Pseudo-diatonic Scales

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Abstract. The generalization of diatonic scales in a given tone system has been investigated by Eytan Agmon (see Agmon 1989, Agmon 1991), John Clough (see Clough 1979, Clough and Myerson 1985), and in relation with microtonality by Gerald Balzano and Mark Gould (see Gould 2000). Recently, Thomas Noll (2006) gave a new synthetic approach of pseudo-diatonic scales (Model $\mathcal{A}$). From our first essay (Jedrzejewski 2002) until the last article (Jedrzejewski 2008), we developed a new model of generalized diatonic scales based on a new arrangement of the Stern-Brocot tree (Model $\mathcal{B}$). With our new definition of diatonicism, we recover Wyschnegradsky' diatonic scales in the quarter tone universe, a concept that he called *diatonicized chromatism* (Wyschnegradsky 1979), studied in (Jedrzejewski 1996) and (Jedrzejewski 2003). In the present article, we point out the differences of the two models ($\mathcal{A}$ and $\mathcal{B}$).

1 Shuffled Stern-Brocot Tree

The Stern-Brocot tree is a binary tree (see Brocot 1862, Stern 1858) of positve rational numbers obtained by inserting the mediant of two adjacents fractions. The mediant of two adjacent fractions p_1/q_1 and p_2/q_2 is defined by (see (Farey 1816)) the Farey sum

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2}$$

Each node of the tree is associated with a word ω of the free monoid $\mathcal{L} = \{S, T\}^*$ of words on the alphabet of two letters S and T. The word ω describes the path in the Stern Brocot to reach the node from the root of the tree: S means a left step and T means a right step. In this way, every positive fraction is encoded as a finite sequence of symbols S and T. At each letter is also associated the two functions $T(x) = x+1$ and $S(x) = x/(x+1)$. At each node, a function is defined by substituting in the word ω of this node the concatenation of letters by the composition of functions. The fraction of this node is the value of the function at $x = 1$. For example, the word $\omega = TS$ means that, starting from the top of the tree at the value 1, we go a step on the right and then a step downwards on the left. The associated function is $TS(x) = T(x/(x + 1)) = (2x + 1)/(x + 1)$ and the value of this function for $x = 1$ is the fraction $3/2$.

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The Shuffled Stern-Brocot tree is a different arrangement of the same binary tree. It starts at $x = 1$, but uses the functions $L(x) = T(x) = x + 1$ for a left downwards step and $R(x) = 1/(x + 1)$ for a right downwards step. Each node is associated with a word of the free monoid $\{L, R\}^*$ of words on the alphabet of two letters L and R. The fraction of the node is the value of the function $\omega(x)$ at $x = 1$.

In the Stern-Brocot tree, a *chromatic number* (a notion introduced in Noll 2006) is a positive rational number $x \neq 1, 1/2, 2/1$, whose associated word ω ends by two different letters (ST or TS). Equivalently, in the shuffled Stern-Brocot tree, a rational number is a chromatic number if and only if its word ω ends by R (see (Jedrzejewski 2008))

2 Construction of Pseudo-diatonic Scales

In model $\mathcal{A}$, a pseudo-diatonic scale is the union of two subsystems: the Agmon subsystem and the Balzano subsystem linked by *concordance.* For each chromatic number $r/(2n + 1)$ with odd denominator, we construct a cycle of pseudo-thirds generated by $r/(2n+1)$ mod 1 (or equivalently by r mod $2n+1$) and a cycle of pseudo-fifths generated by $2r/(2n + 1)$ mod 1 (or equivalently by $2r$ mod $2n + 1$). In this model, a generic pseudo-fifth equals two pseudo-thirds. The generic pseudo-fifth has two successors along the Stern-Brocot tree. The chromatic successor m/N determines the N-chromatic universe into which the Balzano subsystem is embedded as a specific scale.

