UNIFORM TRIADIC TRANSFORMATIONS

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Introduction

Among the most stimulating developments in music theory in the last decade has been the explosion of interest in transformational theory inspired by the work of Hugo Riemann. A recent special issue of the *Journal of Music Theory* devoted to neo-Riemannian theory (vol. 42, no. 2, 1998) testifies to the state of vigorous health that the field now enjoys, and the basic neo-Riemannian operators *P*, *L*, and *R* have made their first appearance in an undergraduate harmony textbook (Roig-Francoli 2003, 863–71).

Its early successes notwithstanding, neo-Riemannian theory is in its infancy, and in its current state suffers from considerable confusion and several very real shortcomings that severely limit its scope and application.[1] Richard Cohn (1998b, 289), for example, has lamented that there is no standardized system for labeling triadic transformations; indeed, the number of different—and sometimes directly conflicting—notational systems in use is substantial. The relation of the "dominant transformation" *D*—an essential feature of the transformational system of Brian Hyer (1995), for example—to those in the *PLR* family seems to have become a source of considerable insecurity for some. What is to be made of the fact that, applied to a major triad, D has the same effect as R followed by L, but applied to a minor triad, D has the same effect as L followed by R? Is D superfluous? And what, for that matter, is "neo-Riemannian"? Is D neo-Riemannian or isn't it? Carol Krumhansl (1998, 265), Fred Lerdahl (2001, 83), and others have explicitly numbered D among the transformations they call "neo-Riemannian," a view tacitly seconded by Cohn (1998a) and prefigured by Hyer; Henry Klumpenhouwer (1994, §5–7 and 2000, 162), on the other hand, not only fails to recognize D as a neo-Riemannian transformation but even expresses reservations about the practice of combining dominant and true neo-Riemannian transformations in the same graph (which he characterizes as a "shift from Riemann to Rameau"). As a matter of semantics, this disagreement is perhaps inconsequential, but it is symptomatic of a fundamental uncertainty of perspective that characterizes much work in the field.

Mathematical uncertainty has also characterized some of this work. The most thorough study of the algebraic structure of triadic transformations to date is that of Hyer. Hyer constructs a group of 144 transformations (including D), complete with "canonical forms" for each of them, and elaborate multiplication tables from which one may determine the product of any two transformations. Hyer's group is an impressive structure, but many have found it confusing, and in any case our understanding of the algebraic properties of triadic transformations remains far from complete. The gymnastics one must perform in order to find canonical forms and calculate products in Hyer's system are unwieldy, and the picture of the group that emerges is ultimately a rather hazy one. Moreover, the system is demonstrably incomplete: several triadic transformations of considerable analytic utility, and whose claims to "neo-Riemannian" status are surely at least as strong as that of D, are simply absent from Hyer's group. The most familiar of these is the "diatonic mediant" transformation M, discussed briefly by David Lewin (1987, 175–80) but not a part of Hyer's formalization.

One of the most glaring deficiencies in neo-Riemannian theory is its fundamental restriction to consonant triads. Neo-Riemannian theory is, in its basic form, a theory all about Forte class 3-11. Needless to say, music contains many other sonorities besides major and minor triads—particularly the chromatic music of late nineteenth-century composers such as Wagner, Liszt, and Franck, often cited as promising territory for neo-Riemannian exploration. Recently Adrian Childs (1998), Edward Gollin (1998), and others have adapted neo-Riemannian models for use with major-minor and half-diminished seventh chords. Childs's and Gollin's formalizations, however, are quite different from each other, and their points of contact with the triadic theory are not clearly defined; neither one, for example, offers a means for analyzing chord progressions containing a mixture of triads and seventh chords.

A number of other objections to the neo-Riemannian approach have also been raised. Its application to standard diatonic progressions is awkward. This observation perhaps relates to the uncomfortable and incomplete integration of the fundamentally diatonic transformations D and M discussed above, and undoubtedly bears some responsibility for the fact that satisfactory applications of neo-Riemannian methodology, even in highly chromatic music, have generally been confined to isolated short passages. The theory is said to disregard the concept of chord roots, which has long been fundamental to tonal theory and is surely relevant even in the repertoire favored by neo-Riemannian theorists. The theory is said to be insufficiently attentive to the distinction between chord and key area, and to hierarchical distinctions in general.

Most of the problematic issues just raised are addressed, directly or indirectly, by the simple algebraic framework proposed on the following pages for the study of triadic transformations. A *uniform triadic transformation*, or UTT, is described in componentwise fashion, through a root-interval approach. Each UTT consists of a *sign* (+ or –, indicating whether the transformation preserves or reverses mode) and two *transposition levels* (integers mod 12: one for major triads, the other for minor, indicating the interval through which the root of a triad is transposed). For example, the leittonwechsel is represented as $L = \langle -, 4, 8 \rangle$, by which we understand that it maps any major triad to a minor triad whose root is four semitones higher (C major to E minor), and maps any minor triad to a major triad whose root is eight semitones higher (C minor to A♭ major). The root-interval description may be contrasted with the approach via common tones and incremental voice leading that has been adopted in most other recent neo-Riemannian studies.[2] There are 288 UTTs altogether: Hyer's 144 and as many more (including the diatonic mediant M). They are multiplied by applying a simple arithmetic formula, not by looking things up in tables. The UTTs form a group $\mathcal{U}$, to be described in Part 1 below, whose structure is a primary object of study in much of the remainder of this paper.

The UTT formalism simultaneously generalizes and simplifies much of the recent work in triadic transformations. It provides a much-needed clarification of what precisely is "Riemann-like" about the behavior of transformations such as P, L, and R: each of these transformations, written as a UTT, has the property that its two transposition levels (4 and 8 in the case of L, as given above) are equal and opposite (mod 12). This property—whatever a transformation does to a major triad, its effect on a minor triad is precisely the opposite—may be regarded as an explicit representation of Riemann's well-known harmonic dualism. UTTs exhibiting this property will be called *Riemannian* transformations. As a UTT, the dominant transformation $D = \langle +, 5, 5 \rangle$ is assuredly *not* Riemannian; the UTT theory, while making this distinction clear, nevertheless allows

Riemannian and non-Riemannian transformations to interact, thereby offering a sort of compromise between the opposing perspectives on the status of D noted above. The Riemannian UTTs form a subgroup $\mathcal{R}$ of the larger group $\mathcal{U}$, and are the subject of Part 2 of this paper.

Among the many attractive properties of the subgroup $\mathcal{R}$ is that it acts on triads in *simply transitive* fashion. That is, for any two given triads, there always exists one and only one Riemannian transformation that maps one to the other. Altogether 24 subgroups of $\mathcal{U}$ have this property; these groups form the subject of Part 3. Six of these groups are cyclic; the generators of the cyclic groups are precisely the 48 elements of maximal order (24) in $\mathcal{U}$—a property of some musical interest, as a number of examples will show. Simply transitive groups also bring a measure of clarity to situations involving apparently redundant transformations, such as D and RL as mentioned above. The selection of one simply transitive group leads to a unique transformational analysis of any triadic progression, but different choices of the group may yield different (yet still meaningful) analyses.

Further algebraic properties of the group $\mathcal{U}$ are explored in Part 4. The behavior of UTTs as permutations of the set of triads is examined, and sheds light on the structure of Hyer's group as well as that of the larger group $\mathcal{U}$. Also presented here are characterizations of many of the subgroups of $\mathcal{U}$, a partial diagram of the complex subgroup lattice of $\mathcal{U}$, and several descriptions of $\mathcal{U}$ in terms of some of these subgroups, including an attractive representation as a *wreath product* of familiar small groups.

The UTT theory admits natural extensions in any of several directions, a few of which are sketched in Part 5. For example, the standard inversion operator I of pitch-class set theory is not a UTT, but I and other such transformations may be adjoined to the group $\mathcal{U}$, generating a larger group $\mathcal{Q}$ of order 1152; the musical and algebraic properties of this large group are discussed here. Also investigated are applications of UTTs to sonorities other than consonant triads: the theory may be applied unchanged to the class of major-minor and half-diminished seventh chords and many other set classes. UTTs *per se* cannot transform one set class to another, but the final sections of the paper suggest methods by which the componentwise representation of transformations may be extended to approach the problem of relating different set classes transformationally.

A few remarks on methodology and terminology are in order before we embark on the program outlined above. It is my hope that the UTT formalism may provide a standardized methodology and nomenclature for triadic transformations, and that it may make transformational theory accessible to a wider audience. I hope also that the second of these aims will not be compromised by the rigorously mathematical nature of the exposition in this article. Proofs of the most significant results are included, and mathematically proficient readers are encouraged to fill in the gaps that remain. Nevertheless, readers who are less mathematically inclined (or less patient) but who are willing, in occasional moments of forbidding technical detail, to suspend their disbelief momentarily should be able to skip the proofs and still follow the proceedings with profit and understanding. Some fundamental algebraic definitions are supplied in the notes; readers desiring further grounding in the relevant concepts are referred to the discussion of group theory in any abstract algebra text (e.g., Dummit and Foote 1999, chapters 1–6), and to the extensive discussion of transformation groups in Lewin 1987. In some instances a more detailed treatment of the ideas presented here may be found in Hook 2002.

The term *neo-Riemannian* is a recent coinage, first appearing in print in Cohn 1996. The prefix *neo-* serves to distinguish today's transformation-oriented approach with Riemann's original perspective, wherein transformations as such were subordinate to notions of harmonic function. In adapting recent terminology for use with UTTs, I have generally avoided this term. As noted above, for UTTs whose transposition levels are equal and opposite I use simply the adjective *Riemannian*. The prefix is abandoned here partly for simplicity, but primarily because the fundamental distinction to be drawn is not between Riemannian and neo-Riemannian but rather between Riemannian and *non*-Riemannian: that is, between the few (24) UTTs that satisfy the Riemannian dualism condition and the much larger number (264) that do not. The point of the adjective is to call attention to the ways in which the behavior of these 24 transformations *is* Riemann-like, not to the readily apparent ways in which it is not.[3]

1. Triads and UTTs

1.1. We represent a *triad* as an ordered pair $\Delta = (r, \sigma)$, where r is the *root* of the triad expressed as an integer mod 12, and σ is a *sign* representing its mode (+ for major, – for minor). Thus $\Delta = (0, +)$ represents C major, while $\Delta = (8, -)$ represents G♯ minor.[4] At times we shall write $r = r_\Delta$ and $\sigma = \sigma_\Delta$ to avoid confusion with other r and σ. The collection of all 24 triads will be denoted Γ. Because all twelve major triads are related by transposition, we say that they belong to the same *transposition class* Γ^+; likewise the twelve minor triads belong to the transposition class Γ^-.

1.2. We shall have occasion to perform arithmetic on the numbers r, following the usual rules of arithmetic mod 12. (The qualifier "mod 12" will usually be omitted.) We shall also have occasion to multiply signs, following the rules

$$++ = +,\ +- = -,\ -+ = -,\ -- = +.$$

Given two triads $\Delta_1 = (r_1, \sigma_1)$ and $\Delta_2 = (r_2, \sigma_2)$, we may consider the relationship between them in terms of the *transposition level* $t = r_2 - r_1$ and the *sign factor* $\sigma = \sigma_1\sigma_2$. The transposition level is simply the directed pitch-class interval from the root of Δ_1 to the root of Δ_2; the sign factor is + if the triads Δ_1 and Δ_2 are of the same mode, – otherwise. We define the Γ-*interval* between the two triads Δ_1 and Δ_2, denoted $\mathrm{int}(\Delta_1, \Delta_2)$ or $\mathrm{int}_\Gamma(\Delta_1, \Delta_2)$, to be the ordered pair (t, σ), where t and σ are the transposition level and sign factor as defined above. Thus, for example, the interval from $(5, +)$ (F major) to $(8, -)$ (G♯ minor) is $(8 - 5, + -) = (3, -)$.

It will be noted that $\mathrm{int}(\Delta_1, \Delta_2)$ is again an element of Γ: in the above example it is the element representing a D♯ minor triad. Nevertheless, we do not claim that "the interval from F major to G♯ minor is D♯ minor"; rather, the interval from F major to G♯ minor is $(3, -)$, the same element of Γ that happens to *represent* D♯ minor (under the canonical identification of $(0, +)$ with C major). The following theorem makes this dual nature of the set Γ more explicit.

1.3. Theorem. The set Γ forms a commutative group with multiplication defined by

$$(t_1, \sigma_1)(t_2, \sigma_2) = (t_1 + t_2, \sigma_1\sigma_2).$$

Moreover, Γ (as a set of triads) forms a generalized interval system (GIS) in which the group of intervals is the *group* Γ and the interval function is the function $\mathrm{int} = \mathrm{int}_\Gamma$ defined above.

This theorem is not essential for most of what follows, and a few brief comments will be given here in lieu of a proof. The integers mod 12 form an additive group $\mathbf{Z}_{12}$, and the set $\{+, -\}$ forms a multiplicative group isomorphic to the additive group $\mathbf{Z}_2$ of integers mod 2. Multiplication in Γ is defined (as in the statement of the theorem) so that Γ is simply the direct product of these two groups;[5] thus Γ is a group isomorphic to $\mathbf{Z}_{12} \times \mathbf{Z}_2$. The identity element is $(0, +)$, and the inverse of (t, σ) is $(-t, \sigma)$. Generalized interval systems are defined in Lewin 1987, 26; the proof of the theorem requires checking the two conditions presented there.

Theorem 1.3 is actually a special case of a general construction described by Lewin (1987, 31–32) whereby the elements of any GIS may be identified with the elements of its group of intervals. In general, one selects a "referential member" of the GIS to be identified with the identity element of the group, and then labels each other element with its interval from the referential member. In this case we have chosen the C major triad as the triad to be identified with the identity interval $(0, +)$. Had a different triad been privileged in this way, then all the triad labels would change (in fact, all would change by the same Γ-interval, the interval from the new referential triad to C major); the intervals between pairs of triads, however, would be unaffected.

1.4. By a *triadic transformation* we mean a transformation on the set Γ, as defined by Lewin (1987, 3)—that is, a mapping from Γ into Γ itself. The word *transformation* by itself, as used here, will refer exclusively to triadic transformations unless noted otherwise. We shall use the convention of left-to-right orthography for transformations; thus X_1X_2 denotes the application of two transformations in the order "X_1, then X_2."[6] If a transformation X maps a triad Δ to a triad Δ', we write $(\Delta)(X) = \Delta'$ (note the transformation written to the *right* of the triad), or, perhaps more suggestively, $\Delta \xrightarrow{X} \Delta'$. We shall use the usual exponential notation to indicate iterated products of a transformation with itself: $X^2 = XX$, $X^3 = XXX, \ldots$.

Lewin defines an *operation* as a transformation for which the mapping from Γ to Γ is one-to-one and onto.[7] Thus X is an operation if for every triad Δ', there is one and only one triad Δ such that $\Delta \xrightarrow{X} \Delta'$. In this case, the *inverse transformation* X^{-1} is well-defined; if $\Delta \xrightarrow{X} \Delta'$, then $\Delta' \xrightarrow{X^{-1}} \Delta$. Some musically interesting transformations are not operations: one could imagine, for example, a transformation that maps each triad to its functional dominant, so that both C major and C minor are mapped to G major (and no triad at all is mapped to G minor). Nevertheless, our primary interest lies in operations, and *we shall generally assume henceforth that all transformations are operations.* The transformations thus form a group $\mathcal{G}$, in which multiplication is defined by composition of mappings (in left-to-right order, as discussed above).

1.5. The number of transformations in the group $\mathcal{G}$ is vast: in fact, it is 24 factorial, a number of 24 digits, for a transformation may be any permutation of the 24-element set Γ. Most of these transformations are musically rather nonsensical, since the action of a transformation on a given triad need not bear any particular relation to its action on any other triad. We shall therefore restrict our attention to transformations exhibiting a certain kind of musical coherence that we call *uniformity*. A triadic transformation is *uniform* if it transforms all major triads "the same way"—that is, through the same Γ-interval—and likewise transforms all minor triads "the same way." More precisely, we say that a transformation U satisfies the *uniformity condition* if for every triad (r, σ) and every transposition level t, if U transforms the triad (r, σ) to the triad (r', σ'), then U transforms the triad $(r + t, \sigma)$ to the triad $(r' + t, \sigma')$. If, for example, U transforms C major to E minor, then in order to be uniform, U must transform D♭ major to F minor, D major to F♯ minor, and so on. Transformations (or, more precisely, operations) satisfying the uniformity condition will be called *uniform triadic transformations*, or UTTs.

Let U be a UTT. Suppose that U transforms the C major triad $(0, +)$ to the triad (t^+, σ^+), and transforms the C minor triad $(0, -)$ to the triad (t^-, σ^-). The uniformity condition then implies that the U-transform of any other major triad $(r, +)$ is $(r, +)(U) = (r + t^+, \sigma^+)$, and that the U-transform of any other minor triad $(r, -)$ is $(r, -)(U) = (r + t^-, \sigma^-)$. The behavior of the UTT U thus appears to be completely determined by four parameters, the signs σ^+ and σ^- and the transposition levels t^+ and t^-. In fact, however, one of the signs is redundant, for the requirement that U be an operation implies that σ^+ and σ^- must be opposites (for otherwise U would transform all triads into triads of mode σ^+, and could not possibly map Γ *onto* Γ). Therefore we may write σ for σ^+ and $-\sigma$ for σ^-. A UTT U is thus completely characterized by *three* parameters: its *sign* $\sigma = \sigma_U$ $(= \sigma^+)$ and its two *transposition levels* $t^+ = t^+_U$ and $t^- = t^-_U$. Therefore we may represent U by the ordered triple $U = \langle\sigma, t^+, t^-\rangle$. We shall reserve the use of the angle brackets $\langle\ \rangle$ specifically for identifying UTTs in this way. We call σ, t^+, and t^- the *components* of U. The componentwise notation $\langle\sigma, t^+, t^-\rangle$ and its equivalence with the uniformity condition are fundamental to the concept of UTTs and will be used throughout this paper.

If $U = \langle\sigma_U, t^+, t^-\rangle$ is a UTT and $\Delta = (r, \sigma_\Delta)$ is a triad, then U acts on Δ by transposing its root upward by either t^+ or t^- semitones, depending on whether σ_Δ is + or − (that is, on whether Δ is major or minor), and by changing its mode if and only if $\sigma_U = -$. More precisely, if σ_Δ is +, the U-transform of Δ is given by $(\Delta)(U) = (r + t^+, \sigma_U)$; on the other hand, if σ_Δ is −, then $(\Delta)(U) = (r + t^-, -\sigma_U)$. The two cases may be combined in the single generally applicable formula

$$(\Delta)(U) = (r + t^{(\sigma_\Delta)}, \sigma_\Delta\sigma_U) \qquad \textit{(action of a UTT on a triad).}$$

(Here, of course, $t^{(\sigma_\Delta)}$ represents t^+ if $\sigma_\Delta = +$, or t^- if $\sigma_\Delta = -$.)

If $\sigma_U = +$, then U maps major triads to major triads and minor to minor, and we say that U is *mode-preserving*. In contrast, a U for which $\sigma_U = -$ maps major to minor and minor to major, and is called *mode-reversing*.

1.6. Many familiar musical relationships can be expressed as UTTs. Consider, for example, the "relative" transformation R, which maps every major triad to its relative minor and vice versa. This is a mode-reversing transformation with $t^+ = 9$ (because, for example, $\mathrm{C} \xrightarrow{R} \mathrm{a}$, transposition level 9) and $t^- = 3$ ($\mathrm{c} \xrightarrow{R} \mathrm{E}\flat$, transposition level 3). (Because of uniformity, every UTT is completely characterized and conveniently described by its action on C major and C minor triads. We shall describe UTTs in this way frequently.) Thus $R = \langle -, 9, 3\rangle$. The following list includes R among several other familiar examples of UTTs.

T_0	$= \langle +, 0, 0\rangle$	(identity transformation):	$\mathrm{C} \xrightarrow{T_0} \mathrm{C}$,	$\mathrm{c} \xrightarrow{T_0} \mathrm{c}$.
T_n	$= \langle +, n, n\rangle$	(pc-set transposition):	$\mathrm{C} \xrightarrow{T_1} \mathrm{D}\flat$,	$\mathrm{c} \xrightarrow{T_1} \mathrm{c}\sharp$;
			$\mathrm{C} \xrightarrow{T_2} \mathrm{D}$,	$\mathrm{c} \xrightarrow{T_2} \mathrm{d}$; etc.
P	$= \langle -, 0, 0\rangle$	(parallel):	$\mathrm{C} \xrightarrow{P} \mathrm{c}$,	$\mathrm{c} \xrightarrow{P} \mathrm{C}$.

$L \;= \langle -, 4, 8\rangle$ (leittonwechsel): $\quad \mathrm{C} \xrightarrow{L} \mathrm{e}, \quad \mathrm{c} \xrightarrow{L} \mathrm{A}\flat.$

$R \;= \langle -, 9, 3\rangle$ (relative): $\quad \mathrm{C} \xrightarrow{R} \mathrm{a}, \quad \mathrm{c} \xrightarrow{R} \mathrm{E}\flat.$

$D \;= \langle +, 5, 5\rangle$ ($= T_5$) (dominant): $\quad \mathrm{C} \xrightarrow{D} \mathrm{F}, \quad \mathrm{c} \xrightarrow{D} \mathrm{f}.$

$M \;= \langle -, 9, 8\rangle$ (diatonic mediant): $\quad \mathrm{C} \xrightarrow{M} \mathrm{a}, \quad \mathrm{c} \xrightarrow{M} \mathrm{A}\flat.$

We have followed the convention established by Lewin (1987, 177) in naming the transformations D and M so that they transform the named triad *to* its tonic, not the other way around; thus the dominant transformation is "what happens" at a V–I cadence.

It should be clear at this point that UTTs may be used to model successions of triads appearing in chord progressions. For example, a circle-of-fifths progression

$$\mathrm{G} \longrightarrow \mathrm{C} \longrightarrow \mathrm{F} \longrightarrow \mathrm{B}\flat$$

may be generated entirely by the dominant transformation D:

$$\mathrm{G} \xrightarrow{D} \mathrm{C} \xrightarrow{D} \mathrm{F} \xrightarrow{D} \mathrm{B}\flat.$$

The same D would do the trick equally well if all the triads in the progression were minor instead of major. The retrograde of this progression may be generated by the UTT $X = \langle +, 7, 7\rangle$:

$$\mathrm{B}\flat \xrightarrow{X} \mathrm{F} \xrightarrow{X} \mathrm{C} \xrightarrow{X} \mathrm{G}.$$

The reader may suspect that $X = D^{-1}$, the inverse of D; this will be verified in Section 1.12 below.

Recent work in neo-Riemannian theory has unearthed many remarkable progressions in nineteenth-century music based on the transformations P, L, and R. (See in particular the many fine examples in the articles by Richard Cohn, 1996 and 1997.) Any of these analyses may be recast easily in the language of UTTs. It must be noted, however, that the UTT that models the relationship between a pair of adjacent triads is not uniquely determined. In the progression $\mathrm{C} \xrightarrow{U} \mathrm{a}$, for instance, U could be the "relative" transformation R, or the "mediant" transformation M, or in fact any UTT of the form $\langle -, 9, n\rangle$. These twelve transformations behave identically when applied to major triads; only their action on minor triads distinguishes them. As we shall see in Part 3 of this paper, simply transitive groups provide a way to choose among the various possible UTTs in any given musical setting.

Our justification of the uniformity condition in terms of "coherence" notwithstanding, it would be a mistake to assume that every musically meaningful triadic transformation is a UTT. The usual inversion operator I (inversion about the pitch class C), for example, is not a UTT: the actions $\mathrm{C} \xrightarrow{I} \mathrm{f}$ and $\mathrm{D}\flat \xrightarrow{I} \mathrm{e}$ represent two different transposition levels (5 and 3, respectively) for major triads, in violation of the uniformity condition. Relationships between UTTs and inversion operations will be examined in Part 5 of this paper.

The following theorem offers additional reformulations of the uniformity condition.

1.7. Theorem. Let U be a triadic transformation (assumed one-to-one and onto). Then the following four conditions are equivalent:

(a) U is a UTT (that is, U satisfies the uniformity condition);

(b) $U = \langle \sigma, t^+, t^- \rangle$ for some σ, t^+, t^-;

(c) U commutes with the transposition T_1 (that is, $UT_1 = T_1U$);

(d) U commutes with every transposition T_n for $n = 0, 1, 2, \ldots, 11$.

Proof. The implication (a) $\Rightarrow$ (b) was established in the discussion in Section 1.5. We shall complete the proof of the theorem by proving the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(b) $\Rightarrow$ (c). Assume that $U = \langle \sigma_U, t^+, t^- \rangle$ as described above. Let $\Delta = (r, \sigma_\Delta)$ be a triad. Then, using the formulas from Sections 1.5 and 1.6,

$$\begin{aligned}
(\Delta)(UT_1) &= ((\Delta)(U))(T_1) \\
&= ((r, \sigma_\Delta)\langle \sigma_U, t^+, t^- \rangle)\langle +, 1, 1 \rangle \\
&= (r + t^{(\sigma_\Delta)}, \sigma_\Delta\sigma_U)\langle +, 1, 1 \rangle \\
&= (r + t^{(\sigma_\Delta)} + 1, \sigma_\Delta\sigma_U) \\
&= (r + 1 + t^{(\sigma_\Delta)}, \sigma_\Delta\sigma_U) \\
&= (r + 1, \sigma_\Delta)\langle \sigma_U, t^+, t^- \rangle \\
&= ((r, \sigma_\Delta)\langle +, 1, 1 \rangle)\langle \sigma_U, t^+, t^- \rangle \\
&= ((\Delta)(T_1))(U) \\
&= (\Delta)(T_1U).
\end{aligned}$$

Hence the mappings UT_1 and T_1U are identical, which is condition (c).

(c) $\Rightarrow$ (d). Suppose $UT_1 = T_1U$. Then

$$UT_2 = UT_1T_1 = T_1UT_1 = T_1T_1U = T_2U,$$

$$UT_3 = UT_2T_1 = T_2UT_1 = T_2T_1U = T_3U,$$

and, continuing in this way, U commutes likewise with $T_4, \ldots, T_{11}$. Of course T_0, the identity transformation, commutes with every transformation automatically.

