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VARIETY AND MULTIPLICITY IN DIATONIC SYSTEMS

John Clough and
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Chords of the diatonic (major scale) set may be construed with either the chromatic set or the diatonic set as referential "universe." For example, {B, D, F} is an instance of the class "diminished triad" and also an instance of the larger class "triad." We shall call the former designation *specific* and the latter *generic*. Thus the class including {C, D, E}, {F, G, A} and {G, A, B} (but not, for instance, {D, E, F}) is specific (two whole steps) and the larger class including all three-note major scale fragments is generic. The specific point of view is the generally adopted one, based on Forte's classification and developed systematically by Browne and in a historical study by Gauldin.[1] The generic perspective was developed by one of us (Clough) and in more recent work by Thomas Zimmerman.[2] Until now the two approaches have remained unconnected, an uncomfortable and peculiar situation given the similarity of the underlying mathematics.

We shall demonstrate here that under the appropriate equivalence relation, there are fundamental connections between the two interpretations. These connections apply to lines as well as chords. We account for these relationships by showing that the embedding of the seven-note diatonic scale in the twelve chromatic notes is one of a special class of theoretical embeddings with properties which imply the stated chordal and linear relationships.

In Part I of the paper, proceeding informally, we exhibit the following relationships for the familiar (seven in twelve) diatonic case:

1. **Partitioning.** Each chord species belongs wholly to a single chord genus.
2. **Cardinality Equals Variety (CV).** Every chord genus comprises a number of species equal to the cardinality of the chord, for cardinalities 1 through 6.
3. **Structure Yields Multiplicity (SM).** Within a particular genus, the number of chords in each species for chord cardinalities 1 through 6 is directly inferable from the generic structure.

In Part II, proceeding more formally, we develop the following general results for scales in chromatic universes of arbitrary size:

4. **Myhill's Property.** A scale in which every generic interval appears in exactly two specific sizes is said to have Myhill's property. Such a scale will exhibit CV and SM for lines. Under certain conditions, CV and SM hold for chords as well. The proof involves the concept of a "generalized circle of fifths."
5. **Uniqueness.** Given c and d coprime, where c is the cardinality of the "chromatic" universe and d is the cardinality of the "diatonic" scale, there exists a scale with Myhill's property, and that scale is unique to within transposition and reduction (deletion of "excess" chromatic notes).
6. **Construction.** There is a simple formula for the construction of such a scale.
7. **Partitioning and Deep Scales.** If a reduced scale (one containing half and whole steps only) has the partitioning property (see above), then $c=2d-1$ or $c=2d-2$, and the diatonic scale is a deep scale.[3] The converse is also true—any reduced scale with c and d as above (all such scales are deep scales) has the partitioning property.

These results significantly enhance and strengthen both of the approaches heretofore adopted (indeed render them practically inseparable though not indistinct) and suggest a new perspective on the enduring beauty and subtlety of the diatonic system, and hence of tonal music. We believe that our theory also has considerable potential for application in the composition of microtonal music.

I

Chord classification: genera and species. The results to be established here require classification of diatonic chords with respect to both the seven- and twelve-pc universes. For the seven-pc (generic) classification, chord structure is reckoned under diatonic equivalence (minor 2nd≡ major 2nd, and so forth) as defined in Clough's previous work. This equivalence relation obviously does not apply to the twelve pc (specific) classification. For both levels of classification, equivalence under transposition (T) is recognized; however, equivalence under inversion with transposition (TI) is not recognized. This scheme is consistent with Clough's work on the generic case. For the specific case, however, it differs from the work of Browne and Gauldin, who invoke the customary broader equivalence relation (T/TI).

We represent generic and specific chord structures in interval normal form, as defined by Clough.[4] Thus, {C,E,G}, taken as a subset of any complete diatonic set containing those three pcs, has the generic structure (223) and the specific structure (435). The notation (223) gives the diatonic lengths of the intervals C–E, E–G, and G–C (reckoning upward); the notation (435) gives their chromatic lengths.

Rotated forms are not considered distinct, so that (223), (232) and (322) are equivalent representations, of which (223) is the normal form by virtue of the largest interval's position at the end of the string. The integers "10," "11," and "12" are preceded by blanks, as for example in (2 10), in order to distinguish "ten" from "one, zero," and so forth.

In Table 1, all non-empty, proper subsets of a single diatonic set (C major) are sorted into equivalence classes as outlined above. The 126 ($=2^7-2$) literal pc sets are listed in columns 3 and 4; column 3 contains the sets of cardinalities 1, 2, and 3; column 4 contains the sets of cardinalities 4, 5, and 6. (Here, and henceforth in this paper, we denote unordered sets of pcs by juxtaposition as in "CDE" instead of by curly brackets.) The pc sets in column 3 are grouped according to genus (column 1) and species (column 2). The pc sets of column 4 are similarly grouped, with genus and species shown in columns 6 and 5 respectively. Altogether, there are eighteen ($=126/7$) genera and sixty-three species.[5] Braces in columns 3 and 4 enclose chords of the same species. Where no brace appears, a species is represented by just one chord. In most cases, diatonic complements are horizontally aligned in columns 3 and 4; however, such alignment cannot be maintained consistently since in many cases chords of the same species do not have complements of the same species.

