

## A Model for Pattern Perception with Musical Applications\*

### Part III: The Graph Embedding of Pitch Structures

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**Abstract.** This is the third paper of a series which begins by treating the perception of pitch relations in musical contexts and the perception of timbre and speech. The preceding papers dealt with those properties of musical scales which allow them to function as reference frames which provide both for the measurement of intervals and for the identification of their elements as scale degrees. The effect of these properties upon the perceptibility of various musical relations and properties has been discussed. Here we extend the treatment to systems of different scales (as exist in many musical cultures) where a listener's recognition of any one scale in the system interacts with his ability to recognize the others. Reading of the two previous papers is required.

#### 16. The Directed Graph, $G$

Thus far we have assumed that a listener has learned only one scale (measuring set) and will classify (measure intervals in) any stimulus (set of pitches) by embedding it in a key of this scale. Now we will consider the classification of stimuli by a listener who has learned *many* scales (e.g. a sophisticated Western listener or an Indian familiar with the enormous number of distinct "ragas" in use). That is, we have dealt with the problem of determining  $x$  in  $P_v(x)$  (Part 2, Sec. 14), given a set of points which is a subset of some  $P_v(x)$ . Now we consider the problem of determining both  $v$  and  $x$  in  $P_v(x)$  when the given points may be a subset of *any* of several given  $P_v(x)$ ,  $v = v_1, v_2, \dots$ .

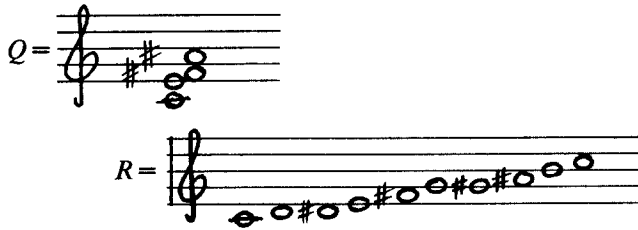
Note that  $P_u(x)$  may be a subset of  $P_v(y)$ ,  $u \neq v$ , where  $x$  may or may not equal  $y$ . Indeed, if we denote by  $P_1(0)$  that  $P_v(x)$  corresponding to all the points

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of  $S$ , then for all  $v, x, P_v(x) \subset P_1(0)$ . That is, the partial orderings of all  $P_v(x)$  by means of set inclusion, define a directed graph,  $G$ , in which connection indicates set inclusion. This graph consists of a Hasse diagram with  $S = P_1(0)$  at the top.

*Example 11.* Let  $m=12$ ,  $\psi(Q)=(4,2,4,2)$ ,  $\psi(R)=(2,1,1,2,1,1,2,1,1)$ ,  $Q = P_u(0)$ ,  $R = P_v(0)$  where 0 corresponds to C.



For what  $x, y$  is  $P_u(x)$  (the chord  $Q$  transposed to begin on the  $x$ th note, counting C as the 0th note) included in  $P_v(y)$  (the scale  $R$  transposed to begin on the  $y$ th note)?

$Q$  has 2-fold symmetry so we need only consider  $x=0,1,2,3,4,5$ ;  $P_u(6) = P_u(0)$  etc.

$R$  has 3-fold symmetry so we need only consider  $y=0,1,2,3$ ;  $P_v(4) = P_v(0)$  etc.

Then

$$P_u(0) \subset P_v(0), P_v(2)$$

$$P_u(1) \subset P_v(1), P_v(3)$$

$$P_u(2) \subset P_v(0), P_v(2)$$

$$P_u(3) \subset P_v(1), P_v(3)$$

$$P_u(4) \subset P_v(0), P_v(2)$$

$$P_u(5) \subset P_v(1), P_v(3)$$

If a line connecting 4 on the lower line to 0 on the upper line indicates that  $P_u(4) \subset P_v(0)$ , the above may be displayed as follows:

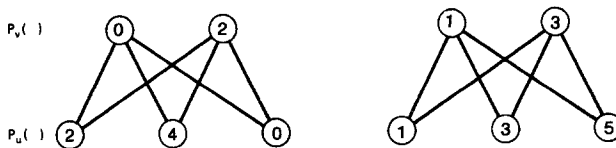


Fig. 1.

Figure 2 is a chart specifying the graph of all proper sets with stability  $\geq 2/3$  (where  $S$  is the integers and  $m=12$ ). In the digital computer program which computed these results, the first point of  $S$  was called  $s_0$  and hence  $s_m = s_{11}$ . (For musical interpretation,  $s_0 = C$ ,  $s_1 = C^\sharp$  or  $D^\flat$ ,  $s_2 = D$ , ... etc.) The chart is read as follows: All sets are listed according to the descending form  $\psi(P)$ ; inverse sets are adjacent and indicated by brackets; each entry,  $y$ ,

Fig. 2. Graph of all proper sets with  $S \geq 2/3$  (where  $S$  is the integers and  $m = 12$ )

| $P_u(x)$ | $v = 24$ | 23   | 22   | 21   | 20   | 19   | 18   | 17   | 16   | 15   | 14   | 13   | 12   | 11   | 10   | 9    | 8    | 7    | 6    | 5    | 4    | 3    | 2    | 1    | $P_v(x)$ |
|----------|----------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|----------|
| 33       | 5551     | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351 | 5351     |
| 32       | 552      | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543  | 543      |
| 31       | 546      | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536  | 536      |
| 30       | 444      | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444  | 444      |
| 29       | 651      | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641  | 641      |
| 28       | 645      | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635  | 635      |
| 27       | 634      | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624  | 624      |
| 26       | 633      | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623      |
| 25       | 633      | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623  | 623      |
| 24       | 5151     | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151 | 5151     |
| 23       | 4341     | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341 | 4341     |
| 22       | 4332     | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332 | 4332     |
| 21       | 4233     | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233 | 4233     |
| 20       | 4223     | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223     |
| 19       | 4223     | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223 | 4223     |
| 18       | 3333     | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333 | 3333     |
| 17       | 5241     | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241 | 5241     |
| 16       | 5142     | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142 | 5142     |
| 15       | 2322     | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322 | 2322     |
| 14       | 4321     | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321 | 4321     |
| 13       | 3313     | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313 | 3313     |
| 12       | 3322     | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322 | 3322     |
| 11       | 3131     | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131 | 3131     |
| 10       | 2222     | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222 | 2222     |
| 9        | 3222     | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222 | 3222     |
| 8        | 3122     | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122 | 3122     |
| 7        | 3212     | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212     |
| 6        | 3212     | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212 | 3212     |
| 5        | 2221     | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221     |
| 4        | 2221     | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221 | 2221     |
| 3        | 2121     | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121     |
| 2        | 2121     | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121 | 2121     |
| 1        | 1111     | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111 | 1111     |

Each entry,  $\gamma$ , indicates that  $P_u(\gamma) \subset P_v(0)$  where  $u$  is indicated by the row label and  $v$  by the column label. For example, the entry 10 in row 16, column 8, indicates that the only transposition of chord number 16 (i.e.,  $CFG^2B^2$ ) which

is included in scale number 8 on C (i.e.,  $CD^*EF^*G^*A^*$ ) is that transposition which has note number 10 ( $A^*$ ) for its bass note (i.e.,  $A^*, D^*, E^*, G^*$ ).

indicates that  $P_u(y) \subset P_v(0)$  where  $u$  is indicated by the row label and  $v$  by the column label. Note that because the periodic case is used,  $P_u(y) \subset P_v(0)$  implies that for all  $k$ ,  $P_u(y+k) \subset P_v(k)$  where the arguments of  $P_u$  and  $P_v$  are reduced modulo the periods of the respective scales.

An example of a diagram displaying a small portion of the above graph is:

$$\begin{aligned}\psi(P_1(x)) &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1); & \psi(P_3(x)) &= (2, 1, 2, 1, 2, 1, 2, 1); \\ \psi(P_{13}(x)) &= 3, 3, 1, 3, 2; & \psi(P_{21}(x)) &= (4, 2, 3, 3); & \psi(P_{31}(x)) &= (5, 3, 4).\end{aligned}$$

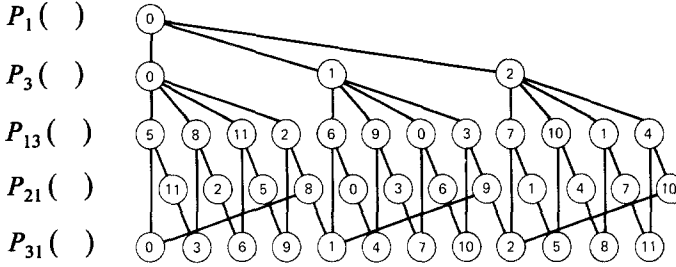


Fig. 3.

To draw a diagram displaying the entire chart would be cumbersome; hence, for convenience, a node on the diagram on the following page (Figure 4) will represent a collection of sets,  $\{P_u(x), x = 0, 1, \dots\}$ , for all permissible values of  $x$  (all keys of scale  $P_u(x)$ ). The numbers along the vertical connecting lines between nodes indicate those values of  $x$  for which  $P_u(x) \subset P_v(0)$  ( $P_u(x)$  appears below  $P_v(x)$ ). Since  $P_u(y+k) \subseteq P_v(k)$  there is no loss of information in this new representation. Inverses are indicated by dotted horizontal lines with arrowheads at each end. Only sets of cardinality five and greater are shown in this diagram.

## 17. Graph Equivalence<sup>1</sup>

Note that all subsets of  $S$  (including those not on graph  $G$ ) are subsets of at least one set on graph  $G$ .<sup>2</sup> Henceforth let a point of graph  $G$  be denoted by " $P_v(x)$ ", and let subsets of  $S$  which may or may not be on the graph be denoted by  $H$ . Then, given  $H$  one can determine all  $v$  and  $y$  for which  $H \subset P_v(y)$ .

To each  $H$  let there correspond a set:

$$V(H) = \{P_v(y) | H \subset P_v(y)\}.$$

Two sets,  $H_1$  and  $H_2$  will be called *graph equivalent* iff  $V(H_1) = V(H_2)$ .

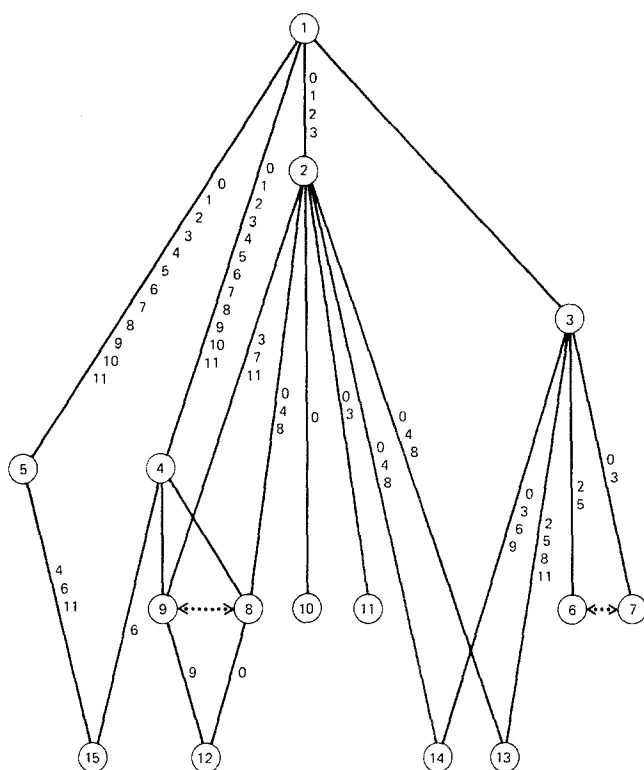
To each  $H$  let there correspond a number

$$I(H) = M - \text{card}(V(H))$$

where  $M$  is the total number of points on graph  $G$ .

