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Local Near Sets: Pattern Discovery in Proximity Spaces

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Abstract The focus of this article is on various approaches to discerning patterns in nonempty sets endowed with a proximity (nearness) relation. Patterns arise in repetitions of some form in the arrangement of the parts of a set. To simplify the steps leading to pattern discovery, an approach inspired by M. Katětov is used, where one proximitises certain parts of a nonempty set, rather than proximitise the whole set. In effect, this is a divide-and-conquer approach to pattern discovery. This leads to a study of patterns that are collections of near sets. An important practical outcome of this approach is the discovery of patterns in proximity spaces.

Keywords Description · Local pattern · Near sets · Proximity space

Mathematics Subject Classification (2010) Primary 54E05; Secondary 03C13 · 54F65

1 Introduction

This article considers approaches to discerning patterns in nonempty sets. Patterns arise in repetitions of some form in the arrangement of the parts of a set. Both descriptive and traditional spatial proximity (nearness) of sets are considered. Perhaps the most pleasing proximity relation is description-based. A description-based proximity relation makes it possible to classify spatially disjoint sets based on the resemblances between members of the disjoint sets. The results in this paper stem from recent work on near sets [1–5] as well as [6–14].

The focus of this work is on Efremovič proximity with properties that take into account near sets as well as remote sets [6,15–17,1]. The traditional view of the nearness of sets is spatial, where sets are near, provided the sets have elements in common. The more recent view of the nearness of sets is descriptive, where sets are near, provided the sets descriptively resemble each other [1,2,11,12,18,19]. A *proximity structure* (X, δ) is obtained by endowing a nonempty set X with a nearness relation δ. This is essentially the Čech view of structure [20, §7A.1].

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The basic approach to pattern discovery in this work is inspired by M. Katětov, where one proximitises certain parts of a nonempty set. Katětov introduced merotopic spaces by topologising parts of a nonempty set [21]. Our perception of nearness is sharpened by considering locally near sets, *i.e.*, proximity of subsets within regions-of-interest either within the same set or between regions-of-interest in disjoint sets. Let $\Im$ be a nonempty set, X a subset of $\Im$, $\mathfrak{P} \in \mathcal{P}^2(X)$ a collection of subsets in X. The collection $\mathfrak{P}$ is a *local pattern*, if and only if, there is a spatial or descriptive repetition in $\mathfrak{P}$. With the proposed approach to near sets, there is a rich yield both in terms of theoretic results and a broad spectrum of applications.

2 Preliminaries

Let X be a nonempty set of points, $\mathcal{P}(X)$ the powerset of X, $\mathcal{P}^2(X)$ the set of all collections of subsets of X. A single point $x \in X$ is denoted by a lowercase letter, a subset $A \in \mathcal{P}(X)$ by an uppercase letter, collection of subsets in $\mathcal{P}^2(X)$ by a round letter such as $\mathcal{B} \in \mathcal{P}^2(X)$. The *closure* of a subset $A \in \mathcal{P}(X)$ (denoted by cl$A$), defined by

$$\mathrm{cl}A = \{x \in X : x\ \delta\ A\},$$

i.e., cl$A$ is the set of all points x in X that are near A. Let δ on a nonempty set X denote a spatial nearness (proximity) relation. For $A, B \in \mathcal{P}(X)$, $A\ \delta\ B$ (reads A is spatially near B), provided $A \cap B \neq \emptyset$, i.e., the intersection of A and B is not empty (cl$A$ and cl$B$ have at least one point in common). The spatial proximity (nearness) relation δ is defined by

$$\delta = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \mathrm{cl}A\ \cap\ \mathrm{cl}B \neq \emptyset\}.$$

$A\ \underline{\delta}\ B$ (reads A far (remote) from B), provided cl$A$ and cl$B$ have no points in common such that $\underline{\delta} = \mathcal{P} \times \mathcal{P} \setminus \delta$. Sets that are far from each other are called *spatially remote* sets. The complement of a set $C \in \mathcal{P}(X)$ is denoted by C^c.

The relation δ is an *Efremovič proximity* [22] (also called an EF-proximity), if and only if, for $A, B, C \in \mathcal{P}(X)$, the following axioms hold.

(**EF**.1) $A\ \delta\ B$ implies A and B are not empty.
(**EF**.2) $A \cap B \neq \emptyset$ implies $A\ \delta\ B$.
(**EF**.3) $A\ \delta\ B$ implies $B\ \delta\ A$ (symmetry).
(**EF**.4) $A\ \delta\ (B \cup C)$, if and only if, $A\ \delta\ B$ or $A\ \delta\ C$.
(**EF**.5) Efremovič axiom:

$$A\ \underline{\delta}\ B \text{ implies } A\ \underline{\delta}\ C \text{ and } B\ \underline{\delta}\ C^c \text{ for some } C \in \mathcal{P}(X).$$

The structure (X, δ) is called an *EF-proximity space* (or, briefly, EF space).

Example 1 Let the set of cells X in Fig. 1 be endowed with a proximity relation δ, where $A, B, C \in \mathcal{P}(X)$. It can be observed that δ satisfies EF space axioms (EF.1) to (EF.4). Let $A \subset C^c$ and observe that

$$\begin{aligned}
&B\ \subset C, \text{ and}\\
&A\ \underline{\delta}\ B, \text{ hence}\\
&B\ \underline{\delta}\ C^c, \text{ and}\\
&A\ \underline{\delta}\ C.
\end{aligned}$$

Hence, δ satisfies axiom (*EF.5)*.

Example 2 **Natural Proximity** [23].

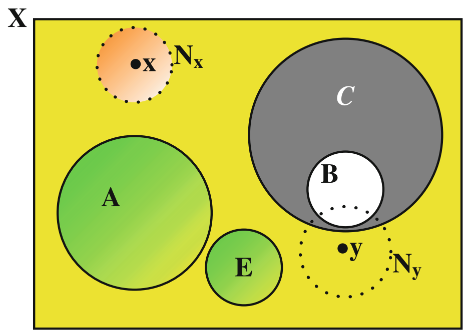


Fig. 1 $\boldsymbol{A\ \underline{\delta}\ B \implies}$
$\mathbf{A\ \underline{\delta}\ C}$ and $\mathbf{B\ \underline{\delta}\ C^c}$ for some $\boldsymbol{C \in \mathcal{P}(X)}$

A *diagonal uniformity* μ on X is a collection of subsets of $X \times X$ called surroundings. For $x \in X$, $A \subset X$, $U \in \mu$, define the following enlargements.

$$U[x] : \{y \in X : (x, y) \in U\}, \text{ and, } U[A] : \bigcup_{x \in A} U[x].$$

Any diagonal uniformity μ induces a natural proximity $\delta = \delta(\mu)$ defined by

$$A\ \delta\ B, \text{ if and only if, } U[A] \cap B \neq \emptyset, \text{ for each } U \in \mu.$$

Let $\varepsilon \in (0, \infty)$ and let $d : X \times X \to \mathbb{R}$ be a metric defined on X. A point y is sufficiently close to a point x, if and only if, $d(x, y) < \varepsilon$. Then, an *open neighbourhood* $N_\varepsilon(x)$ of a point $x \in X$ (for simplicity, we also write N_x as in [24]) is defined by

$$N_x = \{y \in X : d(x, y) < \varepsilon\}.$$

The number ε is usually called the radius of a neighbourhood of a point [25,26]. It is common practice, to call N_x a *spherical neighbourhood* [26, §1–4] or an *open ball* [27, §1.1] of a point x with radius ε such that $d(x, y) < \varepsilon$. A set A is *open*, if and only if, for each $x \in A$, all the points sufficiently near x belong to A [28]. All points y in N_x are sufficiently near x, since $d(x, y) < \varepsilon$. Hence, N_x is an example of an open set. Notice that N_x has a spatial character, since it is a set of all points y that are sufficiently near x, independent of the features of x and y. Hence, N_x is called a *spatial* open neighbourhood to distinguish it from what is known as a *descriptive* open neighbourhood (introduced in the sequel).

Example 3 **Near and Remote Spatial Open Neighbourhoods** [29].
Open neighbourhoods N_x, N_y with centres x, y, respectively, are represented by dotted circles in Fig. 1. N_y and B are near, i.e., $N_y\ \delta\ B$ in Fig. 1, since $\mathrm{cl}(N_y) \cap \mathrm{cl}B \neq \emptyset$, whereas N_x and B are remote, i.e., $N_x\ \underline{\delta}\ B$ in Fig. 1, since $\mathrm{cl}(N_x) \cap \mathrm{cl}B = \emptyset$.

