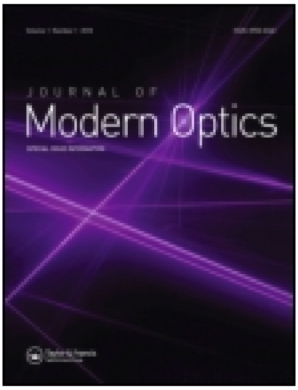

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Optical Generation of Gabor's Expansion Coefficients for Rastered Signals

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Optical generation of Gabor's expansion coefficients for rastered signals

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Abstract. Gabor's expansion of a signal into a discrete set of properly shifted and modulated versions of an elementary signal is introduced; several examples of elementary signals are considered. An arrangement is described which is able to generate Gabor's expansion coefficients of a one-dimensional signal by optical means. An important feature of the optical arrangement is that it accepts the one-dimensional signal on a raster format; hence, the two-dimensional nature of the optical processing system is fully utilized.

1. Introduction

It is sometimes convenient to describe a space signal $\phi(x)$ not in the space domain, but in the frequency domain by means of its *frequency spectrum*, i.e. the Fourier transform $\bar{\phi}(u)$ of the function $\phi(x)$, which is defined by

$$\bar{\phi}(u)=\int \phi(x)\exp[-iux]\,dx. \tag{1}$$

(A bar on top of a symbol will mean throughout that we are dealing with a function in the frequency domain. Unless otherwise stated, all integrations and summations in this paper extend from $-\infty$ to $+\infty$.) The frequency spectrum shows us the *global* distribution of the energy of the signal as a function of frequency. However, one is often more interested in the *local* distribution of the energy as a function of frequency. Geometrical optics, for instance, is usually treated in terms of rays, and the signal is described by giving the directions (i.e. frequencies) of the rays that should be present at a certain point. Hence, we look for a description of the signal that might be called the *local frequency spectrum* of the signal.

The need for a local frequency spectrum arises in other disciplines, too. It arises in music, for instance, where a signal is usually described not by a time function nor by the Fourier transform of that function, but by its *musical score*; indeed, when a composer writes a score, he prescribes the frequencies of the tones that should be present at a certain moment. It arises also in mechanics, where the position and the momentum of a particle are given simultaneously, leading to a description of mechanical phenomena in a *phase space*.

Gabor's signal expansion [1–4] is a candidate for a *discrete* local frequency spectrum. In 1946 Gabor suggested the expansion of a signal into a discrete set of properly shifted and modulated gaussian elementary signals. A quotation from the summary of Gabor's original paper [1] might be useful.

> Hitherto communication theory was based on two alternative methods of signal analysis. One is the description of the signal as a function of time; the other is

Fourier analysis. ... But our everyday experiences ... insist on a description in terms of *both* time and frequency. ... Signals are represented in two dimensions, with time and frequency as co-ordinates. Such two-dimensional representations can be called 'information diagrams', as areas in them are proportional to the number of independent data which they can convey. ... There are certain 'elementary signals' which occupy the smallest possible area in the information diagram. They are harmonic oscillations modulated by a probability pulse. Each elementary signal can be considered as conveying exactly one datum, or one 'quantum of information'. Any signal can be expanded in terms of these by a process which includes time analysis and Fourier analysis as extreme cases.

Although Gabor restricted himself to an elementary signal that had a gaussian shape, his signal expansion holds for rather arbitrarily shaped elementary signals [2–4].

In § 2 of this paper we shall review Gabor's signal expansion, and demonstrate how the expansion coefficients can be determined. Two ways of generating these expansion coefficients optically, will be shown. One of these ways is especially applicable to signals that appear on a rastered format, and will be studied in more detail in §§ 3 and 4.

2. Gabor's signal expansion

We shall represent a space signal $\phi(x)$ not in a pure space domain, nor—by means of its Fourier transform $\bar{\phi}(u)$ for instance—in a pure frequency domain, but in a combined *space–frequency domain* using Gabor's signal expansion. With the help of this expansion, we can express the signal as a superposition of properly shifted and modulated versions of an *elementary signal* $g(x)$, say, yielding

$$\phi(x)=\sum_{mn}\sum a_{mn}g(x-mX)\exp[inUx], \tag{2}$$

where the space shift X and the frequency shift U satisfy the relation $UX=2\pi$. The elementary signal $g(x)$ may be chosen rather arbitrarily; it need not be Gabor's choice of a gaussian signal.

