Interval Preservation in Group- and Graph-Theoretical Music Theories: A Comparative Study

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Interval preservation—wherein intervals remain unchanged among varying musical objects—is among the most basic means of manifesting coherence in musical structures. Music theorists since Milton Babbitt's (1960) seminal publication of "Twelve-Tone Invariants as Compositional Determinants" have examined and generalized situations in which interval preservation obtains. In the course of this investigation, two theoretical contexts have developed: the group-theoretical, as in David Lewin's (1987) Generalized Interval Systems; and the graph-theoretical, as in Henry Klumpenhouwer's (1991) K-net theory. Whereas the two approaches are integrally related—the latter's being particularly indebted to the former—they have also essential differences, particularly in regard to the way in which they describe interval preservation. Nevertheless, this point has escaped significant attention in the literature. The present study completes the comparison of these two methods, and, in doing so, reveals further-reaching implications of the theory of interval preservation to recent models of voice-leading and chord spaces (Cohn 2003, Straus 2005, Tymoczko 2005, among others), specifically where the incorporated chords have differing cardinalities and/or symmetrical properties.

In the group-theoretic approach, we associate an interval i with the action of a member of a group on a set. Then, we say that i is preserved if its conjugation by some operation h is trivial; that is, $i^h = h^{-1}ih = i$. As such, the set of all operations that commute with the members of a group, its centralizer, defines the collection of interval-preserving operations for that group. A canonical example of group-theoretic interval preserving operations is the action of the neo-Riemannian *Schritt/Wechsel* group on the set-class of consonant triads, which preserves intervals that derive from the usual *T/I* group of transposition and inversion operators (Clough 1998). As the action of the latter group on this set-class is regular, it is isomorphic to its centralizer. In particular, the *S/W* group may be generated by an order 12 *Schritt*, which moves major and minor triads equally in opposite directions, and any *Wechsel*.

We may generalize the structure of a centralizer in the group-theoretical approach as follows. First, we have the case of a group H with a transitive action.

Theorem 1. (Dixon and Mortimer, 1996, Let S be a set, x be a point in S, Sym(S) be the symmetric group S, H be a transitive subgroup of Sym(S), and C be the centralizer of H in Sym(S). Then:

1) *C is semiregular, and* $C \cong N_H(H_x)/H_x$*;*
2) *C is transitive if and only if H is regular;*
3) *if C is transitive, then it is conjugate to H in Sym(S) and hence C is regular;*

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4) *C = 1 if and only if H_x is self-normalizing in H (that is, $N_H(H_x) = H_x$);*
5) *if H is abelian, then C = H.*

Next, we examine the special case of a group *D* with an intransitive action, where *D* is a diagonal subgroup of the direct product of its orbit restrictions.

Theorem 2. *Put $D \leq Sym(S)$, and $C = C_{Sym(S)}D$. Further, let R be the set of n D-orbits, and assume that D is a diagonal subgroup of $D^{R_1} \times ... \times D^{R_n}$, where $R_i \in R$. Then, $C = C_{Sym(R_i)}D^{R_i}$ wr Sym(R), for any such $R_i \in R$.*

Finally, we have the generalized structure for an intransitive group *H*.

Theorem 3. Let G = Sym(S). Put H ≤ G, and $C = C_G H$. Further, let $P = \{P_1, ..., P_n\}$ be a partition of H-orbits into unions, such that H^{P_i} is a maximally embedded diagonal subgroup. Then, $C = C_{Sym(P_1)}H^{P_1} \times ... \times C_{Sym(P_n)}H^{P_n}$.

In the graph-theoretical approach, we identify musical objects with a graph's nodes, and intervals with directed arrows that connect those nodes, forming a network. Then, whenever we have some operation h on the nodes of a network, we have also a corresponding conjugation by h on the labels of its arrows; and, as we have observed, h is interval-preserving if $i^h = i$. For example, Figure 1a presents a network N_1, in which the nodes associate with consonant triads (i.e., members of set-class [0,3,7]), and arrows with intervals that derive from the T/I group. Then, as Figures 1b and c are transformations of N_1 by members of the S/W group (the Schritt S_7 and the Wechsel P [Parallel Exchange], respectively), they preserve the intervals of N_1.

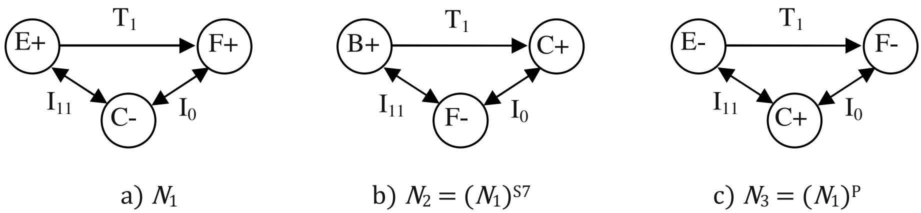
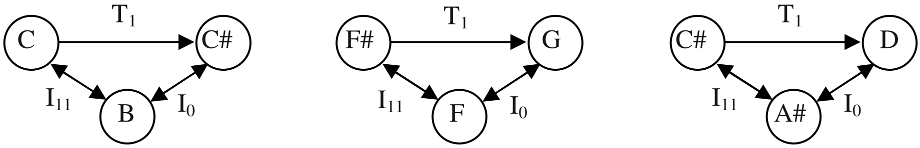


Fig. 1. Three networks with identical T/I-intervallic content (E+ = E major, C- = C minor, etc.)

