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HYPERSCALES AND THE GENERALIZED TETRACHORD

John Clough, John Cuciurean, and Jack Douthett

1. Introduction

We begin with the concept of the Western diatonic scale as the union of two tetrachords—an idea whose historical importance needs no elaboration here. In sections 2 and 3, we first generalize the conjunction of diatonic tetrachords to the arbitrary set composed of two *hypertetrachords* and then explore the interaction of such sets with ambiguity (especially the ambiguous tritone), interval content, and maximal evenness. As a result, we are able to strengthen the definition of *hyperdiatonic* given in a previous paper (Clough and Douthett 1991).

In the final two sections of this paper, we first develop necessary mathematical tools (*consecutive integer sequences, multiplicity sequences*) and then extend the work of a previous paper (Clough et al. 1993) that deals with the similarities between a class of ancient Indian heptatonic scales (the *gramas*) and the Western diatonic scale. We formalize the notion of iterated maximally-even sets given there, develop sets of axioms based on the similarities between the *gramas* and the diatonic, and explore classes of pcsets (we call them hyperscales) that satisfy the axioms. This approach, we feel, yields additional insight to the affinities between the *gramas* and the diatonic.

Central to the present paper (and to our previous paper dealing with the *gramas*) is the concept of *transpositional combination* (*TC*)—Richard Cohn's term for the operation whereby sets are formed by merging smaller sets related by transposition. Indeed, our work may be seen as the development of this concept in a very restricted setting and with very different objectives. While Cohn's (1987, 1991, 1992) work with TC was primarily motivated by objectives in musical analysis, our work is driven mainly by a desire to understand musical scales of diverse cultures in similar terms.[1]

2. Hypertetrachords and Their Scales

We begin with a series of definitions and examples. Where parenthetical numbers appear after definition headings, the definitions are taken from Clough and Douthett (1991). The parenthetical numbers reference sections of that work.

DEFINITION 2.1 (1.1): Given a chromatic universe of c pcs (a c-note chromatic universe), we say c is the *chromatic cardinality.* We represent such a universe by the set $U_c = \{0,1,2, \cdots, c - 1\}$, and, as usual, we assume the integers are assigned to the notes in ascending order.

DEFINITION 2.2 (1.2): To indicate a subset of d pcs selected from U_c, we write $D_{c,d} = \{D_0, D_1, D_2, \ldots, D_{d-1}\}$. Here too, we assume the pcs in $D_{c,d}$ are in ascending order (i.e., $D_0 < D_1 < D_2 < \ldots < D_{d-1}$). We say d is the *diatonic cardinality* of $D_{c,d}$.

In the usual diatonic set $D_{12,7} = \{0,2,4,5,7,9,11\}$, the two transpositionally equivalent tetrachords $H_1 = \{0,2,4,5\}$ and $H_2 = \{0,7,9,11\}$ (i.e., $H_2 = T_7 H_1$) divide the major scale into two parts. The union of these tetrachords is the entire diatonic set ($H_1 \cup H_2 = D_{12,7}$), and their intersection is a singleton ($H_1 \cap H_2 = \{0\}$). Assuming that the tetrachords must divide the diatonic set into two transpositionally equivalent parts whose union is the entire pcset, their intersection must be non-empty (since d is odd). Since the intersection is a singleton (the smallest possible non-empty set), we may say the intersection is "as empty as possible." If the octatonic set $D_{12,8} = \{0,1,3,4,6,7,9,10\}$ is divided into two transpositionally equivalent tetrachords $H_1 = \{0,1,3,4\}$ and $H_2 = \{6,7,9,10\}$ then, as before, their union is the entire pcset, but, unlike the case of the diatonic set, their intersection is empty. Thus, we can still say that the intersection is "as empty as possible." Hence, this phrase means the intersection is either a singleton or empty, depending on whether the cardinality of the parent set is odd or even. We will now generalize this concept.

DEFINITION 2.3 (1.4): Let $D_i, D_j \in D_{c,d}$. Then the *chromatic length* (*clen*) from D_i to D_j, written clen (D_i, D_j), is the smallest non-negative integer congruent to $D_j - D_i$ (mod c).