Partitioning. It is evident from Table 1 that the partitioning of the universe of C-major subsets into species is a refinement of the partition-

Table 1. Partitioning of Diatonic Chords.

1. genus	2. species	3. literal chord	4. literal chord	5. species	6. genus
		B			
		E	FGABCD	(222123)	
		A	BCDEFG	(122124)	
(7)	(12)	D	EFGABC	(122214)	
		G	ABCDEF	(212214)	(111112)
		C	DEFGAB	(212223)	
		F	GABCDE	(221223)	
			CDEFGA		
	(1 11)	BC			
		EF	GABCD	(22125)	
		AB	CDEFG		
(16)		DE	FGABC	(22215)	(11113)
	(2 10)	GA	BCDEF	(12216)	
		CD	EFGAB	(12225)	
		FG	ABCDE	(21225)	
			DEFGA		
		BD			
		EG	ABCDF	(21234)	
	(39)	AC	BDEFG	(32124)	
(25)		DF	EGABC	(32214)	
		GB	ACDEF		(11122)
	(48)	CE	FGABD	(22233)	
		FA	BCDEG	(12234)	
			EFGAC		
		BE			
		EA	BCDFG	(12324)	
		AD	EFGBC	(12414)	
(34)	(57)	DG	ABCEF	(21414)	
		GC	DEFAB	(21423)	(11212)
		CF	GABDE		
	(66)	FB	CDEGA	(22323)	
			FGACD		

Table 1 (continued)

1. genus	2. species	3. literal chord	4. literal chord	5. species	6. genus
(115)	(129) (219) (228)	BCD EFG ABC DEF GAB CDE FGA	ABCD DEFG GABC CDEF FGAB BCDE EFGA	(2127) (2217) (2226) (1227)	(1114)
(124)	(147) (237) (246)	BCE EFA ABD DEG GAC CDF FGB	GBCD CEFG FABC BDEF EGAB ACDE DFGA	(4125) (4215) (3216) (3225)	(2113)
(214)	(327) (417) (426)	BDE EGA ACD DFG GBC CEF FAB	BCDF EFGB ABCE DEFA GABD CDEG FGAC	(1236) (1245) (2145) (2235)	(1123)
(133)	(156) (165) (255)	BCF EFB ABE DEA GAD CDG FGC	ABDE DEGA GACD CDFG FGBC BCEF EFAB	(2325) (2415) (1416) (1425)	(1213)
(223)	(336) (345) (435)	BDF EGB ACE DFA GBD CEG FAC	BDEG EGAC ACDF BDFG BCEG EFAC ABDF	(3234) (3324) (1434) (2334)	(1222)

ing of that universe into genera; each species belongs wholly to a single genus. This phenomenon also shows itself when equivalence holds under T/TI, though it is masked when chords are rostered by consecutive Forte labels, as in Browne.[6] That each chord species belongs wholly to a particular chord genus may seem unsurprising. However, as we shall observe in Part II, such partitioning does not obtain in many systems with characteristics otherwise similar to those of the traditional diatonic-in-chromatic system.

Cardinality equals variety (CV). In the traditional diatonic scale, each numerical interval (second, third, and so forth) appears in two sizes; the scale includes three kinds of triads (excluding the augmented triad); and the diatonic tetrachord (four-note scale fragment) has exactly four species, namely those represented by CDEF, DEFG, EFGA, and FGAB. Thus, two-note chords come in two species, and certain familiar three- and four-note chords come in three and four species, respectively. It turns out that k-note chords come in k species for *all* diatonic chords of 1-6 notes! This is easily verified from Table 1, and it holds here *only* when equivalence is restricted to T alone, not under T/TI equivalence.

Structure implies multiplicity (SM). In reckoning the interval structure of a pc set, we normally make the obvious assignment of unit value to the interval formed by two adjacent pcs. This assumption underlies the sorting procedure that produced Table 1. However, we can sort pc sets into identically the same classes, specifically and generically, by taking the fifth (or fourth) as "unit" interval. By this measure, the specific structure of CEG is (138): C to G = 1 fifth, G to E = 3 fifths, E to C = 8 fifths. (By convention, we count in "upward" fifths, but the choice of direction does not alter the classification.) By fifths-measure, the generic structure of CEG is (133) (here E to C = 3 diatonic fifths), and diatonic triads of all three species have the same generic structure.[7]

Table 2 arrays the data of Table 1 differently, but still displays the partitioning property, which now appears in the relationship between columns 1 and 3. Literal pc sets are not shown, but the number of sets contained in each species is given in column 4. New, with respect to Table 1, are the generic structures reckoned in fifths as described above; these are displayed in column 2. In accord with the above claim regarding identity of classification under different sorting procedures, there is a one-to-one correspondence between columns 1 and 2.

Consulting Table 2, we find that the structure in fifths of each genus (column 2) gives a list of numbers identical to the multiplicities of the various species included in that genus, suitably ordered (column 4). Thus "structure implies multiplicity." How does this remarkable situation come about? We defer the answer to Part II.