<sup>1</sup>The following discussion applies to the directed graph,  $G$ , not to any of its abbreviated versions (as in Figure 3 and 4).

<sup>2</sup> $S \equiv P_1(0)$  is assumed to be on graph  $G$  (see above).



$$\psi(P_1(x)) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$$

$$\psi(P_2(x)) = (2, 1, 1, 2, 1, 1, 2, 1, 1)$$

$$\psi(P_3(x)) = (2, 1, 2, 1, 2, 1, 2, 1)$$

$$\psi(P_4(x)) = (2, 2, 2, 2, 1, 2, 1)$$

$$\psi(P_5(x)) = (2, 2, 2, 1, 2, 2, 1)$$

$$\psi(P_6(x)) = (3, 1, 2, 3, 1, 2)$$

$$\psi(P_7(x)) = (3, 2, 1, 3, 2, 1)$$

$$\psi(P_8(x)) = (3, 1, 2, 2, 2, 2)$$

$$\psi(P_9(x)) = (3, 2, 2, 2, 2, 1)$$

$$\psi(P_{10}(x)) = (2, 2, 2, 2, 2, 2)$$

$$\psi(P_{11}(x)) = (3, 1, 3, 1, 3, 1)$$

$$\psi(P_{12}(x)) = (3, 3, 2, 2, 2)$$

$$\psi(P_{13}(x)) = (3, 3, 1, 3, 2)$$

$$\psi(P_{14}(x)) = (3, 3, 2, 3, 1)$$

$$\psi(P_{15}(x)) = (3, 2, 3, 2, 2)$$

Fig. 4.

$I(H)$  will be called the information value of  $H$  with respect to graph  $G$ . It is equal to the number of points on graph  $G$  to which  $H$  does not belong.

Evidently  $I(H)$  is invariant under musical transposition of  $H$  in the above examples. This is because  $G$  is complete in the sense that  $P_v(y) \in G \rightarrow P_v(y') \in G$  for all  $v, y, y'$ .

Intuitively, information values count those points of graph  $G$  which need *not* be considered when classifying a given subset  $H$  of points in  $S$ . If all points of

graph  $G$  (with the possible exception of  $P_1(0) \equiv S$ )<sup>3</sup> are interpreted as learned “mental reference frames”, any of which may be used by a listener to measure (classify) the intervals of a “signal” or “stimulus” consisting of a string of musical tones,<sup>4</sup> then the information value  $I(H)$ , corresponds to the number of such “reference frames” which are eliminated as possibilities for classifying a “signal” when that portion of the “signal” specified by  $H$  has been heard.

Similarly, graph equivalent sets may be interpreted as different stimuli that can be classified by identical subsets of those “reference frames” which are known to a listener (clearly, graph  $G$  contains only those “reference frames” which are known to the listener,  $P_1(0) \equiv S$  excepted).<sup>5</sup>

## 18. Graph-Sufficient Sets, Graph and Node-Efficiency

A *graph-sufficient set* for graph point  $P_v(y_1)$  is defined as a subset  $H$  of  $P_v(y_1)$  such that

$$H \subset P_w(z) \Rightarrow P_v(y_1) \subset P_w(z) \quad (1)$$

i.e., hearing  $H$  tells us we are in  $P_v(y_1)$  or above it; the only scales (chords) on the graph which contain  $H$  are  $P_v(y_1)$  and its superscales (superchords).

A *node-sufficient set* for a graph point  $P_v(y_1)$  is a subset  $H$  of  $P_v(y_1)$  such that

$$H \subset P_w(z) \Rightarrow P_v(y_1) \subset P_w(z) \quad \text{or} \quad P_w(z) \subset P_v(y_1) \quad (2)$$

Note that (2) differs from (1) only in that it permits  $H$  to be a subset of some  $P_w(z)$  below  $P_v(y_1)$ . A node-sufficient set distinguishes a particular graph point from all graph points incomparable with it.<sup>6</sup> A set can be graph-sufficient for only one graph point but it can be node-sufficient for several. (E.g., all sets are node-sufficient for  $P_1(0) \equiv S$ ).

*Example 12.* Let  $H_1$  be a subset of exactly those  $P_v(x)$  on the graph shown below which are blackened and let  $H_2$  be a subset of exactly those which are

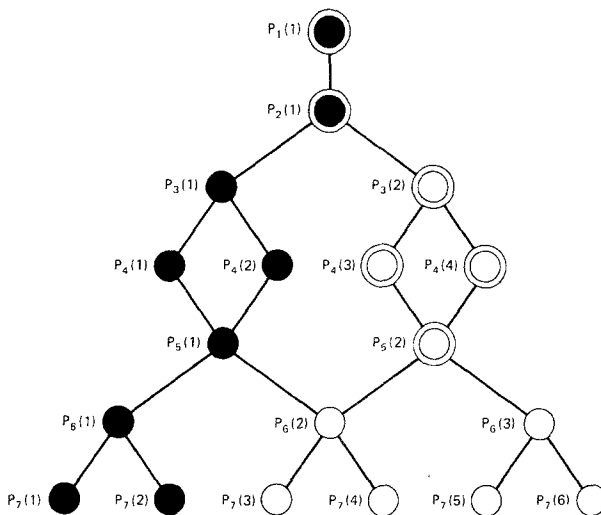
<sup>3</sup>When  $P_1(0)$  does not represent a learned classifier and  $I(H) = M - 1$  (i.e.,  $H$  is a subset of no  $P_v(y)$  other than  $S$ ), the intervals in  $H$  will likely be heard as “mistunings” of elements of some  $P_v(y)$  whose selection from the other graph points will probably depend upon preceding stimuli (see previous discussion of tuning).

<sup>4</sup>This “stimulus” may consist of musical tones simultaneously heard as well as in sequence, provided that these tones are clearly distinguished.

<sup>5</sup> $P_1(1) \equiv S$  is always included in graph  $G$  for convenience, since all graph points are connected to it and since its uniqueness causes it to have no effect on the relative values of different  $I(H)$ .

<sup>6</sup>Here and henceforth “incomparable” means “with respect to inclusion”.

surrounded by a circle:



For the above graph (which is not related to the graph of Figure 2)  $H$  is node-sufficient for exactly  $P_6(1)$ ,  $P_5(1)$ ,  $P_3(1)$ ,  $P_2(1)$  and  $P_1(1)$ , and it is not graph-sufficient for anything.  $H$  is graph-sufficient for exactly  $P_5(2)$  and is node-sufficient for exactly  $P_5(2)$ ,  $P_3(2)$ ,  $P_2(1)$  and  $P_1(1)$ .

Note that if all  $P_v(y)$  are eliminated from graph  $G$  except those which are transpositions of one particular  $P$  and their subsets and supersets, those  $H$ 's which are node-sufficient for each such transposition are sufficient sets as defined in part II, Sec. 14. These will henceforth be called *key-sufficient sets*.

A listener's habits may be such that he will classify a stimulus by the lowest graph point possible (i.e., when  $\bar{n}$  (the number of scale tones per octave) is minimal); e.g., a Western musician listening to classical music would most likely hear a set of tones belonging to a key of a major scale as a subset of such a key although these same tones are also a subset of the twelve-tone chromatic scale. In such a case his classifications will depend upon the graph-sufficient sets in the stimulus. However, the same listener may sometimes classify a stimulus by graph point which is not the lowest possible point on the graph. E.g., the same Western musician would hear a set of tones belonging to keys of both a major and a pentatonic scale as a subset of the key of the major (unless nothing but keys of the pentatonic had been heard in the composition for a long stretch of time). In this case his classification will be according to a node-sufficient (rather than graph-sufficient)<sup>7</sup> set. More will be said of this later.

Also of interest are sets which are subsets of at least one transposition of a particular  $P$  and of no point on graph  $G$  incomparable with all transpositions of  $P$ . These determine  $v$  in  $P_v(y)$ , but need not uniquely fix  $y$  and are called

<sup>7</sup>Note that a node-sufficient set has lower information value than a graph-sufficient set for the same point.

*scale-sufficient* for  $\psi(P)$ . Scale-sufficient sets are applicable to Indian music where any of a large number of scales (the tones in the "Ragas") may be used, but only one mode of each key (that whose tonic coincides with the drone) is used. In the "Alap" portion of an Indian classical performance, scale-sufficient sets are often avoided for a considerable period of time so that the resolution of doubt as to the identity of the "Raga" used assumes dramatic significance.

*Graph, node and scale-minimal* sets are defined (as previously) as sufficient sets of the same type, no proper subset of which satisfies the identical sufficiency condition.

The definitions of *graph efficiency*,  $E^G$ , and *node efficiency*,  $E^N$ , for a graph point  $P_e(x)$ , are similar to the definition of efficiency (cf. part II, Sec. 15) except that the number of graph or node-sufficient sets for  $P_e(x)$  appears in the numerator in place of the number of key-sufficient sets. When  $G$  is *complete* (i.e., closed under transposition)  $E^G$  is the same for all transpositions of a given  $P$  (as is  $E^N$ ).<sup>8</sup>

## 19. A Sample Graph

In the graph specified by Figure 2, Section 16,  $S$  is the integers,  $m=12$ , all  $P_u(x)$  with stability  $\geq 2/3$  are on the graph. Figure 5 (below) shows the containment pattern of all 3, 4 and 5 element subsets  $H$  of  $S$  in those of these graph points with more than six elements or with six elements and a stability of 1. These are familiar to contemporary Western musicians; other points on the graph of figure 2 are subsets of at least one of these and are customarily used as "chords" rather than "scales" in Western music. Hence, were  $P_2(x)$  eliminated from consideration, all subsets of only one of the remaining selected graph points<sup>9</sup> would be node-sufficient<sup>10</sup> for that point.  $P_2$  is relatively low in stability and has a high gradient with respect to the whole-tone scale ( $P_{10}$ ), the twelve-tone scale ( $P_1$ ) and also  $P_{11}$ . Hence  $P_2$  is easily perceived in terms of other (proper) scales. In deference to the contemporary composer, Olivier Messiaen who consciously uses  $P_2$ <sup>11</sup>, it is reluctantly included in Figure 5. All node-sufficient sets are underlined including those which are node-sufficient only if no  $P_2(x)$  is considered (e.g. 6141). Other underlinings are only of *graph-sufficient* sets for  $P_1(0)$  or some  $P_2(x)$ . (It is assumed that listeners will classify node-sufficient sets for  $P_1(0)$  or some  $P_2(x)$  by some lower graph point e.g. some  $P_{11}(x)$  of  $P_{10}(x)$ ).

