} via {11, 2, 5, 7}. The special qualities of 4–27 arise in part from the fact that the members of this set class are nearly symmetrical: each set in Table 5 is in a $P_{1,0}$ relation with the symmetrical diminished seventh 4–28 {2, 5, 8, 11} given in the lowest row of the table. This reflects a tendency, described succinctly by Tymoczko, for the favoured chords of Western music to be those that divide the octave – or (in the case of the triad) part of the octave – almost symmetrically. It is these chords that, like the dominant seventh,

Table 6 Tetrachords capable of $P_{0,1}$ relations

Set class	Interval string	Voice-leading operation	$P_{0,1}$ transformation	T_n/I
4–2 [0, 1, 2, 4]	<1128>	(+1, 0, −1, 0)	{0, 2, 3, 4}	I_4
4–5 [0, 1, 2, 6]	<1146>	(0, 0, +2, −2)	{0, 1, 2, 8}	I_2
4–13 [0, 1, 3, 6]	<1236>	(+2, 0, −2, 0)	{0, 3, 5, 6}	I_6
4–19 [0, 1, 4, 8]	<1344>	(+2, −2, 0, 0)	{0, 3, 4, 8}	I_4
4–19 [0, 1, 4, 8]	<1344>	(0, +1, 0, −1)	{11, 0, 4, 8}	I_0
4–22 [0, 2, 4, 7]	<2235>	(0, 0, +2, −2)	{0, 2, 4, 9}	I_4
4–27 [0, 2, 5, 8]	<2334>	(+2, 0, 0, −2)	{10, 2, 5, 8}	I_{10}

render themselves most open to efficient voice-leading as encapsulated by the $P_{m,n}$ relation.[24]

Set classes 4–2 and 4–19 are special in that one member of the pair of sets in a $P_{2s,0}$ relation resulting from a (0, +1, 0, −1) voice-leading operation will share three invariants with the original set rather than two. For example, 4–19 [0, 1, 4, 8] may be transformed to {0, 1, 5, 9} or {11, 0, 4, 8}. The latter set may be construed as a $P_{2s,0}$ relation ({0, 1} maps onto {11, 0}), or as a $P_{0,1}$ relation in which 1 skips over 0 to 11. Similarly, the transformation of 4–2 [0, 1, 2, 4] to {0, 2, 3, 4} may be seen as a $P_{2s,0}$ (+1, 0, −1, 0) voice-leading operation in which {1, 2} maps to {2, 3} or as a $P_{0,1}$ relation in which 1 moves to 3 skipping over 2.

Table 6 lists tetrachords capable of a $P_{0,1}$ relation ($P_{1,0}$ relations are not possible within tetrachordal set classes), including the two sets just mentioned. As is apparent from the third column, this kind of relation most commonly involves flipping two adjacent members of the interval string that differ by two. It may also arise in the manner described in the context of set classes 4–2 and 4–19 above and similarly by a (+2, 0, −2, 0) voice-leading operation, where one pitch class maps onto an existing one which itself then moves. In the case of 4–13, 1 maps to 3 and 3 maps to 5.

Fourteen tetrachordal set classes have the capacity to form at least one $P_{0,2}$ relation (Table 7). Each form of 4–18 has the capacity to form this relation with nine other members of its set class, and 4–24 has the capacity to form it with four (as with $P_{2,0}$ relations, each similar-motion voice-leading operation will have two forms depending on which dyad is held and which moves). Only 4–17 and 4–18 have the capacity to transform by contrary motion. Set class 4–18, whose string comprises a series of integers that increase incrementally by two with the middle term duplicated, parallels 4–27 in the $P_{2,0}$ family in that each member of this set class may form six $P_{0,2s}$ relations and three $P_{0,2c}$ relations. Set class 4–18 is a close relative of 4–27, being obtainable by a single semitonal perturbation of 4–27 (4–18 [0, 1, 4, 7] is in a $P_{1,0}$ relation with 4–27 {11, 1, 4, 7}).

Table 7 Tetrachords capable of $P_{0,2}$ relations

Set class and prime form	Interval string	$P_{0,2}$ operations*	$P_{0,2}$ transformations		Comment
4–1 [0, 1, 2, 3]	<1119>	(*w*+2, *x*, *y*+2, *z*)	{1, 2, 3, 4}	T_1	3 invariants
			{11, 0, 1, 2}	T_{11}	3 invariants
4–3 [0, 1, 3, 4]	<1218>	(*w*+2, *x*, *y*+2, *z*)	{1, 2, 4, 5}	T_1	
			{11, 0, 2, 3}	T_{11}	
4–7 [0, 1, 4, 5]	<1317>	(*w*+2, *x*, *y*+2, *z*)	{1, 2, 5, 6}	T_1	
			{11, 0, 3, 4}	T_{11}	
4–8 [0, 1, 5, 6]	<1416>	(*w*+2, *x*, *y*+2, *z*)	{1, 2, 6, 7}	T_1	
			{11, 0, 4, 5}	T_{11}	
4–9 [0, 1, 6, 7]	<1515>	(*w*+2, *x*, *y*+2, *z*)	{1, 2, 7, 8}	T_1	
			{11, 0, 5, 6}	T_5	
4–13 [0, 1, 3, 6]	<1236>	(+2, 0, –2, 0)	{0, 3, 5, 6}	I_6	3 invariants
			{10, 1, 3, 4}	I_4	
4–14 [0, 2, 3, 7]	<2145>	(+2, 0, –2, 0)	{0, 4, 5, 7}	I_7	
			{10, 2, 3, 5}	I_5	
4–17 [0, 3, 7, 8]	<3135>	(–2, +2, +2, –2)	{0, 1, 4, 9}	T_9	contrary
4–18 [0, 1, 4, 7]	<1335>	(+2, 0, –2, 0)	{0, 3, 6, 7}	I_7	
			{1, 4, 7, 8}	I_8	3 invariants
		(+2, –2, +2, –2)	{0, 3, 4, 9}	I_4	
			{10, 1, 2, 7}	I_2	
		(0, +2, 0, –2)	{0, 1, 6, 9}	I_1	
			{10, 11, 4, 7}	I_{11}	
		(+2, 0, +2, –4)	{10, 1, 4, 9}	T_9	contrary
		(+4, –2, 0, –2)	{10, 3, 4, 7}	T_3	contrary
		(+2, +2, –2, –2)	{10, 1, 6, 7}	T_6	contrary
4–20 [0, 1, 5, 8]	<1434>	(+2, 0, –2, 0)	{0, 3, 7, 8}	T_7	
			{10, 1, 5, 6}	T_5	
4–23 [0, 2, 5, 7]	<2325>	(0, +2, 0, –2)	{0, 2, 7, 9}	T_7	
			{10, 0, 5, 7}	T_5	3 invariants
4–24 [0, 2, 4, 8]	<2244>	(+2, 0, –2, 0)	{0, 4, 6, 8}	T_4	3 invariants
			{10, 2, 4, 6}	T_2	
		(0, +2, 0, –2)	{0, 2, 6, 10}	T_{10}	
			{10, 0, 4, 8}	T_8	
4–25 [0, 2, 6, 8]	<2424>	(+2, –2, +2, –2)	{0, 4, 6, 10}	T_4	
			{10, 2, 4, 8}	T_2	
4–26 [0, 3, 5, 8]	<3234>	(0, +2, 0, –2)	{0, 3, 7, 10}	T_7	
			{10, 1, 5, 8}	T_5	

*Italics indicate operations on pitch classes, normal type interval strings.

$P_{0,1}$ relations in which a voice skipped over a neighbour one semitone away were referred to above. This 'skipping-over' phenomenon is prevalent among $P_{0,2}$ relations. As Table 7 indicates, the first five sets all form $P_{0,2}$ relations by skipping over. Each of these sets is symmetrical and comprises two semitones separated by intervals that increase in size from interval class 1 (4–1) to interval class 5 (4–9). Their symmetrical construction means that in each case, a $P_{0,1}$ relation may be created either by the first and third notes being perturbed by two semitones, flipping across the second and fourth notes, or by the second and fourth flipping across the first and third in the opposite direction. Given that a tetrachord may be represented as four pitch classes (w, x, y, z), the voice-leading operation is shown as (w+2, x, y+2, z) in each instance, the italics indicating that this is an operation on pitch classes themselves rather than on the interval string. These relations remain predictable from the interval strings, which are all of the form <1, n, 1, (10–n)>, but $P_{0,2}$ relations do not result from flipping their elements. The potential for $P_{0,2}$ relations among the remaining tetrachords in Table 7 is entirely predictable from their interval vectors in the same way that it was with $P_{2,0}$ relations. For example, the<1434> interval string of set class 4–20 [0, 1, 5, 8] contains two non-adjacent entries that differ by two; a retrograde rotation is produced by the operation (+2, 0, –2, 0) and thus a $P_{0,2s}$ relation is possible. In the case of 4–17, two such rotations may be interlocked to form (–2, +2, +2, –2); the interval string is similarly rotated and hence a $P_{0,2c}$ relation is possible. In six other cases, three invariants are produced since $P_{0,2}$ maps a pitch class onto an existing one which is itself displaced onto another.

$P_{0,2}$ relations are, by the relatively crude measure of the total number of semitones perturbed (4), less parsimonious than $P_{2,0}$ relations (mixed $P_{1,1}$ relations may seem to offer a greater degree of parsimony than $P_{0,2}$ but are not possible between tetrachords without effecting a change of set class). However, the degree of parsimony arguably depends on the context in which it is found. So, for example, 4–25 [0, 2, 6, 8] is in a maximally parsimonious relationship with 4–25 {0, 4, 6, 10} if the context is the whole-tone scale 6–35 [0, 2, 4, 6, 8, 10]. Similarly, 4–18 [0, 1, 4, 7] will form a highly parsimonious $P_{0,2}$ relation with 4–18 {0, 3, 6, 7} within the octatonic hexachord 6–Z13 [0, 1, 3, 4, 6, 7]. Arguably, $P_{0,2}$ relations which take place between the steps of easily recognisable gamuts are heard to be more parsimonious than those which take place in collections that are simply formed from the union of the two sets.

