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ON COHERENCE AND SAMENESS, AND THE EVALUATION OF SCALE CANDIDACY CLAIMS

Norman Carey

I. Pitch-Class Sets and Musical Scales

We begin with a view of the universe of pitch classes in 12-tone equal temperament, shown in (a) in Example 1. Pitch class n is represented by the fraction $n/12$, for $n = 0, 1, \ldots 11$. Next, we expand the field of discussion to the set of all real numbers greater than or equal to 0 and strictly less than 1, shown in 1(b). This represents an infinite pitch-class space that can be understood as having been constructed by filling in the blank spaces of 1(a). In this paper, I would like to inquire into this question: which pitch-class subsets of 1(b) might function as scales?

A. The Problem of Musical Scales

The term "scale" is problematic in a number of respects. Both parts of Example 1 carry an implicit assumption that scales fill an octave span. Even if this assumption is set aside, we will still think of scales as replicating the same pattern of steps throughout pitch space at some interval of recurrence. Rahn (1977, 1981) has attempted to define boundaries that would separate scales from non-scalar pitch-class sets. Given the breadth of instantiations and cultural histories of the term scale, I am doubtful that such a boundary can be safely drawn. In this paper, I will accept that

a. The finite pitch-class space of 12-tone equal temperament

0	1/12	2/12	3/12	4/12	5/12	6/12	7/12	8/12	9/12	10/12	11/12	**1**
•	•	•	•	•	•	•	•	•	•	•	•	(◦)

b. The infinite pitch class space defined by the set of real numbers ρ, $0 \leq \rho < 1$

0 ____ **1**

Example 1. Finite and infinite pitch-class spaces.

while the boundary between “scale” and “ordinary pitch-class set” is necessarily porous, it is still possible to formalize the radar we use to decide if we have crossed it. The methods will prove applicable to a variety of actual scales including diatonic, pentatonic, and chromatic systems, as well as many scales in non-Western musics. These methods are also suitable for investigating existing microtonal scales, and for formulating new ones.

The term scale comprises a constellation of related notions. We understand a scale to be a series of pitches bounded by an interval of recurrence, ordinarily the octave. A scale sometimes serves as a kind of tone bank, or pitch repository, providing pitch-class material for scales of smaller cardinalities. The Arabic *maqam* system of 17 notes to the octave comes to mind in this context as well as the 22 *srutis* in Indian music, and, indeed, the Western equal-tempered 12-tone scale. Scales are often conflated with keys, and so take on significance in studies of tonality. In this context we attend to functions of scale degrees and their concomitant hierarchization. The closely related notion, ‘mode,’ also makes its appearance in this context.

Finally, we think of scales as providing primordial melodic stuff. Scales establish a general lower limit on the size of the standard melodic interval, the step, and allow for the most basic melodic distinction, that of the normative melodic step, and the marked element, the skip.

B. Three Factors Governing Scale Candidacy

I would like to propose that there are three main parameters that serve to distinguish scales from the more general category of pitch-class sets. The more complex these factors are, the less likely it is that a given set will function as a scale. These factors are not independent of each other; however, I will attempt to isolate them as much as possible, and then describe their potential interactions. The three factors are cardinality, the existence of generators, and generic intervals.

1. Cardinality

The first two factors have already come under some considerable scrutiny. It is self-evident that the cardinality of a pitch-class set has a direct bearing upon its suitability as a scale. Most of the scales in Rahn's studies are 7-note sets, along with occasional smaller ones. The choice is justifiable from the perspective of Miller's (1957) work on the significance of the value 7±2 as a general limit in the cognition and memory of categories. Although a large number of well-known scales fall within the 7±2 range, scales of other cardinalities will also be considered here. The "pitch repository" type of scale, such as the 12-tone equal-tempered scale, typically exceeds the 7±2 limit. Nevertheless even the largest pitch repository scales seldom exceed 53 pitch classes. I will suggest a generous practical upper limit to be set at 60.[1]

2. Generators

The second factor has to do with the existence of scale generators. The diatonic scale in 12-tone equal temperament is said to be generated by the perfect fifth, or fourth, while the diminished triad is generated by the minor third. What is involved here is really the work of two intervals, the interval of periodicity (octave) and the generator itself. Repeated iterations of the interval of the perfect fifth modulo the perfect octave produce the pitch classes in the diatonic scale. Under this conception, the diatonic scale can be represented on a two-dimensional *Tonnetz*, with octaves extending along one axis, and fifths along the other. (See Carey and Clampitt 1996, 117.) The diatonic in Just Intonation, then, could be represented by a three-dimensional *Tonnetz*, with major thirds providing the third axis. While not generated in a group-theoretic sense, we might still profitably examine structures such as the diatonic under Just Intonation (JI) in this light. If this were to be the case, we would wish to represent our scales in the most parsimonious way with respect to "generators." (Note that we are discussing pitch-space representations. JI *Tonnetze* are often represented in pitch-class space as two-dimensional structures that suppress octave information.) The expression "*p*-limit Just Intonation" where *p* is a prime makes implicit use of this idea of multiple "generators." Thus, in 7-limit JI, perfect fifths, major thirds, and septimal sevenths generate the various tones that fill the octave. I will propose that the more generators it has, the more complex a pitch set might be, and the lower its scalar potential will be. Harry Partch's system, a "pitch repository" type of scale, is a realization of 11-limit JI (Partch 1979). Its 43 tones already betoken a rather high degree of complexity, and its five generators further compound the situation. Partch offers justifications for stopping short of a 13-limit system with six generators (Partch 1979, 120–27). Following Partch, I will propose that scale candidacy is virtually nil when the number of generators is greater than five.

Generators need not be consonances, of course. Dissonant generators (including the interval of periodicity) entail greater complexity, making a pitch-class set less likely to be a scale. Schoenberg also associates dissonance with complexity:

> Dissonances, even the simplest, are more difficult to comprehend than consonances. And therefore the battle about them goes on throughout the length of music history . . . The criterion for the acceptance or rejection of dissonances is not that of their beauty, but rather only their perceptibility.[2]

Generators, then, add complexity to pitch-class sets in two ways: by their number (as delimited) and by their degree of dissonance. Conversely, pitch-class sets that are generated by a small number of consonant intervals are more likely to serve as scales.

3. Generic Intervals

The third factor is comprised of a number of sub-factors that concern the cognitive significance of the generic interval. First made explicit by Clough (1979), the generic interval plays a crucial role in determining scalar utility. It will be useful here to define formally some of the terms that will be used in this study. *Pitch* refers to the "height" aspect of a musical tone. Pitch may be represented by a note name with register (A4), or by a frequency (440 cps), or by the logarithm of the frequency. An ordered interval is comprised of an ordered pair of pitches. The size of an interval is given either by the ratio between the respective frequencies, or the difference between log frequencies. A certain interval is often chosen in order to induce an equivalence relation upon the set of pitches. Referred to here as the *interval of periodicity*, the prototype for this interval is the standard 2:1 octave. Let us say that the log frequency ratio of the interval of periodicity is p. Let r_1 and r_2 represent the log frequency of a pair of pitches. These pitches belong to the same *pitch class* if $\{r_1/p\} = \{r_2/p\}$, where the curly brackets represent the fractional part of the value enclosed. Interval class can be defined in exactly the same way. Two intervals whose logarithmic sizes are i_1 and i_2 belong to the same *interval class* if $\{i_1/p\} = \{i_2/p\}$.[3] More simply, if we let unity represent the logarithmic size of the interval of periodicity, then two pitches belong to the same pitch class if their fractional parts are equal, and similarly with intervals and interval classes.

The specific and generic aspects of intervals will be indicated by the terms *size* and *span*, respectively. It is the specific *size* of the interval that we measured above in frequency ratios or in some logarithmic scale, such as Ellis's cents (Helmholtz 1983, 431). I will use the abbreviation ¢ to express specific interval size in cents. Generic intervals will be indicated by the term *span*. The term *step* signifies a generic interval whose span is one. A "third" has a span of two, a "fourth" of three, and so on. The term "span" averts two potential misunderstandings. Firstly, it avoids an unwanted reference to the labels "third," "fourth," etc., most commonly associated with the diatonic collection. Secondly, unlike these common labels, generic spans can be added or subtracted using arithmetic modulo N, where N is the cardinality of the scale.

Let me propose that a pitch-class set may function as a scale when its generic intervals efficiently organize and encode its specific intervals. Put simply, a scale is that kind of pitch-class set in which it makes sense to think about intervals generically. We generally do not impose the notion of the generic interval upon an ordinary pitch-class set. To think generically is to measure things in scale steps. From an information standpoint, the term "step" is rich in potential meaning, while "whole step" is a condensation of that meaning into a particular. Under what conditions are generic intervals cognitively significant? Put another way, how much information can we reasonably expect generic intervals to be able to encode?

a. Sameness and Difference. Consider the set of steps in the C-major scale: {CD, DE, EF, FG, GA, AB, BC}. Rahn (1991) introduces the notion of *difference*: an instance of difference involves a pair of intervals that have the same generic size but different specific sizes, such as the whole step CD and the half step EF. It will be useful to have a term for the opposite of difference. We will call the appearance of two different specific intervals with the same size and span an instance of *sameness*. In C major, the whole steps CD and DE form an instance of sameness.

Let me propose, as Rahn (1991, 39) implies, that difference associates with complexity. The greater the number of differences, the less likely a pitch-class set will act as a scale. The diatonic under equal temperament contains 56 instances of difference, whereas in JI there are 90. Both have significantly fewer differences than the maximum, which, as I will show, is 126 in the case of 7-note sets. From this perspective, JI endows the diatonic with more complexity than equal temperament does.

In general, scales with *Myhill's Property* (Clough and Myerson 1985) will be relatively low in differences. *Myhill's Property* is said to obtain when each generic class contains specific intervals of exactly two sizes.[4] Clough and Myerson also consider generic intervals in terms of *partitioning*. The generic intervals can be said to partition the specific intervals when intervals whose sizes are the same are always of the same span. In equal temperament, partitioning fails for the diatonic scale due to the presence of the tritone: the augmented fourth and diminished fifth are intervals of the same size, but of different spans. Thus, *Myhill's Property* accrues to the diatonic—under equal temperament—but not partitioning. The situation is somewhat reversed in the diatonic under JI: *Myhill's*

Property is absent, while partitioning abides. Under JI, the diatonic scale exhibits *trivalence*, according to Clampitt (1997). Under Clampitt's definition, a scale is trivalent if each generic class contains specific intervals of exactly three sizes. On the other hand, in any equal-tempered scale, because there is only one specific interval in each generic class, the notion of generic interval becomes redundant. Consequently, equal-interval sets play something of a delimiting role in this study. Their inherent regularity makes them reasonable scale candidates limited only by the first two factors, cardinality and the presence of a consonant generator.

To summarize so far: generic intervals are useful if they contain some, but not too much information. Under this assessment, the specific intervals belonging to each span should be relatively low in difference, and, where different, relatively even in size. In positive terms, generic intervals should provide some order and regularity on the set of specific intervals, as the partitioning property, *Myhill's Property*, and trivalence do.

b. Coherence and Failure. Even with partitioning and/or *Myhill's Property*, the generic classes still may be counterintuitive. Whether one uses the usual labels such as "third" or "fourth" or the "spans" used in this study, these labels imply an ordering. Consequently, we expect that knowing the span of an interval will provide a reliable clue as to its actual (specific) size, although this is not always the case. This leads us to a main issue of this paper, a consideration of the notion of *coherence*. According to Balzano (1980), a scale is coherent if all of its seconds are smaller than all of its thirds, and so on. Rothenberg (1978) defines a scale with this property as "strictly proper," and I have used the term "generically ordered" in the same sense (Carey 1998). Rahn (1991) distinguishes between two types of coherence failure with the terms "ambiguity" and "contradiction." Rothenberg (1978) uses the same terms in the same way. An *ambiguity* exists when two instances of a specific interval have different generic descriptions. The tritones of equal temperament again illustrate: an interval of 600 cents may be construed either as a fourth or as a fifth. In other words, ambiguities (and only ambiguities) cause partitioning to fail. A *contradiction* exists between two intervals, A and B, if A is larger than B, yet B spans more steps than A does. For example, a contradiction exists between the augmented fourth and the diminished fifth of Pythagorean tuning, since the Pythagorean augmented fourth (612¢) is larger than the diminished fifth (588¢). Whereas Agmon (1989) defines coherence as the lack of contradiction only, I will adopt Balzano's stricter convention, and deem a scale to be coherent which contains neither ambiguities nor contradictions.

In a coherent set, if two intervals of identical size are heard, I can be sure that they belong to the same generic class. If a second interval is heard that is larger than the first, I know that one of two possibilities must be true: either the second has a greater span than the first or they have the same span. Ambiguity destroys the possibility of certitude in the first case, contradiction in the second. Either type of coherence failure involves a pair of intervals with conflicting spans and sizes. Conversely, it is the direct or implicit assumption of all of the above studies that coherence is cognitively advantageous, and that assumption will not be challenged here. The rough measure provided by generic interval size most effectively encodes specific interval information when coherence prevails. (See Rothenberg 1978.)

It is obvious that coherence failures are not necessarily fatal to a scale. Scales often have coherence failures, and may be all the more interesting for them. Indeed, the *pycna* of ancient Greek scales are defined in such a way as to guarantee the appearance of ambiguities or contradictions. What I am positing, however, is that coherence failures make a set more complex, and, as I will show, that there is some limit to how many a set can sustain and still be a viable candidate for scale-hood.

II. Maximal Differences and Failures for Sets of Order N

It will be useful to determine the upper limit on the two complexity factors involving generic intervals, difference and coherence failure. Based upon these general results, algorithms will be presented that count the number of differences and failures for any given scale. By comparing maximal and particular values, we derive a pair of quotients that will help to analyze claims of scale candidacy. This topic will be the focus of Part III of this paper.

A. The Heteromorphic Profile of a Set

In his 1991 article, Rahn refers collectively to contradictions, ambiguities, and differences as *heteromorphisms*. Following this, I define the *heteromorphic profile* of a set *S* to be the ordered triple (*c*, *a*, *d*), where *c* is the number of contractions in *S*, *a* the number of ambiguities, and *d* the number of differences.