Table 2. Structure and Multiplicity of Diatonic Chords.

1. genus 2nds	2. genus 5ths	3. species (semitones)	4. no. of sets
(7)	(7)	(12)	7
(16)	(25)	(1 11) (2 10)	2 5
(25)	(34)	(48) (39)	3 4
(34)	(16)	(66) (57)	1 6
(115)	(223)	(129) (219) (228)	2 2 3
(124)	(124)	(246) (147) (237)	1 2 4
(214)	(214)	(417) (426) (327)	2 1 4
(133)	(115)	(156) (165) (255)	1 1 5
(223)	(133)	(336) (345) (435)	1 3 3

Table 2 (continued)

1. genus 2nds	2. 5ths	3. species (semitones)	4. no. of sets
(1114)	(1222)	(2226)	1
		(1227)	2
		(2127)	2
		(2217)	2
(2113)	(2113)	(4125)	2
		(4215)	1
		(3216)	1
		(3225)	3
(1123)	(1123)	(1236)	1
		(1245)	1
		(2145)	2
		(2235)	3
(1213)	(1114)	(2415)	1
		(1416)	1
		(1425)	1
		(2325)	4
(1222)	(1213)	(3324)	1
		(1434)	2
		(2334)	1
		(3234)	3

Table 2 (continued)

1. genus 2nds	2. genus 5ths	3. species (semitones)	4. no. of sets
(11113)	(11122)	(22215)	1
		(12216)	1
		(12225)	1
		(21225)	2
		(22125)	2
(11122)	(11212)	(21234)	1
		(32124)	1
		(32214)	2
		(22233)	1
		(12234)	2
(11212)	(11113)	(12324)	1
		(12414)	1
		(21414)	1
		(21423)	1
		(22323)	3
(111112)	(111112)	(222123)	1
		(122124)	1
		(122214)	1
		(212214)	1
		(212223)	1
		(221223)	2

Diatonic lines. We now consider diatonic lines of pcs from the perspective of the chord relationships established above. By definition, a line is ordered and may contain repetitions. Therefore, from any given pc set we can generate an infinite number of lines. Conversely, we shall say that a line *collapses* into a pc set if the line contains all of and only the pcs of the set. Thus, each of the following lines collapses into CDE:

C - D - E

D - C - E

E - E - C - E - D - D - C

By definition, diatonic transposition preserves the diatonic lengths of intervals. Therefore, the seven diatonic transpositions of a line will all have the same series of generic directed intervals. For example, the line G-B-C-G-G has the series 2-1-4-6, and so do all of its diatonic transpositions, as shown in Table 3.

What of the pattern of *specific* directed intervals? Intuitively, it seems that the species of lines should correspond to the species of sets into which they collapse, and this is indeed the case, as illustrated by the middle two columns of Table 3.

Thus, CV applies to diatonic lines in the sense that the cardinality of a line (= cardinality of the corresponding chord) gives the number of linear species which appear under diatonic transposition of that line. SM also holds; for a given line, the multiplicity of the various linear species which appear under diatonic transposition is given by the generic structure of the corresponding chord, reckoned in fifths. (For the case illustrated in Table 3, that structure is (1114).)

The partitioning property observed for chords, with one exception, also extends to lines. Given a series of specific directed intervals, we can immediately write down the corresponding series of generic directed intervals. The exception is the following: A line comprised entirely of intervals of specific length 6 (or 6 and zero) is generically ambiguous. This is, of course, the familiar and important case of the isolated tritone. The specific line 6-6, for example, corresponds to both generic 3-4 and 4-3. The ambiguity is latent even in the correspondence of chords. We know that the specific dyad (66) corresponds to generic (34); however, we do not know "which" of the 6's corresponds to "3" and which to "4."

In contrast to the case of pc *sets,* CV and SM hold when cardinality = 7 for pc *lines,* as exemplified by the seven different diatonic modes.

II

The first two sections below provide an explanation of CV and SM as exhibited in Part I for the "seven in twelve" case. The method develops

Table 3. Lines Based on the Series 2-1-4-6.

INDIVIDUAL		SPECIES		GENUS	
pc line	pc set	line	chord	line	chord
G - B - C - G - F	{F,G,B,C}	4-1-7-10	(2415)		
C - E - F - C - B	{B,C,E,F}	4-1-7-11	(1416)		
F - A - B - F - E	{E,F,A,B}	4-2-6-11	(1425)		
B - D - E - B - A	{A,B,D,E}			2-1-4-6	(1213)
E - G - A - E - D	{D,E,G,A}				
A - C - D - A - G	{G,A,C,D}	3-2-7-10	(2325)		
D - F - G - D - C	{C,D,F,G}				

