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VECTOR PRODUCTS AND INTERVALLIC WEIGHTING

Steven Block and Jack Douthett

INTRODUCTION

As Forte noted in his original work on set-complexes in music, in the rigorous mathematical sense an interval-class vector is in fact not a vector at all.[1] However, applications of certain vector operations on interval-class vectors can yield useful results for both composer and theorist. We will explore two very distinct applications of vector products; one leads to a "custom designed similarity index," and the other proposes scale structures in microtonal systems.[2] We will accomplish both of these tasks by constructing appropriate weighting vectors. Taking the dot products between a weighting vector and the interval-class vectors allows us to find sets that are closely related with respect to some preconceived set of properties such as interval-class preferences or "evenness".[3] It is the former that relates to a type of "similarity index," and the latter pertains to microtonal systems.

In addition, we consider possible extensions of the interval-class vector (which in effect, measures dyad content) and suggest the constructions of "polychord-content" vectors, that is, vectors that measure trichord content, tetrachord content, etc. of the members of a set-classes.[4] It is then possible to construct weighting vectors to be used in conjunction with these polychord-content vectors in the same way they were applied to interval-class vectors.

RELATING PITCH-CLASS SETS

Traditional measurements of similarity have largely been based on the intervallic content of sets.[5] However, it may be more practical for a composer or theorist to seek out sets which satisfy certain compositional or operational requirements rather than determining similarities among sets in a "static" state and abstract framework. For example, a composer may wish to find a family of sets that have certain intervals suppressed or eliminated and at the same time have others emphasized. In general, similarity measurements are based solely on interval class content and do not allow for the flexibility required above. For the most part, "flexible similarity" has not been explored, though this type of measurement was first implied by John Rahn in 1979.[6] By constructing a weighting vector and taking dot products between this vector and the interval-class vectors, we can uncover sets related by more subtle intervallic properties.

For a simple example, suppose we wish to ascertain sets in which ic5s are emphasized and ic1s suppressed, and we have no particular preference for the other ics; that is, we wish to find sets that have a large number of ic5s, and at the same time, a small number of ic1s. In constructing a weighting vector that will reflect this set of criteria, we will associate positive numbers with ic preferences, negative numbers with undesirable ics, and 0 if there is no preference. Thus, the weight vector

$$W = [-100010]$$

will reflect these desired properties since ic1 is to be suppressed by means of a negative number in the 1st column and ic5 is to be encouraged by means of a positive value in column 5. The resultant dot products between the weighting vector W and the interval-class vectors of all pcsets reveal that the sets with maximum weight, 4, are generated by ic5. This result corresponds with intuition, and when the weights of ic5-generated sets are compared, those with the maximum weight are set-classes 5-35 (02479), 6-32 (024579), and 7-35 (013568A) (table 1). In this case, the weights increase from 1 to 4 as the cardinalities of the sets increase from 2 through 5. This is because ic5 is increased by one in each additional set in the sequence while ic1 remains zero. Note that, while the hexachord and heptachord continue to successively increase ic5 by one, ic1 also increases successively by one, which effectively cancels out the ic5 increase. This results in the weighting equality of these three sets. Then, from cardinalities 8 through 10, ic5

Table 1: A comparison of ic5-generated set-classes and weights with respect to the weighting vector W = [−1 0 0 0 1 0].

Set-Class	Weight	Set-Class	Weight	Set-Class	Weight
2-5	1.00	5-35	4.00	8-23	3.00
3-9	2.00	6-32	4.00	9-9	2.00
4-23	3.00	7-35	4.00	10-5	1.00

continues to successively increase by one, but this is countered by ic1 successively increasing by two. Whence, the weights decrease. On the negative end of the scale, corresponding chromatic sets, 5-1, 6-1, and 7-1 least satisfy the desirable properties and possess the lowest weights.

The results of such weight scaling would not be a as intuitive if desired conditions were more complex. Suppose we seek sets with a large whole-tone content and somewhat less ic3 content. At the same time, we wish to suppress the other ics, in particular, ic6. Moreover, in successively decreasing degrees we endeavor to suppress ics 1, 5, and 4. Such a weighting vector might be constructed as

$$W = [-1\ 1\ .5\ -.75\ -.9\ -1.25].$$ [7]

The sets which fall out of this emphasis are naturally of smaller cardinalities since larger collections are avoided by the negative weighting of ic1 in our weighting vector (table 2). However, the resultant sets would not necessarily be those selected intuitively. While small whole-tone sets (3-6, 4-21) and ic3-generated sets (3-10, 4-28) are included among the strongest representatives, the preference implied by the weighting vector for whole tones places the ic3-generated sets slightly lower. In addition, the whole-tone pentachord occurs somewhat down the list, about 40th place, because its content includes a larger number of suppressed ics than a pentachord like 5-34 (02469) whose relative lack of ic6 and ic4 content cause it to be placed among the top sets satisfying the above requirements. Smaller chromatic sets are weighted heavily because of their ability to generate both of the desired interval classes without generating the suppressed ics.

A composer might desire to work with a pentatonic scale that possesses the above properties but that is not as closely related to diatonic formations as the pentachords (including 5-35) generated by the above weighting vector. In the above listing, the high ic5 content distinguishes the diatonic from the chromatic subsets. Thus, by suppressing ic5 even further, say to the extent of the tritone suppression, 5-35 drops off the top 20 and the pentachord 5-8, (02346), enters the picture as a subjectively more interesting pentatonic scale (table 3).

Table 2: Sets with greatest weights for the weighting vector W = [−1 1 .5 −.75 −.9 −1.25].