(d) $\Rightarrow$ (a). Suppose U is a triadic transformation that commutes with

$$\begin{tikzcd} \Delta \arrow[r, "U"] \arrow[d, "T_n"'] & \Delta' \arrow[d, "T_n"] \\ (\Delta)(T_n) \arrow[r, "U"'] & (\Delta')(T_n) \end{tikzcd}$$

(a) Diagram illustrating commutativity of a UTT U with a transposition T_n

$$\begin{tikzcd} \mathrm{C} \arrow[r, "L"] \arrow[d, "T_1"'] & \mathrm{e} \arrow[d, "T_1"] \\ \mathrm{D}\flat \arrow[r, "L"'] & \mathrm{f} \end{tikzcd}$$

(b) The special case $U = L$, $n = 1$, $\Delta = \mathrm{C}$ major

Figure 1

every T_n. To establish the uniformity condition (as first presented in Section 1.5), we assume that U transforms (r, σ) to (r', σ') and endeavor to prove that U then transforms $(r + t, \sigma)$ to $(r' + t, \sigma')$. From the fact that U commutes with the transposition T_t, it follows that

$$\begin{aligned} (r + t, \sigma)(U) &= ((r, \sigma)(T_t))(U) \\ &= (r, \sigma)(T_tU) \\ &= (r, \sigma)(UT_t) \\ &= ((r, \sigma)(U))(T_t) \\ &= (r', \sigma')(T_t) \\ &= (r' + t, \sigma'), \end{aligned}$$

as desired. ■

The condition $UT_n = T_nU$ of part (d) of the theorem is illustrated by the diagram in Figure 1(a). Starting with any triad Δ, one arrives at the same point regardless of whether one chooses to apply the transformations in the order UT_n (traversing the top and right arrows in the square) or T_nU (traversing the left and bottom arrows). Figure 1(b) illustrates the point in the specific case in which $U = L$ (the leittonwechsel transformation), $n = 1$, and $\Delta = \mathrm{C}$ major. (Whenever two transformations commute, the situation may be represented by such a diagram; mathematicians say that "the diagram commutes.")

1.8. Let $\mathcal{U}$ denote the set of all UTTs. Then $\mathcal{U}$ is a subset of $\mathcal{G}$, the group of all triadic transformations introduced in Section 1.4. The preceding theorem shows that $\mathcal{U}$ consists precisely of those transformations in $\mathcal{G}$ that commute with T_1; $\mathcal{U}$ is therefore the *centralizer* of the element T_1, and as such is automatically a sub*group* of $\mathcal{G}$.[8]

Elements of $\mathcal{U}$ are in one-to-one correspondence with ordered triples $\langle\sigma, t^+, t^-\rangle$ in $\mathbf{Z}_2 \times \mathbf{Z}_{12} \times \mathbf{Z}_{12}$, and the number of such elements is therefore $2 \times 12 \times 12 = 288$. The *group* structure of $\mathcal{U}$ is not that of $\mathbf{Z}_2 \times \mathbf{Z}_{12} \times \mathbf{Z}_{12}$, however, because multiplication is not defined componentwise in $\mathcal{U}$ but rather is inherited from the larger group $\mathcal{G}$, where products are defined by composition of mappings. In this and the following sections, we shall derive formulas for products and inverses of UTTs based on their components.

As an example, consider the UTTs $U = \langle +, 4, 7\rangle$ and $V = \langle -, 5, 10\rangle$; let us calculate the product $UV = \langle\sigma_{UV}, t^+_{UV}, t^-_{UV}\rangle$. When UV acts on a C major triad, the result is

$$(0, +) \xrightarrow{U} (4, +) \xrightarrow{V} (9, -).$$

Thus UV transforms major triads through the Γ-interval $(9, -)$, from which we may deduce that $\sigma_{UV} = -$ and $t^+_{UV} = 9$. When UV acts on a C minor triad, the result is

$$(0, -) \xrightarrow{U} (7, -) \xrightarrow{V} (5, +).$$

(Note that $7 + 10 = 5$, because arithmetic is performed mod 12.) Thus UV transforms minor triads through the Γ-interval $(5, -)$, and $t^-_{UV} = 5$. Hence UV is the UTT $\langle -, 9, 5\rangle$. This product may be calculated by multiplying the signs ($\sigma_{UV} = \sigma_U\sigma_V$) and adding the corresponding transposition levels ($t^+_{UV} = t^+_U + t^+_V$, $t^-_{UV} = t^-_U + t^-_V$), as shown in Figure 2(a).

Now consider the same two UTTs $U = \langle +, 4, 7\rangle$ and $V = \langle -, 5, 10\rangle$ multiplied in the reverse order VU. In this case the actions on C major and C minor triads are $(0, +) \xrightarrow{V} (5, -) \xrightarrow{U} (0, -)$ and $(0, -) \xrightarrow{V} (10, +) \xrightarrow{U} (2, +)$, from which we deduce that $\sigma_{VU} = -$, $t^+_{VU} = 0$, and $t^-_{VU} = 2$, and therefore $VU = \langle -, 0, 2\rangle$. In this case the signs are multiplied as before, but the transposition levels are "cross-added" ($t^+_{VU} = t^+_V + t^-_U$, $t^-_{VU} = t^-_V + t^+_U$), as shown in Figure 2(b).

A little reflection shows that the difference between the procedures in Figures 2(a) and 2(b) stems from the sign of the *first* of the two transformations involved. In the first case the first UTT (U) was mode-preserving, so the second UTT (V) acted on a triad of the same mode as the first, and corresponding transposition levels were therefore applied in succession. In the second case, however, the first UTT (V) was mode-*reversing*, so the second UTT (U) acted on a triad of the *opposite* mode from the

$$+ - = -$$

$$\langle +, 4, 7\rangle\langle -, 5, 10\rangle = \langle -, 9, 5\rangle$$

$$4 + 5 = 9$$

$$7 + 10 = 5$$

(a) The product of two UTTs, the first of which is mode-preserving

$$+ - = -$$

$$\langle -, 5, 10\rangle\langle +, 4, 7\rangle = \langle -, 0, 2\rangle$$

$$5 + 7 = 0$$

$$10 + 4 = 2$$

(b) The product of two UTTs, the first of which is mode-reversing

Figure 2

first, and therefore opposite transposition levels were combined. In general, that is, if $\sigma_U = +$, then $UV = \langle\sigma_U\sigma_V, t^+{}_U + t^+{}_V, t^-{}_U + t^-{}_V\rangle$; if $\sigma_U = -$, then $UV = \langle\sigma_U\sigma_V, t^+{}_U + t^-{}_V, t^-{}_U + t^+{}_V\rangle$. These two formulas may be combined in a single general formula: if $U = \langle\sigma_U, t^+{}_U, t^-{}_U\rangle$ and $V = \langle\sigma_V, t^+{}_V, t^-{}_V\rangle$, then

$$UV = \langle\sigma_U\sigma_V, t^+{}_U + t^{(\sigma_U)}{}_V, t^-{}_U + t^{(-\sigma_U)}{}_V\rangle \qquad \textit{(product formula for UTTs)}.$$

1.9. In the example in the previous section, the products UV and VU turned out to be different, thus demonstrating that multiplication of UTTs is not in general commutative; the group $\mathcal{U}$, that is, is a *non-commutative* (or *non-abelian*) group. In fact, the only transformations in $\mathcal{U}$ that commute with *all* UTTs are the transpositions T_n ($n = 0, 1, \ldots, 11$). That each T_n commutes with every UTT was established in Theorem 1.7. To prove the converse, suppose $U = \langle\sigma, t^+, t^-\rangle$ commutes with every UTT. Then in particular U commutes with $P = \langle -, 0, 0\rangle$. But by the product formula, $UP = \langle -\sigma, t^+, t^-\rangle$ while $PU = \langle -\sigma, t^-, t^+\rangle$, so we must have $t^+ = t^-$. Let us call this common value n, so that either $U = \langle +, n, n\rangle$ or $U = \langle -, n, n\rangle$. If $U = \langle -, n, n\rangle$, then U fails to commute with the UTT $V = \langle +, 0, 1\rangle$, since $UV = \langle -, n + 1, n\rangle$ while $VU = \langle -, n, n + 1\rangle$. Hence $U = \langle +, n, n\rangle = T_n$, as asserted. The *center* of the group $\mathcal{U}$ is therefore the transposition group $\mathcal{T} = \{T_0, T_1, \ldots, T_{11}\}$, a cyclic group clearly isomorphic to $\mathbf{Z}_{12}$, the integers mod 12.[9]

The full story of UTT commutativity is as follows: mode-preserving UTTs always commute; a mode-preserving UTT U and a mode-reversing UTT V commute if and only if U is some transposition T_n; two mode-reversing UTTs $\langle -, m, n\rangle$ and $\langle -, i, j\rangle$ commute if and only if $n - m = j - i$. All these statements follow directly from the product formula.

1.10. The equation for the sign of the product of two UTTs, $\sigma_{UV} = \sigma_U\sigma_V$, may be self-evident, but it carries a deep algebraic significance. This equation asserts that the *sign function*—the mapping that maps each UTT $U = \langle \sigma_U, t^+_U, t^-_U\rangle$ to its sign σ_U—is a *homomorphism* from $\mathcal{U}$ to the two-element multiplicative group $\{+, -\}$. The *kernel* of this homomorphism is the set $\mathcal{U}^+$ consisting of the 144 mode-preserving UTTs (the transformations in $\mathcal{U}$ having positive sign). As such, $\mathcal{U}^+$ is automatically a normal subgroup of $\mathcal{U}$. Because $\langle +, t^+_U, t^-_U\rangle\langle +, t^+_V, t^-_V\rangle = \langle +, t^+_U + t^+_V, t^-_U + t^-_V\rangle$, we see that multiplication of UTTs acts componentwise on the elements of $\mathcal{U}^+$; the group $\mathcal{U}^+$ is therefore isomorphic to $\mathbf{Z}_{12} \times \mathbf{Z}_{12}$, a direct product of two cyclic groups of order 12. (In keeping with the observations of the previous section, $\mathcal{U}^+$ is a commutative group.)[10]

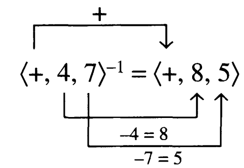
The other 144 UTTs, those that are mode-reversing, form a set $\mathcal{U}^-$, which is not a subgroup of $\mathcal{U}$. (Since the product of two mode-reversing UTTs is mode-preserving, $\mathcal{U}^-$ is not closed under multiplication.) We shall encounter an alternative description of the set $\mathcal{U}^-$ in Section 5.1 of this paper.

1.11. Several simple properties relating to transpositions are direct consequences of the product formula for UTTs. The reader should have little trouble verifying, for example, the formulas $T_mT_n = T_{m+n}$ and $(T_n)^k = T_{kn}$. Also, there is this interesting property: *the square of every mode-reversing UTT is a transposition.* Indeed, every mode-reversing UTT is of the form $U = \langle -, t^+, t^-\rangle$, and therefore

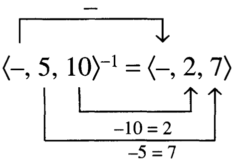
$$U^2 = \langle -, t^+, t^-\rangle\langle -, t^+, t^-\rangle = \langle +, t^+ + t^-, t^+ + t^-\rangle = T_n,$$

where $n = t^+ + t^-$. This number $t^+ + t^-$, the sum of the two transposition levels of the UTT U, will be called the *total transposition* of U and denoted $\tau(U)$. The last result shows, therefore, that if U is a mode-reversing UTT, then $U^2 = T_{\tau(U)}$.

1.12. The identity element of the group $\mathcal{U}$ is of course $T_0 = \langle +, 0, 0\rangle$; we have yet to derive a formula for the inverse of a UTT. As examples, let us calculate the inverses of the UTTs $U = \langle +, 4, 7\rangle$ and $V = \langle -, 5, 10\rangle$. Because $(0, +) \xrightarrow{U} (4, +)$ and $(0, -) \xrightarrow{U} (7, -)$, the inverse transformation U^{-1} must have the effect $(4, +) \xrightarrow{U^{-1}} (0, +)$ and $(7, -) \xrightarrow{U^{-1}} (0, -)$, from which we may conclude that $U^{-1} = \langle +, 8, 5\rangle$. Each transposition level, in other words, is simply inverted mod 12, as illustrated in Figure 3(a). In the case of V, however, the actions $(0, +) \xrightarrow{V} (5, -)$ and $(0, -) \xrightarrow{V} (10, +)$ imply that $(5, -) \xrightarrow{V^{-1}} (0, +)$ and $(10, +) \xrightarrow{V^{-1}} (0, -)$, from which we conclude that $V^{-1} = \langle -, 2, 7\rangle$; in this case the transposition levels are not



(a) Inverse of a mode-preserving UTT



(b) Inverse of a mode-reversing UTT

Figure 3

only inverted but also interchanged, as in Figure 3(b). As was the case with the product formula derived earlier, it is the sign of the UTT that is responsible for the difference in procedure. The inverse of a mode-preserving UTT $\langle +, t^+, t^- \rangle$ is $\langle +, -t^+, -t^- \rangle$, while the inverse of a mode-reversing UTT $\langle -, t^+, t^- \rangle$ is $\langle -, -t^-, -t^+ \rangle$. The general formula

$$\langle \sigma, t^+, t^- \rangle^{-1} = \langle \sigma, -t^{\sigma}, -t^{(-\sigma)} \rangle \qquad \textit{(inverse formula for UTTs)}$$

holds in both cases.

The reader should be able to use the inverse formula to calculate the inverses of the familiar UTTs listed in Section 1.6 above. The inverse of T_n is T_{-n} ($-n$ calculated mod 12). Each of the transformations P, L, and R is its own inverse, for reasons that will be more fully explored in Part 2. The inverses of the dominant and mediant transformations D and M are, respectively, the "subdominant" transformation $D^{-1} = \langle +, 7, 7 \rangle$ and the "submediant" transformation $M^{-1} = \langle -, 4, 3 \rangle$; we shall have more to say about D, M, and their inverses in Parts 3 and 4.

The following theorem summarizes the results of the preceding sections.

1.13. Theorem. The set $\mathcal{V}$ of all uniform triadic transformations is a non-commutative subgroup of $\mathcal{G}$ (the group of all triadic transformations) of order 288. Multiplication in $\mathcal{V}$ is given by the formula $\langle \sigma_U, t^+_U, t^-_U \rangle \langle \sigma_V, t^+_V, t^-_V \rangle = \langle \sigma_U \sigma_V, t^+_U + t^{(\sigma_U)}_V, t^-_U + t^{(-\sigma_U)}_V \rangle$. The identity element is $T_0 = \langle +, 0, 0 \rangle$, and the inverse of $\langle \sigma, t^+, t^- \rangle$ is $\langle \sigma, -t^{\sigma}, -t^{(-\sigma)} \rangle$. The center of $\mathcal{V}$

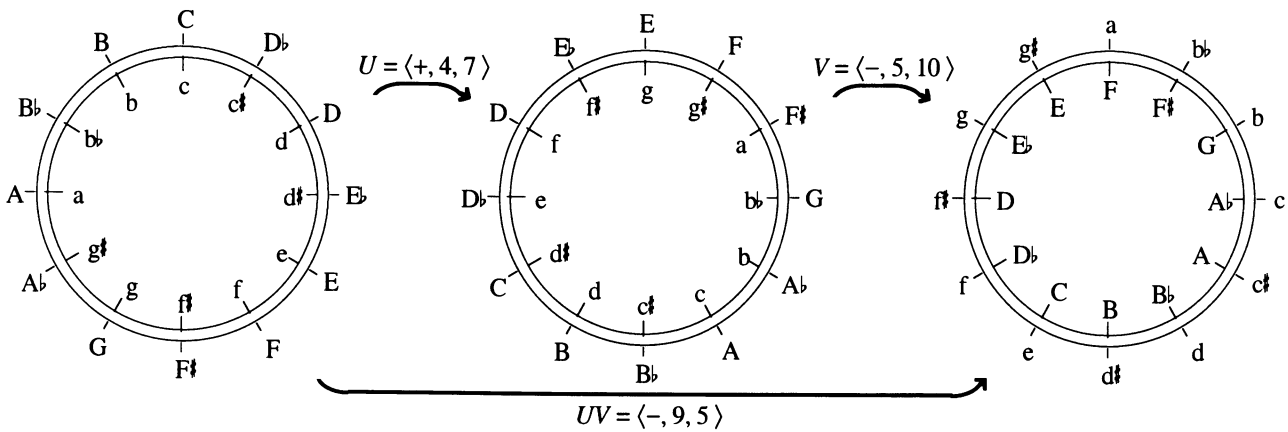


Figure 4. Dual-circle representation of the UTT equation $\langle +, 4, 7\rangle\langle -, 5, 10\rangle = \langle -, 9, 5\rangle$

is the transposition group $\mathcal{T} = \{T_0, T_1, \ldots, T_{11}\}$, a cyclic group of order 12. The mode-preserving UTTs form a commutative subgroup $\mathcal{U}^+$ of order 144, isomorphic to $\mathbf{Z}_{12} \times \mathbf{Z}_{12}$.

1.14. The action of UTTs may be conveniently visualized via a dual-circle representation similar to one developed by John Clough (1998). We define a *dual-circle configuration* to be a pair of circles (conveniently drawn concentrically) each marked off into twelve "hours," with the twelve major triads arranged in order around one circle, ascending by semitones in the clockwise direction, and the twelve minor triads similarly arranged around the other circle. Figure 4 shows three dual-circle configurations. In the first of these, the major triads have been assigned to the outer circle and the minor triads to the inner, and the C major and C minor triads are displayed at the twelve o'clock position—but these are arbitrary choices, not required by the definition.

A UTT $\langle \sigma, t^+, t^- \rangle$ acts on a dual-circle configuration in the following way. The major-triad circle is rotated *counterclockwise* through t^+ positions; the minor-triad circle is rotated *counterclockwise* through t^- positions; finally, if (but only if) $\sigma = -$, the major triads are changed to minor and vice versa. Figure 4 shows the results of applying the UTT $U = \langle +, 4, 7 \rangle$ to the initial configuration shown, and then applying $V = \langle -, 5, 10 \rangle$ to the result. The final configuration is the same as if we had applied the single transformation $\langle -, 9, 5 \rangle$—the product UV as calculated in Section 1.8.

A configuration may be represented in shorthand by an ordered pair of triads (Δ_1, Δ_2), where Δ_1 is the triad occupying the twelve o'clock position in the outer circle and Δ_2 occupies the same position in the inner circle. (Thus one triad in the pair will always be major and the other minor, but they may occur in either order.) In this way all of Figure 4 is represented by the simple notation $(\mathrm{C,c}) \xrightarrow{U} (\mathrm{E,g}) \xrightarrow{V} (\mathrm{a,F})$. The shorthand masks one attractive property of the dual-circle representations, however: the fact that a UTT acting on a configuration always transforms *every* triad to the triad that occupies the same position in the new configuration. In Figure 4, for example, the actions $\mathrm{b} \xrightarrow{U} \mathrm{f}\sharp \xrightarrow{V} \mathrm{E}$ may be read directly from the figure, at the eleven o'clock position on the inner circle.

Let Λ denote the set of all dual-circle configurations. Then the group $\mathcal{U}$ of all UTTs acts on Λ as described above. From the description of this action it is easy to see that if λ_1 and λ_2 are any two configurations in Λ, then there is exactly one UTT U that transforms λ_1 to λ_2; that is, the action of $\mathcal{U}$ on Λ is *simply transitive*.[11] Lewin (1987, 157–58) has shown that simply transitive group actions correspond to generalized interval systems. In this case, Λ forms a GIS whose group of intervals is $\mathcal{U}$. The "interval" from one dual-circle configuration λ_1 to another configuration λ_2 is the unique UTT that transforms λ_1 to λ_2. In Figure 4, for example, the interval between the first two configurations shown is precisely the UTT $U = \langle +, 4, 7 \rangle$.

2. Riemannian UTTs

2.1. Recall from Section 1.11 that the *total transposition* of a UTT $U = \langle \sigma, t^+, t^- \rangle$, denoted $\tau(U)$, is defined to be the sum of the transposition levels:

$$\tau(U) = t^+ + t^-.$$

The mapping τ defines a homomorphism from $\mathcal{U}$ to $\mathbf{Z}_{12}$, the additive group of integers mod 12. To verify this, we must show that $\tau(UV) = \tau(U) + \tau(V)$. In fact,

$$\begin{aligned}\tau(UV) &= \tau(\langle \sigma_U\sigma_V, t^+_U + t^{(\sigma_U)}{}_V, t^-_U + t^{(-\sigma_U)}{}_V \rangle) \\ &= t^+_U + t^{(\sigma_U)}{}_V + t^-_U + t^{(-\sigma_U)}{}_V,\end{aligned}$$

while

$$\tau(U) + \tau(V) = t^+_U + t^-_U + t^+_V + t^-_V;$$

these are the same because $t^{(\sigma_U)}{}_V$ and $t^{(-\sigma_U)}{}_V$ are, in one order or the other, the same as t^+_V and t^-_V.

2.2. The UTTs whose total transposition is 0 are of particular interest. This condition $\tau(U) = 0$, or equivalently $t^+ + t^- = 0$ or $t^+ = -t^-$, we shall call the *Riemannian dualism condition*, and UTTs satisfying it will be called *Riemannian* UTTs. As was suggested in the introduction to this paper, this condition codifies the symmetry inherent in Hugo Riemann's understanding of harmony: whatever a Riemannian transformation does to a major triad, its effect on a minor triad is equal and opposite.[12]

Several Riemannian transformations appear among the familiar UTTs enumerated in Section 1.6. The identity transformation $T_0 = \langle +, 0, 0 \rangle$ and the tritone transposition $T_6 = \langle +, 6, 6 \rangle$ are Riemannian, as are $P = \langle -, 0, 0 \rangle$, $L = \langle -, 4, 8 \rangle$, and $R = \langle -, 9, 3 \rangle$. Altogether there are 24 Riemannian UTTs: twelve mode-preserving transformations of the form $\langle +, n, -n \rangle$ and twelve mode-reversing transformations of the form $\langle -, n, -n \rangle$. In keeping with Riemann's terminology as adopted recently by Klumpenhouwer (1994) and others, we shall refer to UTTs of the former type as *schritts* and those of the latter type as *wechsels*; we shall also use the notations $S_n = \langle +, n, -n \rangle$ and $W_n = \langle -, n, -n \rangle$.[13] Thus every Riemannian UTT is some S_n or W_n; in particular, $T_0 = S_0$, $T_6 = S_6$, $P = W_0$, $L = W_4$, and $R = W_9$. The following identities follow immediately from the product and inverse formulas for UTTs in Part 1:

$$
\begin{aligned}
S_mS_n &= \langle +, m, -m\rangle\langle +, n, -n\rangle = \langle +, m+n, -m-n\rangle = S_{m+n};\\
S_mW_n &= \langle +, m, -m\rangle\langle -, n, -n\rangle = \langle -, m+n, -m-n\rangle = W_{m+n};\\
W_mS_n &= \langle -, m, -m\rangle\langle +, n, -n\rangle = \langle -, m-n, -m+n\rangle = W_{m-n};\\
W_mW_n &= \langle -, m, -m\rangle\langle -, n, -n\rangle = \langle +, m-n, -m+n\rangle = S_{m-n};\\
S_n^{-1} &= \langle +, n, -n\rangle^{-1} = \langle +, -n, n\rangle = S_{-n};\\
W_n^{-1} &= \langle -, n, -n\rangle^{-1} = \langle -, n, -n\rangle = W_n.
\end{aligned}
$$

The *unit schritt* $S_1 = \langle +, 1, 11\rangle$ will be seen to be of considerable theoretical importance, and we shall find it convenient to designate S_1 by the single symbol Q.[14] Thus Q transposes a major triad up a semitone and a minor triad down a semitone ($\mathrm{C} \xrightarrow{Q} \mathrm{D}\flat$, $\mathrm{c} \xrightarrow{Q} \mathrm{b}$). It is easy to see that Q^{12} is the identity transformation T_0, that any other schritt S_n is simply the nth power of the unit schritt Q (that is, $S_n = Q^n$), and that the wechsel W_n is equal to $S_nP = Q^nP$.

The Riemannian UTTs are outnumbered by the non-Riemannian, 264 to 24. In particular, the ten transpositions T_n with $n \neq 0, 6$ and the dominant and mediant transformations $D = \langle +, 5, 5\rangle$ ($= T_5$) and $M = \langle -, 9, 8\rangle$ are *not* Riemannian. It must be remembered that the various special properties of Riemannian UTTs to be derived in the following sections generally do not apply to these other transformations. On the other hand, it bears repeating at this point that the general properties of UTTs presented in Part 1—in particular, the product and inverse formulas—apply to *all* UTTs, Riemannian or otherwise. In principle there is no reason why we should not, for example, multiply L with D, or combine them in the same transformation network, as long as we do not make any Riemannian claims for D.[15]

2.3. The set of all 24 Riemannian UTTs will be denoted $\mathcal{R}$. Thus $\mathcal{R}$ is a subset of $\mathcal{U}$; in fact, $\mathcal{R}$ is by definition the kernel of the homomorphism τ, and is therefore a normal subgroup of $\mathcal{U}$. We shall call $\mathcal{R}$ the *Riemann group*.

The twelve mode-preserving Riemannian UTTs (schritts) form a subset $\mathcal{R}^+$; in fact, $\mathcal{R}^+$ is the intersection of the normal subgroups $\mathcal{R}$ and $\mathcal{U}^+$ of $\mathcal{U}$, and is therefore itself a normal subgroup of $\mathcal{U}$, which we shall call the *schritt group*. The twelve mode-reversing Riemannian UTTs (wechsels) form a subset $\mathcal{R}^-$, which is not a subgroup of $\mathcal{U}$.

Theorem 2.5 below enumerates a number of important properties of Riemannian transformations and the groups $\mathcal{R}$ and $\mathcal{R}^+$. First we digress to prove a preliminary lemma, offering a simple but useful characterization of simply transitive groups of UTTs.

2.4. Lemma. A subgroup $\mathcal{K}$ of $\mathcal{U}$ acts in simply transitive fashion on the set Γ of triads if and only if both of the following conditions are satisfied:

(a) For each sign σ (= + or −) and for every integer m (mod 12), there is exactly one integer n (mod 12) such that the UTT $\langle \sigma, m, n \rangle$ belongs to the subgroup $\mathcal{K}$.

(b) For each σ and every n (mod 12), there is exactly one m (mod 12) such that $\langle \sigma, m, n \rangle$ belongs to $\mathcal{K}$.

Proof. Suppose first that $\mathcal{K}$ is simply transitive. This means, by definition, that for any two triads Δ_1 and Δ_2, there is exactly one U in $\mathcal{K}$ that maps Δ_1 to Δ_2. Given σ and m as in (a), observe that a UTT U is of the form $\langle \sigma, m, n \rangle$ for some n if and only if U maps the C major triad $(0, +)$ to the triad (m, σ). So we let $\Delta_1 = (0, +)$ and $\Delta_2 = (m, \sigma)$. By simple transitivity there is exactly one U in $\mathcal{K}$ mapping Δ_1 to Δ_2, hence exactly one n as in statement (a). The proof of (b) proceeds the same way, using $\Delta_1 = (0, -)$ and $\Delta_2 = (n, -\sigma)$.