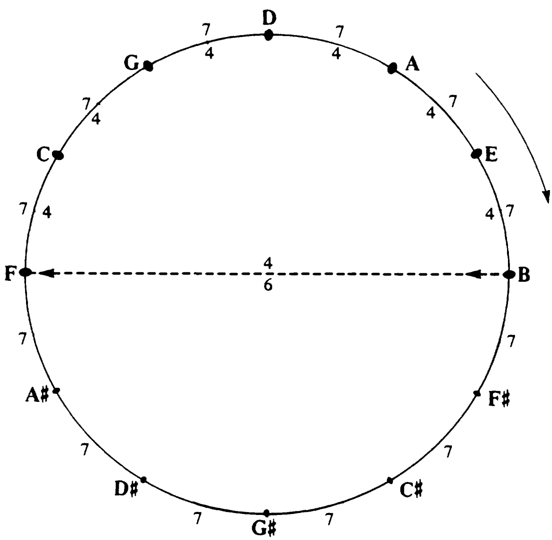


Figure 1. Circle of 5ths

a series of results for lines of pcs containing no repetitions (permutations of chords), first considering the usual diatonic scale and later generalizing to scales in chromatic universes of arbitrary size. These results are then extended to lines with repetitions and, under certain conditions, to chords. The remaining sections of the paper explore the uniqueness and construction of scales and finally chord mapping.

Diatonic lines and the circle of 5ths. To avoid proliferation of terminology, we will first use the term *line* in the more restrictive sense (no repetitions) and then broaden it later. This should not be confusing since the results apply to both cases.

A *line-class* is a set of seven lines related by diatonic transposition. For example, the line-class containing C - D - F, which we denote ⟨C - D - F⟩, is {C - D - F, D - E - G, E - F - A, F - G - B, G - A - C, A - B - D, B - C - E}. Thus, line-class is to line as genus is to chord. On examining the chromatic lengths of intervals, we see that ⟨C - D - F⟩ partitions into three subsets which we call species, after the terminology for chords: {C - D - F, D - E - G, G - A - C, A - B - D}, {E - F - A, B - C - E} and {F - G - B}. The lines in the first species all have intervals of chromatic lengths 2, 3; those of the second species, 1, 4; the lone line of the third species, 2, 4.

We see that the line-class containing the *three*-note line C - D - F partitions into *three* species. This is not a coincidence, as the following theorem shows.

> THEOREM 1. *Cardinality equals variety (CV). For the usual diatonic scale, given any k, $1 \leqslant k \leqslant 7$, and any k-note line, the line-class containing that line embraces exactly k species.*

An unimaginative proof can be carried out by examining all 13,699 lines, grouping them in the 1,957 (= 13,699/7) line classes, and counting the species within each line-class (on the level of chords, we can reduce these numbers to 126 and 18, respectively, as shown in Part I, but we must exclude $k=7$). A more elegant proof, which leads to significant generalizations, is based on the circle of fifths.

In Figure 1, the labels inside the upper semicircle are diatonic lengths; the other labels are chromatic lengths. Measurement is clockwise. Consider again the line C - D - F. The other lines in ⟨C - D - F⟩ are obtained by cycling around the upper semicircle, since the diatonic distances on that semicircle are constant. (See Figure 2.) The three species in this line-class arise from the three possible locations of the "short" fifth, B - F; this interval may appear within the second interval of the three-note line (as in Figure 2, a through d), or the first interval of the three-note line (Figure 2, e and f), or neither (Figure 2g).

More generally, for any k, $1 \leqslant k \leqslant 7$, and any k-note line, we can

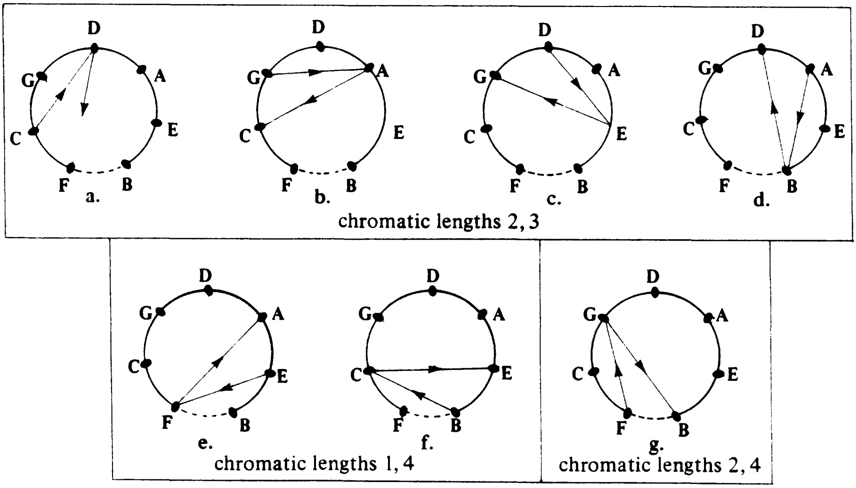


Figure 2. The line-class <C–D–F> and its three species

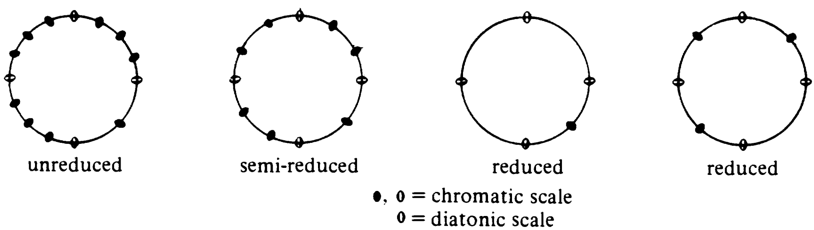


Figure 3. Reduction of a scale

obtain the other lines in the line-class by cycling through, and we can distinguish the species by the position of B - F, for which there are k possibilities. This proves Theorem 1.

