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Author(s): John Clough
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Diatonic Interval Sets and Transformational Structures

JOHN CLOUGH

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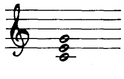

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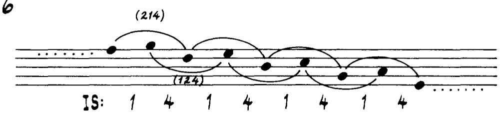

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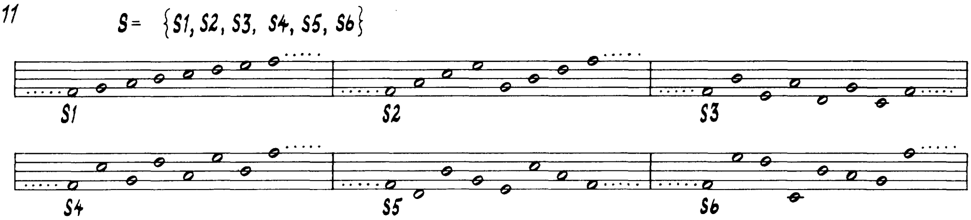

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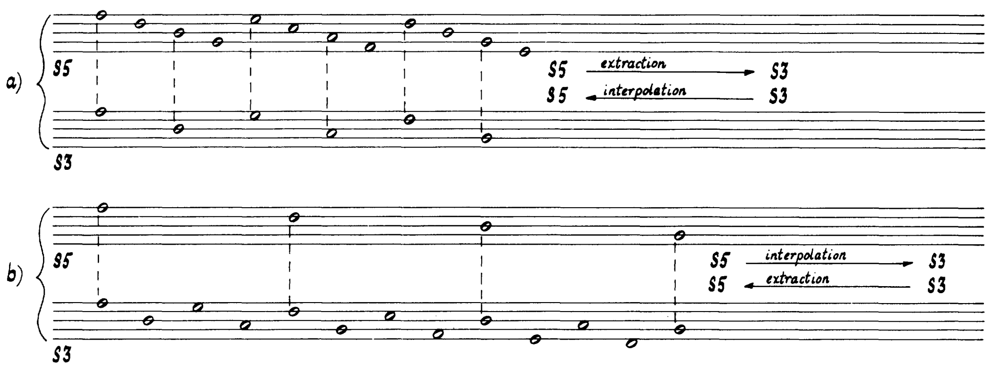

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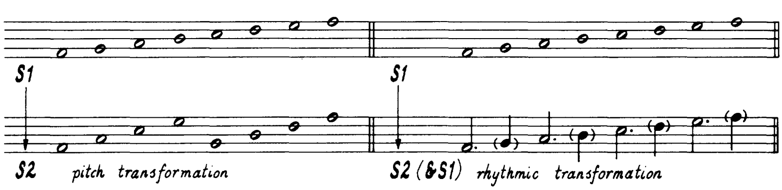
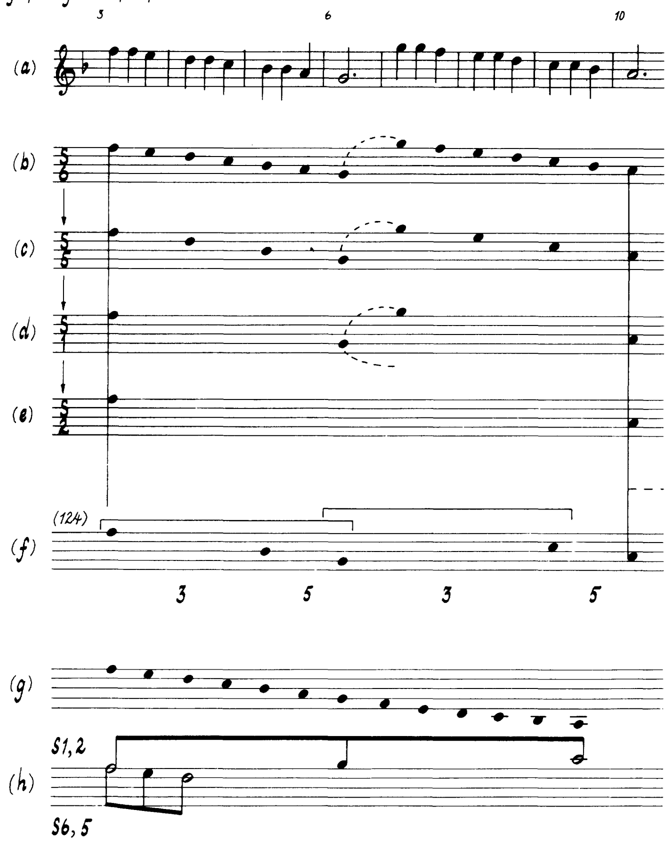