Conversely, suppose that (a) and (b) hold, and let $\Delta_1 = (r_1, \sigma_1)$ and $\Delta_2 = (r_2, \sigma_2)$ be any triads. We wish to show that one and only one U in $\mathcal{K}$ maps Δ_1 to Δ_2. If $\sigma_1 = +$, then U must be of the form $\langle \sigma_2, r_2 - r_1, n \rangle$ for some n; there is a unique such n, hence a unique such U in $\mathcal{K}$, by condition (a). Likewise, if $\sigma_1 = -$, then U must be of the form $\langle -\sigma_2, m, r_2 - r_1 \rangle$ for some m; such a U exists uniquely in $\mathcal{K}$ by condition (b). ■

2.5. Theorem. The Riemann group $\mathcal{R}$ and its subsets $\mathcal{R}^+$ and $\mathcal{R}^-$ have the following properties:

(a) $\mathcal{R}$ is the subgroup of $\mathcal{U}$ generated by the parallel transformation $P = \langle -, 0, 0 \rangle$ and the unit schritt $Q = \langle +, 1, 11 \rangle$.

(b) $\mathcal{R}$ is the subgroup of $\mathcal{U}$ generated by the leittonwechsel $L = \langle -, 4, 8 \rangle$ and the relative transformation $R = \langle -, 9, 3 \rangle$.

(c) $\mathcal{R}$ is isomorphic to $\mathbf{D}_{12}$, the dihedral group of order 24.

(d) $\mathcal{R}$ acts in simply transitive fashion on the set Γ.

(e) $\mathcal{R}^+$ is a cyclic group of order 12, generated by the unit schritt Q.

(f) $\mathcal{R}^+$ acts in simply transitive fashion on each of the transposition classes Γ^+ and Γ^-.

(g) If $U \in \mathcal{R}^-$, then $U^{-1} = U$.

(h) If $U_1, U_2, \ldots, U_k \in \mathcal{R}^-$, then $(U_1 U_2 \cdots U_k)^{-1} = U_k \cdots U_2 U_1$.

(i) If $U_1, U_2, \ldots, U_k \in \mathcal{R}^-$ and k is odd, then $U_1 U_2 \cdots U_k = U_k \cdots U_2 U_1$.

Proof. (a) We have already observed that every Riemannian transformation can be written as either $S_n = Q^n$ or $W_n = Q^nP$; hence P and Q generate the group $\mathcal{R}$.

(b) In light of (a), it suffices to show that P and Q may be expressed as products of L and R. By direct calculation,

$$RL = \langle -, 9, 3\rangle\langle -, 4, 8\rangle = \langle +, 5, 7\rangle = Q^5.$$

Because Q^{12} is the identity T_0, it follows that

$$(RL)^5 = (Q^5)^5 = Q^{25} = (Q^{12})^2Q = Q,$$

and also that

$$(RL)^3R = (Q^5)^3R = Q^{15}R = Q^{12}Q^3R = Q^3R = \langle +, 3, 9\rangle\langle -, 9, 3\rangle = \langle -, 0, 0\rangle = P.$$

(c) The dihedral group $\mathbf{D}_k$ may be defined[16] as the group of order $2k$ generated by x and y, where $x^2 = 1$ (the identity), $y^k = 1$, and $xy = y^{-1}x$. In our case $k = 12$ and the generators are $x = P$ and $y = Q$. Clearly P^2 is the identity T_0, and we have already observed that $Q^{12} = T_0$ also. By the properties of schritts and wechsels in Section 2.2 above,

$$PQ = W_0S_1 = W_{11} = S_{11}W_0 = Q^{11}P = Q^{-1}P.$$

(d) Clearly $\mathcal{R}$ satisfies conditions (a) and (b) of Lemma 2.4. (In both cases, the Riemannian dualism condition holds if and only if $m = -n$.) By the lemma, therefore, the action of $\mathcal{R}$ on Γ is simply transitive.

(e) We have already observed that the elements of $\mathcal{R}^+$ are precisely the schritts $S_1 = Q$, $S_2 = Q^2$, …, $S_{11} = Q^{11}$, and the identity T_0 (= Q^{12}). Thus $\mathcal{R}^+$ is a cyclic group of order 12 generated by Q.

(f) If Δ_1 and Δ_2 are two triads in the same transposition class (Γ^+ or Γ^-), then the unique Riemannian UTT that maps Δ_1 to Δ_2 (which exists by (d)) is mode-preserving, hence an element of $\mathcal{R}^+$. Therefore the action of $\mathcal{R}^+$ is simply transitive on both Γ^+ and Γ^-.

(g) The elements of $\mathcal{R}^-$ are precisely the wechsels W_n. The fact that $W_n^{-1} = W_n$ was observed in Section 2.2 above.

(h) If $U_1, U_2, \ldots, U_k \in \mathcal{R}^-$, then

$$(U_1U_2\cdots U_k)^{-1} = U_k^{-1}\cdots U_2^{-1}U_1^{-1} = U_k\cdots U_2U_1.$$

(The first equality holds in any group; the second follows because, by (g), each U_i is its own inverse.)

(i) The product of an odd number of mode-reversing transformations is mode-reversing. Therefore if $U_1, U_2, \ldots, U_k \in \mathcal{R}^-$ and k is odd, the product $U_1U_2\cdots U_k$ belongs to $\mathcal{R}^-$ as well. Hence by (g) above, $U_1U_2\cdots U_k$ is its own inverse, $(U_1U_2\cdots U_k)^{-1} = U_1U_2\cdots U_k$. But also $(U_1U_2\cdots U_k)^{-1} = U_k\cdots U_2U_1$ by (h); therefore $U_1U_2\cdots U_k = U_k\cdots U_2U_1$. ∎

2.6. The significance of the Riemann group, and in fact most of the properties in the preceding sections, have been recognized before—without, of course, the UTT formalism presented here. Klumpenhouwer (1994), building upon Riemann (1880), enumerates twelve schritts and twelve wechsels forming a system completely equivalent to $\mathcal{R}$; Clough (1998) and others refer to the Riemann group as "the *S*/*W* group."

Cohn (1997) and several subsequent writers have described Riemannian (or "neo-Riemannian") transformations in terms of the "*PLR* family" of operations—all transformations that can be written as products of P, L, and R. Part (b) of the above theorem shows that the inclusion of P in this list is actually redundant: L and R by themselves suffice to generate the Riemann group. In fact, as shown in the proof, $P = RLRLRLR$. (The fact that L and R generate all Riemannian transformations has also been noted previously; see, for example, Gollin 1998, 205, note 13.) On the other hand, allowing the use of P often makes possible much shorter expressions for Riemannian transformations than those involving L and R alone. To write the tritone transposition T_6 in terms of L and R, for example, requires no fewer than twelve factors ($T_6 = (LR)^6 = (RL)^6$); when P is allowed it can be written with only four ($T_6 = PRPR = RPRP$).

There is another familiar musical transformation group that is isomorphic to the dihedral group $\mathbf{D}_{12}$: this is the group $\mathcal{J} = \{T_0, T_1, \ldots, T_{11}, I, IT_1, \ldots, IT_{11}\}$, the "*T*/*I* group" of transpositions and inversions, widely used in pitch-class set theory. (The notation IT_n follows our convention of left-to-right orthography; in pc-set theory this transformation is more often written T_nI.) The group $\mathcal{J}$ is generated by $x = I$ and $y = T_1$. Therefore $\mathcal{R}$ is isomorphic to $\mathcal{J}$ via the mapping $P \to I$, $Q \to T_1$. This isomorphism, too, has been noted before; it figures prominently in Clough 1998, and will be revisited in Part 5 of this paper.

Finally, parts (g) and (h) of Theorem 2.5 are essentially restatements of two theorems of Cohn (1997, 60–61, Theorems 2 and 3). Cohn expresses his results in terms of "odd-numbered compounds" (products of an odd number of factors) of P, L, and R. Clearly these odd-numbered compounds are precisely the wechsels (elements of $\mathcal{R}^-$), while the even-numbered compounds are the schritts (elements of $\mathcal{R}^+$). Framing the results in terms of the group $\mathcal{R}$, as above, simplifies the statements of the theorems, and the UTT formalism provides an elegant and tidy proof.

2.7. Because so many different notations have been used for Riemannian transformations, it seems useful to tabulate all 24 of them, together with the various expressions for them and other names by which they have been known. This is done in Table 1. The first column describes the action of each UTT on C major and C minor triads, the second gives the componentwise notation for each UTT, and the third lists our schritt/wechsel notation (which also appears in Clough 2000).[17] Gollin (1998,

203) also uses the notations S_n and W_n, but he numbers the wechsels differently, as shown in the fourth column. Gollin's system is explicitly dualistic, in that his subscripts identify intervals not between roots but between Riemannian "dual roots"—roots of major triads, fifths of minor triads. Gollin's W_0 is thus the wechsel that preserves the dual root, as in the relationship between C major and F minor—our W_5. The system employed here has the advantage of relating more directly to the UTT components. Of course, the algebraic structure of the group is the same regardless of which numbering one prefers.

The German names in the fifth column originated with Riemann (1880) and have recently been used by Klumpenhouwer (1994, §15–29, and 2000, 168). Riemann and Klumpenhouwer call each transformation an *X-schritt* or *X-wechsel*, where *X* names some interval—the interval between dual roots, and therefore the same interval that appears as the subscript *n* in Gollin's S_n/W_n notation, measured in semitones. For example, the *Ganztonschritt* is S_2, a whole-tone (*Ganzton*) being equal to 2 semitones. (Klumpenhouwer does not use the S_n/W_n notation, but his system of nomenclature and his rigorously dualistic approach would naturally lead him to favor Gollin's wechsel-numbering.) The only one of these names to have gained any currency in English usage is, of course, the *Leittonwechsel*.

The next three columns give the simplest expression (or expressions) for each Riemannian transformation in terms of *P* and *Q* (as in part (a) of Theorem 2.5), in terms of *L* and *R* (as in part (b)), and in terms of *P*, *L*, and *R*. The last of these representations has both advantages and disadvantages in comparison with the others: no more than five factors are ever required, but there are often an assortment of several equally complex expressions from which to choose. In particular, by part (i) of Theorem 2.5, any odd-numbered compound of *P*, *L*, and *R* may be reversed; numerous examples may be found among the wechsels in Table 1. Calculating (and simplifying) the product of two transformations written as *PLR*-compounds, clearly, can be a cumbersome process.

The penultimate column of the table lists the "canonical form" of each Riemannian transformation as it appears in the group of 144 transformations presented by Hyer (1995, 120). Some of these expressions involve D^6, where *D* is the dominant transformation; *D* itself, as noted in Section 2.2 above, is not a Riemannian transformation, but $D^6 = T_6$ is. Hyer's full group, a subgroup of $\mathcal{U}$, will be studied in some detail in Part 4 of this paper.

The final column in Table 1 lists other names by which a few of the Riemannian transformations have been known. The use of *I* for the identity transformation is perhaps unfortunate because of the conflict with the even more common use of *I* for pc-set inversion; Lewin often writes IDENT for the identity. The occasional use of *E* (e.g., Clampitt 1998,

Table 1. The 24 Riemannian UTTs

	Action of transformation		UTT	S_n/W_n	S_n/W_n (Gollin)	Riemann/ Klumpenhouwer	P/Q	L/R	$P/L/R$ (shortest)	Hyer	Other names
Schritts (elements of $\mathcal{R}^+$)	C → C,	c → c	⟨+, 0, 0⟩	S_0	S_0	Identity	(P^0Q^0)	(L^0R^0)	$(P^0L^0R^0)$	(D^0)	T_0, I, ID, IDENT, E
	C → D♭,	c → b	⟨+, 1, 11⟩	S_1	S_1	*Gegenleittonschritt*	Q	$(RL)^5$	$LPRP$, $PLRL$, $RLPL$, $RPLP$	D^6LR	Unit schritt
	C → D,	c → b♭	⟨+, 2, 10⟩	S_2	S_2	*Ganztonschritt*	Q^2	$(LR)^2$	$LRLR$, $RLRP$, $RPRL$	D^6PL	
	C → E♭,	c → a	⟨+, 3, 9⟩	S_3	S_3	*Gegenkleinterzschritt, Gegensextschritt*	Q^3	$(RL)^3$	PR	D^6RP	
	C → E,	c → g♯	⟨+, 4, 8⟩	S_4	S_4	*Terzschritt*	Q^4	$(LR)^4$	LP	LP	
	C → F,	c → g	⟨+, 5, 7⟩	S_5	S_5	*Gegenquintschritt*	Q^5	RL	RL	RL	
	C → F♯,	c → f♯	⟨+, 6, 6⟩	S_6	S_6	*Tritonusschritt*	Q^6	$(LR)^6$, $(RL)^6$	$PRPR$, $RPRP$	D^6	T_6, octatonic pole
	C → G,	c → f	⟨+, 7, 5⟩	S_7	S_7	*Quintschritt*	Q^7	LR	LR	LR	
	C → A♭,	c → e	⟨+, 8, 4⟩	S_8	S_8	*Gegenterzschritt*	Q^8	$(RL)^4$	PL	PL	
	C → A,	c → d♯	⟨+, 9, 3⟩	S_9	S_9	*Kleinterzschritt, Sextschritt*	Q^9	$(LR)^3$	RP	RP	
	C → B♭,	c → d	⟨+, 10, 2⟩	S_{10}	S_{10}	*Gegenganztonschritt*	Q^{10}	$(RL)^2$	$LRPR$, $PRLR$, $RLRL$	D^6LP	
	C → B,	c → c♯	⟨+, 11, 1⟩	S_{11}	S_{11}	*Leittonschritt*	Q^{11}	$(LR)^5$	$LPLR$, $LRLP$, $PLPR$, $PRPL$	D^6RL	

Wechsels (*elements of* $\mathcal{R}^-$)

$C \to c$,	$c \to C$	$\langle -, 0, 0 \rangle$	W_0	W_7	*Quintwechsel*	P	$(RL)^3R$	P	P	PAR
$C \to c\sharp$,	$c \to B$	$\langle -, 1, 11 \rangle$	W_1	W_8	*Gegenterzwechsel*	QP, PQ^{11}	$(LR)^3L$	LPR, RPL	D^6LRP	P', SLIDE, S
$C \to d$,	$c \to B\flat$	$\langle -, 2, 10 \rangle$	W_2	W_9	*Kleinterzwechsel, Sextwechsel*	Q^2P, PQ^{10}	RLR	RLR	D^6PLP	
$C \to d\sharp$,	$c \to A$	$\langle -, 3, 9 \rangle$	W_3	W_{10}	*Gegenganztonwechsel*	Q^3P, PQ^9	$(LR)^5L$	PRP	D^6R	
$C \to e$,	$c \to A\flat$	$\langle -, 4, 8 \rangle$	W_4	W_{11}	*Leittonwechsel*	Q^4P, PQ^8	L	L	L	LT
$C \to f$,	$c \to G$	$\langle -, 5, 7 \rangle$	W_5	W_0	*Seitenwechsel*	Q^5P, PQ^7	$(RL)^4R$	PLR, RLP	RLP	L', N (*nebenverwandt*), stride
$C \to f\sharp$,	$c \to F\sharp$	$\langle -, 6, 6 \rangle$	W_6	W_1	*Gegenleittonwechsel*	Q^6P, PQ^6	$(LR)^2L$	RPR	D^6P	
$C \to g$,	$c \to F$	$\langle -, 7, 5 \rangle$	W_7	W_2	*Ganztonwechsel*	Q^7P, PQ^5	$(RL)^2R$	LRP, PRL	LRP	R'
$C \to g\sharp$,	$c \to E$	$\langle -, 8, 4 \rangle$	W_8	W_3	*Gegenkleinterzwechsel, Gegensextwechsel*	Q^8P, PQ^4	$(LR)^4L$	LPL, PLP	PLP	H (hexatonic pole)
$C \to a$,	$c \to E\flat$	$\langle -, 9, 3 \rangle$	W_9	W_4	*Terzwechsel*	Q^9P, PQ^3	R	R	R	REL
$C \to b\flat$,	$c \to D$	$\langle -, 10, 2 \rangle$	W_{10}	W_5	*Gegenquintwechsel*	$Q^{10}P$, PQ^2	$(RL)^5R$	$LPRPR$, $LRPRP$, $PLRLR$, $PRLRP$, $PRPRL$, $RLPLR$, $RLRLP$, $RPLPR$, $RPRPL$	D^6L	*Tonnetz* pole
$C \to b$,	$c \to D\flat$	$\langle -, 11, 1 \rangle$	W_{11}	W_6	*Tritonuswechsel*	$Q^{11}P$, PQ	LRL	LRL	D^6RLP	

327) probably derives from the common use of e for the identity in group theory. Lewin (1987, 178) uses PAR, LT, and REL for P, L, and R, and also introduces SLIDE for W_1 (C $\rightarrow$ c♯, c $\rightarrow$ B). Cohn (1998b, 289–90) shortens SLIDE to S at the risk of some confusion: this S is a wechsel, not a schritt, and both Lewin (1992) and Klumpenhouwer (1994) have used S for the (non-Riemannian) "subdominant" transformation D^{-1}. Cohn also uses the notation N (*nebenverwandt*) for W_5; the term is from Weitzmann 1853, and the relation is studied in detail in Cohn 2000. Klumpenhouwer (2000, 163) points out that the *nebenverwandt* corresponds precisely to what Goetschius (1900, 114) called the *stride*: "a perfect fifth downward from any major keynote, and upward from any minor keynote, with a change of mode." (This apparently dualistic definition notwithstanding, Goetschius's approach to harmony was not primarily Riemannian in its conception.)

The *hexatonic pole* relation W_8 is studied at length in Cohn 1996; the notation H, attributed to Robert Cook, appears in Cohn 1998b. This transformation, equivalent to LPL or PLP, maps any triad to the triad farthest removed from it on a hexatonic LP- (or PL-) cycle. Analogously, S_6 ($= T_6 = D^6$), equivalent to $PRPR$ or $RPRP$, maps any triad to the most distant point on an octatonic PR- (or RP-) cycle, and therefore may be called the *octatonic pole*. Because W_{10} requires a longer PLR-expression than any other Riemannian transformation, this wechsel requires the greatest number of moves on the Riemannian *Tonnetz* for its representation, and has accordingly been dubbed the *Tonnetz pole* by Scott Murphy (2001).

The symbols P', L', and R', introduced in Morris 1998, highlight a certain duality between these three wechsels and P, L, and R. Taking P' as an example, we note that P' (our W_1, Lewin's SLIDE) may be written as either LPR or RPL, with L and R in either arrangement about a central P. (Again by part (i) of Theorem 2.5, $LPR = RPL$.) Applied to any triad, P' preserves the third but alters the root and the fifth—precisely the opposite of P, which alters only the third. Corresponding remarks apply to L' and R'.

2.8. The Riemannian dualism condition takes a simple geometric form when UTTs act on dual-circle configurations as described in Section 1.14. Under the action of a Riemannian UTT, the two circles in any initial configuration are effectively rotated through the same number of positions, but in opposite directions. Consequently, if we consider the pitch-class numbers (mod 12) associated with the roots of the triads at corresponding positions on the two circles, we observe that, when the UTT is applied, one of the numbers increases and the other decreases by the same amount, so that the sum of the two is unchanged. This root-sum-preservation property of Riemannian UTTs is illustrated in Figure 5. Here the Riemannian UTT $W_5 = \langle -, 5, 7 \rangle$ acts on the initial dual-circle

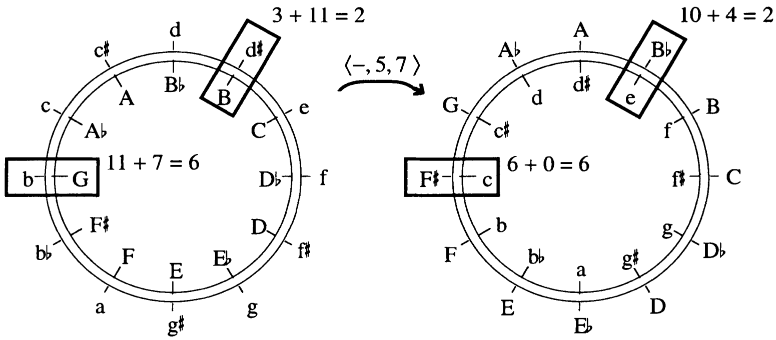


Figure 5. The root-sum-preservation property of the action of Riemannian UTTs on dual-circle configurations

configuration (d, B♭), producing the final configuration (A, d♯) (in the shorthand of Section 1.14). In both the initial and final configurations, root-sum calculations are shown at the one o'clock and nine o'clock positions, although any other clock positions could have been chosen. The reader may check that the UTTs diagrammed previously in Figure 4, which are not Riemannian, do not have this property.

2.9. We conclude Part 2 by introducing a variant of the total transposition function, which gives rise to another subgroup, closely related to $\mathcal{R}$. If $U = \langle \sigma, t^+, t^- \rangle$ is a UTT, we define the *skew transposition* of U, denoted $\tau^*(U)$, to be

$$\tau^*(U) = \begin{cases} \tau(U) = t^+ + t^- & \text{if } \sigma = +, \\ \tau(U) + 6 = t^+ + t^- + 6 & \text{if } \sigma = -. \end{cases}$$

In section 2.1 we showed that the total transposition τ is a homomorphism from $\mathcal{U}$ to $\mathbf{Z}_{12}$; using this result, it is a straightforward matter to prove that τ^* is also a homomorphism. The kernel of τ^* is a normal subgroup $\mathcal{R}^*$ of $\mathcal{U}$, which we call the group of *skew-Riemannian* transformations. The mode-preserving transformations in $\mathcal{R}^*$ are the same as those in $\mathcal{R}$, namely the schritts $S_n = \langle +, n, -n \rangle$. The mode-reversing transformations in $\mathcal{R}^*$, however, are *skew-wechsels* of the form $W_n^* = \langle -, n, -n + 6 \rangle$, a sort of "tritone substitution" for the wechsels $W_n = \langle -, n, -n \rangle$ in $\mathcal{R}$. Clearly $\mathcal{R}^*$ satisfies conditions (a) and (b) of Lemma 2.4; therefore $\mathcal{R}^*$, like $\mathcal{R}$, acts on Γ in simply transitive fashion. In Part 3 of this paper we shall see how $\mathcal{R}$ and $\mathcal{R}^*$ figure into a rich network of simply transitive subgroups of $\mathcal{U}$.

3. Simply Transitive Groups of UTTs

3.1. As noted in Section 1.6, the UTT that transforms one given triad to another is not uniquely determined. Given the equation $(\Delta_1)(U) = \Delta_2$ and the triads Δ_1 and Δ_2, we can determine the sign of U and one of its transposition levels, but the other transposition level is free to vary; in general, therefore, there are twelve possible transformations U satisfying the given equation. If, however, we know that U must belong to a certain simply transitive subgroup of the UTT group $\mathcal{U}$, then the situation is clarified, for simple transitivity ensures that exactly one such U acts as specified by the equation. Simple transitivity is thus an attractive property for a transformation group to have, and such groups are in many ways more convenient to work with than larger groups such as $\mathcal{U}$ itself.

Two simply transitive subgroups of $\mathcal{U}$ were introduced in Part 2, namely the groups $\mathcal{R}$ and $\mathcal{R}^*$ of Riemannian and skew-Riemannian UTTs, respectively. It should be clear that every simply transitive group of UTTs must, like $\mathcal{R}$ and $\mathcal{R}^*$, have exactly 24 elements. (There are 24 different triads to which the C major triad, for example, can be mapped by a transformation, and in a simply transitive group there must be exactly one transformation to accomplish each of these actions.) Not every order-24 subgroup of $\mathcal{U}$ is simply transitive, however. As an example of one that is not, consider all UTTs of the forms $\langle +, n, 0\rangle$ and $\langle +, n, 6\rangle$, as n ranges through the integers mod 12. These transformations form a group, but it cannot possibly act in simply transitive fashion on Γ since it contains no mode-reversing transformations.

In this part of the paper, we shall identify and categorize all the simply transitive subgroups of $\mathcal{U}$.

3.2. Given integers a and b (mod 12), we define three subsets of $\mathcal{U}$ as follows. First, $\mathcal{K}^+(a)$ is the set of all mode-preserving UTTs of the form $\langle +, n, an\rangle$ as n ranges through the integers mod 12. Next, $\mathcal{K}^-(a, b)$ is the set of all mode-reversing UTTs of the form $\langle -, n, an + b\rangle$. Finally, $\mathcal{K}(a, b)$ is the set of all UTTs of both of these types; that is, $\mathcal{K}(a, b) = \mathcal{K}^+(a) \cup \mathcal{K}^-(a, b)$. It should be clear that, for fixed a and b, $\mathcal{K}^+(a)$ and $\mathcal{K}^-(a, b)$ are each of cardinality 12, and therefore $\mathcal{K}(a, b)$ is of cardinality 24. It is also easy to see that every set $\mathcal{K}^+(a)$ is a group. We shall find that $\mathcal{K}(a, b)$ is a group only for certain values of a and b, and that all the simply transitive subgroups of $\mathcal{U}$ are among these sets $\mathcal{K}(a, b)$.

3.3. Lemma. $\mathcal{K}(a, b)$ is a subgroup of $\mathcal{U}$ if and only if the numbers a and b satisfy $a^2 = 1$ and $ab = b$ (mod 12).

Proof. First suppose $\mathcal{K}(a, b)$ is a group. The UTTs $\langle -, 0, b\rangle$ and $\langle +, 1, a\rangle$ belong to $\mathcal{K}^-(a, b)$ and $\mathcal{K}^+(a)$ respectively and therefore to $\mathcal{K}(a, b)$; therefore their product $\langle -, 0, b\rangle\langle +, 1, a\rangle = \langle -, a, b + 1\rangle$ must belong to

$\mathcal{K}(a, b)$ as well, and therefore to $\mathcal{K}^-(a, b)$. It follows that $\langle -, a, b + 1\rangle$ is of the form $\langle -, n, an + b\rangle$ for some n. Hence $n = a$ and $a^2 + b = b + 1$, so that $a^2 = 1$. Also, $\langle -, 1, a + b\rangle$ is in $\mathcal{K}^-(a, b)$ and therefore in $\mathcal{K}(a, b)$, so $\langle -, 1, a + b\rangle\langle -, 0, b\rangle = \langle +, b + 1, a + b\rangle$ must belong to $\mathcal{K}^+(a)$, and must therefore be of the form $\langle +, n, an\rangle$. Hence $a(b + 1) = a + b$, so that $ab = b$.