Set-Class	Prime Form	IC Vector	Weight
3-6	(024)	[020100]	1.25
4-10	(0235)	[122010]	1.10
3-7	(025)	[011010]	0.60
3-2	(013)	[111000]	0.50
4-21	(0246)	[030201]	0.25
4-22	(0247)	[121110]	−0.05
4-11	(0135)	[121110]	−0.15
4-23	(0257)	[021030]	−0.20
4-2	(0124)	[221100]	−0.25
3-10	(036)	[002001]	−0.25
5-35	(02479)	[032140]	−0.35
5-23	(02357)	[132130]	−0.45
4-1	(0123)	[321000]	−0.50
4-28	(0359)	[004002]	−0.50
4-26	(0358)	[012120]	−0.55
5-34	(02469)	[032221]	−0.55
5-2	(01235)	[332110]	−0.65
5-1	(01234)	[432100]	−0.75

We believe that the flexibility of the weighting vector is one of its greatest attributes. The user decides in advance what characteristics are desired in a set and constructs the weighting accordingly. If the user requests more general characteristics such as a preference of ic1 and suppression of the tritone without specifying considerations for other intervalic content, the results are indeed intuitive in that the largest and most chromatic sets naturally fall out of such broad stipulations (table 4). However, the designer of the weighting vector may make specific demands on the preferred chromatic sets by expressing, for example, a preference of ic1 and ic3, an avoidance of the larger interval classes, and neutrality for ic2. Then the results are not quite as obvious (table 5a). While these stipulations now render prominent all the smaller chromatic sets (as would be expected), sets that are not overtly chromatic, such as 4-10 (0235) and 5-10 (01346), are also among the most preferred. Without these stipulations, larger sets will most naturally have the greater weights, but once we impose these stipulations, the larger sets will lose importance, since they contain too many instances of undesired ics. In fact, many of the previously high ranking sets (table 4) are now relegated to low rankings (table 5b). The diatonic heptachord, 7-35, which one might suspect is

Table 3: Sets with greatest weights for the adjusted weighting vector $W = [-1\ 1\ .5\ -.75\ -1.25\ -1.25]$.

Set-Class	Prime Form	IC Vector	Weight
3-6	(024)	[020100]	1.25
4-10	(0235)	[122010]	0.75
3-2	(013)	[111000]	0.50
3-7	(025)	[011001]	0.25
4-21	(0246)	[030201]	0.25
3-10	(036)	[002001]	−0.25
4-2	(0124)	[221100]	−0.25
4-1	(0123)	[321000]	−0.50
4-11	(0135)	[121110]	−0.50
4-28	(0369)	[004002]	−0.50
5-1	(01234)	[432100]	−0.75
5-8	(02346)	[232201]	−0.75
4-3	(0134)	[212100]	−0.75
4-22	(0247)	[021120]	−0.75
5-2	(01235)	[332110]	−1.00
4-12	(0236)	[112101]	−1.00
3-8	(026)	[010101]	−1.00
3-1	(012)	[210000]	−1.00

among these undesired sets, is, however, not quite as poorly represented, since its chromatic and ic3 content somewhat offsets the undesired perfect interval content.

A weighting vector can also be used to access a "similarity in function" among sets; that is, it relates sets on the basis of interval content with respect to the ability of these sets to fulfill the particular objectives of the designer/composer. Thus, in the context of the properties called for in one of the previous examples (table 3), sets are being related on the basis of their high ic2 and ic3 content, while suppressing the other ics, in particular ic5 and ic6. One interpretation of these results is that pcsets 5-1, 5-8, 4-3, and 4-22 are all equally effective in achieving the above requirements since they have the same weight. Note that using the similarity relations proposed by Allen Forte,[8] neither pair of pentachords or tetrachords would be considered related. Moreover, his Kh relations illustrate that there is no single K/Kh complex which includes all four of these sets, 4-22 being the odd man out. In view of this, we conclude that, although these sets are not particularly similar in the static sense, they are in fact very similar with respect to the criterion established by the designer/composer.

Table 4: Sets with greatest weights for the weighting vector W = [10 0 0 0 0 −10], a more general weighting vector.

Set-Class	Prime Form	IC Vector	Weight
11-1	(0123456789A)	[AAAAA5]	50.00
10-1	(0123456789)	[988884]	50.00
9-1	(012345678)	[876663]	50.00
8-1	(01234567)	[765442]	50.00
7-1	(0123456)	[654321]	50.00
6-1	(012345)	[543210]	50.00
10-2	(012345678A)	[898884]	40.00
10-3	(012345679A)	[889884]	40.00
10-4	(012345689A)	[888984]	40.00
10-5	(012345789A)	[888894]	40.00
9-2	(012345679)	[777663]	40.00
9-3	(012345689)	[767763]	40.00
9-4	(012345789)	[766773]	40.00
8-2	(01234568)	[665542]	40.00
8-3	(01234569)	[656542]	40.00
8-4	(01234578)	[655552]	40.00
8-7	(01234589)	[645652]	40.00
7-2	(0123457)	[554331]	40.00

In general, a unique weighting vector is constructed from each sets of interval class requirements. Thus, similarity based on a weighting vector cannot be compared with other similarity measurements since other measurements are independent from any preconceived set of constraints. However, it is possible to construct a weighting vector that can mimic other similarity measurements. In Isaacson's study on similarity relations, he examines various similarity functions of the set 3-1 and compares this set with all the sets whose cardinalities are 3 through 9.[9] A weighting vector can be constructed to reflect the same relations. We suggest a weighting vector akin to

$$W = [1\ .5\ -2\ -2\ -2\ -2]$$

in order to stress that 3-1 has twice as many ic1s as ic2s, and since no other interval classes are included in this set, we suppress these by the entering −2 in the corresponding places. This results in a hierarchical listing of pcsets similar to 3-1 (table 6). It is interesting to note that, while our list is naturally not the same as Isaacson's, Lewin's, or Rahn's (as determined in Isaacson's article), a comparison of these hierarchical orderings illustrates that the order of the sets in Lewin's and Rahn's lists (figure 1) are similar to the orderings in our list (table 6).