Diminutions Between a Set and its $P_{m,n}$ Relation

The foregoing discussions of $P_{2,0}$ and $P_{0,2}$ relations assume that both voices move at the same time. However, by moving each voice in turn it is possible to interpolate a diminution between the two sets in a $P_{2,0}$ or $P_{0,2}$ relation, thus prolonging the transformation effected. In the process, a set which is different to the set class under transformation will emerge. Taking the nine different $P_{2,0}$

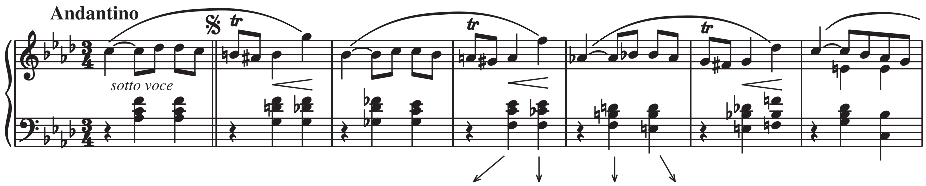
Table 8 'Diminutions' of $P_{2,0}$ relations within set class 4–27

4–27B transforms to 4–27A or B?	Root transformations	Diminutions
A	7	4–28, 4–23
A	10	4–28, 4–24
A	1	4–28, 4–20
A	0	4–26y, 4–25
A	9	4–26y, 4–26z
A	6	4–25, 4–26z
B	3	4–28, 4–26y
B	6	4–28, 4–25
B	9	4–28, 4–26z

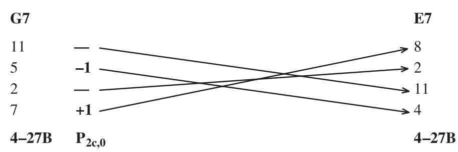
relations of which 4–27 is capable as an example, pairs of tetrachords are revealed by moving each voice individually (these are shown on the right-hand side of Table 8). The members of the pairs come from seven different classes, and each pairing is unique. Among these diminutions, 4–28, the diminished seventh, predominates with six appearances. This is unsurprising given that four different forms of the dominant seventh may be derived by perturbing each pitch class of any diminished seventh by a semitone. (For example, semitonal perturbations of 4–28 {1, 4, 7, 10} generate 4–27 {0, 4, 7, 10}, {1, 3, 7, 10}, {1, 4, 6, 10} and {1, 4, 7, 9}.) The other sets in the right-hand column of Table 8 all have clear roles in tonal and/or extended tonal music: 4–25 (French sixth); 4–20 (major triad with major seventh); 4–23 (dominant seventh with suspended fourth);[25] 4–26 (minor triad with minor seventh, which appears in two forms shown as y and z); and 4–24 (whole-tone tetrachord). This list contains seven of the eight set classes that may be derived by a single semitonal perturbation of 4–27A or 4–27B. The missing one is 4–18, a rather less tonal set than the others and, as noted above, the one with the highest number of $P_{0,2}$ relations.

The concept of another set class emerging as a diminution between the original and $P_{m,n}$-related versions of 4–27, can lead to interesting harmonic interpretations of nineteenth-century music. Chopin's Mazurka Op. 68 No. 4 provides an example (Ex. 10). In bars 4–5 an $F^{ø7}$ chord forms a $P_{2,0}$ relation with E^7 (a progression that anticipates Wagner's *Tristan*), but a diminished seventh (4–28) intervenes because the perturbations take place one after the other. This concept of diminution is somewhat different to the classic Schenkerian models, not least in prolonging a progression between two dissonant chords whose status in the foreground is the result of a relationship that is not dependent on scale step. However, the voice-leading operation has an obvious connection to one of the most familiar of diminutions – the suspension – in that one voice is held back in a progression, thus forming a verticality between the original and

Ex. 10 Chopin, Mazurka in F minor, Op. 68 No. 4, bars 1–7



F7	$P_{2s,0}$	$F^{ø}7$	$P_{1,0}$	(°7)	$P_{1,0}$	E7
9	**–1**	8	**0**	8	—	8
3	—	3	**–1**	2	—	2
0	**–1**	11	**0**	11	—	11
5	—	5	**0**	5	**–1**	5
4–27B		**4–27A**		**4–28**		**4–27B**



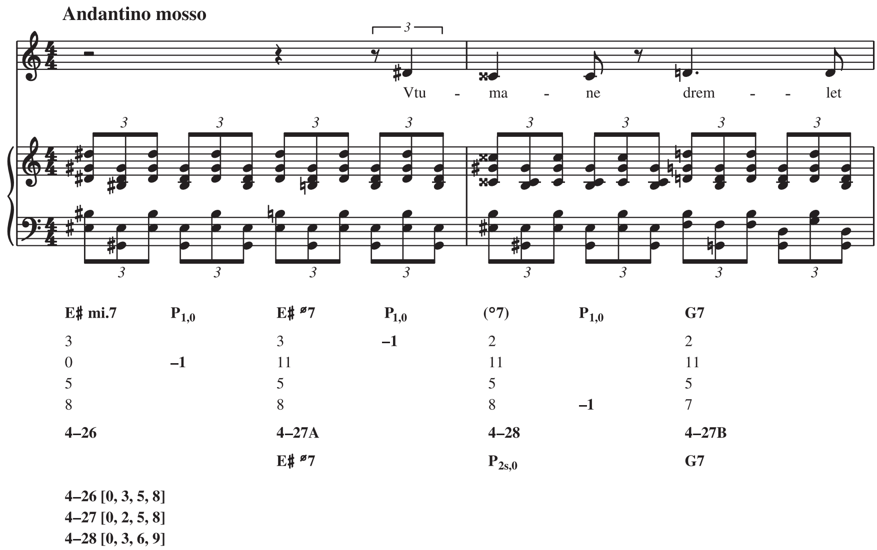
4–27 [0, 2, 5, 8]
4–28 [0, 3, 6, 9]

its goal that is of a different and distinct type. The progression described in this example is nested within a larger $P_{2,0}$ relation between G^7 and E^7 (as upper and lower neighbours encircling the tonic F). As with Ex. 5 above, it is once again evident that the foreground employment of parsimonious voice-leading within set class 4–27 is matched by its significance in the middleground, but here with the addition of foreground diminution.

The opening bars of 'Elegy', the fifth song in Musorgsky's *Sunless* cycle (Ex. 11), provides another example of this kind of process. Here the underlying tonality is uncertain, as though Musorgsky is trying to capture the misty darkness suggested by the opening words. This passage also reflects a tendency for coherence in Musorgsky's music to rest on parsimonious rather than functional progressions, often involving augmented sixths. Each chord progresses to the next by a single downward semitone. The first chord is E♯ minor with a minor seventh (set class 4–26 which, after 4–28, is the most frequently occurring set in the list of $P_{1,0}$ perturbations of 4–27 in Table 6 above). This opening chord may be regarded as a pre-perturbation of $E♯^{ø7}$, which shares pitch classes and similarities of layout with Wagner's *Tristan* chord.[26] $E♯^{ø7}$ progresses to G^7 halfway through bar 2 via a diminished seventh on E♯, another diminution. (Beyond the example, G^7 is followed by E major, which, had the seventh been included, would have formed yet another $P_{2,0}$ relation.)

The opening of Chopin's Prelude in E minor Op. 28 No. 4 provides a more extended example (Ex. 12). After the first bar, the harmonic language is almost entirely tetrachordal; seventh chords and $P_{2,0}$ relations, some with diminutions, are pervasive. Although it might be argued that the relations described below

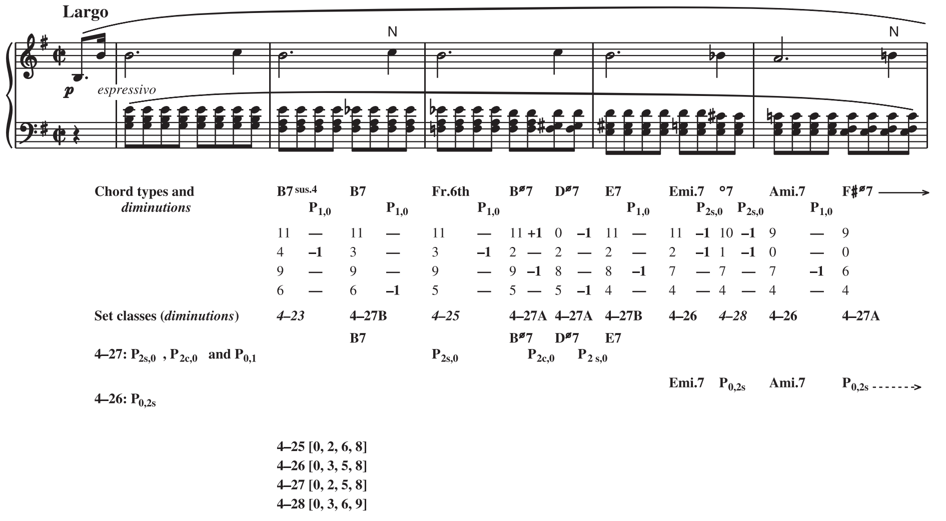
Ex. 11 Musorgsky, *Sunless*, 'Elegy', bars 1–2

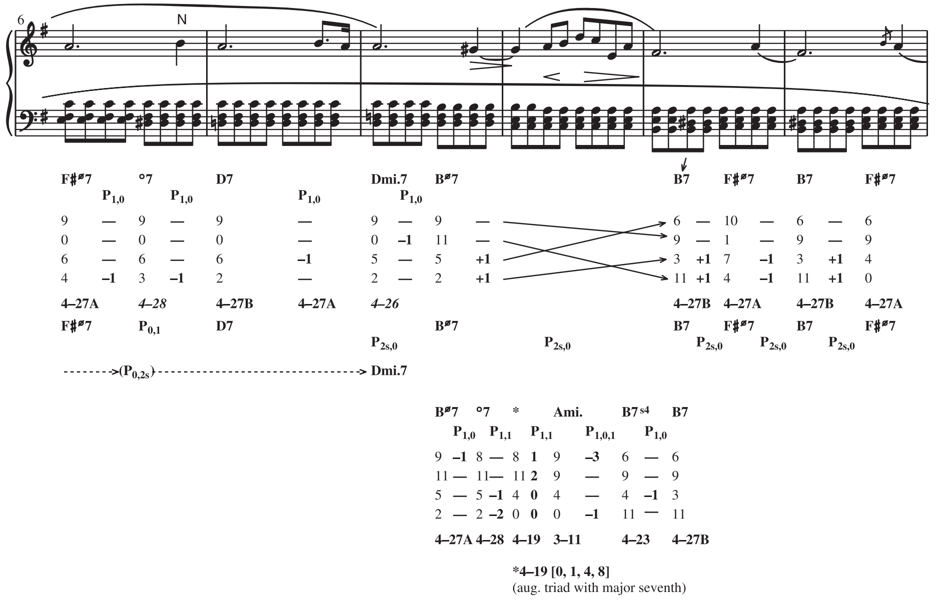


are simply the effect of suspensions, auxiliaries and passing notes, anyone who has attempted a more conventional Schenkerian analysis of this piece will recognise that due to the overlapping of the resolution of one dissonant event by the onset of another, the distinction between the 'essential' and the 'decorative' is often hard to make. The penultimate line of the analysis beneath the score in Ex. 12 indicates the presence of eight $P_{2,0}$ relations and one $P_{0,1}$ relation involving set class 4–27.

It will be noted that two of these $P_{2,0}$ relations involve diminutions: 4–25 (bar 3) and 4–26 (bar 8). The single $P_{0,1}$ relation (bars 6–7) involves the diminution 4–28 (that is, the whole-tone moves take place in two semitone steps). The forms of 4–27 which underlie much of this passage form an interlocking group of relations, shown in the central column of Table 9. It will be seen that the relations are mainly $P_{2s,0}$, but there is one instance (bar 3) of the much rarer $P_{2c,0}$ between two successive half-diminished sevenths. Not surprisingly for a work in E minor, F♯^{ø7} and B^7 are at the heart of this sequence, and the passage under examination ends with these two chords. To the right and left of the table are the diminutions. The important role of 4–25, 4–26 and 4–28 has been noted above; 4–23 prefaces B^7 at the beginning of bar 2 and in bar 10. The move to the neighbour note C in the right hand at the end of bar 2 creates 4–28 {F♯, A, C, E♭}, the partner of 4–23 in the first $P_{2,0}$ relation, albeit a little out of sequence. Of the diminutions in Table 9, only 4–24 is not used as a chord in this passage.