3 Descriptively Near and Remote Sets

In the study of patterns, a descriptive form of EF-proximity is useful [2]. Let X be a nonempty set endowed with a descriptive proximity relation δ_Φ, $x \in X$, $A, B \in \mathcal{P}(X)$, and let $\Phi = \{\phi_1, \ldots, \phi_i, \ldots, \phi_n\}$, a set of probe functions $\phi_i : X \to \mathbb{R}$ that represent features of each x, where $\phi_i(x)$ equals a feature value of x. Let $\Phi(x)$ denote a feature vector for the object x, i.e., a vector of feature values that describe x, where

$$\Phi(x) = (\phi_1(x), \ldots, \phi_i(x), \ldots, \phi_n(x)).$$

A feature vector provides a description of an object. Let $A, B \in \mathcal{P}(X)$. Let $\mathcal{Q}(A)$, $\mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. For example,

$$\mathcal{Q}(A) = \{\Phi(a) : a \in A\},$$

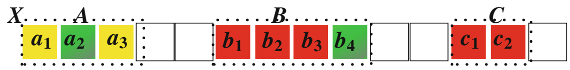


Fig. 2 $\Phi = \{colour\ probe\ fns\}$, $\text{cl}A \underset{\Phi}{\cap} \text{cl}B = \{a_2, b_4\}$, $\text{cl}A \underset{\Phi}{\cap} \text{cl}C = \{\}$

The expression $A\ \delta_{\Phi}\ B$ reads A *is descriptively near* B. The descriptive proximity of A and B is defined by

$$A\ \delta_{\Phi}\ B \Leftrightarrow \mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset.$$

Descriptive remoteness of A and B (denoted by $A\ \underline{\delta}_{\Phi}\ B$) is defined by

$$A\ \underline{\delta}_{\Phi}\ B \Leftrightarrow \mathcal{Q}(A) \cap \mathcal{Q}(B) = \emptyset.$$

Early informal work on the descriptive intersection of disjoint sets based on the shapes and colours of objects in the disjoint sets is given by N. Rocchi [30, p. 159]. The *descriptive intersection* $\cap_{\Phi}$ of A and B is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

The descriptive intersection will be nonempty, provided there is at least one element of A with a description that matches the description of a least one element of B. That is, a nonempty descriptive intersection of sets A and B is a set containing $a \in A$ and $b \in B$ such that $\Phi(a) = \Phi(b)$. Observe that A and B can be disjoint and yet $A\ \cap_{\Phi}\ B$ can be nonempty. In finding subsets $A, B \in \mathcal{P}(X)$ that are descriptively near, one considers descriptive intersection of the closure of A and the closure of B. That is, $\text{cl}A\ \cap_{\Phi}\ \text{cl}B$ implies $A\ \delta_{\Phi}\ B$. The descriptive proximity (nearness) relation δ_{Φ} is defined by

$$\delta_{\Phi} = \left\{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \text{cl}A \underset{\Phi}{\cap} \text{cl}B \neq \emptyset\right\}.$$

Example 4 **Descriptive Intersection of Disjoint Sets**

The coloured and white squares in Fig. 2 represent cells in a weave. A *cell* in a fabric is that part of a weave strand that overlaps another weave strand. The parallel strands of each layer in a weave are perpendicular to those strands in the other layer, making the cells square [31]. Choose Φ to be a set of probe functions representing weave cell colours. Let the set of cells X in Fig. 2 be endowed with δ_{Φ}. Notice that sets $A, B \in \mathcal{P}(X)$ are disjoint but the descriptive intersection is nonempty. That is, $\text{cl}A\ \cap_{\Phi}\ \text{cl}B = \{a_2, b_4\}$. Similarly, for $B, C \in \mathcal{P}(X)$, $\text{cl}B\ \cap_{\Phi}\ \text{cl}C = \{b_1, b_2, b_3, c_1, c_2\}$.

The descriptive remoteness of A and B (denoted by $A\ \underline{\delta}_{\Phi}\ B$) such that $\underline{\delta}_{\Phi} = \mathcal{P}(X) \times \mathcal{P}(X) \backslash \delta_{\Phi}$ is defined by

$$A\ \underline{\delta}_{\Phi}\ B \Leftrightarrow \text{cl}A \underset{\Phi}{\cap} \text{clcl}B = \emptyset.$$

Example 5 **Descriptively Remote Disjoint Sets**

Choose Φ to be a set of probe functions representing weave cell colours. In Fig. 2, sets $A, C \in \mathcal{P}(X)$ are disjoint. In addition, there are no cells in A with descriptions that resemble cells in C. Hence, the descriptive intersection is empty. That is, $A\ \underline{\delta}_{\Phi}\ C$ (A and C are remote), since $\text{cl}A\ \cap_{\Phi}\ \text{cl}C = \emptyset$.

A binary relation δ_{Φ} is a *descriptive EF-proximity*, provided the following axioms are satisfied for $A, B, C \in \mathcal{P}^2(X)$.

($\boldsymbol{EF_{\Phi}}$.1) $A\ \delta_{\Phi}\ B$ implies $A \neq \emptyset$, $B \neq \emptyset$.
($\boldsymbol{EF_{\Phi}}$.2) $A \underset{\Phi}{\cap} B \neq \emptyset$ implies $A\ \delta_{\Phi}\ B$.

($\boldsymbol{EF_\Phi}$.3) $A\ \delta_\Phi\ B$ implies $B\ \delta_\Phi\ A$ (descriptive symmetry).
($\boldsymbol{EF_\Phi}$.4) $A\ \delta_\Phi\ (B \cup C)$, if and only if, $A\ \delta_\Phi\ B$ or $A\ \delta_\Phi\ C$.
($\boldsymbol{EF_\Phi}$.5) Descriptive Efremovič axiom:

$$A\ \underline{\delta}_\Phi\ B \text{ implies } A\ \underline{\delta}_\Phi\ C \quad \text{and} \quad B\ \underline{\delta}_\Phi\ C^c \text{ for some } C \in \mathcal{P}(X).$$

The structure (X, δ_Φ) is a *descriptive EF-proximity space* (or, briefly, *descriptive EF space*).

Example 6 **Descriptive EF Space** Let the set of points X in Fig. 1 be endowed with a descriptive proximity relation δ_Φ, where $A, B, C \in \mathcal{P}(X)$. It can be observed that δ_Φ satisfies EF space axioms (EF$_\Phi$.1) to (EF$_\Phi$.4). Let $A \subset C^c$ and observe that

$B \subset C$,

$C\ \underline{\delta}_\Phi\ C^c$

$A \subset C^c$ and

$A\ \underline{\delta}_\Phi\ B$, hence

$B\ \underline{\delta}_\Phi\ C^c$, and

$A\ \underline{\delta}_\Phi\ C$.

Hence, δ_Φ satisfies axiom (*EF_Φ.5*).

Descriptive EF-proximity is useful in describing, analysing and classifying the parts within a single set or the parts in either near or remote sets. Applications of descriptive EF-proximity are given in the sequel (see, also, [1,2,4,5]).

3.1 Descriptive Open Neighbourhoods

This section briefly introduces open neighbourhoods defined in terms of the description of neighbourhood points. Let Φ be a set of probe functions that represent the features of points in a nonempty set X. Let (X, δ_Φ) be a descriptive EF space and let $\Phi(x), \Phi(y)$ denote feature vectors of length n for $x, y \in X$. Then the Manhattan distance d between descriptions $\Phi(x), \Phi(y)$ is defined by

$$d(\Phi(x), \Phi(y)) = \sum_{i=1}^{n} |\phi_i(x) - \phi_i(y)|\,.$$

Let $\varepsilon \in (0, \infty)$. A *descriptive open neighbourhood* of a point $x \in X$ (denoted $N_{\Phi(x)}$) is defined by

$$N_{\Phi(x)} = \{y \in X : d(\Phi(x), \Phi(y)) = 0 \text{ and } |x - y| < \varepsilon\}\,.$$

In other words, the points in a descriptive open neighbourhood are descriptively *indistinct*. In that case, each descriptively distinct point belongs to a different descriptive open neighbourhood. It is possible to compare descriptive open neighbourhoods.

For $A, B \in \mathcal{P}(X)$ and metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the Čech pseudometric $D_\Phi : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ is defined by

$$D_\Phi(A, B) = \inf\{d(\Phi(a), \Phi(b)) : a \in A, b \in B\}.$$

For $\varepsilon > 0$, descriptive open neighbourhoods $A, B \in \mathcal{P}(X)$ are *sufficiently near*, provided $D_\Phi(A, B) < \varepsilon$.

Example 7 **Remote Descriptive Open Neighbourhoods**

In Fig. 3, let X be endowed with a descriptive EF proximity relation δ_Φ. Choose Φ to be a set containing a probe function representing greylevel intensity. Let A and B in Fig. 3 represent descriptive open neighbourhoods. Neighbourhoods A and B are examples of descriptively remote sets, since all pixels in B have zero intensity (the pixels are black) and all pixels in A have greylevel intensity greater than zero, i.e., $A\ \underline{\delta}_\Phi\ B$.

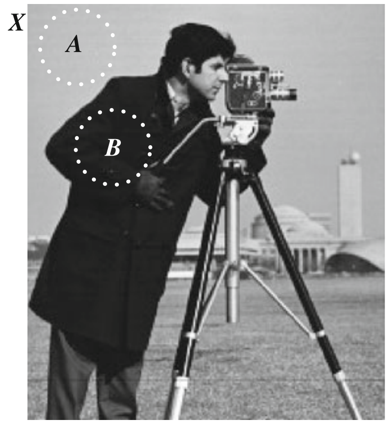


Fig. 3 Remote Descriptive Open Neighbourhoods $\boldsymbol{A}, \boldsymbol{B}$

4 Hausdorff Raster Spaces

Recall that *distinct points* belong to disjoint neighbourhoods in a Hausdorff T_2 space. It is helpful to distinguish between spatial and non-spatial distinctness. Let X be a nonempty set. Points $x, y \in X$ are *spatially distinct*, provided x and y are in different locations (separated spatially). Points x and y are considered *descriptively distinct*, provided x and y have different descriptions.