Each row corresponds to an  $H_u(0)$  whose  $\psi(H_u(0))$  is shown in the leftmost column. The headings of columns 2-7 specify  $\psi(P)$  corresponding to  $P_2(x)$ ,

<sup>8</sup>When  $G$  is complete  $E^G$  and  $E^N$  may be computed without finding corresponding sufficient sets (which in this case, can be easily extracted). See subsequent computation paper.

<sup>9</sup>None of which is now contained in another.

<sup>10</sup>But not necessarily graph-sufficient (e.g.,  $P_{24}(0)$  (see Figure 2), for which  $\psi(P)=(5,1,5,1)$ , is graph-sufficient for itself, but not for  $P_3(0)$  ( $\psi(P_3)=(2,1,2,1,2,1,2,1)$ ), for which it is node-sufficient.

<sup>11</sup>In his book, "The Technique of My Musical Language" [15], he lists  $P_2(x)$  as one of his "modes of limited transposition". It is not clear, however, that it is heard as a "scale" as defined here (i.e., a "mental reference frame").



$P_3(x)$ ,  $P_4(x)$ ,  $P_5(x)$ ,  $P_{11}(x)$  and  $P_{10}(x)$  respectively.  $P_1(0)$  is not shown since all  $H_u(x)$  are subsets of it. Each table entry,  $y$ , indicates that  $H_u(0) \subset P_v(y)$  where  $u$  is determined by the row label and  $v$  by the column label. (Note that  $H_u(0) \subset P_v(y) \Leftrightarrow H_u(z) \subset P_v(y+z)$  with appropriate reductions of  $z$  and  $y+z$ ). The number in the rightmost column is the total number of entries in all columns except that corresponding to  $P_2(x)$ . These numbers roughly correspond inversely to information values.

|       | 211211211 | 21212121 | 2222121 | 2221221   | 313131 | 222222 | 0 |
|-------|-----------|----------|---------|-----------|--------|--------|---|
| 10 11 | 0         |          |         |           |        |        | 0 |
| 921   | 1         | 0        | 0 3     | 0 5       |        |        | 5 |
| 912   | 2         | 1        | 110     | 310       |        |        | 5 |
| 831   | 0 1       | 0        | 0       |           | 0      |        | 3 |
| 822   | 0 2       |          | 4 6 8   | 1 6 8     |        | 0      | 7 |
| 813   | 1 2       | 0        | 0       |           | 1      |        | 3 |
| 741   | 0 1       |          | 3       | 0 5       | 0      |        | 4 |
| 732   | 0         | 1        | 1 810   | 1 3 810   |        |        | 8 |
| 723   | 1         | 1        | 1 310   | 0 3 510   |        |        | 8 |
| 714   | 0 1       |          | 8       | 1 8       | 0      |        | 4 |
| 651   | 0         | 0        | 0       | 0         |        |        | 3 |
| 642   | 0 2       | 1        | 4 610   | 6         |        | 0      | 6 |
| 633   | 2         | 0 1      | 010     | 0         |        |        | 5 |
| 624   | 0 2       | 0        | 0 4 6   | 6         |        | 0      | 6 |
| 615   | 0         | 1        | 10      | 0         |        |        | 3 |
| 552   | 2         |          | 1 6 8   | 1 3 6 810 |        |        | 8 |
| 543   | 1 2       | 0        | 1 3     | 3 510     | 1      |        | 7 |
| 534   | 1 2       | 0        | 6 8     | 1 6 8     | 1      |        | 7 |
| 444   | 0 1 2     |          | 0 4 8   |           | 0 1    | 0      | 6 |
| 9111  |           |          |         |           |        |        | 0 |
| 8211  | 0         |          |         |           |        |        | 0 |
| 8121  | 1         | 0        | 0       |           |        |        | 2 |
| 8112  | 2         |          |         |           |        |        | 0 |
| 7311  | 0         |          |         |           |        |        | 0 |
| 7221  | 1         |          | 3       | 0 5       |        |        | 3 |
| 7212  |           | 1        | 110     | 310       |        |        | 5 |
| 7131  | 0 1       |          |         |           | 0      |        | 1 |
| 7122  | 0         |          | 8       | 1 8       |        |        | 3 |
| 7113  | 1         |          |         |           |        |        | 0 |
| 6411  | 0         |          |         |           |        |        | 0 |
| 6321  |           | 0        | 0       | 0         |        |        | 3 |
| 6312  | 2         | 1        | 10      |           |        |        | 2 |
| 6231  | 0         | 0        | 0       |           |        |        | 2 |
| 6222  | 0 2       |          | 4 6     | 6         |        | 0      | 4 |
| 6213  | 2         | 0        | 0       |           |        |        | 2 |
| 6141  | 0         |          |         | 0         |        |        | 1 |
| 6132  | 0         | 1        | 10      |           |        |        | 2 |
| 6123  |           | 1        | 10      | 0         |        |        | 3 |
| 6114  | 0         |          |         |           |        |        | 0 |
| 5511  |           |          |         |           |        |        | 0 |
| 5421  | 1         | 0        | 3       | 5         |        |        | 3 |
| 5412  | 2         |          | 1       | 310       |        |        | 3 |
| 5331  | 1         | 0        |         |           |        |        | 1 |
| 5322  | 2         |          | 6 8     | 1 6 8     |        |        | 5 |
| 5313  | 1 2       | 0        |         |           | 1      |        | 2 |
| 5241  | 1         |          | 3       | 5         |        |        | 2 |
| 5232  |           |          | 1 8     | 1 3 810   |        |        | 6 |
| 5223  | 1         |          | 1 3     | 3 510     |        |        | 5 |
| 5214  | 1         |          | 8       | 1 8       |        |        | 3 |
| 5151  |           | 0        |         |           |        |        | 1 |
| 5142  | 2         |          | 6       | 6         |        |        | 2 |
| 5133  | 2         | 0        |         |           |        |        | 1 |
| 5124  | 2         | 0        | 6       | 6         |        |        | 3 |
| 4431  | 0 1       |          | 0       |           | 0      |        | 2 |
| 4422  | 0 2       |          | 4 8     |           |        | 0      | 3 |
| 4413  | 1 2       |          | 0       |           | 1      |        | 2 |
| 4341  | 0 1       |          |         | 0 5       | 0      |        | 3 |

Fig. 5.

|       | 211211211 | 21212121 | 2222121 | 2221221 | 313131 | 222222 |
|-------|-----------|----------|---------|---------|--------|--------|
| 4332  | 0         | 1        | 810     | 10      |        | 4      |
| 4323  | 1         | 1        | 10      | 0 510   |        | 5      |
| 4242  | 0 2       | 1        | 410     |         | 0      | 4      |
| 4233  | 2         | 1        | 010     | 0       |        | 4      |
| 3333  |           | 0 1      |         |         |        | 2      |
| 81111 |           |          |         |         |        | 0      |
| 72111 |           |          |         |         |        | 0      |
| 71211 | 0         |          |         |         |        | 0      |
| 71121 | 1         |          |         |         |        | 0      |
| 71112 |           |          |         |         |        | 0      |
| 63111 |           |          |         |         |        | 0      |
| 62211 | 0         |          |         |         |        | 0      |
| 62121 |           | 0        | 0       |         |        | 2      |
| 62112 | 2         |          |         |         |        | 0      |
| 61311 | 0         |          |         |         |        | 0      |
| 61221 |           |          |         | 0       |        | 1      |
| 61212 |           | 1        | 10      |         |        | 2      |
| 61131 | 0         |          |         |         |        | 0      |
| 61122 | 0         |          |         |         |        | 0      |
| 61113 |           |          |         |         |        | 0      |
| 54111 |           |          |         |         |        | 0      |
| 53211 |           |          |         |         |        | 0      |
| 53121 | 1         | 0        |         |         |        | 1      |
| 53112 | 2         |          |         |         |        | 0      |
| 52311 |           |          |         |         |        | 0      |
| 52221 | 1         |          | 3       | 5       |        | 2      |
| 52212 |           |          | 1       | 310     |        | 3      |
| 52131 | 1         |          |         |         |        | 0      |
| 52122 |           |          | 8       | 1 8     |        | 3      |
| 52113 | 1         |          |         |         |        | 0      |
| 51411 |           |          |         |         |        | 0      |
| 51321 |           | 0        |         |         |        | 1      |
| 51312 | 2         |          |         |         |        | 0      |
| 51231 |           | 0        |         |         |        | 1      |
| 51222 | 2         |          | 6       | 6       |        | 2      |
| 51213 | 2         | 0        |         |         |        | 1      |
| 51141 |           |          |         |         |        | 0      |
| 51132 |           |          |         |         |        | 0      |
| 51123 |           |          |         |         |        | 0      |
| 51114 |           |          |         |         |        | 0      |
| 44211 | 0         |          |         |         |        | 0      |
| 44121 | 1         |          | 0       |         |        | 1      |
| 44112 | 2         |          |         |         |        | 0      |
| 43311 | 0         |          |         |         |        | 0      |
| 43221 | 1         |          |         | 0 5     |        | 2      |
| 43212 |           | 1        | 10      | 10      |        | 3      |
| 43131 | 0 1       |          |         |         | 0      | 1      |
| 43122 | 0         |          | 8       |         |        | 1      |
| 43113 | 1         |          |         |         |        | 0      |
| 42411 | 0         |          |         |         |        | 0      |

Fig. 5. (cont'd.)

|       | 211211211 | 21212121 | 2222121 | 2221221 | 313131 | 222222 |
|-------|-----------|----------|---------|---------|--------|--------|
| 42321 |           |          | 0       | 0       |        | 2      |
| 42312 | 2         | 1        | 10      |         |        | 2      |
| 42231 | 0         |          | 0       |         |        | 1      |
| 42222 | 0 2       |          | 4       |         | 0      | 2      |
| 42213 | 2         |          | 0       |         |        | 1      |
| 42141 | 0         |          |         | 0       |        | 1      |
| 42132 | 0         | 1        | 10      |         |        | 2      |
| 42123 |           | 1        | 10      | 0       |        | 3      |
| 41412 | 2         |          |         | 10      |        | 1      |
| 41331 | 1         |          |         |         |        | 0      |
| 41322 | 2         |          | 8       |         |        | 1      |
| 41313 | 1 2       |          |         |         | 1      | 1      |
| 41232 |           |          | 8       | 10      |        | 2      |
| 41223 | 1         |          |         | 510     |        | 2      |
| 41133 | 2         |          |         |         |        | 0      |
| 33321 |           | 0        |         |         |        | 1      |
| 33312 |           | 1        |         |         |        | 1      |
| 33231 | 0         | 0        |         |         |        | 1      |
| 33222 | 0         |          | 4 6     | 6       |        | 3      |
| 33132 | 0         | 1        |         |         |        | 1      |
| 32322 |           |          | 6       | 1 6 8   |        | 4      |

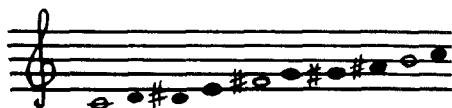
Fig. 5. (cont'd.)