In model $\mathcal{B}$, a large ME* scale A is a non-degenerated well-formed scale of generator k whose complement A^c has no adjacent pitch classes (k is a non-negative integer). A large ME* scale A ($k \geq 2$) is *tightly generated* if there is no non-degenerated well-formed k-generated scale W between A^c and A ($A^c \subset W \subset A$). In the N-tone equal temperament ($N \geq 12$), a *generalized diatonic scale* (or simply a diatonic scale) is k-generated large ME* with generator $k > 2$ minimizing the difference of cardinality $\#A - \#A^c$ (condition of minimality). If there is no such scale ($N = 15$ is the only known example so far), the trivial 2-generated scale might be chosen as the generalized diatonic scale. So for each N, there is always one generalized diatonic scale.

In order to compare the pseudo-diatonic scales of model $\mathcal{A}$ with the generalized diatonic scales of model $\mathcal{B}$, we implement an algorithm for searching the scales. For each chromatic number m/N of the shuffled Stern-Brocot tree, we look at its predecessor m_1/N_1 and construct the generic cycle of fifths as part of the (generalized) Balzano subsystem

$$U = \{m_1 k \mod N_1,\ k = 0, 1, ..., N_1 - 1\}$$

The rational number $m_2/N_2 = (m - m_1)/(N - N_1)$ is the Farey complement of m_1/N_1 satisfying

$$\frac{m}{N} = \frac{m_1}{N_1} \oplus \frac{m_2}{N_2}$$

With m_2/N_2, we complete the (generalized) Balzano subsystem in terms of the cycle:

$$V = \{m_2 k \mod N_2,\ k = 0, 1, ..., N_2 - 1\}$$

The permutation $\sigma(x) = x \mod N$ induces an embedding of U and V in the chromatic universe $\mathbb{Z}_N$. This algorithm produces the two types of scales A and B. The N_1 white keys are the set $\sigma(U)$ and the N_2 black keys are a transposition of $\sigma(V)$. In model $\mathcal{A}$, the generator $m_1 = 2r$ is even, and the number of white keys $N_1 = 2n+1$ is always odd. Moreover, the rational number $m_1/2N_1 = r/N_1$ must be a chromatic number. In model $\mathcal{B}$, the generalized diatonic scale is determined by the condition of minimality. In both models, the chromatic number m/N and the number $(N - m)/N$ leads to the same scales.

3 Comparison of the Two Models

Doubles of chromatic numbers are not chromatic. In model $\mathcal{A}$, since the chromatic number $m_1/2N_1$ ends by R, the word of the generic pseudo-fifth m_1/N_1 ends by L. If ω is the word of m_2/N_2 (the black keys), the chromatic number m/N of the N-chromatic universe ends by LR. In model $\mathcal{B}$, for $12 \leq N \leq 42$, the generalized diatonic scales have the same associated words ωLR if $N \neq 13, 14, 15, 18, 21$. For these values ($N = 13, 14, 15, 18, 21$), there are no pseudo-diatonic scales in model $\mathcal{A}$. This suggests to compare the two models only in this case ($N \neq 13, 14, 15, 18, 21$). A computer program shows the following results (see also the following table).