Example 8 **Spatially Distinct Points**

Let the spatial EF space (X, δ) be represented by Fig. 1. In Fig. 1, spatial open neighbourhoods are represented by N_x, N_y. Points $x, y \in X$ in Fig. 1 are spatially distinct, since x and y have different coordinates. Observe that $\{x\} \ \underline{\delta} \ \{y\}$, since $\{x\} \cap \{y\} = \emptyset$. Hence, points x and y are remote. Since all points in N_x, N_y spatially distinct, N_x, N_y are examples of spatially distinct neighbourhoods.

Example 9 **Descriptively Distinct Points**.

Choose Φ to be a set of probe functions that represent pixel intensities. Let the spatial EF space (X, δ_Φ) be represented by Fig. 3, where X contains descriptive open neighbourhoods A and B. All points $a \in A, b \in B$ in Fig. 1 are descriptively distinct, since a and b have different greylevel intensities. Observe that $\mathcal{Q}(A) \ \underline{\delta}_\Phi \ \mathcal{Q}(B)$, since $\mathcal{Q}(A) \cap \mathcal{Q}(B) = \emptyset$. Hence, neighbourhoods A and B are descriptively remote.

Lemma 1 *In a spatial EF proximity space, a nonempty set is spatially near itself.*

Lemma 2 *In a descriptive EF proximity space, a nonempty set is descriptively near itself.*

Proof Let (X, δ_Φ) be a nonempty descriptive proximity space. Let $A \in \mathcal{P}(X)$. Then, $A \ \delta_\Phi \ A$ for $A \in \mathcal{P}(X)$, since $\mathcal{Q}(A) \cap \mathcal{Q}(A) \neq \emptyset$. □

Lemmas 1 and 2 provide simple truths about the nearness of a set to itself and also prove useful in the defining what are known as spatial and descriptive patterns (see Sect. 8).

Lemma 3 *A point is descriptively near itself.*

Lemma 4 *In a descriptive EF proximity space, spatially distinct points can be descriptively near or remote.*

Theorem 1 *Descriptively remote points are spatially remote points.*

Theorem 2 *Spatially distinct neighbourhoods are remote.*

It is possible for spatially distinct neighbourhoods to be either descriptively near or remote.

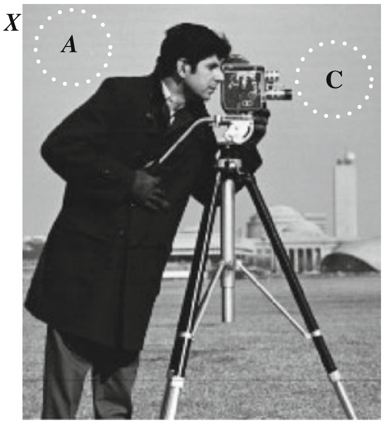


Fig. 4 Remote Descriptive Open Neighbourhoods A, C

Example 10 **Spatially Distinct and Descriptively Remote Neighbourhoods**.
Let the spatial EF proximity space (X, δ) and descriptive EF proximity space (X, δ_Φ) be represented by Fig. 4. That is, the set X of picture elements in Fig. 4 is endowed with the spatial proximity δ as well as the descriptive proximity δ_Φ. Let $A, C \in \mathcal{P}(X)$ be descriptive open neighbourhoods such that all picture elements in both neighbourhoods have matching descriptions. Notice that A, C are also examples of spatially distinct neighbourhoods, since all points $a \in A, c \in C$ are spatially distinct. Similarly, observe that neighbourhoods $A, B \in \mathcal{P}(X)$ represented in Fig. 3 are spatially distinct neighbourhoods that are descriptively remote, i.e., $A \not\delta_\Phi B$, since $\mathrm{cl}A \cap_\Phi \mathrm{cl}B = \emptyset$.

The generalisation of the line of reasoning in Example 10 leads to the result in Theorem 3.

Theorem 3 *Spatially distinct neighbourhoods can be descriptively near or remote.*

Let (X, δ_Φ) be a descriptive EF proximity space. Descriptive open neighbourhoods $A, B \in \mathcal{P}(X)$ are *descriptively remote neighbourhoods*, if and only if, $\mathrm{cl}A \cap_\Phi \mathrm{cl}B = \emptyset$. Given $a \in A, b \in B$, we know that $\Phi(a) \neq \Phi(b)$ (a and b are descriptively distinct). This leads to the following result.

Theorem 4 *In a descriptive EF proximity space, descriptively distinct points belong to descriptively remote neighbourhoods.*

From Theorem 4, observe that descriptively remote neighbourhoods are disjoint neighbourhoods. That is, descriptively remote neighbourhoods have no points in common.

Corollary 1 *In a descriptive EF proximity space, descriptively remote neighbourhoods have no points in common.*

Proof Immediate from Theorem 4. □

Let (X, δ_Φ) be a descriptive EF proximity space and let Φ be a set of probe functions representing features of points in X. The set X is a *descriptive Hausdorff space*, if and only if, descriptively distinct points belong to disjoint descriptive open neighbourhoods. Let $x, y \in X$ such that $d(\Phi(x), \Phi(y)) > 0$. By definition, points x, y are descriptively distinct. From Theorem 4, the x, y belong to descriptively remote neighbourhoods. Hence, the descriptive open neighbourhoods of x and y are disjoint. This leads to the following result.

Theorem 5 *A descriptive proximity space X is a descriptive Hausdorff (T_2) space.*

Recall that a *raster* is a rectangular pattern of scanning lines followed by an electron beam on a television screen or computer monitor. A *raster image* is a 2D array of numbers. Each number in a raster image is a pixel intensity in one of the lines in the raster. The points in a raster image are descriptively distinct by virtue of their individual intensities. From Corollary 1, descriptively distinct points in a raster image belong to disjoint descriptive neighbourhoods. Hence, from Theorem 5, obtain the following result.

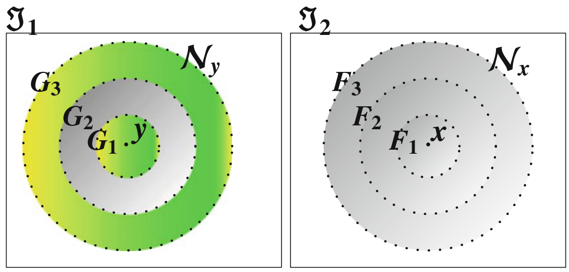


Fig. 5 Near Neighbourhood Filters $\mathcal{N}_y\ \delta_\Phi\ \mathcal{N}_x$

Theorem 6 *Every raster image is a descriptive Hausdorff space.*

This result has important implications in the study of pictures (see, *e.g.*, [2,12,17,32–34]). An Example of a Hausdorff raster space is shown in the picture in Fig. 4. An application of Hausdorff raster spaces is given in terms of the detection of patterns in pictures in [1].

5 Near Neighbourhood Filters and Camouflage Detection

Recall that a filter on a set X is a nonempty collection $\mathcal{F}$ of $\mathcal{P}(X)$ that has the following properties: $\emptyset \notin \mathcal{F}$; if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$; and if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. The collection of all neighbourhoods of a point $x \in X$ is called a *neighbourhood filter* (denoted $\mathcal{N}_x$). For example, sample neighbourhood filters $\mathcal{N}_x, \mathcal{N}_y$ are shown in Fig. 5, i.e.,

$$\mathcal{N}_x = \{F_1, F_2, F_3 : F_1 \subset F_2 \subset F_3\},$$
$$\mathcal{N}_y = \{G_1, G_2, G_3 : G_1 \subset G_2 \subset G_3\}.$$

A maximal filter on X is called an *ultrafilter* on X. A subcollection $\mathcal{B}$ in a filter $\mathcal{F}$ is a *filter base*, provided every member of $\mathcal{F}$ contains some element of $\mathcal{B}$. For example, in Fig. 5, F_1 is a filter base for the filter $\mathcal{N}_x$ and G_1 is a filter base for $\mathcal{N}_y$. Let $\varepsilon \in (0, \infty)$ and let δ_Φ denote a descriptive proximity defined in terms of Φ, a set of probe functions representing filter point features. Neighbourhood filters $\mathcal{N}_x, \mathcal{N}_y$ are *descriptively sufficiently near* neighbourhood filters (denoted $\mathcal{N}_x\ \delta_\Phi\ \mathcal{N}_y$), if and only if, $D_\Phi(A, B) < \varepsilon$ for some $A \in \mathcal{N}_x$, $B \in \mathcal{N}_y$ [12,29,35]. For example, in Fig. 5,

$$\mathcal{N}_x\ \delta_\Phi\ \mathcal{N}_y,\ \text{since } D(G_2, F_2) = 0 \text{ for } G_2 \in \mathcal{N}_y,\ F_2 \in \mathcal{N}_x.$$

That is, the description of points in G_2 matches the description of points in F_2. Hence, $\mathcal{N}_x\ \delta_\Phi\ \mathcal{N}_y$.