In general, the discrete set of shifted and modulated elementary signals $g(x-mX)\exp[inUx]$ may not be orthogonal, which implies that *Gabor's expansion coefficients* a_{mn} cannot be determined in the usual way. It is possible, however, to find a function $\gamma(x)$, say, that is *bi-orthonormal* [2–4] to the set of elementary signals in the sense

$$\int g(x)\gamma^*(x-mX)\exp[-inUx]\,dx=\begin{cases}1 & \text{for } m=n=0,\\ 0 & \text{elsewhere.}\end{cases} \tag{3}$$

(An asterisk will denote throughout complex conjugation.) With the help of this bi-orthonormal function, the expansion coefficients follow readily via

$$a_{mn}=\int \phi(x)\gamma^*(x-mX)\exp[-inUx]\,dx. \tag{4}$$

The integral in equation (4) can be interpreted as the *cross-ambiguity function* of the signal $\phi(x)$ and the function $\gamma(x)$. Hence, all the optical arrangements [5–8] that are designed to display the cross-ambiguity function of two signals, can be used to generate Gabor's expansion coefficients in an optical way.

As an illustration of Gabor's signal expansion, we consider—like Gabor did—the case of a gaussian elementary signal, and display the absolute values of Gabor's expansion coefficients a_{mn} for some simple signals . (See also Bastiaans [2] for more mathematical details.) In figure 1 we have sketched the array of expansion coefficients for the Dirac function $\phi(x)=X\delta(x)$; the area of a dot is proportional to the absolute value of the corresponding expansion coefficient. We see that for $x=0$ all frequencies are present, while there is almost no contribution from gaussian elementary signals that are not located around the point $x=0$. The array of coefficients for the constant $\phi(x)=1$ has been sketched in figure 2. We see that the frequency $u=0$ is present at all points x, while there is almost no contribution from other frequencies. The expansion coefficients for a step function [$\phi(x)=0$ for $x<0$, $\phi(x)=1$ for $x>0$] have been sketched in figure 3. Note that for $m\to-\infty$ all the coefficients become zero, while for $m\to+\infty$ the array of coefficients equals the one shown in figure 2. These examples show that Gabor's signal expansion can indeed be considered as a (discrete) local frequency spectrum of the signal.

Equations (2)–(4) can be formulated in a more convenient form, where the summations and the integrations have been replaced by simple multiplications [2–4]. To find these simple forms, we introduce a function $\tilde{a}(x,u)$ associated to the expansion coefficients a_{mn}, and functions $\tilde{\phi}(x,u)$, $\tilde{g}(x,u)$ and $\tilde{\gamma}(x,u)$ associated to the functions $\phi(x)$, $g(x)$ and $\gamma(x)$, respectively.

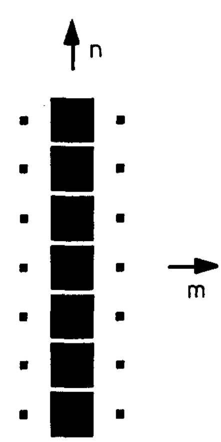

Figure 1. Gabor's expansion coefficients for a Dirac function.

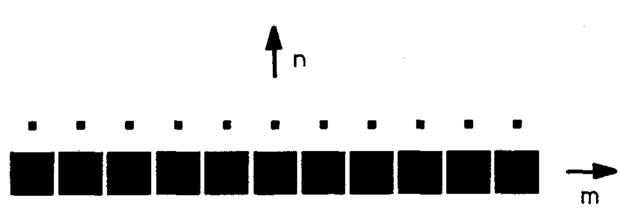

Figure 2. Gabor's expansion coefficients for a constant.