COROLLARY 1. *Structure yields multiplicity (SM). For the usual diatonic scale, within any particular line-class, the multiplicities of the various species correspond to the diatonic distances between consecutive elements, measured in fifths.*

Proof. From Figure 2 we see that if the short fifth is included in the second interval of the three-note line, there are four possible lines because the diatonic length of the second interval is equivalent to four fifths. If the short fifth is included in the first interval of the line, there are two possible lines because the diatonic length of that interval is equivalent to two fifths. Finally, if the short fifth is included in neither of the two intervals, there is one possible line because the interval of closure from the last note of the line to the first note has a diatonic length of one fifth.

More generally, for any k-note line, each of the k possible locations for the short fifth yields a number of species equal to its length, measured in fifths. Hence SM holds for the usual diatonic scale.

Myhill's property and its consequences. We now generalize. Instead of the usual set of twelve chromatic notes, we consider an abstract (finite) chromatic set of c notes. We select a subset to be the diatonic set; let its cardinality be d. We label the diatonic notes $D_0, D_1, \ldots, D_{d-1}$. It is clear how we define chord, line, interval, diatonic and chromatic lengths, genus, line-class, and species. A scale is said to have property CV (Cardinality = Variety) if for every k with $1 \leqslant k \leqslant d$, and for any k-note line, the line-class containing that line embraces exactly k species. This is the property of the usual diatonic scale asserted by Theorem 1. We wish to determine conditions under which a scale has property CV.

We define the *spectrum* of an interval I to be the set of all chromatic lengths of intervals in $\langle I \rangle$. If a scale has CV, then every interval in particular has a two-element spectrum. John Myhill conjectured to us that the converse is true.[8] We shall say that a scale has MP (Myhill's property) if every interval has a two-element spectrum. We shall prove that MP implies CV by constructing for any scale with MP a "generalized circle of fifths".

First we introduce a convenient simplification. If a scale has MP, then in particular the spectrum of the interval $D_0 - D_1$ is a two-element set. If these two elements are consecutive integers, then we say the scale is *semireduced.* If the spectrum of $D_0 - D_1$ is $\{1,2\}$—that is, if between each pair of consecutive diatonic notes there is either exactly one chromatic note or none at all—then we say the scale is *reduced*. Clearly, any reduced scale is semireduced. Given any non-semireduced scale with MP, there is at least one corresponding semireduced (indeed reduced) scale obtained by deleting non-diatonic notes in the obvious way(s), as illustrated in Figure 3. A semireduced scale so obtained still has MP and as is easily verified, has CV if and only if the original scale does. Note that *reduced* and *semireduced* are defined only for scales with MP. From here on, we assume that all scales are semireduced.

We say that a scale has CP (Consecutivity Property) if each interval has a spectrum consisting of consecutive integers.

LEMMA 1. *Every semireduced scale has CP.*

Proof. Choose h, $0<h<d$, and consider the interval $D_0 - D_h$. If the spectrum of this interval contains only one integer, the lemma is trivially true. Otherwise, there exist i and j with $j-i=h$ and $|D_i - D_j| \neq |D_{i+1} - D_{j+1}|$, where the absolute value sign denotes chromatic length, and the subscripts are to be reduced (mod d), if necessary, to bring them into the range $\{0, 1, \ldots, d-1\}$. But since the scale is semireduced, $|D_{i+1} - D_{j+1}| - |D_i - D_j| = |D_j - D_{j+1}| - |D_i - D_{i+1}| = \pm 1$. Thus no two consecutive terms in the sequence $|D_0 - D_h|, |D_1 - D_{h+1}|, \ldots, |D_{d-1} - D_{h-1}|$ differ by more than one, and the elements of the spectrum are consecutive.

LEMMA 2. *Given h, $0<h<d$, the sum of the chromatic lengths of the d intervals of diatonic length h is ch.*

Proof. The number of semitones in the chromatic scale is, by definition, c, and each of these semitones is contained in precisely h intervals of diatonic length h.

The following number-theoretic lemma is crucial to the construction of the generalized circle of fifths.

LEMMA 3. *Let $(c,d)=1$; then there exists a non-negative integer c', $c'<d$, such that $cc' \equiv -1 \pmod d$. (The notation $(c,d)=1$ means that c and d are coprime; their greatest common divisor is one.)*

We omit the proof; a more general theorem is proved in the early chapters of nearly every introductory number theory textbook.