Let (X, δ) be an EF-proximity space, $A, B \in \mathcal{P}(X)$. B is a *proximal neighbourhood* A (denoted $A \ll B$), if and only if, $A\ \underline{\delta}\ (X - B)$ [6, §3, p. 15], [17]. This means that A is *strongly contained* in B. Further, B is a *descriptive proximal neighbourhood* of A (denoted $A \ll_\Phi B$), if and only if, $A\ \underline{\delta}_\Phi\ (X - B)$, i.e., A is descriptively remote (far) from $X - B$.

Example 11 **Proximal Neighbourhoods**

Let $X = \mathfrak{I}_1 \cup \mathfrak{I}_2$ in Fig. 6 and let δ denote an EF-proximity on X. Then, for example, $F_1 \ll F_3$ (F_3 is a proximal neighbourhood of F_1 in Fig. 6). Again, for example, $G_1 \ll_\Phi G_3$ (G_3 is a descriptive proximal neighbourhood of G_1 in Fig. 6.

Theorem 7 Disjoint Proximal Neighbourhoods *[17, §2.2].*
Any two remote sets have disjoint proximal neighbourhoods.

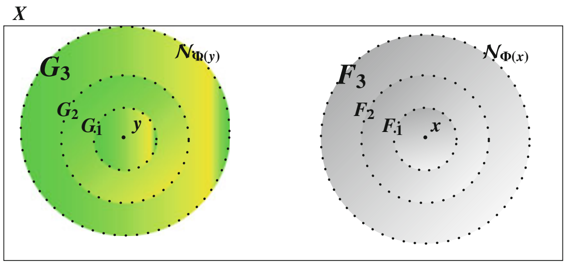


Fig. 6 Descriptively Remote Neighbourhood Filters

Fig. 7 Natural Camouflage Example: Dragonfly

Example 12 **Sample Disjoint Proximal Neighbourhoods**.

Let X in Fig. 6 be endowed with an EF-proximity relation δ_Φ. Choose Φ to be a set of probe functions that measure average greyscale intensity and various colours. Assume neighbourhoods $N_{\Phi(x)} \cap_\Phi N_{\Phi(y)} = \emptyset$ (descriptively remote) with proximal neighbourhoods $G_1 \ll G_2$ and $F_1 \ll F_2$, respectively. Since $N_{\Phi(x)}, N_{\Phi(y)}$ are descriptively remote (i.e., $\Phi(a) \neq \Phi(b)$ for each $a \in N_{\Phi(x)}, b \in N_{\Phi(y)}$), $G2$ and $F2$ are disjoint descriptive proximal neighbourhoods.

The observations in Example 12 generalise for any pair of descriptively remote sets and lead to the following result.

Theorem 8 **Disjoint Descriptive Proximal Neighbourhoods**.

Any two descriptively remote sets have disjoint descriptive proximal neighbourhoods.

Proof Let (X, δ_Φ) be a descriptive EF proximity space with descriptively remote sets A, B. Then $A \cap_\Phi B$ is empty. Let A, B contain proximal neighbourhoods E, E', respectively. Since A, B are descriptively remote, then E, E' are disjoint descriptive proximal neighbourhoods. □

Example 13 **Remote Neighbourhood filters**.

In Fig. 6, $G_1 \in \mathcal{N}_y$ and $F_1 \in \mathcal{N}_x$ are disjoint sets. From Theorem 7, G_1 and F_1 have disjoint proximal neighbourhoods. In this example, $G_1 \ll G_3$ and $F_1 \ll F_3$. It can also be observed that G_3 and F_3 are descriptively remote sets in $\mathcal{N}_y, \mathcal{N}_x$, respectively. Then, from Theorem 8, G_3 and F_3 have disjoint descriptive proximal neighbourhoods. This can be seen from the fact that $G_3 \cap_\Phi F_3$ is empty.

Descriptive proximal neighbourhoods play an important role in detecting discrepancies in comparing camouflage neighbourhood filters. We illustrate this in terms of a comparison between a camouflaged dragonfly centered in neighbourhood filter $\boldsymbol{\mathcal{N}_x}$ and a similar setting favoured by dragonflies centered in $\boldsymbol{\mathcal{N}_y}$ (see Fig. 8).

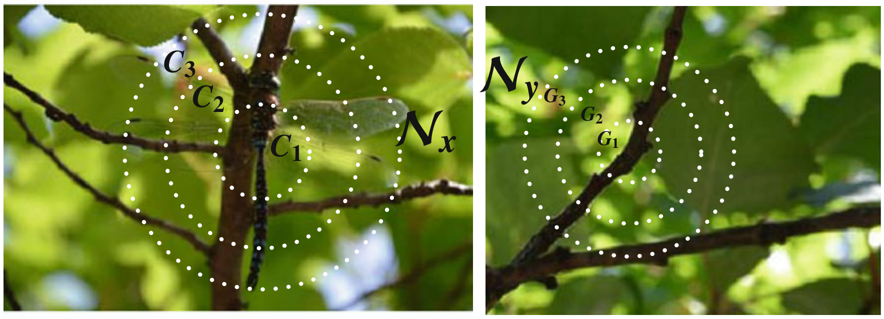


Fig. 8 Camouflage Neighbourhood Filters

The first step in the detection of camouflage in Fig. 7 is to consider a pair of disjoint neighbourhood filters $\mathcal{N}_x$, $\mathcal{N}_y$ superimposed on the zoomed versions of camouflaged object and non-camouflaged object (in this case, image of a dragonfly on a tree branch and image of a similar tree branch). This is done in Fig. 8. Next, let $X = \mathcal{N}_x \cup \mathcal{N}_y$ be an EF-proximity space and let δ_Φ be a descriptive EF-proximity on X.

Let Φ contain probe functions representing pixel colours (for simplicity, shape descriptors are omitted from Φ). Assume that neighbourhoods $C_2 \in \mathcal{N}_x$, $G_2 \in \mathcal{N}_\dagger$ are descriptively remote sets. From Theorem 8, we can expect to find a pair of descriptive proximal neighbourhoods. Observe that C_2 is a descriptive proximal neighbourhood of C_1, i.e., $C_1 \ll_\Phi C_2$. In addition, G_2 is a descriptive proximal neighbourhood of G_1 ($G_1 \ll_\Phi G_2$). These proximal neighbourhoods are descriptively disjoint in the sense that the descriptions of points in one proximal neighbourhood are not the same as the descriptions of the points in the other proximal neighbourhood. From a camouflage point of view, this is an indication that a camouflaged object has been detected in the region occupied by the coarser subsets in the neighbourhood filters. This leads to local proximal neighbourhood patterns applicable in the study of camouflage (see Example 21).

6 Descriptive Point Clusters

In an EF-proximity space X, a *cluster* $\mathcal{C}$ is a collection of subsets of X satisfying the following conditions.

(**cluster**.a) If $A, B \in \mathcal{C}$, then A is near B.
(**cluster**.b) If A is near every E in $\mathcal{C}$, then A belongs to $\mathcal{C}$.
(**cluster**.c) If $A \cup B$ belongs to $\mathcal{C}$, then either A or B belongs to $\mathcal{C}$.

Then a *point cluster* $\mathcal{C}_x$ is the collection of all sets close to x. Let X be endowed the descriptive EF-proximity relation δ_Φ. A *descriptive cluster* $\mathcal{C}_\Phi$ is a collection of subsets of X satisfying the following conditions.

(**cluster**.a) If $A, B \in \mathcal{C}_\Phi$, then $A\ \delta_\Phi\ B$, i.e., A is *descriptively near* B.
(**cluster**.b) If $A\ \delta_\Phi\ E$ for every E in $\mathcal{C}_{\Phi(x)}$, then $\mathcal{Q}(A)$ belongs to $\mathcal{Q}(\mathcal{C}_\Phi)$.
(**cluster**.c) If $A \cup B$ belongs to $\mathcal{C}_\Phi$, then either $\mathcal{Q}(A)$ or $\mathcal{Q}(B)$ belongs to $\mathcal{Q}(\mathcal{C}_\Phi)$.

The collection of all sets descriptively close to the description of a fixed point x is a *descriptive point cluster* (denoted by $\mathcal{C}_{\Phi(x)}$).

The application of descriptive point clusters takes advantage of the connection between compact EF proximity spaces and point clusters discovered by S. Leader [36] (see, also, [6,17]). An EF space is *compact*, if and only if, every open cover of the space has a finite subcover.

Theorem 9 Leader [36]. *An EF-proximity space X is compact, if and only if, every cluster from X is a point cluster.*

This leads to the following result for compact descriptive EF-proximity spaces.

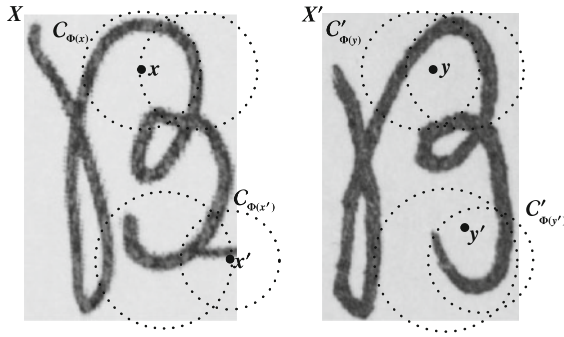


Fig. 9 Point Clusters $\mathcal{C}_{\Phi(x)}, \mathcal{C}_{\Phi(x')}$ and $\mathcal{C}'_{\Phi(y)}, \mathcal{C}'_{\Phi(y')}$

Theorem 10 *A descriptive EF-proximity space X is compact, if and only if, every descriptive cluster from X is a descriptive point cluster.*

Proof Symmetric with the proof of Theorem 9 given in [36]. □

The *descriptive intersection of point clusters* is defined by

$$\mathcal{C}_{\Phi(x)} \underset{\Phi}{\cap} \mathcal{C}'_{\Phi(y)} = \left\{ A \underset{\Phi}{\cap} E : A \in \mathcal{C}_{\Phi(x)}, E \in \mathcal{C}'_{\Phi(y)} \right\}.$$

Example 14 **Forgery Test**.