Conversely, suppose $a^2 = 1$ and $ab = b$. To prove that $\mathcal{K}(a, b)$ is a subgroup, we must show that all possible products and inverses of elements of $\mathcal{K}(a, b)$ belong to $\mathcal{K}(a, b)$. We therefore consider the various cases of products and inverses of the UTTs in $\mathcal{K}^+(a)$ and $\mathcal{K}^-(a, b)$. For products there are four cases:

(1) $\langle +, m, am\rangle\langle +, n, an\rangle = \langle +, m + n, am + an\rangle$
$= \langle +, (m + n), a(m + n)\rangle \in \mathcal{K}^+(a)$.

(2) $\langle +, m, am\rangle\langle -, n, an + b\rangle = \langle -, m + n, am + an + b\rangle$
$= \langle -, (m + n), a(m + n) + b\rangle \in \mathcal{K}^-(a, b)$.

(3) $\langle -, m, am + b\rangle\langle +, n, an\rangle = \langle -, m + an, am + b + n\rangle$. Now $a(m + an) + b = am + a^2n + b = am + n + b = am + b + n$ (using $a^2 = 1$), so this product is of the proper form and belongs to $\mathcal{K}^-(a, b)$.

(4) $\langle -, m, am + b\rangle\langle -, n, an + b\rangle = \langle +, m + an + b, am + b + n\rangle$. Now $a(m + an + b) = am + a^2n + ab = am + n + b = am + b + n$ (using $a^2 = 1$ and $ab = b$), so this product belongs to $\mathcal{K}^+(a)$.

For inverses there are two cases:

(1) $\langle +, n, an\rangle^{-1} = \langle +, -n, -an\rangle = \langle +, (-n), a(-n)\rangle \in \mathcal{K}^+(a)$.

(2) $\langle -, n, an + b\rangle^{-1} = \langle -, -(an + b), -n\rangle$. Now $a(-(an + b)) + b = -a^2n - ab + b = -n$ (using $a^2 = 1$ and $ab = b$), so this inverse belongs to $\mathcal{K}^-(a, b)$. ■

3.4. The condition $a^2 = 1$ is satisfied only for $a = 1, 5, 7$, and 11. (These are precisely the *units*, or multiplicatively invertible elements, of $\mathbf{Z}_{12}$.) If $a = 1$ then the condition $ab = b$ is automatically satisfied. For the other values of a, the allowable values of b are different in each case. The following is the complete list of the groups $\mathcal{K}(a, b)$:

$\mathcal{K}(1, 0), \mathcal{K}(1, 1), \mathcal{K}(1, 2), \ldots, \mathcal{K}(1, 11)$

$\mathcal{K}(5, 0), \mathcal{K}(5, 3), \mathcal{K}(5, 6), \mathcal{K}(5, 9)$

$\mathcal{K}(7, 0), \mathcal{K}(7, 2), \mathcal{K}(7, 4), \mathcal{K}(7, 6), \mathcal{K}(7, 8), \mathcal{K}(7, 10)$

$\mathcal{K}(11, 0), \mathcal{K}(11, 6)$

Of the 24 groups in this list, the last two are already familiar: from the definition of $\mathcal{K}(a, b)$ it follows immediately that $\mathcal{K}(11, 0)$ is the Riemann group $\mathcal{R}$ while $\mathcal{K}(11, 6)$ is the group $\mathcal{R}^*$ of skew-Riemannian UTTs. The group $\mathcal{K}(1, 0)$ is also particularly easy to describe: it consists of ($\mathcal{K}^+(1)$) all transpositions $T_n = \langle +, n, n \rangle$ and ($\mathcal{K}^-(1, 0)$) their mode-reversing counterparts $T_nP = \langle -, n, n \rangle$. That is, the elements of $\mathcal{K}(1, 0)$ are the *mode-independent* UTTs, those for which the transposition levels for major and minor triads are the same. The other groups $\mathcal{K}(1, b)$ are related to $\mathcal{K}(1, 0)$ in somewhat the same way that $\mathcal{R}^*$ is related to $\mathcal{R}$: namely, they all contain the same mode-preserving UTTs—in this case the transpositions—but the mode-reversing UTTs are different for each value of b.

For any given a and b, the UTTs comprising $\mathcal{K}(a, b)$ may be enumerated directly from the definition of the group in Section 3.2. Some UTTs, it may be noted, belong to many of these groups: in fact, the identity transformation T_0 and the tritone transposition T_6 belong to every group in the above list. Some other UTTs, such as $\langle +, 0, 1 \rangle$, do not belong to any $\mathcal{K}(a, b)$. Every mode-reversing UTT $\langle -, m, n \rangle$ belongs to exactly one of the twelve groups $\mathcal{K}(1, b)$, namely $b = n - m$. The largest number of groups $\mathcal{K}(a, b)$ to which any mode-reversing UTT can belong is four, one for each possible value of a. For example, the wechsels $P = W_0 = \langle -, 0, 0 \rangle$ and $T_6P = W_6 = \langle -, 6, 6 \rangle$ both belong to $\mathcal{K}(1,0)$, $\mathcal{K}(5,0)$, $\mathcal{K}(7,0)$, and $\mathcal{K}(11,0)$, while the skew-wechsels $W_0{}^* = \langle -, 0, 6 \rangle$ and $W_6{}^* = \langle -, 6, 0 \rangle$ both belong to $\mathcal{K}(1,6)$, $\mathcal{K}(5,6)$, $\mathcal{K}(7,6)$, and $\mathcal{K}(11,6)$.

The following lemma, dealing specifically with the groups $\mathcal{K}(1, b)$, is preparatory to our main result (Theorem 3.6 below), which identifies the simply transitive subgroups of $\mathcal{U}$ as precisely the groups $\mathcal{K}(a, b)$ and categorizes them according to some basic algebraic properties.

3.5. Lemma. Suppose U belongs to $\mathcal{K}^-(1, b)$, and suppose that for some k, $U^k = T_1 = \langle +, 1, 1 \rangle$. Then the group $\mathcal{K}(1, b)$ is cyclic and is generated by U.

Proof. It is clear that $U^{2k} = T_2$, $U^{3k} = T_3$, …; that is, U generates all the transpositions, which are all the elements of $\mathcal{K}^+(1)$. So to prove that U generates all of $\mathcal{K}(1, b)$ it suffices to show that any mode-reversing $V = \langle -, m, m + b \rangle$ in $\mathcal{K}^-(1, b)$ is some power of U. Since U belongs to $\mathcal{K}^-(1, b)$, U is itself of the form $U = \langle -, n, n + b \rangle$ for some n. We calculate

$$\begin{aligned} VU^{-1} &= \langle -, m, m + b \rangle \langle -, n, n + b \rangle^{-1} \\ &= \langle -, m, m + b \rangle \langle -, -n - b, -n \rangle \\ &= \langle +, m - n, m - n \rangle \\ &= T_{m-n}. \end{aligned}$$

But we have observed that every transposition is a power of U; hence for some j, $VU^{-1} = U^j$. Therefore $V = U^{j+1}$ is a power of U, as desired. ∎

3.6. Theorem. (a) The simply transitive subgroups of $\mathcal{U}$ are precisely the groups $\mathcal{K}(a, b)$ with $a^2 = 1$ and $ab = b$. These groups satisfy the following additional properties:

(b) $\mathcal{K}(a, b)$ is commutative if and only if $a = 1$.

(c) $\mathcal{K}(a, b)$ is cyclic if and only if $a = 1$ and b is odd.

(d) $\mathcal{K}(a, b)$ is a normal subgroup of $\mathcal{U}$ if and only if $a = 11$.

Proof. (a) We first show that $\mathcal{K}(a, b)$ satisfies the two conditions of Lemma 2.4, and is therefore simply transitive. Condition (a) of that lemma requires that (1) for each m there is exactly one n such that $\langle +, m, n\rangle \in \mathcal{K}(a, b)$, and (2) for each m there is exactly one n such that $\langle -, m, n\rangle \in \mathcal{K}(a, b)$. Condition (b) requires that (3) for each n there is exactly one m such that $\langle +, m, n\rangle \in \mathcal{K}(a, b)$, and (4) for each n there is exactly one m such that $\langle -, m, n\rangle \in \mathcal{K}(a, b)$. Of these four requirements, (1) and (2) are obvious from the definition of $\mathcal{K}(a, b)$: the unique n for (1) is $n = am$, and for (2) $n = am + b$. For (3) and (4) we must show that, given n, the equations (3) $n = am$ and (4) $n = am + b$ each have a unique solution for m. In fact, because $a^2 = 1$, (3) is equivalent to $m = an$, while (4) is equivalent to $m = a(n - b)$, uniquely determining m in each case.

Now suppose, conversely, that $\mathcal{K}$ is a simply transitive subgroup of $\mathcal{U}$. We wish to show that $\mathcal{K}$ is one of the groups $\mathcal{K}(a, b)$. By Lemma 2.4 there is a unique a such that $\langle +, 1, a\rangle \in \mathcal{K}$, and also a unique b such that $\langle -, 0, b\rangle \in \mathcal{K}$. We shall show that $\mathcal{K} = \mathcal{K}(a, b)$.

Let us denote $\langle +, 1, a\rangle$ by U_a and $\langle -, 0, b\rangle$ by V_b. For each m, because both U_a and V_b are elements of $\mathcal{K}$, the transformations $U_a{}^m = \langle +, m, am\rangle$ and $U_a{}^m V_b = \langle -, m, am + b\rangle$ must belong to $\mathcal{K}$ as well. By Lemma 2.4 again, these are the only transformations of the form $\langle +, m, n\rangle$ and $\langle -, m, n\rangle$ in $\mathcal{K}$. The elements of $\mathcal{K}$ are therefore precisely the same as the elements of $\mathcal{K}(a, b)$, so $\mathcal{K} = \mathcal{K}(a, b)$, as asserted.

(b) With U_a and V_b defined as above, we calculate $U_a V_b = \langle -, 1, a + b\rangle$ while $V_b U_a = \langle -, a, b + 1\rangle$. Obviously these will be the same only if $a = 1$; thus commutativity of $\mathcal{K}(a, b)$ implies $a = 1$. Conversely, suppose $a = 1$. The elements of $\mathcal{K}(1, b)$ are of the forms $\langle +, n, n\rangle$ and $\langle -, n, n + b\rangle$; the observations of Section 1.9 imply that all possible pairs of such UTTs commute.

(c) Cyclic groups are always commutative, so by (b) the only possible cyclic groups $\mathcal{K}(a, b)$ are those with $a = 1$. We show that $\mathcal{K}(1, b)$ is cyclic if and only if b is odd.

If b is odd, then $1 - b$ is even. Let n be a number such that $2n = 1 - b$ (mod 12), and let $U = \langle -, n, n + b\rangle$, which belongs to $\mathcal{K}(1, b)$. Then $U^2 = \langle +, 2n + b, 2n + b\rangle = \langle +, 1, 1\rangle = T_1$. By Lemma 3.5, the group $\mathcal{K}(1, b)$ is cyclic and is generated by U.

Now suppose that there exists a cyclic generator U for the group $\mathcal{K}(1, b)$. Clearly U is mode-reversing, since otherwise it could not generate the mode-reversing elements of $\mathcal{K}(1, b)$; U is therefore of the form $U = \langle -, n, n + b \rangle$. Then $U^2 = \langle +, 2n + b, 2n + b \rangle = T_{2n+b}$. If b is even, then $2n + b$ is even as well, from which it follows that the powers of U^2—the even powers of U—are all transpositions of the form T_m with even m; the odd powers of U, meanwhile, are mode-reversing UTTs. But this means that $T_1 = \langle +, 1, 1 \rangle$, while an element of $\mathcal{K}(1, b)$, cannot be a power of U at all. From this contradiction we conclude that b must be odd.

(d) Suppose $\mathcal{K}(a, b)$ is a normal subgroup of $\mathcal{U}$, and let $W = \langle -, 1, a + b \rangle$, which is an element of $\mathcal{K}(a, b)$. Then UWU^{-1} must belong to $\mathcal{K}(a, b)$ for every UTT U. In particular, let $U = \langle +, 0, 1 \rangle$. Then

$$\begin{aligned} UWU^{-1} &= \langle +, 0, 1\rangle\langle -, 1, a + b\rangle\langle +, 0, 11\rangle \\ &= \langle -, 1, a + b + 1\rangle\langle +, 0, 11\rangle \\ &= \langle -, 0, a + b + 1\rangle. \end{aligned}$$

This is of the proper form for a mode-reversing transformation in $\mathcal{K}(a, b)$ only if $a(0) + b = a + b + 1$, which implies that $a + 1 = 0$, or $a = 11$. Hence the only candidates for normal subgroups of $\mathcal{U}$ among the groups $\mathcal{K}(a, b)$ are those with $a = 11$. In fact, as noted in Section 3.4, there are only two such groups, $\mathcal{K}(11, 0) = \mathcal{R}$ and $\mathcal{K}(11, 6) = \mathcal{R}^*$, both of which were shown to be normal subgroups of $\mathcal{U}$ in Part 2. ■

3.7. In the following sections we shall study the elements of maximal order in the group $\mathcal{U}$. These are of both algebraic and musical interest, and turn out to be intimately connected with the cyclic groups $\mathcal{K}(1, b)$ with b odd.[18]

We know from Section 1.10 that $\mathcal{U}^+$, the group of mode-preserving UTTs, is isomorphic to $\mathbf{Z}_{12} \times \mathbf{Z}_{12}$; hence the maximal order of any mode-preserving UTT is 12. Familiar examples of mode-preserving UTTs of order 12 include the semitone transpositions T_1 and T_{11}, as well as the dominant transformation $D = T_5$ and its inverse $D^{-1} = T_7$. If U is a mode-reversing UTT, then U^2 is mode-preserving; therefore no UTT can possibly have an order greater than 24. But by part (c) of Theorem 3.6, there exist cyclic subgroups of $\mathcal{U}$ of order 24, namely the groups $\mathcal{K}(1, b)$ with b odd, and any generator of such a group is an element of order 24 in $\mathcal{U}$. In fact, these generators are the *only* UTTs of order 24, and they are quite easy to identify, as the following theorem shows.

3.8. Theorem. There are exactly 48 elements of order 24 in $\mathcal{U}$; they are the UTTs of the form $\langle -, m, n \rangle$ such that $m + n$ is equal to 1, 5, 7, or 11. Such a transformation $U = \langle -, m, n \rangle$ generates the simply transitive group $\mathcal{K}(1, b)$, where $b = n - m$.

Proof. For each $m = 0, 1, \ldots, 11$, there are four possible values of n such that $m + n$ will equal 1, 5, 7, or 11; hence it is clear that there are 48 UTTs $\langle -, m, n\rangle$ of the type described. Let $U = \langle -, m, n\rangle$ be such a UTT; let $b = n - m$, and let $c = m + n$. Observe that U belongs to the group $\mathcal{K}(1, b)$. Since c is the sum of the two transposition levels of U, it follows by Section 1.11 that $U^2 = T_c$, from which $U^{2c} = (U^2)^c = \langle +, c^2, c^2\rangle$. By hypothesis, c is equal to 1, 5, 7, or 11. Observe that $c^2 = 1 \pmod{12}$ in every case; therefore $U^{2c} = \langle +, 1, 1\rangle$. By Lemma 3.5, U generates the group $\mathcal{K}(1, b)$, and is therefore of order 24.

It remains only to show that every UTT of order 24 is of this type. Suppose V is of order 24 in $\mathcal{U}$. As noted in Section 3.7 above, all mode-preserving UTTs are of order 12 or less; hence V is mode-reversing and can be written $V = \langle -, m, n\rangle$ for some m and n. Let $c = m + n$. The even powers of V are $V^2 = \langle +, c, c\rangle$, $V^4 = \langle +, 2c, 2c\rangle, \ldots, V^{24} = \langle +, 12c, 12c\rangle = T_0$. Because V is of order 24, these are all distinct; therefore the numbers c, $2c, \ldots, 12c$ $(= 0)$ exhaust the integers mod 12. This is possible only if c is co-prime to 12; that is, only if $c = 1, 5, 7$, or 11. ■

3.9. For any given odd value of b, there are 12 mode-reversing transformations $\langle -, m, n\rangle$ in $\mathcal{K}^-(1, b)$. Eight of these satisfy $m + n = 1, 5, 7$, or 11; the other four satisfy $m + n = 3$ or 9. It then follows from Theorem 3.8 that each of the six cyclic groups $\mathcal{K}(1, b)$ may be generated by any one of eight different elements, altogether accounting for the 48 order-24 UTTs described in the theorem.[19]

UTTs of order 24 are of considerable musical interest. When such a transformation is applied repeatedly, the resulting chain of triads will cycle through all 24 major and minor triads before returning to its starting point.[20] Two familiar examples of such transformations are the mediant transformation $M = \langle -, 9, 8\rangle$ and its inverse, the submediant transformation $M^{-1} = \langle -, 4, 3\rangle$, either of which generates the group $\mathcal{K}(1, 11)$ according to Theorem 3.8. This group $\mathcal{K}(1, 11)$ will play an important role in a characterization of the structure of the entire group $\mathcal{U}$ in Part 4 below; we shall designate this group by the letter $\mathcal{M}$.[21]

The group $\mathcal{M}$ has been discussed briefly by Lewin (1987, 179–80), and cycles of triads generated by M have been studied by Cohn (1997, 36–37). Such a chain occurs in the scherzo of Beethoven's Ninth Symphony (mm. 143–171):

$$\mathrm{C} \xrightarrow{M} \mathrm{a} \xrightarrow{M} \mathrm{F} \xrightarrow{M} \mathrm{d} \xrightarrow{M} \mathrm{B}\flat \xrightarrow{M} \cdots \xrightarrow{M} \mathrm{A}.$$

This chain is nineteen triads long, only five short of a complete cycle.

Table 2 lists this chain along with several other examples of triad chains generated by order-24 UTTs, chosen from the musical literature. Such chains are rarely prolonged to the extent of the M-chain in the Ninth Symphony example, although the remarkable $\langle -, 5, 2\rangle$-chain from Beethoven's

Table 2. Chord progressions and tonal cycles generated by order-24 UTTs

Chord progressions Source	Progression	UTT (U)	U^2	Group
Bach, Violin Concerto in A Minor, I, mm. 88–94	e→E→a→A→d→···	$\langle -, 5, 0\rangle$	T_5	$\mathcal{K}(1, 7)$
Mozart, Requiem, *Confutatis*, mm. 10–12	C→c→G→g→D→···	$\langle -, 0, 7\rangle$	T_7	$\mathcal{K}(1, 7)$
Beethoven, String Quartet, Op. 18, No. 6, IV, mm. 20–28	B→e→F♯→b→C♯→···	$\langle -, 5, 2\rangle$	T_7	$\mathcal{K}(1, 9)$
Beethoven, Symphony No. 3, I, mm. 178–186	c→A♭→c♯→A→d	$\langle -, 5, 8\rangle$	T_1	$\mathcal{K}(1, 3)$
Beethoven, Symphony No. 9, II, mm. 143–171	C→a→F→d→B♭→···	$\langle -, 9, 8\rangle = M$	T_5	$\mathcal{K}(1, 11) = \mathcal{M}$
Liszt, "Wilde Jagd," mm. 180–184	E♭→g→D→f♯→D♭→···	$\langle -, 4, 7\rangle$	T_{11}	$\mathcal{K}(1, 3)$

Tonal cycles Source	Key sequence	UTT (U)	U^2	Group
Bach, *Well-Tempered Clavier*	C→c→C♯→c♯→D→···	$\langle -, 0, 1\rangle$	T_1	$\mathcal{K}(1, 1)$
Chopin, Preludes, Op. 28	C→a→G→e→D→···	$\langle -, 9, 10\rangle$	T_7	$\mathcal{K}(1, 1)$
Liszt, *Transcendental Etudes*	C→a→F→d→B♭→···	$\langle -, 9, 8\rangle = M$	T_5	$\mathcal{K}(1, 11) = \mathcal{M}$

Quartet, Op. 18, No. 6, comes close (sixteen triads). For examples that circumnavigate the entire cycle of 24 triads, we may turn to several well-known *collections* of pieces that traverse the complete cycle in systematic ways. The organizing principles for such cycles can be described using order-24 UTTs and the corresponding simply transitive groups; a few are listed at the bottom of Table 2.[22]

The requirement in Theorem 3.8 that $\tau(U)$ must equal 1, 5, 7, or 11 has a natural musical interpretation, well illustrated by the examples in Table 2. If U is mode-reversing, then U^2 is the transposition T_c, where $c = \tau(U)$; if U is of order 24, then T_c must be of order 12. But, as is widely recognized, the only transpositions of order 12 are T_1, T_5, T_7, and T_{11}; that is, the only intervals that generate a complete 12-note aggregate are ascending and descending semitones (T_1, T_{11}) and fifths (T_5, T_7). Table 2 displays U^2 for each listed triad chain; of course, U^2 may also be read directly from a progression by taking only every other triad (for instance, only the major triads). Of the four possibilities for U^2, the descending semitone T_{11}, as in the Liszt "Wilde Jagd" example, is surely the rarest.[23]

3.10. For any given triadic progression, the choice of any one of the simply transitive groups $\mathcal{K}(a, b)$ determines a unique transformational analysis of the progression. The descent by diatonic thirds from the scherzo of the Ninth Symphony, for example, was analyzed in Section 3.9 above as an M-chain. Cohn (1997, 36–37), however, has described the same progression as an alternation of the transformations R and L:

$$\mathrm{C} \xrightarrow{R} \mathrm{a} \xrightarrow{L} \mathrm{F} \xrightarrow{R} \mathrm{d} \xrightarrow{L} \mathrm{B}\flat \xrightarrow{R} \mathrm{g} \xrightarrow{L} \cdots$$

The difference may be explained as the result of choosing a different simply transitive group. In the first analysis, the UTT labels were drawn from the group $\mathcal{M} = \mathcal{K}(1, 11)$; in Cohn's analysis the transformations were chosen from the Riemann group $\mathcal{R} = \mathcal{K}(11, 0)$. Here is still another analysis, drawn this time from $\mathcal{K}(1, 0)$, the group of mode-independent UTTs discussed in Section 3.4:

$$\mathrm{c} \xrightarrow{T_9P} \mathrm{a} \xrightarrow{T_8P} \mathrm{F} \xrightarrow{T_9P} \mathrm{d} \xrightarrow{T_8P} \mathrm{B}\flat \xrightarrow{T_9P} \mathrm{g} \xrightarrow{T_8P} \cdots$$

Each of the three representations has its virtues. The analysis in $\mathcal{K}(1, 0)$ relates the fairly obvious fact that the progression is composed of mode-reversing transformations with transposition levels alternating between 9 and 8. The group $\mathcal{M}$ is the only one in which the entire progression is described in terms of a single transformation, an attractive characteristic; on the other hand, for that very reason, the analysis in $\mathcal{M}$ fails to convey the fact that two different types of motion do indeed occur in alternation. The Riemannian analysis brings the many well-known and attractive properties of the transformations R and L into play, but fails to distinguish between this descending-thirds progression and its dual, a sequence of *ascending* thirds, analyzed (in $\mathcal{R}$) in precisely the same way:

$$\mathrm{c} \xrightarrow{R} \mathrm{E}\flat \xrightarrow{L} \mathrm{g} \xrightarrow{R} \mathrm{B}\flat \xrightarrow{L} \mathrm{d} \xrightarrow{R} \mathrm{F} \xrightarrow{L} \cdots$$

In $\mathcal{M}$, this last progression is not an M-chain but an M^{-1}-chain; in $\mathcal{K}(1, 0)$ it is an alternation between T_3P and T_4P.

Any triadic progression thus lends itself to a variety of different analyses—in principle, 24 of them, one for each simply transitive group.[24] In any particular case, some of these analyses are likely to be more meaningful than others. The many examples appearing in the substantial neo-Riemannian literature in recent years are, in general, progressions particularly well-suited to analysis in the group $\mathcal{R}$; for example, an "*RPL* loop" (Cohn 1997, 42–46)—

$$\mathrm{f} \xrightarrow{R} \mathrm{A}\flat \xrightarrow{P} \mathrm{g}\sharp \xrightarrow{L} \mathrm{E} \xrightarrow{R} \mathrm{c}\sharp \xrightarrow{P} \mathrm{D}\flat \xrightarrow{L} \mathrm{f}$$

—or a "hexatonic cycle":

$$\mathrm{A}\flat \xrightarrow{P} \mathrm{g}\sharp \xrightarrow{L} \mathrm{E} \xrightarrow{P} \mathrm{e} \xrightarrow{L} \mathrm{C} \xrightarrow{P} \mathrm{c} \xrightarrow{L} \mathrm{A}\flat$$

This last progression might also, however, be profitably analyzed in either of the two groups $\mathcal{K}(1, 8)$ or $\mathcal{K}(7, 8)$: in either case it is generated by a single UTT, namely $\langle -, 0, 8 \rangle$. The situation is similar to the M-chain above, although no order-24 UTTs are involved.[25]

Because the mediant transformation M connects triads related by diatonic thirds, and because M^2 is the dominant transformation D, the group $\mathcal{M}$ is often an appropriate choice for the analysis of diatonic progressions, in which small (positive and negative) powers of M are likely to appear most frequently:

$$\mathrm{I} \xrightarrow{M^2} \mathrm{IV} \xrightarrow{M^{-4}} \mathrm{V} \xrightarrow{M^3} \mathrm{vi} \xrightarrow{M} \mathrm{IV} \xrightarrow{M} \mathrm{ii} \xrightarrow{M^{-5}} \mathrm{V} \xrightarrow{M^2} \mathrm{I}$$

A Riemannian analysis of such a progression would appear rather nonsensical.

Lewin's observation (1987, 157–58) that simply transitive group actions correspond to generalized interval systems bears recalling here. Each of the 24 simply transitive subgroups of $\mathcal{U}$ gives rise to its own notion of "interval" on the set Γ—that is, its own conception of the "distance" between two triads. The "interval" from the C major triad $(0, +)$ to the D minor triad $(2, -)$, for example, is T_2P in the group $\mathcal{K}(1, 0)$; $W_2 = RLR$ in the Riemann group $\mathcal{R}$; $W_2{}^* = \langle -, 2, 4 \rangle$ in the group $\mathcal{R}^*$; or M^3 in the group $\mathcal{M}$. In a highly subjective sense, the small exponent in the expression M^3 (relative to 24, the order of M) suggests that C major and D minor are fairly "close" to each other in the group $\mathcal{M}$, while the relative complexity of the expression RLR suggests that they are moderately "distant" as measured in $\mathcal{R}$.