Table 5a: Sets with the greatest weights for the weighting vector W = [5 0 4.5 −5 −7.5 −10].

Set-Class	Prime Form	IC Vector	Weight
5-1	(01234)	[432100]	24.00
6-1	(012345)	[543210]	21.00
4-1	(0123)	[321000]	19.50
4-3	(0134)	[212100]	14.00
5-2	(01235)	[332110]	11.50
3-1	(012)	[210000]	10.00
4-2	(0124)	[221100]	9.50
3-2	(013)	[111000]	9.50
7-1	(0123456)	[654321]	8.00
5-3	(01245)	[322210]	6.50
4-10	(0235)	[122010]	6.50
6-2	(012346)	[443211]	6.00
3-3	(014)	[101100]	4.50
4-4	(0125)	[211110]	2.00
5-4	(01236)	[322111]	1.50
5-10	(01346)	[223111]	1.00
5-8	(02346)	[232201]	−1.00
4-12	(0236)	[112101]	−1.00

However, Isaacson's list differs considerably from the others suggesting that his function might not reflect similarity as accurately. Again, while emphasizing that the weighting vector measures similarity only with respect to a set's ability to match abstract compositional requirements, even with a relatively simple weighting vector as the above, with 4 entries the same, 55 distinctions are being made among sets of cardinality 3 through 9.

To this point, we have seen how a weighting vector can be constructed to generate sets with particular interval content preferences. Now we will extend this approach so we can generate sets with specific trichord content preferences. To do this, we first construct 12-place vectors that reflect the trichord content of all the pcsets whose cardinalities are 4 through 10. For each set, the first entry, V1, of its trichord-content vector will be the total number of included 3-1 subsets, V2 included 3-2 subsets, . . . V12 included 3-12 subsets.[10] Thus 4-1 would be represented as

[2 2 0 0 0 0 0 0 0 0 0 0]

since there are precisely two 3-1 and two 3-2 trichord subsets included in 4-1. Similarly, 4-9 will have a vector with the number 4 in

Table 5b: Sets with the least weights for the weighting vector W = [5 0 4.5 −5 −7.5 −10].

Set-Class	Prime Form	IC Vector	Weight
10-6	(012346789A)	[888885]	−74.00
10-5	(012345789A)	[888894]	−71.50
10-4	(012345689A)	[888984]	−69.00
10-2	(012345678A)	[898884]	−63.00
9-8	(01234678A)	[676764]	−63.00
9-9	(01235678A)	[676683]	−63.00
9-12	(01245689A)	[666963]	−63.00
8-25	(0124678A)	[464644]	−62.00
9-5	(012346789)	[766674]	−60.50
6-35	(02468A)	[060603]	−60.00
10-3	(012345679A)	[889884]	−59.50
10-1	(0123456789)	[988884]	−59.00
8-9	(01236789)	[644464]	−57.00
8-16	(01235789)	[554563]	−57.00
8-24	(0124568A)	[464743]	−57.00
7-33	(012468A)	[262623]	−56.00
9-11	(0235679A)	[667773]	−56.00
9-4	(012345789)	[766773]	−55.00

the fifth entry since the only type of trichordal subset of 4-9 is 3-5, of which there are four. Note that, in each of these examples, the parent set has a total of four trichordal subsets. In general, a set whose cardinality is d will have

$$\binom{d}{3} = \frac{d(d-1)(d-2)}{6}$$

trichordal subsets.[11] Thus, all tetrachords will have four trichordal subsets. In addition, these vectors differentiate z-related sets. For example, the trichord-content vector components of the all-interval tetrachords differ in the 2nd and 3rd place and the 7th and 8th place.

Now we construct a 12-place weighting vector based on a preconceived set of trichord content preferences and take the dot product between this vector and all the trichord-content vectors. As before, this will result in a hierarchical list of pc sets with respect to the set of desired content preferences.

Suppose the 7-note diatonic scale (7-35) is our ideal large set. We then seek to reinforce its properties, its trichord content vector being

$$[0\ 4\ 0\ 4\ 2\ 3\ 8\ 2\ 5\ 1\ 6\ 0]$$

Table 6: Sets with the greatest weights for the weighting vector $W = [1\ .5\ -2\ -2\ -2\ -2]$ similarity to set-class 3-1.

Set-Class	Prime Form	IC Vector	Weight
3-1	(012)	[210000]	2.50
4-1	(0123)	[321000]	2.00
3-2	(013)	[111000]	−0.50
5-1	(01234)	[432100]	−0.50
3-6	(024)	[020100]	−1.00
4-2	(0124)	[221100]	−1.00
3-3	(014)	[101100]	−3.00
3-4	(015)	[100110]	−3.00
3-5	(016)	[100011]	−3.00
3-7	(025)	[011010]	−3.50
3-8	(026)	[010101]	−3.50
3-9	(027)	[010020]	−3.50
4-3	(0134)	[212100]	−3.50
4-4	(0125)	[211110]	−3.50
4-5	(0126)	[210111]	−3.50
4-6	(0127)	[210021]	−3.50
5-2	(01235)	[332110]	−3.50
4-10	(0235)	[122101]	−4.00
4-11	(0135)	[121110]	−4.00
4-21	(0246)	[030201]	−4.50
6-1	(012345)	[543210]	−5.00
3-10	(036)	[002001]	−6.00
3-11	(037)	[001110]	−6.00
3-12	(048)	[000300]	−6.00
4-7	(0145)	[201210]	−6.00
4-8	(0156)	[200121]	−6.00
4-9	(0167)	[200022]	−6.00
5-3	(01245)	[322210]	−6.00
5-4	(01236)	[322111]	−6.00

The entries in our weighting vector will correspond to the eight distinct values of the trichord-content vector on a scale from -3 to $+5$: all 0s in the trichord-class vector are shifted to -3; all 1s are shifted to -2; etc., thus producing the weighting vector

$$W = [-3\ 1\ -3\ 1\ -1\ 0\ 5\ -1\ 2\ -2\ 3\ -3].$$

Many of the high-ranking sets generated by this weighting vector are expected (table 7a). For the most part, the top 16 sets shown here are

Figure 1: Isaacson's ICVSIM, Lewin's REL2, and Rahn's AMEMB2 comparing all sets to SC3-1 (from Isaacson's "Similarity of Interval-Class Content between Pitch-Class Sets: The ICVSIM Relation, JMT 34/1).