LEMMA 4. *Let a scale have MP, with $(c,d)=1$. Let c' be as in the preceding lemma. With one exception, then, the intervals of diatonic length c' all have chromatic length $d'=(cc'+1)/d$; the exception has chromatic length $d'-1$. Note that for the usual diatonic scale, $c=12$, $d=7$, $c'=4$, $d'=7$, and the exceptional interval is the diminished fifth, B - F, with chromatic length $d'-1=6$.*

Proof. By Lemma 2, the sum of the chromatic lengths of the d intervals of diatonic length c' is cc'. By definition, $cc' = dd' - 1$, so we have d integers summing to $dd' - 1$. By MP there are exactly two distinct integers among these d integers, and by Lemma 1 they are consecutive; hence $d - 1$ of these integers are d', and the other is $d' - 1$.

We can now label the diatonic set in such a way that $|D_0 - D_{c'}| = |D_{c'} - D_{2c'}| = \ldots = |D_{(d-2)c'} - D_{(d-1)c'}| = d'$, $|D_{(d-1)c'} - D_{dc'}| = d' - 1$, the subscripts being read (mod d). Thus we have constructed a generalized circle of fifths, as shown in Figure 4.

LEMMA 5. *A scale with MP and $(c,d) = 1$ has CV.*

Proof. The argument from the circle of fifths given for Theorem 1 goes over to the generalized circle of fifths.

We now show that the hypothesis $(c,d) = 1$ is superfluous.

THEOREM 2. *MP implies CV.*

Proof. Clearly it suffices to show that MP implies $(c,d) = 1$. Suppose to the contrary a scale has MP and $(c,d) = r > 1$. Consider the line-class $\langle D_0 - D_{d/r} \rangle$. There are d intervals in this line-class of total chromatic length $\frac{d}{r}c$ (by Lemma 2). Thus we have d integers summing to $\frac{c}{r}d$, and $\frac{c}{r}$ is an integer. These d integers cannot all be $\frac{c}{r}$—if they were the spectrum of $D_0 - D_{d/r}$ would have only one element $\frac{c}{r}$, violating MP. Thus at least one of the integers exceeds $\frac{c}{r}$, and at least one falls short. To satisfy CP then, $\frac{c}{r}$ must be in the spectrum but then, the spectrum has at least three elements, violating MP.

COROLLARY 2. *MP implies SM.*

Proof. The argument from Corollary 1 applies to Lemma 5 and Theorem 2.

Since CV and SM always go hand in hand where MP obtains, we shall hereafter omit specific mention of SM.

If we now redefine *line* to be a finite list of elements of the diatonic set, repetitions permitted, and define the cardinality of a line to be the number of distinct elements in the list (so the cardinality of C - D - F - D - D is 3), then we see that if the scale admits a generalized circle of fifths, then CV holds, for the repetitions do not introduce any possible new locations for the short interval $d' - 1$. Therefore, a line of cardinality k gives rise to k species, and Theorem 2 holds for lines in general.

The situation for chords is a bit more delicate. In the usual diatonic

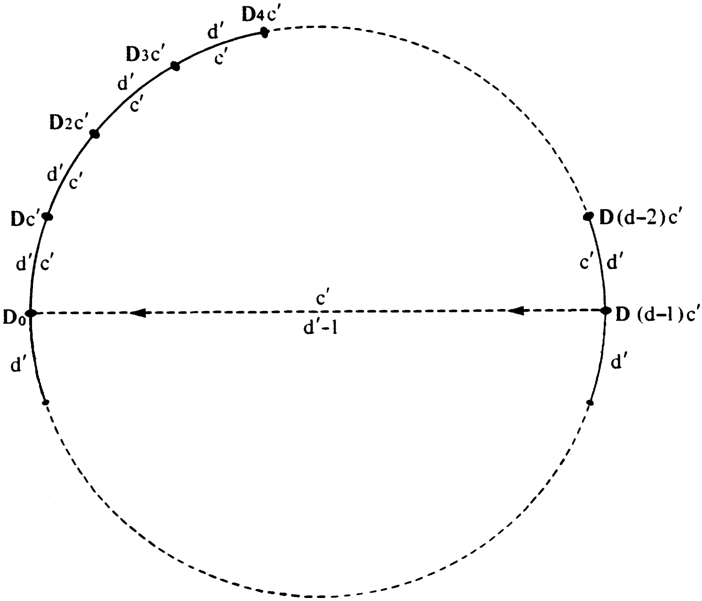
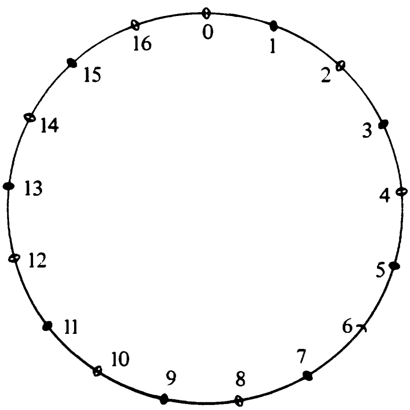


Figure 4. Generalized Circle of 5ths



d=9, c=17
Diatonic scale: 0, 2, 4, 6, 8, 10, 12, 14, 16.
Chords }0, 6, 12{, }2, 8, 14{ and }4, 10, 16{
all have the structure (566).