The prominence of 4–26 in this passage is particularly interesting. This set class is capable of both $P_{2c,0}$ and $P_{0,2s}$ relations, and, as was apparent in Table 8, Ex. 12 Chopin, Prelude in E minor, Op. 28 No. 4, bars 1–11





it is equal in importance to 4–28 as a diminution between $P_{2,0}$ relations within set class 4–27. As shown in the lowest line of the analysis, bars 4–5 contain a remarkable example of a $P_{0,2}$ relation involving 4–26, in which 4–28 acts as a diminution between the two forms, dividing the whole-tone motions between B and A and between D and C into whole-tone steps, while E and G are held.

Table 9 Summary of $P_{2,0}$ and $P_{0,1}$ transformations in Chopin, Prelude in E minor, Op. 28 No. 4, bars 1–11

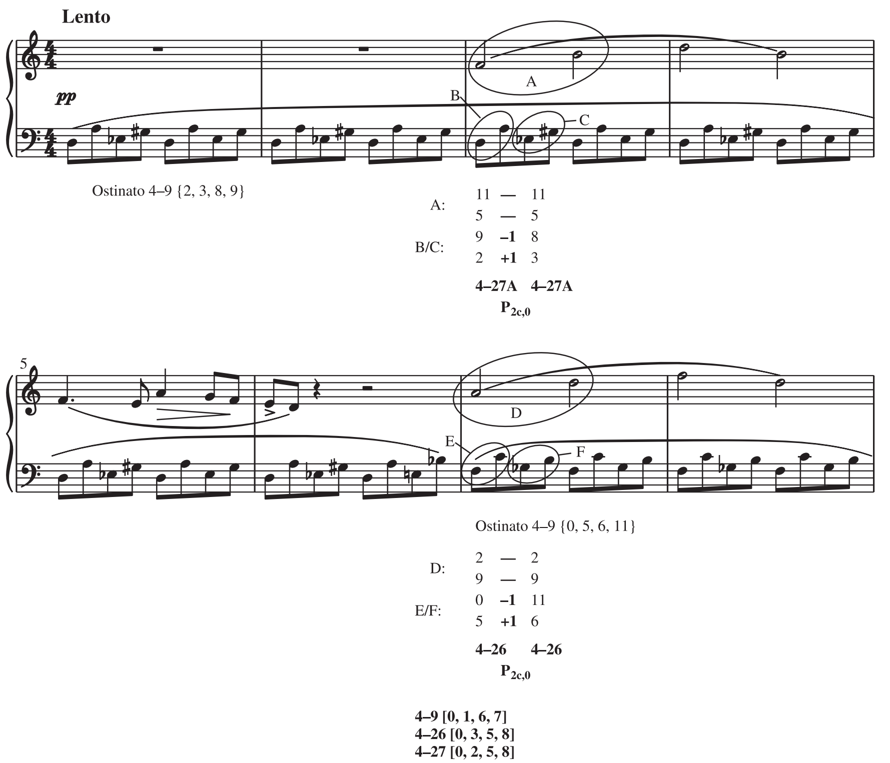
	D7 {F♯, A, C, D}	
	$\mathbf{P_{0,1}}$	4–28 {F♯, A, C, E♭}
	F♯$^{\varnothing}$7 {F♯, A, C, E}	
4–23 {E, F♯, A, B}	$\mathbf{P_{2s,0}}$	4–28 {F♯, A, C, E♭}
	B7 {B, D♯, F♯, A}	
4–25 {F, A, B, D♯} (Fr6)	$\mathbf{P_{2s,0}}$	4–26 {B, D, F♯, A}
	B$^{\varnothing}$7 {B, D, F, A}	
4–26 {D, F, A, C}	$\mathbf{P_{2c,0}}$	4–28 {B, D, F, A♭}
	D$^{\varnothing}$7 {D, F, A♭, C}	
4–24 {D, E, A♭, C}	$\mathbf{P_{2s,0}}$	4–28 {B, D, F, A♭}
	E7 {E, G♯, B, D}	

This 4–26 $P_{0,2}$ relation is nested between E^{7}and F♯$^{\varnothing 7}$, a progression achieved less parsimoniously by holding E in the bass while three voices move down by whole step. Much more speculatively, as shown in the lowest line of the example, an expanded $P_{0,2}$ relation between forms of 4–26 with a $P_{0,1}$ relation involving 4–28 as diminution nested within it may also be identified between the first chords of bars 5 and 8.

Before leaving this example, it is worth examining the progression between bars 8 and 10, where a more complex series of diminutions arises. The final chord of bar 8 is B$^{\varnothing 7}$ and the first of bar 10 is B^7; these two chords are in a $P_{2,0}$ relation. Between them come a series of diminutions which operate differently to the much simpler cases discussed above. In the registral positioning actually adopted (as opposed to the abstract $P_{2,0}$ relation just identified), Chopin's progression from a first inversion of B$^{\varnothing 7}$ to B^7 in root position is hardly parsimonious – in fact, it involves three voices moving down by a minor third and one by a whole tone. The effect is to create a kind of space or gap filled by parsimonious voice-leading between four intervening sets of different classes, here regarded as diminutions. (The sequence of $P_{m,n}$ relations, including the forms of 4–27 which frame box 7, is $P_{1,0}$, $P_{1,1}$, $P_{1,1}$, $P_{1,0,1}$, $P_{1,0}$, in which $P_{1,0,1}$ represents the perturbation of the upper voice by a minor third in the move from 3–11 to 4–23.) The sonority heard at the beginning of bar 9, the augmented triad with major seventh (4–19), is perhaps more familiar in hexatonic and atonal contexts, but here it has a prominent role, one clearly justified by the voice-leading.[27]

In the context of early twentieth-century music, set class 4–19 has a propensity to make appearances in passages of parsimonious voice-leading. Edward Gollin has demonstrated how the potential of this set class to form $P_{2,0}$ relations is variously exploited by Scriabin (Piano Sonata No. 8, 1912–13), Ravel (the

Ex. 13 Bartók, 'Painful Wrestling', *Ten Easy Piano Pieces*, bars 1–8



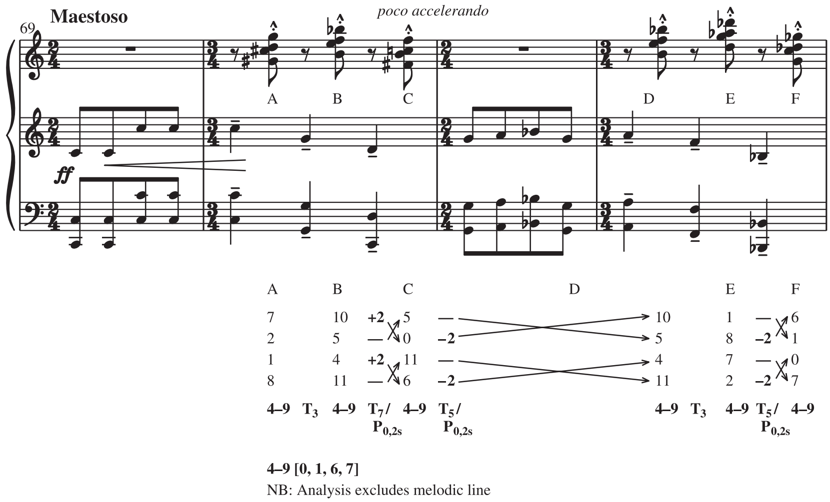
'Forlane' from *Le tombeau de Couperin*, 1914–17) and Bartók (*Bluebeard's Castle*, 1911).[28] 'Painful Wrestling', the second piece from Bartók's *Ten Easy Piano Pieces* (1908), has an interesting series of $P_{2,0}$ relations that bring together 4–9, 4–26 and 4–27 (Ex. 13). As indicated beneath the example, bars 3 and 4 may be regarded as a combination of the $P_{2c,0}$-related pair 4–27 {2, 5, 9, 11} and {3, 5, 8, 11}, with the common dyad (F–B) in the right hand and the dyads that belong only to one form of 4–27 (D–A and E♭–G♯) alternating in the left hand. The latter dyads form 4–9 {2, 3, 8, 9}, the ostinato arising from the semitonal perturbation of the opening fifth D–A. In bars 7–8, a $P_{2c,0}$ relation holds between 4–26 {0, 2, 5, 9} and {2, 6, 9, 11}. Again, the non-shared dyads form the ostinato, now 4–9 {0, 5, 6, 11}; the right hand in bar 7 begins with A–D, thus preserving the importance of this dyad. This analysis complements one based on inclusion, in which the instances of 4–9, 4–26 and 4–27 would be referred to underlying octatonic hexachords (6–30 {2, 3, 5, 8, 9, 11} and 6–Z50 {0, 2, 5, 6, 9, 11} in bars 3–4 and 6–7 respectively). These octatonic hexachords interpenetrate the D-Dorian scale suggested by the opening fifth and the predominance of white notes.[29]

Ex. 14 Bartók, 'Dedication', *Ten Easy Piano Pieces*, bars 1–8

Particularly notable here is the sharp distinction between the interval vectors of the important tetrachords: set class 4–9 (200022) contrasts with 4–27 (012111) ('R_0' in Forte's measures of similarity)[30] and with 4–26 (012120). Yet if the measure of distance is not similarity of interval vectors but voice-leading parsimony, then 4–27 and 4–9 are $P_{1,1}$ related (consider, for example, 4–27 {0, 4, 7, 10} and 4–9 {4, 5, 10, 11}). Remaining with the *Ten Easy Pieces*, the 'Dedication' which precedes the collection opens with four notes that slowly arpeggiate a D major triad with major seventh: a single semitonal perturbation of 4–27 to 4–20 (Ex. 14). The aggregate set in bars 5–8 is 6–Z29 {1, 4, 6, 7, 10, 11}, the union of a $P_{2,0}$ related pair: 4–27B {1, 4, 6, 10} (the F♯^{7} arpeggio in the right hand), and 4–27A {1, 4, 7, 11} (C♯^{ø7}, in which the two notes 'perturbed' from F♯^{7} are placed prominently in the left hand). The latter half-diminished chord is certainly less evident than the former; nevertheless the passage serves to highlight the way in which aspects of $P_{2,0}$ relations between forms of 4–27 may be evident in larger collections.

The music of Bartók contains some further examples of $P_{2,0}$ relations within tetrachordal set classes. In the eighth of his *Improvisations on Hungarian Peasant Songs*, Op. 20 (1920), the final statement of the melody is counterpointed by chords built exclusively from the six forms of 4–9 (the chord choice always excludes the melody note). The chords are initially deployed in groups of three, the first two of which are shown in Ex. 15. In the groups illustrated in this example and the one that follows, one pair of chords (labelled B–C and E–F in Ex. 15) are in a $P_{0,2s}$ relation, and the other pair (A–B and D–E) have mutually exclusive pitch content. Furthermore, the last chord in the first group and the first in the second (that is, chords C and D) are also in the relation $P_{0,2s}$. As the piece reaches its final climax (bars 76–80), there is a succession of nine chords, with all pairs generating $P_{0,2}$ relations. The voice-leading here

Ex. 15 Bartók, *Improvisations on Hungarian Peasant Songs*, Op. 20 No. 8, bars 69–73



is fractured because the tetrachords are transposed exactly and voices that may have progressed by step are displaced by an octave. However, Bartók retains pitch classes 5 and 11 – which are common to chords B, C and D in the same register – whereas between chords D and E he preserves the register of the moving voices but displaces the fixed ones. Despite these instances of registral disruption, the successive chordal pairs clearly divide into two groups. The first contains pairs with no common tones, whereas the second contains pairs which hold two pitch classes invariant while moving the others by interval class 2.