*Illustration of the Use of the Table.* Let it be required to find those “scales” (=transpositions of  $P_1 - P_5$  and  $P_{10} - P_{11}$ ) of which (1) {C, F#, B} and (2) {B, D, F#, G} are subsets.

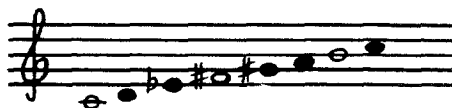
Ad (1). The  $\psi(P)$  for CF# B is 651. Since this is lexicographically later than its cyclic permutations 516 and 165, it's already in descending form, and we look for the row (row 11) of the table which has 651 standing in its leftmost column. (Entries in this column are arranged in order of length, and in reverse lexicographic order between entries of the same length.) Then (reading row 11 from left to right)

$$\{C, F\#, B\} \subset P_2(0), P_3(0), P_4(0) \text{ and } P_5(0)$$

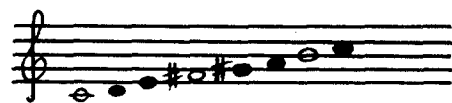
i.e., it is included in Messiaen's mode  $P_2$  starting on C:



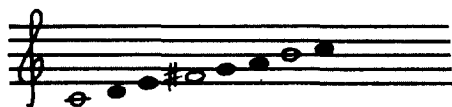
$P_3$  (“string of pearls”) starting on C:



$P_4$  starting on C (a cyclic permutation of A melodic minor):



$P_5$  starting on C (C Lydian, a cyclic permutation of G major):



and trivially  $P_1$  (one chromatic scale, omitted from the table because everything is included in it).

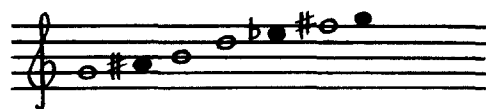
The number 3 written in the rightmost column of line 11 of the table indicates that there are 3 possible  $P_u(x)$  in which {C F# B} is included (not counting the trivial  $P_1$ 's and the dubious  $P_2$ 's). If there is only one such  $P_u(x)$  (as in rows 5331 and 61221 of the table) then  $H$  is node-minimal for that one (if  $P_2$  is excluded from the graph); if there are more (as in rows 53211 and 61311) it is node-minimal for  $P_1$  ("atonal" in a precise sense): in either case the row is underlined. The numbers in the right-hand column roughly correspond inversely to information values.

Ad (2). The  $\psi(P)$  for B D F# G is 3 4 1 4. Its descending form is 4 3 4 1 which we find in its place in column 1 of the table. This corresponds to the position G B D F#, so  $0 \equiv G$ . Reading the entries opposite 4 3 4 1 from left to right, we see:

Under  $P_2$ , 0 and 1, i.e., Messiaen's mode on G or A<sup>b</sup>.

Under  $P_5$ , 0 and 5, i.e., the Lydian mode on G=0 or C=5 (=D and G major respectively).

Under  $P_{11}$  (a symmetric mode unknown to Messiaen) 0, i.e.,  $P_{11}$  on G



On the table below the first two entries are graph efficiencies<sup>12</sup> for  $P_1(0)$  and any  $P_2(x)$ ; the rest are node efficiencies for any  $P_3(x)$ ,  $P_4(x)$ ,  $P_5(x)$ ,  $P_{11}(x)$  and  $P_{10}(x)$ . ( $\psi P$ 's are shown on the left):

| $\psi(P)$                         | if all $P_2(x)$ are on graph | if all $P_2(x)$ are excluded from graph |
|-----------------------------------|------------------------------|---|
| (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) | .5517                        | .4646                                   |
| (2, 1, 1, 2, 1, 1, 2, 1, 1)       | .6190                        | —                                       |
| (2, 1, 2, 1, 2, 1, 2, 1)          | .7464                        | .6786                                   |
| (2, 2, 2, 2, 1, 2, 1)             | .9592                        | .8844                                   |
| (2, 2, 2, 1, 2, 2, 1)             | .9524                        | .9143                                   |
| (3, 1, 3, 1, 3, 1)                | .8000                        | .8000                                   |
| (2, 2, 2, 2, 2, 2)                | 1.0000                       | 1.0000                                  |

If we consider a graph that consists only of  $P_1(0)$  and all points whose  $\psi(P)$  corresponds to the "major scale" (2, 2, 2, 1, 2, 2, 1) and to the "melodic minor

<sup>12</sup>Note that a graph-sufficient set for  $P_v(y)$  cannot be classified by subsets of  $P_v(y)$ . Hence graph efficiencies are used instead of node efficiencies for  $P_1(0)$  and all  $P_2(x)$ .

scale" (2,2,2,2,1,2,1), node efficiencies are as follows:

(2,2,2,2,1,2,1) .7932

(2,2,2,1,2,2,1) .9143

Note that  $\psi(P)$  which are low in efficiency (computed using key-sufficient sets—see Figure 2) may be high in graph or node efficiency. Hence scales which may be too redundant (Part II, Section 15) to be greatly used in diatonic music (e.g., (2,2,2,2,2,2), (2,1,2,1,2,1,2,1)) may be extensively used in music which freely makes use of all twelve tones and their subsets. Examples can be found in Ravel, Stravinsky, Frank Martin, Bartok, etc. (Ravel mixes the "whole tone scale" with the "major" and "minor" scales; Stravinsky's first movement of the "Symphony of Psalms" uses (2,1,2,1,2,1,2,1) extensively, etc.)

Observe that among the  $H_u(x)$  with three elements (triads), those whose  $\psi(P)$  are (10,1,1), (8,3,1), (6,5,1) have the highest information values. The frequency of use of such triads in twentieth century music is well documented; hence "minor" chords and chords in fourths. A particularly startling example is Anton Webern's "Piano Variations" (Opus 27). The entire composition consists of a succession of graph-sufficient sets for the "twelve tone scale" (1,1,1,1,1,1,1,1,1,1,1,1), although the "serial" technique of composition does not guarantee this.<sup>13</sup>

If we retain the notion that a "cadence" must uniquely determine key and tonality (Section 15 at the end), and add that it must also determine scale, "cadences" for "non-diatonic" music (i.e., using keys of all scales on the graph) can be constructed. Here a sample cadence for  $P_4(0) = C, D, E, F^\#, G^\#, A, B$  ( $\psi(P) = (2,2,2,2,1,2,1)$ ), when E is the tonic, will be constructed. Notice (from Figure 5) that  $\{C, E, F^\#, G^\#, B\}$  (4,2,2,3,1) is as small a minimal node-sufficient set as can be found for  $P_4(0)$ . Let us also take advantage of the fact that the interval  $(G^\#, C)$  is ambiguous in this scale, and therefore "resolve" it to an unambiguous interval (as F-B to E-C in C major). The final chord will contain E and B so that the tonic, E, is reinforced by the resulting difference tone (assuming conventional timbres). Notice that  $P_4(0)$  is conventionally heard as "A minor", not as a mode with E as a tonic, as in this example:

### Example 13



<sup>13</sup>A careful analysis of the rules of the "serial" (i.e., "twelve tone") technique will show that they prejudice the composer in favor of writing successions of graph-sufficient sets for the "twelve-tone scale" and avoiding tonal centers (modes). Examination of works of well-known "serial" composers (e.g. Webern) will show that such is done in excess of the demands of the rules of the technique.

Similar examples can be constructed using the chart in Figure 5. Of course, if a node-sufficient set contains more than one node-minimal subset the identification of the scale is strengthened. A common method is to use all the tones of a scale in a cadence. Oliver Messiaen's music abounds with such examples. Here is an excerpt from the third song, "Dance du Bebe-Pilule" of his "Chants de Terre et de Ciel".<sup>14</sup> The scale is  $P_3(1) = D^b, E^b, E, F^\#, G, A, B^b, C$  ( $\psi(P) = (2, 1, 2, 1, 2, 1, 2, 1)$ ) where  $E^b$  is the tonic:

*Example 14.*

Notice that, with appropriate labeling of node-sufficient (or graph-sufficient) sets, it is possible to extend the traditional "figured bass" system to apply to non-diatonic music.

On such a basis new methods of teaching musical "ear training" can also be suggested.

Condition (c) at the end of Section 15 pertains to "color changes" between "chords" in diatonic cadences. The following discussion is relevant to such differences and similarities between sets of tones in non-diatonic music.

## 20. Image Distance

Consider two sets,  $H_1$  and  $H_2$ , which are not graph equivalent. If the graph  $G$  is altered by the removal of certain points, these sets will become graph equivalent. The minimum number,  $X$ , of such points which must be removed from graph  $G$  for  $H_u(x)$  and  $H_v(y)$  to become graph equivalent is equal to the cardinality of the symmetric difference between their respective  $V(H_1)$  and  $V(H_2)$  (Section 17, at the beginning):

$$X = \text{card}(V(H_1) \cup V(H_2)) - (V(H_1) \cap V(H_2)).$$

In order to arrive at a measure for the disturbance which must be induced in that portion of the graph  $G$  which is pertinent to the classification of  $H_1$  and  $H_2$ , we divide  $X$  (above) by the cardinality of that portion (i.e.,  $\text{card}(V(H_1) \cup V(H_2))$ ). A number between zero and one results which we call the *image*

<sup>14</sup>Elkan-Vogel Co., Philadelphia, Pa. (U.S.A.). Copyright by Durand & Cie, 1939.

distance,  $\bar{I}(H_1, H_2)$ , of  $H_1$  and  $H_2$ :

$$\bar{I}(H_1, H_2) = 1 - \frac{\text{card}(V(H_1) \cap V(H_2))}{\text{card}(V(H_1) \cup V(H_2))}. \quad (1)$$

In computing the cardinalities in the above expression all elements of the sets being evaluated are, in effect, assigned equal weight (i.e., 1). This corresponds to assigning equal likelihood to all sets on the graph. This is rarely the case in actual music, where some scales and/or keys are more likely to occur than others. However, (1) above is easily modified by assigning different weights (altering the cardinality function so that a set may be counted more than once) to each point on the graph. Such weighting, however, will not alter the ordering of the image distances between any two pairs,  $H_u(x)$  and  $H_v(y)$ , in the examples which will follow. Note that image distance provides a more comprehensive relation between “chords” than the common criteria of “the number of common tones” and “the relation between roots”. By use of the chart in Figure 5 the reader can verify that the well-known order of similarity relations between triads in the same and different keys can be derived from image distance (e.g., {CEG} is less similar to {F<sup>#</sup> A<sup>#</sup> C<sup>#</sup>} than to {DFA}). If such similarity relations are intended to apply only to diatonic music, all scales except the major ( $\psi(P) = (2, 2, 2, 1, 2, 2, 1)$ ) and melodic minor ( $\psi(P) = (2, 2, 2, 2, 1, 2, 1)$ ) should be eliminated from the graph. For most nondiatonic music it is probably appropriate to eliminate  $P_2(x)$  ( $\psi(P) = (2, 1, 1, 2, 1, 1, 2, 1, 1)$ ) for all  $x$ .