Ex. 16 Beethoven: Symphony No. 7, III, mm. 3-18

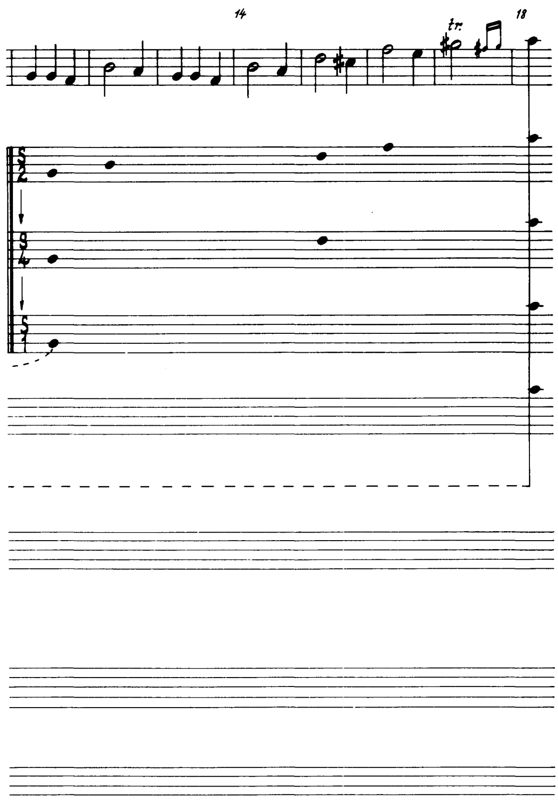


I. Introduction

In an earlier, related paper,[1] I posited a diatonic universe of seven pitch classes in which all intervals of the same general name are regarded as equivalent. Feeling that the results would speak for themselves, I neglected to present a prior rationale for the equivalence relation. As the scope of this work enlarges, however, the need for such a rationale becomes clear, and I begin with that here. Following definitions of technical terms, the major parts of the paper develop a series of abstractions, focusing upon the serial attributes of the diatonic system, and upon the hierarchical properties of diatonic sets generated by a single interval. Finally, I offer a brief example of musical analysis based on the hierarchical model.

II. Intervallic Equivalence

Given an incomplete notation of a piece or excerpt, from which



certain details have been excised, the musically literate reader can, despite the incompleteness, study some aspects of structure, extract musical meaning from the notation, and experience something "musical" though inchoate.

Consider Example 1. The absence of clef and key signature permits five posssible realizations, disregarding transpositions and assuming a suppressed traditional key signature (possibly no sharps or flats). Notwithstanding the ambiguity, we can discover important facts about the structure of the example, among them that the opening triad is confirmed by subsequent events, whose pitch and temporal characteristics both play a role in that confirmation.

Neither previous familiarity with the "actual" or complete form of an incomplete notation, nor precise durational information are necessary conditions of such a reading. The absence of these two factors in Example 2 does not prevent our hearing something of its structure. Beyond the triple groupings on the surface, our reading of Example 2 might center on the absence of triads on the surface, or the dual stepwise ascent, or other details, but in any case we are not at a loss to discover significant properties of the example in spite of the lack of precise intervallic information, and lack of a "piece" reference.

It is possible to invent presentations which suppress other aspects of the pitch information. We could, for example, omit the staff altogether and rely on approximate vertical location to indicate pitch, as in certain graphic scores. In the present study, however, we will be concerned with the effects of suppression, in diatonic contexts, of precisely that information which is conveyed by clef and key signature plus suppression of registral placement. Put more technically, this means that we regard diatonic intervals of the same general name (prime, 2nd, 3rd, etc.) and their corresponding compound intervals, as equivalent, and that we deal with just seven pitch classes instead of many more diatonic pitches of specific register.

It is as though the pitch information (or really the interval information) were given in the form of a succession of scale degrees where the designation scale degree 1 is arbitrarily assigned to the first note without any connotation of tonality or scale type. Thus, 1, 3, 5, 4, 3, 2, 1, could represent the first seven notes of Example 1, or Example 3 or an infinity of other presentations. Obviously the opening 1, 3, 5, remains a triad succession, root-third-fifth, and the closing 3, 2, 1, remains a descending scale segment, regardless of registral placement of the individual notes. We may say that these characteristics persist in the absence of precise information about intervals and register; or, what amounts to the same thing, we may say that in the *presence* of such precise information, the characteristics are invariant under transposition of the whole or registral displacement of individual notes. We will come presently to a more precise definition of *interval* which embodies this kind of invariance.

At this point it is natural to pose the question: Why suppress information? Surely not even the smallest detail is without potential significance. What madness is this which would disregard obviously significant information like precise interval size? Is not the invention of musical theory tenuous enough without self-imposed myopia?

An analogy may be useful. In the branch of mathematics known as *topology,* two solid objects may be regarded as equivalent if by twisting, squeezing, wrinkling, etc. (but not cutting or joining) one can be deformed into the same shape as the other. Thus two objects of dissimilar appearance, such as a doughnut and an LP record, may be topologically equivalent. An important point is that the dimensions and smoothness of the objects so compared are totally irrelevant. Topology shows that there are important structural properties of solid objects which have nothing to do with dimensions and smoothness. Indeed it seeks to discover precisely those attributes of the object which reside in qualities *other than* dimension and smoothness.

While this analogy proves nothing about the musical case, it may serve to underline the notion that musical constructs may be usefully analyzed and compared on the basis of less than complete information. I argue further, and hope to show in this paper, that certain pitch-structural features can be more clearly observed when intervallic information is distilled in a particular way. I believe it is a useful form of inquiry to ask: Exactly what things are revealed through study of a limited set of pitch-interval characteristics?