The opening of the 'Chorale' from Stravinsky's *Symphonies of Wind Instruments* (1920) is not obviously characterised by the kinds of parsimonious voice-leading discussed above (Ex. 16). Nonetheless, perturbations of both triad and dominant seventh do in fact underlie its harmonic syntax, a subject to which I shall return at the end of this article. Indeed, these sonorities might be regarded as background phenomena, never appearing unadorned but instead serving as the source of important surface detail. The 'Chorale' opens with 5–31 {2, 5, 7, 8, 11} and 6–Z29 {0, 2, 5, 7, 8, 11} in alternation. Set class 5–31 {2, 5, 7, 8, 11} represents the union of 4–27B {2, 5, 7, 11} and 4–28 {2, 5, 8, 11}, the latter tetrachord arising from the effects of diminution when pitch class 7 is perturbed to 8 in the course of the $P_{2c,0}$ relation between 4–27B {2, 5, 7, 11} and 4–27A [0, 2, 5, 8] (that is, G^7 and $D^{ø7}$). The following set class, 6–Z29 {0, 2, 5, 7, 8, 11}, represents the union of these two forms of 4–27. As shown in Ex. 16, the voice-leading which generates this hexachord can be interpreted as arising from the splitting in two of pitch class 7: it is retained at the same time as moving to pitch class 8 (a pitch class already present in the bass). Similarly, pitch class 11 moves to 0 in an upper voice but is held in the middle register.

Ex. 16 Stravinsky, 'Chorale', *Symphonies of Wind Instruments*, bars 1–4

2	(–2)	
		0
11	–1	(*11*)
7	+1	(*7*)
5	—	5
(*8*)		8
2	—	2
4–27B	$\mathbf{P_{2c,0}}$	**4–27A**
{2, 5, 7, 11}		**{0, 2, 5, 8}**
(*5–31 {2, 5, 7, 8, 11}*)		(*6–Z29 {0, 2, 5, 7, 8, 11}*)
(Union of 4–27B {2, 5, 7, 11} and 4–28 {2, 5, 8, 11})		(Union of 4–27B {2, 5, 7, 11} and 4–27A {0, 2, 5, 8})

4–27 [0, 2, 5, 8]
4–28 [0, 3, 6, 9]
5–31 [0, 1, 3, 6, 9]
6–Z29 [0, 1, 3, 6, 8, 9]

$P_{m,n}$ Relations Between Pentachords and Hexachords

$P_{m,n}$ relations exist within set classes of larger cardinality, and the potential of pentachords and hexachords to form these relations will now be considered. As shown in Table 10, of the 38 pentachordal set classes, 26 are capable of at least one of the six types of relation with which this article is concerned: $P_{1,0}$, $P_{2s,0}$, $P_{2c,0}$, $P_{0,1}$, $P_{0,2s}$ and $P_{0,2c}$. These types are indicated by asterisks together with the equivalent T_n and T_n/I operations and invariant set classes. As will be explored below, pentachords are rich in potential for $P_{1,0}$, $P_{0,1}$ and $P_{2s,0}$ relations but are notable both for the absence of $P_{2c,0}$ relations and for a single $P_{0,2c}$ relation. The vast majority of relations invert the original set because, as with tetrachords, the voice-leading operations result in a rotation of the retrograde of the interval string.

Thirteen pentachords have the capacity to engage in two or more relations. The most fecund sets are 5–21 and 5–35. The latter has two $P_{1,0}$ relations, resulting in transformations to T_5 and T_7, and two $P_{2s,0}$ relations, resulting in transformations to T_2 and T_{10}. In this respect, set class 5–35, the pentatonic scale, mirrors the behaviour of its complement 7–35, the diatonic scale. Single semitonal perturbations of the fourth and seventh degrees effect changes of key in either the sharp or flat direction (T_7 or T_5 transpositions respectively); similar-motion perturbations of the tonic and fourth degrees upwards effect a T_2 'sharp' transposition and perturbations of the third and seventh downwards, a T_{10} 'flat' transposition.

Table 10 Possible $P_{m,n}$ relations within pentachordal set classes

Set class	Interval string	Invariant set class	T_n/I	$P_{1,0}$	$P_{0,1}$	$P_{2s,0}$	$P_{0,2s}$	$P_{2c,0}$	$P_{0,2c}$
5–2 [0, 1, 2, 3, 5]	<11127>	3–1	I_4			★			
5–3 [0, 1, 2, 4, 5]	<11217>	4–7	I_5	★					
5–5 [0, 1, 2, 3, 7]	<11145>	4–1	I_3	★					
5–6 [0, 1, 2, 5, 6]	<11316>	4–8	I_6		★				
5–7 [0, 1, 2, 6, 7]	<11415>	4–6	I_2		★	★			
5–10 [0, 1, 3, 4, 6]	<12126>	3–10	I_6			★			
5–13 [0, 1, 2, 4, 8]	<11244>	3–1	I_2				★		
		4–24	I_4		★	★			
5–16 [0, 1, 3, 4, 7]	<12135>	4–3	I_4		★				
5–19 [0, 1, 3, 6, 7]	<12315>	4–9	I_7	★					
5–20 [0, 1, 3, 7, 8]	<12414>	4–20	I_3	★					
		4–8	I_8		★				
5–21 [0, 1, 4, 5, 8]	<13134>	3–12	I_4				★		
		3–12	I_8				★		
		4–7	I_5	★					
		4–20	I_1	★					
5–22 [0, 1, 4, 7, 8]	<13314>	3–12	T_8				★		
5–23 [0, 2, 3, 5, 7]	<21225>	4–23	I_7	★					
5–24 [0, 1, 3, 5, 7]	<12225>	3–6	I_6			★			
5–25 [0, 2, 3, 5, 8]	<21234>	4–10	I_5	★					
5–26 [0, 2, 4, 5, 8]	<22134>	3–12	I_8			★			
		3–10	I_{10}						★
5–27 [0, 1, 3, 5, 8]	<12234>	3–9	I_4			★			
		3–6	I_6				★		
5–28 [0, 2, 3, 6, 8]	<21324>	3–10	I_6				★		
	<23124>	4–25	I_8		★				
5–29 [0, 1, 3, 6, 8]	<12324>	3–9	I_2			★			
		3–9	I_4				★		
5–30 [0, 1, 4, 6, 8]	<13224>	4–24	I_0		★	★			
		3–9	I_2				★		
		3–12	I_8			★			
5–31 [0, 1, 3, 6, 9]	<12333>	4–28	I_6		★	★			
		4–28	I_9	★					
5–32 [0, 1, 4, 6, 9]	<13233>	4–17	I_4	★					
		3–10	I_9			★			
5–33 [0, 2, 4, 6, 8]	<22224>	4–21	T_2		★				
		4–21	T_{10}		★				
5–34 [0, 2, 4, 6, 9]	<22233>	3–6	T_2			★			
		3–6	T_{10}			★			
5–35 [0, 2, 4, 7, 9]	<22323>	3–9	T_2			★			
		3–9	T_{10}			★			
		4–23	T_5	★					
		4–23	T_7	★					
5–Z38 [0, 1, 2, 5, 8]	<11334>	3–1	I_2			★			

Table 11 $P_{m,n}$ voice-leading operations between pentachords

$P_{m,n}$ relation	Voice-leading operation	Number of instances
$P_{1,0}$	(+1, −1, 0, 0, 0)	12
$P_{0,1}$	(+2, −2, 0, 0, 0)	10 (4 shared with $P_{2s,0}$)
$P_{2s,0}$	(+1, −1, +1, −1, 0)	17 (3 shared with $P_{0,1}$)
	(+1, 0, −1, 0, 0)	
$P_{0,2s}$	(+2, 0, −2, 0, 0)	2
	(+2, −2, +2, −2, 0)	3
(skip over in one voice)	(+2, −2, +1, 0, −1)	2
(skip over in both voices)	(+1, −1, 0, 0, 0)	1
$P_{2c,0}$		0
$P_{0,2c}$ (skip over in one voice)	(+1, 0, −1, −2, +2)	1

Note: voice-leading operations are subject to rotation.

Set classes 5–21 (which also engages in four relations) and 5–30 (which engages in three) are significant because of their connections with larger symmetrical collections. $P_{1,0}$ relations between forms of 5–21 are possible within a single hexatonic gamut. Thus 5–21 [0, 1, 4, 5, 8] may transform to {0, 1, 4, 5, 9} by an adjacent step within 6–20 [0, 1, 4, 5, 8, 9], reflecting the way in which Riemann's 'P' and 'L' triadic transformations (for example, C+, C–, A♭+, G♯–) generate hexatonic collections.[31] Set class 5–30 [0, 1, 4, 6, 8] is a gapped segment of the 'even' whole-tone scale {0, 4, 6, 8} plus pitch class 1 from the 'odd' whole-tone scale. (Similarly, set class 5–34, notable for its two $P_{2s,0}$ relations, has a four-note symmetrical 'core' from the 'even' interval class 2 cycle: {0, 2, 4, 6} plus one pitch class from the 'odd' cycle.)

As with tetrachords, $P_{m,n}$ relations between pentachords are in most cases predictable from their interval strings. These must be capable of rotation when one of the voice-leading operations listed in Table 11 is applied (these may also be rotated). Interval strings of sets that produce $P_{1,0}$ relations must have two adjacent entries which differ by one; $P_{2s,0}$ relations require at least one pair of non-adjacent entries in the interval string to differ by one. The precise T_n/I relationship between pentachords in $P_{m,n}$ relations is determined by the T_n/I operator needed to map the invariant trichord or tetrachord onto itself (see column 3 of Table 10). For example, 5–10 [0, 1, 3, 4, 6] is in a $P_{2s,0}$ relation with 5–10 (I_6) {0, 2, 3, 5, 6}. The invariant set is 3–10 [0, 3, 6], which maps onto itself at I_6. (The interval string for 5–10 is <12126>, which retrogrades and rotates to <21216>.) Because invariant sets must map onto themselves, they are invariably symmetrical.[32]

With pentachords and hexachords the invariant subsets may be used to predict the ability of set classes to form $P_{m,n}$ relations, complementing our ability to predict these relationships from interval strings. Each of the five trichords that maps onto itself under one or more T_n/I operators will form an invariant subset within one or more pentachordal $P_{2,0}$ or $P_{0,2}$ relations. Furthermore, eleven of the fifteen tetrachords that map onto themselves will form an invariant subset within one or more pentachordal $P_{1,0}$ or $P_{0,1}$ relations. The ability of a symmetrical trichord to become the invariant element in a $P_{2,0}$ or $P_{0,2}$ relation may be evaluated in the manner indicated by the following illustration.