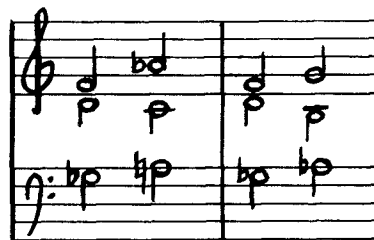
More sensitive discriminations than the familiar criteria for similarity between triads are thus provided. The following less obvious examples are chosen so that the number of common tones in each corresponding pair is the same, so that “positions” of corresponding “chords” are identical, and so that the effect of tonality is minimized.  $P_2(x)$  for all  $x$  is eliminated from the graph (although its inclusion does not alter the ordering of the different  $\bar{I}$ ),  $P_1(0)$  is included (hence  $\bar{I} \neq 1$  and distinctions result between pairs of sets whose intersections belong only to  $P_1(0)$ ), and no weighting of different graph points is used. In both examples the second pair has a larger image distance than the first. Example 15 is about as subtle a case as can be constructed in twelve-tone equal temperament:

Example 15.



$$\begin{aligned} \bar{I} &= 2/5 & \bar{I} &= 6/7 \\ (\alpha) &= (8, 1, 3) \text{ on C} \\ (\beta) &= (6, 5, 1) \text{ on C} \\ (\gamma) &= (6, 5, 1) \text{ on D}^\flat \end{aligned}$$

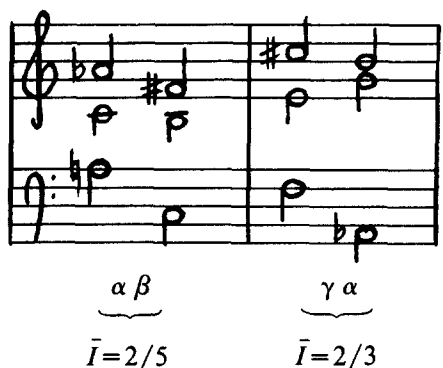
Example 16.



$$\begin{aligned} \bar{I} &= 2/3 & \bar{I} &= 6/7 \\ (\alpha) &= (8, 3, 1) \text{ on G}^\flat \\ (\beta) &= (8, 1, 3) \text{ on C} \\ (\gamma) &= (8, 1, 3) \text{ on B} \end{aligned}$$

The first pair of Example 15 cannot be compared with the first pair of Example 16 without alteration, because both pairs are subsets of  $P_3(0)$  ( $\psi(P_3) = (2, 1, 2, 1, 2, 1, 2, 1)$ ) so that hearing one pair facilitates classification of the next. This can be overcome by transposing (changing the key memberships) of one of the pairs, but then difficulties in avoiding differences between the pairs with respect to dissonance and tonal implication arise. Also, sequences must be avoided, since these generate expectations which facilitate acceptance of the last chord in the sequence. One must attempt to equalize the pairs with respect to all properties except image distance. About the best that can be done in this respect when comparing the first pair of each of the above examples is shown below. However, the voice leading in the first pair is not so smooth as in the second, thereby tending to increase the sense of difference between the first pair relative to the second. (This prejudicial condition can be reversed by exchanging the position of the B and C in the second chord of the first pair).

*Example 17.*



|                             |   |            |
|-----------------------------|---|------------|
| $(\alpha) = (8, 1, 3)$ on C | { | example 15 |
| $(\beta) = (6, 5, 1)$ on C  |   | first pair |
| $(\gamma) = (8, 3, 1)$ on F | { | example 16 |
| $(\alpha) = (8, 1, 3)$ on B |   | first pair |
|                             |   | transposed |

Of course,  $\bar{I}$  remains relevant if each of a pair,  $H_1$  and  $H_2$ , appears in a melodic sequence instead of in a chord progression.

## 21. Graph Distance

Just as the image distance between a pair,  $H_1$  and  $H_2$ , derives from those points by which each of such pairs may be classified, the graph distance between a pair of points on the graph,  $P_u(x)$  and  $P_v(y)$ , derives from those graph points which may be classified by each of such pairs. Graph distance is a relation between keys of scales and is thus relevant to musical "modulation".

A graph point (a particular key of a scale),  $P_u(x)$  is often selected as a classifier for some  $H = H_1 \cup H_2 \cup \dots$ ; each of which  $H_i$  may be classified by some graph point which is a subset of  $P_u(x)$ . (This will occur when the "stimuli",  $H_1, H_2, \dots$ , are contiguous, occupy little time, and when the union of graph points by which each of the  $H_i$  are classified forms a set which is graph-sufficient for  $P_u(x)$ .) Hence a graph point is characterized by the other graph



points which are subsets of it.<sup>15</sup> Accordingly, we define  $W(u, x) = \{P_v(y) | P_v(y) \subset P_u(x)\}$ . Note that  $W(u, x)$  derives from set inclusion in the direction opposite to that of  $V(H)$  (Section 17).

Then the *graph distance*,  $\bar{D}(u, x, w, z)$ , between two graph points,  $P_u(x)$  and  $P_w(z)$ , is defined (analogously to  $\bar{I}(H_1, H_2)$ ) as

$$\bar{D}(u, x, w, z) = 1 - \frac{\text{card}(W(u, x) \cap W(w, z))}{\text{card}(W(u, x) \cup W(w, z))}.$$

Graph distances are intended to correspond to degrees of perceived “similarity”<sup>16</sup> between keys of different musical “scales”. For example, the “major scale” (C, D, E, F, G, A, B) appears to be more similar (in this sense) to the “melodic minor scale” (A, B, C, D, E, F<sup>#</sup>, G<sup>#</sup>), than does the “whole tone scale” (C, D, E, F<sup>#</sup>, G<sup>#</sup>, A) (although each pair has the same number of common tones). Since our notion of “scale” corresponds to “equivalence class”  $\|\alpha_{ij}\|$ , and since a given  $\|\alpha_{ij}\|$  may have more than one member  $\psi(P)$ , each with corresponding points on graph  $G$ , for certain applications it may be necessary to restrict graph  $G$  to sets with a single  $\psi(P)$  in each  $\|\alpha_{ij}\|$ . That is, when two distinct sets are the same *key*, *mode*, and *scale* (but different *tunings*) (Section 12), they are perceived as “mistunings” of each other. It is thus inappropriate to consider both such sets as lying on the same graph.<sup>17</sup> Note, however, that when two such sets are low<sup>18</sup> on graph  $G$  and when the image distance between them is large, it is unlikely that they will be heard as “mistunings” of each other.<sup>19</sup> Thus it is inappropriate to place two distinct  $P$  in the same  $\|\alpha_{ij}\|$  on graph  $G$  when they are high on the graph and are of the same key and mode. This occurs in its worst form when graph  $G$  is complete (i.e. closed under transposition) and two distinct  $\psi(P)$  in the same  $\|\alpha_{ij}\|$  have points on the graph.

Sometimes it is useful to speak of the “distance” between  $\psi(P)$  and  $\psi(Q)$  rather than between  $P_i$  and  $Q_i$ , some pair of their respective graph points. (i.e. “Is the major scale more similar to the melodic minor scale than the “whole tone scale?”) Thus the *scale distance*,  $\bar{S}(u, v)$  is defined as

$$\bar{S}(u, v) = \min_{x, y} \bar{D}(u, x, v, y)$$

The number (or numbers),  $(y - x)$ , which corresponds to the above minimum, shows the relative keys which correspond to maximal similarity. (When

<sup>15</sup>For example, the “major scale” is characterized by its triads, seventh chords, etc.

<sup>16</sup>Note that our conception of “scale” makes no distinction between “modes” (tonal centers) of a key of such a scale. Hence the “similarity” which is referred to above is distinct from that similarity which results from relations between tonics of keys of scales (except when the discussion to follow applies).

<sup>17</sup>Such problems often arise when  $m > 12$  (e.g.,  $m = 31$ ).

<sup>18</sup>We use the convention that sets appear above their subsets on graph  $G$  (see Figure 3).

<sup>19</sup>There is one such case when  $m = 12$ ;  $P = (C, E, G^b, B^b)$  and  $Q = (C, F, G^b, B)$  (for which  $\Psi(P) = (4, 2, 4, 2)$  and  $\Psi(Q) = (5, 1, 5, 1)$ ) are in the same equivalence class, but there is little perceived similarity because classification ordinarily occurs higher on the graph by different graph points in each case.

graph  $G$  is complete,  $x$  (or  $y$ ) may of course be arbitrarily set without affecting the value of  $\bar{S}(u, v)$ .) In the musical application we may also choose which tonic ("mode")<sup>20</sup> should be assigned to each of such a pair of keys of scales to achieve greatest similarity: When there is more than one tone common to both (keys of scales), assign the tonic of *both* to that tone which is best supported by the difference tones and harmonics generated by the tones of each of the pair of keys (cf Part I, Section 2). (These harmonics and difference tones are dependent upon the timbres being used.)

It is now known that a wide range of inharmonic residues have definite pitch<sup>21</sup> (i.e., a sensation of pitch occurs even when the partials are not integer multiples of a fundamental), and in such cases extremely unfamiliar intervals may sound "pure" or "constant"<sup>22</sup> and familiar "consonant" intervals may sound "dissonant" or "impure".<sup>23</sup> Sometimes a tone which is well supported as a tonic in both of the keys (above) cannot be found, such that footnote 24, Section 7 of Part I is pertinent, and the above procedure does not apply.

Note that implicit in the above interpretation is the assumption that the perception of differential similarities between different pairs of scales (learned "mental reference frames") is always dependent upon those other scales which have been learned by the listener and such of these whose use he may anticipate in a particular situation; e.g., the same listener's expectations will differ (as will graph  $G$ ) when listening to classical Western music and to twentieth century Western music.

## 22. The Effect of the Graph upon the Tuning of Scales

Previous to this, we have discussed restrictions on "mistunings" of the elements of a proper subset such that it retains its propriety (Part 1, Section 8) and also restrictions on such mistunings so that any proper or improper scale retains its identity (Part 2, Section 10). The relevance of these different restrictions to a particular  $P_u(x)$  depends upon the structure of the graph in which it is embedded.

Consider a graph which contains one and only one point,  $P$ , which is a proper set. Note that when axioms 2.2 and 2.3 apply (as they do in Western music) and the elements of the scale are distributed so that adjacent pairs form equal "intervals" (in  $P \times P$ ),  $\bar{R}$  (the union of the ranges of all the elements) covers  $S$  ((9) of Part 1, Section 4).<sup>24</sup> Note also that the cardinality of a proper  $P$  and its corresponding code are identical. Thus it is possible to classify any

<sup>20</sup>"Mode" is here used to indicate the particular tonic of a scale in the sense that the different "church modes" indicate different tonics in the major scale.

<sup>21</sup>See J. F. Schouten, R. J. Ritsma and B. Lopex Cardozo, "Pitch of the Residue", [16] and also J. E. Evetts, "The Subjective Pitch of a Complex Inharmonic Residue", [17].

<sup>22</sup>We are here referring to "acoustical consonance" as described by Helmholtz (in terms of coincidence of harmonics and beats), not to qualities of intervals deriving from context, (e.g., ambiguity, etc.).

<sup>23</sup>In fact, consistent alterations in timbre (especially of this type), when coupled to change in pitch, can alter the initial ordering (see Part 1, Section 2).