This posture does not advocate the permanent eschewal of information in musical analysis. We do not give up anything in making a selective inquiry; we merely set something aside. For example, it may be appropriate to consider the note group in Example 4 as any of the following:

1) 3-note C major triad in close position with middle C as bass
2) C major triad
3) triad
4) chord of 3 pitches
5) chord of 3 pitch classes

depending upon what we wish to assert about the function of the note group. In considering it as a chord of 3 pitches in contrast, say, to some surrounding chords of 17 pitches, we reserve the option of calling it a C major triad in relation, say, to a following F major triad.

Whenever we regard all steps in the diatonic system as equivalent (and, by extension, all 3rd's as equivalent, all 4th's as equivalent, etc.), we also reject the view of the diatonic system as a subset of the 12-pitch-class universe. That is, we disregard the potential chromatic notes between certain step-related note pairs, and equate the "next-to" or step function with unit distance.

In some contexts, the same concept applies to universes of less than seven pitch classes. For example, when a triad is arpeggiated for a relatively long timespan with minimal interference (pitches outside the triad) we may think of a universe of three pitch classes with "steps" between adjacent pitches of the triad, disregarding (in the limited context) the exact interval sizes.

The point is simply this: In order to characterize clearly the multiple functions of an event, it may be useful to make various selections of its attributes.

III. Definitions

The term *pitch class* (PC) will be employed in the usual sense; however a universe of *seven* PC's is posited. There are no special restrictions on the structural characteristics of systems modelled by the seven-PC universe: Any seven note scalar system, diatonic or otherwise (including the system of seven equal-tempered pitches per octave) may be so modelled. However, for the sake of simplicity, and because this study will focus primarily upon the diatonic case, abstract examples with specific pitches are based on the C-major scale, with PC numbers as follows: C = 0, D = 1, E = 2,. . . . . ., B =6.

I use the term *interval* to mean the directed distance from one pitch class (PC) to another, measured in ascending diatonic steps.[2] Each step, whether whole or half, is a unit distance. Thus the interval from C to C is zero; from C to D is 1; from C to E is 2; from C to F is 3 (not 2½); etc. There are seven different intervals: 0, 1, 2, 3, 4, 5, 6. Example 5 shows a string of notes and the interval numbers from each note to the next. The application of this definition will be restricted here to contexts involving a single major or minor scale, but, as implied by the definition of PC, this is not a necessary constraint.

An *interval series* (IS) is simply a succession of intervals. An IS may contain any number of intervals. It may contain immediate repetitions. It may contain the zero interval; however IS containing the zero interval are excluded from the present study. An IS is notated as a string of interval numbers preceded by "IS:", or as a string of interval numbers written with no intervening spaces (for example, 34112). The *order* of intervals in an IS is, by definition, an essential feature, and is given by the normal left-to-right reading of the notation.

An IS is useful as an abstraction from a series of PC's: It displays the intervals between adjacent PC's in order, and therefore contains all of the information regarding the *structure* of the PC series, while omitting the particular transpositional level.

An important special kind of IS is the equal-interval series (=IS), an IS consisting of a string of the same interval. Disregarding the degenerate case of the zero interval, there are six classes of =IS, corresponding to the six non-zero intervals. (Two =IS of the same class may differ in length.) It is well known that an =IS will produce the seven PC's of the diatonic scale before repeating any of them; however, I believe that the implications of this fact have not been fully understood. In large part, this article is a study of those implications.

By *chord* I mean the interval structure of an unordered set of PC's or any of its transpositions. The structure is indicated by a sequence of numbers enclosed in parentheses always adding to seven in the diatonic case, listing the intervals between successive PC's, given all PC's arranged in ascending order within an octave, with the first PC repeated at the end. Thus the set {C, E, G} or {0, 2, 4} forms the chord (223). Since *chord,* by definition, refers to an unordered set, the rotations of (223), namely (232) and (322), designate exactly the same chord as (223). An important special case is the chord consisting of all seven diatonic PC's (1111111), which I call the *total diatonic* (TD). Since, for any given diatonic context, there is just one collection of PC's which forms the TD, I use the term TD to refer to that collection itself as well as to its interval structure. The complete roster of diatonic chords is illustrated in my previous article.[3]

The =IS is a special case of a more general class of IS, namely those IS consisting entirely of a repeated interval or a repeated sub-series of intervals. The IS 121212 and the IS 213213213, for example, belong to this more general class. I call an IS such as these a repeating IS (RIS), and call the repeated interval or shortest repeated sub-series of intervals a *generator.* As usual, I will say that such an interval or sub-series *generates* a particular chord. Thus, interval 2 generates chords as shown in the following table. (The "null chord" is omitted.)