3.11. We conclude Part 3 by giving a more complete description of the commutative but non-cyclic groups $\mathcal{K}(1, b)$ with b even. Such a group consists of the twelve transpositions $T_n = \langle +, n, n \rangle$ and twelve mode-reversing UTTs of the form $\langle -, m, m + b \rangle$. Since b is even, we may choose k so that $2k = -b \pmod{12}$. Then $k + b = -k$, so the wechsel $W_k = \langle -, k, -k \rangle$ belongs to $\mathcal{K}(1, b)$; the mode-reversing UTTs in $\mathcal{K}(1, b)$, described above, are simply equal to W_k multiplied by the various transpositions T_n.[26]

It follows from these observations that the group $\mathcal{K}(1, b)$ is generated by this wechsel W_k and the transposition T_1. Every element of $\mathcal{K}(1, b)$ can be written uniquely as $W_k{}^i T_1{}^j$ where i is either 0 or 1 and j is an integer mod 12. (We may recall that W_k commutes with T_1, and also that $W_k{}^2$ is the identity transformation T_0. The former follows from Theorem 1.7, the latter from part (g) of Theorem 2.5.) The group $\mathcal{K}(1, b)$ is therefore isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_{12}$: the transformation $W_k{}^i T_1{}^j$ in $\mathcal{K}(1, b)$ corresponds with the ordered pair (i, j) in $\mathbf{Z}_2 \times \mathbf{Z}_{12}$.

Clough (1998, 303–5) has made brief reference to these groups $\mathcal{K}(1, b)$. Having presented a geometric model for transposition and inversion similar to the dual-circle representations in Section 1.14 above, Clough proceeds to apply his model to the "*T/I* group" of transpositions and inversions and to the "S/W group" of schritts and wechsels (our group $\mathcal{R}$). At the end of the article, he mentions that the operators can be recombined to form a "T/W group" and an "S/I group," both isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_{12}$. Clough's "$T/W$ group" does not, in fact, consist of the twelve transpositions and twelve wechsels: that set, after all, does not form a group, inasmuch as the product of two wechsels is in general a schritt, not a transposition. As Clough describes the construction of this group, however, it is generated by the transposition T_1 and a single wechsel. The above discussion demonstrates that this construction can yield any of the six groups $\mathcal{K}(1, b)$ with b even. Specifically, if the chosen wechsel is W_k, the group will be $\mathcal{K}(1, -2k)$. Likewise several different "S/I groups" may be constructed, depending on the choice of the inversion used as a generator. These groups too act in simply transitive fashion on triads, but they are not among the groups $\mathcal{K}(a, b)$. In fact, these "S/I groups" are not subgroups of $\mathcal{U}$ at all, since the inversion operators are not UTTs. Groups containing inversion operators as well as UTTs will be studied in Part 5.

4. The Structure of the UTT Group

4.1. We turn next to a detailed study of the algebraic structure of the group $\mathcal{U}$ and its subgroups.[27] In the discussion below, we shall make use of a standard construction in group theory, the product of two subgroups. (For more details, see Dummit and Foote 1999, 94–95.) If H and K are subgroups of a group G, then the product HK is defined as the set of all products hk, as h ranges through the elements of H and k ranges through the elements of K. The set HK is not necessarily a subgroup, although if either H or K is a *normal* subgroup of G it follows that HK is a subgroup; in this case, moreover, HK is equal to KH. In the case of finite groups, regardless of whether HK is a subgroup or not, its cardinality is given by the formula

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$$

Theorem 4.3 below offers several characterizations of the structure of $\mathcal{U}$. The theorem refers to the UTT $B = \langle +, 0, 1\rangle$ and the group $\mathcal{B}$ generated by B (a subgroup of $\mathcal{U}$). It should be clear that $B^n = \langle +, 0, n\rangle$, and therefore that $\mathcal{B}$ is a cyclic group of order 12 isomorphic to $\mathbf{Z}_{12}$ (the integers mod 12). The proof of the theorem relies on the following lemma, which is easily verified using the definitions of the groups involved:

4.2. Lemma.

(a) If $\mathcal{K}$ is any of the simply transitive subgroups $\mathcal{K}(a, b)$, then $\mathcal{B} \cap \mathcal{K} = \{T_0\}$; that is, the only UTT that belongs to both $\mathcal{B}$ and $\mathcal{K}$ is the identity T_0. In particular, $\mathcal{B} \cap \mathcal{R} = \mathcal{B} \cap \mathcal{M} = \{T_0\}$.

(b) $\mathcal{R} \cap \mathcal{M} = \{T_0, T_6\}$.

4.3. Theorem.

(a) If $\mathcal{K}$ is any of the simply transitive subgroups $\mathcal{K}(a, b)$, then $\mathcal{U} = \mathcal{BK}$. Every UTT U in $\mathcal{U}$ can be written uniquely as $U = B^iV$ for some i satisfying $0 \le i \le 11$ and some V in $\mathcal{K}$.

(b) $\mathcal{U} = \mathcal{BR}$. Every U in $\mathcal{U}$ can be written uniquely as $U = B^iV$ for some i satisfying $0 \le i \le 11$ and some V in $\mathcal{R}$.

(c) If V is any element of order 24 in $\mathcal{U}$, then $\mathcal{U}$ is generated by B and V. Every U in $\mathcal{U}$ can be written uniquely as $U = B^iV^j$ for some i and j satisfying $0 \le i \le 11$, $0 \le j \le 23$.

(d) $\mathcal{U} = \mathcal{BM}$. Every U in $\mathcal{U}$ can be written uniquely as $U = B^iM^j$ for some i and j satisfying $0 \le i \le 11$, $0 \le j \le 23$.

(e) $\mathcal{U}$ is generated by $B = \langle +, 0, 1\rangle$ and $P = \langle -, 0, 0\rangle$.

(f) $\mathcal{U} = \mathcal{MR}$. Every U in $\mathcal{U}$ can be written as $U = M^iV$ for some i satisfying $0 \le i \le 23$ and some V in $\mathcal{R}$; in fact, for each U there are exactly two such representations, with exponents i differing by 12.

Proof. Let $\mathcal{K}$ be one of the groups $\mathcal{K}(a, b)$. From the formula given in Section 4.1 for the cardinality of the product of two subgroups, and from the intersection formula of Lemma 4.2(a), it follows that

$$|\mathcal{BK}| = \frac{|\mathcal{B}| \cdot |\mathcal{K}|}{|\mathcal{B} \cap \mathcal{K}|} = \frac{12 \cdot 24}{1} = 288.$$

But $\mathcal{BK}$ is a subset of $\mathcal{U}$, and $\mathcal{U}$ itself has exactly 288 elements; hence $\mathcal{BK} = \mathcal{U}$. Because $\mathcal{B}$ consists precisely of the powers of B, it follows that every element of $\mathcal{U}$ can be written as B^iV for some i and V as described in part (a) of the theorem. As i ranges through the 12 available exponents and V ranges through the 24 elements of $\mathcal{K}$, $12 \cdot 24 = 288$ products are formed, which must all be different in order to exhaust the elements of $\mathcal{U}$; hence each such representation is unique.

Parts (b), (c), and (d) are special cases of (a). For (b) we use $\mathcal{K} = \mathcal{R} = \mathcal{K}(11, 0)$; for (c) we use the group $\mathcal{K}(1, b)$ generated by V according to Theorem 3.8. The mediant transformation M is one such element of order 24, so (d) follows from (c).

Having shown in (b) that $\mathcal{U}$ is generated by $B = \langle +, 0, 1\rangle$ and the elements of $\mathcal{R}$, and recalling from Theorem 2.5(a) that $\mathcal{R}$ is generated by $P = \langle -, 0, 0\rangle$ and $Q = \langle +, 1, 11\rangle$, in order to prove (e) we need only show that Q can be expressed in terms of B and P. One can verify easily that $BP = \langle -, 0, 1\rangle$, $(BP)^2 = \langle +, 1, 1\rangle$, and therefore $(BP)^2B^{10} = \langle +, 1, 11\rangle = Q$.

By the product cardinality formula and Lemma 4.2(b),

$$|\mathcal{MR}| = \frac{|\mathcal{M}| \cdot |\mathcal{R}|}{|\mathcal{M} \cap \mathcal{R}|} = \frac{24 \cdot 24}{2} = 288.$$

Hence $\mathcal{MR} = \mathcal{U}$, and every UTT can be expressed in the form $U = M^iV$ as described in (f). Suppose some U has two such representations, say $U =$

$M^iV = M^jW$, where $0 \le i, j \le 23$ and both V and W belong to $\mathcal{R}$. Multiplying this equation on the left by M^{-j} and on the right by V^{-1} yields $M^{i-j} = WV^{-1}$. But M^{i-j} belongs to $\mathcal{M}$ while WV^{-1} belongs to $\mathcal{R}$, and by Lemma 4.2(b) again the only elements common to these two groups are T_0 $(= M^0)$ and T_6 $(= M^{12})$. Therefore the exponents i and j can differ only by 0 or 12. If $i = j$, then of course $V = W$ and the two representations are the same. So if we restrict i to the range $0 \le i \le 11$, we obtain $12 \cdot 24 = 288$ different UTTs of the form M^iV with V in $\mathcal{R}$; it follows that every element of $\mathcal{U}$ has one representation of this form, and a second representation M^jW where $j = i + 12$, $M^j = T_6M^i$, and $W = T_6V$. ■

4.4. A word of caution about the subgroup products in the preceding theorem may be prudent. Part (a) of this theorem does *not* say that the group $\mathcal{U}$ is isomorphic to the direct product $\mathcal{B} \times \mathcal{K}$, nor is any similar claim made in part (b), (d), or (f). In fact, $\mathcal{U}$ is not isomorphic to any such direct product. A more general construction, the *semidirect product*, however, is applicable in some of these cases; this thread will be pursued in Section 4.15 below.

The Riemannian UTTs appearing in parts (b) and (f) can of course be expressed in any of the formats presented in Part 2 for such transformations (recall Table 1). For instance, combining Theorem 4.3(b) with the "P/Q" column of Table 1, we deduce that every UTT can be expressed in a unique way as $B^iP^jQ^k$, for some exponents i, j, and k satisfying $0 \le i \le 11$, $j = 0$ or 1, and $0 \le k \le 11$.

Although the theorem seems to give privileged status to the UTT $B = \langle +, 0, 1 \rangle$, a variety of other UTTs could equally well have been chosen to serve the same purpose. First, since the group $\mathcal{B}$ is generated by any one of the UTTs B^5, B^7, or B^{11}, any of those could fill the role of B; also, we could interchange the two transposition levels and work instead with $\bar{B} = \langle +, 1, 0 \rangle$ (or its 5th, 7th, or 11th power). The number of ways in which the group $\mathcal{U}$ may be generated by a pair of its elements, as in part (e), is vast.[28]

Part (f) of the theorem, the representation of $\mathcal{U}$ in terms of $\mathcal{M}$ and $\mathcal{R}$, is less elegant than the statements in parts (a)–(d) because of the non-uniqueness of the expression M^iV (which stems from the fact that two groups of order 24 are involved, rather than one of order 24 and one of order 12). This result has been included here because of its close connection with the work of Brian Hyer, to be explored in the following sections.

4.5. We say that a UTT $U = \langle \sigma, t^+, t^- \rangle$ is *even* (or, more fully, *even in the sense of total transposition*) if its total transposition $\tau(U) = t^+ + t^-$ is an even number; U is *odd (in the sense of total transposition)* if $\tau(U)$ is an odd number. The familiar UTTs $T_n = \langle +, n, n \rangle$, $P = \langle -, 0, 0 \rangle$, $Q = \langle +, 1, 11 \rangle$, $L = \langle -, 4, 8 \rangle$, $R = \langle -, 9, 3 \rangle$, and $D = \langle +, 5, 5 \rangle$ are all even, while $M = \langle -, 9, 8 \rangle$ and $B = \langle +, 0, 1 \rangle$ are odd. The Riemannian UTTs, whose total transposition is 0 by definition, are all even. By Theorem 3.8, all UTTs of order 24 (of which M is one example) have total transposition 1, 5, 7, or 11; thus all order-24 UTTs are odd. Every even UTT is of order 12 or less.

We know from Section 2.1 that τ is a homomorphism from $\mathcal{U}$ to $\mathbf{Z}_{12}$; that is, $\tau(UV) = \tau(U) + \tau(V)$ for all U and V. It follows that the product of two UTTs of the same parity (both even or both odd) is always even, while the product of two UTTs of opposite parity (one even and one odd) is always odd.

4.6. There is a second sense in which UTTs may be classified as even or odd; this approach is based on the cycle structure of the transformation as a permutation of the 24 triads. Every triadic transformation—every element of the gigantic group $\mathcal{G}$ introduced in Section 1.4—may be represented as a product of disjoint cycles whose lengths total 24 or less. For example, the UTT $U = \langle -, 0, 8\rangle$ breaks down into four 6-cycles (cycles of length 6):

$$\begin{array}{lllllllllllll}
\mathrm{C} & \rightarrow & \mathrm{c} & \rightarrow & \mathrm{A}\flat & \rightarrow & \mathrm{g}\sharp & \rightarrow & \mathrm{E} & \rightarrow & \mathrm{e} & \rightarrow & \mathrm{C}, \\
\mathrm{D}\flat & \rightarrow & \mathrm{c}\sharp & \rightarrow & \mathrm{A} & \rightarrow & \mathrm{a} & \rightarrow & \mathrm{F} & \rightarrow & \mathrm{f} & \rightarrow & \mathrm{D}\flat, \\
\mathrm{D} & \rightarrow & \mathrm{d} & \rightarrow & \mathrm{B}\flat & \rightarrow & \mathrm{b}\flat & \rightarrow & \mathrm{F}\sharp & \rightarrow & \mathrm{f}\sharp & \rightarrow & \mathrm{D}, \\
\mathrm{E}\flat & \rightarrow & \mathrm{d}\sharp & \rightarrow & \mathrm{B} & \rightarrow & \mathrm{b} & \rightarrow & \mathrm{G} & \rightarrow & \mathrm{g} & \rightarrow & \mathrm{E}\flat.
\end{array}$$

In this case the cycle structure simply reveals the four hexatonic systems studied in detail by Cohn (1996). In permutation theory it is common to represent this structure more compactly in a form such as

$$(\mathrm{C}, \mathrm{c}, \mathrm{A}\flat, \mathrm{g}\sharp, \mathrm{E}, \mathrm{e})(\mathrm{D}\flat, \mathrm{c}\sharp, \mathrm{A}, \mathrm{a}, \mathrm{F}, \mathrm{f})(\mathrm{D}, \mathrm{d}, \mathrm{B}\flat, \mathrm{b}\flat, \mathrm{F}\sharp, \mathrm{f}\sharp)(\mathrm{E}\flat, \mathrm{d}\sharp, \mathrm{B}, \mathrm{b}, \mathrm{G}, \mathrm{g}).$$

Each cycle, demarcated with parentheses in this notation, is "cyclic" in the sense that each object in the list (in this case each triad) is mapped to the following object, and the object listed last in a cycle is mapped to the one listed first.

Such a representation may be broken down further into a product of (not necessarily disjoint) 2-cycles, each of which simply interchanges two triads. For example, the 6-cycle (C, c, A♭, g♯, E, e), which appears in the above representation of U, may be written as a product of five 2-cycles:

$$(\mathrm{C}, \mathrm{c}, \mathrm{A}\flat, \mathrm{g}\sharp, \mathrm{E}, \mathrm{e}) = (\mathrm{C}, \mathrm{c})(\mathrm{C}, \mathrm{A}\flat)(\mathrm{C}, \mathrm{g}\sharp)(\mathrm{C}, \mathrm{E})(\mathrm{C}, \mathrm{e}).$$ [29]

(In general, any n-cycle may be written as a product of $(n-1)$ 2-cycles.) In this way the entire transformation U may be written as a product of twenty 2-cycles.

The 2-cycle representation of a permutation is not unique, but it is a standard theorem of permutation theory that if a permutation can be written as a product of an even number of 2-cycles, then *every* 2-cycle representation of that permutation has an even number of 2-cycles; if one 2-cycle representation is odd, then *every* such representation is odd.[30] Permutations of these two types are known as *even* and *odd* permutations, respectively, and accordingly we shall refer to UTTs as *even* or *odd in the sense of permutation theory.*

We have now given two definitions of even and odd UTTs. The following theorem shows that the two are in fact equivalent.

4.7. Theorem. A UTT is even in the sense of total transposition if and only if it is even in the sense of permutation theory.

Proof. Let U be a UTT. In Section 4.4 we observed that U can be expressed as $B^iP^jQ^k$ for some exponents i, j, and k. In the sense of total transposition, $B = \langle +, 0, 1\rangle$ is odd, while $P = \langle -, 0, 0\rangle$ and $Q = \langle +, 1, 11\rangle$ are even. The product P^jQ^k, therefore, is a product of even UTTs, and is itself therefore even by the product parity rule mentioned at the end of Section 4.5. So the parity of $U = B^iP^jQ^k$ (in the sense of total transposition) is the same as that of B^i, which is in fact the same as the parity of the number i. We show now that this is also the parity of U in the sense of permutation theory.

The transformation B cyclically permutes the minor triads while leaving the major triads unchanged; that is, B is the 12-cycle (c, c♯, d, …, b). As discussed in Section 4.6 above, a 12-cycle may be written as a product of eleven 2-cycles; hence B is odd in the sense of permutation theory. The parallel transformation P is clearly a product of twelve 2-cycles, (C, c)(D♭, c♯)(D, d) ⋯ (B, b), so P is even in this sense. Finally, the unit schritt Q is a product of two 12-cycles, (C, D♭, D, …, B)(c, b, b♭, …, c♯), and therefore may be written as a product of twenty-two 2-cycles, so Q is also even. So in the sense of permutation theory also, the parity of $U = B^iP^jQ^k$ is the same as that of B^i, which is the same as the parity of the exponent i. ■

4.8. In light of Theorem 4.7, we may speak simply of *even* or *odd* UTTs with no ambiguity. Of the 288 UTTs, exactly half, or 144, are even. Because products and inverses of even UTTs are even, the even UTTs form a subgroup $\mathcal{H}$ of $\mathcal{U}$. Because $\mathcal{H}$ is of index 2 in $\mathcal{U}$, it is automatically a normal subgroup.[31] Another normal subgroup of $\mathcal{U}$ of order 144 was introduced in Section 1.10, namely the group $\mathcal{U}^+$ of all mode-preserving UTTs. Whether $U = \langle \sigma, t^+, t^-\rangle$ belongs to $\mathcal{U}^+$ depends only on the sign σ; whether U belongs to $\mathcal{H}$ depends only on the transposition levels t^+ and t^-. The mode-preserving even UTTs form a smaller normal subgroup, $\mathcal{H}^+ = \mathcal{H} \cap \mathcal{U}^+$, of order 72.

Some clues to the structure of the subgroup $\mathcal{H}$ may be gleaned from Theorem 4.3. If a UTT U is expressed in the form $U = B^iV$ for some V in

$\mathcal{R}$, as in Theorem 4.3(b), then U is even if and only if the exponent i is even. Hence the group $\mathcal{H}$ is generated by B^2 and the Riemannian UTTs. Theorem 2.5, it will be recalled, offers various options for generators of $\mathcal{R}$; we may conclude, for instance, that $\mathcal{H}$ is generated by B^2, L, and R, or by B^2, P, and Q. Recalling Theorem 4.3(e) and its proof, one might expect that the inclusion of Q in the last list is redundant, but it is not, as we shall see in Theorem 4.10 below.

Similar reasoning based on Theorem 4.3(f) shows that $\mathcal{H}$ is generated by M^2 and the Riemannian UTTs. But M^2 is the dominant transformation D (= T_5), so every even UTT may be expressed in terms of D and Riemannian UTTs. (As Theorem 4.3(f) indicates, however, such expressions are not unique.) The powers of D form the transposition group $\mathcal{T} = \{T_0, T_1, \ldots, T_{12}\}$, so we may conclude that $\mathcal{H} = \mathcal{T}\mathcal{R}$. In representing even UTTs as products of powers of D and Riemannian UTTs, D could be replaced by any other generator of the cyclic group $\mathcal{T}$, such as T_1.

4.9. This last description of the group $\mathcal{H}$ makes clear that $\mathcal{H}$ is precisely the transformation group studied by Brian Hyer (1995). Hyer's Figure 4, the multiplication table for his group, represents each of his 144 transformations as a power of D multiplied by some expression involving P, L, and R. As we know from Section 2.7 and Table 1, such *PLR*-expressions for Riemannian UTTs are not unique. Hyer, however, chooses "canonical forms" for twelve of the 24 Riemannian transformations, and obtains his 144 transformations by multiplying these twelve (on the left) by the powers of D (from $D^0 = T_0$ to $D^{11} = D^{-1}$). Canonical forms for all 24 Riemannian transformations are not needed, because, in accordance with Theorem 4.3(f), this would result in two different representations for each transformation, with "Riemannian parts" differing by a factor of $M^{12} = D^6 = T_6$. By listing canonical forms for only one of each pair of T_6-related Riemannian transformations, Hyer eliminates this redundancy.[32] Hyer's canonical forms may be seen in the "Hyer" column of Table 1, along with expressions involving D^6 for the other twelve Riemannian transformations. Table 1 may also be used to convert Hyer's notation to UTT notation.

Hyer's formalization is impressive but rather untidy—and not only because of the omission of the mediant transformation M and the other odd UTTs. The twelve canonical forms are, of course, not a complete enumeration of the Riemann group $\mathcal{R}$; in fact, they are not even a *subgroup* of $\mathcal{R}$. For example, although P and R both appear in the list of canonical forms, their product PR does not, nor does any *PLR*-expression that equals PR. To calculate the product of P and R in Hyer's system, one must look up the transformations in the multiplication table: one reads that D^mP times D^nR equals $D^{m+n+6}RP$; in particular, P times R is represented by the expression D^6RP. In comparison with the formula for multiplying two UTTs, this process seems quite opaque.[33]

The following theorem describes another sort of "complexity" of the group $\mathcal{H}$; this time the complexity is algebraically real, not a function of Hyer's notation. We know from parts (c), (d), and (e) of Theorem 4.3 that the full UTT group $\mathcal{U}$ can be generated by two elements in a variety of ways. For the subgroup $\mathcal{H}$, several *three*-element generating sets were exhibited in Section 4.8; this turns out to be the best we can do.[34]

4.10. Theorem. The group $\mathcal{H}$ is not generated by any two of its elements.

Proof. The simplest group that cannot be generated by two elements is the eight-element group $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, the direct product of three copies of the additive group $\mathbf{Z}_2 = \{0, 1\}$. (Any two non-identity elements of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ generate a four-element subgroup.) We shall construct a homomorphism f from $\mathcal{H}$ *onto* $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. Once we have done so, the theorem is proved: indeed, if some two UTTs U and V generated $\mathcal{H}$, then their images $f(U)$ and $f(V)$ would generate $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

We define f via three components α, β, and γ. First, α is simply the sign function:

$$\alpha(\langle \sigma, t^+, t^- \rangle) = \begin{cases} 0 & \text{if } \sigma = +, \\ 1 & \text{if } \sigma = -. \end{cases}$$

For the definition of β, note first that if $\langle \sigma, t^+, t^- \rangle$ belongs to $\mathcal{H}$, then t^+ and t^- are necessarily of the same parity; every even UTT is either "even-even" or "odd-odd." So we can define

$$\beta(\langle \sigma, t^+, t^- \rangle) = \begin{cases} 0 & \text{if } t^+ \text{ and } t^- \text{ are even}, \\ 1 & \text{if } t^+ \text{ and } t^- \text{ are odd}. \end{cases}$$

Finally, if U belongs to $\mathcal{H}$, then $\tau(U)$ is even, so either $\tau(U) = 0 \bmod 4$ or $\tau(U) = 2 \bmod 4$. We define

$$\gamma(U) = \begin{cases} 0 & \text{if } \tau(U) = 0 \bmod 4, \\ 1 & \text{if } \tau(U) = 2 \bmod 4. \end{cases}$$

It is easily verified that α, β, and γ are homomorphisms. The desired mapping f from $\mathcal{H}$ to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ is simply the direct product of α, β, and γ; that is, $f(U) = (\alpha(U), \beta(U), \gamma(U))$ for every U in $\mathcal{H}$. To show that f maps $\mathcal{H}$ *onto* $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, it suffices to observe that the eight UTTs $\langle +, 0, 0 \rangle$, $\langle +, 0, 2 \rangle$, $\langle +, 1, 1 \rangle$, $\langle +, 1, 3 \rangle$, $\langle -, 0, 0 \rangle$, $\langle -, 0, 2 \rangle$, $\langle -, 1, 1 \rangle$, and $\langle -, 1, 3 \rangle$, all elements of $\mathcal{H}$, are mapped to all eight possible triples $(\alpha(U), \beta(U), \gamma(U))$. ■

4.11. The group $\mathcal{U}$ is of sufficient size and complexity to make an exhaustive study of all its subgroups impractical. In this section, however, we present a large family of subgroups of $\mathcal{U}$, including most of those discussed previously.

These groups are most easily defined by generalizing the definition given in Section 3.2 for the simply transitive groups $\mathcal{K}(a, b)$. Let d be a divisor of 12; that is, $d = 1, 2, 3, 4, 6$, or 12. Let a and b be integers mod d. We define $\mathcal{K}_d^+(a)$ as the set of all UTTs of the form $\langle +, m, n\rangle$ such that $n = am \pmod d$; $\mathcal{K}_d^-(a, b)$ as the set of all UTTs $\langle -, m, n\rangle$ such that $n = am + b \pmod d$; and $\mathcal{K}_d(a, b)$ as the union of $\mathcal{K}_d^+(a)$ and $\mathcal{K}_d^-(a, b)$. For every m, it is easy to see that there are $12/d$ values of n such that $\langle +, m, n\rangle$ belongs to $\mathcal{K}_d^+(a)$ and also $12/d$ values of n such that $\langle -, m, n\rangle$ belongs to $\mathcal{K}_d^-(a, b)$. Therefore the sets $\mathcal{K}_d^+(a)$ and $\mathcal{K}_d^-(a, b)$ are each of cardinality $144/d$, and $\mathcal{K}_d(a, b)$ is of cardinality $288/d$. Following the proof of Lemma 3.3, one can show that $\mathcal{K}_d(a, b)$ is a group if and only if the numbers a and b satisfy $a^2 = 1 \pmod d$ and $ab = b \pmod d$.