Figure 5. Case of CV for lines but not for chords

scale, CV holds except for $k=7$—there is only one seven-note chord. In general, problems arise when the cardinality of the chord is not coprime with d, the cardinality of the diatonic set. As an illustration, consider the scale with parameters $d=9$, $c=17$ (Figure 5). This scale has MP, so it admits a generalized circle of 5ths. However CV fails for the chord genus (333), which has cardinality 3 but only one species, (566). It seems that the best one can say is that, if a scale has MP, then CV holds except for certain chords of cardinality not coprime with d.

Uniqueness and construction of scales with CV and SM. Given parameters c, d with $(c,d)=1$, we show that there exists a scale with these parameters for which CV holds; moreover it is essentially unique. We consider uniqueness first.

THEOREM 3. *Let $S=\{C_0, C_1, \ldots, C_{c-1}\}$ and $S'=\{C'_0, C'_1, \ldots, C'_{c-1}\}$ be chromatic scales of c notes, each containing a diatonic set of d notes with CV. Then there exists an integer j such that C_i is in the diatonic set of S if and only if C'_{i+j} is in the diatonic set of S'; here the subscripts are to be interpreted (mod c).*

Proof. Since the scales share parameters c and d, they also share c′ and d′. By the construction of the generalized circle of fifths, there are notes C_k in S and C'_h in S′ such that the diatonic set of S is $\{C_k, C_{k+d'}, \ldots, C_{k+(d-1)d'}\}$ and the diatonic set of S′ is $\{C'_h, C'_{h+d}, \ldots, C'_{h+(d-1)d'}\}$. Now simply let $j=h-k$.

Concerning the existence of scales with CV and given parameters, we first show how to construct the usual diatonic scale. Here $c=12$ and $d=7$. Write down the multiples of $\frac{12}{7}$: 0, $1\frac{5}{7}$, $3\frac{3}{7}$, $5\frac{1}{7}$, $6\frac{6}{7}$, $8\frac{4}{7}$, $10\frac{2}{7}$, 12, ... Then erase the fractions, leaving 0, 1, 3, 5, 6, 8, 10, 12, ... Interpret this sequence as the positions of the diatonic notes and the omitted integers as the positions of the non-diatonic notes, and you obtain a scale with CV; if you identify position 0 with the note B, you recover the C major scale.

We will prove that this procedure works quite generally. First we need to quote another well-known lemma from elementary number theory.

LEMMA 6. *If r, s, t are integers, r divides st, and $(r,s)=1$, then r divides t.*

THEOREM 4. *Given c and d with $(c,d)=1$, let $a_k=\left[\frac{kc}{d}\right]$, $k=0, 1, 2, \ldots$. Then the integers a_k are the positions of the notes of the diatonic set in a scale with CV with parameters c and d.*

Remark. The square brackets are the standard mathematical notation for what was described above as "erasing the fractions"; thus, $\left[1\frac{5}{7}\right]=1$, $\left[3\frac{3}{7}\right]=3$, and so forth.

Proof. It suffices to show that the scale so constructed has MP and is semireduced, that is, that for every j, $1 \leqslant j < d$, the set $\{a_{k+j} - a_k : k = 0, \pm 1, \pm 2 \ldots\}$ is a set of two consecutive integers. Since, for all x, $x - 1 < [x] \leqslant x$, and since $a_{k+j} - a_k = \left[\frac{(k+j)c}{d}\right] - \left[\frac{kc}{d}\right]$, we have $\frac{(k+j)c}{d} - 1 - \frac{kc}{d} < a_{k+j} - a_k < \frac{(k+j)c}{d} - \frac{kc}{d} + 1$. Thus for fixed j there is an open interval (in the mathematical sense) of length 2 containing the spectrum of the interval (in the musical sense) of diatonic length j. Such a mathematical interval contains at most two integers; so the spectrum is at most a two-element set. Now suppose the spectrum is a one-element set. Then there is an integer, call it e, such that, for all k, $a_{k+j} - a_k = e$. Then $de = \sum_{n=1}^{d} (a_{k+nj} - a_{k+(n-1)j}) = a_{k+dj} - a_k = \left[\frac{(k+dj)c}{d}\right] - \left[\frac{kc}{d}\right] = cj$, so d divides cj. By Lemma 6, d divides j. But $1 \leqslant j < d$ by assumption, yielding a contradiction. Thus, the spectrum is a two-element set so the scale has MP; the two elements are evidently consecutive integers, so the scale is semireduced.

Partitioning. In addition to having the nice property CV, the usual diatonic scale has the partitioning property as shown early in Part I. It is easy to see that for many scales with CV (for chords as well as lines), chord species do not fit neatly within chord genera. For example, in the scale with $c = 6$, $d = 5$, chords of species (123) appear within both of the genera (113) and (122). Under what conditions does a scale with MP also exhibit the partitioning property? A partial answer is provided by the following theorem.