Let $\Im, \Im'$ be digital images containing handwriting specimens and select $X \in \mathcal{P}(\Im)$, $X' \in \mathcal{P}(\Im')$, respectively, where X, X' are compact descriptive EF spaces. In addition, let $\mathcal{C}_{\Phi(x)}, \mathcal{C}'_{\Phi(y)}$ denote descriptive point clusters in $\mathcal{P}^2(X), \mathcal{P}^2(X')$, respectively (see Fig. 9). Further, limit X, X' to a pair of cursive characters in handwriting samples.

We know that when the cursive characters in an original handwriting sample are compared with characters in a supposed duplicate, the duplicate is, in fact, a forgery, if point clusters containing the original and its copy are remote from each other. In other words, $\Im'$ is a forgery of the handwriting in $\Im$, provided a descriptive point cluster $\mathcal{C}_{\Phi(x)}$ in $\Im$ is remote (far from) the intersection of an original handwriting sample $\mathcal{C}_{\Phi(x)}$ and forged copy $\mathcal{C}'_{\Phi(y)}$, i.e.,

$$\mathcal{C}_{\Phi(x)} \, \underline{\delta}_{\Phi} \left(\mathcal{C}_{\Phi(x)} \underset{\Phi}{\cap} \mathcal{C}'_{\Phi(y)} \right).$$

That is, if the point clusters for the original and copy contain corresponding points that do not have matching descriptions, then a potential forgery is detected. Using this idea, a forgery test reduces to checking if

$$\mathcal{C}_{\Phi(x)} = \left(\mathcal{C}_{\Phi(x)} \underset{\Phi}{\cap} \mathcal{C}'_{\Phi(y)} \right).$$

Example 15 **Handwriting Forgery Detection**.

If two handwriting samples reveal a forgery, then set $\mathcal{C}_{\Phi(x)} \cap_{\Phi} \mathcal{C}'_{\Phi(y)}$ does not equal $\mathcal{C}_{\Phi(x)}$. That is, $\mathcal{C}_{\Phi(x)}$ contains points that are not descriptively near one or more points in $\mathcal{C}'_{\Phi(y)}$.

There are two handwritten round Bs in Fig. 9. To test, for example, whether the point cluster $\mathcal{C}'_{\Phi(y)}$ is a forgery of the upper part of the round B in the point cluster in $\mathcal{C}_{\Phi(x)}$ in Fig. 9, it is necessary to define the set of features Φ in terms of one or more shape descriptors. For this example, probe functions for roundness and gradient direction are chosen (*cf.*, [37, p. 582, p. 299, resp.]). It can be observed that $\mathcal{C}'_{\Phi(y)}$ contains a forgery of the upper part of the round B in the point cluster in $\mathcal{C}_{\Phi(x)}$ If we consider only roundness and gradient direction of the picture elements in the pair of descriptive point clusters $\mathcal{C}_{\Phi(x)}, \mathcal{C}'_{\Phi(y)}$, the forgery test reveals that

$$\mathcal{C}_{\Phi(x)} \neq \left(\mathcal{C}_{\Phi(x)} \underset{\Phi}{\cap} \mathcal{C}'_{\Phi(y)} \right).$$

The discrepancies between the round Bs in the two point clusters in Fig. 9 is visually evident, if we consider the roundness shape descriptor (i.e., the top of the B in $\mathcal{C}_{\Phi(x)}$ is more round than the top of the B in $\mathcal{C}'_{\Phi(y)}$). Again, for example, the gradient (slope) directions of the tops of the pair of Bs is markedly different. Hence, the intersection of the two point clusters will not equal the set of descriptions in $\mathcal{C}_{\Phi(x)}$). This leads to local point cluster patterns applicable in the study of handwriting forgery (see Example 22).

The proposed approach to forgery detection is analogous to the principal step in the approach proposed by J.-J. Brault and R. Plamondon [38], namely, weighting the perceptual importance of every signature point according to specific neighbouring points. By contrast, the proposed approach checks for the remoteness of point clusters defined relative to handwriting character points. This approach leads to a number of nearness measures useful in forgery detection. Here is one example. Let $\mathcal{C}_{\Phi(x)}, \mathcal{C}'_{\Phi(y)}$ be point clusters on regions-of-interest in handwriting specimens X and X'. Here is a nearness measure $\mu : \mathcal{P}^2(\mathbb{R}) \times \mathcal{P}^2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\mu(\mathcal{C}_{\Phi(x)}, \mathcal{C}'_{\Phi(y)}) = \frac{T}{D}, \text{ where}$$

$$T = \sum_{\substack{A \in \mathcal{C}_{\Phi(x)} \\ B \in \mathcal{C}'_{\Phi(y)}}} \left| A \underset{\Phi}{\cap} B \right|,$$

$$D = \sum_{\substack{A \in \mathcal{C}_{\Phi(x)} \\ B \in \mathcal{C}'_{\Phi(y)}}} |A \cup B| \cdot$$

That is, $|A \cap_{\Phi} B|$ is a count of the number of descriptively matching points such that A is in cluster $\mathcal{C}_{\Phi(x)}$ and B is in cluster $\mathcal{C}'_{\Phi(y)}$.

The proposed approach to forgery detection implicitly assumes the availability of digital images containing handwriting samples that are compared. Hence, the work on handwriting forgery detection is part of a larger research area called *digital forensics* [39]. The constraint-based approach in digital forensics uses physical or geometric relationships that are inherent in a digital image with the assumption that the principal points that are perceptually estimated should be close to the center of a photo. Then constraints such as photometric consistency of illumination in shadows are useful in detecting suspicious shadows. It is conjectured that the descriptive point cluster approach defined relative to the illumination intensity in shadows is effective in detecting image forgeries.

7 Admissible Covers

Let (X, δ) be an EF-proximity space and let $(a_n), (b_n)$ be sequences in X. For simplicity, a_n also denotes sequence (a_n). Let a_i denote the i^{th} member of a_n (denoted by $a_i \in a_n$). Then we define

$(a_n) \approx (b_n) \Leftrightarrow$ for each infinite subset $M \subset \mathbb{N}$, $\{a_n : n \in \mathbb{N}\}\ \delta\ \{b_n : n \in \mathbb{N}\}$.

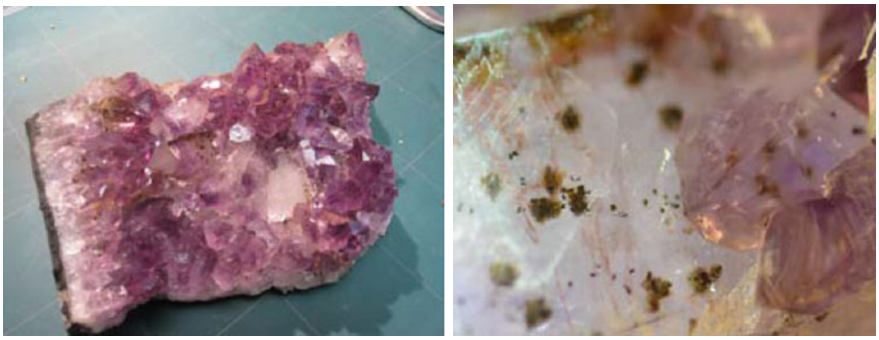
The star of a_n with respect to a cover $\mathcal{U}$ of X (denoted $\mathrm{St}(a_n, \mathcal{U})$ is defined by

$\mathrm{St}(a_n, \mathcal{U}) = \{a_i \in a_n : \{a_i\} \cap \mathcal{U} \neq \emptyset\}$.

The descriptive star of a_n with respect to a cover $\mathcal{U}$ of X (denoted $\mathrm{St}(a_n, \mathcal{U})$ be defined by

$$\mathrm{St}_{\Phi}(a_n, \mathcal{U}) = \left\{ a_i \in a_n : \{a_i\} \underset{\Phi}{\cap} \mathcal{U} \neq \emptyset \right\}.$$

A cover $\mathcal{U}$ of X is called *admissible* if, and only if, $(a_n) \approx (b_n)$ implies $b_n \in St(a_n, \mathcal{U})$ for some $n \in \mathbb{N}$. A cover is *descriptively admissible*, if and only if, $(a_n) \approx_{\Phi} (b_n)$ implies $b_n \in St_{\Phi}(a_m, \mathcal{U})$ for some $n \in \mathbb{N}$. Let $\mathcal{F}$ denote a



a Brasilian amethyst **b** Surface microfossils

Fig. 10 Sample surface amethyst microfossils

family of all descriptively admissible covers of X. Let $A, B \in \mathcal{F}$. Then $A\ \delta_\Phi\ B$ in X (A is descriptively near B), if and only if,

for each $\mathcal{U} \in \mathcal{F}$, $\mathrm{cl}B \underset{\Phi}{\cap} \mathrm{cl}(St(A, \mathcal{U})) \neq \emptyset$.