• The pseudo-diatonic scales of model $\mathcal{A}$ are not uniquely determined by the chromatic cardinality N since — for example — there are two pseudo-diatonic scales for $N = 42$. In model $\mathcal{B}$, for each N, there is by definition only one generalized diatonic scale.
• The pseudo-diatonic scale of model $\mathcal{A}$ do not always exist for each N (in contrast to model $\mathcal{B}$), even though m_1 is even. In the quarter tone universe ($N = 24$), there is no pseudo-diatonic scale, because the number $m_1/2N_1 = 3/13$ is not a chromatic number. In model $\mathcal{B}$, the pseudo-fifth has 13 quarter tones. It is composed of two different types of pseudo-thirds. In model $\mathcal{A}$, the two pseudo-thirds are always of the same size, determined by the number $r/N_1 = m_1/2N_1$.
• In model $\mathcal{A}$, the number N_1 of white keys is always odd and the generator m_1 is always even. In model $\mathcal{B}$, the number of notes N_1 of a generalized diatonic scale is sometimes odd and sometimes even. Moreover, contrary to model $\mathcal{A}$, the number N_1 can be even and the generator m_1 odd ($N =$ 17, 25, 27, 29, 37, 39, 41) or the number N_1 is odd and the generator is also odd ($N = 36$).
• If m_1 is even, the scales of the two models are different in two cases; (1) if the number $m_1/2N_1$ is not a chromatic number ($N =$ 24, 28, 32, 35, 40) and (2) if the pseudo-diatonic scale does not satisfy the condition of minimality. In this case, m_1 is even in model $\mathcal{A}$ and odd in model $\mathcal{B}$ ($N =$ 29, 37, 39, 41).
• If, for both models, the pseudo-fifths or their complements (v) are evaluated in cents $1200 \log_2(v)$ it turns out, that the range of the pseudo-fifth in model

$\mathcal{A}$ varies between 660 and 1057 cents and in model $\mathcal{B}$ it varies between 630 and 978 cents. This shows that the pseudo-fifth must be understand as a new interval of the N-chromatic universe, and should not literally be compared with the tempered fifth (700 cents).

N	Model	$m/N = m_1/N_1 \oplus m_2/N_2$	ω	Pseudo-Fifths	Pseudo-Third
12	$\mathcal{A}, \mathcal{B}$	$7/12 = 4/7 \oplus 3/5$	R^3	700	$2/7 = RL^2R$
16	$\mathcal{A}, \mathcal{B}$	$7/16 = 4/9 \oplus 3/7$	$RLRL$	675	$2/9 = RL^3R$
17	$\mathcal{B}$	$5/17 = 3/10 \oplus 2/7$	RL^2R	847	–
19	$\mathcal{A}, \mathcal{B}$	$7/19 = 4/11 \oplus 3/8$	RLR^2	758	$2/11 = RL^4R$
20	$\mathcal{A}, \mathcal{B}$	$11/20 = 6/11 \oplus 5/9$	R^3L^2	660	$3/11 = RL^2R^2$
22	$\mathcal{A}, \mathcal{B}$	$17/22 = 10/13 \oplus 7/9$	R^2L^2R	927	$5/13 = RLR^3$
23	$\mathcal{A}, \mathcal{B}$	$7/23 = 4/13 \oplus 3/10$	RL^2RL	835	$2/13 = RL^5R$
24	$\mathcal{B}$	$11/24 = 6/13 \oplus 5/11$	$RLRL^3$	650	$(3/13 = RL^3RL)$
25	$\mathcal{B}$	$9/25 = 5/14 \oplus 4/11$	RLR^2L	768	–
26	$\mathcal{A}, \mathcal{B}$	$7/26 = 4/15 \oplus 3/11$	RL^2R^2	877	$2/15 = RL^6R$
27	$\mathcal{B}$	$5/27 = 3/16 \oplus 2/11$	RL^4R	978	–
28	$\mathcal{B}$	$15/28 = 8/15 \oplus 7/13$	R^3L^4	643	$(4/15 = RL^2R^2L)$
29	$\mathcal{A}$	$17/29 = 10/17 \oplus 7/12$	R^3LR	703	$5/17 = RL^2RLR$
29	$\mathcal{B}$	$9/29 = 5/16 \oplus 4/13$	RL^2RL^2	828	–
30	$\mathcal{A}, \mathcal{B}$	$7/30 = 4/17 \oplus 3/13$	RL^3RL	920	$2/17 = RL^7R$
31	$\mathcal{A}, \mathcal{B}$	$11/31 = 6/17 \oplus 5/14$	RLR^2L^2	774	$3/17 = RL^4R^2$
32	$\mathcal{A}$	$27/32 = 16/19 \oplus 11/13$	R^2L^4R	1013	$8/19 = RLRLR^2$
32	$\mathcal{B}$	$15/32 = 8/17 \oplus 7/15$	$RLRL^5$	638	$(4/17 = RL^3RL^2)$
33	$\mathcal{A}, \mathcal{B}$	$7/33 = 4/19 \oplus 3/14$	RL^3R^2	945	$2/19 = RL^8R$
34	$\mathcal{A}, \mathcal{B}$	$25/34 = 14/19 \oplus 11/15$	R^2LR^2L	882	$7/19 = RLR^2LR$
35	$\mathcal{B}$	$11/35 = 6/19 \oplus 5/16$	RL^2RL^3	823	$(3/19 = RL^5RL)$
36	$\mathcal{B}$	$17/36 = 9/19 \oplus 8/17$	$RLRL^6$	633	–
37	$\mathcal{A}$	$7/37 = 4/21 \oplus 3/16$	RL^4RL	973	$2/21 = RL^9R$
37	$\mathcal{B}$	$13/37 = 7/20 \oplus 6/17$	RLR^2L^3	778	–
38	$\mathcal{A}, \mathcal{B}$	$29/38 = 16/21 \oplus 13/17$	$R^2L^2RL^2$	916	$8/21 = RLR^4$
39	$\mathcal{A}$	$17/39 = 10/23 \oplus 7/16$	$RLRL^2R$	677	$5/23 = RL^3R^3$
39	$\mathcal{B}$	$16/39 = 9/22 \oplus 7/17$	$RLRLRL$	708	–
40	$\mathcal{A}$	$7/40 = 4/23 \oplus 3/17$	RL^4R^2	990	$2/23 = RL^{10}R$
40	$\mathcal{B}$	$19/40 = 10/21 \oplus 9/19$	$RLRL^7$	630	$(5/21 = RL^3RL^3)$
41	$\mathcal{A}$	$25/41 = 14/23 \oplus 11/18$	R^5L	732	$7/23 = RL^2RL^2R$
41	$\mathcal{B}$	$13/41 = 7/22 \oplus 6/19$	RL^2RL^4	820	–
42	$\mathcal{A}, \mathcal{B}$	$11/42 = 6/23 \oplus 5/19$	$RL^2R^2L^2$	886	$3/23 = RL^6R^2$
42	$\mathcal{A}$	$37/42 = 22/25 \oplus 15/17$	R^2L^6R	1057	$11/25 = RLRL^2R^2$