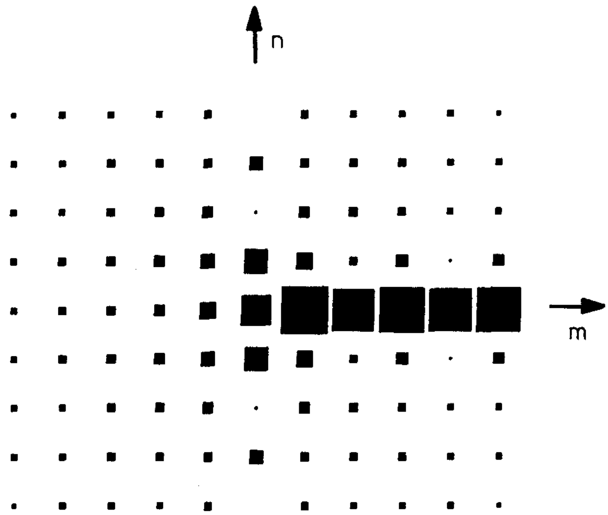

Figure 3. Gabor's expansion coefficients for a step function.

The function $\tilde{a}(x,u)$ is defined by a Fourier series with coefficients a_{mn}:

$$\tilde{a}(x,u)=\sum_{mn}\sum a_{mn}\exp\left[-i(muX-nUx)\right]. \tag{5 a}$$

Note that the function $\tilde{a}(x,u)$ is periodic in x and u, with periods X and U, respectively. The inverse relationship has the form

$$a_{mn}=\frac{1}{XU}\int\int_{XU}\tilde{a}(x,u)\exp\left[i(muX-nUx)\right]dx\,du, \tag{5 b}$$

where the integrations extend over one period X and one period U, respectively.

The function $\tilde{\phi}(x,u)$ is defined by

$$\tilde{\phi}(x,u)=\sum_{m}\phi(x+mX)\exp\left[-imuX\right]. \tag{6 a}$$

Note that this function is periodic in u, with period U, and quasi-periodic in x, with quasi-period X. Equation (6 a) provides a means of representing a one-dimensional space function by a two-dimensional space-frequency function on a rectangle with finite area $UX=2\pi$ [9]. The inverse relationship has the form

$$\phi(x+mX)=\frac{1}{U}\int_{U}\tilde{\phi}(x,u)\exp\left[imuX\right]du, \tag{6 b}$$

where, again, the integration extends over one period U. It will be clear that the variable x in equation (6 b) can be restricted to an interval of length X, with m taking on all integer values. The functions $\tilde{g}(x,u)$ and $\tilde{\gamma}(x,u)$, associated to the functions $g(x)$ and $\gamma(x)$, respectively, are defined in a similar way.

With the help of the functions $\tilde{a}(x,u)$, $\tilde{\phi}(x,u)$, $\tilde{g}(x,u)$ and $\tilde{\gamma}(x,u)$, we can rewrite equations (2)–(4) as

$$\tilde{\phi}(x,u)=\tilde{a}(x,u)\tilde{g}(x,u), \tag{7}$$

$$X\tilde{g}(x,u)\tilde{\gamma}^*(x,u)=1 \tag{8}$$

and

$$\tilde{a}(x,u)=X\tilde{\phi}(x,u)\tilde{\gamma}^*(x,u), \tag{9}$$

respectively. In fact we have now formulated another way of finding Gabor's expansion coefficients a_{mn}: from the signal $\phi(x)$ we derive the associated function $\tilde{\phi}(x,u)$ via equation (6 *a*), we multiply this function by the known function $X\tilde{\gamma}^*(x,u)$ to find the function $\tilde{a}(x,u)$ according to equation (9), and the expansion coefficients a_{mn} follow from the function $\tilde{a}(x,u)$ with the help of the inversion formula (5 *b*). This way of finding Gabor's expansion coefficients is perfectly applicable to a signal that appears on a *raster format* for which the raster width equals X: for such a rastered signal $\phi(x)$, the associated function $\tilde{\phi}(x,u)$ can be generated by a simple, one-dimensional, optical Fourier transformation. An optical arrangement to generate the expansion coefficients for such rastered signals is described in the next section.