Example 2 shows the intervals that can be derived from the Japanese pentatonic, the *In* scale, the best known mode of which is *hirajoshi*. The generic labels given for each row are the "spans" discussed earlier. The scale is trivalent: each generic interval comes in three sizes. Example 3 shows the information for the same scale in matrix form. The heteromorphic profile of the *In* scale is (4,1,30). That is, the scale has thirty differences, and exhibits both kinds of failure. There are four contradictions. One of them is shown between the "third" with specific size three, and the "step" with specific size four. The single ambiguity in the scale is also shown on the array.[5]

By contrast, as shown in Example 4, the heteromorphic profile of the

Example 2. Intervals in the *In* scale

Generic Intervals		Specific Intervals				
	4	8	11	8	10	11
	3	7	7	7	6	9
	2	5	6	3	5	5
	1	1	4	2	1	4

Ambiguity – "third" is same size as "fourth".
Contradiction – "step" is larger than "third".

Number of contradictions: 4
Number of ambiguities: 1
Number of differences: 30
Heteromorphic Profile: (4,1,30)

Example 3. The *In* scale matrix

		Specific Intervals				
Generic Intervals	4	9	10	10	9	10
	3	7	7	8	7	7
	2	4	5	5	5	5
	1	2	2	3	2	3

Number of contradictions: 0
Number of ambiguities: 0
Number of differences: 20
Heteromorphic Profile: (0,0,20)

Example 4. Matrix for the anhemitonic pentatonic

anhemitonic pentatonic, the usual black-key pentatonic, is (0,0,20). From the point of view of both coherence failure and difference, the usual pentatonic is simpler than the *In* scale, but in order to compare these results more intelligently, we will need to know the maximum number of differences and failures for a scale of cardinality *N*.

B. The Interval Matrix

Example 5 shows a generalization of the interval matrix of the previous examples.[6] The sums in each cell illustrate the principle governing the formation of the matrix. This five-note interval matrix is generalized to *N* notes in the obvious way. The row labels on the left identify intervals by generic span.[7] Clearly, because there are $N-1$ generic intervals, and because each row contains *N* specific intervals, there are $N(N-1)$ intervals altogether. Because differences and failures involve pairs of intervals, we first need to determine the total number of pairs of distinct intervals in the matrix. Theorem 1 gives this result.

NB: The following proofs require four auxiliary propositions involving summation of series. These are listed in Appendix A as Lemmas 1–4. Appendix B lists the theorems proven in this paper.

Let IP(*N*) represent the total number of interval pairs in an interval matrix.

THEOREM 1. $\mathrm{IP}(N) = \dfrac{(N^2-N)(N^2-N-1)}{2}$.

		Specific Intervals				
Generic Intervals	4	A+B+C+D	B+C+D+E	C+D+E+A	D+E+A+B	E+A+B+C
	3	A+B+C	B+C+D	C+D+E	D+E+A	E+A+B
	2	A+B	B+C	C+D	D+E	E+A
	1	A	B	C	D	E

Example 5. A general interval matrix for $N = 5$

PROOF: We have just shown above that there are $N(N-1)$ or $(N^2 - N)$ intervals in the matrix altogether. Assume any ordering on these intervals. Then the first interval forms $(N^2 - N) - 1$ pairs with the remaining intervals, the second interval forms $(N^2 - N) - 2$ pairs, that is, with all but itself and the first, and so on. Therefore,

$$\mathrm{IP}(N) = \left(N^2 - N - 1\right) + \left(N^2 - N - 2\right) + \ldots + 1 \tag{1.1}$$

$$\mathrm{IP}(N) = 1 + 2 + \ldots + \left(N^2 - N - 1\right) \tag{1.2}$$

$$\mathrm{IP}(N) = \sum_{i=1}^{N^2-N-1} (i) \tag{1.3}$$

We employ Lemma 2, substituting $(N^2 + N - 1)$ for n, and this leads directly to our result:

$$\mathrm{IP}(N) = \sum_{i}^{N^2-N-1} i = \frac{\left(N^2 - N - 1\right)\left(N^2 - N\right)}{2} \tag{1.4}$$

QED. The second column of Appendix C shows the results for the number of interval pairs for N from 3 to 25.

C. Differences

I will now show the limit on the maximal number of differences for a scale of cardinality N. Let $\mathrm{MD}(N)$ represent this limit.

THEOREM 2. $\mathrm{MD}(N) = \dfrac{(N)(N-1)(N-1)}{2}$

PROOF. Differences are to be found among the specific intervals within each of the $N-1$ non-zero generic intervals. Assume that all of the places in an interval array for a scale of cardinality N are different. Then step A forms a difference with the remaining $N-1$ steps, B with the $N-2$ excluding itself and A, and so on, all the way to the one difference formed by the next-to-last and the last steps. This value is 1+2+...+(N–1), which, as Lemma 2 shows, is equal to $\frac{(N-1)N}{2}$. Since the same value is repeated in each of the $N-1$ rows, the result follows:

$$\mathrm{MD}(N) = (N-1)\left(\sum_{i}^{N-1} i\right) = (N-1)\frac{((N-1)(N))}{2} = \frac{(N)(N-1)(N-1)}{2} \tag{2.1}$$

QED. Consequently, the maximal number of differences for a scale of cardinality N compared to the total number of interval pairs is slightly more than $1/N$.

COROLLARY 2a. $\frac{\mathrm{MD}(N)}{\mathrm{IP}(N)} \approx \frac{1}{N}$

$$\frac{\mathrm{MD}(N)}{\mathrm{IP}(N)} = \frac{(N)(N-1)(N-1)}{2} \div \frac{(N^2-N)(N^2-N-1)}{2} \tag{2a.1}$$

$$= \frac{2(N^2-N)(N-1)}{2(N^2-N)(N^2-N-1)} = \left(\frac{1}{N} + \frac{1}{N(N^2-N-1)}\right) \tag{2a.2}$$

Then we can say, as N increases, $\frac{\mathrm{MD}(N)}{\mathrm{IP}(N)}$ approaches $\frac{1}{N}$. This is significant: the role that difference will play in the question of scale candidacy diminishes with increasing cardinality. Difference provides information only about pairs of intervals belonging to the same generic class. While useful, difference (or sameness) will not be as powerful a determinant of scale candidacy as coherence.

The sixth column of Appendix C shows the results for the maximal number of differences for N from 3 to 25.

D. Coherence Failures

The result regarding the maximal number of coherence failures comes in two stages. Some interval pairs in the matrix are immune to failure, and so next we find PF(N), the number of interval pairs which may potentially fail. These potential failures cannot all fail simultaneously: as I will show, the existence of a failure involving some interval pair can constrain another pair not to fail. This leads to the main result, SF(N), the maximum number of simultaneous failures for sets of cardinality N.

a.

Generic Intervals		Specific Intervals				
	4	A+B+C+D	B+C+D+E	C+D+E+A	D+E+A+B	E+A+B+C
	3	A+B+C	B+C+D	C+D+E	D+E+A	E+A+B
	2	A+B	B+C	C+D	D+E	E+A
	1	A	B	C	D	E

b.

Generic Intervals		Specific Intervals				
	4	A+B+C+D	B+C+D+E	C+D+E+A	D+E+A+B	E+A+B+C
	3	A+B+C	B+C+D	C+D+E	D+E+A	E+A+B
	2	A+B	B+C	C+D	D+E	E+A
	1	A	B	C	D	E

c.

Generic Intervals		Specific Intervals				
	4	A+B+C+D	B+C+D+E	C+D+E+A	D+E+A+B	E+A+B+C
	3	A+B+C	B+C+D	C+D+E	D+E+A	E+A+B
	2	A+B	B+C	C+D	D+E	E+A
	1	A	B	C	D	E

Example 6. Potential failures with origins A, A+B, and A+B+C.
a. Origin A and 6 potential failure targets
b. Origin A+B and 5 potential failure targets
c. Origin A+B+C and 3 potential failure targets

1. Potential Failures.

In order to count potential failures, we will refer to the interval with the smaller generic span as the *origin* and the interval with the larger span as the *target*. There are two types of interval pairs for which failure is impossible. By definition, failures do not occur between intervals of the same span. Failures are also impossible when an origin is one of its target's summands. (If X is the size of an origin, and $X + Y$ is the size of a target, then $X \geq X + Y$ is impossible, since interval sizes are always positive.) Example 6(a) shows that step interval A as origin can fail with six potential targets. All of the other intervals are either other steps, with which interval A cannot fail, or intervals that contain A in their sums. Similarly, Example 6(b) shows that there are five potential failures for the interval whose size is A+B. Finally, Example 6(c) shows that there are three potential failures that originate in the interval whose size is A+B+C. Thus, the number of potential failures that include intervals in the first column as origins is 14.

The *degree of failure* is the difference between the spans of the origin and target of an interval exhibiting coherence failure. For example, origin A+B (span 2) forms a potential second-degree failure with B+C+D+E (span 4). Example 6(a) shows three potential first degree failures, two of the second degree, and one of the third; Example 6(b) has three potential first degree failures and two of the second degree; and Example 6(c) has three potential first degree failures. Consequently, there are 3x3 potential first-degree failures shown in all, 2x2 of the second degree, and 1x1 of the third, that is $3^2 + 2^2 + 1^2 = 14$ failures whose origins lie in the first column. Because there are five columns in all, there are 70 (= 5 x 14) potential failures for a 5-note scale.

A few conventions and definitions prepare the theorems to follow:

Let N be an integer ≥ 3. Given: an $N \times (N - 1)$ interval matrix for a scale of cardinality N. Each cell in the matrix contains a sum that represents the specific (logarithmic) size of an interval. By construction, an interval in row r is the sum of r consecutive step intervals, beginning with the step at the base of its column. Thus, any given step appears as a summand r times in row r. A *failure* is an interval pair of unlike spans in which the size of the interval with the smaller span (the *origin* of the pair) is equal to or greater than the interval with larger span (the *target*). Let PF(N) represent the potential number of failures for a scale of cardinality N.

THEOREM 3. $\mathrm{PF}(N) = \dfrac{N(N-1)(N-2)(2N-3)}{6}$

PROOF. Let k be an integer, $1 \leq k \leq N - 2$. Consider failures of degree $N - 1 - k$ for any given value of k. The failure degrees range from 1 (when $k = N - 2$) to $N - 2$ (when $k = 1$): clearly, the least degree of failure is 1 and the greatest degree is $N - 2$, which is the difference between the largest span $(N - 1)$ and the smallest (1). Intervals in a target row each have $N - 1 - k$ more summands than those in the origin row. Any given origin appears as a summand once in its own row, twice in the row above, three times in the row above that, and so on.

Each origin forms k potential failures of degree $N - 1 - k$, because each origin is itself a summand in all but k intervals in the target row. (3.1)

There are k row pairs that differ by $N - 1 - k$. These are rows 1 and $N - k$, 2 and $N - k + 1\ldots$, up to k and $N - 1$. (3.2)

There are N origins in each row pair. (3.3)

Multiplying the results of (3.1)–(3.3) yields a total of $(k)(k)N$ potential failures of degree $N - 1 - k$.

Therefore the total number of potential failures is the sum of $N(k^2)$ for all k from 1 to $N - 2$. That is,

$$\mathrm{PF}(N) = N\left(\sum_{k=1}^{N-2} k^2\right) \tag{3.4}$$

We can use Lemma 3 in order to clear the summation. Substituting $N - 2$ for n in Lemma 3 gives,

$$\mathrm{PF}(N) = N\left(\frac{(N-2)(N-1)(2N-3)}{6}\right) = \frac{N(N-1)(N-2)(2N-3)}{6} \tag{3.5}$$

QED. The third column of Appendix C shows the results for the number of potential failures for N from 3 to 25.

2. Maximal Failures

What is the maximum number of simultaneous failures for a scale of cardinality N? Let $\mathrm{SF}(N)$ represent this maximum.

THEOREM 4. $\mathrm{SF}(N) = \dfrac{N(N-1)(N-2)(3N-5)}{24}$

PROOF. We assume the $N \times (N - 1)$ interval matrix as in Theorem 3. Again, let k be an integer, $1 \le k \le N - 2$.

First we show that, for all k, the maximum number of failures for the $(N - 1 - k)$th degree is $k(1+2+...+k)$ or, $\frac{k^2(k+1)}{2}$. We begin with the largest failure degree, $N - 2$.

Let $k = 1$. The maximal number of $N - 2$ order failures is $1 = 1(1)$.

Each step is a summand in all but one specific interval of span $N - 1$. If a given step interval is the origin of a failure of degree $N - 2$, then that step is larger than the sum of all of other steps, so, *a fortiori*, that step is larger than any other step, precluding any other $N - 2$ order failure. Then there is at most one failure between rows 1 and $N - 1$. The number of pairs of rows that differ by $N - 2$ is 1, and 1x1 = 1.

Let $k = 2$. The maximal number of $N - 3$ order failures is $6 = 2(2+1)$.

Each step is a summand in all but two intervals of span $N - 2$. Therefore, there can be no more than two steps that may simultaneously function as origins of failures of this degree. Any given step interval, then can only fail with at most 2 targets at this degree. If a step interval fails with both of its potential targets of span $N - 2$, then that step interval is again larger than any other step. The second largest step may still fail with the one remaining target of span $N - 2$ that contains neither itself nor the largest step as summands. This same analysis would apply to the other pair of rows that differ by $N - 3$, namely rows 2 and $N - 1$. Thus, the number of row pairs that differ by $N - 3$ is 2. Each of these row pairs can have up to 2+1 = 3 failures: 2x3 = 6.

Let $k = 3$. The maximal number of $N - 4$ order failures is $18 = 3(3+2+1)$.

No more than three steps may simultaneously function as origins of failures of this order since each step appears as a summand in all but three intervals of span $N - 3$. The maximum number of failures involving a single step is three. Following previous arguments, if some step interval fails with three it must be the largest step. This restricts the second largest step to at most 2 failures, whose targets are the remaining $N - 3$ intervals that contain neither of the largest steps as summands. The appearance of these failures restricts any other step interval to at most 1 failure for the same reasons. The number of pairs of rows that differ by $N - 4$ is 3. Each of these row pairs can have up to 3+2+1 = 6 failures: 3x6 = 18.