RIS	*chord*	
2	(25)	
22	(223)	(triad)
222	(1222)	(7th chord)
2222	(11122)	
22222	(111112)	
222222	(1111111)	(total diatonic)

And the sub-series 12 generates chords as follows:

RIS	*chord*
12	(124)
1212	(11212)
121212	(111112)
12121212	(1111111)

IV. Serial Properties of the Diatonic System

Are there compositions which contain diatonic serialism? Yes, but relatively few. Stravinsky's *Sonata for Two Pianos* is one example that comes to mind. If, by diatonic serialism, we mean something resembling the complex structures of classical 12-note serialism and contemporary pieces in that tradition, it is possible that there are no pieces which instance diatonic serialism. There may be good reason for this. There is, in any case, a construct in traditional music which, when applied in diatonic contexts, is inherently serial, though in a weaker sense than that of classical 12-note serialism. The construct I refer to is the *sequence*. I use the term here in the slightly unusual sense of intervallic sequence only, not neccesarily associated with registral and rhythmic factors. Also, unless otherwise noted, I shall be referring to *melodic* sequence.

More precisely, by a *sequence* I mean any PC series corresponding to a RIS where the interval(s) of the generator has (have) a non-zero sum. (The requirement of the non-zero sum simply ensures that transposition takes place; otherwise there would be no sequence.) Example 6 shows a sequence formed by the RIS .....14141414..... (Note that the overlapping trichords formed are all of (124) or (214) structure. Trichord chains are treated more fully in "Aspects...".)

What are the senses in which a sequence may be called "serial", beyond the fact that, by definition, it contains a repeated interval pattern? That is, by what procedures of counting can we discover serial or quasi-serial enumerations of the seven PC's in a diatonic sequence? I suggest three analytical procedures by which we can attempt to decompose a sequence so that all of its PC's are assigned to subsequence consisting of the TD with no duplications. When this is possible, I will say that the sequence is a *serial process* under the particular analytical procedure.

Procedure number 1: *decomposition* into parallel lines. Whatever the number of intervals in the generator, the sequence is decomposed into that number of lines by distributing the PC's, one after another in rotation to the separate lines. The sequence in Example 6 is decomposed into two lines (there being two intervals in its generator) by assigning the 1st, 3rd, 5th, etc. PC's to one line and the 2nd, 4th, 6th, etc. PC's to the other line. Each of the resulting lines corresponds to an =IS, a string of interval 3's. In this manner *any* sequence may be decomposed into lines each of which corresponds to an =IS. Since all =IS cycle through the TD without repetition, any diatonic sequence qualifies as a serial process under decomposition.

Note that exact sequences in the *twelve*-PC universe do not necessarily generate the total *chromatic;* thus they are not necessarily serial under decomposition in the full sense that all diatonic sequences are. (As explained later, this difference is due to the fact that the number 7 is prime, while 12 is non-prime.)

Procedure number 2: *straightforward enumeration*. Some sequences may be completely segmented into uninterrupted spans of the TD (possibly overlapping) without repetition. For reasons already mentioned, any sequence corresponding to an =IS is a serial process under this procedure. Example 7 shows an enumeration where the generator is the interval 2. The example also shows that repeated transposition of (223) (the traditional triad) by interval 6, which corresponds to the illustrated melodic sequence, is a serial process. (More later on the transposition of chords.)

Curiously, no generator consisting of two intervals yields a serial process under straightforward enumeration.

I have not systematically investigated all generators of three through six intervals, but, from what I have been able to discover, those which yield a serial process under straightforward enumeration are sparsely distributed. Example 8 shows that the generator 233 yields a serial process under this procedure. The slurs in the example illustrate the fact that for a generator of n intervals, overlapping spans of seven PC's comprising the TD begin on every nth PC. Again an associated chord succession is shown.

Obviously there are many generators of seven intervals which yield a serial process under straightforward enumeration. Any ordered set of the seven distinct PC's, followed by repeated transpositions of itself at the same non-zero interval, can be formed only by a generator consisting of seven intervals.

If a generator yields a serial process under straightforward enumeration, then so does its inversion and all cyclic permutations of the generator and its inversions. For example, 544, the inversion of 233, yields a serial process; therefore 445, 454, 332, and 323 also yield a serial process.

Procedure number 3: *grouping*. Under grouping, for a generator of n intervals, the PC series is first grouped into consecutive non-overlapping cells of n PC's. The PC's within any cell may then be re-ordered in any desired manner. Finally, when the same PC occurs in two or more successive cells, all instances of that PC except the first are deleted within the succession of cells containing the PC repetition. Effectively, this procedure regards the output of the generator as a series of parellel chords, permitting any ordering of PC's within each chord, and invoking the concept of "tied" PC's to delete repetitions. Example 9 illustrates this procedure for the generator 215, and shows that it yields a serial process under grouping.

It turns out that for generators yielding a serial process under grouping, the PC series which is "distilled" (that which remains after the permitted re-orderings and deletions) corresponds in every case to an =IS, and is therefore "strict" in the sense that not only may it be segmented into spans of the TD containing no repetition, but in addition each segment contains the 7 PC's in the same order.

Since analysis by grouping amounts to studying the serial process in parallel chords, the extent to which the diatonic system exhibits the serial process under grouping can most conveniently be illustrated by tabulating, for all 16 chords of 2-5 PC's in the diatonic universe, and for all 6 intervals, the incidence of serial process under grouping. This tabulation is given in Example 10, and as shown there, the serial process under grouping is quite pervasive. Note, for example, that traditional triads (223) and seventh chords (1222) yield the serial process under grouping for *any* interval of transposition.