Set class 3–12 [0, 4, 8] maps onto itself at I_8 and may thus be arranged into two inversionally related, sum 8, interval-class 4 cycles:

3–12 (0):	0	4	8
3–12 (I_8):	8	4	0
Sum:	8	8	8

For this to be extended to form a $P_{2,0}$ or $P_{0,2}$ relation, two further pitch classes must be added to each form of 3–12. Each supplementary vertical pair must sum to 8 (otherwise the result will be two different pentachordal set classes), and it must also be possible to form pairs with a difference of one for a $P_{2s,0}$ relation, or two for a $P_{0,2s}$ relation. In the case of 3–12, it can be seen that the addition of 2 and 5 to 3–12 (0) and 3 and 6 to 3–12 (I_8) fulfils the conditions necessary for a $P_{2s,0}$ relation between two forms of 5–24:

5–24 (0):	0	4	8	2	5
5–24 (I_8):	8	4	0	6	3
Sum:	8	8	8	8	8

Notice that the added pitch classes do not continue the interval cycles formed by the symmetrical trichords (the resulting pentachords are rarely symmetrical) and that the potential semitonal voice-leading crosses over the sum 8 dyads (the sum 8 dyads are 2 + 6 and 5 + 3; the voice-leading is 2 → 3 and 5 → 6).

$P_{0,2s}$ relations operate in a similar way, and crossing over is again in evidence:

5–13 (2):	*0*	*1*	*2*	4	8
5–13 (I_2):	*2*	*1*	*0*	10	6
Sum:	2	2	2	2	2

(The invariant trichord, 3–1, is shown in italics.)

Self-evidently, crossing over allows the two dyadic sums to balance, a correspondence which enables a large number of possibilities for semitonal and whole-tone similar-motion voice-leading within same-sum dyads.

This crossing-over feature is not replicated by contrary motion, hence it becomes possible to understand why $P_{2c,0}$ relations are impossible between pentachords. The pairs of dyads which result from contrary motion by semitone will always have odd sums, yet all the possible invariant trichords have T_n/I operators that are even and therefore unable to combine with odd-sum dyads to form $P_{2c,0}$ relations. $P_{0,2c}$ relations are possible within pentachordal set classes because there is a small number (6) of dyadic pairs which have even sums, but given the very limited possibilities under which they occur (the dyads additional to the invariant trichord must come from interval class 4 cycles), it is not surprising that only a single instance, 5–26, exits:

5–26 (0):	*2*	*5*	*8*	6	10
5–26 (I_{10}):	*8*	*5*	*2*	4	0
Sum:	10	10	10	10	10

(The invariant trichord, 3–10, is shown in italics.)

In the case of $P_{1,0}$ and $P_{0,1}$, relations which hold a tetrachord invariant, the same principles apply, except that only a single dyad with the same sum as the invariant tetrachord mapped onto itself by inversion is needed. The two pitch classes in this additional dyad will differ either by one or two depending on whether the voice-leading is by semitone or whole tone. As an example, take the $P_{1,0}$ relation between 5–32 (I_4) {0, 3, 4, 7, 10} and 5–32 (3) {0, 3, 4, 7, 9}. The common subset is 4–17 [0, 3, 4, 7]. This tetrachord maps onto itself at I_7; the remaining dyad required to produce a $P_{1,0}$ relation between two forms of 5–32 must sum to 7 and its constituent pitch classes must differ by one, as is the case with 10 → 9 below:

5–32 (0):	*0*	*3*	*4*	*7*	10
5–32 (3):	*7*	*4*	*3*	*0*	9
Sum:	7	7	7	7	7

(The invariant tetrachord, 4–17, is shown in italics.)

The reasons for the substantial number of $P_{1,0}$ and $P_{0,1}$ relations should be apparent from the foregoing discussion. Fifteen tetrachords are capable of mapping onto themselves, three in multiple ways. All but one of these set classes can form an invariant set class within a $P_{1,0}$ and $P_{0,1}$ relation between pentachords. The sole exception is 4–26, the prime form of which maps onto itself at I_8. Because it is impossible to divide 8 into two integers that differ by one or two other than 3 and 5 (which are already part of the prime form of the invariant set) no $P_{1,0}$ or $P_{0,1}$ relation is possible. In comparison, only five trichords are capable of mapping onto themselves under T_n/I, and only one trichord (3–12) can do this in more than one way. Pentachordal $P_{2s,0}$ and $P_{0,2s}$ relations exploit this short list to the full.

Returning to interval strings and to Tables 10 and 11, it will be noted that there are three single whole-tone perturbations that are also compounds of semitonal perturbations. For example, in the case of 5–7 [0, 1, 2, 6, 7], the simultaneous semitonal similar-motion perturbations of 6 → 7 and 7 → 8 have the same result as a $P_{0,1}$ relation, in which 6 skips over 7 to 8. There are also three distinctive whole-tone similar- or contrary-motion relations in which one voice skips over and the other does not. In the case of 5–22 [0, 1, 4, 7, 8], similar-motion perturbation results in 5–22 {0, 3, 4, 8, 9} (1 → 3 and 7 → 9,

Table 12 $P_{0,2,s}$ transformation of 5–21 [0, 1, 4, 5, 8] to 5–21 {11, 0, 3, 4, 8}

5–21 [0, 1, 4, 5, 8] to {11, 0, 3, 4, 8}	Interval string	Voice-leading operations
5–21 {0, 1, 4, 5, 8}	<13134>	
5–32 {11, 1, 4, 5, 8}	<23133>	(+1, 0, 0, 0, –1)
5–22 {11, 0, 4, 5, 8}	<14133>	(–1, +1, 0, 0, 0)
5–32 {11, 0, 3, 5, 8}	<13233>	(0, –1, +1, 0, 0)
5–21 {11, 0, 3, 4, 8}	<13143>	(0, 0, –1, +1, 0)
Net		(0, 0, 0, +1, –1)

skipping over 8); the voice-leading operation that produces this relation when applied to the interval string <13314> is (+2, –2, +1, 0, –1). This rather odd-looking operation is a compound of (+2, –2, 0, 0, 0), which produces the single whole-tone motion without skip-over, and (0, 0, 0, +1, 0, –1), which produces semitone similar motion; the motion in this case has the same effect as a whole-tone skip-over. In the case of the $P_{0,2s}$ relation between 5–21 [0, 1, 4, 5, 8] and 5–21 {11, 0, 3, 4, 8}, two voices skip over: 1 crosses 0 to 11 and 5 crosses 4 to 3, yet the operation is effected by a voice-leading operation (0, 0, 0, +1, –1) normally associated with a single semitone relation. As indicated in Table 12, in order to understand this correspondence, each skip-over must be viewed as the compound of two semitonal motions; thus there are four intermediate stages in this relation, several of which cancel each other out, thereby leaving a net operation of (0, 0, 0, +1, –1).

There is a strong connection between the pentachords listed in Table 10 and extended tonal music. For example, the four pentachords in the list of what Pople terms 'Standard Chords' are all pentachords found in Table 10.[33] Furthermore, they are all capable of sustaining more than one $P_{m,n}$ relation:

Dominant major ninth	(G, A, B, D, F)	5–34
Dominant minor ninth	(G, A, B, D, F)	5–31
Major added ninth	(C, D, E, G, A)	5–35
Minor ninth	(D, E, F, A, C)	5–27

A progression between two dominant major ninths (5–34) on roots a whole tone apart will form a $P_{2,0}$ relation. An example is bars 27–29 of Ravel's piano piece *A la manière de . . . Chabrier* (1913), where D^9 is followed by E^9.

Altogether, 32 of the 50 hexachordal set classes are capable of forming $P_{m,n}$ relations. These sets are listed in Table 13, with the types of relation indicated by asterisks. The equivalent T_n/I operations are given together with the set classes of the invariant tetrachords and pentachords. Six hexachords can form two relations and four can form three relations. However, the most fecund set classes, with the potential to form four or more relations, are 6–21 and 6–27 (4),

Table 13 Possible $P_{m,n}$ relations within hexachordal set classes

Set class	Interval string	Invariant set class	T_n/I	$P_{1,0}$	$P_{0,1}$	$P_{2s,0}$	$P_{0,2s}$	$P_{2c,0}$	$P_{0,2c}$	Comments
6–Z3	<111216>	5–Z12	I_6		⋆	⋆				
[0, 1, 2, 3, 5, 6]										
6–5	<111315>	4–9	I_7				⋆			
[0, 1, 2, 3, 6, 7]										
6–Z10	<121125>	4–17	I_7			⋆				
[0, 1, 3, 4, 5, 7]										
6–Z11	<112125>	4–23	I_7				⋆			
[0, 1, 2, 4, 5, 7]										
6–Z12	<112215>	4–9	I_7			⋆				
[0, 1, 2, 4, 6, 7]		4–21	I_6				⋆			double skip over
6–Z14	<121134>	4–7	I_5					⋆		
[0, 1, 3, 4, 5, 8]										
6–15	<112134>	4–24	I_4				⋆			double skip over
[0, 1, 2, 4, 5, 8]		4–3	I_6				⋆			
		4–7	I_5			⋆				
6–16	<131124>	4–8	I_6						⋆	
[0, 1, 4, 5, 6, 8]										
6–Z17	<112314>	4–6	I_2				⋆			
[0, 1, 2, 4, 7, 8]		4–9	I_3			⋆				
		4–9	T_6						⋆	
6–18	<113214>	4–6	I_2			⋆				
[0, 1, 2, 5, 7, 8]		4–9	I_3				⋆			
		4–23	I_7				⋆			double skip over
		4–8	I_8			⋆				
		4–9	T_6					⋆		
6–Z19	<121314>	4–7	I_{11}				⋆			single skip over
[0, 1, 3, 4, 7, 8]		4–19	T_4						⋆	double skip over
		4–19	T_8						⋆	double skip over
		4–20	I_3						⋆	double skip over
		5–Z17	I_4		⋆	⋆				five invariants
		4–17	I_7				⋆			double skip over
		5–22	I_8		⋆	⋆				five invariants
6–21	<211224>	4–21	T_{10}						⋆	single skip over
[0, 2, 3, 4, 6, 8]		5–8	I_6		⋆					
		5–33	I_8		⋆	⋆				
		4–21	T_2				⋆			single skip over
6–22	<112224>	4–21	T_2						⋆	single skip over
[0, 1, 2, 4, 6, 8]		4–21	T_{10}						⋆	single skip over
		5–15	I_2		⋆		⋆			five invariants
6–Z24	<121224>	5–Z17	I_4		⋆		⋆			five invariants
[0, 1, 3, 4, 6, 8]		4–10	I_7			⋆				
6–Z25	<122124>	5–Z12	I_6		⋆					
[0, 1, 3, 5, 6, 8]		4–26	I_8			⋆				
		4–10	I_{11}				⋆			

Table 13 *Continued*

Set class	Interval string	Invariant set class	T_n/I	$P_{1,0}$	$P_{0,1}$	$P_{2s,0}$	$P_{0,2s}$	$P_{2c,0}$	$P_{0,2c}$	Comments
6–27	<121233>	4–3	I_4			*				
[0, 1, 3, 4, 6, 9]		4–28	I_6			*				
		4–28	I_3				*			
		4–10	I_7				*			
6–Z29	<123213>	4–28	T_6					*		
[0, 1, 3, 6, 8, 9]										
6–30	<123123>	4–9	I_1			*				
[0, 1, 3, 6, 7, 9]										
6–31	<122313>	4–24	I_{10}				*			
[0, 1, 3, 5, 8, 9]		4–19	T_4					*		
		4–19	T_8			*				
		4–17	I_5			*				
		4–26	I_8			*				
		4–7	I_9			*				
6–32	<221223>	4–23	T_2					*		
[0, 2, 4, 5, 7, 9]	<212232>	4–23	T_{10}					*		
6–33	<212223>	4–22	T_2					*		
[0, 2, 3, 5, 7, 9]		4–22	T_{10}					*		
		4–10	I_5			*				
		4–23	I_7			*				
		4–23	I_9			*				
6–34	<122223>	4–21	T_2					*		
[0, 1, 3, 5, 7, 9]		4–24	T_4					*		
		4–25	T_6					*		
		4–24	T_8					*		
		4–21	T_{10}					*		
		5–34	I_0		*					
		4–24	I_2			*				
		4–25	I_4			*				
		4–24	I_6			*				
		4–21	I_8			*				
		5–33	I_{10}		*					
6–Z36	<111135>	5–1	I_4		*					
[0, 1, 2, 3, 4, 7]		4–1	I_5				*			
6–Z39	<211134>	4–1	I_7			*				
[0, 2, 3, 4, 5, 8]										
6–Z40	<111234>	4–1	I_3				*			
[0, 1, 2, 3, 5, 8]		4–6	I_4			*				
6–Z41	<111324>	4–6	I_3			*				
[0, 1, 2, 3, 6, 8]		4–6	I_4						*	
6–Z43	<113124>	4–8	I_6				*			
[0, 1, 2, 5, 6, 8]		4–25	I_8				*			
		4–7	I_7			*				