It is therefore a straightforward matter to enumerate all the groups $\mathcal{K}_d(a, b)$. The smallest of these are the groups $\mathcal{K}_{12}(a, b)$, which are, in fact, precisely the simply transitive groups $\mathcal{K}(a, b)$ studied previously. At the other end of the spectrum, there is only a single group $\mathcal{K}_1(a, b)$, namely $\mathcal{K}_1(0, 0)$, which is the group $\mathcal{U}$ itself. (Every integer equation mod 1 is trivially satisfied, so every UTT belongs to $\mathcal{K}_1(0, 0)$.) From the definition of $\mathcal{K}_d(a, b)$ one sees easily that the elements of $\mathcal{K}_2(1, 0)$ are precisely the even UTTs, so $\mathcal{K}_2(1, 0) = \mathcal{H}$. If $\mathcal{K}_d(a, b)$ is a group, then $\mathcal{K}_d^+(a)$ is a subgroup, and several of these subgroups are familiar to us: $\mathcal{K}_1^+(0) = \mathcal{U}^+$; $\mathcal{K}_2^+(1) = \mathcal{H}^+$; $\mathcal{K}_{12}^+(1) = \mathcal{T}$ (the transposition group $\{T_0, T_1, \ldots, T_{11}\}$); and $\mathcal{K}_{12}^+(11) = \mathcal{R}^+$ (the schritt group).[35]

Many of the other groups $\mathcal{K}_d(a, b)$ and $\mathcal{K}_d^+(a)$, however, are new to us. The lattice in Figure 6 shows all 45 of the groups $\mathcal{K}_d(a, b)$ and the corresponding subgroups $\mathcal{K}_d^+(a)$. The largest groups appear at the top of the lattice, the smallest at the bottom. Two groups are connected by a line whenever the smaller is contained within the larger; a simple criterion for determining this is given in Lemma 4.12 below. All subgroups of the four groups $\mathcal{K}_{12}^+(a)$ are also shown, at the bottom of the figure. These four groups are all cyclic, so the same is true of their subgroups.[36]

The proof of the following lemma is straightforward, and is omitted here.

4.12. Lemma. Let d and d' be divisors of 12; let a and b be integers mod d, and let a' and b' be integers mod d'. Then $\mathcal{K}_{d'}(a', b')$ is a subgroup of $\mathcal{K}_d(a, b)$ if and only if the following three conditions are satisfied:

(a) d' is a multiple of d;
(b) $a = a' \pmod d$;
and (c) $b = b' \pmod d$.

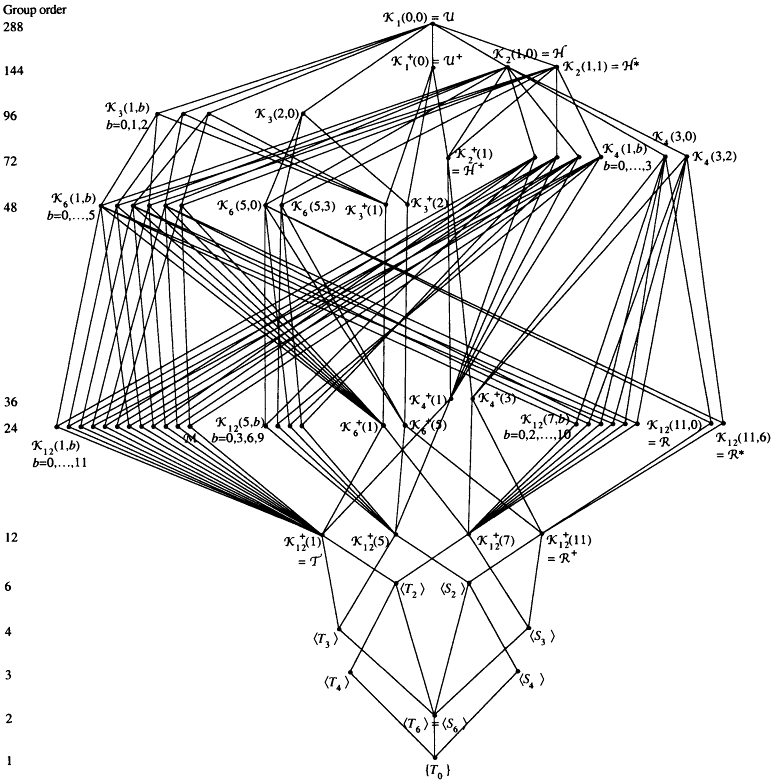


Figure 6. Lattice of the subgroups $\mathcal{K}_d(a, b)$ and $\mathcal{K}_d^+(a)$

4.13. For each divisor d of 12, the set $\mathcal{K}_d(d-1, 0)$ is a group of special interest, which we shall denote $\mathcal{X}_d$. By definition, $\mathcal{K}_d(a, 0)$ consists of all UTTs $U = \langle \sigma, m, n \rangle$ such that $n = am \pmod{d}$; when $a = d - 1$, this is equivalent to $m + n = 0 \pmod{d}$, or to the statement that $\tau(U)$ is a multiple of d. Therefore $\mathcal{X}_d = \tau^{-1}(d\mathbf{Z}_{12})$; that is, $\mathcal{X}_d$ is the inverse image under the homomorphism τ of the subgroup of $\mathbf{Z}_{12}$ generated by d. (For example, a UTT U belongs to $\mathcal{X}_3$ if and only if $\tau(U)$ is equal to 0, 3, 6, or 9.) The groups $\mathcal{X}_1 = \mathcal{K}_1(0,\ 0) = \mathcal{U}$ (all UTTs), $\mathcal{X}_2 = \mathcal{K}_2(1,\ 0) = \mathcal{H}$ (even UTTs), and $\mathcal{X}_{12} = \mathcal{K}_{12}(11,\ 0) = \mathcal{R}$ (Riemannian UTTs) are familiar, but $\mathcal{X}_3 = \mathcal{K}_3(2,\ 0)$, $\mathcal{X}_4 = \mathcal{K}_4(3,\ 0)$, and $\mathcal{X}_6 = \mathcal{K}_6(5,\ 0)$ have not been discussed previously.

If d is an *even* divisor of 12 (that is, 2, 4, 6, or 12), then there is a "skew" counterpart to $\mathcal{X}_d$, namely $\mathcal{X}_d^* = \mathcal{K}_d(d-1, d/2)$. This group comprises all mode-preserving U with $\tau(U) = 0 \pmod{d}$ and all mode-reversing U with $\tau(U) = d/2 \pmod{d}$. Of the four skew groups, $\mathcal{X}_{12}^* = \mathcal{K}_{12}(11, 6)$ is the familiar group $\mathcal{R}^*$ of skew-Riemannian UTTs, but $\mathcal{X}_2^* = \mathcal{K}_2(1, 1)$, $\mathcal{X}_4^* = \mathcal{K}_4(3, 2)$, and $\mathcal{X}_6^* = \mathcal{K}_6(5, 3)$ are new. The first of these is of particular interest: $\mathcal{X}_2^*$, which we may also denote $\mathcal{H}^*$, consists of all mode-preserving even UTTs and all mode-reversing odd UTTs.

If d is any divisor of 12, we may define a function τ_d from $\mathcal{U}$ to $\mathbf{Z}_d$ (the integers mod d) by the equation $\tau_d(U) = \tau(U) \bmod d$. If d is even, we may also define

$$\tau_d^*(U) = \begin{cases} \tau(U) \bmod d & \text{if } \sigma_U = +, \\ \tau(U) + d/2 \bmod d & \text{if } \sigma_U = -. \end{cases}$$

Because τ is a homomorphism (from Section 2.1), it follows easily that τ_d and τ_d^* are also homomorphisms. The kernels of τ_d and τ_d^* are the groups $\mathcal{X}_d$ and $\mathcal{X}_d^*$, respectively; therefore both $\mathcal{X}_d$ and $\mathcal{X}_d^*$ are normal subgroups of $\mathcal{U}$. In fact, of the 45 groups $\mathcal{K}_d(a, b)$, only these ten subgroups (six groups $\mathcal{X}_d$ and four groups $\mathcal{X}_d^*$) are normal. To see this, note first that if $\mathcal{K}_d(a, b)$ is normal in $\mathcal{U}$, an argument similar to the proof of Theorem 3.6(d) shows that $a = d - 1$. Then, examining the list of groups $\mathcal{K}_d(a, b)$, we find that the only such groups are $\mathcal{K}_d(d-1, 0) = \mathcal{X}_d$ and $\mathcal{K}_d(d-1, d/2) = \mathcal{X}_d^*$.

Finally, it is worth noting that the only commutative groups among the various $\mathcal{K}_d(a, b)$ are the twelve groups $\mathcal{K}(1, b) = \mathcal{K}_{12}(1, b)$ identified in Theorem 3.6(b). The proof given there shows that if $\mathcal{K}_d(a, b)$ is commutative, then $a = 1$. If $d < 12$, then the UTTs $U = \langle +, 1, d+1 \rangle$ and $V = \langle -, 0, b \rangle$ both belong to $\mathcal{K}_d(1, b)$. But $UV = \langle -, 1, b+d+1 \rangle$ while $VU = \langle -, d+1, b+1 \rangle$, so clearly $\mathcal{K}_d(1, b)$ cannot be commutative unless $d = 12$.

4.14. One other family of subgroups of $\mathcal{U}$, not included among the groups $\mathcal{K}_d(a, b)$, deserves brief mention. If d is a divisor of 12, let $\mathcal{Y}_d$ be the set of all UTTs $\langle \sigma, m, n \rangle$ such that both m and n are divisible by d. It is easy to see that $\mathcal{Y}_d$ is a group; it is, in fact, a subgroup of $\mathcal{X}_d$. (The definition of $\mathcal{X}_d$ requires only that the sum $m + n$ be a multiple of d; this is certainly true if both m and n are multiples of d.) The largest of the groups $\mathcal{Y}_d$ is $\mathcal{Y}_1 = \mathcal{U}$; the smallest is $\mathcal{Y}_{12}$, which consists only of the two elements $\langle +, 0, 0 \rangle = T_0$ and $\langle -, 0, 0 \rangle = P$. In general, the order of $\mathcal{Y}_d$ is $288/d^2$.

The behavior of the groups $\mathcal{Y}_d$ is easily appreciated in musical terms. The group $\mathcal{Y}_3$, for example, consists of the 32 UTTs each of whose transposition levels is 0, 3, 6, or 9. This group may be thought of as acting on the set of the eight (major and minor) triads whose roots are C, E♭, F♯, and A (or either of the other two "octatonic families" of triads). If UTTs in $\mathcal{Y}_3$ are applied in any combination to a triad in this set, the result will always belong to the same octatonic family. Likewise, $\mathcal{Y}_2$ may be regarded as acting on a 12-triad "whole-tone family," $\mathcal{Y}_4$ on a 6-triad "hexatonic family," and $\mathcal{Y}_6$ on a "tritone family" such as {C, c, F♯, f♯}. These groups are very similar in structure to $\mathcal{U}$ itself, but act on restricted sets of triads.

If d is an *even* divisor of 12, there is also a skew group $\mathcal{Y}_d^*$, consisting of all mode-preserving UTTs $\langle +, m, n\rangle$ such that $m = n = 0 \pmod d$ and all mode-reversing UTTs $\langle -, m, n\rangle$ such that $m = n = d/2 \pmod d$. For example, $\mathcal{Y}_2^*$ is a group of order 72 consisting of (in the language of the proof of Theorem 4.10) all mode-preserving even-even UTTs and all mode-reversing odd-odd UTTs. This group may be regarded as acting on a set of twelve triads consisting of the major triads rooted in one whole-tone scale and the minor triads rooted in the other. Similarly, $\mathcal{Y}_4^*$ is a group of order 18 acting on a set such as {C, d, E, f♯, A♭, b♭}.

Many other groups may be formed as intersections of various groups among $\mathcal{K}_d(a, b)$, $\mathcal{Y}_d$, and $\mathcal{Y}_d^*$. We give just two examples. First, $\mathcal{Y}_2^* \cap \mathcal{R} = \mathcal{Y}_2^* \cap \mathcal{K}_{12}(11, 0)$ is a group of order 12 consisting of all the even-numbered schritts ($S_0, S_2, \ldots, S_{10}$) and all the odd-numbered wechsels ($W_1, W_3, \ldots, W_{11}$). This group is, actually, a simply transitive subgroup of $\mathcal{Y}_2^*$ (acting on the restricted set of triads discussed above in conjunction with $\mathcal{Y}_2^*$). Secondly, $\mathcal{Y}_2 \cap \mathcal{X}_4^* = \mathcal{Y}_2 \cap \mathcal{K}_4(3, 2)$ is a group of order 36 consisting of all mode-preserving UTTs $\langle +, m, n\rangle$ such that m and n are even and $m + n = 0 \pmod 4$, along with all mode-reversing UTTs $\langle -, m, n\rangle$ such that m and n are even and $m + n = 2 \pmod 4$.

4.15. In general, if a group can be written as a product of two of its subgroups, say $G = HK$, where H and K are both *normal* subgroups of G whose intersection consists only of the identity element of G, then we may conclude that G is isomorphic to the direct product $H \times K$. (See Dummit and Foote 1999, 173.) Theorem 4.3 offers several characterizations of $\mathcal{U}$ as products of two of its subgroups, but, as noted in Section 4.4, these are not direct-product representations; in no case are both subgroups normal in $\mathcal{U}$. Mathematicians have devised a weaker form of group product, known as a *semidirect product*, that applies in some of these situations. Representations of $\mathcal{U}$ as semidirect products are the subject of these final sections of Part 4.

The theory of semidirect products is considerably less elegant than the theory of direct products. This theory will only be sketched here; for details, see Dummit and Foote 1999, 177–82. The main difficulty is the fact that if H and K are groups, then their semidirect product, conventionally notated $H \rtimes K$, is not uniquely determined. It depends, in general, on the construction of a homomorphism from K into the group of automorphisms of H—that is, an action of K on H—which corresponds to conjugation of elements of H by elements of K. (That is, an element k of K acts on H by mapping each element h to the element which, in the semidirect product, will be equal to khk^{-1}.)

In a semidirect product $G = H \rtimes K$, the subgroup H must be normal in G, but K need not be. In fact, for G to be isomorphic to a semidirect product $H \rtimes K$ of two subgroups, it suffices to verify the following three conditions:

(a) H is a normal subgroup of G;

(b) $HK = G$;

and (c) $H \cap K = \{1\}$ is the trivial subgroup consisting only of the identity element of G.

Of the subgroups mentioned in Theorem 4.3, $\mathcal{R}$ and $\mathcal{R}^*$ are the only normal subgroups of $\mathcal{U}$, hence the only candidates for H by condition (a) above. By Theorem 4.3(a), $\mathcal{U} = \mathcal{B}\mathcal{R}$ $(= \mathcal{R}\mathcal{B})$ and $\mathcal{U} = \mathcal{B}\mathcal{R}^*$ $(= \mathcal{R}^*\mathcal{B})$, which takes care of (b) with $\mathcal{B}$ playing the role of K. Condition (c) follows from Lemma 4.2(a). It follows that $\mathcal{U}$ is isomorphic to a semidirect product $\mathcal{R} \rtimes \mathcal{B}$, and also to a semidirect product $\mathcal{R}^* \rtimes \mathcal{B}$.

4.16. Another description of the structure of $\mathcal{U}$ relates to an established construction in group theory. Recall that $\mathcal{U}^+$ is the group of order 144 (isomorphic to $\mathbf{Z}_{12} \times \mathbf{Z}_{12}$) consisting of all mode-preserving UTTs, and let $\mathcal{P} = \{T_0, P\}$ be the group of order 2 (isomorphic to $\mathbf{Z}_2$) generated by the parallel transformation $P = \langle -, 0, 0\rangle$. (This group appeared in Section 4.14 as $\mathcal{Y}_{12}$.) Then $\mathcal{U}^+$ is a normal subgroup of $\mathcal{U}$ (by Section 1.10); $\mathcal{U}^+\mathcal{P} = \mathcal{U}$ (because every mode-reversing UTT is UP for some mode-preserving U); and clearly $\mathcal{U}^+ \cap \mathcal{P} = \{T_0\}$. As described in Section 4.15, it follows that $\mathcal{U}$ is isomorphic to a semidirect product $\mathcal{U}^+ \rtimes \mathcal{P}$, or $(\mathbf{Z}_{12} \times \mathbf{Z}_{12}) \rtimes \mathbf{Z}_2$.

This representation is useful because it describes a meaningful aspect of the behavior of the group $\mathcal{U}$. As noted in Section 4.15, a semidirect product $H \rtimes K$ involves an action of the group K on the group H by conjugation. When $\mathcal{P} = \{T_0, P\}$ acts on $\mathcal{U}^+$ by conjugation, the identity T_0 of course fixes every element of $\mathcal{U}^+$, while P maps a UTT $U = \langle \sigma, t^+, t^- \rangle$ to the UTT $PUP^{-1} = PUP = \langle -, 0, 0\rangle\langle \sigma, t^+, t^-\rangle\langle -, 0, 0\rangle = \langle \sigma, t^-, t^+\rangle$.[37] Informally speaking, that is, $\mathbf{Z}_2$ acts on $\mathbf{Z}_{12} \times \mathbf{Z}_{12}$ by interchanging the two factors $\mathbf{Z}_{12}$.

A more general construction is possible. Let K be a subgroup of the permutation group $\mathbf{S}_n$ (the group of all $n!$ permutations of n objects), acting on a direct product H^n of n copies of a group H by permuting the factors. The resulting semidirect product $H^n \rtimes K$ is a group of order $|H|^n \cdot |K|$, known as the *wreath product* of H and K and denoted $H \wr K$. (See Dummit and Foote 1999, 189.) In our case $n = 2$ and the permutation group $K = \mathbf{S}_2$ is isomorphic to $\mathbf{Z}_2$; we have shown that the UTT group $\mathcal{U}$ is isomorphic to the wreath product $\mathbf{Z}_{12} \wr \mathbf{Z}_2$. The groups $\mathcal{Y}_d$ introduced in Section 4.14 are also wreath products; in fact, $\mathcal{Y}_d$ is isomorphic to $\mathbf{Z}_{12/d} \wr \mathbf{Z}_2$.

5. Extensions and Generalizations

The UTT methodology may be extended in a variety of directions, a few of which are explored briefly in this part of the paper. In Sections 5.1–5.5 we investigate connections between UTTs and inversion operators, first via links with the work of Jonathan Kochavi, then by extending the group $\mathcal{U}$ to a larger group that contains the usual inversion operator of pitch-class set theory. In subsequent sections we remark on the applicability of the UTT theory to set classes other than triads, to serial structures, and in mod-n systems where n is something other than 12. Finally, in Sections 5.12–5.14 we pursue the possibility that transformations need not be restricted to just two transposition classes at a time. Many of these ideas are only sketched here; more details may be found in Hook 2002.

5.1. Kochavi (1998, 308) defines a *contextually-defined inversion operator* (CIO) to be a transformation on pitch-class sets of a certain designated set class, mapping each set in that class to a set to which it is inversionally related. Inasmuch as UTTs are defined only on major and minor triads (set class 3-11), it is clear that the mode-reversing UTTs (elements of $\mathcal{U}^-$) are CIOs. The converse does not hold—that is, not every CIO is a mode-reversing UTT—for two reasons: first because Kochavi allows CIOs on other set classes besides triads, and also because nothing in Kochavi's definition guarantees that the uniformity condition of Section 1.5 is satisfied. Kochavi's most significant results, however, deal specifically with those CIOs that commute with all transpositions T_n. By Theorem 1.7 this condition ensures uniformity for triadic transformations; hence the triadic CIOs that commute with transpositions are precisely the mode-reversing UTTs.

Let I denote the standard inversion operator of pc-set theory, inversion about the pitch class C (= 0). We shall write I_n for the transformation IT_n (elsewhere more commonly called T_nI, in right-to-left orthography); thus $I = I_0$. As noted in Section 1.6, I and I_n are not UTTs. If two pitch classes a and b are related by I_n, then $a + b = n$. If two *triads* (r, σ) and (r', σ') are related by I_n, then the *root* of one triad (r) inverts to the *fifth* of the other ($r' + 7$), so that $r + r' + 7 = n$; also, of course, $\sigma' = -\sigma$.

Now suppose a mode-reversing UTT $U = \langle -, t^+, t^- \rangle$ acts on the triad $\Delta = (r, \sigma)$. Then $(\Delta)(U)$, being a triad of mode $-\sigma$, is identical with $(\Delta)(I_n)$ for some n. Then $(\Delta)(I_n) = (\Delta)(U) = (r, \sigma)\langle -, t^+, t^- \rangle = (r + t^\sigma, -\sigma)$. The observations in the preceding paragraph then imply the following theorem:

5.2. Theorem. If $U = \langle -, t^+, t^- \rangle$ is a mode-reversing UTT and $\Delta = (r, \sigma)$ is a triad, then $(\Delta)(U) = (\Delta)(I_n)$, where $n = 2r + t^\sigma + 7$.

5.3. The formula in the above theorem corresponds to what Kochavi (1998, 310) calls the *indexing function* of the CIO U. The formula for n depends not only on U but also on the triad Δ; this is what is "contextual" about Kochavi's CIOs. Kochavi formulates his indexing functions, however, not in terms of the *root* of the triad but in terms of a *canonical representative*. The most convenient choice of a canonical representative for a triad, to match Kochavi's formalization, is the root of a major triad ($\text{rep}(r, +) = r$) or the fifth of a minor triad ($\text{rep}(r, -) = r + 7$). To recast Theorem 5.2 in these terms, it is necessary to consider the two transposition classes of triads (major and minor) separately. Suppose Δ is a triad with canonical representative $\text{rep}(\Delta) = m$. If $\Delta = (r, +)$ is a major triad, then $m = r$ and the theorem gives the indexing function as $n = 2r + t^{\sigma} + 7 = 2m + t^{+} + 7$. On the other hand, if $\Delta = (r, -)$ is a minor triad, then $m = r + 7$, $r = m + 5$, and the indexing function is $n = 2r + t^{\sigma} + 7 = 2(m + 5) + t^{-} + 7 = 2m + t^{-} + 5$. These formulas are in accord with Kochavi's Theorem 1, which states that for any CIO that commutes with transpositions, the indexing function on each transposition class is given by a formula of the form $n = 2m + c$, where m is the canonical representative of the triad (or other set) and c is a constant (possibly a different constant for each transposition class). Kochavi's theorem has the advantage of generality, as it applies not only to triads (although, as we shall see in Sections 5.6–5.7, many of our results about UTTs, including analogs of Theorem 5.2, are applicable to other set classes as well). Theorem 5.2, on the other hand, has the virtues of simplicity (a single formula holds for both transposition classes) and concreteness (the constant c is given explicitly in terms of the components of U). Using Theorem 5.2, one may readily calculate the indexing function of a wechsel W_n or any other mode-reversing UTT.

One of Kochavi's later results (1998, 317) states, in the above notation, that if the indexing function of U is given by $2m + c_1$ on one transposition class and $2m + c_2$ on the other, then $(\Delta)(U^2) = (\Delta)(T_n)$, where $n = c_1 + c_2$. This, too, accords with our knowledge of UTTs. The above discussion shows that if $U = \langle -, t^{+}, t^{-} \rangle$, then the constants c_1 and c_2 are $t^{+} + 7$ and $t^{-} + 5$, respectively. Hence $n = c_1 + c_2 = t^{+} + t^{-} + 12 = t^{+} + t^{-}$ is the total transposition of U, and we know from Section 1.11 that $U^2 = T_n$.

5.4. Next we show, following a suggestion of John Clough, how the UTT formalism may be extended to include I and other inversion operators. As motivation for the method, consider Figure 7, a dual-circle representation of the behavior of I on triads. The diagram correctly depicts the action C $\rightarrow$ f, c $\rightarrow$ F associated with I; the action is non-uniform, however, because triads do not ascend by semitones clockwise around the right-hand pair of circles, as required by the definition of a dual-circle configuration in Section 1.14. In fact, the action is *anti-uniform*: tri-

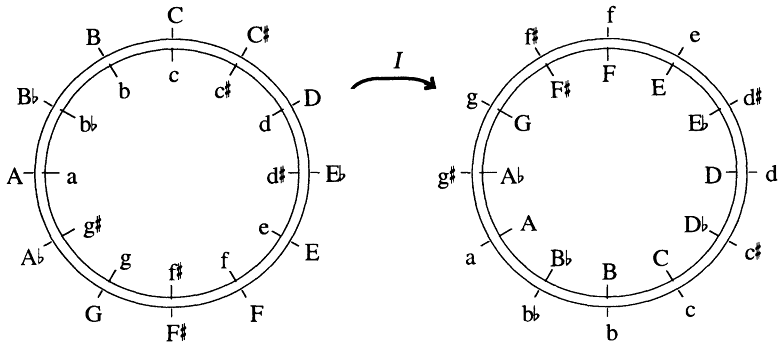


Figure 7. Dual-circle representation of the action of the inversion operator I on triads

ads proceed in ascending order *counterclockwise*. The circles, that is, have been not only rotated but also *reflected*.