Example 10 Table showing incidence of the serial process under grouping, for diatonic chords in sequential transposition.

✓ = serial process under grouping

	INTERVAL OF TRANSPOSITION					
CHORD	1	2	3	4	5	6
(1 6)	✓	✓			✓	✓
(2 5)		✓	✓	✓	✓	
(3 4)	✓		✓	✓		✓
(1 1 5)	✓	✓	✓	✓	✓	✓
(1 2 4)			✓		✓	✓
(2 1 4)	✓	✓		✓		
(1 3 3)	✓	✓	✓	✓	✓	✓
(2 2 3)	✓	✓	✓	✓	✓	✓
(1 1 1 4)	✓	✓	✓	✓	✓	✓
(2 1 1 3)	✓	✓		✓		
(1 1 2 3)			✓		✓	✓
(1 2 1 3)	✓	✓	✓	✓	✓	✓
(1 2 2 2)	✓	✓	✓	✓	✓	✓
(1 1 1 1 3)	✓	✓	✓	✓	✓	✓
(1 1 1 2 2)		✓	✓	✓	✓	
(1 1 2 1 2)	✓		✓	✓		✓
	1	2	3	4	5	6

These definitions of "serial process" and three associated analytical processes may seem arbitrary or unfounded to many. Of course there are other definitions of process and analysis which might be invoked, some which, I can imagine, might lead to more significant results than these. Also, the results presented here are incomplete and informal, even within the definitions proposed. (I hope to find formal proofs for some of the assertions.) Notwithstanding these imperfections, I hope that many readers will agree that the diatonic system is inherently serial in unsuspected ways. Certain roots of 12-note serialism are, I believe, deeply embedded in the essence of diatonicism.

V. Equal-Interval Series and Hierarchical Structures

The group M

The set of non-zero interval numbers in the 7-PC universe forms a mathematical group under multiplication modulo 7. We denote this group by the symbol M:

$$M = \{1, 2, 3, 4, 5, 6\} \text{ under } \cdot \text{ (mod 7)}$$

The identity element is 1, and inverses are as follows:

$$\begin{aligned} &1 \cdot 1 \equiv 1 \\ &2 \cdot 4 \equiv 4 \cdot 2 \equiv 1 \\ &3 \cdot 5 \equiv 5 \cdot 3 \equiv 1 \\ &6 \cdot 6 \equiv 1 \qquad\qquad \text{(all products mod 7)} \end{aligned}$$

The group is abelian (commutative). The group property depends upon the primeness of the number 7, for the set of consecutive integers {1, 2, 3,. . ., n-1} forms a group under multiplication modulo n if and only if n is prime.

The application of multiplicative operators has been studied in the 12-PC case by Howe[4] and Lewin[5] and others, and in the general equal-tempered case by Gamer.[6] While there are similarities in the underlying mathematics of such operators in all such cases, the universe modelled in the diatonic case differs obviously from the cases cited above in that it is non-equal-tempered.

If we apply any one of the integers {1, 2, 3, 4, 5, 6} as a multiplier (mod 7) over each of the intervals in an IS, we obtain a transformation of the series. I denote the operator by the symbol Mx where x is the multiplier. For example given an IS, A = 16243, the operator M2 applied to A yields a transformation of A as follows:

$$M2(A) = 2{\cdot}1\ 2{\cdot}6\ 2{\cdot}2\ 2{\cdot}4\ 2{\cdot}3 = 25416$$

It follows from the preceding that the operators {M1, M2, M3, M4, M5, M6} form a group isomorphic to M with operation "followed by" and with inverses corresponding to M. For example, the inverse of M2 is M4, and M2 followed by M4 = M1. Using the IS "A" from above for purposes of illustration:

$$M4(M2(A)) = 4{\cdot}2\ 4{\cdot}5\ 4{\cdot}4\ 4{\cdot}1\ 4{\cdot}6$$
$$= 16243 = A$$

The set S

As noted in the section on definitions, the set of all =IS divides naturally into six classes (excluding the zero-interval case). To each of these six classes corresponds a class of PC series, generated by a particular non-zero interval. I denote these classes of PC series by the symbols S1, S2, S3, S4, S5 and S6, where Sn is generated by interval n, and represent the entire set of classes by the symbol S:

$$S = \{S1, S2, S3, S4, S5, S6\}$$

We can think of each of the six classes as a kind of proto-series extending infinitely in both directions, containing as sub-series the infinity of possible instances of that class.

S1: .....0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6.....
S2: .....0, 2, 4, 6, 1, 3, 5, 0, 2, 4, 6, 1, 3, 5.....
S3: .....0, 3, 6, 2, 5, 1, 4, 0, 3, 6, 2, 5, 1, 4.....
S4: .....0, 4, 1, 5, 2, 6, 3, 0, 4, 1, 5, 2, 6, 3.....
S5: .....0, 5, 3, 1, 6, 4, 2, 0, 5, 3, 1, 6, 4, 2.....
S6: .....0, 6, 5, 4, 3, 2, 1, 0, 6, 5, 4, 3, 2, 1.....

An instance of each member of S is given in Example 11. (Each PCS in Example 11 is of length eight: this is done merely to show that repetition occurs only after the TD is stated.)