Table 13 *Continued*

Set class	Interval string	Invariant set class	T_n/I	$P_{1,0}$	$P_{0,1}$	$P_{2s,0}$	$P_{0,2s}$	$P_{2c,0}$	$P_{0,2c}$	Comments
6–Z44	<113133>	4–19	T_4						*	
[0, 1, 2, 5, 6, 9]		4–19	T_8						*	
		5–Z37	I_2		*					
		4–20	I_3				*			
		5–22	I_6		*					
		4–7	I_7				*			
		4–17	I_{11}				*			
6–Z46	<112233>	4–20	I_3			*				
[0, 1, 2, 4, 6, 9]										
6–Z47	<112323>	4–6	I_2			*				
[0, 1, 2, 4, 7, 9]		5–35	I_4		*	*				
		4–6	I_{11}				*			
6–Z49	<121323>	4–25	T_6						*	
[0, 1, 3, 4, 7, 9]										
6–Z50	<132123>	4–9	T_6			*				
[0, 1, 4, 6, 7, 9]										

6–18 and 6–33 (5), 6–31 (6), the Z-related pair 6–Z19/6–Z44 (7) and 6–34 (11). Although these set classes are similar to 4–27 and 5–35/7–35 in their fecundity, they differ, with the exception of 6–33 (a subset of 7–35), in being more characteristic of atonal music (the pair 6–Z19/6–Z44 is particularly common in atonal contexts). Set class 6–34 (Scriabin's 'mystic chord'), although a superset of 4–27, is not a subset of the diatonic collection; its significance will be explored below. In its prime form, 6–31 [0, 1, 3, 5, 8, 9] only differs from that of the hexatonic scale 6–20 [0, 1, 4, 5, 8, 9] in the displacement of a single pitch class (3) by a semitone, and similarly from the diatonic hexachord 6–35 {0, 1, 3, 5, 8, 10}, where pitch class 10 is displaced. Set class 6–31 is rich in potential for parsimonious voice-leading, being capable of one $P_{2c,0}$ relation, one $P_{0,2s}$ relation and four $P_{2s,0}$ relations.

The majority of $P_{m,n}$ relations between hexachords are equivalent to inversions, and the principles that determine the possibility of these relations are broadly the same as those discussed in relation to tetrachords and pentachords. All tetrachords and pentachords with the capacity to map onto themselves at some value of T_n/I are present as invariant subsets between the hexachords included in Table 13, while the operators listed here are in large part those which map the invariant tetrachords and pentachords onto themselves. Fifteen tetrachordal set classes have the ability to map onto themselves; all but three do so only under inversion. All ten pentachordal set classes which map onto themselves do so by inversion only; the preponderance of inversions in the list is therefore hardly surprising. Although there is a vast array of operators, I_7 occurs most frequently, reflecting the frequency with which 4–9, 4–17 and 4–23 occur in the list of tetrachords held invariant between hexachords in $P_{m,n}$ relations. As was the case in relations between pentachords, the dyads required in addition to the invariant set when invariance is the product of inversion must have the same sums when the original and transformed sets are aligned. Again, this makes possible a large number of similar-motion relations, but few by contrary motion. As will be examined below, contrary motion between hexachords is associated with transposition, where the additional dyads are no longer required to have the same sums.

In contrast to pentachords, a significant number (24) of $P_{m,n}$ relations between hexachords are equivalent to transpositions. In a small number of cases where these relationships involve 4–9, 4–25 and 4–28 as invariant set classes, this reflects the fact that the invariant set can map onto itself by transposition alone as well as by inversion. But in the vast majority this is not the case, and a relation arises because a tetrachordal set class is represented at two levels of transposition within the particular hexachord; this tetrachord will then be invariant when it is transposed by the operator required to map one form of the tetrachord onto the other. So, for example, 6–34 [0, 1, 3, 5, 7, 9] is the superset of 4–21 (T_1) {1, 3, 5, 7} and 4–21 (T_3) {3, 5, 7, 9}. Self-evidently, a T_2 transposition of 6–34 to {2, 3, 5, 7, 9, 11} will hold 4–21 (T_3) {3, 5, 7, 9} invariant.

Table 13 includes eighteen relationships of this type involving eight different hexachordal set classes (6–Z19/Z44, 6–21, 6–22, 6–31, 6–32, 6–33 and 6–34) and five invariant tetrachordal set classes (4–19, 4–21, 4–22, 4–23 and 4–24). All are equivalent to transpositions by even numbers, and the invariant tetrachords are not ones which have the ability to map onto themselves under transposition (although they nearly all have the capacity to do so under inversion, the exceptions being 4–19 and 4–22). The reason for the prevalence of T_{even} is that the invariant tetrachords and the hexachords of which they are subsets favour interval class 2 or interval class 4 cycles, as in the example of 6–34 and 4–21 above. This tetrachord and 4–24 [0, 2, 4, 8] are built entirely from the interval class 2 cycle; 4–19 is a superset of 3–12, the augmented triad (an interval class 4 cycle); 4–19 is invariant within $P_{m,n}$ relations between forms of 6–Z19 and 6–Z44, which maximise interval class 4 in their interval vector.

Transposition equates to rotating an unretrograded interval string, and although it is not universally the case among set classes (as was demonstrated by 4–14), contrary motion equates with transposition rather than inversion. In the T_2 $P_{2c,0}$ relation between 6–34 [0, 1, 3, 5, 7, 9] and 6–34 {2, 3, 5, 7, 9, 11}, which holds the 4–21 {3, 5, 7, 9} invariant described above, the contrary-motion voice-leading is $0 \rightarrow 11$ and $1 \rightarrow 2$, dyads of different sums. Pitch class 11 arises from the transposition of pitch class 9, a member of the original invariant set 4–21 (3) which is itself replaced by the transposition of pitch class 7 to 9:

6–34 (0):	1	*3*	*5*	*7*	*9*	0
6–34 (2):	*3*	*5*	*7*	*9*	11	2

This feature, in which a member of the original invariant set is transposed to another pitch class that may lie a semitone or whole tone away from a non-invariant member of the original set, makes possible a series of contrary-motion $P_{2c,0}$ and $P_{0,2c}$ relationships through transposition. For comparison, 4–21 {3, 5, 7, 9} will also map onto itself under inversion; however, this relation is linked to two different forms of 6–34 and to similar motion by semitone:

6–34 (2):	*3*	*5*	*7*	*9*	11	2
6–34 (I_{10}):	*9*	*7*	*5*	*3*	1	10
Sums:	*0*	*0*	*0*	*0*	0	0

The relationship between the interval string of the invariant tetrachord and its hexachordal supersets in a $P_{m,n}$ relation will now be considered. A tetrachord comprises a four-element interval string, the direction of which is unchanged by transposition. Any hexachordal superset will split one or more intervals within the tetrachord's interval cycle. Set class 4–21 has the interval string <2226>. If the transpositions of its superset 6–34, discussed above, are rotated such that they begin with the invariant tetrachord, then in 6–34 (0), the final 6 is split into <312>; in 6–34 (2) this is rotated to <231> (–1, +2, –1). (In terms of pitch classes, {9, 0, 1, 3} becomes {9, 11, 2, 3}.) The interval strings of the two forms of 6–34 proceed in the same direction (6–34 (0) <222312>; 6–34 (2) <222231>). By comparison, in the $P_{2s,0}$ relation between 6–34 (2) and 6–34 (I_{10}) that holds the same tetrachord invariant, interval 6 first appears as <231>, then as <123>; in other words, the split is retrograded, as is the string of the transformed hexachord. (The pitch classes concerned are {9, 10, 1, 3} and {9, 11, 2, 3}.)

In the above examples, one interval in the invariant tetrachord is split by two pitch classes. In the $P_{2c,0}$ relation between 6–31 (0) [0, 1, 3, 5, 8, 9] and 6–31 (4) {0, 1, 4, 5, 7, 9} the invariant set is 4–19 (I_1) {0, 1, 5, 9}. The latter set has the interval string <1443>. The first form of 6–31 splits this string in the following way: <14<22>4<31>3>; the second, <14<31>4<22>3> (where the small number strings split the preceding large number). Thus, as the smaller numbers indicate, the substring <2231> *rotates* to <3122> with a (+1, –1, –1, +1) voice-leading operation, and the interval string of the hexachord as a whole is also rotated: <122313> becomes <131223>.

As summarised in Table 13, the most frequently occurring type of voice-leading is by semitone similar motion; but as with tetrachords, $P_{1,0}$ relations are not possible. This is because all possible invariant pentachords map onto themselves at even values of I, whereas the dyad formed by the additional pitch classes required to complete the two $P_{m,n}$ related hexachords must, to achieve voice-leading by a semitone, be odd. The $P_{0,1}$ relation column in Table 13 indicates that, as with pentachords, single whole-tone and semitone similar-motion

Table 14 $P_{m,n}$ voice-leading operations between hexachords

$P_{m,n}$ relation (total)	Voice-leading operation	Number of instances
$P_{1,0}$ (0)		
$P_{0,1}$ (14)	(0, 0, 0, 0, +2, −2)	14 (5 shared with $P_{2s,0}$; 2 shared with $P_{0,2s}$)
$P_{2s,0}$ (34)	(0, +1, 0, −1, 0, 0)	16 (5 shared with $P_{0,1}$)
	(+1, −1, 0, +1, −1, 0)	7
	(0, 0, +1, −1, +1, −1)	11
$P_{0,2s}$ (25)	(0, +2, 0, −2, 0, 0)	11 (2 shared with $P_{0,1}$)
	(0, −2, +2, 0, −2, +2)	2
	(0, +2, −2, +2, −2, 0)	3
(skip over in one voice)	(0, +2, 0, −2, 0, 0)	1
	(+1, 0, −1, 0, −2, +2)	1
(skip over in two voices)	(+1, 0, 0, 0, −1, 0)	2
	(+1, 0, −1, +3, 0, −3)	1
	(+1, 0, −1, 0, +4, −4)	1
	(+1, +2, −2, −1, +3, −3)	1
	(+1, −1, +2, +1, −1, −2)	1
	(+2, −1, +1, −2, +3, −3)	1
$P_{2c,0}$ (13)	(+1, −1, 0, −1, +1, 0)	8
	(+1, −1, 0, 0, −1, +1)	4
	(0, 0, 0, −1, +2, −1)	1
$P_{0,2c}$ (12)	(0, −2, +2, 0, −2, +2)	3
	(0, 0, +2, −2, −2, +2)	3
(skip over in two voices)	(3, −1, +1, −2, +2, −3)	1
	(+2, −1, +3, −2, +1, −3)	1
	(+1, −1, +3, −2, +2, −3)	1
(skip over in one voice)	(−1, 0, +1, 0, +2, −2)	1
	(−1, −1, 0, −2, +1, +3)	1
	(+1, +1, 0, +2, −1, −3)	1

Note: The voice-leading operations may be rotated.

operations may have the same net result. For example, 6–Z3 [0, 1, 2, 3, 5, 6] may transform to {0, 1, 3, 4, 5, 6}, either by 2 mapping onto 3 and 3 onto 4, or through whole-tone voice-leading from 2 to 4. Such skipping over is common among hexachords.