To allow reflections in our formalization, we introduce two new signs into each transformation, ρ^+ and ρ^-, the *reflection factors* for major and minor triads. These factors ρ^+ and ρ^- indicate whether the action of the transformation on each transposition class is uniform ($\rho = +$, circle rotated only) or anti-uniform ($\rho = -$, circle rotated and reflected). We shall refer to these generalized transformations as *quasi-uniform triadic transformations*, or QTTs. To be precise, a QTT is an ordered 5-tuple

$$U = \langle \sigma, t^+, t^-, \rho^+, \rho^- \rangle,$$

where σ, ρ^+, and ρ^- are signs and t^+ and t^- are integers mod 12. A QTT $U = \langle \sigma_U, t^+_U, t^-_U, \rho^+_U, \rho^-_U \rangle$ acts on a triad $\Delta = (r_\Delta, \sigma_\Delta)$ in the following stages. First, as usual, the root r_Δ is incremented by the appropriate transposition level $t^{(\sigma_\Delta)}{}_U$. Next, *if* the appropriate reflection factor $\rho^{(\sigma_\Delta)}{}_U$ is $-$, then the (new) root is replaced by its negative (mod 12); this step effects the reflection of the circle. Finally, *if* the sign σ_U is $-$, then the mode σ_Δ is changed to $-\sigma_\Delta$. These steps are formalized in the following equation:

$$(\Delta)(U) = (\rho^{(\sigma_\Delta)}{}_U(r_\Delta + t^{(\sigma_\Delta)}{}_U), \sigma_\Delta\sigma_U) \qquad \textit{(action of a QTT on a triad).}$$

If the reflection factors are both +, then the action of U is the same as the action of the UTT $\langle \sigma, t^+, t^- \rangle$; we may therefore identify any UTT $\langle \sigma, t^+, t^- \rangle$ with the QTT $\langle \sigma, t^+, t^-, +, + \rangle$. The QTTs form a group $\mathcal{Q}$ of order $2 \times 12 \times 12 \times 2 \times 2 = 1152$, of which $\mathcal{U}$ is a subgroup. Products and inverses in $\mathcal{Q}$ are given by the following equations, where $U = \langle \sigma_U, t^+_U, t^-_U, \rho^+_U, \rho^-_U \rangle$ and $V = \langle \sigma_V, t^+_V, t^-_V, \rho^+_V, \rho^-_V \rangle$:

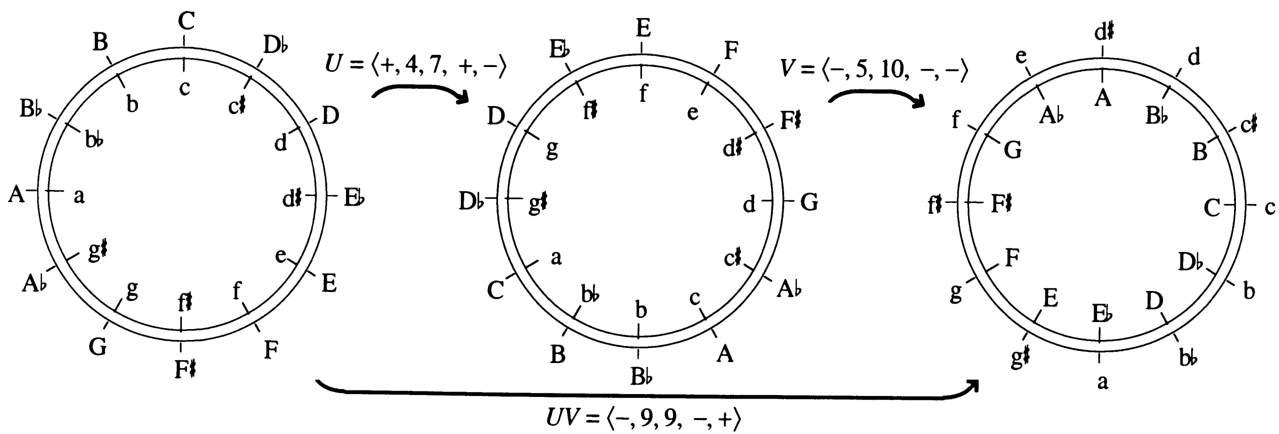


Figure 8. Dual-circle representation of the QTT equation
$\langle +, 4, 7, +, - \rangle \langle -, 5, 10, -, - \rangle = \langle -, 9, 9, -, + \rangle$

$$UV = \langle \sigma_U\sigma_V,\ t^+{}_U + \rho^+{}_U t^{(\sigma_U)}{}_V,\ t^-{}_U + \rho^-{}_U t^{(-\sigma_U)}{}_V,\ \rho^+{}_U \rho^{(\sigma_U)}{}_V,\ \rho^-{}_U \rho^{(-\sigma_U)}{}_V \rangle;$$
$$\langle \sigma, t^+, t^-, \rho^+, \rho^- \rangle^{-1} = \langle \sigma, -\rho^{\sigma} t^{\sigma}, -\rho^{(-\sigma)} t^{(-\sigma)}, \rho^{\sigma}, \rho^{(-\sigma)} \rangle.$$

These formulas, too, reduce to the familiar versions from Sections 1.8 and 1.12 when all reflection factors are +. A dual-circle representation of the product formula is given in Figure 8. Comparison of Figure 8 with Figure 4 shows that even the transposition levels of a product may be affected when reflections are introduced.

5.5. Among the simplest QTTs that are not UTTs are the simple "flips"

$$F^+ = \langle +, 0, 0, -, + \rangle,$$

which is non-uniform only on major triads, and

$$F^- = \langle +, 0, 0, +, - \rangle,$$

which is non-uniform only on minor triads. Their product is the double flip

$$F^+F^- = \langle +, 0, 0, -, - \rangle.$$

These flips are *not* inversions, as they are mode-preserving. Any QTT $\langle \sigma, t^+, t^-, \rho^+, \rho^- \rangle$ may be written as the UTT $\langle \sigma, t^+, t^-, +, + \rangle$ multiplied on the right by F^+ and/or F^- as needed to account for ρ^+ and ρ^-; it follows that the group $\mathcal{Q}$ is generated by the UTTs together with F^+ and F^-.

Among the true inversions, the QTT

$$J = \langle -, 0, 0, -, - \rangle$$

is the simplest. J exchanges triads in the pattern C ↔ c, D♭ ↔ b, D ↔ b♭, …, B ↔ c♯. Other inversions are given by

$$J_n = \langle -, n, n, -, - \rangle = T_nJ,$$

as n ranges through the integers mod 12; thus $J = J_0$. The usual inversion operator I, which inverts pitch classes about C and therefore exchanges the triads C ↔ f, D♭ ↔ e, …, is precisely J_7, and the familiar $I_n = IT_n$ is the same as J_{7-n}. (Note the differing order of factors in the equations $J_n = T_nJ$ and $I_n = IT_n$.) Thus the inversions $J_0, J_1, \ldots, J_{11}$ are simply a permutation of $I_0, I_1, \ldots, I_{11}$, and the familiar "T/I group" $\mathcal{J} = \{T_0, \ldots, T_{11}, I_0, \ldots, I_{11}\}$ may equivalently be written $\mathcal{J} = \{T_0, \ldots, T_{11}, J_0, \ldots, J_{11}\}$. In working with QTTs, the J_n notation has advantages that will soon become clear.[38] As an exercise, the interested reader may use the equation $I_n = J_{7-n}$ and the formula for the action of a QTT to provide an alternate proof of Theorem 5.2.

We have remarked previously (in Section 2.6) that the "S/W group" $\mathcal{R}$ and the "T/I group" $\mathcal{J}$ are both isomorphic to the dihedral group $\mathbf{D}_{12}$ and are therefore isomorphic to each other. One such isomorphism is given by $S_n \to T_n$, $W_n \to J_n$. The duality between schritts/wechsels and transpositions/inversions is evident in these equations:

$$\begin{aligned} S_m S_n &= S_{m+n}, & T_m T_n &= T_{m+n}; \\ S_m W_n &= W_{m+n}, & T_m J_n &= J_{m+n}; \\ W_m S_n &= W_{m-n}, & J_m T_n &= J_{m-n}; \\ W_m W_n &= S_{m-n}, & J_m J_n &= T_{m-n}; \\ S_n^{-1} &= S_{-n}, & T_n^{-1} &= T_{-n}; \\ W_n^{-1} &= W_n, & J_n^{-1} &= J_n. \end{aligned}$$

The equations on the left were given in Section 2.2; those on the right follow directly from the product and inverse formulas for QTTs. (Using I_n instead of J_n would necessitate some sign changes in the subscripts and ruin the perfect correspondence. The mapping $S_n \to T_n$, $W_n \to I_n$ is an anti-isomorphism, not an isomorphism.)

This duality, in fact, extends well beyond the groups $\mathcal{R}$ and $\mathcal{J}$. A dualizing operator ~ may be defined on the entire group $\mathcal{Q}$ as follows: if $U = \langle \sigma, t^+, t^-, \rho^+, \rho^- \rangle$, then $U^\sim = \langle \sigma, t^+, -t^-, \sigma\rho^+, \sigma\rho^- \rangle$. The operator ~ has the properties $(UV)^\sim = U^\sim V^\sim$ and $U^{\sim\sim} = U$; in technical terms, ~ is an involution in the automorphism group of $\mathcal{Q}$. For any QTT U, we may regard $U^\sim$ as the "dual" of U, another QTT whose role in the structure of the group $\mathcal{Q}$ is identical to that of U (but whose action on triads may be very different). Via this operator, schritts are dual to transpositions ($S_n^\sim = T_n$, $T_n^\sim = S_n$) and wechsels are dual to inversions ($W_n^\sim = J_n$, $J_n^\sim = W_n$). All QTTs with $\sigma = +$ and $t^- = 0$ or 6 are self-dual (that is, $U^\sim = U$).[39]

The structure of the group $\mathcal{Q}$ is incredibly rich, and a detailed study of it would take us much too far afield here. A few interesting subgroups of $\mathcal{Q}$ are worth mentioning, however. The image of $\mathcal{U}$ under the dualizing operator ~ is another group $\mathcal{U}^\sim$ of order 288, isomorphic to $\mathcal{U}$, consisting of all QTTs of the forms $\langle +, t^+, t^-, +, + \rangle$ and $\langle -, t^+, t^-, -, - \rangle$: that is, all mode-preserving uniform transformations and all mode-reversing anti-uniform transformations. For every statement about algebraic relationships among UTTs in the group $\mathcal{U}$, a corresponding dual statement holds among dual-UTTs in $\mathcal{U}^\sim$. All QTTs $\langle \sigma, t^+, t^-, \rho^+, \rho^- \rangle$ for which $\rho^+ = \rho^-$ form a group of order 576, dual to itself, that includes both $\mathcal{U}$ and $\mathcal{U}^\sim$. Another group of order 576 consists of all QTTs for which the product of the three signs σ, ρ^+, and ρ^- is + (that is, either all three are + or exactly one is +). The simply transitive subgroups of $\mathcal{Q}$ include not only the familiar subgroups $\mathcal{K}(a, b)$ of $\mathcal{U}$, but also the transposition/inversion group $\mathcal{J}$, Clough's "*S/I* groups" (mentioned in Section 3.11), and a variety of other groups of order 24.

The following commutativity properties of QTTs are of interest, and demonstrate aspects of the duality mentioned above. Just as the center of the group $\mathcal{U}$ is the transposition group $\mathcal{T}$, the center of $\mathcal{U}^{\sim}$ is the schritt group $\mathcal{R}^{+}$; that is, the only elements of $\mathcal{U}^{\sim}$ that commute with all other elements of $\mathcal{U}^{\sim}$ are the twelve schritts S_n. The QTTs that commute with the transposition T_1 (or, equivalently, with all transpositions) are precisely the UTTs; by duality, the QTTs that commute with the schritt S_1 (or, equivalently, with all schritts) are precisely the elements of $\mathcal{U}^{\sim}$. Although both the schritt/wechsel group $\mathcal{R}$ and the transposition/inversion group $\mathcal{I}$ are dihedral and hence non-commutative, elements of $\mathcal{R}$ always commute with elements of $\mathcal{I}$. In fact, a stronger statement is true: the *only* QTTs that commute with all transpositions and inversions are the schritts and wechsels, and vice versa. Each of the two subgroups $\mathcal{R}$ and $\mathcal{I}$, that is, is the centralizer of the other within $\mathcal{Q}$.

5.6. We have presented UTTs (and QTTs) as transformations acting on major and minor triads, but algebraically, they act only on ordered pairs (r, σ), which could represent various other sorts of musical objects. In particular, the set Γ of these ordered pairs could stand for any set class of asymmetrical pc-sets: asymmetry ensures that the class has 24 distinct representatives, in two transposition classes of 12 pc-sets each, with the members of one class related to the members of the other by inversion. To complete the identification of Γ with such a set class, we need only (arbitrarily) designate one of the two transposition classes as Γ^+ and the other as Γ^- (thus defining the sign σ of each set), and give some method for specifying the "root" (r) of each set in the class (which, following Kochavi, might more accurately be termed a *canonical representative*).[40] The entire UTT theory, complete with Riemannian transformations and simply transitive subgroups, carries over without change to any such set class.[41]

Set class 4-27 [0258], consisting of major-minor and half-diminished seventh chords, is perhaps the most obvious candidate for study in this way. For these chords we may take the word "root" in the usual sense and take major-minor as the "positive" mode. Then the UTT $M = \langle -, 9, 8\rangle$, for example, transforms $\text{CMm}^7 \rightarrow \text{a}^{ø7}$, $\text{c}^{ø7} \rightarrow \text{A}\flat\text{Mm}^7$. The description of this transformation as "mediant" now seems somewhat obscure, but it still generates a cycle of 24 different chords, and still generates the simply transitive group $\mathcal{K}(1, 11)$.

Adrian Childs (1998) and Edward Gollin (1998), in two closely related articles, have developed transformational theories for seventh chords, both now easily reinterpreted in terms of UTTs. Childs focuses on transformations with the smoothest possible voice-leading; the transformation shown in Figure 9, for example, he calls $S_{3(2)}$, indicating that two voices move by half-step in *similar* motion, the two notes held constant form interval class *3*, and the two moving voices form interval class *2*. Gollin,

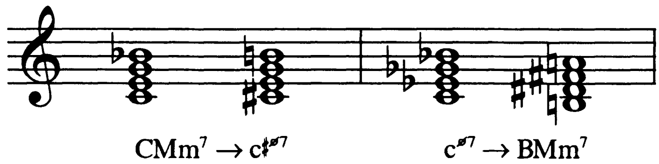


Figure 9. A seventh-chord transformation: Childs's $S_{3(2)}$, Gollin's $I^{\text{ii}}_{\text{iii}}$, or the UTT $\langle +, 1, 11 \rangle = W_1$

inspired by the triadic theories of Moritz Hauptmann (1853), labels the elements of a seventh chord *i*, *ii*, *iii*, and *iv*, upward from the root of a major-minor seventh but downward from the seventh of a half-diminished seventh; this same transformation is then designated $I^{\text{ii}}_{\text{iii}}$, indicating an *inversion* in which elements *ii* and *iii* are mapped onto each other. As a UTT, this transformation maps $(0, +) \rightarrow (1, -)$ and $(0, -) \rightarrow (11, +)$, and is therefore equal to $\langle +, 1, 11 \rangle$, the wechsel W_1.

The transformations obtained by Childs and Gollin, along with their UTT equivalents, are listed in Table 3. Dualism is implicit in Childs's system and explicit in Gollin's; all the transformations in both systems, therefore, are Riemannian, and in fact each system contains a sufficient number of such transformations to generate the Riemann group $\mathcal{R}$.[42] Although the algebraic structure is the same whether the group acts on triads or seventh chords, the voice-leading properties, and hence the musical viability, of many of these transformations are quite different in the four-voice setting offered by seventh chords. This point is an important one: *voice-leading properties of transformations are not intrinsic to the algebraic structure of the transformation group.* When the wechsel W_6 acts on triads (C → f♯, c → F♯), for instance, there are no common tones and only one voice moves by half-step; when W_6 acts on seventh chords, however (CMm^7 → f♯^{ø7}, c^{ø7} → F♯Mm^7), there are two common tones and the other two voices move by half-step. The schritt S_4, on the other hand, is smoother on triads than on seventh chords.[43]

5.7. David Lewin's analysis of Stockhausen's *Klavierstück III* (Lewin 1993, 16–67) offers an opportunity to illustrate the use of UTTs on another set class. Lewin studies transformations acting on the five-note set class 5-4 [01236], consisting of a chromatic tetrachord and an isolated note. Lewin attributes special significance to two inversionally related forms of this pentachord, $P = \{8, 9, 10, 11, 2\}$ and $p = \{11, 10, 9, 8, 5\}$, and to the inversion relating them. The two are related by I_7, but Lewin decides that they are more meaningfully related by a contextually-defined inversion he calls J (no relation to the QTT J introduced in Section 5.5), which inverts any set in this class so as to preserve its chromatic tetrachord. He then constructs a transformation group generated by J and the transpositions T_n, and derives several properties of the structure of this group.

In fact, Lewin's transformations may be regarded as UTTs acting on set class 5-4, and his group is a familiar one. If we take the sign of P to be + and the "root" of a pentachord in this class to be its isolated note, then $P = (2, +)$ while $p = (5, -)$; it is then easy to see that Lewin's J is precisely the wechsel $W_3 = \langle -, 3, 9 \rangle$. The group generated by J and the transpositions is the simply transitive group $\mathcal{K}(1, 6)$ from Part 3. The properties that Lewin derives of his transformations (for example, J commutes with T_n) and of his group (for example, it is commutative but not cyclic) correspond exactly with what we know to be true of UTTs and the group $\mathcal{K}(1, 6)$.[44]

5.8. There is no real need to apply UTTs to inversionally symmetric sets. Such a set has only a single "mode," so the sign and the second transposition level of a UTT acting on such a set are irrelevant. A UTT therefore collapses to a single transposition level, and the group $\mathcal{U}$ collapses to a cyclic group $\mathbf{Z}_{12}$ for a set such as the diminished triad [036], or something even smaller for a set which also has transpositional symmetry (for example, $\mathbf{Z}_3$ for the fully-diminished seventh chord [0369]). The only set class that is transpositionally but not inversionally symmetric is 6-30 [013679], the "Petrushka chord"; because every set in this

Table 3. Seventh-chord transformations in Childs 1998 and Gollin 1998, and their representations as UTTs

Action		Childs	Gollin	UTT	
$\mathrm{CMm}^7 \rightarrow \mathrm{E}\flat\mathrm{Mm}^7$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{a}^{ø7}$	$C_{3(4)}$		$\langle +, 3, 9 \rangle$	$= S_3$
$\mathrm{CMm}^7 \rightarrow \mathrm{F}\sharp\mathrm{Mm}^7$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{f}\sharp^{ø7}$	$C_{6(5)}$		$\langle +, 6, 6 \rangle$	$= S_6\ (= T_6)$
$\mathrm{CMm}^7 \rightarrow \mathrm{AMm}^7$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{d}\sharp^{ø7}$	$C_{3(2)}$		$\langle +, 9, 3 \rangle$	$= S_9$
$\mathrm{CMm}^7 \rightarrow \mathrm{c}^{ø7}$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{CMm}^7$	$S_{2(3)}$	$I^{\mathrm{i}}_{\mathrm{iv}}$	$\langle -, 0, 0 \rangle$	$= W_0\ (= P)$
$\mathrm{CMm}^7 \rightarrow \mathrm{c}\sharp^{ø7}$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{BMm}^7$	$S_{3(2)}$	$I^{\mathrm{ii}}_{\mathrm{iii}}$	$\langle -, 1, 11 \rangle$	$= W_1$
$\mathrm{CMm}^7 \rightarrow \mathrm{d}^{ø7}$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{B}\flat\mathrm{Mm}^7$		$I^{\mathrm{i}}_{\mathrm{i}}$	$\langle -, 2, 10 \rangle$	$= W_2$
$\mathrm{CMm}^7 \rightarrow \mathrm{e}^{ø7}$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{A}\flat\mathrm{Mm}^7$		$I^{\mathrm{ii}}_{\mathrm{iv}} = I^{\mathrm{iii}}_{\mathrm{iii}}$	$\langle -, 4, 8 \rangle$	$= W_4\ (= L)$
$\mathrm{CMm}^7 \rightarrow \mathrm{f}\sharp^{ø7}$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{F}\sharp\mathrm{Mm}^7$	$S_{4(3)}$	$I^{\mathrm{i}}_{\mathrm{ii}}$	$\langle -, 6, 6 \rangle$	$= W_6$
$\mathrm{CMm}^7 \rightarrow \mathrm{g}^{ø7}$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{FMm}^7$	$S_{3(4)}$	$I^{\mathrm{iii}}_{\mathrm{iv}}$	$\langle -, 7, 5 \rangle$	$= W_7$
$\mathrm{CMm}^7 \rightarrow \mathrm{a}^{ø7}$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{E}\flat\mathrm{Mm}^7$	$S_{5(6)}$	$I^{\mathrm{i}}_{\mathrm{iii}}$	$\langle -, 9, 3 \rangle$	$= W_9\ (= R)$
$\mathrm{CMm}^7 \rightarrow \mathrm{b}\flat^{ø7}$,	$\mathrm{c}^{ø7} \rightarrow \mathrm{DMm}^7$	$S_{6(5)}$	$I^{\mathrm{ii}}_{\mathrm{ii}} = I^{\mathrm{iv}}_{\mathrm{iv}}$	$\langle -, 10, 2 \rangle$	$= W_{10}$

class is invariant under T_6, transposition levels mod 6 (rather than mod 12) suffice, and a group of UTT-like transformations of order $2 \times 6 \times 6 = 72$ results. In the terminology of Section 4.16, this group is isomorphic to the wreath product $\mathbf{Z}_6 \wr \mathbf{Z}_2$. It is also isomorphic to the group $\mathcal{Y}_2'$ of Section 4.14.

5.9. Forms of a twelve-tone row may also be represented by ordered pairs (r, σ), for instance by letting the positive "mode" correspond to prime forms of the row and the negative mode to inverted forms (or alternatively to retrograde forms). Jack Douthett (2001), in an analysis of Webern's Concerto for Nine Instruments, Op. 24, has shown that UTTs are useful in describing the structure of Webern's "derived rows"—rows consisting of multiple copies of one shorter series, such as a trichord or tetrachord—and moreover that the structure of a composition may reflect the behavior of certain groups of UTTs, including small cyclic groups and some of the simply transitive groups. A somewhat similar analysis of the first movement of Webern's String Quartet, Op. 28, appears in Hook 2002, 128–35; the entire movement is generated by a small number of transformations of the B-A-C-H tetrachord on which the row is based.

5.10. Instead of working with integers mod 12 in the formalization of triads and UTTs, one could work with integers mod n. A set $\Gamma^{(n)}$ of $2n$ "triads" (r, σ) is the result, along with a group $\mathcal{U}^{(n)}$ of $2n^2$ transformations $\langle \sigma, t^+, t^- \rangle$, where r, t^+, and t^- are now integers mod n. (Those who wonder what a "triad" is in the mod-n case are reminded that any asymmetrical set class will work.)

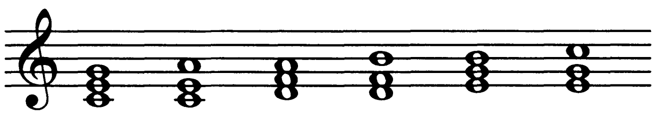
Most of the results of Parts 1–4 of this paper apply also in the more general case, but a few modifications are necessary; here is a brief summary. The main results of Part 1 carry over unchanged. The Riemann group $\mathcal{R}^{(n)}$, consisting of n schritts and n wechsels, is isomorphic to the dihedral group $\mathbf{D}_n$, and the other results of Part 2 carry over as well, except that it is not clear how to define L and R in the general setting, so Theorem 2.5(b) must be rephrased. (It can be shown that wechsels W_j and W_k together generate $\mathcal{R}^{(n)}$ whenever $j - k$ is co-prime to n.) Groups $\mathcal{K}^{(n)}(a, b)$ can be defined, and the characterization of simply transitive groups in Theorem 3.6(a) remains valid, but of course the values of a and b satisfying the conditions $a^2 = 1$, $ab = b$ (mod n) will vary considerably depending on n. Part (c) of that theorem holds only if n is even; if n is odd, then *all* the commutative groups $\mathcal{K}^{(n)}(1, b)$ are cyclic. Part (d) holds with 11 replaced by $n - 1$; for even values of n there are two such normal subgroups, $\mathcal{R}^{(n)}$ and $\mathcal{R}^{(n)*}$, but for odd n, $\mathcal{R}^{(n)}$ is the only one. The elements of maximal order $(2n)$ in $\mathcal{U}^{(n)}$ are the UTTs $\langle -, t^+, t^- \rangle$ for which $t^+ + t^-$ is co-prime to n; the number of such transformations is $n \cdot \varphi(n)$, where $\varphi(n)$ denotes the number of integers mod n that are co-prime to n. The appearance of the subgroup lattice in Figure 6, of course, depends strongly on n: the lattice will be most complex for values of n which, like $n = 12$, have many divisors. Algebraically, $\mathcal{U}^{(n)}$ is isomorphic to the wreath product $\mathbf{Z}_n \wr \mathbf{Z}_2$.

5.11. Perhaps the most obvious application of mod-n UTTs is in the construction of a transformational theory in equal-tempered systems with any desired number of notes to the octave.[45] An application of more immediate relevance to most music theorists, however, involves using the mod-7 system to study diatonic structures. The group $\mathcal{U}^{(7)}$ is considerably simpler than the familiar $\mathcal{U}^{(12)}$, not only because it is smaller (there are 98 mod-7 UTTs) but also because 7 is a prime number. The only simply transitive subgroups of $\mathcal{U}^{(7)}$ are the seven groups $\mathcal{K}^{(7)}(1, b)$, all of which are cyclic, and the Riemann group $\mathcal{R}^{(7)} = \mathcal{K}^{(7)}(6, 0)$. Almost all of the mode-reversing mod-7 UTTs—42 out of 49, all except the wechsels—are of maximal order (14).[46]

In a diatonic universe, there is no distinction between major and minor triads (a triad mod 7 belongs to set class [024], which is symmetrical), but the signs appearing in the formalization of UTTs may be used to model other sorts of binary oppositions. Clough (2000), for example, has applied mod-7 UTTs to the study of diatonic sequences, such as the ascending 5-6 sequence shown in Figure 10(a). Here the "+" chords are in root position, the "–" chords in first inversion. The chord root r corresponds to scale degree $r + 1$ in C major. (Recall that r ranges from 0 to 6, not from 1 to 7.) The sequence is generated by the mod-7 UTT $\langle -, 5, 3\rangle$; this is one of the order-14 UTTs in $\mathcal{U}^{(7)}$ (that is, it will cycle through all 14 triads before returning to its starting point), and a generator of the cyclic group $\mathcal{K}^{(7)}(1, 5)$. In the "Pachelbel" sequence of Figure 10(b), all chords are in root position; the +/– opposition is simply one of voicing (note the alternating 10ths and 12ths between outer voices). Here the generating UTT is $\langle -, 4, 1\rangle$, and the group is $\mathcal{K}^{(7)}(1, 4)$. Clough (2000) studies several mod-7 transformation groups, all of which may be interpreted as subgroups of $\mathcal{U}^{(7)}$ (or, in the case of groups involving inversion operators, subgroups of $\mathcal{Q}^{(7)}$).

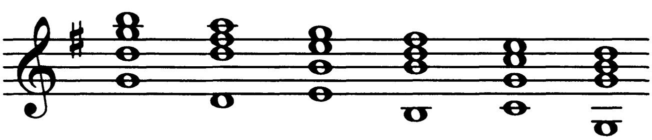
5.12. The idea that the sign σ associated with a musical object $\Delta = (r, \sigma)$ may represent something other than the mode of a triad suggests a further extension of the theory. In the triadic case there are only two options for σ (+ and –) because there are only two types of triads (major and minor). More generally, we might wish to consider objects of *several* types simultaneously, allowing transformations free rein among (for example) major triads, minor triads, and several varieties of seventh chords. In this scenario we may designate the types by *type-indices* 1, 2, …, k rather than signs + and –. Exactly what the types represent need not be specified in general; a type designation may include information about chord inversion or voicing (as in the examples in the previous section), or other properties such as pitch ordering or duration.

In this general setting, (r, σ) denotes the object whose type-index is σ and whose root (or other canonical representative) is r. We are then interested in transformations of the form $U = \langle \sigma_U; t_1, t_2, \ldots, t_k \rangle$. Here σ_U is neither a sign nor a type-index but a *permutation* of the type-indices $1, 2, \ldots, k$ (a member of the symmetric group $\mathbf{S}_k$). If σ_U maps the index i to the index j, then U transforms objects of type i to objects of type j, via the appropriate transposition level t_i. All such transformations form a group of order $12^k \cdot k!$, isomorphic to the wreath product $\mathbf{Z}_{12} \wr \mathbf{S}_k$.