This section briefly introduces two types of metrisable EF-proximity spaces useful in many applications such as the study of sequences of points in microfossil images. There is an aspect of descriptive admissible covers that is attractive in the study of microfossil images. This attractiveness stems from the fact that one need only consider the cardinality of the descriptive intersection. Given a cover $\mathcal{U}$ of X, $A, B \in \mathcal{P}(X)$, and the star of A with respect to cover $\mathcal{U}$ of X ($St(A, \mathcal{U})$). Descriptive intersections are easy to determine and the count of the number of members of that are descriptively near each other is computationally simple. A measure of the degree of nearness $\mu_{\delta_\Phi} : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}$ is defined by

$$\mu_{\delta_\Phi}(A, B) = \frac{\left| B \underset{\Phi}{\cap} \mathrm{St}(A, \mathcal{U}) \right|}{|B \cup \mathrm{St}(A, \mathcal{U})|}.$$

Then $\mathrm{St}(A, \mathcal{U})\ \delta_\Phi\ B$, if and only if, $\mu_{\delta_\Phi}(A, B) > 0$.

7.1 Admissible Covers in Micropalaeontology

An application of admissible covers is given in this section in terms of micropalaeontology, where there is interest in microfossils in the study of the environment as well as in mineral and fossil fuel exploration (see, *e.g.*, [40]). One of the principal advantages of the proposed approach to the study of microfossils is that a very fine-grained form of microfossil comparison, analysis and classification. Fine-grained comparison of microfossil structures results from a consideration of the spatial as well as descriptive admissible covers in terms of the nearness or the remoteness of sequences of points (pixels) of interest in the study of microscope subimages.

Micropalaeontology is a branch of science concerned with fossil animals and plants at the microscopic level [41]. During the past three decades, significant advances have been made in the understanding of microscopic life and their fossil counterparts. The current completeness of the microfossil record means that the Phanerozoic era (0.01 MYA to 540 MYA) and parts of the Preterozoic era (540 MYA to 2500 MYA) can be dated using microfossils. In addition, microfossils are indispensable in the study of sedimentary basins, providing a biostratigraphical and palaeoecological framework as well as a measure of the maturity of hydrocarbon-prone rocks. A good overview of microfossils as environment indicators in marine shales is given by Ellison [42] and as indicators of glacial drift [43].

Sample microfossils near the surface of a Brasilian amethyst are shown in Fig. 10a. A small group of surface ostracod microfossils [41], looking like bits of pepper and estimated to be from the Devonian era (350-415 MYA),

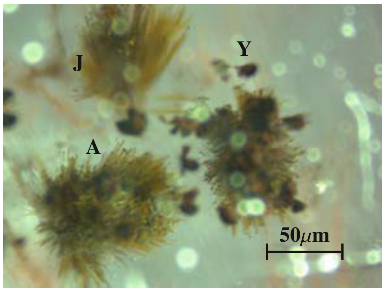


Fig. 11 Zoomed microfossil image

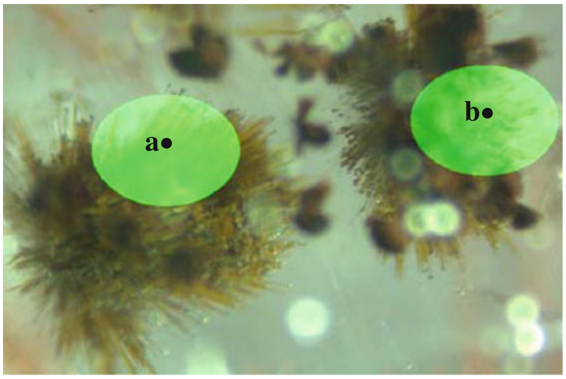


Fig. 12 Microfossils Intensity Neighbourhoods

are located in the north central part of the amethyst in Fig. 10b and a subgroup of the same microfossils in Fig. 11. There are about 33,000 living and fossil ostracod species, originally marine and probably benthic (fauna and flora found on the bottom of a sea or lake). Some adapted to a semi-terrestial life, living in damp soil and leaf litter. A small group of surface of the same microfossils can be seen more clearly on a 50μm scale in Fig. 10b, obtained with an AxioCam Erc5s camera on a Zeiss Discovery V8 stereo zoom microscope using a Zeiss CL 6000 LED light. This section briefly presents an application of two types metrisable EF-proximity spaces in the study of sequences of points in microfossil images.

Example 16 **Microfossil Descriptive Neighbourhoods**.
Let X be a set of points in a microfossil image (see Fig. 11). Let $\Phi = \left\{\phi_g\right\}$, where $\phi_g(x)$ equals the greylevel intensity of $x \in X$ and $\varepsilon \in (0, \infty)$. Assume that the microfossil image has a descriptive admissible cover $\mathcal{U}$ of X that is a collection of open descriptive neighbourhoods $N_{\Phi(x)}$ of points. For simplicity, only two descriptive neighbourhoods of points are shown in Fig. 12, namely, $N_{\Phi(a)},\ N_{\Phi(b)}$. A translucent mask has been superimposed over the points in each of these two intensity neighbourhoods. It is easy to verify that many pairs of pixels from these two neighbourhoods have matching greylevel intensities. Hence, Hence, $\mu_{\delta_\Phi}(N_{\Phi(a)},\ N_{\Phi(b)}) > 0$.

Example 17 **Gradient-based Microfossil Neighbourhoods**.
Let X be a set of points in a microfossil image. Assume that the microfossil image a descriptive admissible cover $\mathcal{U}$ of X that is a collection of open descriptive neighbourhoods of points. Let $\Phi = \{\phi_\Delta\}$, where $\phi_\Delta(x)$ equals the gradient direction of $x \in X$ and $\varepsilon \in (0, \infty)$. In this example, we consider only the gradient direction of the pixels along the setae (*e.g.*, stiff hair-like or bristle-like structures protruding from the bodies of each adult ostracod such as those shown in Fig. 13). Let $\varepsilon = 0.1$. Four gradient direction-based neighbourhoods are shown in Fig. 13, namely,

(Δ.1) $N_{\Phi(a_1)}$ with centre a_1,
(Δ.2) $N_{\Phi(a_2)}$ with centre a_2,

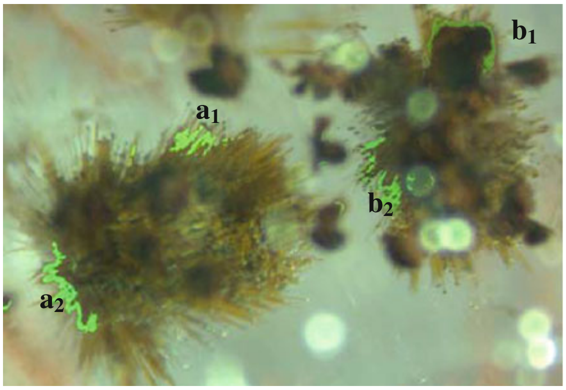


Fig. 13 Four Microfossil Gradient Neighbourhoods

(Δ.3) $N_{\Phi(b_1)}$ with centre b_1,
(Δ.4) $N_{\Phi(b_2)}$ with centre b_2.

Neighbourhood $N_{\Phi(a_2)}$ contains the largest number of setae such that $d(\phi(a_2), \phi(y)) < \varepsilon$ for $y \in N_{\Phi(a_2)}$. Observe that neighbourhood $N_{\Phi(b_2)}$ contains the next largest number of setae with pixels with gradient direction close to the gradient direction of b_2. It is a straightforward task to verify that the gradient direction of the setae in these two neighbourhoods are sufficiently near, i.e., in many cases, $d(\phi(a_2), \phi(b_2)) < \varepsilon$. Hence, given a cover $\mathcal{U}$ of X with $A, B \in \mathcal{P}(X)$,

$$\mathrm{cl}B \underset{\Phi}{\cap} \mathrm{cl}(St(A, \mathcal{U})) \neq \emptyset,$$

implies $A \;\delta_\Phi\; B$ and $\mu_{\delta_\Phi}(A, B) > 0$. This leads to local admissible cover patterns applicable in Micropalaeontology and the study of microfossils in fossil fuel exploration and in gaining knowledge of glacial drift and changes in the environment (see Example 23).

8 Local Patterns in Proximity Spaces

This section briefly introduces patterns in proximity spaces. The proposed approach in the study of proximally near patterns stems from recent work on visual patterns in a topology of digital images [5]. There are two basic forms of nearness patterns to consider, namely, spatial and descriptive.