Transformations within S, by M

The six members of S are transformationally related through M, as follows: Since each member of S corresponds to a string of identical intervals, if we select A and B, both members of S (A = B is permitted), we can find an M operator which will transform A into B. Suppose A = Sx and B = Sy. We need only solve

$$x \cdot m \equiv y \pmod 7 \qquad (1)$$

where $m \in M$. (Because of the group properties, if x and y are in M, the equivalence $x \cdot m \equiv y$ will always have a unique solution m in M.) For example, to transform S2 into S1, we first solve

$$2 \cdot m \equiv 1 \pmod 7$$

finding m = 4. Now applying M4 to S2, we obtain S1:

S2: ...0, 2, 4, 6, 1, 3, 5, 0...
M4↓
S1: ...0, 1, 2, 3, 4, 5, 6, 0...

The complete set of transformations within S by means of M is nothing more or less than the multiplication table of M:

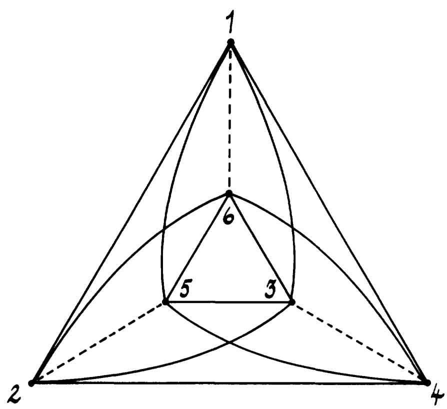
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

The products within the matrix show the results of applying M operators to members of S. For example, the table shows that $5 \cdot 2 \equiv 3 \pmod 7$; therefore $S5 \xrightarrow{M2} S3$. And since $3 \cdot 4 \equiv 5 \pmod 7$, $S3 \xrightarrow{M4} S5$. This pair of transformations, illustrating the reversibility of any given transformation, is shown in musical notation in Example 12.

Generators of M

The results of multiplication in M (essentially the multiplication table of the group) are displayed in graphic form in Example 13. Each of the six numbered dots represents an element of the group. Each line connecting a pair of dots represents a multiplication as follows: curved lines represent multiplication by 3 or 5 ($\equiv 3^{-1}$), depending upon direction. For example, the line connecting elements 2 and 6 indicates $2 \cdot 3 \equiv 6$ and $6 \cdot 5 \equiv 2 \pmod 7$. Similarly, sides of triangles represent multiplication by 2 or 4 ($\equiv 2^{-1}$), depending upon direction, and dashed lines indicate multiplication by 6 (in either direction). The "line" connecting each dot to itself represents multiplication by 1.

ex. 13



We can see from Example 8 that the element 3 generates the entire group: $3^{-1} \equiv 3$, $3^2 \equiv 2$, $3^3 \equiv 6$, $3^4 \equiv 4$, $3^5 \equiv 5$, $3^6 \equiv 3^0 \equiv 1$. (Higher powers of 3 are redundant: $3^7 \equiv 3^1$, etc.) The element 5 ($\equiv 3^{-1}$) also generates the entire group. The elements 2 and 4 generate the subgroup {1, 2, 4}; the element 6 generates the subgroup {1, 6}; and the identity element generates the trivial subgroup {1}.

By extension, each dot in Example 13 may be taken to represent a member of S, while lines connecting dot-pairs represent application of the M operators.

Extraction and interpolation

By *extraction* I mean a process of selection from a PC series, without re-ordering, which creates a new series. For example, if we select the 1st, 2nd and 4th PC's from the series 0, 2, 3, 5, 6, we obtain 0, 2, 5, a new series. This process is obviously similar to *reduction* in the music-analytic sense, although no musical criteria are (at this stage) associated with extraction. By *regular* extraction, I mean the process of selecting every 2nd, or every 3rd, or every 4th, etc., PC from a series. For example, if we select every 2nd PC from the series 0, 2, 3, 5, 6, 1, 2, we obtain the new series 0, 3, 6, 2. There are, of course, many musical processes which bring about the regular emphasis of every 2nd, or every 3rd, etc., PC in a series.

When regular extraction is performed on a member of S, the resulting "new" series is always again a member of S. Which member of S depends upon the member of S with which we begin and the "rhythm" of extraction (whether every 2nd, or every 3rd, etc.). For example, if we extract every 2nd PC from S2 we obtain S4:

$$\begin{array}{lllllllllllllll} \text{S2:} & \ldots 0, & 2, & 4, & 6, & 1, & 3, & 5, & 0, & 2, & 4, & 6, & 1, & 3, & 5, 0\ldots \\ \downarrow & & & & & & & & & & & & & & \\ \text{S4:} & \ldots 0, & & 4, & & 1, & & 5, & & 2, & & 6, & & 3, & 0\ldots \end{array}$$

Note that since both series are considered to be infinitely extended in both directions. this result does not depend upon with which PC of S2 the process of extraction begins.[7]

As applied to members of S, the process of regular extraction is isomorphic to the application of M operators over S: The ordinal number of the extraction rhythm (every 2nd, 3rd, etc.) is simply converted into an M operator yielding the same result. Thus, for the above case: $\text{S2} \xrightarrow{\text{M2}} \text{S4}$. For transformation by extraction, the relation between the two series is given by formula (1) above, where m is a multiplier corresponding to the ordinal number of the extraction rhythm.