<sup>24</sup>In fact, it is sufficient that these "intervals" form a repeating sequence of two magnitudes only ((8), *ibid*).

element of  $P$  as a “mistuning” of an element of another proper set,  $\bar{P}$ , with the same cardinality whose points are equally spaced. When several elements of  $P$  are simultaneously presented,  $S$ -proper modifications apply, and the interpretation of elements of  $P$  as mistuned elements of  $\bar{P}$  is restricted. Note, however, that  $R$  (the union of all  $S$ -proper modifications)  $\equiv \bar{R}$  (Section 5), and since  $R \equiv S$ , great latitude remains. Thus, when tonality is a factor, it is usually possible to adjust the tuning of the elements of  $\bar{P}$  so that each temporary tonic is reinforced (by different tones and harmonics) as it occurs (this often happens in performances of “free twelve-tone” music). In this fashion nearly all unfamiliar proper scales with the same number of elements may appear on first hearing to be different tunings of a single proper scale (provided, of course, there is only one point on the graph). This phenomenon is familiar to Western musicians who have experimented with exotic tunings of seven tone scales—when such scales are proper they often tend at first hearing to sound like mistuned major scales (or modes of such scale).<sup>25</sup> This also accounts for why it is sometimes erroneously stated by unsophisticated listeners that the pattern of the type of pentatonic scale in China and Thailand can be represented by the black keys of the piano. (The Thai pentatonic scale is extracted from a seven tone equal temperament system, not a twelve.)

Suppose  $P$  is no longer the only point on the graph. Since it now is necessary to be able to distinguish between  $P$  and the other graph points, the above discussion no longer applies. Even if all the graph points are different keys of the same scale, sufficient sets for each such key must be identifiable. That is, each mode of the scale must be distinct and hence each column of  $\|\alpha_{ij}\|$  must be distinct. Hence, in music where one of many scales and keys (or modes) of such scales may be used, mistunings of elements of a particular  $P$  are restricted according to  $E$ -ranges or  $SE$  modifications both when  $P$  is proper and improper (Section 13). When graph points are keys of different scales, it is necessary to express each  $\psi(P)$  in the same number of units per cycle,  $m$  (the cardinality of  $P_1(0)$ ). The larger  $m$  is, the more stringent may be the restrictions on the mistunings of the elements of the individual graph points. That is, all graph-sufficient sets must now maintain their distinctness. (It is worth noting that, when experimenting with unfamiliar synthetically constructed musical scales, apparent resemblances between different proper scales of a given cardinality tend to disappear as soon as different keys of such scales are used in a musical composition).<sup>26</sup> Thus the entire graph of mental reference frames available to a listener and relevant to his expectations at a given time influences the limits of permissible deviations in tuning.

### 23. Propriety, Redundancy, the Graph and Musical Form

In general, musical form derives from a number of symmetries between different portions of a composition ranging from very small to very large units (e.g.,

<sup>25</sup>Of course, this effect can be easily overcome if sufficiently exotic tone colors are chosen—especially those utilizing inharmonic partials.

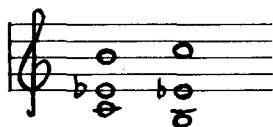
<sup>26</sup>This has been reported to me by musicians who have experimented with equipment for producing such synthetic scales.

motifs, phrases, sections, etc.). To a large extent the types of such symmetries are bounded by the materials used which determine the properties with respect to which symmetries (equivalence) can occur. Helmholtz writes concerning the consequences of “definiteness and certainty in the measurement of intervals for our sensation”:<sup>27</sup>

Upon this reposes also the characteristic resemblance between the relations of the musical scale and of space, a resemblance which appears to me of vital importance for the particular effects of music. It is an essential character of space that at every position within it like bodies can be placed, and like motions can occur. Everything that is possible to happen in one part of space is equally possible in every other part of space and is perceived by us in precisely the same way. This is the case also with the musical scale. Every melodic phrase, every chord, which can be executed at any pitch, can be also executed at any other pitch in such a way that we immediately perceive the characteristic marks of their similarity.

Such a property (which amounts to a set of tones retaining their identity as a “Gestalt” when “inversion” (permutation of the order of the elements) occurs, as we have seen, is possessed only by proper scales, in fact only by strictly proper scales. Hence Western music using the major scale makes extensive use of modal transpositions of motifs. Since the major scale is also very high in efficiency, melodies sometimes depend upon phrases where doubt is not immediately resolved. Also, since the major scale has many proper subsets (see Part I, Section 8 and Figure 1, Part II, Section 13) and many intervals with roots (see the discussion preceding Figure 1) melodies make use of principal-auxiliary tone relationships determined harmonically (harmonic and non-harmonic tones), rhythmically, and/or by the ambiguity of the tritone. Notice again that the “figured bass” system is composed of the proper subsets of the major and minor scales. Hence these retain their identity when “inverted”. This is not the case with improper sets. E.g.

*Example 18.*



Note again that a strictly proper set which is low in efficiency (or graph-efficiency, if appropriate) retains motivic symmetry as one of its principal compositional resources. Hence it is not surprising that the twelve-tone system uses motivic material as its chief resource and that the historical progression from modal to chromatic music in the West has been characterized by an increased dependence upon harmonic material and motivic symmetry.<sup>28</sup> It is

<sup>27</sup>Herman Helmholtz, *On the Sensation of Tone* . . . , [1], page 370.

<sup>28</sup>Clearly, music written purely in the whole-tone scale would be even more motivically dependent, since harmonic resources are sparser.

also of interest to note that the existence of an ambiguous interval in a proper scale with high stability compensates for a loss in similarity between certain few modal transpositions of motifs by facilitating partition into principal and dependent tones and by increasing the effect of cadences (Section 16 and the remarks following Figure 5).

In the case of improper scales not all the above resources are available. Motivic similarities are not possible between all portions of the scale without severe distortion (which, of course, can be deliberately utilized). Also, the requirement of instantaneous measurement of intervals necessitates a partition into principal and dependent tones and/or the firm fixing of a tonic (Section 8). Since sufficient sets with as few elements as possible facilitate identification of the tonic and/or principal tones, low efficiency (high redundancy) is extremely important for the use of improper scales.<sup>29</sup> In fact, low efficiency is of greater significance to the use of improper scales than high efficiency is for proper scales, since only a small part of the compositional resources of proper scales are sacrificed by low efficiency. However, failure to identify the tonic and/or principal tones of an improper scale until many tones had been heard would decimate compositional materials. That is, the principal resources of melodic form when improper scales are employed are those symmetries which depend not upon similar interval relations, but upon similarities between sequences of principal and dependent tones. (Also, the non-invertability of improper scales and the "tense" quality of contradictory intervals have expressive potential which can be exploited by the use of symmetries and distinctions derived from such properties.) The above speculations are confirmed by the ethno-musico-logical investigations undertaken thus far.

The symmetries referred to above can be quite subtle, such as harmonic sequence in Western music and nuclear theme in Javanese music.<sup>30</sup> Many more abstract relations (and relations between relations) can be found in sophisticated music. Just as the choice of materials determines those symmetries which are recognizable at lower hierarchical levels (e.g., the motivic level), the choice of such symmetries in turn circumscribes the relations at the next higher level (e.g. phrase). Upon such decisions the musical characteristics of a culture and of individual style depend. By way of analogy it is of interest to note that many of the properties of spoken languages are consequences of the choice of materials. For example, the fact that Chinese words generally consist of only one syllable coupled with the variety of words in the commonly used vocabulary may necessitate the superimposition of sliding tones (which Chinese uses) in order to avoid ambiguity.

Here we have studied those relations which apply at the "phonemic level" of musical languages and some relations which apply at the next hierarchical levels (i.e., "motif" and "phrase" level). Many of the relations which may apply between units at the level above the phonemic level are circumscribed by the points on the graph *G*, of expected "reference frames". As we have seen, the graph *G*, is utilized to define equivalences between sets of tones with respect to properties defined on the graph. The removal or addition of graph points alters

<sup>29</sup>Unless, of course, the tonic is fixed in advance by a drone.

<sup>30</sup>See Mantle Hood, "The Nuclear Theme..." [5].

such equivalences (upon which available techniques of musical composition depend).

Note that at any hierarchical level in a musical composition the identification of the units themselves (e.g., "motif", "phrase", "sentence", "section") depends upon the properties with respect to which symmetries or equivalences occur. In unfamiliar music these properties (as well as the identity of the units dependent upon such properties) are often difficult to determine. (For example, it is very difficult for a Western listener to extract the "nuclear theme" from Javanese music.) Work is in progress which treats the problem of symmetries at all musical hierarchical levels. This consists of an adaptive model and algorithm which attempts to reduce actual music to a symbolic form (which represents the relevant symmetries, relations and properties) which, in turn, is used for synthesis. The reconstructed music is subject to feedback provided by a listener who indicates where the synthesis has been successful. The algorithm then alters the symbolic representation and convergence is attempted by repeating the procedure iteratively. This work will be described in another paper.

Computer programs which perform the computations described in this paper are available on request, as are tables of such computations for unfamiliar "reference frames". These are discussed in the next paper in this series.

## 24. Description of Equipment and Proposed Experiments

Only a very brief description will be given here of experiments and equipment for testing the theory. The equipment is nearly complete and experiments are expected to begin shortly. Ethnomusicological testing of the theory, however, has already yielded positive results and such information is available on request.

Several reference structures ("keys" of "scales" where the tonic is not fixed—henceforth called "structures") can be constructed in which a particular interval is acoustically identical in all, but is ambiguous in some cases and unambiguous in others. It is predicted that, for a single subject who learns two structures, the perception of such an interval common to both will be more difficult and less accurate after manipulating and listening to that structure in which the particular interval is ambiguous. A keyboard controlled device is presented to the subject which produces all tones in a particular structure and *no others*. The keyboard is arranged so that no information other than a correspondence between direction and the raising and lowering of pitch is provided. The structures used are, of course, unfamiliar to the listener (the keyboard device is capable of producing *any* set of tones with less than thirty-two elements within an octave). The subject is asked to manipulate the device until he feels he is able to anticipate any pitch from the surrounding pitches he produces. This is checked by asking him to tune (adjust the knob on) an oscillator with continuously variable frequency to a given pitch when deprived of the freedom to produce it on the keyboard. This is accomplished by disconnecting the lever on the keyboard from the output of its corresponding oscillator. Alternatively, the subject may be tested by being required to select the missing pitch from several alternatives presented to him.

Several tests are made with different frequencies eliminated in this manner. When these tasks are successfully performed the subject is removed from the keyboard. The experimenter then produces a sequence forming an interval which is either “ambiguous” or “unambiguous” for the particular structure. One of the two component tones is sustained and the subject is asked to match the other tone by adjusting a variable-frequency oscillator initially set at the frequency of the sustained tone. Alternatively, he is tested by being required to select the tone from among several alternatives. His performance is measured for accuracy and speed. The experiment is later repeated with the structure exchanged for another in which the particular interval under consideration is now ambiguous if it previously was unambiguous or vice versa. The structures being used are selected so that their stabilities ( $\bar{S}$ ) are as nearly equivalent as possible.