ICVSIMSets		**REL2**	**Sets**	**AMEMB2Sets**	
0.00	3-1	1.000	3-1	1.000	3-1
0.37	5-4 5-5	0.9107	4-1	0.889	4-1
0.50	4-1 4-2 4-4 4-5	0.8326	5-1	0.833	3-2
0.58	3-2 6Z3 6Z4 6-5	0.8047	3-2 4-2	0.778	4-2
	6Z12 6Z36 6Z37	0.7696	6-1	0.769	5-1
	6Z41 7-4 7-5	0.7634	5-2	0.692	5-2
0.69	5-2 5-3 5-6 5-9	0.7198	6-2	0.667	4-3 4-4 4-5 4-6
	5Z12 5Z36 8-5	0.7182	7-1		4-10 4-11 6-1
0.76	4-3 4-6 4-11 4Z15	0.7071	4-3 4-4 4-5 4-6	0.615	5-3 5-4 5-5 5-8
	4Z29	0.7054	5-3 5-4 5-5		5-9
0.82	3-3 3-4 3-5 6-2	0.6814	5-8 5-9	0.611	6-2
	6-9 6Z10 6Z11	0.6801	7-2	0.583	7-1
	6Z17 6Z39 6Z40	0.6798	6z3 6z4 6z36 6z37	0.556	4-12 4-13 4-14
	6Z43 7-6 7-9	0.6755	8-1		4z15 4-16
	7Z12 7-15 7-19	0.6667	4-10 4-11		4z29 6z3 6z4
	7Z36	0.6633	6-8 6-9		6-8 6-9 6z36
0.90	5-1 5-8 5-10 5-11	0.6504	7-3 7-4 7-5		6z37
	5-13 5-14 5-15	0.6452	8-2	0.542	7-2
	5Z18 5-19 5Z38	0.6395	9-1	0.538	5-6 5-7 5-10
	8-6 8-9 8Z15	0.6381	7-8 7-9		5-11 5z12 5-13
	8Z29	0.6325	6-5 6z6 6-7 6z38		5-14 5-15 5-23
0.96	4-7 4-8 4-10 4-12	0.6298	5-6 5-7		5-24 5z36
	4-13 4-14 4-16 9-5	0.6234	5-10 5-11 5z12	0.516	8-1
1.00	3-6 3-7 3-8 6Z6		5-13 5-14 5-15	0.500	3-3 3-4 3-5 3-6
	6Z13 6-18 6Z38		5z36 6z10 6z11		6-5 6z6 6-7
	6Z42 7-2 7-3 7-7		6z12 6z39 6z40		6z10 6z11
	7-8 7-10 7-13 7-14		6z41		6z12 6-21 6-22
	7Z18 7Z28 7Z38	0.6219	8-3 8-4 8-5 8-6		6z38 6z39
1.07	5-7 5-16 5-20 5-24	0.6166	7-6 7-7		6z40 6z41 7-3
	5-28 8-1 8-2 8-4	0.6146	9-2		7-4 7-5 7-8 7-9
	8-8 8-12 8-13 8-13	0.6123	8-10 8-11	0.484	8-2
	8-16	0.6083	7-10 7-11 7z12	0.462	5-16 5z17
1.12	4-9 4-18 9-1 9-8		7-13 7-14 7-15		5z18 5-19 5-20
1.15	3-9 3-11 6-1 6-8		7z36		5-25 5-26 5-27
	6-15 6-16 6Z23	0.6055	9-6		5-28 5-29 5-30
	6Z24 6Z45 6Z46	0.5973	8-21		5z37 5z38 9-1
	7-1 7-11 7-16 7-20	0.5963	6-21 6-22	0.458	7-6 7-7 7-10
	7-24	0.5962	8-7 8-8 8-9		7-11 7Z12
1.21	5Z17 5-25 5-26	0.5057	9-3 9-4 9-5		7-13 7-14 7-15
	5-29 5-30 5Z37	0.5903	7-23 7-24		7-23 7-24 7-33
	8-3 8-11 8-18 8-25	0.5890	8-12 8-13 8-14		7Z36
			8z15 8-16 8z29		

larger diatonic sets. However, despite the fact that we sought to reinforce the trichordal content of 7-35, 8-23 is the set best suited to the characteristics reflected by the weighting vector. This result is primarily due to the heavy weight we give to 3-7, 3-9, and 3-11, along with the suppression of 3-12 in our weighting vector.[12] In addition, relatively small sets such as the three hexachords and the pentachord 5-35 are highly ranked in our list. This pentachord is present because of its abundant trichords 3-7 and 3-9 and its lack of suppressed trichords.

It is interesting that a small shift in weighting will yield quite different results. Suppose we decide to suppress chromaticism even further by suppressing 3-1, 3-2 and 3-3 to a greater extent while still seeking diatonic elements, thus producing a vector like

$$W = [-5\ -3\ -4\ 1\ -1\ 0\ 5\ -1\ 2\ -2\ 3\ -3].$$

Only three entries have been changed. By comparing the previous results (table 7a) with our new results (table 7b), we see that the diatonic sets are represented by both weightings. However, our injunction against too much chromaticism has now brought the diatonic heptachord, 7-35, to the top. Additionally, a much larger percentage of smaller sets (about 50%), cardinalities 4 through 6, are among the top sets satisfying these very specific requirements.