By *interpolation* I mean a process of introducing additional PC's between the PC's of a given series to form a new series. Here we are concerned only with a specific kind of interpolation, namely *regular* interpolation in some member of S. By regular interpolation I mean the introduction of additional PC's so as to form some fixed number of intervals in replacement of each of the intervals in the original series. If S2 is the original series, and we wish to form, say, four intervals to replace each one of the original intervals, then the new series will take the form

...0, *, *, *, 2, *, *, *, 4, *, *, *, 6, *, *, *
1, *, *, *, 3, *, *, *, 5, *, *, *, 0, *, *, *,...

where the asterisks represent the new PC's to be introduced. As in the case of extraction, interpolation corresponds to a musical process, in this case that of introducing (in actual composition or in the imagination) notes at a lower structural level and with particular kinds of rhythmic and intervallic constraints, or in a word, a particular form of *elaboration*.

The remarkable and important fact here is that, given a replacement "ratio" (such as four "new" intervals to replace each "old" interval), it is always possible to interpolate so that the resulting intervals in the new series are *equal*. Given the interval to be "decorated" (call it y) and the number of new intervals to be formed (call it m), we find the interval of interpolation (call it x) as follows:

$$x \equiv y \cdot m^{-1} \pmod 7 \qquad (2)$$

where m^{-1} is the inverse of m in M. For example, in the above case

$$x \equiv 2 \cdot 4^{-1} \equiv 2 \cdot 2 \equiv 4 \pmod 7$$

and we can now form the new series (PC's of the original series are underlined):

$$\ldots\underline{0}, 4, 1, 5, \underline{2}, 6, 3, 0, \underline{4}, 1, 5, 2, \underline{6}, 3, 0, \ldots$$

The relation between extraction and interpolation is now apparent: They are inverse operations (in a mathematical as well as musical sense). Formula (2) above is easily derived from formula (1):

$$\begin{aligned} x \cdot m &\equiv y && (1) \\ x \cdot m \cdot m^{-1} &\equiv y \cdot m^{-1} \\ x &\equiv y \cdot m^{-1} && (2) \end{aligned}$$

(all products mod 7)

Obviously, given any two of x, y and m, we can solve for the unknown. m is expressed in terms of x and y as follows: $m \equiv x^{-1} \cdot y \pmod 7$.

Extraction, interpolation, and M, related

Given a set of values in M which satisfied $x \cdot m \equiv y \pmod 7$, we can interpret the solution to express either interpolation or extraction. As shown in Example 14a, $5 \cdot 2 \equiv 3 \pmod 7$ implies that the transformation S5→S3 is achieved by extraction of every 2nd PC from S5, or that the transformation S3→S5 is achieved by replacement of each interval in S3 by 2 equal intervals. In like manner, Example 14b shows the implications of $3 \cdot 4 \equiv 5 \pmod 7$, which are the pair of transformations opposite to those in Example 14a. Regular extraction and regular interpolation are isomorphic to one another, and to the application of M operators over S! One-to-one mappings among the three systems are as follows:

M operator	extraction rhythm	number of interpolated intervals
M1	1st	1
M2	2nd	4
M3	3rd	5
M4	4th	2
M5	5th	3
M6	6th	6

Since the three systems are isomorphic, the properties of all three depend upon the primeness of the number 7, as previously noted for M.

Reversibility of structural pairings in S

Suppose that we are given two members of S to be set in relationship so that one is structurally higher than the other in the sense of being the result of extraction from the other or, inversely, that one is structurally lower than the other in the sense of being the result of interpolation in the other. From the above we can see that, whatever the two given members of S, either may assume the structurally higher role or the structurally lower role. In no case will the particular choice of a pair from S determine the structural relationship. Given A and B in S, we can move by either extraction or interpolation from A to B and vice versa. For example, the transformation S5→S6 can be accomplished as follows:

by extraction:

```
S5:  . . . 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 . . .
 ↓
S6:  . . . 0       6       5       4 . . .
```

by interpolation:

```
S5:  . . . 0   5   3   1 . . .
 ↓
S6:  . . . 0 6 5 4 3 2 1 0 . . .
```

In the case of S5→S6 by extraction, S6 is at the higher level (or, in musical parlance, it is relatively structural); in the case of S5→S6 by interpolation, S6 is at a lower level (or, in musical parlance, it is decorative, with respect to S5). Thus any two members of S may fold into one another in a curious way: The relationship of structural member to embellishing member is reversible!

Rhythm and pitch operators related

Returning to the isomorphism among extraction, interpolation, and M, we note an additional curious point: The application of an M operator to a member of S is a *pitch* operation in the sense that for each PC in the original series we obtain exactly one PC in the new series. In this respect it is like other pitch operations such as transposition. On the other hand, extraction and interpolation are essentially *rhythm* operations: They are conceived as reductions of elaborations of a source series, in which a many-to-one or one-to-many relationship obtains between the PC's of the source series and the PC's of the target series. In this view, certain pitch operations and certain rhythm operations are identical to one another in structural effect. Example 15 shows a simple case.