3. Optical arrangement for generating Gabor's expansion coefficients

Let us consider the optical arrangement depicted in figure 4. A plane wave of monochromatic laser light is normally incident upon a transparency situated in the input plane. The transparency contains the signal $\phi(x)$ in a rastered format. With X

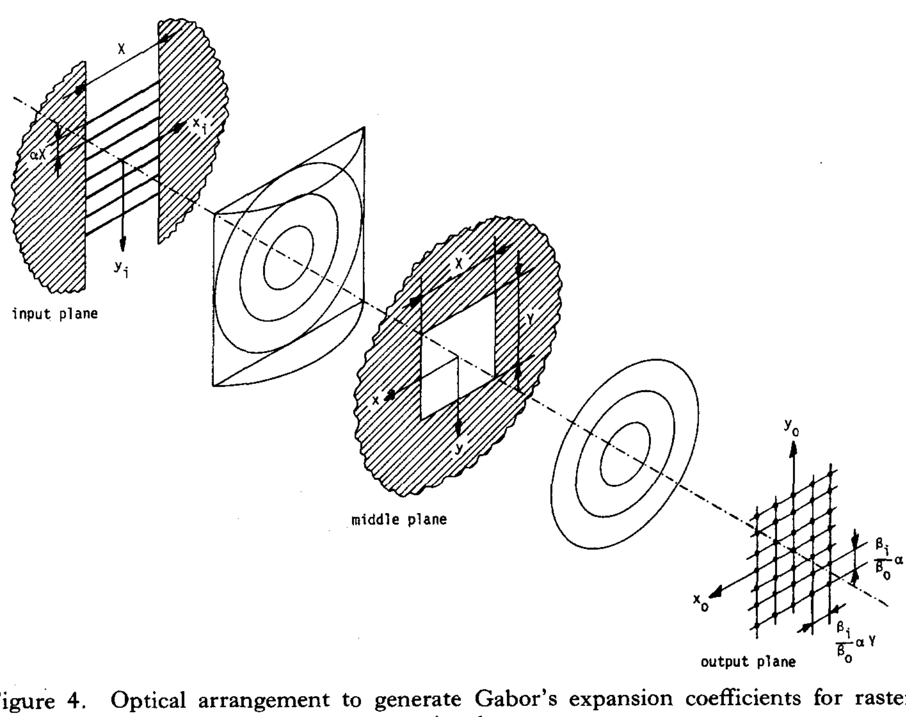

Figure 4. Optical arrangement to generate Gabor's expansion coefficients for rastered signals.

being the width of this raster and αX being the spacing between the raster lines, the light amplitude $f_i(x_i, y_i)$ just behind the transparency reads

$$f_i(x_i, y_i) = \mathrm{rect}\,(x_i/X) \sum_m \phi(x_i + mX)\delta(y_i - m\alpha X), \tag{10}$$

where rect (t) represents the rectangular function

$$\mathrm{rect}\,(t) = \begin{cases} 1 & \text{for } -\frac{1}{2} < t \leqslant \frac{1}{2}, \\ 0 & \text{elsewhere.} \end{cases} \tag{11}$$

An astigmatic optical system between the input plane and the middle plane performs a Fourier transformation in the y-direction and an ideal imaging (with inversion) in the x-direction. Such an astigmatic system can be realized as shown, for instance, using a combination of a spherical and a cylindrical lens. The astigmatic operation results in the light amplitude

$$\begin{aligned} f_1(x, y) &= \iint \exp\,[-i\beta_i y y_i]\delta(x - x_i) f_i(x_i, y_i)\, dx_i\, dy_i \\ &= \mathrm{rect}\,(x/X)\tilde{\phi}(x, \alpha\beta_i y) \end{aligned} \tag{12}$$

just in front of the middle plane; the parameter β_i contains the effects of the wave number of the laser light and the focal length of the spherical lens.

A transparency with amplitude transmittance

$$m(x, y) = \mathrm{rect}\,(x/X)\,\mathrm{rect}\,(y/Y) X\tilde{\gamma}^*(x, \alpha\beta_i y), \tag{13}$$

where $Y = U/\alpha\beta_i$, is situated in the middle plane. Just behind this transparency the light amplitude takes the form

$$\begin{aligned} f_2(x, y) &= m(x, y) f_1(x, y) \\ &= \mathrm{rect}\,(x/X)\,\mathrm{rect}\,(y/Y)\tilde{a}(x, \alpha\beta_i y). \end{aligned} \tag{14}$$