Let $k = 4$. The maximal number of $N - 5$ order failures is $40 = 4(4+3+2+1)$.

By the same arguments, and so on for the remaining values of k up to $N - 2$. It therefore holds that the maximum number of failures of degree k is $k(1+2+...+k)$. Invoking Lemma 2, $k(1+2+...+k) = \frac{k^2(k+1)}{2}$. We need to find the total for all failure degrees, and thus the maximum number of simultaneous failures is the sum of $\frac{k^2(k+1)}{2}$ for all k from 1 to $N - 2$, that is, a.

	Specific Intervals				
	A	B	C	D	E
fifths (4)	30	15	23	27	29
fourths (3)	28	14	7	19	25
thirds (2)	24	12	6	3	17
seconds (1)	16	8	4	2	1

Heteromorphic Profile: (25,0,40)
Maximal Failures for $N = 5$: 25
Maximal Differences for $N = 5$: 40
Maximal Failure level: 3

b.

	f.d.	Origin		Target	
		r/c		r/c	
1	1	1A	16	2B	12
2	1	1A	16	2C	6
3	1	1A	16	2D	3
4	1	1B	8	2B	6
5	1	1B	8	2C	3
6	1	1C	4	2C	3
7	1	2A	24	3B	14
8	1	2A	24	3C	7
9	1	2A	24	3D	19
10	1	2B	12	3C	7
11	1	2E	17	3A	14
12	1	2E	17	3B	7
13	1	3A	28	4B	15
14	1	3A	28	4C	23
15	1	3A	28	4D	27
16	1	3D	19	4B	15
17	1	3E	25	4B	15
18	1	3E	25	4C	23
19	2	1A	16	3B	14
20	2	1A	16	3C	7
21	2	1B	8	3C	7
22	2	2A	24	4B	15
23	2	2A	24	4C	23
24	2	2E	17	4B	15
25	3	1A	16	4B	15

f.d.: Failure Degree. r/c: row/column

Example 7. "Powers of Two" 5-note scale with maximal failures.
a. Scale matrix
b. Origins and targets of the 25 contradictions

$$\mathrm{SF}(N) = \sum_{k=1}^{N-2} \frac{k^2(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{N-2}\left(k^3+k^2\right) \tag{4.1}$$

$$= \frac{1}{2}\left(\sum_{k=1}^{N-2} k^3 + \sum_{k=1}^{N-2} k^2\right) \tag{4.2}$$

We can use Lemma 4 to simplify $\sum_{k=1}^{N-2} k^3$, and Lemma 3 for $\sum_{k=1}^{N-2} k^2$. In both cases, we substitute $N-2$ for n, and this gives,

$$\frac{1}{2}\left(\sum_{k=1}^{N-2} k^3 + \sum_{k=1}^{N-2} k^2\right) = \frac{1}{2}\left(\left(\frac{(N-2)(N-1)}{2}\right)^2 + \left(\frac{(N-2)(N-1)(2N-3)}{6}\right)\right) \tag{4.3}$$

$$= \frac{1}{2}\left(\frac{(3(N-2)(N-2)(N-1)(N-1)) + (2(N-2)(N-1)(2N-3))}{12}\right) \tag{4.4}$$

$$= \frac{1}{2}\left(\frac{((N-2)(N-1))((3N^2-9N+6)+(4N-6))}{12}\right) \tag{4.5}$$

$$\mathrm{SF}(N) = \frac{N(N-1)(N-2)(3N-5)}{24} \tag{4.6}$$

QED. The fourth column of Appendix C shows the results of this expression for N from 3 to 25.

a. Demonstration of Maximal Failure with N = *5.* Let us now construct a scale with maximal failure for 5 notes. Theorem 4 asserts that this would have 25 failures, since $\frac{N(N-1)(N-2)(3N-5)}{24} = \frac{5 \cdot 4 \cdot 3 \cdot 10}{24} = 25$. Example 7(a) shows the interval matrix for a five-note pitch-class set with 25 failures. The sizes of the steps in this scale are decreasing powers of two. The table in Example 7(b) illustrates the presence of these 25 failures. This scale also displays maximal difference. Theorem 2 shows that this number is $\frac{(N)(N-1)(N-1)}{2} = \frac{5 \cdot 4 \cdot 4}{2} = 40$.

b. Maximal Ambiguities? Note that all of the failures shown in Example 7 are contradictions. A scale with maximal failures could also have some ambiguities; however, as should be clear, they cannot *all* be ambiguities.[8] An unanswered question at this point is whether the maximum number of ambiguities is determinable for a set of cardinality N. On the basis of empirical tests, I will speculate that scales with the following characteristics maximize ambiguities: 1) the scale has just two step sizes, the larger being exactly twice the size of the smaller 2) the distribution of the step sizes is minimally even, that is, all of the larger steps are adjacent, as are the smaller 3) the number of larger step intervals is slightly less than half the cardinality of the scale. The matrix for a seven-note scale with 46 ambiguities is shown in Example 8. I do not know of any other 7-note scale with more ambiguities.

E. Interpreting Results

Our three results can be combined to show how failures affect sets according to cardinality. The three corollaries below are given without proofs. We continue to assume $N \geq 3$.

COROLLARY 4a. $\lim_{N\to\infty}\left(\frac{\mathrm{PF}(N)}{\mathrm{IP}(N)}\right) = \frac{2}{3}$ (strictly increasing from $\frac{1}{5}$)

COROLLARY 4b. $\lim_{N\to\infty}\left(\frac{\mathrm{SF}(N)}{\mathrm{PF}(N)}\right) = \frac{3}{8}$ (strictly increasing from $\frac{1}{3}$)

COROLLARY 4c. $\lim_{N\to\infty}\left(\frac{\mathrm{SF}(N)}{\mathrm{IP}(N)}\right) = \frac{1}{4}$ (strictly increasing from $\frac{1}{15}$)

These results inform us of the following: Corollary 4a shows that as N approaches infinity, the number of potential failures approaches 2/3 of the number of interval pairs. Convergence is fairly rapid. Corollary 4b shows that no more than 3/8 of the potential failures for any scale can fail simultaneously. This value is relatively constant for any value of N, in that the maximal number of failures is never less than 3/9 the number of the potential failures nor greater than 3/8. Finally, Corollary 4c establishes the fact that the maximum number of simultaneous failures never exceeds one-

		Specific Intervals						
G	**6**	9	8	8	8	9	9	9
E	**5**	8	7	6	6	7	8	8
N	**4**	7	6	5	4	5	6	7
E	**3**	6	5	4	3	3	4	5
R I	**2**	4	4	3	2	2	2	3
C	**1**	2	2	2	1	1	1	1

Scale W	[0246789] in 10-tone equal-temperament
Step interval pattern of W	2 2 2 1 1 1 1
Heteromorphic Profile of W	(13,46,92)

Example 8. A 7-note scale with 46 ambiguities

fourth the total number of interval pairs. All of these results indicate that the question of coherence is significant for scales of any cardinality. Earlier we had noted that the significance of difference converges on the infinitesimal as cardinality increases.

III. Coherence and Sameness Quotients

Now that the maximal number of differences and the maximal number of simultaneous failures are known, it would be useful to have a method of counting the actual number of differences or failures in any scale. We begin with failures. Example 9 shows a fragment of Pascal code that uses four nested "for" loops in order to step through each of the PF(N) possible failures. While there are considerably more interval pairs in total, IP(N), it is only necessary to check through the smaller number, PF(N), of potential failures. The code could be further factored to stop checking when various maxima have been reached, but running the code as shown helps to serve as empirical verification of our proofs.

Let S represent a set of cardinality N. We let AF(S) represent the actual number of failures in S, which we utilize to form the *coherence quotient of S* or CQ(S):[9]

DEFINITION. $\mathrm{CQ}(S) = 1 - \dfrac{\mathrm{AF}(S)}{\mathrm{SF}(N)}$.

We postulate that sets with higher values will be more "scale-like" than those with lower values. If S has no failures, then AF(S) = 0, CQ(S) = 1, and S is coherent. Conversely, if S has maximal failures, then AF(S) = SF(N), and CQ(S) = 0. With this measure, we may investigate scales, scale systems, and even abstract scale structures according to their relative degree of ambiguity and contradiction. We begin by taking care of the trivial case of equal-interval sets.

COROLLARY 4d. If E_N is an equal-tempered set of cardinality N, then CQ(E_N) = 1. (Equal-interval sets are always coherent.)

PROOF. Let μ be the size of the step interval. Let T be an integer, $1 \leq T < N$. Then all intervals of span T have the size $T\mu$. (4d.1)

Let J be an integer, $1 < J < N - T$. Then all intervals of span $T + J$ are larger than those of span T: $(T + J)\mu > T\mu$ because J, and μ are both greater than 0. QED.

Equivalently, equal-interval sets have no ambiguities or contradictions. (AF(E_N) = 0.)

The maximal number of differences for a set of order N has also been determined (See Theorem 2). As postulated earlier, we will continue to

```
const
max = 20

type
intarray = array[ 1 .. max , 1 .. max ] of integer; { for demonstration  - can
                                                       also be real numbers }

var
theArray  : intarray;   { rotational  array – interval  matrix }
x : integer;            {theArray  counter – origin  column }
y : integer;            {theArray  counter – origin  row}
p : integer;            { theArray counter – target  column}
q : integer;            { theArray counter – target  row }
i : integer;            { loop counter }
N : integer;            { cardinality  of scale }
theLev : integer;       { tracks  current failure  degree }
maxLev : integer;       { records highest  failure  degree }
concount : integer;     { tracks the number of contradictions }
ambcount : integer;     { tracks the number of failures }

begin
    x := 1;
    y := 1;
    maxlev := 0;
    for theLev := 1 to (N - 2) do                    { all failures;  all degrees }
    begin
        for y := 1 to (N - theLev - 1) do            { setting origin column }
        begin
            for x := 1 to N do                       { setting origin row }
            begin
                for i := 1 to (N - theLev - 1) do    { cycling through targets  }
                begin
                    p := (x+i) mod  N;
                    if p = 0 then
                        p := N
                    q := y + theLev;
                    if  theArray[x,y] >= theArray[p,q] then
                    begin
                        maxlev := theLev;
                        if (theArray[x,y] > theArray[p,q]) then
                            concount := concount + 1
                        else
                            ambcount := ambcount + 1;
                    end;
                end;
            end;
        end;
    end;
end;
```

Example 9. Pascal code for counting failures surmise that sets with fewer differences tend to outrank those with more in terms of scale candidacy. The procedure for determining the number of differences for a given scale is shown in Example 10. Let AD(S) stand for the number of differences in S. This value allows us to form a *sameness quotient of* S SQ(S):

DEFINITION. $$\mathrm{SQ}(S) = 1 - \frac{\mathrm{AD}(S)}{\mathrm{MD}(N)}$$

COROLLARY 4e. If E_N is an equal-tempered set, then SQ(E_N) = 1. Equal-interval sets have no differences. (AD(E_N) = 0.)

PROOF. The proof is immediate from line 4d.1 above. Unlike Corollary 4d, the converse of 4e is also true: given set S, if SQ(S) = 1, then S is an equal-interval set.

The coherence and sameness quotients of equal-interval sets both equal one. This appears to promote equal-interval sets to the top of the scalar heap, and indeed, equal-interval scales are commonly featured in scale repertories. We will return later to the question later and refine our interpretation of these indices.

Here is how the two quotients may form similarity relations. Let S and T be pitch- class sets.

DEFINITION. $$\mathrm{COH}(S,T) = 1 - \left|\mathrm{CQ}(S) - \mathrm{CQ}(T)\right|$$

DEFINITION. $$\mathrm{SAM}(S,T) = 1 - \left|\mathrm{SQ}(S) - \mathrm{SQ}(T)\right|$$

The range and power of these measures is noteworthy. They are capable of comparing sets not only of different cardinalities, but indeed, even sets belonging to different scalar universes. For the remainder of this paper, however, the focus will be on the coherence and sameness quotients themselves, and not on the similarity relations derived from them.[10]

We are now in a position to assess different pitch-class sets, scales, and scale systems in terms of coherence and sameness. The first type to be investigated is the class of well-formed scales, defined in Carey and Clampitt 1989. The second consists of the 7-note sets studied in Rahn 1991. In this context, we can consider the 72 *melakarta ragas* of South Indian music theory.

A. Well-formed Scales

Well-formed scales are generated scales whose generator exhibits an invariant generic size. As a benefit of this invariance we are able to refer to the interval of about 7/12 of an octave that generates the diatonic scale as the "fifth." Even though that same interval generates the Guidonian

```
procedure countdif (theArray : intarray; N: integer; var difcount : integer);
var
x : integer;                              {theArray counter – origin column }
y : integer;                              {theArray counter – origin row}
p : integer;                              { theArray counter – target column}
q : integer;                              { theArray counter – target row }
difs : integer;
begin
    difs := 0;
    for y := 1 to (N - 1) do                   {gets all rows}
    begin
        q := y;
        for x := 1 to (N - 1) do          {gets all starting points in each row}
        begin
            for p := (x+1) to (N) do  {gets all diffs from each
                                             starting point}
            begin
            if theArray[x,y] <> theArray[p,q] then
                difs := difs +1;
            end;
        end;
    end;
                                          end.
```

Example 10. Pascal code for counting differences

hexachord, it does not exhibit generic invariance in that set; sometimes the generator appears as a "fifth" and sometimes a "fourth." Therefore, while generated, the Guidonian hexachord is not well-formed. The universe of well-formed scales is comprised of two sub-types: equal-interval scales (*degenerate* well-formed scales) and scales with Myhill's Property (*non-degenerate* well-formed scales). This study shows that in a well-formed scale both the number of failures and the number of differences are significantly lower than the maximums. That is, any arbitrary well-formed scale displays rather high coherence and sameness quotients. Corollaries 4d and 4e have already shown that degenerate well-formed scales are always coherent and have no differences. We turn our attention to the other sub-type, the *non-degenerate* well-formed scales, that is, those with Myhill's Property. Well-formed scales require a generator, as just described, and another interval, commonly the octave, to function as the interval of periodicity. Each such pair of intervals determines a hierarchy of well-formed scales, whose cardinalities are the denominators of the continued fraction of the log ratio of the generating pair.