Table 14 lists the voice-leading operations that result in $P_{m,n}$ relations between hexachords and the total number of instances. It will be noted that they now take a great variety of forms, particularly where whole-tone skip-overs are involved. Set class 6–Z19 is particularly rich in $P_{0,2c}$ relations which involve either one or two voices skipping over, as well as a $P_{0,1}$ relation that skips over. By contrast, 6–Z19's 'Z partner', 6–Z44, is capable of seven distinct relations by whole-tone voice-leading (two single, two whole-tone contrary and three whole-tone similar, all without skipping over), emphasising the fact that, despite their identical interval vectors, these set classes behave in very different ways. (The richness of non-crossing whole-tone relationships generated by 6–Z44 is predicable from its interval string: <113133>, in contrast to that for 6–Z19 <121314>.) As with pentachords, some voice-leading operations are deceptively simple; in practice they are compounds in the manner of 5–21 (see again Table 12). The $P_{0,2s}$ relations formed within set classes 6–Z11 and 6–Z43 by a simple (+1, 0, –1, 0, 0, 0) operation on their interval strings are an exact parallel with 5–21.

The six-note 'standard chords' identified by Pople are listed below:

Dominant ♯11	(G, A, B, C♯, D, F)	6–34
Dominant ♯11 with minor ninth	(G, A, B, C♯, D, F)	6–30
Dominant minor thirteenth	(G, A, B, D, E, F)	6–Z28
Whole-tone dominant ♯5	(G, A, B, C♯, D♯, F)	6–35
Diminished chord with added sixth	(B, D, F, G, A, B)	6–27
Diminished chord with added sixth	(as ♯5) (E, F♯, A, B, C, D)	6–27
Augmented ninth as 10	(G, B, B, D, E, F)	6–31
Whole-tone dominant	(G, A, B, D, E, F)	6–35

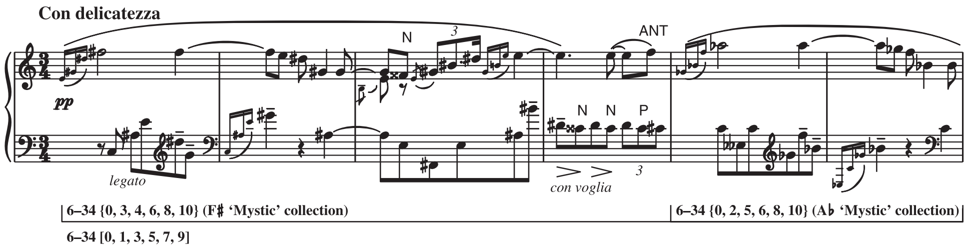
All of the above chords, except the whole-tone dominant chord 6–35, are capable of forming $P_{m,n}$ relations; the list contains 6–34, the dominant ♯11, identified above as the most fecund set. To this list should be added two seven-note set classes which are themselves capable of forming $P_{m,n}$ relations and which contain two of the six-note sets listed above:

Dominant major 13th (the acoustic scale)	(G, A, B, C♯, D, E, F)	7–34	contains	6–34 and 5–34
Dominant major 13th with minor ninth	(G, A, B, C♯, D, E, F)	7–31	contains	6–30 and 5–31

Set class 6–34 stands alongside 4–27 as one rich in possibilities for $P_{m,n}$ relations. Any form of this hexachord is capable of five $P_{2c,0}$ relations: T_2, T_4, T_6, T_8 and T_{10}, and six $P_{2s,0}$ relations: I_0, I_2, I_4, I_6, I_8 and I_{10}. It will be noticed that in each case the transpositions increase by increments of 2. Furthermore, within the group of $P_{2c,0}$ relations, each member is in a $P_{2s,0}$ relation with each of the others; within the $P_{2s,0}$ relations, each member of the group is in a $P_{2c,0}$ relation with each of the others. Moreover, each $P_{2c,0}$ relation is matched by a $P_{0,2c}$ relation. For example, [0, 1, 3, 5, 7, 9] is in both these relations with {11, 2, 3, 5, 7, 9}, depending on whether pitch class 0 moves to 11 and 1 to 2, or 1 moves to 11 and 0 to 2. Each $P_{2s,0}$ relation is matched by a $P_{0,1}$ relation; for example [0, 1, 3, 5, 7, 9] is in both a $P_{2s,0}$ and a $P_{0,1}$ relation with {0, 3, 5, 7, 9, 11}.

Set class 6–34's combination of $P_{m,n}$ relations and systematic transposition across the T_2 cycles provides a kind of synthetic tonality not unrelated to that based on 7–35. This potential has been most notably exploited by Scriabin. As Pople's analysis using his *Tonalities* software has demonstrated, Scriabin's *Feuillet*

Ex. 17 Scriabin, *Feuillet d'Album*, Op. 58, bars 1–6



d'Album, Op. 58, with the exception of a single crotchet beat, may be analysed as the prolongation of four forms of 6–34, the 'mystic chord', within overwhelmingly acoustic gamuts. The four forms are shown below, together with the two unused forms that complete the cycle of T_2 (F♯ is taken as 'tonic' because it begins and ends the work and is predominant within it):[34]

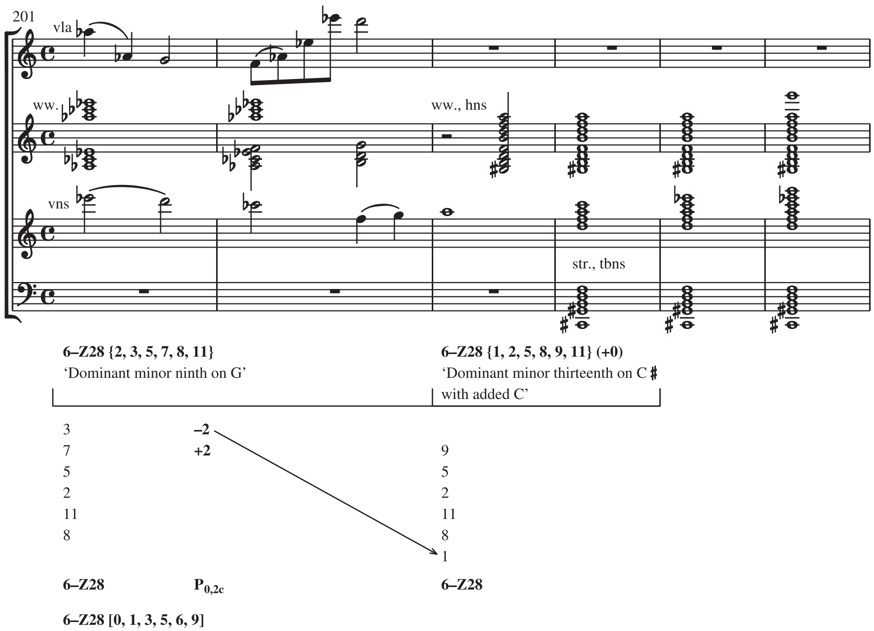
A. Mystic chord on F♯	6–34 {0, 3, 4, 6, 8, 10}	(0)
B. Mystic chord on A♭	6–34 {0, 2, 5, 6, 8, 10}	(2)
C. Mystic chord on E	6–34 {1, 2, 4, 6, 8, 10}	(10)
D. Mystic chord on C	6–34 {0, 2, 4, 6, 9, 10}	(6)
Not used:	6–34 {0, 2, 4, 7, 8, 10}	(4)
	6–34 {0, 2, 4, 6, 8, 11}	(8)

Ex. 17 shows the opening six bars of this piano piece. Bars 1–2 and 5–6 use only the F♯ and A♭ collections respectively. Bars 3–4 remain in the F♯ collection, but Scriabin introduces several notes that are arguably decorative (the neighbour-note roles of F$^{\times}$ and C$^{\times}$ are made clear by their notation) or, in the case of F♮, that anticipate the collection which follows.

Mahler's Tenth Symphony (1910) provides an example of 6–Z28 engaging in a $P_{0,2c}$ relation (Ex. 18). Again the analysis is produced by Pople's *Tonalities* software, which here identifies the chordal succession as 'Dominant minor on G' succeeded by a 'Dominant minor 13th on C♯ with added C'.[35] If, as the software suggests, the C is an added note, then the progression between the 'essential' chords is one between 6–Z28 {2, 3, 5, 7, 8, 11} and 6–Z28 {1, 2, 5, 8, 9, 11}, in which 3 and 7 move to 1 and 9 by contrary motion. The 7–9 motion is explicit in the violins; 3 is the highest note of the first chord, and 1 is the bass of the next.

This survey has demonstrated that the most familiar $P_{m,n}$ relation, the $P_{2,0}$ DOUTH2, is only the most often remarked-upon manifestation of a whole network of relations. $P_{2,0}$ relations of this type are easily found in nineteenth-century music, notably that of Chopin and Wagner, but, as has been made clear, such progressions represent only a small proportion of the potential progressions of this type, even between tetrachords. An intensive examination

Ex. 18 Mahler, Symphony No. 10, Adagio, bars 201–6



of the music of Bartók, Debussy, Ravel, Scriabin and others may well reveal a range of $P_{m,n}$ relations within a variety of set classes. However, as is well known, compositional priorities for many composers changed at the beginning of the twentieth century. A number of significant composers (not least the otherwise contrasting figures of Schoenberg and Debussy) moved from a preoccupation with parsimony and stepwise voice-leading between a relatively restricted range of third-based harmonic types to a concern with the similarities and differences between a much wider range of harmonic collections of varied cardinality. In the case of Schoenberg, these collections were atonal (and complementation became a more important principle in harmonic organisation than parsimony). In the case of Debussy (and, for that matter, Stravinsky), they were, most commonly, the octatonic, whole-tone, modal, pentatonic and chromatic scales. Such changes in musical language have for the most part been reflected in analytical tools such as pitch-class set theory and pitch-class set genera. Measures of inclusion, similarity and difference are the stock-in-trade of these modes of analysis rather than the study of degrees of perturbation, although valuable studies of atonal voice-leading have been made by Lewin and Joseph Straus in particular.[36]

There is evidence that although obvious $P_{m,n}$ relations between pitch-class sets of the same class will occur in the non-tonal repertoire, perturbation by a semitone or tone (or even the splitting of a tone, such that a single tone gives rise to two others a semitone or whole tone away) may also be a significant generator of relationships between sets of differing sizes and classes. Callender's study of the network of relationships between 6–34, 6–35, 6–Z49, 7–34, 7–31 and 8–28 in music by Scriabin is significant in this regard.[37] The kinds of relationship Callender explores include the means by which 6–35 and 6–Z49 may be produced from 6–34 through the perturbation of a single pitch class by a semitone. 7–35, a superset of 6–34, may also be produced from 6–35 by splitting a pitch class. Earlier in this article there was some discussion of the opening chords in the 'Chorale' from Stravinsky's *Symphonies of Wind Instruments*, in which I noted that the opening hexachord is the union of one form of 4–27 and its $P_{2,0}$ transform. In the 'Chorale', neither the triad nor the dominant seventh appears as an unadorned sonority, yet 3–11 and 4–27 arguably lie in the background (in the broad sense of the word). Not only do some of the chords have 3–11 and 4–27 as subsets, but nearly all of them may also be generated from the triad or dominant seventh through the perturbation of one or two pitch classes by a semitone or whole tone. Towards the end of the 'Chorale', 4–26 {2, 4, 7, 11} becomes prominent both as a chord in its own right, and as a subset of 5–27 {0, 2, 4, 7, 11} (the final chord). Set class 4–26 {2, 4, 7, 11} is a single semitonal perturbation of 4–27 {2, 5, 7, 11}, one of the two dominant sevenths that came together in the first chord, and of 4–27 {2, 4, 8, 11}. Set class 4–26 is also the union of a triad and its R transform (see again Ex. 9).