The diatonic system as an inherently levelled structure

Because of the reversible roles of members of S in structural relationship with one another, a rich variety (actually an infinite variety) of hierarchies constructed from members of S is possible. In many of these, the same member of S appears at different structural levels. For example, we can start with the above case of S5→S6 by extraction, and add another extraction from S6, obtaining S5 at a higher structural level:

```
S5:  . . .0               5               3
S6:       0       6       5       4       3       2
S5:  . . .0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 . . .
```

Enlarging the hierarchy still further, we obtain S6 at a lower level by interpolation in S5:

```
S5:       0             5             3
S6:       0      6      5      4      3      2
S5:       0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0
S6:  . . .0654321065432106543210654321065432106543210. . .
```

As the example stands now, S5 and S6 both appear at two structural levels. If we wish to obtain a uniform rhythm of extraction/interpolation throughout the hierarchy (2:1 extraction (corresponding to M2), or 1:2 interpolation (corresponding to M4)), we can insert S3 in the middle:

```
S5:   0                   5                   3
S6:   0         6         5         4         3         2
S3:   0    3    6    2    5    1    4    0    3    6    2
S5:   0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0
S6: ...06543210654321065432106543210654321065432 10...
```

Given any member of S as a "seed" and the "growth" process of regular extraction/interpolation, the resultant hierarchy, however rich and extensive, is forever closed: Owing to the closure property of M, it cannot move outside S. Therefore, as the number of levels in a hierarchy is increased, the probability that the same member of S will appear at two different levels increases rapidly; in hierarchies of more than six levels, there must be such a repetition at different levels. The important fact is this: To the extent that members of S are found in the presence of regular growth processes, the diatonic system inherently tends toward hierarchies in which the same features appear at different structural levels. The conditions supporting this tendency are simple and, I believe most would agree, prevalent in musical literature.

The exploration of possible hierarchies constructed from members of S need not detain us long here; two further observations will suffice: First, *any* hierarchy composed of members of S is theoretically possible, since, for any transformation Sx→Sy, there is an appropriate M operator. (See the group multiplication table above.) Second, if the M operator is consistent between all adjacent members of S in a hierarchy, the hierarchy will have a repetition rate, or period, of 1, 2, 3 or 6, depending upon the M operator, as given in the following table (which corresponds to the information given in the section on "Generators").

M operator	period
M1	1
M2 or M4	3
M3 or M5	6
M6	2

VI. A Brief Analysis

In Example 16, staff (a) quotes mm. 3-18 of the first violin part of Beethoven's Symphony No. 7, III. Staves (b), (c), (d) and (e) show the hierarchical structure in S for mm. 3-10 (an alternative to staff (c) is shown on staff (f)); staves (b), (c) and (d) show the hierarchical structure in S for mm. 11-18. (On staff (b), mm. 12, 14, 15 and 16, the selection of the "appoggiaturas" B♭, D and F, instead of their resolutions is justified on the basis of precedent in mm. 3-11.) The dashed line beneath staff (e) indicates the completion of S1 in m. 18, in the same register as m. 3 and m. 7. An inversional relationship between pairs in S, suggested by mm. 3-10 plus m. 18, is shown on staves (g) and (h). Several facets exhibited here, including the inversional pairs in S, the linkage by common PC of different hierarchical structures, and the interval 2 from F to A (presaged in mm. 1-2 and affirmed by the emphasis on A in the measures following the excerpt, and achieved by distinct interpolations between the note-pairs of its component interval 1's (F to G in mm. 1-10, and G to A in mm. 11-18)), serve as basis for developments later in this movement.

In conclusion I wish to emphasize the intent of this brief analysis. Confronted with the Beethoven excerpt as an object of analysis, the reader would, without benefit of the present study, surely discover (or at least attend to the possibility of) something akin to the hierarchical structure shown in my analysis (though no doubt differing in analytical representation). The intent, however, is not to enlighten or inform with respect to this musical example, or even to propose an analytical approach to comparable musical examples. Rather, the intent is to instantiate the modelling of an aspect of the diatonic system—the inherent tendency to form hierarchies of Sn's—an aspect thoroughly pervasive and abundantly evident in diverse musics, yet, I believe, incompletely understood as an aspect of system.

NOTES

[1] "Aspects of Diatonic Sets", *Journal of Music Theory* 23/1 (Spring 1979), pp. 45-61.

[2] The choice of *ascending* direction is arbitrary. The use of descending direction would partition all ordered pairs of PC's into classes of intervals in exactly the same way, and would have no effect upon *structures* formed among the intervals.

[3] "Aspects. . . . , p. 48.

[4] Hubert Howe, "Some Combinational Properties of Pitch Structures", *Perspectives of New Music* 4/1 (Fall-Winter 1965), pp. 45-61.

[5] David Lewin, "On Certain Techniques of Re-Ordering in Serial Music", *Journal of Music Theory* 10/2 (Winter 1966), pp. 276-287.

[6] Carlton Gamer, "Some Combinational Resources of Equal-Tempered Systems," *Journal of Music Theory* 11/1 (Spring 1967), pp. 32-59.

[7] For 12-tone rows, the extraction process and its relation to multiplicative operators is treated in Lewin, op. cit.