1. Coherence and Failure in Well-formed Scales—Preview

How many coherence failures does a well-formed scale have, and under what circumstances is a well-formed scale coherent? A few notational conventions will help to get us on our way. Let Z stand for the set of integers. The *class of well-formed scales* WF(N,g) includes all well-formed scales with N notes whose larger step interval occurs g times. For example, WF(7,5) includes all members of Easley Blackwood's "recognizable diatonic tunings," including the diatonic under equal temperament, Pythagorean tuning, or meantone temperament (Blackwood 1985). The class WF(5,2) contains the usual pentatonic. Following Blackwood, let R represent the log ratio between the larger step interval and the smaller one in a well-formed scale. Let wfs(N,g,R) be a well-formed scale with parameters N, g, and R. Then wfs(N,g,R) $\in$ WF(N,g).

Below is a summary of the results that are proven in the theorems of the following section.

A.1 If $g = N - 1$ then wfs(N,g,R) is always coherent.

A.2 If $1 \leq R < 2$ then wfs(N,g,R) is always coherent.

A.3 If $R > 2$ then wfs(N,g,R) *may* contain contradictions.

A.4 If $R \geq 2$ and $R \in Z$ then wfs(N,g,R) *may* contain ambiguities.

Conversely, if the well-formed scale contains ambiguities then $R \in Z$, and if it contains contradictions then R > 2. Result A.1 is rather uncommon, and will be designated as a special case.

If N is fixed, the potential for failure increases if R increases, or if g decreases, or both. An increase in R signifies that the size of the larger step is increasing with respect to the smaller one. A decrease in g signifies an increase in the population of smaller step intervals. Regardless of the value of R, wfs(N,N–1,R) is always coherent, that is, CQ(wfs(N,N–1,R)) = 1. (This is a restatement of the "special case," A.1.) All scales in WF(N,N–2) have exactly one failure when R is greater than or equal to 2. This class would include all of those in Clough and Douthett's "hyperdiatonics," as well as those in Agmon's "diatonic systems." Let wfs(N,N–2,R) represent a member of WF(N,N–2). Then CQ(wfs(N,N–2,R)) $\geq 1 - \dfrac{1}{\mathrm{SF}(N)}$. That is, any member of WF(N,N–2) has a very high coherence quotient.

2. Coherence and Failure in Well-formed Scales—Theorems

The following two theorems will determine the number of failures in a well-formed scale. (In the next section we take up the question of difference and sameness.) First we will need some propositions from well-formed scale theory.

Let N represent the cardinality of a well-formed scale.

Let T be an integer, $0 < T < N$. T will represent the generic spans of intervals in a well-formed scale.

Let μ_T and ν_T represent the log sizes of the two specific intervals of span T. Without loss of generality, assume that $\mu_T > \nu_T$.

Let $R = \frac{\mu_T}{\nu_T} > 1$

Let g represent the number of times that the larger step interval, μ_1, appears in one "octave" of the scale.

Five lemmas concerning well-formed scales are required for the following proofs. These are proven in Carey 1998, and found in Appendix A, beginning with Lemma 5.

a. Ambiguities in Well-formed Scales. Let #A(N,g,R) represent the total number of ambiguities in a well-formed scale with parameters N, g, and R:

THEOREM 5.

If $R \in Z$ and $2 \leq R < \frac{N-1}{g} + 1$, then

$$\#\mathrm{A}(N,g,R) = \frac{((N-1)-g(R-1))((N)-g(R-1))((N+1)-g(R-1))}{6}$$

PROOF: The first part of the proof shows that, if ambiguities exist in wfs(N,g,R), R must be an integer as restricted. We begin by constructing an ambiguous pair. We will assume that $\mu_T > \nu_T$ for all T. For any given value of T, let J represent a positive integer, such that $T + J < N$.

Therefore, some ambiguous pair of specific intervals must be of the form,

$$\mu_T = \nu_{T+J} \tag{5.1}$$

Making use of Lemmas 7.1 and 7.2 we have

$$(T-x_T)\mu_1 + (x_T)\nu_1 = (T+J-x_{T+J}-1)\mu_1 + (x_{T+J}+1)\nu_1 \tag{5.2}$$

or, collecting terms,

$$\mu_1(x_{T+J} - x_T + 1 - J) = \nu_1(x_{T+J} - x_T + 1) \tag{5.3}$$

Now μ_{T+1} is larger than μ_T by a step. Then $x_{T+1} - x_T$ = either 1 or 0 according to whether $\mu_{T+1} - \mu_T = \nu_1$ or μ_1, respectively. The same is true of $x_{T+2} - x_{T+1}$, and so on, up to and including $x_{T+J} - x_{T+J-1}$. Then $x_{T+J} - x_T$ is at most J. Let $K = x_{T+J} - x_T + 1$. Then,

$$0 \leq K \leq J + 1 \tag{5.4}$$

Substituting K for $x_{T+J} - x_T + 1$ we can rewrite (5.3) as,

$$\mu_1 (K - J) = \nu_1 (K) \tag{5.5}$$

$$\frac{\mu_1}{\nu_1} = \frac{K}{K-J} \tag{5.6}$$

Now, by assumption $\frac{\mu_1}{\nu_1} > 1$ and by definition $J > 0$, therefore $K > J$.

But line (5.4) shows that $K \leq J + 1$, and so it must be that $K = J + 1$. Substituting $J + 1$ for K,

$$\frac{\mu_1}{\nu_1} = \frac{(J+1)}{(J+1)-J} = J + 1 \tag{5.7}$$

By definition, $\frac{\mu_1}{\nu_1} = R$, so

$$R = J + 1 \tag{5.8}$$

Equation (5.8) shows that in order for ambiguities to exist in a well-formed scale, R must be an integer, that is $R \in Z$. The degree of the ambiguity is $J = R - 1$. Furthermore, since by definition $J \geq 1$, then it must also be true that

$$R \geq 2 \tag{5.9}$$

We now show the upper bound for R.

The value $x_{T+J} - x_T$ in equation (5.3) testified to the presence of an unbroken string of v_1 step intervals. Furthermore, because $x_{T+J} - x_T = K - 1 = J$, then J is the length of that string of v_1 step intervals. Let x be a real number, and let $[x]$ be a function that returns the greatest integer less than or equal to x. Lemma 8 shows that the longest string of step v_1 intervals is $\left[\frac{N}{g} \right]$. Therefore,

$$J \leq \frac{N}{g} \tag{5.10}$$

We have constructed the ambiguous pair, $\mu_T = v_{T+J}$. Lemmas 9.1 and 9.2 show us that their multiplicities are $Tg_{\mathrm{mod}N}$ and $(-(T + J))g_{\mathrm{mod}N}$ respectively. We take the sum of these multiplicities:

$$\left(\#\left(\mu_T\right)+\#\left(v_{T+J}\right)\right)=\left(Tg_{\bmod N}+\left(-(T+J)\right)g_{\bmod N}\right)\equiv -Jg_{\bmod N} \qquad (5.11)$$

But we showed in (5.10) that $J \leq \frac{N}{g}$. Therefore, $N > N - Jg \geq 0$, and so, with strict equality,

$$\#\left(\mu_T\right)+\#\left(v_{T+J}\right)=N-Jg \qquad (5.12)$$

So the sum of the multiplicities of the ambiguous pair will be $N - Jg$. Now by Lemmas 9.1 and 9.2, in conjunction with Lemma 5, the multiplicities of the μ_T run through all values from 1 to $N - 1$, and similarly for v_T. The least value for the sum of a pair of multiplicities is $\#\left(\mu_{g^{-1}{}_{\bmod N}}\right)+\#\left(\nu_{-g^{-1}{}_{\bmod N}}\right) = 1+1=2$, therefore we must further restrict our sum to values such that $N - Jg > 1$. Substituting $R - 1$ for J, $N - (R - 1)g > 1$, or, $R<\frac{N-1}{g}+1$. Combining this result with (5.9) we have both bounds for R:

$$2 \leq R < \frac{N-1}{g}+1 \qquad (5.13)$$

How many pairs of intervals $\left(\left(\mu_T\right),\left(\nu_{T+J}\right)\right)$ are there such that $\#\left(\mu_T\right)+\#\left(\nu_{T+J}\right)=N-Jg$? We see that

$$(1)+(N-Jg-1)=N-Jg$$

$$(2)+(N-Jg-2)=N-Jg$$

$$\vdots$$

$$(N-Jg-1)+(1)=N-Jg$$

Clearly, then, there are $N - 1 - Jg$ such pairs in all.

In other words, ambiguities involve specific intervals starting with the most rare. As we noted, the smallest significant value for $N - Jg$ is 2. This value results in a single ambiguity, and involves the two specific intervals that are singletons. If $N - Jg = 3$, the multiplicities of the interval pairs are 1 and 2, and 2 and 1. This will result in 4 (=(1×2) + (2×1)) ambiguities in all. If $N - Jg = 4$, the multiplicities are 1 and 3, 2 and 2, and 3 and 1. In this case, there are ten ambiguities, because (1×3) + (2×2) + (3×1) = 10.

Thus, for integer R as constrained in (5.13):

$$\#A(N,g,R)= 1(N-Jg-1)+2(N-Jg-2)+\ldots+1(N-Jg-1)(1) \tag{5.14}$$

$$= \sum_{i=1}^{(N-1)-Jg} i(N-Jg-i) \tag{5.15}$$

$$= \sum_{i=1}^{(N-1)-Jg} i(N-Jg) - \sum_{i=1}^{(N-1)-Jg} i^2 \tag{5.16}$$

Invoking Lemmas 1 and 2 to simplify the first summation in (5.16) and Lemma 3 for the second, we have:

$$= \frac{(N-1-Jg)(N-Jg)(N-Jg)}{2} - \frac{(N-1-Jg)(N-Jg)\big(2(N-1-Jg)+1\big)}{6} \tag{5.17}$$

Multiplying out and collecting terms gives,

$$\#A(N,g,R) = \frac{\big((N-1)-Jg\big)\big((N)-Jg\big)\big((N+1)-Jg\big)}{6} \tag{5.18}$$

Replacing $R - 1$ for J gives the result:

$$\#A(N,g,R) = \frac{\big((N-1)-g(R-1)\big)\big((N)-g(R-1)\big)\big((N+1)-g(R-1)\big)}{6} \tag{5.19}$$

QED

COROLLARY 5a. *The upper bound on R.*
Theorem 5 proves that in order for ambiguities to exist in a well-formed scale, R must be an integer constrained according to (5.13), repeated here as (5a.1):

$$2 \le R < \frac{N-1}{g} + 1 \tag{5a.1}$$

We can adjust the far right side of inequality (5a.1) to make the conditions still more precise by exploiting the fact that $R \in Z$. When g does not divide $N - 1$ evenly, then for integer R, $R \le \left\lceil \frac{N-1}{g} \right\rceil + 1$. When g does divide $N - 1$ evenly, then $\frac{N-1}{g}$ itself is an integer, and so, $R \le \frac{N-1}{g}$. We can combine these results by defining a "ceiling function." If x is a real number, let $\lceil x \rceil$ represent the greatest integer strictly less than x. That is, if $x > [x]$, then $\lceil x \rceil = [x]$. If $x = [x]$, then $\lceil x \rceil = x - 1$.

$$2 \le R \le \left\lceil \frac{N-1}{g} \right\rceil + 1 \tag{5a.2}$$

Knowing that $J = R - 1$ is the degree of ambiguity, we also have

$$1 \le J \le \left\lceil \frac{N-1}{g} \right\rceil \tag{5a.3}$$

From (5a.3) we can deduce that if $g \ge \frac{N-1}{2}$, the degree of ambiguity (J) will be at most 1. This is the case in over half of all well-formed scale classes of cardinality N. If a well-formed scale contains contradictions as well as ambiguities, the largest degree of failure will accrue to the ambiguities. As Corollary 6d will show, this same result carries over to failures generally: if $G > \frac{N-1}{2}$, the degree of failure in wfs(N,G,R) is never greater than one.

b. Contradictions in Well-formed Scales

Let $j = \min\left(\lceil R-1 \rceil, \left\lceil \frac{N-1}{g} \right\rceil\right)$. Let #C($N$,$g$,$R$) represent the total number of contradictions in a well-formed scale with parameters N, g, and R:

THEOREM 6.

$$\#\mathrm{C}(N,g,R) = \frac{j\big(4(N-1)(N)(N+1) - g(j+1)\big(2\big(3N^2 - Ng(2j+1) - 1\big) + g^2(j)(j+1)\big)\big)}{24}$$

PROOF. Theorem 5 showed that R is forced to take on integer values in order for ambiguities to arise. No such restriction is assumed in the present discussion. R is a real number greater than 1.

We assume, temporarily, that $N - 1 - (R-1)g > 0$. Theorem 5 shows that, when $R = 2$ then $J = 1$, and so, by 5.19,

$$\#\mathrm{A}(N,g,2) = \frac{\big(N-(1)g-1\big)\big(N-(1)g\big)\big(N-(1)g+1\big)}{6} \tag{6.1}$$

When R is strictly between 2 and 3, then R is no longer an integer, and so there are no ambiguities. However, the pairs of intervals that had been counted in (6.1) as ambiguities have now become contradictions: the chain of reasoning in lines (5.1)–(5.8) is reversible, so that, if we begin by replacing (5.8) with $R > J + 1$, and reason backwards, then (5.1) will become $\mu_T > \nu_{T+J}$, that is, the interval pair (μ_T, ν_{T+J}) forms a contradiction. Then (6.1) counts the number of contradictions when $3 > R > 2$.

If $R = 3$, then

$$\#\mathrm{A}(N,g,3) = \frac{(N-(2)g-1)(N-(2)g)(N-(2)g+1)}{6} \tag{6.2}$$

The contradictions counted in (6.1) exist whenever $R > 2$, and so are still present with $R = 3$. When R is strictly between 3 and 4, again there are no ambiguities. The ambiguities counted in (6.2) become contradictions when $R > 3$, and the number of contradictions is now the sum of (6.1) and (6.2). When $R = 4$, the number of contradictions remains the sum of (6.1) and (6.2), and

$$\#\mathrm{A}(N,g,4) = \frac{(N-(3)g-1)(N-(3)g)(N-(3)g+1)}{6} \tag{6.3}$$

When R is strictly between 4 and 5, the number of contradictions is the sum of (6.1), (6.2), and (6.3), and so on, as R increases.