In another series of tests, tones *not* in the structure are produced, and the subject is required to match them on a variable frequency oscillator. We expect errors to occur in the direction of those tones of the structure in whose range the tone being matched lies, and such errors are expected to decrease as the boundaries of the range is approached.

The results of the above test may be checked against those of another series of tests wherein the subject is required to identify the tones not in the structure with tones in the structure which are “most similar”. Such identification is expected to differ when the same tone is presented in the context of two distinct structures (even when tones in both structures which are adjacent to the tone to be identified are identical in both cases), provided the tone to be identified lies in the range of different tones in each structure.

Of course, the timbre of the tones used in the above experiments are of central importance. The equipment will be capable of producing a large variety of timbres. Nearly all the potentialities of a large electronic organ and of an “electronic music synthesizer” are available simultaneously by the adjustment of keyboard controls. Inharmonic partials (non-integer multiples of the frequency of a given tone) can also be produced by the depression of a single key on the keyboard. Such resulting timbres can differ for each tone in a key of a scale. In this way the construction of cases where axiom 2.3 is violated will be attempted (see the fifth paper in this series). In the experiment timbre will be adjusted for each structure (which will often contain “irrational” frequency ratios) so as to minimize its resemblance to other familiar structures (e.g., any key of a “major scale”) and so that the construction of the initial ordering (as described in Part 1, Section 2) is as easy as possible.

Of course, all predictions tested by the experiments must take into account a listener-dependent minimum pitch discrimination,  $\epsilon$ . In a series of tones each of which differs from the previous tone by less than  $\epsilon$ , intransitivities in the relation of apparent equivalence (between tones) may occur, (e.g.,  $x \equiv y$ ,  $y \equiv z$  and  $x \not\equiv z$ ). Techniques for dealing with this situation are developed in the following section and are applied to the computation of experimental predictions.

The mathematical model also predicts a discrete alteration in the stability ( $\bar{S}$ ) of a structure and in the identity of its ambiguous intervals when certain amounts of mistuning of the tones forming the structure has taken place (see

Section 25, which follows). Thus the experiment previously described is repeated with the different structures replaced by such different tunings of the same structure ( $\bar{S}$  now differs for both tunings).

Another set of experiments will be performed to check predictions of the model concerning the function of sufficient sets. A particular structure (both proper and improper sets are used) is connected to the keyboard and the subject is instructed to manipulate and listen to the keyboard as before. He is then removed from the keyboard and informed that he will be presented with a sequence of tones belonging to this structure. However, he is also told that the absolute pitch location of the structure will be altered between each sequence (i.e. it may be in any key of the tuning of the scale). He is asked whether another tone added at the end of the sequence belongs to the same structure (key of tuning of the scale) as the preceding tones. He is presented with some sequences which, exclusive of the last tone, contain sufficient sets for a key and to which this tone does indeed belong, and some to which it does not. Some sequences are also presented which do not contain sufficient sets. The subject's responses are checked for identity with the predictions of the model.

These experiments are repeated with a particular element of the reference frame mistuned *after* the subject is removed from the keyboard. The limits of such mistuning which permit identifications of final tones of the sequence are checked against the computed *E*-range. These *E*-ranges are computed first for the key of the scale used and then are enlarged as much as possible so that the sufficient sets for each key remain distinct from those of other keys (which, by definition, are on the graph). Both mistunings are tested, as are mistunings which exceed such limits.

The experiments are again repeated with *several* elements of each structure mistuned after the subject is removed from the keyboard. This mistuning is varied as each sequence is presented. The variations are at first within the limits defined by *SE*-modifications and then exceed such limits. The cases where the subject produces both correct and incorrect identifications are correlated with the various mistunings.

A final repetition of the experiment is performed when more than one scale and its keys are on the graph. That is, before being tested, the subject is permitted to learn more than one unfamiliar structure (scale). He is then interrogated by being presented with sequences selected from keys of any such scale.

One of the by-products of the acceptance test for a subject (wherein he is required to produce missing tones in a reference structure to demonstrate that he has learned it) is the length of time required for such learning to take place. This will be correlated with the stability and propriety of the reference structures learned, and with the component intervals and the tone colors used.

Note that in these experiments, equipment is used which permits the subject to learn by *producing* stimuli. This is deliberate and derives from well-known theories and experimental results indicating that both visual and language learning is facilitated by such methods. Musical examples are well known. The teaching of musical dictation is much facilitated if the student is previously taught to sight-sing. Note also that in the acceptance test, when a subject is asked to produce a tone that has been eliminated from a learned reference



frame, he is provided with feedback to his response other than just "correct" or "incorrect". He may subsequently check himself by producing the correct tone (which is immediately reconnected to the keyboard).

The equipment used in the experiments also is useful as a musical instrument capable of exploiting new musical materials. The variety of pitches and tone colors available as well as the construction of its keyboards are suited to this purpose. These aspects of the equipment will be described in a subsequent paper covering details of the musical application.

## 25. The Use of a Listener—Dependent $\varepsilon$ .

For application to the perception of tones we must consider that it is impossible for a listener to order two intervals (or determine that they are distinct) when these differ by less than some small  $\varepsilon$  (dependent upon the listener and the timbre being used). This  $\varepsilon$  is an interval selected from  $S \times S$  and may therefore be compared with any interval in  $P \times S$ . Accordingly when addition is defined as in Part I, (2.4) we define  $P$  to be *strictly  $\varepsilon$ -proper* when

$$|\alpha_{i+1,j} - \alpha_{i,k}| \geq \varepsilon \text{ for all } i, j, k.$$

Analogously the  $\varepsilon$ -stability of  $P$  is defined as

$$\bar{S}_\varepsilon = 1 - \text{card} \left\{ (i, j) \mid |\alpha_{ij} - \inf_k (\alpha_{i+1,k})| < \varepsilon \vee |\alpha_{ij} - \sup_k (\alpha_{i-1,k})| < \varepsilon \right\} / n(n-1) \\ (i = 1, \dots, n-1; j = 1, \dots, n)^{31}$$

( $n$  is the period and  $m$  is the number of units per octave.)

If  $\|\alpha_{ij}\|$  is strictly proper and  $\varepsilon > T$  ( $T$  in the sense of Part I, Section 3) clearly  $\|\alpha_{ij}\|$  is not strictly  $\varepsilon$ -proper. However,  $\varepsilon$  will usually be used when a specific  $\|\delta_{ij}\|$  has been selected, and for such cases only we define the *average tolerance*

$$\bar{T} = \left( \sum_{i=0}^{n-1} T_i \right) / mn$$

Note that if  $\bar{S}_\varepsilon$  is computed with a fixed  $\varepsilon$ ,  $\bar{S}_\varepsilon$  will be decreased from this computed value when  $\varepsilon$  is increased to the point where  $\varepsilon \geq T_\varepsilon$ , where  $T_\varepsilon$  is defined as follows:

If  $T_i < \varepsilon$ ,  $T_i^\varepsilon$  is computed thus: Define  $V_i = \max_j (\alpha_{ij})$ ,  $D_{i+1} = \min (\alpha_{i+1,j})$ . Delete from the row  $i+1$  all entries  $\alpha_{i+1,j}$  for which  $\alpha_{i+1,j} - V_i < \varepsilon$ ; call the

<sup>31</sup> Note that a single element of row  $i$  may be within  $\varepsilon$  of elements in both rows  $i-1$  and  $i+1$ . Hence the symbol " $\vee$ " (or) appears in the formula for  $\bar{S}_\varepsilon$  so that such elements are not counted twice.

smallest one remaining  $\alpha_{i+1,j}$  and set  $b = \alpha_{i+1,j} - V_i$ . Now delete from the row  $i$  all entries  $\alpha_{ij}$  for which  $D_{i+1} = \alpha_{ij} < \epsilon$ ; call the largest remaining one  $\alpha_{ij}$  and set  $c = D_{i+1} - \alpha_{ij}$ .  $T_i^\epsilon = \min(b, c)$ . If  $T_i \geq \epsilon$ ,  $T_i^\epsilon = T_i$ ,  $T_\epsilon = \min(T_i^\epsilon)$ .

$T_i^\epsilon$  will be called the  $\epsilon$ -row tolerance and  $T_\epsilon$  the minimum  $\epsilon$ -tolerance. It is to be understood that  $T_i^\epsilon$  and  $T_\epsilon$  will be intervals in  $P \times S$  unless a particular  $\|\alpha_{ij}\|$  is specified, in which case they will assume numerical values.

$T_\epsilon$  corresponds to the pitch discrimination required of a particular listener for the computed value of  $\bar{S}$  to apply to his perceptions.

For the application to the perception of tones, which usually involves an octave cycle, we will assume that all  $\|\delta_{ij}\|$  which are members of the same  $\|\alpha_{ij}\|$  are drawn from sets,<sup>32</sup>  $S$ , such that the same interval corresponds to an octave cycle for all  $\|\delta_{ij}\| \in \|\alpha_{ij}\|$  and  $\epsilon$  is the same interval for all  $\|\delta_{ij}\| \in \|\alpha_{ij}\|$ . Then  $\epsilon$  is a constant fraction of an octave; that is for all  $\|\delta_{ij}\| \in \|\alpha_{ij}\|$ ,  $\epsilon/m$  is constant.<sup>33</sup>

Hence the structure of the equivalence classes is affected by considering two intervals which differ by less than  $\epsilon$  as "equal" (note that this is *not* an equivalence relation since it is possible that  $i_1 = i_2$ ,  $i_2 = i_3$  and  $i_1 \neq i_3$ ):

(a) All  $\|\alpha_{ij}\|$  will have a finite number of members since when  $m$  is so large that

$$\epsilon > \min_{ijkl} (\delta_{ij} - \delta_{kl} | \delta_{ij} \neq \delta_{kl}) \quad (1)$$

the ordering of the  $\delta_{ij}$  is altered so that  $\|\delta_{ij}\|$  is no longer a member of  $\|\alpha_{ij}\|$ .

(b) Those  $\|\alpha_{ij}\|$  where  $\psi(P)^*(K \text{ and } m \text{ are minimum})$  is such that (1) applies will be empty. (See Section 11, Part II.)

(c) Some  $\|\alpha_{ij}\|$  will acquire additional members in cases where  $m$  is such that marginally unequal intervals become equal so as to cause the ordering to coincide with that of  $\|\alpha_{ij}\|$ .

Since, when we consider two intervals which differ by less than  $\epsilon$  as equal, we are no longer dealing with an equivalence relation, the equivalence class notation used thus far fails (e.g.  $\psi(P) = (3, 2, 2, 2, 2)$  and  $\epsilon = 1$ ).

To remedy this we define two scales as  $\epsilon$ -equivalent if and only if whenever two intervals in the first scale differ by  $\epsilon$ , so also do the corresponding ones in the second and vice versa. The following notation realizes this in a way convenient for automatic computation and the classes remain equivalence classes; i.e.,

$$(\|\alpha_{ij}^1\| \equiv \|\alpha_{ij}^2\|) \wedge (\|\alpha_{ij}^2\| \equiv \|\alpha_{ij}^3\|) \Rightarrow \|\alpha_{ij}^1\| \equiv \|\alpha_{ij}^3\|.$$

Let  $\beta(P)$  be a vector containing the subscripts of all the terms of the first  $n-1$  rows of  $\|\delta_{ij}\|$ . The order in which these subscripts appear will be the same as the ascending order of the values  $(\delta_{ij})$  to which each such subscript corresponds. When  $\delta_{ij} = \delta_{kl}$ ,  $ij$  precedes  $kl$  if  $i < k$  or if  $i = k$  and  $j < l$ . All such subscripts corresponding to equal values in  $\|\delta_{ij}\|$  will be enclosed by brackets.