All of the above suggests that the weighting vector has applications with respect to compositional and analytical intent. If applied with careful consideration, it has the potential of revealing properties of sets with respect to some rather varied criteria and should point the way towards comparing sets on the basis of operations and transformations which they may be best suited for. We also point out that the techniques we have discussed can continue to be extended, and weighting vectors can be used in conjunction with not only dyad-content (interval-class) and trichord-content vectors, but quite generally with polychord-content vectors, thus discovering previous hidden content and similarity properties of set-classes. The need for more complete descriptions of the properties of individual set-classes should be obvious.

MICROTONAL SCALE STRUCTURE

Now we shift our attention to extending certain structural properties that exist in the 12-note universe to microtonal universes. A theory that extends the "maximally even" property of many well known sets in the 12-note universe to sets in other universes was first advanced by Clough and Douthett.[13] These well known sets include the tritone, augmented triad, diminished seventh chord, the anhemitonic

Table 7a: List of sets satisfying diatonic requirements of the weighting vector W = [−3 1 −3 1 −1 0 5 −1 2 −1 3 −3] acting on the trichord-content vector of this set.

Set-Class	Prime Form	TC Vector	Weight
8-23	(0123578A)	[262664A46280]	72.00
7-35	(013568A)	[040423825160]	70.00
9-9	(0123678A)	[486AA5C873A1]	65.00
6-32	(024579)	[020202604040]	54.00
8-22	(012356A)	[264644965271]	53.00
9-7	(0123457A)	[4A8885B865A1]	50.00
7-23	(0234579)	[242323724150]	47.00
8-26	(0134578A)	[166643845481]	64.00
6-33	(023579)	[010112523130]	38.00
8-14	(01245679)	[356763844271]	37.00
7-29	(0124679)	[132352633250]	36.00
7-27	(0124579)	[134422624151]	36.00
6-25	(013568)	[030321412130]	34.00
9-6	(01234568A)	[5A8867AC5382]	33.00
7-34	(013468A)	[042223663241]	33.00
5-35	(02479)	[000001403020]	32.00

pentatonic (black keys), whole-tone, diatonic, and octatonic scales. Using completely different techniques, several microtonal theorists have constructed scales that belong to subfamilies of the family of all maximally even sets. (We shall discuss this later in some detail.) Moreover, in a recent paper, Clough, Douthett, Ramanathan, and Rowell discovered the maximally even property in the grama (scale) structure of the ancient Indian 22-note microtonal system.[14] All this gives precedence for exploring the concept of evenness in more detail.

Clough and Douthett constructed an algebraic algorithm that generated what they called "maximally even sets." For each cardinal family, there is a unique set-class whose members are maximally even. Although the accuracy of this terminology is supported by good intuition, Clough and Douthett were without a measurement to compare sets in a given cardinal family to verify their intuition. By constructing a weighting vector, we can rank all the sets in a given cardinal family with respect to their evenness. In doing so, we justify the term "maximally even sets" as defined by Clough and Douthett. Moreover, this measurement allows us to extend other familiar sets that are "close-to-maximally even" in the 12-note universe to microtonal universes.

Perhaps the most intuitive evenness measurement was first suggested by Douthett and Entringer and is based on the chord lengths

Table 7b: List of sets satisfying diatonic requirements of the weighting vector W = [−5 −3 −4 1 −1 0 5 −1 2 −1 3 −3] acting on the trichord-content vector of this set.

Set-Class	Prime Form	TC Vector	Weight
7-35	(013568A)	[040423825160]	54.00
6-32	(024579)	[020202604040]	46.00
8-32	(0123578A)	[262664A46280]	42.00
5-35	(02479)	[000001403020]	32.00
6-33	(023579)	[010112523130]	30.00
7-23	(0234579)	[242323724150]	25.00
6-25	(013568)	[030321412130]	22.00
8-22	(012356A)	[264644965271]	21.00
6-26	(013578)	[020422222040]	20.00
5-27	(01358)	[010201201030]	20.00
7-27	(0124579)	[134422624151]	20.00
9-9	(0123678A)	[486AA5C873A1]	19.00
7-29	(0124679)	[132352633250]	18.00
5-23	(02357)	[020101201030]	17.00
4-26	(0358)	[000000200020]	16.00
7-34	(013468A)	[042223663241]	15.00

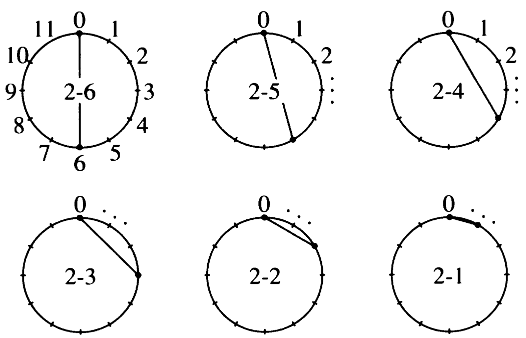

Figure 2. Distribution of dyads about octave circles.