That is, for some integer j,

$$\#\mathrm{C}(N,g,R) = \begin{cases} 1(N-1g-1)+2(N-1g-2)+ \ldots\ldots\ldots\ldots +(N-1g-1)(1) \\ +1(N-2g-1)+2(N-2g-2)+ \ldots\ldots\ldots +(N-2g-1)(1) \\ +1(N-3g-1)+2(N-3g-2)+ \ldots\ldots +(N-3g-1)(1) \\ \vdots \\ +1(N-jg-1)+2(N-jg-2)+ \ldots +(N-jg-1)(1) \end{cases} \tag{6.4}$$

The value j is related to J in the previous theorem. Whereas J depicts the degree of ambiguity, here j will depict the largest degree of contradiction. As (6.4) demonstrates, if $N - jg - 1 > 0$, then it must be that $j \leq \left\lceil \frac{N-1}{g} \right\rceil$. Furthermore we have seen in the demonstration in (6.1)–(6.3) that the largest degree of failure is $[R-1]$. If $R \geq \frac{N-1}{g} + 1$, then $j = \left\lceil \frac{N-1}{g} \right\rceil$. Thus, if we wished to count ambiguities and contradictions together, we would set the value j to $\min\left([R-1], \left\lceil \frac{N-1}{g} \right\rceil\right)$; however, we would like to omit ambiguities from the count and so we have:

$$j = \min\left(\lceil R-1\rceil, \left\lceil \frac{N-1}{g} \right\rceil\right) \tag{6.5}$$

Then (6.4) can be expressed as the following compound summation:

$$\#\mathrm{C}(N,g,R) = \sum_{k=1}^{j} \sum_{i=1}^{N-1-gk} i(N-gk-i) \tag{6.6}$$

Lines (5.14)–(5.17) from Theorem 5 show the solution for a similar expression and so we may utilize this result here, substituting *Jg* from the earlier expression with *gk* from (6.6):

$$\#\mathrm{C}(N,g,R) = \sum_{k=1}^{j} \frac{(N-1-gk)(N-gk)(N+1-gk)}{6} \tag{6.7}$$

This could also be expressed as:

$$\#\mathrm{C}(N,g,R) = \sum_{k=1}^{j} \frac{(N-gk)^3-(N-gk)}{6} \tag{6.8}$$

On the other hand, it is often preferable not to have to rely upon a recursive solution, and a closed expression can be determined as follows, clearing the expression of the index, *k*:

Multiplying out terms from (6.7) we get,

$$= \frac{1}{6}\left(\sum_{k=1}^{j} (N^3-N) - (3N^2gk) + (3Ng^2k^2) - (g^3k^3) + (gk)\right) \tag{6.9}$$

Invoking Lemmas 1–4 as appropriate gives,

$$= \frac{1}{6}\left((j(N-1)(N)(N+1)) - \left(\frac{j(j+1)(3N^2g)}{2}\right) + \left(\frac{j(2j+1)(j+1)(3Ng^2)}{6}\right) - \left(\frac{g^3(j^4+2j^3+j^2)}{4}\right) + \left(\frac{gj(j+1)}{2}\right)\right) \tag{6.10}$$

Finally, multiplying out and collecting terms yields our result,

$$= \frac{j\left(4(N-1)(N)(N+1) - g(j+1)\left(2\left(3N^2 - Ng(2j+1) - 1\right) + g^2(j)(j+1)\right)\right)}{24} \tag{6.11}$$

QED. If we are interested in the total number of failures in a well-formed scale, we replace *j* with *J*, where $J = \min\left([R-1], \left\lceil \frac{N-1}{g} \right\rceil\right)$. Substituting *J* for *j* in (6.7) will then count the total number of failures in the well-formed scale. Let #F(N,g,R) be the total number of failures in a well-formed scale, ambiguities plus contradictions. Then rewriting (6.7) and (6.11) with J substituting for j we get,

$$\#\mathrm{F}(N,g,R) = \sum_{k=1}^{J} \frac{(N-1-gk)(N-gk)(N+1-gk)}{6} \tag{6.12}$$

$$= \frac{J\left(4(N-1)(N)(N+1) - g(J+1)\left(2\left(3N^2 - Ng(2J+1)-1\right)+g^2(J)(J+1)\right)\right)}{24} \tag{6.13}$$

c. Well-formed Scales with High Coherence Quotient. Although failures may exist in well-formed scales, their propagation is constrained by a number of factors. Well-formed scales in which N > 3 will always have fewer failures than the maximum demonstrated in Theorem 4. When failures do exist, they are likely only to be of the first degree. Indeed, well-formed scales are often coherent. There are two conditions under which they are always coherent. The first condition involves the size of R, and the other the value of g.

Let α be a real number, $1 \le \alpha < 2$. Corollary 6a shows that the set wfs(N,g,α) has no failures.

COROLLARY 6a. *Well-formed scales in which R is between* 1 *and* 2 *are coherent.*

$$\mathrm{AF}(\mathrm{wfs}(N,g,\alpha)) = \#\mathrm{F}(N,g,\alpha) = 0$$

PROOF. If $1 \le \alpha < 2$, then

$$J = \min\left([R-1], \left\lceil \frac{N-1}{g} \right\rceil\right) = [\alpha - 1] = 0 \tag{6a.1}$$

Since J is the first factor in the polynomial (6.13), then the value of the entire polynomial is 0, that is AF(wfs(N,g,α)) = 0.

QED. Equivalently, CQ(wfs(N,g,α)) = 1.

COROLLARY 6b. *Well-formed scales in which* $g = N - 1$ *have no failures.*

$$\mathrm{AF}(\mathrm{wfs}(N,N-1,R)) = \#\mathrm{F}(N,N-1,R) = 0$$

(This is the special case A.1.)

PROOF. If $g = N - 1$, then

$$J = \min\left([R-1], \left\lceil \frac{N-1}{g} \right\rceil\right) = \left\lceil \frac{N-1}{N-1} \right\rceil = 0 \tag{6b.1}$$

whatever value R takes. Therefore, the polynomial expression in (6.13) again reduces to 0. That is, AF(wfs(N,N–1,R)) = 0.

QED. Then CQ(wfs(N,N–1,R)) = 1.

An important class of well-formed scales is WF(N, N–2). This class includes WF(7,5), which, in turn, includes all of the generated tunings of the diatonic scale, in particular, the diatonic in Pythagorean tuning, equal temperament, and meantone tunings. Balzano's scales (Balzano 1980) are either members of WF(N, N–2) or of WF(N,2).[11] All members of Clough and Douthett's "hyperdiatonics" (Clough and Douthett 1991), as well as those in Agmon's "diatonic systems" (Agmon 1996) belong to this class. The hyperdiatonics contain 2 smaller steps and N–2 larger steps. The larger steps are twice as big as the smaller steps. Using the conventions we have adopted, the hyperdiatonics can be characterized as well-formed scales of the type wfs(N,N–2,2). As Clough and Douthett prove, the hyperdiatonic sets contain a single ambiguity. We can generalize that result here. Sets wfs(N,N–2,R) contain at most a single failure. When $R = 2$, the failure will be an ambiguity. When R is greater than 2, a contradiction.

COROLLARY 6C. *The maximum number of failures in* WF(N,N-2) *is* 1.

AF(wfs(N,N–2,R)) ≤ 1

There are two cases to consider. If $R = \alpha$ from above, then, by Corollary 6a, #F(N,N–2,α) = 0, and the corollary is true. Now let β be a real number, $\beta \geq 2$. Consider the case of wfs(N,N–2,β). We first determine J. Because $N \geq 3$ we have,

$$J = \min\left(\lceil \beta - 1 \rceil, \left\lceil \frac{N-1}{N-2} \right\rceil\right) = \left\lceil \frac{N-1}{N-2} \right\rceil = 1 \tag{6c.1}$$

Substituting N–2 for g and 1 for J in equation (6.12) we have,

$$\#\mathrm{F}(N,N-2,\beta) = \frac{(N-1-(N-2))(N-(N-2))(N+1-(N-2))}{6}$$

$$= \frac{1 \cdot 2 \cdot 3}{6} = 1 \tag{6c.2}$$

QED. When $\beta = 2$, the failure is an ambiguity; when $\beta > 2$, a contradiction.

In the cases just discussed, there will be at most a single failure, and the degree of that failure is also 1. When $g < \frac{N}{2}$, more failures may arise.

Nevertheless the degree of failure remains 1 for at least half the values of g.

Let G be an integer, $\frac{N-1}{2} \leq G < N-1$.

COROLLARY 6d. *Given* wfs(N,G,R), *the degree of failure* $J \leq 1$.

PROOF. If $1 \leq R < 2$, then Corollary 6a applies, and so $J = 0$, and the present corollary is true. Then assume $R = \beta$ from above, that is, $R \geq 2$. Let x be an integer, $-1 \leq x < N-2$. Then $G = \frac{N+x}{2}$. (When N is odd, x will also be odd, and even when even.) It follows that

$$1 < \frac{N-1}{\frac{N+x}{2}} \leq 2 \qquad (6d.1)$$

which is the same as,

$$\left\lceil \frac{N-1}{\frac{N+x}{2}} \right\rceil = 1 \qquad (6d.2)$$

Therefore,

$$J = \min\left([\beta-1], \left\lceil \frac{N-1}{G} \right\rceil\right) = \left\lceil \frac{N-1}{\frac{N+x}{2}} \right\rceil = 1 \qquad (6d.3)$$

Then by (6d.3) and Corollary 6a,

$$J \leq 1. \qquad (6d.4)$$

QED. Corollaries 6a–6d provide considerable information regarding the relative coherence of well-formed scales. While coherence failures are a potentiality, they are limited in a number of ways. As long as R is between 1 and 2, the well-formed scale wfs(N,g,R) is coherent. As long as g is equal to or greater than $\frac{N-1}{2}$, the degree of failure is at most 1. To find greater numbers and degrees of failure in a well-formed scale, one must look for cases in which $R > 2$, and $g < \frac{N-1}{2}$. The next corollary reveals the well-formed scales with the lowest coherence quotients. As a look back at (6.12) will show, failures are maximized in well-formed scales of cardinality N when $g = 1$ and $R \geq N-1$.

d. The Lowest Coherence Quotient for a Well-formed Scale

Let ξ be a real number, $\xi \geq N-1$.

COROLLARY 6e. *For any given N*, $\max(\#F(N,g,R)) = \#F(N,1,\xi) = \dfrac{(N-2)(N-1)(N)(N+1)}{24}$

That is, failures are maximized in well-formed scales of cardinality N when $g = 1$, and $R \geq N-1$.

PROOF: For sets wfs$(N,1,\xi)$ it is necessarily true that J equals $N-2$, the largest possible value for J: By definition, $\xi \geq N-1$, and so, $[\xi-1] \geq N-2$. Therefore,

$$J = \min\left([\xi-1], \left\lceil\frac{N-1}{1}\right\rceil\right) = \min\left([\xi-1], \lceil N-1\rceil\right) = N-2 \qquad (6e.1)$$

Furthermore, the value $g = 1$ gives the largest set of values in the numerator of (6.12).

Plugging the values $J = N - 2$ and $g = 1$ into (6.13) reveals that the maximal number of failures for a well-formed scale with N notes is

$$\max(\#F(N,g,R)) = \#F(N,1,\xi) = \frac{(N-2)(N-1)(N)(N+1)}{24} \qquad (6e.2)$$

QED. Therefore, because $SF(N) = \dfrac{N(N-1)(N-2)(3N-5)}{24}$, the coherence quotient of a well-formed scale with maximal failure is $1 - \dfrac{N+1}{3N-5} = \dfrac{2N-6}{3N-5}$. When $N = 3$, the coherence quotient of a well-formed scale with maximal failures is 0. As N increases, the quotient increases asymptotically towards $\dfrac{2}{3}$.

Although well-formed, wfs$(N,1,\xi)$ is not likely to be given much attention as a scale candidate. The Forte set 7-1 (0123456) is such a set, an example of wfs(7,1,6). The value g here is 1, remembering that g counts the number of times the larger step interval occurs. The larger step is the tritone and the smaller step is the semitone, and so, $R = 6$. When $N = 7$, $g = 1$, and $R = 6$, the conditions for Corollary 6e are met. Consequently, the scale has the maximum number of failures for a well-formed 7-note scale. Corollary 6e shows us that the number of failures is $\dfrac{5\cdot 6\cdot 7\cdot 8}{24} = 70$. In the general case the maximum number of failures for this cardinality is twice as much: $SF(7) = 140$. Therefore, the CQwfs $(7,1,6) = 1 - \frac{70}{140} = 0.5$.

3. Sameness and Difference in Well-formed scales

The number of differences in (non-degenerate) well-formed scales of a given cardinality is a function of the cardinality alone, and, as such, is unaffected by the values g or R. Let #D(N) represent the total number of differences in such a scale with cardinality N:

THEOREM 7. $\#\mathrm{D}(N) = \dfrac{(N-1)(N)(N+1)}{6}$

PROOF. By Myhill's Property, there are two interval sizes in each row. If the multiplicities of the two specific intervals in a given row are x and y, then the number of differences in that row is xy. Lemmas 9.1–2 state that $\#(\mu_T)=(Tg_{\mathrm{mod}N})$, and $\#(\nu_T)=(-Tg_{\mathrm{mod}N})$. Because N and g are relatively prime (see Lemma 5), then over all $N-1$ rows we will find some permutation of multiplicities 1 and $N-1$, 2 and $N-2$, etc., up to $N-1$ and 1. (Each well-formed scale class has a different permutation, but because we are concerned only with the sum, the order of the permutations is irrelevant.)