Finally, a two-dimensional Fourier transformation is performed between the middle plane and the output plane. Such a Fourier transformation can be realized as shown, for instance, using a spherical lens. The light amplitude in the output plane then takes the form

$$\begin{aligned} f_o(x_o, y_o) &= \frac{1}{XY} \iint \exp\,[-i\beta_o(x_o x - y_o y)] f_2(x, y)\, dx\, dy \\ &= \sum_{mn}\sum a_{mn}\,\mathrm{sinc}\,(\beta_o x_o/\alpha\beta_i Y - n)\,\mathrm{sinc}\,(\beta_o y_o/\alpha\beta_i X - m), \end{aligned} \tag{15}$$

where $\mathrm{sinc}\,(t) = \sin(\pi t)/\pi t$; the parameter β_o, again, contains the effects of the wave number of the light and the focal length of the spherical lens. We conclude that Gabor's expansion coefficients appear on a rectangular lattice of points:

$$a_{mn} = f_o(n\alpha\beta_i Y/\beta_o, m\alpha\beta_i X/\beta_o). \tag{16}$$

We remark that it is not an essential requirement that the input transparency consists of Dirac functions. When we replace the practically unrealizable Dirac functions $\delta(y - m\alpha X)$ by realizable functions $d(y - m\alpha Y)$, say, then equation (10) reads

$$f_i(x_i, y_i) = \mathrm{rect}\,(x_i/X) \sum_m \phi(x_i + mX)\, d(y_i - m\alpha X), \tag{17}$$

and the light amplitude $f_1(x, y)$ just in front of the middle plane takes the form

$$f_1(x, y) = \mathrm{rect}\,(x/X)\bar{\phi}(x, \alpha\beta_i y)\bar{d}(\beta_i y). \tag{18}$$

The additional factor $\bar{d}(\beta_i y)$ can easily be compensated for by means of a transparency in the middle plane. Note that the special case $d(y) = \mathrm{sinc}\,(y/\alpha X)/\alpha X$, and thus $\bar{d}(\beta_i y) = \mathrm{rect}\,(y/Y)$, has the advantage that all the light from the input plane will fall inside the rectangle $\mathrm{rect}\,(x/X)\,\mathrm{rect}\,(y/Y)$ in the middle plane.

4. Examples of elementary signals

We shall consider some examples of elementary signals $g(x)$ and determine their associated functions $\tilde{g}(x, u)$. Since the function $\tilde{g}(x, u)$ is periodic in u, with period U, and quasi-periodic in x, with quasi-period X, we can confine ourselves to the interval $(-\frac{1}{2}X < x \leqslant \frac{1}{2}X,\ -\frac{1}{2}U < u \leqslant \frac{1}{2}U)$. Therefore, we shall only consider the truncated version $\tilde{g}_T(x, u) = \tilde{g}(x, u)\,\mathrm{rect}\,(x/X)\,\mathrm{rect}\,(u/U)$ of the associated function $\tilde{g}(x, u)$. For any elementary signal $g(x)$ we shall determine the corresponding amplitude transmittance $m(x, y)$ of the transparency in the middle plane.

4.1. *Space-limited elementary signal*

As a first example we consider the rectangular elementary signal $g(x) = \mathrm{rect}\,(x/X)$. The truncated version $\tilde{g}_T(x, u)$ of its associated function $\tilde{g}(x, u)$ reads $\tilde{g}_T(x, u) = 1$. The amplitude transmittance $m(x, y)$ of the transparency in the middle plane takes the form

$$m(x, y) = \mathrm{rect}\,(x/X)\,\mathrm{rect}\,(y/Y), \tag{19}$$

whose practical realization is obvious.

We remark that Gabor's signal expansion using a rectangular elementary signal represents, in fact, a well-known way of expanding a signal: we simply consider the signal in successive intervals of length X, and describe the signal in each interval by means of a Fourier series.

This example of a rectangular elementary signal can easily be generalized to an arbitrary elementary signal $g(x)$ that is limited to the interval $(-\frac{1}{2}X < x \leqslant \frac{1}{2}X)$; the truncated associated function $\tilde{g}_T(x, u)$ reads $\tilde{g}_T(x, u) = g(x)$, and the amplitude transmittance $m(x, y)$ takes the form

$$m(x, y) = \mathrm{rect}\,(x/Y)\,\mathrm{rect}\,(y/Y)\frac{1}{g(x)}. \tag{20}$$

4.2. *Band-limited elementary signal*

Our second example is the band-limited elementary signal $g(x) = \mathrm{sinc}\,(x/X)$. Its Fourier transform $\bar{g}(u) = X\,\mathrm{rect}\,(u/U)$ is limited to the frequency-interval $(-\frac{1}{2}U < u \leqslant \frac{1}{2}U)$, and its truncated associated function reads $\tilde{g}_T(x, u) = \exp\,[iux]$. The amplitude transmittance of the transparency in the middle plane takes the form

$$m(x, y) = \mathrm{rect}\,(x/X)\,\mathrm{rect}\,(y/Y)\exp\,[i\alpha\beta_i xy]. \tag{21}$$

The exponential factor $\exp[i\alpha\beta_i xy]$ in equation (21) can easily be realized by means of a combination of a spherical lens and a cylindrical lens that is oriented under 45° with the x and the y axis.