Let (X, δ) be a spatial EF proximity space, $\mathcal{B} \in \mathcal{P}^2(X),\ B \in \mathcal{B}$, and $A \in \mathcal{P}(X)$. A *spatial nearness pattern* is a mapping $\mathfrak{P} : \mathcal{P}(X) \to \mathcal{P}^2(X)$ defined by

$$\mathfrak{P}(A) = \mathcal{B} : A \;\delta\; B \text{ for } B \in \mathcal{B}.$$

That is, a set $B \in \mathcal{B}$ belongs to a spatial pattern, provided B is spatially near A. The set A is called a motif. The *motif* of a pattern is a nonempty set containing one or more parts that are repeated in the sets of the pattern. That is, one or more parts of a motif can be found in each of the other sets in the pattern. A *spatial remoteness pattern* is a mapping $\underline{\mathfrak{P}} : \mathcal{P}(X) \to \mathcal{P}^2(X)$ defined by

$$\underline{\mathfrak{P}}(A) = \mathcal{B} : A \;\underline{\delta}\; B \text{ for } B \in \mathcal{B}.$$

That is, the set A has no points in common with subsets $B \in \mathcal{B}$.

Corollary 2 *If $\mathfrak{P}(A) = \mathcal{B}$ is a pattern in a spatial EF space, then motif $A \in \mathcal{B}$.*

Proof Immediate from Lemma 1. $\square$

Let (X, δ_Φ) be a descriptive EF proximity space with $A \in \mathcal{P}(X)$, $\mathcal{B} \in \mathcal{P}^2(X)$. Choose Φ to be a set of probe functions that represent features of $x \in X$. A *descriptive nearness pattern* is a mapping $\mathfrak{P}_\Phi : \mathcal{P}(X) \to \mathcal{P}^2(X)$ defined by

$\mathfrak{P}_\Phi(A) = \mathcal{B} : A\ \delta_\Phi\ B$ for $B \in \mathcal{B}$.

That is, a set $B \in \mathcal{B}$ belongs to a descriptive pattern, provided B is descriptively near motif A. A *descriptive remoteness pattern* is a mapping $\underline{\mathfrak{P}}_\Phi : \mathcal{P}(X) \to \mathcal{P}^2(X)$ defined by

$\underline{\mathfrak{P}}_\Phi(A) = \mathcal{B} : A\ \underline{\delta}_\Phi\ B$ for $B \in \mathcal{B}$.

That is, the set A is descriptively remote from subsets $B \in \mathcal{B}$.

Corollary 3 *If $\mathfrak{P}_\Phi(A) = \mathcal{B}$ is a pattern in a descriptive EF space, then motif $A \in \mathcal{B}$.*

Proof Immediate from Lemma 2. □

Examples of patterns abound in spatial and descriptive proximity spaces. Two brief examples are given next (more examples appear in the sequel).

Example 18 **Spatial Pattern**.
Assume that the set X in Fig. 1 is endowed with an EF spatial proximity δ. The set N_y in Fig. 1 is the motif of a spatial pattern. That is, the spatial pattern $\mathfrak{P}(N_y) = \mathcal{B}$, where $\mathcal{B} = \left\{N_y, B, C\right\}$, since $N_y\ \delta\ N_y$, $N_y\ \delta\ B$ $(\mathrm{cl}(N_y) \cap \mathrm{cl}B \neq \emptyset)$, and $N_y\ \delta\ C$ $(\mathrm{cl}(N_y) \cap \mathrm{cl}C \neq \emptyset)$.

Example 19 **Descriptive Pattern**.
Assume that the set X in Fig. 15 is endowed with an EF descriptive proximity δ_Φ. Choose Φ to be a set of probe functions that represent behaviour features of $x \in X$. For example, picture element x has the feature *braiding*, provided x belongs to that part of X depicting *braiding hair*.

Let $A, B_1, B_2 \in \mathcal{P}(X)$ be subsets containing picture elements representing braiding behaviour in Fig. 15. In Fig. 15, sets A, B_1 represent braiding being done by hands and their reflection in the mirror represented by B_2. The set A is the motif of the descriptive pattern. Then a sample descriptive pattern is $\mathfrak{P}_\Phi(A) = \mathcal{B}$, where $\mathcal{B} = \{A, B_1, B_2\}$, since $A\ \delta_\Phi\ A$, $A\ \delta_\Phi\ B_1$ $(\mathrm{cl}A \cap_\Phi \mathrm{cl}(B_1) \neq \emptyset)$ and $A\ \delta_\Phi\ B_2$ $(\mathrm{cl}A \cap_\Phi \mathrm{cl}(B_2) \neq \emptyset)$.

8.1 Other Descriptive Proximity Patterns

In the physical world, there is a profusion of descriptive proximity patterns. We give some examples in this section.

Example 20 **Other Descriptive Proximity Patterns**.
Assume that the set X in Fig. 14 is endowed with a descriptive EF proximity δ_Φ. Choose Φ to be a set of probe functions representing pixel greyscale intensity and colour. Sets A, B_1, B_2, B_3, B_4 represent descriptive open neighbourhoods in Fig. 14. Let A be a descriptive pattern motif. Observe that $A\ \delta_\Phi\ B_i, 1 \leq i \leq 4$. Hence, $\mathfrak{P}_\Phi(A) = \{A, B_1, B_2, B_3, B_4\}$ is an example of a local descriptive proximity pattern.

Example 21 **Camouflage Pattern**.
In Fig. 8, if we assume that the camouflage is perfect, then $\mathfrak{P}_\Phi(A) = \{G_1, \mathcal{N}_x\}$, where, for example, $G_1 \cap_\Phi \mathcal{N}_x \neq \emptyset$. However, it is safe to assume that no camouflage is perfect. Hence, one can expect to find a descriptive remoteness pattern $\underline{\mathfrak{P}}_\Phi(A) = \{\mathcal{B}\}$. In this example, choose $A = C_3$ and let $\mathcal{N}_{\Phi(y)}$ to obtain $\underline{\mathfrak{P}}_\Phi(C_3) = \{\mathcal{B}\}$.

Example 22 **Forgery Pattern**.
In Fig. 9, let $A = \mathcal{C}_{\Phi(x)}$, then obtain the forgery pattern $\underline{\mathfrak{P}}_\Phi(A) = \{\mathcal{B}\}$ with point cluster $\mathcal{C}'_{\Phi(x)} \in \mathcal{B}$. This is an example of a descriptive remoteness pattern, since one can find points in A with descriptions that do not match the description of one or more points in $\mathcal{C}'_{\Phi(x)} \in \mathcal{B}$.

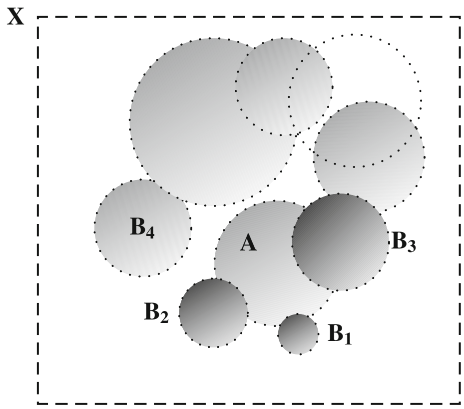


Fig. 14 Sample local proximity patterns

Fig. 15 Descriptive proximity pattern (Punch 1869)

Example 23 **Micropalaeontology Pattern**.
Again, for example, the descriptive neighbourhood N_{ϕ,a_2} and $\mathcal{B} = \{N_{\phi(b_1)}, N_{\phi(b_2)}\}$ in Fig. 13 define the micropalaeontology pattern $\mathfrak{P}_\Phi(N_{\phi(a_2)}) = \{N_{\phi(a_2)}, \mathcal{B}\}$, since $N_{\phi(a_2)} \cap_\Phi \mathcal{B} \neq \emptyset$. From these observations, we can derive descriptive neighbourhood patterns useful in the study of camouflage.

8.2 Local Star Patterns

Local spatial and descriptive covering patterns can be identified in a number of instances of covers on proximity spaces. Let (X, δ), (X, δ_Φ) be spatial and descriptive proximity spaces, respectively. Let $\mathcal{C}, \mathcal{C}'$ be covers of X. Cover $\mathcal{C}$ *refines* cover $\mathcal{C}'$ (denoted $\mathcal{C} < \mathcal{C}'$), if and only if, each set A in $\mathcal{C}$ is contained in some set B in $\mathcal{C}'$. Choose Φ to be a set of probe functions used to describe members of X. Cover $\mathcal{C}$ *descriptively refines* cover $\mathcal{C}'$ (denoted $\mathcal{C} <_\Phi \mathcal{C}'$), if and only if, the set of descriptions $\mathcal{Q}(A)$ of each set A in $\mathcal{C}$ is contained $\mathcal{Q}(B)$ of some set B in $\mathcal{C}'$, i.e.,

$$\mathcal{C} \underset{\Phi}{<} \mathcal{C}' \Leftrightarrow \mathcal{Q}(A) \subset \mathcal{Q}(B) \text{ for each } A \in \mathcal{C} \text{ for some } B \in \mathcal{C}'.$$

The *star* of $A \in \mathcal{P}(X)$ with respect to a cover $\mathcal{C}$ (denoted $\mathrm{St}(A, \mathcal{C})$) is defined to be

$$\mathrm{St}(A, \mathcal{C}) := \{B \in \mathcal{C} : A \cap B \neq \emptyset\},$$

i.e., $\mathrm{St}(A, \mathcal{C})$ is the collection of subsets $B \in \mathcal{C}$ that are near A and A refines $\mathcal{C}$. Then one can observe $\mathfrak{P}(A) = \{A, \mathrm{St}(A, \mathcal{B})\}$ is a *local star pattern* on X, provided $A \;\delta\; B$ for at least one B in $\mathcal{C}$.