between pairs of pcs in a pcset on the octave circle.[15] Consider the family of pcsets whose members have cardinality 2 (figure 2). Intuitively, the tritone represents the most even distribution of points about the octave circle. The sequence of sets 2-6, 2-5, 2-4, 2-3, 2-2, and 2-1 ranks the sets from most even to least even. The corresponding chord lengths between the pcs of the dyads go from longest to shortest. Thus, we can determine which of two dyads is most evenly distributed by comparing the lengths of the chords connecting their pcs; the longer the chord, the more even the distribution. This measurement of evenness can be extended to other cardinal families. Consider the family of pcsets whose members have cardinality 4. For ease in calculation, we assume the octave circle is a unit circle, and we measure the chord lengths between every pair of pcs in the pcset and add them together. We will call this sum the ***weight*** of the pcset. Then a ***maximally even set*** in this family has the maximum weight and a ***minimally even set*** has the minimum weight. The maximally and minimally even tetrachords are the diminished seventh chords and the tetrachord clusters respectively (figure 3). Douthett and Entringer have shown that this definition of maximally even sets is equivalent to the definition given by Clough and Douthett. It is not difficult to see that minimally even sets are, in general chord clusters. Moreover, z-related pairs have the same evenness ranking. In these all interval tetrachords (figure 4), the chord lengths AB = A′B′, AC = C′D′, AD = B′D′, BC = A′C′, BD = A′D′, and CD = B′C′. Thus

$$\mathrm{AB} + \mathrm{AC} + \mathrm{AD} + \mathrm{BC} + \mathrm{BD} + \mathrm{CD} =$$
$$\mathrm{A'B'} + \mathrm{A'C'} + \mathrm{A'D'} + \mathrm{B'C'} + \mathrm{B'D'} + \mathrm{C'D'}.$$

It is clear that this measurement is invariant under transposition and inversion.

We proceed to construct a 6-place weighting vector that will allow us to compute the weight (total chord length) of a set. The first coordinate of this vector is the length of a chord connecting a half-step; the second coordinate is the length of a chord connecting a whole-step; and so on. With a little trigonometry it is a simple matter to compute the coordinates of the weighting vector:

$$W_{12} = [w_1\ w_2\ w_3\ w_4\ w_5\ w_6]$$

where $w_k = 2\sin(k\pi/12)$. The subscript 12 indicates the weighting vector is designed for the 12-note universe.[16] Now we can compute the weight of a given pcset by calculating the dot product between its interval-class vector and the weighting vector. Using this technique, the weight of the diminished seventh chord is

$$[0\ 0\ 4\ 0\ 0\ 2] \cdot [.52\ 1.00\ 1.41\ 1.73\ 1.93\ 2] = 9.64,$$

and the weight of the tetrachord cluster is

$$[3\ 2\ 1\ 0\ 0\ 0] \cdot [.52\ 1.00\ 1.41\ 1.73\ 1.93\ 2] = 4.97.$$

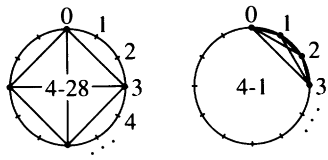

Figure 3: Distribution of a diminished seventh chord and a tetrachord cluster about octave circles.

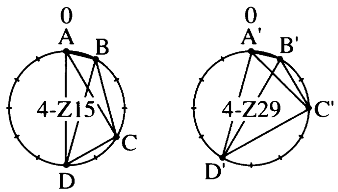

Figure 4: Distribution of the all interval tetrachords about octave circles.

These weights constitute the maximum and minimum weights for the family of sets with cardinality 4. Since the two all interval tetrachords have the same interval-class vector, they have the same weight:

$$[1\ 1\ 1\ 1\ 1\ 1] \cdot [.52\ 1.00\ 1.41\ 1.73\ 1.93\ 2] = 8.59.$$

It is easy to extend this construction to other universes. If c is the size of the universe, then the weighting vector is

$$W_c = [w_1\ w_2\ \ldots\ w_{[c/2]}]$$

where $w_k = 2\sin(k\pi/c)$ and the function $[c/2]$ is the floor function (also known as the greatest integer function).

In the 12-note universe, the weighting vector reveals that heptachords that rank just below the maximally even diatonic set also appear frequently in musical structures. The three heptachords that rank just under the diatonic set are, in order of evenness; 7-34, the ascending melodic minor; 7-33, the whole-tone plus one; and 7-32, the harmonic minor.

Although Clough and Douthett were the first to suggest an evenness factor in scale construction, they were not the first to construct these sets. It is surprising to discover that several theorists exploring microtonal scale structure, Yasser, Balzano, Clough and Myerson, and Agmon, have, without a notion of evenness or measurement

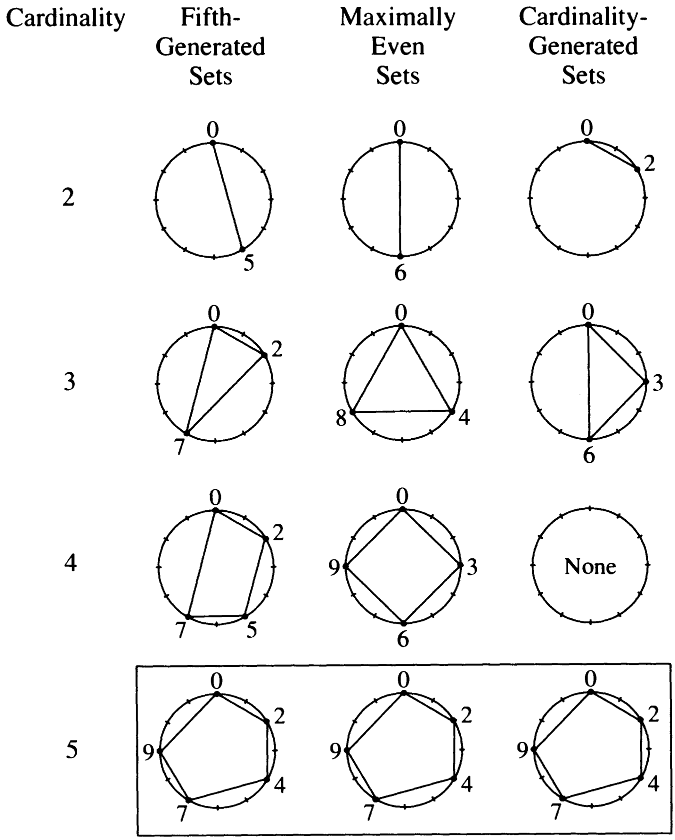

Figure 5: Fifth-generated, maximally even, and cardinality-generated sets.

thereof, constructed subfamilies of maximally even sets.[17] This observation allows us to make some interesting comparisons in the works of these theorists.