Then, the number of differences is

$$\#\mathrm{D}(N) = 1(N-1) + 2(N-2) + \ldots + (N-1)(1) \tag{7.1}$$

Let i be an integer, $1 \le i < N$. As a summation, (7.1) is expressed as:

$$\#\mathrm{D}(N) = \sum_{i=1}^{N-1} i(N-i) = \sum_{i=1}^{N-1} iN - i^2 \tag{7.2}$$

Invoking Lemmas 2 and 3

$$\#\mathrm{D}(N) = \frac{N(N-1)N}{2} - \frac{(N-1)(N)(2N-1)}{6} \tag{7.3}$$

Multiplying out and collecting terms brings us our result,

$$\#\mathrm{D}(N) = \frac{(N-1)(N)(N+1)}{6} \tag{7.4}$$

QED.

By definition, $\mathrm{SQ}(S) = 1 - \dfrac{\mathrm{AD}(S)}{\mathrm{MD}(N)}$. In the current case, $\mathrm{AD}(S) = \#\mathrm{D}(N)$. Then $\dfrac{\mathrm{AD}(S)}{\mathrm{MD}(N)} = \dfrac{\#\mathrm{D}(N)}{\mathrm{MD}(N)} = \dfrac{\frac{(N-1)(N)(N+1)}{6}}{\frac{(N)(N-1)(N-1)}{2}} = \dfrac{N+1}{3(N-1)}$. Therefore,

$$\mathrm{SQ}(\mathrm{wfs}(N,g,R)) = 1 - \frac{N+1}{3(N-1)} = \frac{2(N-2)}{3(N-1)} \tag{7.5}$$

As cardinality increases, the well-formed scale sameness quotient increases asymptotically from 1/3 towards 2/3. The sameness quotient of any 3-note well-formed scale is 1/3, and for a 7-note well-formed scale it has already reached 5/9. Well-formedness provides a significantly lower limit on differences, as it does for failures. On the other hand, because the

Forte Number	Prime Form	CQ(S)	Failure Degree	Contra-dictions	Ambi-guities	Differ-ences	72 *Mela-kartas*	19 *Mela-kartas*
7-35	013568a	0.9929	1	0	1	56	6	6
7-34	013468a	0.9286	1	0	10	72	4	
7-32	0134689	0.8714	1	0	18	82	7	3
7-30	0124689	0.8286	1	2	22	86	7	2
7-22	0125689	0.8143	1	4	22	84	3	1
7-31	0134679	0.7786	1	4	27	84	3	
7-29	0124679	0.7714	1	4	28	92	6	1
7-33	012468a	0.7500	1	0	35	80	2	
7-20	0125679	0.7429	1	10	26	90	4	4
7-28	0135679	0.7214	1	6	33	96	2	
7-21	0124589	0.7143	1	7	33	90	4	1
7-27	0124579	0.6857	2	9	35	98	4	
7-19	0123679	0.6786	2	16	29	92	5	
7-26	0134579	0.6643	2	13	34	98	1	
7-25	0234679	0.6643	2	19	28	92		
7-7	0123678	0.6500	2	22	27	84	2	1
7-15	0124678	0.6357	2	16	35	96	1	
7-38	0124578	0.6357	2	27	24	90	1	
7-37	0134578	0.6071	2	34	21	84		
7-24	0123579	0.5929	2	19	38	102	3	
7-18	0123589	0.5929	2	23	34	100	2	
7-17	0124569	0.5786	3	26	33	96	1	
7-14	0123578	0.5500	2	25	38	102	1	
7-13	0124568	0.5429	3	38	26	90		
7-23	0234579	0.5071	2	26	43	104		
7-16	0123569	0.5000	3	32	38	102	1	
7-11	0134568	0.5000	3	43	27	92		
7-1	0123456	0.5000	5	69	1	56		
7-12	0123479	0.4929	2	41	30	96	1	
7-6	0123478	0.4929	3	45	26	90	1	
7-36	0123568	0.4857	3	38	34	100		
7-5	0123567	0.4643	3	55	20	84		
7-4	0123467	0.4071	3	57	26	90		
7-8	0234568	0.3929	3	51	34	96		
7-9	0123468	0.3857	3	46	40	104		
7-10	0123469	0.3857	3	46	40	104		
7-2	0123457	0.3786	4	62	25	86		
7-3	0123458	0.3571	3	67	23	86		

Example 11. Seven-note pitch-class sets ordered by coherence quotient

sameness quotient is constant for any well-formed scale of cardinality N, the coherence quotient provides more information about these scales.

Appendix C shows a listing of the main results of this paper for values of N from 3 to 25.

B. 5- and 7-note Set Classes of 12-tone Equal Temperament

The previous section demonstrated that the coherence and sameness quotients of well-formed scales are significantly higher than average. In this section, we will further assess the ability of the coherence quotient in particular to discern scale-like sets. The sets to be investigated are the 5- and 7-note set classes of 12-tone equal temperament. In order to test the efficacy of the quotient further, we will study the 7-note set classes in the light of the 72 *melakarta ragas* of South Indian music theory. As we will see, the coherence quotients of the *melakartas* are also strongly inclined toward higher values.

Example 11 shows the thirty-eight 7-note set classes in the 12-tone equal-tempered universe ordered by coherence quotient. The example also displays the maximal failure degree, and the actual number of contradictions, ambiguities and differences, for each set. (The last two columns display information concerning the *melakartas*, to be discussed below. The first conveys the number of times each set class is represented in the complete list of seventy-two *melakartas*, and the other shows the number of instances in a reduced set of nineteen *melas*.) The range of coherence values is broad, although the lowest coherence quotient is 0.3571 for set class 7-3. The average coherence quotient for the 7-note set classes, consequently, is .6117, slightly above the mean of 0.5. Equal temperament, itself free from failure, limits, to some extent, the number of failures of its larger set classes.

The coherence quotients of some set classes are identical, and we would like some way of distinguishing scale candidacy claims in these cases, if possible. I will use two additional factors to make finer distinctions. First, I will assume that, given two sets with equal coherence values, a larger failure degree reduces scalar potential. If sets share coherence values and failure degrees, I will then declare the set with fewer contradictions to be the more likely scale. Rather than introducing numerical adjustments based upon these refinements, I simply list the sets in the order imposed thereby.

A few examples will demonstrate. The sets 7-1 (0123456) and 7-11 (0134568) share a coherence quotient of 0.5. While neither of them appears to be the most attractive of scale candidates, on the basis of its 5th degree failure, 7-1 is granted even less scalar potential than 7-11, whose greatest failure degree is 3. On the other hand, 7-11 (0134568), and 7-16 (0123569) share not only a coherence quotient of 0.5, but also a failure degree of 3. Here we assert, as Rahn does, that contradictions are more problematic than ambiguities, and this allows for an ordering on these two sets. The only pair of set classes that is completely indistinguishable in any of these ways is 7-9 (0123468) and 7-10 (0123469). The number of contradictions, ambiguities, degree of failure, and even differences is exactly the same in both sets.

As I have just indicated, Rahn (1991) treats ambiguity and contradiction as ontologically quite different.[12] Implicitly, Rahn's initial ordering with respect to coherence is entirely upon the basis of contradiction. Rahn proposes that the four contradiction-free set classes are special, and that these four may be ordered with respect to increasing complexity as 7-35, 7-34, 7-32, 7-33. And while he eschews the "platonism" of various numerically-based set-theoretic approaches that concern "cardinalities," "multiplicities," and so on, it is unclear how what he himself calls "degrees" of complexity are to be ascertained otherwise.

I do not ascribe the same importance to Rahn's differentiation of failure types, although I do acknowledge some distinction. Most importantly, ambiguity causes the breakdown of partitioning and contradiction does not. Nevertheless, from a perceptual standpoint, ambiguities and contradictions share an essential property: the loss of the ability to predict generic span upon the basis of specific size alone.

The three set classes with the highest coherence quotients appear among Rahn's privileged four. These are 7-35 (diatonic), 7-34 (melodic minor ascending), and 7-32 (harmonic minor). It is with respect to the remaining contradiction-free set class, 7-33 (012468a), that the findings of this study differ most strongly from those of Rahn's, a direct result of our differing methodologies. It is here, too, that I believe the coherence quotient provides a better method of deciding claims of scale candidacy. Note that the coherence quotient of 7-33 is significantly lower than those of 7-32, 7-34, and 7-35. The coherence quotient proximately associates the diatonic (7-35) with its two minor mode variants (7-34 and 7-32), and distances these three from the "whole-tone plus one" collection (7-33), a set that does not have scalar significance in common-practice tonality. There are four sets whose coherence quotient is greater than that of 7-33, and less than that of 7-32. The four intervening sets, 7-29 (0124679), 7-31 (0134679), 7-22 (0125689), and 7-30 (0124689), have claims to scalehood as strong as or stronger than 7-33 does. All four of the intervening sets have more *melakartas* associated with them than 7-33, which itself has the fewest out of the first nine set classes listed. Three of the four intervening sets are also represented in the reduced set of nineteen *melakartas* while 7-33 is not. Furthermore, 7-22 is the set class that underlies the so-called "Hungarian minor" scale, G A B♭ C♯ D E♭ F♯ G.

The *melakarta* system itself is shown in Appendix D. First proposed in the seventeenth century by Venkatamakhin, it is organized according to a very clear-cut combinatorial procedure. There are seven pitch classes in each *mela*. Pitch classes 0 and 7 appear in all seventy-two. Scales in the first half contain pitch class 5 but not 6, whereas those in the second half have pitch class 6 and not 5. The remaining four pitch classes arise by choosing two notes among pitch classes 1–4 and two more from pitch classes 8–11. Every possibility is utilized, resulting in $2 \times 6 \times 6 = 72$ combinations in all. Venkaṭamakhin also proposed that the complete set of seventy-two could be abbreviated into a set of nineteen. The index numbers of the scales in the reduced set are listed in bold in Appendix D.[13]

The *melakarta ragas* are understood as prototypes, and each *mela* can be realized in numerous *janya ragas* of actual use. The particularities of the *janyas* involve practices such as omitting notes in either ascent or descent, and the use of *vakras*, what Kaufmann refers to as "zig-zags." The most popular *melas* show the largest number of *janyas*, and, while the number of *janyas* cannot be so firmly fixed, according to Kaufmann, *melas* 8, 15, 20, 22, 28, and 29 are the most popular, as evinced by the great numbers of *janyas* that attend them. While the *melas* form the basis of the *janyas*, for our purposes the *melas* may be considered to be instantiations of 7-note set classes. Just as an abundance of *janyas* testify, according to Kaufmann, to the popularity of certain *melas*, so the *melas* themselves help to show which 7-note set classes are the best scale candidates.[14]

The combinatorial system that gives rise to the *melakartas* produces a healthy range of scales that still strongly favors coherence. It is not a question of the total suppression of coherence failures, but restraining their proportion to within a certain limit. Ambiguities and contradictions add spice to the mix, creating a state of pleasant complexity. The lowest coherence quotient of a *melakarta raga* is 0.4929 and, as Example 11 shows, the population of *melakartas* is thickest in the set classes with the larger coherence quotients. (Only three of the *melakarta ragas* exhibit a coherence quotient of 0.5 or lower.) Every set class with a coherence quotient greater than 2/3 is represented by at least two *melakartas*. Fifty-seven of the seventy-two *melakartas* belong to this region. From another perspective, the seven set classes with the highest coherence account for fully half of the *melakartas*. While there are six *melakartas* that represent 7-35, there are seven in the cases of 7-32, and 7-30, but this needs some commentary. Neither 7-32 nor 7-30 are invariant under inversion, therefore there are 14 possible octave species for each. The diatonic, 7-35, is invariant under inversion, and so there are only 7 possibilities. (Inverting the diatonic modes will not result in additional modes.) That is, almost all of the modal possibilities of 7-35 are found in the *melas* (all but Locrian, in fact), while only half of the possibilities for the other two sets are found. Furthermore, Kaufmann maintains that *melas* 8, 15, 20, 22, 28, and 29 are the most popular. All of these are diatonic modes except *mela* 15, which is a mode of 7-22.

The *melakartas* are also strongly coordinated with failure degree.

While all possible degrees of failure, from one to five, are found in the 7-note set classes, the maximal failure degree of a *melakarta* is three. There are multiple *melakartas* for each of the eleven set classes with first-degree failure, for a total of 48 *melakartas* in this group. Out of the thirteen set classes with second-degree failure, ten of them are represented by *melakartas*, and there are 21 in this group, less than half the number of those with first-degree failures. While there are fourteen set classes with a degree of failure three or greater, only 3 of them support *melakartas*, and in each case, they are sets with third-degree failure, and each with only a single *melakarta* representative. The results with respect to the reduced set of nineteen *melakartas* tell the same tale, but even more strongly. Thirteen of the nineteen have a coherence quotient of greater than 0.75, and only one has a failure degree of greater than one.

According to Kaufmann, the fewer *janyas* a particular *mela* supports, the more "artificial," it is likely to be. Rahn (1981) proposes that the property he refers to as "bisection" provides a way to distinguish the common from the artificial. A 7-in-12-note set is bisected if all of its "fifths" have one of three specific sizes, namely, six, seven, or eight. Clearly, the diatonic is a member of the bisected group. As it turns out, the property of bisection is, for the 7-note set classes (or the *melakartas*), identical to the property of having a coherence quotient greater than 0.72, with the exception of a single set class.[15] Set classes listed below 7-28 in Example 11 have a coherence quotient less than 0.72. All of these sets, with the exception of 7-15, do *not* exhibit the bisector property. *Melakarta* 38, an instance of set class 7-15, has a coherence quotient of 0.6357. According to Kaufmann's survey, *mela* 38 is relatively poor in *janya* forms. For these reasons, it may be more likely to be considered an "artificial" *mela*. In light of this single discrepancy, coherence is slightly better than bisection at distinguishing common from artificial *melas*. On the other hand, I am not prepared to endorse the limit 0.72 that emerges in this context. Set classes 7-21 and 7-7 account for six *melakartas* that Rahn would call artificial, but also are the set classes of two of the "reduced" set of nineteen *melakartas*. It is not reasonable to conclude that the reduced set should contain artificial scales.