<sup>32</sup>In most cases it may be assumed that all  $\|\delta_{ij}\| \in \|\alpha_{ij}\|$  are drawn from the same set  $S$ .

<sup>33</sup>It may well be that  $\epsilon$  is a function of  $\bar{n}$  (the number of scale-notes per octave). In such case,  $\bar{n}$  is constant for all members of an equivalence class, and  $\epsilon$  may be experimentally determined.

When not separated by brackets, terms of  $\beta(P)$  will be separated by dots (“.”).

*Example 19.*  $\psi(P) = (3, 2, 3, 2, 2)$ ,  $\varepsilon = 0$

$$\|\delta_{ij}\| = \begin{bmatrix} 3 & 2 & 3 & 2 & 2 \\ 5 & 5 & 5 & 4 & 5 \\ 8 & 7 & 7 & 7 & 7 \\ 10 & 9 & 10 & 9 & 10 \end{bmatrix} \quad \|\alpha_{ij}\| = \begin{bmatrix} 2 & 1 & 2 & 1 & 1 \\ 4 & 4 & 4 & 3 & 4 \\ 6 & 5 & 5 & 5 & 5 \\ 8 & 7 & 8 & 7 & 8 \end{bmatrix}$$

$$\beta(P) = (1, 2, 1, 4, 1, 5)(1, 1, 1, 3)2, 4(2, 1, 2, 2, 2, 3, 2, 5)(3, 2, 3, 3, 3, 4, 3, 5) \\ 3, 1(4, 2, 4, 4)(4, 1, 4, 3, 4, 5)$$

If we use the symbol “ $\underline{\varepsilon}$ ” to denote that two intervals differ by less than  $\varepsilon$ ,  $[(p_i p_j) \underline{\varepsilon} (p_k p_l)] \wedge [(p_k p_l) \underline{\varepsilon} (p_q p_r)] \wedge [(p_i p_j) \underline{\varepsilon} (p_q p_r)] \Rightarrow [(p_i p_j) \leq (p_k p_l) \leq (p_q p_r)] \vee [(p_q p_r) \leq (p_k p_l) \leq (p_i p_j)]$ . Hence,  $\beta(P)$  can be modified so that brackets enclose all terms which are  $\underline{\varepsilon}$  and the properties of an equivalence class are retained. In Example 19, if  $\varepsilon = 1$ ,

$$\beta(P) = (1, 2, 1, 4, 1, 5)(1, 1, 1, 3)(2, 4)2, 1, 2, 2, 2, 3, 2, 5) \\ (3, 2, 3, 3, 3, 4, 3, 5)(3, 1)(4, 2, 4, 4)4, 1, 4, 3, 4, 5)$$

where the first left-hand bracket pairs with the first right-hand bracket, the second left-hand with the second right-hand, etc.

Using this notation different  $\psi(P)$ 's can be mapped into corresponding  $\beta(P)$ 's using the relation of “ $\underline{\varepsilon}$ ” and these  $\beta(P)$ 's mechanically compared for identity.

When  $T_\varepsilon \geq 0$  only the first  $\frac{n(n-1)}{2}$  terms of  $\beta(P)$  are needed,<sup>34</sup> and the absence of one bracket in the final pair will indicate equality of the last group of terms to a term not shown. Also, the row subscript of  $\delta_{ij}$  can be omitted in these cases if a partition sign “|” is used to separate rows. Hence the above example becomes:

$$\beta(P) = (2, 4, 5(1, 3)|(4)1, 2, 3, 5).$$

If any particular element of  $P$ , say  $P_k$ , is altered by an amount,  $\varphi_k$ , there will exist some maximum positive and negative values of  $\varphi_k$  for which  $P$  will remain in the same equivalence class, i.e.,  $E$ -range, see Section 10. If, as above, the relation of “ $\underline{\varepsilon}$ ” is used instead of “ $=$ ” in the mapping, these maxima will be altered.

Note that if  $\varphi_k$  is added to  $p_k$  only certain elements of  $\delta_{ij}$  are altered. Since  $\delta_{ij} = p_i + j - p_j$ , these are  $\delta_{ik}$  and  $\delta_{i, k-i}$  for all  $i < n$ . If  $\varphi_k$  is positive, all  $\delta_{ik}$  are

<sup>34</sup> $T_\varepsilon > \varepsilon$  guarantees that no terms from the remainder of  $\|\delta_{ij}\|$  enter into the total ordering of the first  $\frac{n(n-1)}{2}$  terms—see Part I, Section 6.

reduced and all  $\delta_{i,k-i}$  are increased. If  $\varphi_k$  is negative, the reverse is true. Let  $\varphi_k^+$  and  $\varphi_k^-$  represent the maximum positive and negative values respectively which  $\varphi_k$  can assume without altering the equivalence class membership of  $P$ . An examination of all possibilities yields the following formula:

$$\varphi_k^\pm = \min_{\max} \left\{ \begin{array}{l} \min_{\max} \left\{ \begin{array}{l} 1/2(\delta_{lk} - \delta_{i,k-i} \pm \epsilon) |\delta_{i,k-i} - \delta| \leq \epsilon \\ 1/2(\delta_{lk} - \delta_{i,k-i} \mp \epsilon) |\delta_{lk} - \delta_{i,k-i}| > \epsilon \end{array} \right. \\ \min_{\max} \left\{ \begin{array}{l} \delta_{ij} - \delta_{i,k-i} \pm \epsilon |\delta_{i,k-i} - \delta_{ij}| \leq \epsilon \\ \delta_{lk} - \delta_{ij} \pm \epsilon |\delta_{ij} - \delta_{lk}| \leq \epsilon \\ \delta_{ij} - \delta_{i,k-i} \mp \epsilon |\delta_{ij} - \delta_{i,k-i}| > \epsilon \\ \delta_{lk} - \delta_{ij} \mp \epsilon |\delta_{lk} - \delta_{ij}| > \epsilon \end{array} \right. \end{array} \right.$$

where the upper sign (“+” or “-” or “min” or “max”) corresponds to  $\varphi_k^+$  and the lower to  $\varphi_k^-$ . The formula<sup>35</sup> is deliberately written redundantly to indicate the procedure for efficient computation.

The above indicates that each equivalence class when the relation “ $\epsilon$ ” is used has both a maximum and minimum pitch discrimination necessary for its apprehension.  $\epsilon$  corresponds to this maximum; i.e., intervals which differ by less than  $\epsilon$  must *not* be distinguished. Let  $\Phi = \min_k (|\varphi_k^+|, |\varphi_k^-|)$ . Then  $\Phi$  corresponds to the minimum; i.e., intervals which differ by more than this amount *must* be distinguished.

We now define an  $\epsilon$ -sufficient set  $Q_\epsilon$ , as previously, except that the meaning of “subset” is altered: A set of points  $Q^*$  is called an  $\epsilon$ -subset of  $P$  if all points of  $Q^*$  correspond to points of  $P$  biuniquely in such a way that the difference between any pair of points in  $Q^*$  and the difference between the corresponding pair of points in  $P$  differ by no more than  $\epsilon$ .

An  $\epsilon$ -minimal set is an  $\epsilon$ -sufficient set which contains no proper subset<sup>36</sup> which is  $\epsilon$ -sufficient.

$\epsilon$ -efficiency,  $E_\epsilon$  is defined similarly to efficiency except that  $\epsilon$ -sufficient sets are used.

Also,  $E_\epsilon^G$  and  $E_\epsilon^N$  are the same as  $E^G$  and  $E^N$  respectively, except that  $\epsilon$  is involved in the determination of the graph and node-sufficient sets respectively.

## References

1. H. L. F. HELMHOLTZ, *On the sensations of tone as a physiological basis for the theory of music*, (translated by Alexander J. Ellis, 1885) Peter Smith, New York, 1948.
2. P. BOOMSLITER AND W. CREEL, The long pattern hypothesis in harmony and hearing, *J. Music Theory* 5 (1961), 2-31.

<sup>35</sup>The above formula, strictly speaking, in some cases should be “ $\varphi_k^+ < \text{some quantity}$ ”, or “ $\varphi_k^- > \text{some quantity}$ ,” indicating that the conditions are not satisfied at equality. However, for simplicity this is omitted.

<sup>36</sup>“Proper” in its usual meaning in terms of set inclusion!

3. J. C. R. LICKLIDER, Three auditory theories, in *Psychology: a study of a science*, (ed. by S. Koch) McGraw-Hill, New York, 1959, 41–144.
4. J. KUNST, *Music in Java*, Martinus Nijhoff, The Hague, 1949.
5. MANTLE HOOD, *The Nuclear Theme as a Determinant of Patet in Javanese Music*, J. B. Wolters—Groningen, Djakarta, 1954.
6. MANTLE HOOD, Slendro and Pelog redefined, *Selected Reports*, Institute of Ethnomusicology, U.C.L.A., 1966.
7. D. ROTHENBERG, *A mathematical model for the perception of redundancy and stability in musical scales*. Paper read at Acoustical Society of America, New York, May 1963, Also: Technical Reports to Air Force Office of Scientific Research 1963-1969, (grants and contracts AF-AFOSR881-65, AF49(638)-1738 and AF-AFOSR68-1596)
8. G. M. STRATTON, Vision without inversion of the retinal image, *Psych. Rev.* **4** (1897), 341–360 and 463–481.
9. R. H. THOULESS, Phenomenal regression to the real object, *Brit. Jour. Psychol.*, **21** (1931), 339–359.
10. M. VON SENDEN, *Raum-und Gestaltauffassung bei operierten Blindgeborenen vor und nach der Operation*, Barth, Leipzig, 1932.
11. CARROL C. PRATT, Comparison of tonal distance and bisection of tonal intervals larger than an octave, *Jour. of Experimental Psychol.*, **11** (1928), 77–87 and 17–36.
12. H. MUNSTERBURG, Vergleichen der Tondistanzen, *Beitrage zur Experimentelle Psychology*, **4** (1892), 147–177.
13. POINCARÉ, *The Value of Science*, Dover Publications, New York, 1954.
14. P. HINDEMITH, *The Craft of Musical Composition*, Associated Music Publishers, New York, 1941.
15. OLIVIER MESSIAEN, *Technique de mon Langage Musical*, Alphonse Leduc, Paris, 1944.
16. J. F. SCHOUTEN, R. J. RITSMA, AND B. LOPEZ CARDOZO, Pitch of the Residue, *Jour. of the Acoustical Society of America*, **34** (1962), Part 2, 1418–1424.
17. J. E. EVETTS, The Subjective Pitch of a Complex Inharmonic Residue, unpublished report, Pembroke College, England, 1958.

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