The sets constructed by each of these theorists have certain properties that parallel those inherent in the anhemitonic pentatonic and diatonic sets. Fifth generation is one such property. We observe an interesting relationship between the fifth-generated family of sets (figure 5, column 1) and the maximally even family (figure 5, column 2), cardinalities 2 through 10. There are only two sets common to both families; they are the anhemitonic pentatonic and diatonic sets. Key-

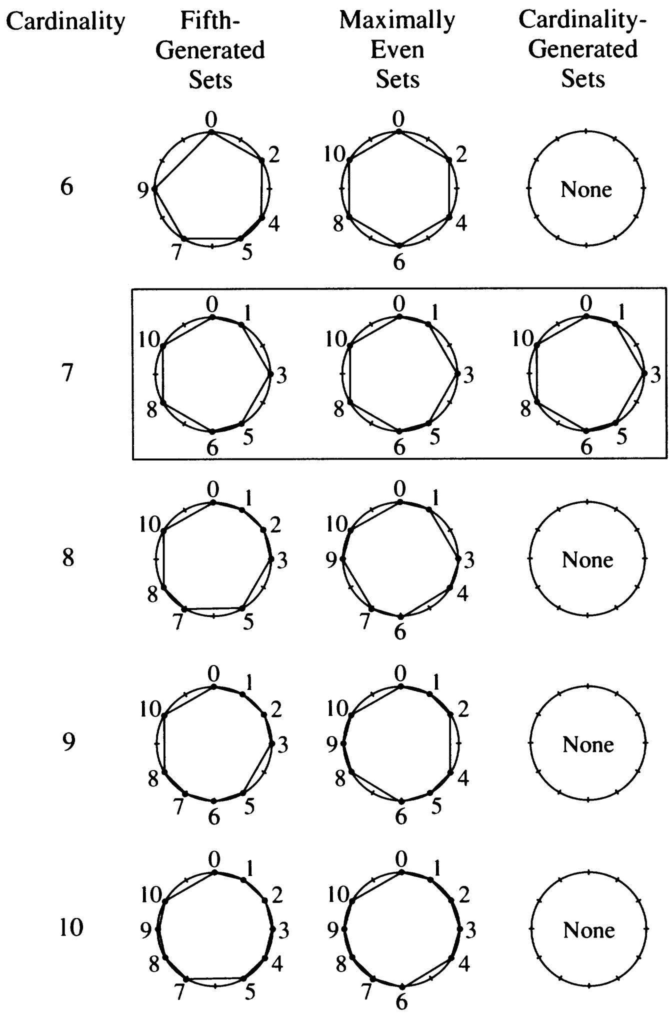

Figure 5: (continued)

board construction reflects this relationship; that is, the black and white keys form two complementary sets that are both fifth-generated and maximally even. Interestingly, 12 is not the only chromatic cardinality in which such keyboard construction has been suggested. The patterns of black and white keys in Yasser's keyboards reflect precisely the same relationship.

Another property shared by the anhemitonic pentatonic and diatonic sets is ***cardinality-generation;*** that is, their set cardinalities are the chromatic lengths of their generators: 5 half-steps is a generator for the pentatonic set, and 7-half steps, a generator for the diatonic set. Now, observe the relationship between the family of maximally even sets (figure 5, column 2) and the family of sets that are cardinality-generated (figure 5, column 3). Again, only the anhemitonic pentatonic and diatonic sets are common to both families. If this property is extended to other chromatic universes, the following is true: (1) if the cardinality of the universe can be expressed as the product of two consecutive integers, then we have precisely the relationship that exists in Balzano's sets and their complements; (2) if we require the size of the chromatic universe be divisible by 4, then the case is precisely that of Agmon's sets and their complements as well as the (hyper)pentatonic and (hyper)diatonic sets of Clough and Douthett.

Clough and Myerson were interested in generalizing certain combinatorial properties inherent in the diatonic set. Their approach led them to an algebraic algorithm that generates a subclass of maximally even sets, namely, those pcsets where the chromatic and pcset cardinalities are coprime.

Although little is known about the precise relationship, there is a strong link between maximally even sets and continued fraction approximations, in particular, approximations to the pure fifth.[18] This approach will produce fifth-generated maximally even sets. Preliminary work suggests that, in at least one instance other than the 12-note universe, these sets are also cardinality-generated; that is, they possess all three properties of the usual pentatonic and diatonic sets illustrated in figure 5.

The above discussion suggests that these weighting vectors can be used to construct scale structure in microtonal universes. One possible scenario is as follows:

A chromatic universe and two complementary maximally even sets within that universe can be selected. Using an approach essentially opposite to that of Yasser, a synthetic set of partials harmonious with the chosen sets can be constructed. Finally, using vector products, subfamilies of close-to-maximally even sets from the cardinal families of the chosen sets are found. At this point, it would be the composer's turn to devise a method of composition, perhaps using the new synthetic tonality with a redefined consonance, or using other maximally even or close-to-maximally even sets within the universe, or a set-class approach, or a combination of these. The rules belong to the composer!