We remark that for a signal which is band-limited to the frequency interval $(-\frac{1}{2}U<u\leqslant\frac{1}{2}U)$, Gabor's signal expansion reduces to the well-known sampling theorem.

This example can easily be generalized to an arbitrary elementary signal $g(x)$ that is band-limited to the frequency interval $(-\frac{1}{2}U<u\leqslant\frac{1}{2}U)$; the truncated associated function $\tilde{g}_T(x,u)$ reads $\tilde{g}_T(x,u)=\exp[iux]\bar{g}(u)/X$, and the amplitude transmittance $m(x,y)$ takes the form

$$m(x,y)=\mathrm{rect}(x/X)\,\mathrm{rect}(y/Y)\exp[i\alpha\beta_i xy]\frac{X}{\bar{g}(\alpha\beta_i Y)}. \tag{22}$$

4.3 *Gaussian elementary signal*

As our final example we consider the gaussian elementary signal

$$g(x)=\exp[-\pi(x/X)^2].$$

The associated function $\tilde{g}(x,u)$ reads

$$\tilde{g}(x,u)=\exp[-\pi(x/X)^2]\theta_3(\pi u/U+i\pi x/X),$$

where $\theta_3(\cdot)$ is a *theta function* [10] with *nome* $q=\exp[-\pi)$. The amplitude transmittance of the transparency in the middle plane takes the form

$$m(x,y)=\mathrm{rect}(x/X)\,\mathrm{rect}(y/Y)\exp[\pi(x/X)^2]\frac{1}{\theta_3(\pi z)}, \tag{23}$$

where, for convenience, we have set $z=y/Y+ix/X$.

We remark that Gabor's choice of a gaussian elementary signal has a serious drawback. Since $\theta_3(\pi z)$ has a zero for $z=\frac{1}{2}+\frac{1}{2}i$, the transmittance function $m(x,y)$ must have a pole and, hence, cannot be realized exactly.

5. Conclusion

We have described how Gabor's expansion coefficients can be generated by optical means. Our optical arrangement requires the signal on a raster format. Such a raster format fully utilizes the two-dimensional nature of the optical system, its parallel processing features, and the large space-bandwidth product possible in optical processing. Our technique exhibits a resemblance to folded spectrum techniques [11], where space-bandwidth products in the order of 3×10^5 are reported (see, for example, Casasent [11], § 8.3).

Local frequency spectra, like Gabor's signal expansion, may have an application in speech processing [8], where speech recognition and speaker identification are important problems. With out set-up, and a space-bandwidth product of 3×10^5, we might be able to process speech fragments of about 1 min.

On présente le développement de Gabor d'un signal en un système discret d'éléments modulés et décalés d'un signal élémentaire; plusieurs exemples de signaux élémentaires sont considérés. On décrit un arrangement qui est capable d'engendrer les coefficients du développement de Gabor d'un signal unidimensionnel par des moyens optiques. Une caractéristique importante de l'arrangement optique est qu'il adapte le signal unidimensionnel au format de la surface sensible d'un tube de télévision; ainsi, la nature bidimensionnelle du système de traitement optique est pleinement utilisée.

Gabors Entwicklung eines Signals in einen diskreten Satz geeignet verschobener und modulierter Versionen eines elementaren Signals wird eingeführt; einige Beispiele elementarer Signale werden betrachtet. Es wird eine Anordnung beschrieben, die Gabors Entwicklungskoeffizienten eines eindimensionalen Signals optisch erzeugt. Eine wichtige Eigenschaft dieser Anordnung ist, daß sie das eindimensionale Signal auf einem Raster aufnimmt; folglich wird die zweidimensionale Natur des optischen Systems voll genutzt.

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