The table gives for each N-chromatic universe (first column) the Farey relation (third column) needed to construct the pseudo-diatonic scale of model $\mathcal{A}$ or the generalized diatonic scale of model $\mathcal{B}$ (given in the second column). The last column provides the chromatic number $(m_1/2N_1)$ for the pseudo-third of model $\mathcal{A}$. Some numbers $(m_1/2N_1)$ in parenthesis are not chromatic numbers. The word ω associated with the generic pseudo-fifth is given in the fourth column. The value in cents of the pseudo-fifth is in column 5.

Some scales, available in both models, have a chromatic number $2/(2n+3)$ associated with a word of the form RL^nR, $n \geq 2$. In this case, it is easy to show that the pseudo-diatonic and the diatonic scale of models $\mathcal{A}$ and $\mathcal{B}$ are the same. If $n = 2k$, $(k = 1, 2, ...)$ is even, in the $(7k+5)$-chromatic universe, this scale has $(4k+3)$ white keys and $(3k+2)$ black keys,

$$\frac{7}{7k+5} = \frac{4}{4k+3} \oplus \frac{3}{3k+2}$$

and if $n = 2k+1$ is odd, the scale has, in the $(7k+9)$-chromatic universe, $(4k+5)$ white keys and $(3k+4)$ black keys:

$$\frac{7}{7k+9} = \frac{4}{4k+5} \oplus \frac{3}{3k+4}$$

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