I would like to briefly turn to the 5-note set classes of 12-tone equal temperament. Example 12 shows the coherence quotients for the 5-note set classes, together with their complements. The range of coherence quotients is greater in the 5-note sets than in the 7-note sets. Set class 5-4 (01236) has 23 failures out of a possible 25, giving it a coherence quotient of 0.08. Still, there are no 5-note pitch-class sets with maximal failure. I will hypothesize, but not prove, that the smallest equal-tempered universe that contains a 5-note set with maximal failure is 16, whereas a 64-note equal temperament is required for a 7-note pitch-class set with maximal failures.[16]

The coherence quotients of a 5-note set class and its complement differ, on average, by less than 15%. Example 13 shows this coordination more explicitly. Most intriguingly, there are seven pairs of complementary set classes whose values are extremely close, and in all cases, the coherence values are high. These are identified on the top right half of Example 13. Among this set are found a very rich sampling of some of the most prevalent 5- and 7-note scales in the repertory of world musics. Example 14 presents these sets as scales.

How does all of this serve to address our original question: what are the factors that tend to promote or depress the likelihood that a pitch-class set will serve as a scale?

IV. Conclusions and Prospect

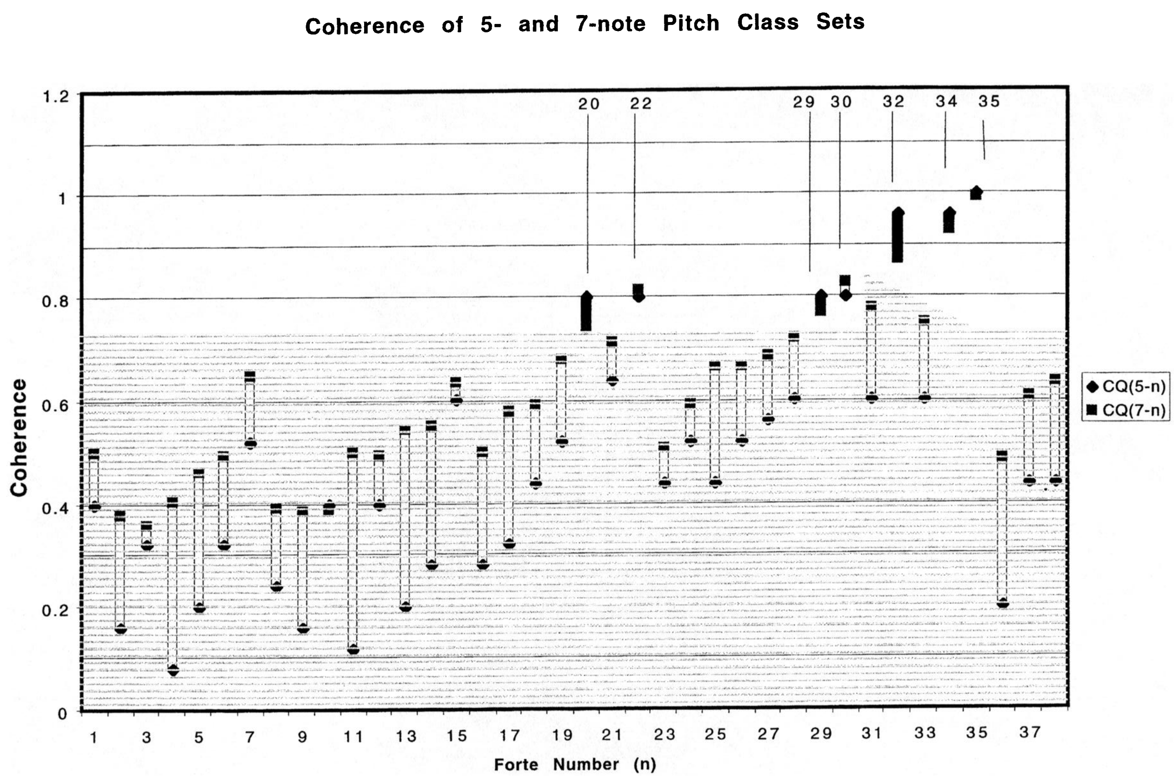
Example 15 summarizes the conditions that govern scale candidacy. What I am proposing is that small and relatively featureless pitch-class sets are the best scale candidates. In the language of information theory, scales carry relatively little information, and are fairly rich in redundancies. Symmetries or other invariances generally abound. Scales form the substratum of musical works. In order for the works themselves to manifest characteristics distinct from one another, it is an advantage for the substratum to be rather bland. A substructure that calls too much attention to itself marks everything with its own stamp, diminishing the range of individual compositional character.

On the other hand, some degree of complexity is also advantageous, and this deprives equal-interval sets of their apparent privileged status. This study has sought to explore how much complexity a scale can tolerate. I will review the results of this study by reintroducing the three factors described earlier: cardinality, generators, and generic intervals. Much of what there is to say about the last topic can be indicated by a consideration of the step intervals of the scale. Indeed, a scale may be completely characterized by a simple ordered listing of its step intervals.

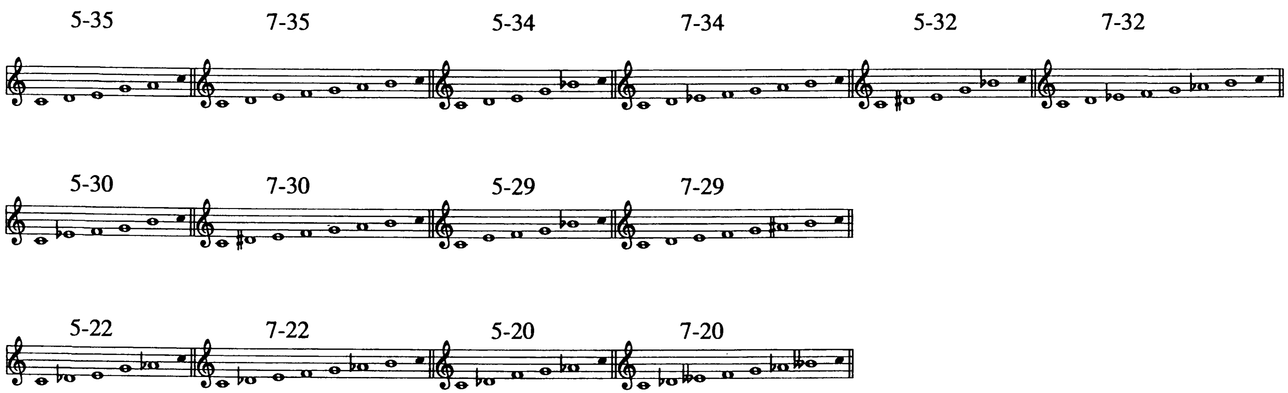
Each of these factors can serve to increase complexity. Complexity increases with the number pitch classes, and with the number and relative dissonance of generating intervals. For any cardinality, the greater the number of step interval types (i.e., more "difference"), the higher the complexity. Nevertheless, if the step intervals are still close to one another in size, it is possible to create a scale with even maximum difference, and no coherence failures. If the step intervals are markedly different from each other in size, the conditions for coherence failures are established, adding a new layer of complexity. Yet, if these very different step sizes are randomly distributed with respect to each other, there will be fewer coherence failures than if each size is successively larger (or successively smaller) than its predecessor. This seems to be why segments of the over-

Forte Number	Prime Form	CQ(5–n)	CA(7–n)	Prime Form	Forte Number
5-1	01234	0.4000	0.5000	0123456	7-1
5-2	01235	0.1600	0.3786	0123457	7-2
5-3	01245	0.3200	0.3571	0123458	7-3
5-4	01236	0.0800	0.4071	0123467	7-4
5-5	01237	0.2000	0.4643	0123567	7-5
5-6	01256	0.3200	0.4929	0123478	7-6
5-7	01267	0.5200	0.6500	0123678	7-7
5-8	02346	0.2400	0.3929	0234568	7-8
5-9	01246	0.1600	0.3857	0123468	7-9
5-10	01346	0.4000	0.3857	0123469	7-10
5-11	02347	0.1200	0.5000	0134568	7-11
5-12	01356	0.4000	0.4929	0123479	7-12
5-13	01248	0.2000	0.5429	0124568	7-13
5-14	01257	0.2800	0.5500	0123578	7-14
5-15	01268	0.6000	0.6357	0124678	7-15
5-16	01347	0.2800	0.5000	0123569	7-16
5-17	01348	0.3200	0.5786	0124569	7-17
5-18	01457	0.4400	0.5929	0123589	7-18
5-19	01367	0.5200	0.6786	0123679	7-19
5-20	01568	0.8000	0.7429	0125679	7-20
5-21	01458	0.6400	0.7143	0124589	7-21
5-22	01478	0.8000	0.8143	0125689	7-22
5-23	02357	0.4400	0.5071	0234579	7-23
5-24	01357	0.5200	0.5929	0123579	7-24
5-25	02358	0.4400	0.6643	0234679	7-25
5-26	02458	0.5200	0.6643	0134579	7-26
5-27	01358	0.5600	0.6857	0124579	7-27
5-28	02368	0.6000	0.7214	0135679	7-28
5-29	01368	0.8000	0.7714	0124679	7-29
5-30	01468	0.8000	0.8286	0124689	7-30
5-31	01369	0.6000	0.7786	0134679	7-31
5-32	01469	0.9600	0.8714	0134689	7-32
5-33	02468	0.6000	0.7500	012468a	7-33
5-34	02469	0.9600	0.9286	013468a	7-34
5-35	02479	1.0000	0.9929	013568a	7-35
5-36	01247	0.2000	0.4857	0123568	7-36
5-37	03458	0.4400	0.6071	0134578	7-37
5-38	01258	0.4400	0.6357	0124578	7-38

Example 12. Coherence quotients in 5- and 7-note set classes



Example 13. Pairs of complementary set classes



Example 14. Highly coherent 5- and 7-note set classes as scales

Properties of Scale Candidates	*Unlikely Scale Candidates*
Less Complex, More Redundant	More Complex, Less Redundant
I Cardinality	
Fewer pitch classes	More pitch classes
II Generating Intervals	
• Fewer • Consonant	• More • Dissonant
III Step Intervals	
• Few step types • Step intervals of similar sizes • More random distribution of step sizes	• Many types of steps • Sizes of step intervals very diverse • Step sizes increase (or decrease) monotonically

Example 15. Complexity factors

tone series seldom satisfy our notion of scale. The step sizes of overtone segments are all different, and, particularly lower in the series, the difference in sizes can be great. Furthermore, intervals in the overtone series appear in decreasing size order, promoting contradictions and ambiguities. Perhaps the single most powerful condition concerns what I have called the degree of coherence failure. I know of no pitch-class set of cardinality N that serves as a scale in which the degree of failure is greater than $N/2$.

Systems that are higher in entropy are more stable. Stability (redundancy) may be an inherently useful property in scales, but there is some information content. In terms of information, we may not be saying much, but we do know what we are saying. I believe that the information content of scales is related to their capacity to promote a tonic. This, however, becomes the topic of another paper.

NOTES

This article is dedicated to the memory of John Clough. A version of the article was presented at Toronto 2000: *Musical Intersections,* a joint conference involving the Society for Music Theory and fourteen other societies. The author wishes to thank Robert Morris, whose questions inspired this topic and who offered helpful suggestions along the way. The author also wishes to thank the two anonymous readers for their insights and suggestions.

1. Daniélou describes the system of sixty *lüs* as the basis of Chinese tuning systems (Daniélou 1995, 42ff.).
2. Cited in Tenney (1988, 2).
3. Note, however, that under these definitions, intervals that are inversions of each other in general do not belong to the same class. Under the more common definition of interval class, intervals whose logarithms are i_1 and i_2 are in the same interval class if $\{i_1/p\} = \{i_2/p\}$, or if $\{i_1/p\} = 1 - \{i_2/p\}$.
4. It is important to point out the difference between the way Myhill's Property (MP) is defined by Clough (and various co-authors), and how the definition is modified in well-formed scale theory. Myhill's Property is discussed in Clough and Douthett 1991 (103) with respect to Maximally Even (ME) sets. If a ME set has MP, then the two specific interval sizes in each generic class differ by exactly one chromatic step. There is no presumption of an underlying chromatic in well-formed scale theory, thus the difference between the larger step and the smaller step may be greater than, equal to, or less than the size of the smaller step. Nevertheless, the difference between the two specific interval sizes in any generic class is still a constant. (See Lemma 6.)
5. As Clough and Douthett (1991) prove in the context of Maximally Even sets, a single ambiguity must take the form of an octave-bisecting tritone that appears in two generic guises.
6. Rothenberg (1975) makes use of a matrix of this type in his work on "strictly proper" systems.
7. Columns of the matrix are not discussed in this paper; however, as the reader may note, with the addition of a bottom row of zeros they may be construed as the modes of the scale in the sense of "octave species." In other words, these interval matrices are related to rotational arrays.
8. Assume the contrary, that is, that a scale S of cardinality $N > 3$ with maximal failure and no contradictions may exist. Now if S has maximal failures then some step interval fails with an interval of span $N - 1$. But then it must form contradictions with two intervals of span $N - 2$.
9. Rothenberg (1975 II, 354) proposes a "stability measure" whose purpose is similar to our coherence quotient. The stability measure, however, is only defined on sets without contradictions, and distinguishes them according to cardinality and number of ambiguities.
10. Coherence in particular may be a significant factor in other contexts. Robert Morris (2000) has explored the role of ambiguity and contradiction in a variety of rhythmic patterns found in South Indian music.
11. Balzano's scales are defined as having cardinality d in an equal-tempered c-note universe, where for integer $k \geq 3$, $\mathrm{d} = 2k + 1$, and $\mathrm{c} = k(k + 1)$. In fact, the only Balzano scale that is a member of WF(N,N–2) is the 7-in-12 diatonic, wfs(7,5,2).

There are d – 2 (here five) larger steps, and 2 smaller ones, and $R = 2$. In all of the other Balzano scales, there are d – 2 smaller steps and 2 larger ones. The value R in these will be equal to $\frac{(d-5)(d-3)}{16}$. Clearly, as d increases, R increases without bound. While, as we will see, scales in WF(N,N–2) have high coherence quotients, those in WF(N,2) do not, particularly when R is large. Thus, despite Balzano's assertion that coherence is a significant scalar property, just two of the scales he proposes exhibit it. When d = 9, $R = 1.5$, and the coherence conditions of Corollary 6a are met. All Balzano scales of larger cardinality are not coherent. For a study of the relationships between these various scale types, see Clough, Engebretsen, and Kochavi 1999.