C: $(0, +) \rightarrow (5, -) \rightarrow (1, +) \rightarrow (6, -) \rightarrow (2, +) \rightarrow (0, -)$

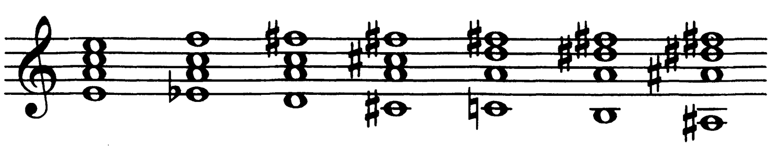
(a) A diatonic sequence generated by the mod-7 UTT $\langle -, 5, 3 \rangle$



G: $(0, +) \rightarrow (4, -) \rightarrow (5, +) \rightarrow (2, -) \rightarrow (3, +) \rightarrow (0, -)$

(b) A diatonic sequence generated by the mod-7 UTT $\langle -, 4, 1 \rangle$

Figure 10



am$^6_4$ FMm$^4_2$ DMm^7 f♯m$^6_4$ DMm$^4_2$ BMm^7 d♯m$^6_4$

$(9, 1) \xrightarrow{U} (5, 2) \xrightarrow{U} (2, 3) \xrightarrow{U} (6, 1) \xrightarrow{U} (2, 2) \xrightarrow{U} (11, 3) \xrightarrow{U} (3, 1)$

Figure 11. An "omnibus" progression generated by a transformation $U = \langle \sigma; 8, 9, 4 \rangle$ in a three-type system

$$
\begin{tikzcd}
\textit{Triads:} & \mathrm{a} \arrow[rrr, dashed, "S_3"] \arrow[dr, "L\subset"'] & & & \mathrm{f} \arrow[rrr, dashed, "S_3"] \arrow[dr, "L\subset"'] & & & \mathrm{d}\sharp \\
\textit{Seventh chords:} & & \mathrm{FMm}^7 \arrow[r, "S_9"'] & \mathrm{DMm}^7 \arrow[ur, "\supset L"'] \arrow[rr, dashed, "T_0"'] & & \mathrm{DMm}^7 \arrow[r, "S_9"'] & \mathrm{BMm}^7 \arrow[ur, "\supset L"'] &
\end{tikzcd}
$$

Figure 12. Cross-type analysis of the omnibus progression

Such a system can be used to model, for example, the "omnibus" progression shown in Figure 11. There are three types of chords in this progression: we use type 1 to signify minor triads in second inversion (m^6_4), type 2 for major-minor seventh chords in third inversion (Mm^4_2), and type 3 for major-minor seventh chords in root position (Mm^7). Let σ be the permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ on the type-indices, and let U be the transformation $\langle\sigma; 8, 9, 4\rangle$. Repeated application of U to any chord of any of the three specified types produces an omnibus; the one shown in Figure 11 results when the starting chord is (9, 1) (am^6_4).

5.13. An alternative approach to a transformational analysis of the omnibus makes use of *cross-type transformations*. A *cross-type transformation* is a mapping that transforms objects of one type into objects of a different type. Such transformations are not strictly permitted according to Lewin's definition of transformations (1987, 3), but there are no mathematical impediments to their construction, and in fact they have appeared sporadically in the literature. A familiar example is the mapping from chromatic pitch space to chromatic pc-space that maps every pitch to its corresponding pitch class.[47]

In the current situation, there are two types of objects and two transposition classes of each: triads (major or minor) and seventh chords (major-minor or half-diminished). In the transformation network of Figure 12, triads are shown in the top row, seventh chords in the bottom. UTTs may be applied to triads (S_3 in Figure 12), or to seventh chords (S_9 and T_0) in the manner demonstrated in Section 5.6. The omnibus progression zigzags back and forth between the two rows, following the solid arrows. The crucial links are the diagonal arrows, which are cross-type transformations. In relating triads to seventh chords it is natural to consider the *inclusion transformation*, denoted $\subset$, which maps every major or minor triad to the unique major-minor or half-diminished seventh chord that contains it ($\mathrm{C} \xrightarrow{\subset} \mathrm{CMm}^7$, $\mathrm{c} \xrightarrow{\subset} \mathrm{a}^{ø7}$). Thus the action $\mathrm{a} \xrightarrow{L\subset} \mathrm{FMm}^7$ in Figure 12 represents the composite of the triadic action $\mathrm{a} \xrightarrow{L} \mathrm{F}$ and the cross-type action $\mathrm{F} \xrightarrow{\subset} \mathrm{FMm}^7$. The inverse of $\subset$, denoted $\supset$, is the seventh-chord-to-triad transformation that maps each seventh chord to the unique major or minor triad that it contains ($\text{CMm}^7 \xrightarrow{\supset} \text{C}$, $\text{c}^{ø7} \xrightarrow{\supset} \text{e}\flat$); the transformation $\supset L$, which also appears in Figure 12, is the inverse of $L\subset$.

From the above description of the action of $\subset$, it may be noted that $\subset$ is equivalent to the UTT $\langle +, 0, 9\rangle$ coupled with a type change.[48] The type change commutes with UTTs; in effect it is computationally inert, amounting only to a reinterpretation of the musical meaning associated with the mathematical object (r, σ). It is therefore a simple matter to carry out calculations involving $\subset = \langle +, 0, 9\rangle$ and its inverse $\supset = \langle +, 0, 3\rangle$. One can easily verify, for example, the transformational identity implied by the trapezoidal loops of arrows in Figure 12:

$$(L\subset)S_9(\supset L) = \langle -, 4, 8\rangle\langle +, 0, 9\rangle\langle +, 9, 3\rangle\langle +, 0, 3\rangle\langle -, 4, 8\rangle = \langle +, 3, 9\rangle = S_3.$$

5.14. The reader may object that neither of the two analyses of the omnibus just presented reveals very much about the progression. Why, for instance, is this particular progression so effective and widely used, while progressions generated by other, apparently similar transformations are nonsensical in comparison? The answer, of course, has to do with the specific properties of the transformations $L\subset$, S_9, and $\supset L$ appearing in Figure 12 (or of the three components of U in Figure 11, which amount to the same thing). It is simple to verify that each of these transformations admits the possibility of extremely smooth voice-leading; the seventh-chord UTT S_9, in fact, is the same as Childs's $C_{3(2)}$, one of the transformations he derived precisely on the basis of their voice-leading characteristics.

The important point, though, is not that the transformational theory answers such questions but that it provides a framework in which they may be asked. Progressions like the omnibus have previously been resistant to transformational analysis because of the variety of chord types involved. Techniques like those suggested in Sections 5.12 and 5.13, more fully developed, may offer a methodology for studying such issues.

NOTES

This paper represents a considerable condensation of Chapters 1–6 of the author's dissertation (Hook 2002). Earlier versions of this paper were presented at conferences of Music Theory Midwest (Indianapolis, May 1999) and the Society for Music Theory (Atlanta, November 1999). The author gratefully acknowledges the many valuable comments received on those occasions. Conversations with John Clough and Jack Douthett have been particularly beneficial. Others whose comments and correspondence have been helpful include David Clampitt, Richard Cohn, Darrell Haile, Eric Isaacson, Jonathan Kochavi, and David Neumeyer.

1. Several of these objections, and others, have been articulated by Fred Lerdahl (2001, 83–85).
2. Both root-interval and voice-leading perspectives may be found in the writings of Riemann and other theorists of his time. Root-interval organization is clearly present in Riemann 1880, often cited as Riemann's closest approach to modern neo-Riemannian techniques. Nora Engebretsen (2001) has traced Riemann's attempts to reconcile the two outlooks; see also Cohn 1998a, 174–75.
3. The more general term "neo-Riemannian theory" aptly describes much recent work such as Cohn's, but I am uncomfortable with its use as a blanket expression embracing UTTs—first because, once again, only a small minority of UTTs behave in ways that have anything to do with the theories of Riemann. There are, I believe, many pitfalls in waving a person's name on too broad a banner, particularly when some of the eponym's ideas are controversial or discredited (a point borne out all too well by the example of Schenkerian analysis). A more neutral term lets those who may take exception to some aspects of the original scholar's work accept more readily that the methods may be valuable anyhow, while encouraging the adherents of the theory to concede that there may be more than one valid approach. In the present situation I suggest that UTTs be considered part of "harmonic transformation theory" rather than "neo-Riemannian theory."
4. Readers familiar with the work of Riemann and with Klumpenhouwer 1994 may legitimately wonder if we should take, as the referential pitch for a minor triad, not its root in the traditional sense but rather its Riemannian "dual root"—that is, its fifth. By this approach, the ordered pair $(0, -)$ would represent a minor triad with dual root C—that is, an F minor triad. In fact, the system that would arise from such a foundation is, both algebraically and functionally, completely equivalent to the one presented here. The same triadic transformations arise in either case, although any given transformation may have different "coordinates." If the system dealt only with Riemannian transformations such as P, L, and R, the dual-root system might be marginally more convenient. But the system presented here encompasses non-Riemannian transformations as well, and moreover most music theorists today identify triads more readily with traditional roots than with Riemannian dual roots; the notation has been chosen accordingly. The implications of the choice of a coordinate system are studied in some detail in Hook 2002, 78–84, including a formula for converting UTT notation from one coordinate system to another.
5. In general, if G and H are groups, then the *direct product* $G \times H$ is the set of all ordered pairs (g, h) with g in G and h in H. This direct product also forms a group,

with multiplication defined componentwise: $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$. See Dummit and Foote 1999, 18.

6. Composition of mappings is always associative; that is, $(XY)Z = X(YZ)$, so we may simply write XYZ for either one. It is not, however, commutative; that is, XY may differ from YX.
7. X is *one-to-one* (or *injective*) if it always maps different triads to different triads, *onto* (or *surjective*) if every triad occurs as the X-transform of some triad. Actually, because the set of triads is finite, a triadic transformation is one-to-one if and only if it is onto. See Dummit and Foote 1999, 2.
8. See Dummit and Foote 1999, 50. A more direct proof that $\mathcal{U}$ is a group is straightforward; in fact, this follows from the product and inverse formulas for UTTs derived below.
9. The *center* of a group G, the subset consisting of those elements in G that commute with every other element of G, is always a subgroup. See Dummit and Foote 1999, 51.
10. A mapping f from a group G to a group H is a *homomorphism* if it preserves group structure in the sense that $f(xy) = f(x)f(y)$ for all x and y in G. The *kernel* of the homomorphism f is the set K consisting of all a in G such that $f(a)$ is the identity element of H. If f is a homomorphism, then K is automatically a subgroup of G; in fact, it is a *normal* subgroup, which means that xax^{-1} is an element of K for all x in G and a in K. See Dummit and Foote 1999, pp. 37, 76–83.
11. *Transitivity* refers to the fact that such a U always exists; *simple* transitivity to the uniqueness of such U. See Lewin 1987, 157; Dummit and Foote 1999, 117.
12. Klumpenhouwer (1994, §5) recognized the significance of this dualism condition when he wrote of the leittonwechsel that "it changes the modality of a major triad to its parallel minor and transposes a major third up, while it changes the modality of a minor triad to its parallel major and transposes a major third *down*. The reversal of direction that accompanies the reversal in modality is the imprint of dualist thinking." The dualist ideology infused all of Riemann's mature writings on harmony (e.g., Riemann 1893), but, as Klumpenhouwer notes, it was in an early work (Riemann 1880) that he most closely approached modern "neo-Riemannian" methods.
13. As discussed in Section 2.7 below, some authors employ slightly different versions of the S_n / W_n notation. Also, some authors, respecting the German origin of the terms, capitalize the words *Schritt* and *Wechsel* and use the German plurals *Schritte* and *Wechsel*; in view of the increasingly frequent appearances of these terms in English, I have chosen to treat them (and the compound *leittonwechsel*) as ordinary English nouns.
14. The letter S might seem a more logical choice, but, as will be noted in Section 2.7, other authors have already used S for at least two other triadic transformations.
15. Cohn's assertion (1998a, 172) that the transformation D is "redundant" in the context of P, L, and R "since it is produced by a composition of L followed by R" is, at the very least, misleading: as UTTs, $D = \langle +, 5, 5 \rangle$ while $LR = \langle +, 7, 5 \rangle$ $(= S_7)$. The non-Riemannian D and the Riemannian LR yield the same result only when applied to minor triads.
16. The dihedral group $\mathbf{D}_k$, a group of order $2k$, is the group of symmetries (rotations and reflections) of a regular k-sided polygon. (The element x represents a reflection, y a rotation.) See Dummit and Foote 1999, 23–28. The notation for dihedral

groups is not quite standardized; most authors call this group $\mathbf{D}_k$, but a few, including Dummit and Foote, call it $\mathbf{D}_{2k}$.

17. Clough presents the product formulas for schritts and wechsels from Section 2.2 above, and also studies the subgroups of the S/W group.
18. The *order* of an element x in a group is the smallest n such that x^n is the identity. See Dummit and Foote 1999, 20.
19. In general, if x generates a cyclic group of order N, then x^k will also generate the same group whenever the number k is co-prime to N (has no prime factors in common with N). The number of generators of any cyclic group of order N is therefore $\varphi(N)$ (Euler's φ-function), the number of integers mod N that are co-prime to N. In the current setting $N = 24$ and $\varphi(N) = 8$. See Dummit and Foote 1999, 58–59.
20. It is *not* generally true that every chain formed by a UTT of order n will cycle through n different triads before repeating. For example, the UTT $\langle +, 3, 4\rangle$ is of order 12, but generates no triad cycles of length 12: it generates cycles of major triads of length 4 and cycles of minor triads of length 3. In the case of a UTT U of order 24, however, every cycle will be of length 24. Indeed, by Section 1.11, $U^2 = T_c$, where $c = \tau(U) = 1, 5, 7,$ or 11; therefore U^2 always generates triad cycles of length 12, and U generates triad cycles of length 24.
21. As we observed in Section 1.1, note 4, the numerical representation of a UTT depends on the choice of a "coordinate system" for representing triads. Briefly, a UTT whose representation in one coordinate system is U will be represented in any other system by a *conjugate* of the form ZUZ^{-1}, where Z is the *coordinate transformation* from one system to the other. In a coordinate system other than the one adopted here, a transformation such as M may have different coordinates, and may in fact generate a different subgroup. Subgroups related by coordinate change are always conjugates of each other in $\mathcal{U}$, however, and are therefore isomorphic. In the case of $\mathcal{M}$, the conjugate subgroups are precisely the other cyclic groups $\mathcal{K}(1, b)$ with b odd. Jack Douthett (2001) has shown that the 24 simply transitive subgroups $\mathcal{K}(a, b)$ fall into eight conjugacy classes. See also Hook 2002, 78–84.
22. Several later composers, including Scriabin and Shostakovich, followed Chopin's key sequence in their own tonal cycles. Liszt completed only the first twelve in his projected cycle of 24 Transcendental Etudes—the "flat keys," ending with No. 12, "Chasse-neige," in B♭ minor. Half a century later, the Russian composer Sergei Lyapunov completed the cycle left unfinished by Liszt, writing twelve Transcendental Etudes, Op. 11, in the "sharp keys" from F-sharp major to E minor, following the key sequence established by Liszt.
23. I am indebted to John Clough for the "Wilde Jagd" example. Clough has introduced the term *uniform flip-flop circle* (UFFC) for a cycle like any of those just mentioned: a closed cycle of triads, alternating in mode and alternating between two transposition levels. Clough 2000 develops the theory of UFFCs in some detail, and also studies the relation between UFFCs and UTTs.
24. The idea of analyzing one succession of triads from the point of view of several different transformation groups has been explored previously in Lewin 1993, Klumpenhouwer 1994, Clampitt 1998, and Clough 2000.
25. The hexatonic cycle is another example of a uniform flip-flop circle as defined by Clough. The generator $\langle -, 0, 8\rangle$ is precisely the transformation that Cohn (1996, 19) refers to as "T_1", a one-step transposition in a hexatonic generalized interval system.

26. Each of these groups $\mathcal{K}(1, b)$ actually contains *two* wechsels, corresponding to the two choices of k, differing by 6, for which $2k = -b$. The twelve wechsels are thus evenly distributed among the six groups of this type. In the following discussion, the choice of the wechsel W_k in the group $\mathcal{K}(1, b)$ is arbitrary; either of the two available choices works equally well.
27. I am grateful to Jack Douthett for pointing out several of the results in Part 4. Douthett was the first to study the structure of $\mathcal{U}$ from the point of view of generators and relations, and to investigate the subgroups that I have called $\mathcal{X}_d$ and $\mathcal{X}_d{}^*$ in Section 4.13; he also constructed some subgraphs of the subgroup lattice that appears in Figure 6. The wreath product representation of $\mathcal{U}$ in Section 4.16 was first noticed by Peter Steinbach and communicated to me by Douthett.
28. Group theorists often describe the structure of a group by specifying a system of *generators and relations*. This consists of (1) a set of elements that together generate the group, and (2) a set of algebraic relations satisfied by the generators, with the property that any other relations among the generators can be deduced from those specified. (See Dummit and Foote 1999, 216–22.) The description of the dihedral group $\mathbf{D}_k$ presented in the proof of Theorem 2.5 is a simple example of such a system. For a group with the size and complexity of $\mathcal{U}$, many different presentations by generators and relations are possible. For example, $\mathcal{U}$ may be described as the group generated by two elements x and y satisfying $x^{12} = 1$, $y^2 = 1$, and $(xy)^2 = (yx)^2$. The UTT B plays the role of the generator x, while P corresponds to y.
29. To understand the action of this transformation, read the 2-cycles from left to right. Suppose we wish to apply the transformation to C (the C major triad). The first 2-cycle maps C to c, and c is unaffected by the four remaining 2-cycles; hence the effect is C → c. Now suppose we start instead with c. The first 2-cycle maps c to C, which is then mapped to A♭ by the second 2-cycle and unchanged thereafter; hence c → A♭. In the same way we can deduce A♭ → g♯, g♯ → E, E → e, and e → C.
30. See Dummit and Foote 1999, 108–10. In permutation theory 2-cycles are generally known as *transpositions*, as they simply "transpose" two elements, but for obvious reasons it seems prudent to avoid this use of the term in a musical context.
31. See Dummit and Foote 1999, 92. Alternatively one can prove that $\mathcal{H}$ is normal directly from the definition of a normal subgroup, or from the fact that $\mathcal{H}$ is the inverse image under the homomorphism τ of the normal subgroup $2\mathbf{Z}_{12} = \{0, 2, 4, 6, 8, 10\}$ of $\mathbf{Z}_{12}$.
32. To be precise, Hyer's canonical forms are representatives of the twelve cosets of the quotient group $\mathcal{R}/\{T_0, T_6\}$. The multiplication table for this quotient group is Hyer's Figure 5.
33. This aspect of Hyer's system has confused many readers. David Kopp (1995, 268), for instance, misreads Hyer's Figure 5 as saying (erroneously) that $PR = RP$, when in fact it says (correctly) only that PR and RP belong to the same coset in $\mathcal{R}/\{T_0, T_6\}$. Kopp's dissertation develops an alternative system of eight transformations. When these are recast as UTTs, it is easy to see that all eight are even and that they generate the group $\mathcal{H}$; ultimately, therefore, Kopp's system is equivalent to Hyer's.
34. That a group may require more generators than a larger group of which it is a subgroup may seem counterintuitive, but such a situation is in fact rather common. In fact, *every* finite group, no matter how many generators it requires, is a subgroup

of the symmetric group $\mathbf{S}_n$ (the group of permutations of n objects) for some sufficiently large n, and $\mathbf{S}_n$ is always generated by two elements, for example the 2-cycle (1, 2) and the n-cycle $(1, 2, \ldots, n)$.

35. Jack Douthett has pointed out another noteworthy property of the schritt group $\mathcal{R}^+$: this is precisely the *commutator subgroup* $\mathcal{U}'$ of $\mathcal{U}$, defined as the subgroup generated by all elements of the form $UVU^{-1}V^{-1}$ in $\mathcal{U}$ (known as *commutators*). (See Dummit and Foote 1999, 171.) It is easy to see that every commutator in $\mathcal{U}$ is both Riemannian and mode-preserving; hence $\mathcal{U}' \subseteq \mathcal{R}^+$. But $\mathcal{R}^+$ is generated by $Q = \langle +, 1, 11\rangle$, which is equal to the commutator $UVU^{-1}V^{-1}$, where $U = \langle +, 0, 11\rangle$ and $V = \langle -, 1, 0\rangle$; hence $\mathcal{U}' = \mathcal{R}^+$.
36. In the representation of the subgroups of $\mathcal{K}_{12}{}^+(a)$, angle brackets are used, as they frequently are in group theory, to denote the cyclic group generated by an element; for example, $\langle T_3\rangle$ is the group $\{T_0, T_3, T_6, T_9\}$.
37. This UTT, obtained from U by exchanging the two transposition levels, may simply be called the *conjugate* of U and denoted by $\bar{U}$ or, often more conveniently, U^-. The conjugate $\bar{B} = \langle +, 1, 0\rangle$ of the UTT $B = \langle +, 0, 1\rangle$ was mentioned in Section 4.4. The conjugation operator $^-$ possesses many interesting properties; for example, $(UV)^- = U^-V^-$, and $U^{--} = U$.
38. The numbering of inversions, unlike that of transpositions, is arbitrary, as it depends on privileging a particular axis of reflection. The reader may recall similar remarks in Section 2.7 about the various methods of numbering the wechsels W_n.
39. The operator ~ is not the only involution-automorphism of $\mathcal{Q}$. Another, itself a sort of dual to ~, is the operator ^ given by $\langle \sigma, t^+, t^-, \rho^+, \rho^-\rangle^\wedge = \langle \sigma, -t^+, t^-, \sigma\rho^+, \sigma\rho^-\rangle$. Still another is an extension of the conjugation operator $^-$ mentioned in note 37, defined for all QTTs by $\langle \sigma, t^+, t^-, \rho^+, \rho^-\rangle^- = \langle \sigma, t^-, t^+, \rho^-, \rho^+\rangle$. While ~ maps S_n to T_n and W_n to J_n, ^ maps S_n to T_{-n} and W_n to J_{-n}; $^-$ maps S_n to S_{-n} and W_n to W_{-n} while fixing T_n and J_n. Various relations hold among the operators ~, ^, and $^-$; for instance, $U^{\sim -} = U^{-\wedge}$. It can be shown that, starting with any QTT, the maximum number of different QTTs that can be formed by applying any combination of ~, ^, and $^-$ is eight.
40. One could choose, for example, the note corresponding to 0 in the prime form of the set. The only requirement is that within each transposition class, the "roots" must map onto each other as the set is transposed. "Roots" of sets of opposite "mode" (sign) need not necessarily map onto each other by inversion; in fact, this does not happen in our standard formulation of the triadic case.
41. The first "T" in the abbreviation "UTT" is, of course, obsolete if the objects being transformed are not triads. Nevertheless, triads remain the normative example, and inasmuch as the more general transformations continue to reflect the algebraic structure of the triadic case, we shall continue to refer to them as UTTs.
42. Gollin (1998, 203–4) points out the isomorphism between his group and the schritt/wechsel group for triads, and asserts that the same group structure applies to "any asymmetrical tetrachordal (or trichordal) set class." In fact, the restriction to trichords and tetrachords is unnecessary. Gollin's alternative numbering system for wechsels was discussed in Section 2.7.
43. Two passages from Rimsky-Korsakov's opera *Christmas Eve* are analyzed in Hook 2002, 164–70. One passage is triadic; the other comprises seventh chords. The two are algebraically identical, but the seventh-chord excerpt displays much smoother voice leading.

44. The only transpositions appearing in Lewin's transformation networks are T_0, T_1, T_6, and T_{11}. The symmetry of the subscripts suggests the possibility of replacing transpositions with schritts, and using the group $\mathcal{R}$ rather than $\mathcal{K}(1, 6)$. Such a Riemannian analysis is presented in Hook 2002, 102–4. It is difficult to make a case for the superiority of either of the two analyses over the other; the two are equally efficient in the sense that the same number of different transformations is required in either case in order to account for all the relations observed by Lewin.

45. Cohn (1997, 12–21) remarks that it is possible to define transformations P, L, and R with appropriately smooth voice-leading properties only for certain values of n; after $n = 12$, the next good candidates are $n = 18$ (third-tone tuning) and $n = 24$ (quarter-tone tuning). Balzano (1980) has studied the algebraic properties of the 12-note system from a perspective rather different from Cohn's, concluding that the next similar system is the one with *twenty* notes. Both of these viewpoints seem to be at odds with those who have singled out *nineteen*-note tuning as compositionally most useful because of its acoustic properties (it contains well-tuned thirds and fifths and a recognizable diatonic scale); see, for example, Blackwood 1985.

These discrepancies could perhaps be taken as confirmation of Cohn's contention that consonant triads and the mod-12 universe they inhabit are "over-determined": they have, in effect, more desirable properties than we have any right to expect from one simple construction. On the other hand, it could also be argued that Cohn's requirements for maximally smooth voice-leading unfairly favor small values of n (such as $n = 12$). In a 19-note setting, in fact, transformations mimicking the diatonic behavior of P, L, and R are easily defined; R fails Cohn's smoothness criterion because it displaces one note by three chromatic steps rather than one or two. But three chromatic steps in the 19-note scale form a diatonic whole tone, which is actually *smaller* than the whole tone in the 12-note scale, through which the usual R moves one of its pitches.

46. This property, which follows from the results mentioned in Section 5.10, relates also to the "serial" nature of diatonic interval sequences noted by Clough (1979–80).

47. Other examples of cross-type transformations include Soderberg's (1998–99) WARP functions, Santa's (1999) MODTRANS, and Douthett and Steinbach's (1998) P^*, L^*, and R^*. Callender's (1998) $S_{(x)}$ relation implicitly defines a cross-type transformation equivalent to $L\subset$, described below. In Hook 2000, definitions of *cross-type transposition* and *cross-type inversion* are formulated, generalizing Lewin's transposition and inversion operators to mappings from one GIS to another, and several examples are given.

48. For a fuller discussion see Hook 2002, 114–18. The UTT representation of the inclusion transformation depends, as usual, on the note chosen as the "root" of each type of object. In a purely Riemannian development of the theory, the "root" of a minor triad would be the note we call its fifth, the "root" of a half-diminished seventh chord would be its seventh, and the transformation $\subset$ would be conveniently represented by the identity T_0. As we have noted in other similar contexts (see note 4), the Riemannian notation would be less transparent in dealing with non-Riemannian transformations, and in any case the various systems are equivalent in their computational and analytical potential.

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