NOTES

1. See endnote 6 in Allen Forte's "A Theory of Set-Complexes for Music," *Journal of Music Theory* 8 (1964): 136–183.
2. We define a ***vector product*** as follows:

   Let $A = (a_1,a_2,\cdots,a_n)$ and $B = (b_1,b_2,\cdots,b_n)$.
   Then the vector product between A and B is

$$A \cdot B = \sum_{k=1}^{n} a_k b_k = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n.$$

3. John Clough and Jack Douthett's paper "Maximally Even Sets," *Journal of Music Theory* 35 (1991): 93–173 defined ***maximally even sets,*** but they were without a measurement to compare the "evenness" of sets to justify their definition. We construct an "evenness" weighting vector that will enable us to compare sets in the same cardinal family.
4. David Lewin's "Forte's Interval Vector, My Interval Function, Regener's Common Note Function," *Journal of Music Theory* 21 (1977): 194–237 was the first to explore trichord content with respect to set-classes.
5. Eric J. Isaacson's "Similarity of Interval-Class Content Between Pitch-Class: The ICVSIM Relation," *Journal of Music Theory* 34 (1990): 1–28 contains a useful summary of the literature on similarity relations as well as his own proposed measurement.
6. John Rahn. "Relating Sets," *Perspectives of New Music* 18 (1979–80): 483–98.
7. A weighting vector might use any values: small, large, negative, positive, real numbers, rational numbers, or integers. During the course of this discussion, various types of vectors are used as a demonstration of this property, beginning with the first vector which simply used three expressions: negative, positive and neutral. A scaling could be sought, but the results would be the same if the proportions remain the same; a weighting vector of

$$W = [1\ 1\ -1\ 1\ -1\ -1]$$

   will produce the same hierarchical ordering as that of

$$W = [5\ 5\ 1.25\ 5\ 1.25\ 1.25] \text{ or } W = [100\ 100\ 25\ 100\ 25\ 25].$$

   In the practical usage of applying a weighting vector to ic vectors, however, the general scale from -3 to $+3$, using rational numbers with no more than two places works well.

8. Allen Forte. *The Structure of Atonal Music* (New Haven: Yale University Press, 1973), 200–208.
9. Isaacson, 1–28.
10. Lewin, 194–237.
11. The notation $\binom{n}{m}$, read "n choose m," represents the number of distinct subsets of cardinality m included in a set of cardinality n. It is well known that

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

Thus, for $d \geq 3$,

$$\binom{d}{3} = \frac{d!}{3!(d-3)!} = \frac{1 \cdot 2 \cdot 3 \cdots d}{(1 \cdot 2 \cdot 3)(1 \cdot 2 \cdot 3 \cdots (d-3))} = \frac{d(d-1)(d-2)}{6}.$$

12. The result suggests that the analyst needs to develop a more accurate description of the 7–35 set with respect to its tetrachord properties.
13. Clough and Douthett, 93–173.
14. J. Clough, J. Douthett, N. Ramanathan, and L. Rowell. "Early Indian Heptatonic Scales and Recent Diatonic Theory," *Music Theory Spectrum* 15 (1993): 36–58.
15. In ongoing research, Jack Douthett and Roger Entringer have constructed several systems for measuring evenness. Some are based on chord lengths between pcs on the octave circle (as is the one we will use), others on arc lengths between pcs, and still others are based on the ***interval spectrums*** of a set as defined in J. Clough and G. Myerson's paper "Variety and Multiplicity in Diatonic Systems," *Journal of Music Theory* 29 (1985): 249–270 and adopted by Clough and Douthett in their paper "Maximally Even Sets". There are four criteria used to determine a good evenness measurement: (1) the measurement should be invariant under transposition and inversion (i.e. Every set in a given set-class has the same weight); (2) for a given cardinal family, chromatic clusters should attain either the minimum or maximum weight; (3) if the set cardinality divides the chromatic cardinality, the sets that are evenly distributed (e.g. the augmented triad, diminished seventh chord, etc.) should attain the maximum or minimum weight respectively; and (4) no other set in the cardinal family should attain either the minimum or maximum weight. Note that (3) and (4) are dependent on the set cardinality dividing the chromatic cardinality, but the systems Douthett and Entringer have constructed are consistent with respect to minimally and maximally even sets even when this condition is not met. As it turns out, a minimally even set is a chord cluster, and a maximally even set is the same as that defined by Clough and Douthett. Thus, although the rankings of sets may vary from one system to another, the sets with extreme weights are always minimally and maximally even sets.
16. Another evenness weighting vector of particular interest is

$$V_{12} = [v_1\ v_2\ v_3\ v_4\ v_5\ v_6]$$

where $v_k = \frac{1}{k_2} + \frac{1}{(12-k)^2}$. In this case, for a given cardinal family, the extremal sets are still minimally and maximally even sets, but the weights are reversed; that is, a minimally even set will have minimum weight with respect to W_{12} but maximum weight with respect to V_{12}, and visa versa for a maximally even set. Given that the weights are reversed, minimally are maximally even sets are the same with respect to both vectors. However, excluding the extremal sets, the evenness ranking of a set with respect to V_{12} may be different from its ranking with respect to W_{12} (although in general, the rankings do not differ by much). The evenness ranking with respect to V_{12} is of particular interest, for if Forte had listed z-related pairs together, the vector V_{12} would generate Forte's set-class table.

17. J. Yasser, *A Theory of Evolving Tonality* (New York: American Library of Musicology 1932). G. Balzano, "The Group-theoretic Description of 12-fold and Microtonal Systems," *Computer Music Journal* 4 (1980): 66–84. J. Clough and G. Myerson, "Variety and Multiplicity in Diatonic Systems," *Journal of Music Theory* 29 (1985): 249–270. E. Agmon, "A Mathematical Model of the Diatonic System," *Journal of Music Theory* 33 (1989): 1–25.
18. In his doctoral dissertation, *Rhythmic Implications of Diatonic Theory : A Study in Scott Joplin's Ragtime Piano Works* (State University of New York, Buffalo: 1992), Marc Wooldridge explores some of these relationships. In addition, N. Carey and D. Clampitt's "Aspects of Well Formed Scales," *Music Theory Spectrum* 11 (1989): 187–206 contains a unique approach relating group automorphisms and continued fraction convergents of the pure fifth. This approach could also prove useful in understanding these connections.