12. More than one-third the values presented in Example 11 as contradictions, ambiguities, and differences are different from those given in Figure 4 in Rahn 1991. The values shown here are derived from the algorithms shown in Examples 9 and 10.
13. The names of these nineteen are not the same as their cognates in the complete list. See Kaufmann, 1972, xx–xxi. There is a discrepancy between the way scale *Saamavaraali* is represented on page xx and how it is categorized on xxi. According to the former, it is a member of 7-30, according to the latter, 7-19. In that the representation on page xx is given in both Western pitch notation and Sanskrit note names, I have chosen to place it in 7-30. The analysis that follows is little affected by the choice.
14. Although this technique may be challenged as culturally myopic, one possible criticism, involving the question of tuning, is fairly easily addressed. How tuning operates in South Indian music is a subject Indian theorists themselves have long disputed; however, regardless of which model we choose, whether it be a tuning like Just Intonation, or Pythagorean tuning, or equal temperament, or even a division of the octave into 22 equal-tempered *srutis*, the resultant coherence quotients would not be significantly altered. See Morris 2001, 75 for a discussion of these and other related issues.
15. Rahn says that there are 27 *melakartas* that are not bisected, and this is correct. However, four of his identifications are incorrect: *melakarta* 24 and 66 *are* bisected sets, and *melakartas* 54 and 60 are *not.*
16. To generalize the hypothesis: The smallest equal-tempered universe in which there is a set class of order N with maximal failure is 2^{N-1}. One such set class has pitch classes [0, 1, 2, 4, …, 2^{N-2}]. The following 5-note scale scale in 16-tone equal temperament exhibits maximal failure: 0¢, 75¢, 150¢, 300¢, 600¢. The set has 25 failures, 5 of which are ambiguities. Its coherence quotient is 0. I do not believe that there is an equal temperament of fewer notes that can accommodate a 5-note scale with maximal failure.

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Part III: The graph embedding of pitch structures *Mathematical Systems Theory* 12: 73–101.

APPENDIX A—LEMMAS

Summation Lemmas

1 **Repeated addition is multiplication.**

LEMMA 1. $\sum_{i=1}^{n} a = na$

2 **The sum of the first n digits.**

LEMMA 2. $\sum_{i=1}^{n} i = (1+2+\ldots+n) = \frac{n(n+1)}{2}$

3 **The sum of the first n squares.**

LEMMA 3. $\sum_{i=1}^{n} i^2 = \left(1^2 + 2^2 + \ldots + n^2\right) = \frac{n(n+1)(2n+1)}{6}$

4 **The sum of the first n cubes.**

LEMMA 4. $\sum_{i=1}^{n} i^3 = \left(1^3 + 2^3 + \ldots + n^3\right) = \left(\frac{n(n+1)}{2}\right)^2$

Well-formed scale Lemmas

Page number references in parentheses refer to Carey 1998.

DEFINITIONS: Let N represent the cardinality of a non-degenerate well-formed scale; Let T be a positive integer, $1 \leq T < N$. Let μ_T and ν_T represent, respectively, the larger and smaller interval size of span T. Let g represent the number of times that the larger step interval, μ_1 appears in each octave. Finally, if x is a real number, then let $[x]$ represent the "integral part" of x, the greatest integer less than or equal to x.

5 **N and g have no common factors greater than 1** (129).

LEMMA 5. $\gcd(N,g) = 1$

We would also say that N and g are *relatively prime*.

6 **The difference between specific intervals of the same span** (128). The difference between the larger and smaller intervals of the same generic size is the same as the difference between the larger and smaller step intervals:

LEMMA 6. $\mu_T - \nu_T = \mu_1 - \nu_1$

7 **Intervals and step content** (130). The intervals μ_T and ν_T are made up of some combination of μ_1 and ν_1 step intervals. Let x_T represent the ν_1 content of μ_T.

LEMMA 7.1. $\mu_T = (T - x_T)\mu_1 + (x_T)\nu_1$

LEMMA 7.2. $\nu_T = (T - x_T - 1)\mu_1 + (x_T + 1)\nu_1$

(Note that subtracting the value in 7.2 from 7.1 gives the value in Lemma 6.)

8 **Maximal lengths of strings of step intervals** (100). The step interval ν_1 never occurs more than $\left\lceil \frac{N}{g} \right\rceil$ times in succession. Let $str(\nu_1)$ represent some consecutive string of ν_1 step intervals. Let $L(str(\nu_1))$ be a function that returns the length of that string as its value.

LEMMA 8. $\max\left(L\left(str(\nu_1)\right)\right) = \left\lceil \frac{N}{g} \right\rceil$

9 **Multiplicity of step intervals** (130). Let $\#(\mu_T)$ (read "the multiplicity of μ_T") be a function whose returned value is the number of occurrences of the larger intervals of span T and similarly for $\#(\nu_T)$.

LEMMA 9.1. $\#(\mu_T) = Tg_{\text{mod}N}$

LEMMA 9.2. $\#(\nu_T) = -Tg_{\text{mod}N}$

Note that, consistent with its definition, $g = \#(\mu_1)$.

APPENDIX B—THEOREMS

Theorem 1: Total Number of Interval Pairs in an N-note Scale

$$\mathrm{IP}(N) = \frac{(N^2-N)(N^2-N-1)}{2}$$

Theorem 2: Maximal Number of Differences in an N-note Scale

$$\mathrm{MD}(N) = \frac{(N)(N-1)(N-1)}{2}$$

Theorem 3: Number of Interval Pairs That May Fail

$$\mathrm{PF}(N) = \frac{N(N-1)(N-2)(2N-3)}{6}$$

Theorem 4: Maximal Number of Simultaneous Failures

$$\mathrm{SF}(N) = \frac{N(N-1)(N-2)(3N-5)}{24}$$

Theorem 5: Number of Ambiguities in wfs(N,g,R)

$$\#\mathrm{A}(N,g,R) = \frac{((N-1)-g(R-1))((N)-g(R-1))((N+1)-g(R-1))}{6}$$

Theorem 6: Number of Contradictions in wfs(N,g,R)

$$\#\mathrm{C}(N,g,R) = \sum_{k=1}^{j} \frac{(N-1-gk)(N-gk)(N+1-gk)}{6}, \quad j=\min\left(\lceil R-1\rceil, \left\lceil \frac{N-1}{g}\right\rceil\right)$$

Theorem 7: Number of Differences in Well-formed N-note Scale

$$\#\mathrm{D}(N) = \frac{(N+1)(N)(N-1)}{6}$$

Corollary 6e: Maximal Failures for Well-formed N-note Scale

$$\max(\#\mathrm{F}(N,g,R)) = \frac{N(N-1)(N-2)(N+1)}{24}$$

APPENDIX C—RESULTS

Results for $3 \le N \le 25$

N	IP(N)	PF(N)	SF(N)	max (#F(N,g,R))	MD(N)	#D(N)
3	15	3	1	1	6	4
4	66	20	7	5	18	10
5	190	70	25	15	40	20
6	435	180	65	35	75	35
7	861	385	140	70	126	56
8	1540	728	266	126	196	84
9	2556	1260	462	210	288	120
10	4005	2040	750	330	405	165
11	5995	3135	1155	495	550	220
12	8646	4620	1705	715	726	286
13	12090	6578	2431	1001	936	364
14	16471	9100	3367	1365	1183	455
15	21945	12285	4550	1820	1470	560
16	28680	16240	6020	2380	1800	680
17	36856	21080	7820	3060	2176	816
18	46665	26928	9996	3876	2601	969
19	58311	33915	12597	4845	3078	1140
20	72010	42180	15675	5985	3610	1330
21	87990	51870	19285	7315	4200	1540
22	106491	63140	23485	8855	4851	1771
23	127765	76153	28336	10626	5566	2024
24	152076	91080	33902	12650	6348	2300
25	179700	108100	40250	14950	7200	2600

APPENDIX D—*Melakarta ragas*

Number/Name	Pitch Classes												step intervals	Forte Number
1 *Kanakangi*	0	1	2			5	7	8	9			12	1132113	20
2 *Ratnangi*	0	1	2			5	7	8		10		12	1132122	29
3 *Ganamurthi*	0	1	2			5	7	8			11	12	1132131	19
4 *Vanaspati*	0	1	2			5	7		9	10		12	1132212	27
5 *Manavati*	0	1	2			5	7		9		11	12	1132221	24
6 *Tanarupi*	0	1	2			5	7			10	11	12	1132311	12
7 *Senavati*	0	1		3		5	7	8	9			12	1222113	30
8 *Hanumathodi*	0	1		3		5	7	8		10		12	1222122	35
9 *Dhenuka*	0	1		3		5	7	8			11	12	1222131	30
10 *Natakapriya*	0	1		3		5	7		9	10		12	1222212	34
11 *Kokilapriya*	0	1		3		5	7		9		11	12	1222221	33
12 *Rupavati*	0	1		3		5	7			10	11	12	1222311	24
13 *Gayakapriya*	0	1			4	5	7	8	9			12	1312113	21
14 *Vakulabharanam*	0	1			4	5	7	8		10		12	1312122	32
15 *Mayamalavagoula*	0	1			4	5	7	8			11	12	1312131	22
16 *Chakravakam*	0	1			4	5	7		9	10		12	1312212	32
17 *Suryakantam*	0	1			4	5	7		9		11	12	1312221	30
18 *Hatakambari*	0	1			4	5	7			10	11	12	1312311	19
19 *Jhankaradvani*	0		2	3		5	7	8	9			12	2122113	29
20 *Natabhairavi*	0		2	3		5	7	8		10		12	2122122	35
21 *Kiravani*	0		2	3		5	7	8			11	12	2122131	32
22 *Kharakharapriya*	0		2	3		5	7		9	10		12	2122212	35
23 *Gaurimanohari*	0		2	3		5	7		9		11	12	2122221	34
24 *Varunapriya*	0		2	3		5	7			10	11	12	2122311	27
25 *Mararanjani*	0		2		4	5	7	8	9			12	2212113	27

26 *Charukesi*	0		2		4	5		7	8		10		12	2212122	34
27 *Sarasangi*	0		2		4	5		7	8			11	12	2212131	32
28 *Harikambodi*	0		2		4	5		7		9	10		12	2212212	35
29 *Dhirashankarabharanam*	0		2		4	5		7		9		11	12	2212221	35
30 *Naganandhini*	0		2		4	5		7			10	11	12	2212311	29
31 *Yagapriya*	0			3	4	5		7	8	9			12	3112113	17
32 *Ragavardhini*	0			3	4	5		7	8		10		12	3112122	27
33 *Gangeyabhushani*	0			3	4	5		7	8			11	12	3112131	21
34 *Vagadishvari*	0			3	4	5		7		9	10		12	3112212	29
35 *Sulini*	0			3	4	5		7		9		11	12	3112221	30
36 *Chalanata*	0			3	4	5		7			10	11	12	3112311	20
37 *Salagami*	0	1	2				6	7	8	9			12	1141113	7
38 *Jalarnavam*	0	1	2				6	7	8		10		12	1141122	15
39 *Jhalavarali*	0	1	2				6	7	8			11	12	1141131	7
40 *Navanitam*	0	1	2				6	7		9	10		12	1141212	38
41 *Pavani*	0	1	2				6	7		9		11	12	1141221	14
42 *Raghupriya*	0	1	2				6	7			10	11	12	1141311	6
43 *Gavambodhi*	0	1		3			6	7	8	9			12	1231113	19
44 *Bhavapriya*	0	1		3			6	7	8		10		12	1231122	29
45 *Shubhapantuvarali*	0	1		3			6	7	8			11	12	1231131	20
46 *Shadvidhamargini*	0	1		3			6	7		9	10		12	1231212	31
47 *Suvarnangi*	0	1		3			6	7		9		11	12	1231221	28
48 *Divyamani*	0	1		3			6	7			10	11	12	1231311	18
49 *Dhavalambari*	0	1			4		6	7	8	9			12	1321113	18
50 *Namanarayani*	0	1			4		6	7	8		10		12	1321122	28
51 *Kamavardhini*	0	1			4		6	7	8			11	12	1321131	20
52 *Ramapriya*	0	1			4		6	7		9	10		12	1321212	31

(continued)

APPENDIX D (continued)

Number/Name	Pitch Classes												step intervals	Forte Number
53 *Gamanasrama*	0	1			4	6	7		9		11	12	1321221	29
54 *Visvambari*	0	1			4	6	7			10	11	12	1321311	19
55 *Syamalangi*	0		2	3		6	7	8	9			12	2131113	19
56 *Shanmukhapriya*	0		2	3		6	7	8		10		12	2131122	30
57 *Simhendra Madhyamam*	0		2	3		6	7	8			11	12	2131131	22
58 *Hemavati*	0		2	3		6	7		9	10		12	2131212	32
59 *Dharmavati*	0		2	3		6	7		9		11	12	2131221	32
60 *Nitimati*	0		2	3		6	7			10	11	12	2131311	21
61 *Kantamani*	0		2		4	6	7	8	9			12	2221113	24
62 *Rishabapriya*	0		2		4	6	7	8		10		12	2221122	33
63 *Latangi*	0		2		4	6	7	8			11	12	2221131	30
64 *Vachaspati*	0		2		4	6	7		9	10		12	2221212	34
65 *Mechakalyani*	0		2		4	6	7		9		11	12	2221221	35
66 *Chitrambari*	0		2		4	6	7			10	11	12	2221311	30
67 *Sucharitra*	0			3	4	6	7	8	9			12	3121113	16
68 *Jyotisvarupini*	0			3	4	6	7	8		10		12	3121122	26
69 *Dhatuvardhini*	0			3	4	6	7	8			11	12	3121131	21
70 *Nasika Bhushani*	0			3	4	6	7		9	10		12	3121212	31
71 *Kosalam*	0			3	4	6	7		9		11	12	3121221	32
72 *Rasikapriya*	0			3	4	6	7			10	11	12	3121311	22

Numbers in boldface: Venkatamakhin's 19