THEOREM 5. *For any reduced scale, if each chord species is unambiguous in its generic membership, then $c = 2d - 1$ or $c = 2d - 2$. The converse is also true.*

Proof. For any diatonic scale, the number of dyad genera is $\left[\frac{d}{2}\right]$. By definition, reduced scales have MP. By MP, if d is odd, each dyad genus includes two species; if d is even, there is a single exceptional dyad genus that includes only one species. By hypothesis, these species are all distinct; thus, the number of distinct dyad species in the diatonic scale is $2\left[\frac{d}{2}\right]$ if d is odd, $2\left[\frac{d}{2}\right] - 1$ if d is even. The number of distinct dyad species in the diatonic scale cannot exceed the number of chromatic dyads, which is $\left[\frac{c}{2}\right]$. So we have $2\left[\frac{d}{2}\right] \leqslant \left[\frac{c}{2}\right]$ if d is odd, $2\left[\frac{d}{2}\right] - 1 \leqslant \left[\frac{c}{2}\right]$ if d is even. For any reduced scale, $d > \frac{c}{2}$. The theorem follows from these inequalities. Note that $c = 2d - 2$, d even, is impossible here since all reduced scales have $(c,d) = 1$.

Conversely, for each c and d with $c = 2d - 1$ or $c = 2d - 2$, Theorem 3 states that there is an essentially unique reduced scale with those parameters, and Theorem 4 enables one to construct it; it is easy to check that generic membership is unambiguous in the family of scales so constructed.

Another question now arises, for with parameters c, d as above, we get a reduced scale which is also a deep scale as studied by Gamer. Thus all reduced scales with the partitioning property are deep scales. Is the converse also true? No, the deep scale 0, 1, 3, 4 in the chromatic scale of seven notes is one counterexample among many. The net result is that the familiar diatonic scale is one of a quite special class of scales that possess all three of the following attributes: the partitioning property, CV, and deepness. This combination of attributes may be the source of less well defined qualities–subtlety, power, adaptability, and indestructibility–with which we have always associated the traditional scale. Be this as it may, we believe that our theory provides a means of investigating certain features of musical structure as yet overlooked or poorly understood.

NOTES

1. Richmond Browne, "Tonal Implications of the Diatonic Set," *In Theory Only* 5/6-7 (1981): 3-21; Robert Gauldin, "The Cycle-7 Complex: Relations of Diatonic Set Theory to the Evolution of Ancient Tonal Systems," *Music Theory Spectrum* 5 (1983): 39-55.
2. John Clough, "Aspects of Diatonic Sets," *Journal of Music Theory* 23 (1979): 45-61, and "Diatonic Interval Sets and Transformational Structures," *Perspectives of New Music* 18/2 (1980): 461-482; Thomas Zimmerman, "A Theory of Diatonic Sets and its Application to Selected Works of Ludwig von Beethoven" (Ph.D. diss., University of Michigan, 1983).
3. Defined by Carlton Gamer in "Some Combinational Resources of Equal-Tempered Systems," *Journal of Music Theory* 11 (1967): 32-59.
4. Clough, "Aspects," pp. 46-47. A slightly different normal form was previously given by Eric Regener, "On Allen Forte's Theory of Chords," *Perspectives of New Music* 13/1 (1974): 191-212.
5. The number 63 is partially explained as follows. The 18 genera divide into 1, 3, 5, 5, 3 and 1 chords of cardinalities 1-6, respectively, the symmetry resulting from diatonic complementation. As demonstrated later, the multiplicity of a species equals its cardinality. Therefore we have $(1 \cdot 1)+(3 \cdot 2)+(5 \cdot 3)+(5 \cdot 4)+(3 \cdot 5)+(1 \cdot 6)=63$.
6. Browne, "Tonal Implications," pp. 16-19. For example, the mapping of trichords and tetrachords based on T/TI is as follows:

genus (Forte)	species	genus (Forte)
3-2	(115) (1114)	4-10
3-6		4-11
		4-21
3-4	(124) (1123)	4-13
3-7		4-14
3-8		4-22
		4-Z29
3-5	(133) (1213)	4-8
3-9		4-16
		4-23
3-10	(223) (1222)	4-20
3-11		4-26
		4-27

From the mapping based on T/TI, and the fact that, as long as we remain within the same universe, chords have unique complements, it follows that the same specific tetrachord may appear as the "complement" (with respect to the seven-pc diatonic set) of two different specific trichords if and only if the two trichords map to the same generic trichord, and similarly for other complementary cardinalities. Thus, in Browne's statement that "some tetrachords act as complements to strikingly different trichords" (p. 15), "strikingly" applies to differences within the same generic class only, as for example to the qualitative differences among the three triads. Given the mapping based on T/TI and the fact that the number of different chords increases with cardinality (through cardinality 5), such cases are inevitable and, as Browne suggests, interesting.

Further on the matter of complementation, Browne's statement that "repeated subgroups . . . *always* possess different . . . complements" (p. 14) is contradicted by the following cases: F and B, BE and FC, ABE and FGC (all with respect to C major). However, the equivalence of chords under T alone holds no advantage here, as exceptions to Browne's rule would be evident in Table 1 also.

7. In both the generic and specific cases, the identity of classification is due to one-to-one reflexive mapping of the set of integers {0, 1, 2, . . . , n–1} under multiplication by m modulo n, with m and n coprime. In the generic case, 4 (the fifth) is prime to 7; in the specific case, 7 (the perfect fifth) is prime to 12.
8. We are indebted to our colleague, John Myhill, for bringing us together to work on the problems of this paper.