Another situation exists for cases in which the action of the group of intervals is merely transitive, and herein lies an important distinction between the two approaches. Whereas a network *N* and its transform under a member of the centralizer will always have the same intervallic content, these are not the only networks that preserve those intervals. A canonical example appears in the K-classes of K-net theory (O'Donnell 1998), as illustrated in Figure 2. As the action of the *T/I* group is transitive on pitch-classes—but not simply transitive—its centralizer consists only of the center of the group: the subgroup generated by T_6 (Peck 2004). Hence, the K-nets in Figures 2a and b have the same intervals. Nevertheless, the entire K-class (consisting of all networks with identical intervallic content) for Figure 2a contains twelve networks. For instance, whereas Figure 2c does indeed contain the same intervals as the previous two, it does not obtain from either by a well-formed group-theoretical operation on pitch-classes: we cannot define a permutation on a set that sends all its members in opposite directions simultaneously. It derives rather via some quasi-*Schritt*, S'_1.



a) N_1 b) $N_2 = (N_1)^{T6}$ c) $N_3 = (N_1)^{S'1}$

Fig. 2. Three strongly isographic K-nets

The K-class structure is possible because the nodes of a network are defined by members of a group of intervals in relation to a given point, not by permutations of a set. Accordingly, each node in the network is unique, even if any two or more associate with the same member of the set, just as the members of the group are unique. Then, as Cayley's Theorem demonstrates,

Theorem 4. ("Cayley's Theorem") (Dixon and Mortimer, 1996, 6). *Every group G is isomorphic to a subgroup of the symmetric group on G.*

Any group G is isomorphic to a regular representation on itself. Hence, the centralizer of any Cayley representation is isomorphic to the representation, and we may define $|G|/|G_x|$ networks with the same intervals (where $|G|$ is the size of the group, and $|G_x|$ is the size of a point stabilizer in the group). In the example of a K-class, the *T/I* group contains twenty-four members, and each pitch-class x is stabilized by two members of the group (T_0 and I_{2x}); consequently, the K-class contains 24/2 = 12 members.

We may generalize these methods to groups with intransitive actions (i.e., those with more than one orbit). In the group-theoretical approach, the overall centralizer is a direct product of orbit centralizers, which, in the special case of a group with a diagonal action, may also be a wreath product that permutes the orbit centralizers (Dixon and Mortimer 1996, 109). A canonical example of this latter situation occurs in Uniform Triadic Transformations (Hook 2002), which preserve transpositional intervals among consonant triads. In the graph-theoretical approach, however, the resulting Cayley representation allows us to consider intervals in any network as deriving from a diagonal group, therefore always permitting permutations of constituent orbits. As such, for any intransitive network N with m connected components and n orbits, we may construct $n!/(n-m)! \cdot |G|/|G_x|$ networks with the same intervallic content. This structure enables us ultimately to describe interval-preserving operations among all pitch-class sets, regardless of their cardinalities and/or symmetrical properties, thus applying them to recent geometric models of all chords.

References

Babbitt, M.: Twelve-Tone Invariants as Compositional Determinants. The Musical Quarterly 46, 46–59 (1960)

Clough, J.: A Rudimentary Model for Contextual Transposition and Inversion. Journal of Music Theory 42(2), 297–306 (1998)

Cohn, R.: A Tetrahedral Graph of Tetrachordal Voice-Leading Space. Music Theory Online 9.4. (2003)

Dixon, J.D., Mortimer, B.: Permutation Group Theory. Springer, New York (1996)

Hook, J.: Uniform Triadic Transformations. Journal of Music Theory 46, 57–126 (2002)

Klumpenhouwer, H.: A Generalized Model of Voice-Leading for Atonal Music. Ph.D. diss., Harvard University (1991)

Lewin, D.: Generalized Music Intervals and Transformations. Yale University Press, New Haven (1987)

O'Donnell, S.: Klumpenhouwer Networks, Isography, and the Molecular Metaphor. Intégral 12, 53–79 (1998)

Peck, R.: Centers and Centralizers: Commutativity in Group-Theoretical Music Theory. In: Presentation to the 27th Annual Meeting of the Society for Music Theory, Seattle, Washington (2004)

Straus, J.N.: Atonal Pitch Space. In: Presentation to the 28th Annual Meeting of the Society for Music Theory, Cambridge, Massachusetts (2005)

Tymoczko, D.: A Map of All Chords. In: Presentation to the 28th Annual Meeting of the Society for Music Theory, Cambridge, Massachusetts (2005)