The idea that important sonorities arise from the union of chords and their $P_{m,n}$ relations was remarked upon in the context of the opening hexachord of the 'Chorale'; it receives further support in this passage in that the first chord used after the introductory bars is 4–17 (bar 7), the major-minor tetrachord, which, as shown in Ex. 9, is the union of a triad and its P transform (the chord first appears without its fifth, but subsequently emerges in complete form in bar 11). This is a chord frequently represented in the work of both Stravinsky and Bartók. It may also be generated by perturbing one note of a dominant seventh by a semitone and another by a whole tone. The chords which then follow in bars 7–9 may each be generated in a similar way through perturbation of 4–27 by a semitone and a whole tone, so each stands in a similar relation to a notional underlying dominant seventh.

Next to exact repetition, pitch-class proximity is the most important means of generating relationships between harmonic configurations. When present, $P_{m,n}$ relations create a very strong sense of identity between pitch-class sets. Even when set classes change, parsimonious voice-leading remains a significant force in creating harmonic coherence.[38] As has been shown, interval strings provide a useful way of understanding parsimonious voice-leading, as well as permitting easy identification of the capability of set classes to form $P_{1,0}$, $P_{0,1}$ and $P_{2,0}$ relations. However, where $P_{0,2}$ relations involve skipping over, these are much less easily predicted, and recognition of the T_n/I equivalents of the $P_{m,n}$ relations requires study of the behaviour of the invariant subsets. As was stated at the outset, understanding of the relationships studied here will be enriched not only through a clearer understanding of their place within a more generalised theory, but also through more analytical work focused on their deployment in the repertoire.

NOTES

The author wishes to acknowledge the invaluable assistance given by Steven Jan in developing the ideas in this paper.

1. The principal text on neo-Riemannian analysis remains the special issue of the *Journal of Music Theory*, 42/ii (1998).
2. Adrian P. Childs, 'Moving Beyond Neo-Riemannian Triads: Exploring a Transformational Model for Seventh Chords', *Journal of Music Theory*, 42/ii (1998), pp. 181–93; Richard Cohn, 'Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions', *Music Analysis*, 15/i (1996), pp. 9–40, and 'Neo-Riemannian Operations, Parsimonious Trichords, and Their "Tonnetz" Representations', *Journal of Music Theory*, 41/i (1997), pp. 1–66; Robert Gauldin, 'The DOUTH2 Relation as a Dramatic Signifier', *Music Analysis*, 20/ii (2001), pp. 179–92; and David Lewin, 'Cohn Functions', *Journal of Music Theory*, 40/ii (1996), pp. 181–216.
3. Examples may be found in the special issue of the *Journal of Music Theory*, 42/ii (1998). See also Clifton Callender, Ian Quinn and Dmitri Tymoczko, 'Generalized Chord Spaces' (http://music.princeton.edu/~dmitri/chordspaces.pdf, accessed 25 September 2007), for an overview.
4. Dimitri Tymoczko, 'The Geometry of Musical Chords', *Science*, 313 (2006), pp. 72–4; Callender, Quinn and Tymoczko, 'Generalized Chord Spaces'.
5. Callender, Quinn and Tymoczko, 'Generalized Chord Spaces', pp. 3–7.
6. Lewin, 'Cohn Functions', p. 207.
7. Jack Douthett and Peter Steinbach, 'Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition', *Journal of Music Theory*, 42/ii (1998), pp. 241–64.
8. Throughout this article the superscript '^{ø}7' indicates a half-diminished seventh, while '7' alone indicates a dominant seventh.
9. Childs, 'Moving Beyond Neo-Riemannian Triads', has also developed a useful set of identifiers for these relations which defines each in terms of the held and moving intervals. For example, S3(4) indicates that a minor third (ic 3) is held

while a major third (ic 4) moves in similar motion. C3(4) indicates that the same intervals are held and move, but in contrary motion. No pcs are shared between the intervals involved. However, Childs is only concerned with $P_{2,0}$ relations between forms of 4–27 [0, 2, 5, 8], whereas the investigation here includes $P_{0,1}$ and extends to sets of other cardinalities for which it would be difficult if not impossible to employ his system of classification.

10. See Childs, 'Moving Beyond Neo-Riemannian Triads'; Lewin, 'Cohn Functions'; and Gauldin, 'The DOUTH2 Relation'.

11. In these discussions an equivalence of I and T transformations has been assumed. It must be recognised, however, that half-diminished and dominant sevenths have distinctive voice-leading tendencies. See Richard Bass, 'Half-Diminished Functions and Transformations in Late-Romantic Music', *Music Theory Spectrum*, 23/i (2001), pp. 41–60.

12. Douthett and Steinbach, 'Parsimonious Graphs', p. 256. The figures placed in rectangular enclosures are the sums (modulo 12) of the pitch classes that constitute the seventh chords at each point.

13. Bass, 'Half-Diminished Functions', p. 48.

14. The octatonic collection 8–28 {0, 2, 3, 5, 6, 8, 9, 10} contains the two 'dominant seventh' chords, but caution must always be emphasised when speaking of octatonicism in Musorgsky, since this scale was never used overtly by him.

15. Richard Cohn, 'Introduction to Neo-Riemannian Theory: A Survey and Historical Perspective', *Journal of Music Theory*, 42/ii (1998), p. 169.

16. Lewin, 'Cohn Functions', pp. 205–9.

17. Anthony Pople, 'Styles and Languages at the Turn of the Century', in Jim Samson (ed.), *The Cambridge History of Nineteenth-Century Music* (Cambridge: Cambridge University Press, 2001), p. 608.

18. Brian Hyer has noted that in the *Musik-Lexikon*, Riemann records that he first used the terms 'dominant', 'parallel', 'relative' and 'Leittonwechsel' in *Harmony Simplified* (1895). See Hyer, 'Reimag(in)ing Riemann', *Journal of Music Theory*, 39/i (1995), p. 137, n. 10.

19. Cohn, 'Neo-Riemannian Operations', p. 5.

20. See Cohn, 'Maximally Smooth Cycles', pp. 12–13, and the extensive discussion of this issue in Cohn, 'Neo-Riemannian Operations'.

21. Lewin, 'Cohn Functions', pp. 181–4.

22. Cohn, 'Maximally Smooth Cycles', pp. 16–17, describes how chaining T_5 or T_7 transpositions of diatonic sets in this way (or alternatively chaining L and P transformations of triads) serves to constitute 'maximally smooth cycles'.

23. As this voice-leading operation represents additions to and subtractions from pitch classes, it is shown in italics in order to distinguish it from the operators representing additions to and subtractions from interval classes. The latter will play a much more significant role in this paper.

24. Tymoczko, 'Geometry of Musical Chords', p. 73.

25. See Ex. 7, bar 8, for examples of 4–20 and 4–23 as independent verticalities.

26. Lewin, 'Cohn Functions', pp. 205–8, explores the DOUTH2 properties of the *Tristan* chord; some of the relationships he considers are shared with this passage.

27. This tetrachord is also of course familiar as that formed by the 'And with his stripes' motive in Handel's *Messiah*.

28. Edward Gollin, 'Some Transformational Pathways into the Post-Tonal Frontier'; unpublished paper presented at the Mannes Seminar, New York in 2003.

29. Pitch classes 4, 7 and 10 in bars 5–6 may be regarded as a diatonic interaction with the octatonic scale at this point.

30. Allen Forte, *The Structure of Atonal Music* (New Haven, CT and London: Yale University Press, 1973), pp. 46–50.

31. See Cohn, 'Maximally Smooth Cycles', Fig. 5, p. 25.

32. In a number of cases invariant subsets appear transposed with respect to their prime forms when embedded within the prime form of a pentachord or hexachord which is capable of a $P_{m,n}$ relation. For this reason, the T_n/I operators listed in Table 10 do not always match those required to map the prime form of the invariant subset onto itself.

33. These chordal definitions are those used in the formulation of Anthony Pople's *Tonalities* analysis software. See his 'Using Complex Set Theory for Tonal Analysis: an Introduction to the *Tonalities* Project', *Music Analysis*, 23/ii–iii (2004), pp. 153–94.

34. See Clifton Callender, 'Voice-Leading Parsimony in the Music of Alexander Scriabin', *Journal of Music Theory*, 42/ii (1998), pp. 221–3, for a discussion of $P_{2,0}$ relations between T_{even}-related 'mystic' collections.

35. The analysis in terms of a progression between two hexachords is one found in the *Tonalities* files left after Pople's untimely death in 2003.

36. David Lewin, 'Some Ideas About Voice-Leading between PC Sets', *Journal of Music Theory*, 42/i (1998), pp. 15–72; Joseph N. Straus, 'Uniformity, Balance, and Smoothness in Atonal Voice-Leading', *Music Theory Spectrum*, 25/ii (2003), pp. 305–52.

37. Callender, 'Voice-Leading Parsimony', pp. 219–33.

38. On the psychological basis of the importance of pitch proximity, see Carol M. Krumhansl, 'Perceived Triad Distance: Evidence Supporting the Psychological Reality of Neo-Riemannian Transformation', *Journal of Music Theory*, 42/ii (1998), pp. 265–79.

ABSTRACT

It is well known (from Riemann, Lewin, Cohn and others) that two voices of a triad (set class 3–11) may be held and one moved 'parsimoniously' by a tone or semitone to produce another triad; or two voices of a dominant and half-diminished seventh (set class 4–27) may be held while two voices are displaced by a semitone to produce another dominant or half-diminished seventh (Lewin's

'DOUTH2' relation). This article takes 4–27 as its starting point and first explores examples of such parsimonious voice-leading between sets of the same class in nineteenth-century music (notably Brahms, Chopin and Musorgsky). The discussion introduces the idea of a diminution existing between forms of 4–27 when the two voices do not proceed simultaneously. The capability of tetrachords of other classes as well as set classes of larger cardinality to engage in parsimonious voice-leading either by semitone or whole tone is then explored (with examples drawn from the music of Bartók, Mahler, Scriabin and Stravinsky). The type and degree of parsimony are indicated by the $P_{m,n}$ designations developed by Douthett and Steinbach, and the article examines the ways in which the capacity of sets to behave in this manner may be predicted from interval strings and invariant subsets.