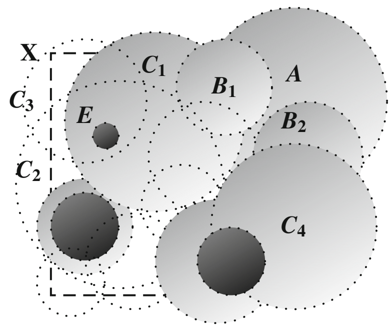


Fig. 16 Local star patterns

Example 24 **Local Star Pattern**.
In Fig. 16, let the set X be endowed with an EF proximity δ, $A \in \mathcal{P}(X)$, $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be a cover of X. $\mathrm{St}(A, \mathcal{B}) = \{B_1, B_2\}$, since $A\ \delta\ B_1$ (since $\mathrm{cl}A \cap \mathrm{cl}(B_1) \neq \emptyset$) and $A\ \delta\ B_2$ (since $\mathrm{cl}A \cap \mathrm{cl}(B_2) \neq \emptyset$). Hence, $\mathfrak{P}(A) = \{A, \mathrm{St}(A, \mathcal{B})\}$ is a local star pattern. Again, for example, the star of a_n with respect to the cover $\mathcal{U}$ of X in Fig. 13 defines the micropalaeontology spatial star pattern $\mathfrak{P} = \{a_n, \mathrm{St}(a_n, \mathcal{U})\}$.

The *descriptive star* of $A \in \mathcal{P}(X)$ with respect to a cover $\mathcal{C}$ (denoted $\mathrm{St}_\Phi(A, \mathcal{C})$) is defined to be

$$\mathrm{St}_\Phi(A, \mathcal{C}) := \left\{ B \in \mathcal{C} : A \underset{\Phi}{\cap} B \neq \emptyset \right\},$$

i.e., $\mathrm{St}_\Phi(A, \mathcal{C})$ is the collection of subsets $B \in \mathcal{C}$ descriptively near A and A descriptively refines $\mathcal{C}$. The collection $\mathfrak{P}_\Phi(A) = \{A, \mathrm{St}_\Phi(A, \mathcal{B})\}$ is a *local descriptive star pattern* on X, provided $A\ \delta_\Phi\ B$ for at least one B in $\mathcal{B}$.

Example 25 **Local Descriptive Star Pattern**.
In Fig. 16, let the set X be endowed with an EF proximity δ_Φ, motif $A \in \mathcal{P}(X)$, and let $\mathcal{C} = \{C_1, C_2, C_3, C_4, \ldots, C_n\}$ be a cover of X. Choose Φ to be a set of probe functions for greylevel intensity and colour for pixels in X. Then $\mathrm{St}_\Phi(A, \mathcal{C}) = \{C_1, C_4\}$, since $\mathrm{cl}A \cap_\Phi \mathrm{cl}(C_1) \neq \emptyset$ and $\mathrm{cl}A \cap_\Phi \mathrm{cl}(C_4) \neq \emptyset$. Hence, $\mathfrak{P}_\Phi(A) = \{A, \mathrm{St}_\Phi(A, \mathcal{C})\}$ is a local descriptive star pattern. Again, for example, the star of b_n with respect to the cover $\mathcal{U}$ of X in Fig. 13 defines the micropalaeontology descriptive star pattern $\mathfrak{P}_\Phi = \left\{N_{\phi,a_1}, \mathrm{St}_\Phi(N_{\phi,a_1}, \mathcal{U})\right\}$, since $\mathrm{cl}(N_{\phi,a_1}) \cap_\Phi \mathrm{cl}(N_{\phi,b_2}) \neq \emptyset$ for neighbourhood $N_{\phi,b_2} \in \mathcal{U}$.

A cover $\mathcal{C}$ is a *star refinement* of a cover $\mathcal{D}$ (denoted by $\mathcal{C} *\!< \mathcal{D}$), provided

$\{\mathrm{St}(A, \mathcal{C}) : A \in \mathcal{C}\}$ refines $\mathcal{D}$.

The collection $\mathfrak{P}(A) = \{\mathrm{St}(A, \mathcal{C}), \mathcal{D}\}$ is a *local star refinement pattern* on X, provided $\mathrm{St}(A, \mathcal{C}) \subseteq \mathcal{D}$.

Example 26 **Local Star Refinement Pattern**.
Let (X, δ) be the EF space with cover $\mathcal{B}$ from Example 24 and let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a cover of X. $\mathfrak{P}(A) = \{\mathrm{St}(A, \mathcal{B}), \mathcal{C}\}$ is a star refinement pattern, since $\mathrm{St}(A, \mathcal{B}) \subseteq \mathcal{C}$.

A cover $\mathcal{C}$ is a *descriptive star refinement* of a cover $\mathcal{D}$ (denoted by $\mathcal{C} *\!<_\Phi \mathcal{D}$), provided

$\{\mathrm{St}_\Phi(A, \mathcal{C}) : A \in \mathcal{C}\}$ descriptively refines $\mathcal{D}$.

That is, the descriptions of the members of $\mathrm{St}_\Phi(A, \mathcal{C})$ are a subset of the descriptions of the members of $\mathcal{D}$ (denoted $\mathrm{St}(A, \mathcal{C}) \subseteq_\Phi \mathcal{D}$). The collection $\mathfrak{P}_{*<_\Phi}(A) = \{\mathrm{St}(A, \mathcal{C}), \mathcal{D}\}$ is a *local star refinement pattern* on X, if and only if, $\mathrm{St}(A, \mathcal{C}) \subseteq_\Phi \mathcal{D}$.

Example 27 **Local Descriptive Star Refinement Pattern**.
Let (X, δ) be the EF space with cover $\mathcal{B}$ from Example 24 and let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a cover of X. $\mathfrak{P}_{*<_\Phi}(A) = \{\mathrm{St}(A, \mathcal{B}), \mathcal{C}\}$ is a star refinement pattern, since $\mathrm{St}(A, \mathcal{B}) \subseteq_\Phi \mathcal{C}$.

From Example 15, the pair of descriptive point clusters $\mathcal{C}_{\Phi(x)}, \mathcal{C}'_{\Phi(y)}$ defines a local descriptive star refinement pattern, since it is reasonable to expect that the cover of a forged cursive character will be rather similar descriptively to the original cursive character. It is evident that this is the case in Example 15. Hence, the descriptive star refinement pattern $\mathfrak{P}_{*<_\Phi}(A)$ defined by

$$\mathfrak{P}_{*\underset{\Phi}{<}}(A) = \left\{\text{St}(A, \mathcal{C}'_{\Phi(y)}), \mathcal{C}_{\Phi(x)}\right\}$$

is an example of a descriptive point cluster-based forgery pattern.

From the foregoing observations about local patterns, we obtain the following results.

Theorem 11 *Local patterns in an EF space are collections of near sets.*

Lemma 5 *Let $(X, \delta), (X, \delta_\Phi)$ be spatial and descriptive EF-spaces, respectively, with nonempty sets $A, B \in \mathcal{P}(X)$, $A \cap B \neq \emptyset$. Then $A \cap B \subseteq A \cap_\Phi B$.*

Proof Let $A, B \in \mathcal{P}^2(X)$ and assume $A \cap B \neq \emptyset$. If $x \in A \cap B$, then, by definition, $\Phi(x) \in \mathcal{Q}(A)$ and $\Phi(x) \in \mathcal{Q}(B)$. Assume $x \in A \backslash A \cap B$, $y \in B \backslash A \cap B$ such that $\Phi(x) = \Phi(y)$. Then $x, y \in A \cap_\Phi B$. Hence, $A \cap B \subseteq A \cap_\Phi B$. □

Given two nonempty sets A and B, if $A \subseteq B$, then A is *coarser* that B (A is *strictly coarser* than B, provided $A \subset B$) and B is termed *finer* (or *strictly finer*) than A.

Theorem 12 *Spatial nearness patterns are coarser than descriptive nearness patterns.*

Proof Immediate from Lemma 5. □

Theorem 13 *Spatial star patterns are coarser than descriptive star patterns.*

Proof Let $(X, \delta), (X, \delta_\Phi)$ be spatial and descriptive EF-spaces, respectively, $A \in \mathcal{P}(X)$, $\mathcal{C} \in \mathcal{P}^2(X)$. From Lemma 5, $\text{St}(A, \mathcal{C}) \subseteq \text{St}_\Phi(A, \mathcal{C})$. Hence, $\mathfrak{P}(A) = \{A, \text{St}(A, \mathcal{C})\}$ is coarser than $\mathfrak{P}_\Phi(A) = \{A, \text{St}_\Phi(A, \mathcal{C})\}$. □

Similarly, we obtain the following result.

Theorem 14 *Spatial star refinement patterns are coarser than descriptive star refinement patterns.*

9 Conclusion

This paper presents a proximity space approach to the solution of the local pattern recognition problem, considered in the context of either spatial or descriptively near (or remote) sets. The standard spatial Efremovič (EF) proximity space is extended to a descriptive EF proximity space, thanks to the introduction of feature-vector descriptions of the members of sets and description-based extensions of set intersection and union. Pattern recognition is simplified by proximitising the parts of a set rather than proximitising the whole set, leading to the discovery of local patterns in very complex sets such as digital images. Several practical applications of the proposed approach to pattern recognition are given.

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