DEFINITION 2.4 (1.5): Let $D_i, D_j \in D_{c,d}$. Then the *diatonic length* (*dlen*) from D_i to D_j, written dlen(D_i, D_j), is the smallest non-negative integer congruent to $j - i$ (mod d). If dlen$(D_i, D_j) = 1$ (i.e., $j = i + 1$(mod d)), the interval is called a *step-interval*, and the pair (D_i, D_{i+1}) is called a *step-dyad* in $D_{c,d}$.

DEFINITION 2.5: Let S be a subset of $D_{c,d}$. Then S is *step-related* (to $D_{c,d}$) if all but possibly one of the step-dyads in S are step-dyads in $D_{c,d}$.

Note that if S is a step-related proper subset of $D_{c,d}$ then there is precisely one step-dyad in S that is not a step-dyad in $D_{c,d}$.

EXAMPLE 2.1: Let $D_{12,9} = \{0,2,4,5,6,7,8,10,11\}$, $S = \{4,5,6\}$, and $S^* = \{2,4,6\}$. Then S is step-related (to $D_{12,9}$) since the only step-dyad in S that is not a step-dyad in $D_{12,9}$ is the interval from 6 to 4. S^*, however, is not step-related since the intervals from 4 to 6 and from 6 to 2 are step-dyads in S^* but not step-dyads in $D_{12,9}$.

DEFINITION 2.6: A pcset $D_{c,d}$ has a *hypertetrachordal structure* if there exist two transpositionally equivalent subsets H_1 and H_2, *both* step-related to $D_{c,d}$, whose union is the entire parent set and whose intersection is as empty as possible; that is, $H_2 = T_n H_1$ for some n, $H_1 \cup H_2 = D_{c,d}$, and $H_1 \cap H_2$ contains at most one element.

Thus, in the case of the usual diatonic set, the usual tetrachords are step-related hypertetrachords. Note that this definition does not preclude hypertetrachords from having other than 4 pcs.

EXAMPLE 2.2: Consider the usual pentatonic, $D_{12,5} = \{0,2,5,7,10\}$. The sequence of step-intervals (i.e., of their clens) is (23232). The underscored parts are the sequences of step-intervals of a pair of hypertetrachords. The hypertetrachords are $H_1 = \{0,2,5\}$ and $H_2 = \{5,7,10\}$. Note that $H_2 = T_5 H_1$, $H_1 \cup H_2 = D_{12,5}$, and $H_1 \cap H_2 = \{5\}$. Thus, all the conditions of Definition 2.6 are satisfied. Whence, $D_{12,5}$ has a hypertetrachordal structure.

At this point, it may be instructive to note a connection with established pcset theory. Transpositional (T) and, especially, inversional (I) symmetries are among the indispensable concepts of that theory. It is well known that if (and only if) a pcset is I-symmetrical, it may be decomposed into two proper I-related subsets whose union is the original pcset: This was shown by John Rahn (1980, 93). If the number of pcs is odd, the two subsets must intersect; if even, they may or may not intersect. The situation is somewhat different in the case of T-symmetry, and perhaps not as well known. To begin with, the cardinality of pcsets with non-trivial T-symmetry (self mapping under T_n, $n \neq 0$) is never coprime with twelve (assuming the usual chromatic universe of 12 pcs); thus, pcsets of 1, 5, 7, or 11 pcs are ruled out. As in the case of I-symmetry, non-trivial T-symmetry implies decomposition into two (or possibly more) proper, T-related subsets. But in contrast to the inversional case, the converse of the statement does not hold: ignoring decomposition into one-note subsets, there are pcsets which lack non-trivial T-symmetry and which nevertheless decompose into proper T-related subsets. This is, in fact, the case with many pcsets composed of hypertetrachords, including the usual diatonic.

We conclude this section with another definition and a theorem involving step-related hypertetrachords. This theorem will prove useful in the process of developing our hyperscale axioms.

DEFINITION 2.7: Assume $D_{c,d}$ has a hypertetrachordal structure. The clen(s) between the hypertetrachords will be called the *gap(s)*; that is, a gap is a non-zero clen from the "end" of one hypertetrachord to the "beginning" of the other, when the hypertetrachords are listed as scale segments. If the intersection of these hypertetrachords is a singleton then the note in that intersection will be called the *hinge.*

EXAMPLE 2.3: For hypertetrachords H_1 ={0,2,4,5} and H_2={0,7,9,11} (={7,9,11,0}) in the C major scale, the hinge is pc 0 and the gap is 2; that is, the clen from F to G or 5 to 7 is 2.

EXAMPLE 2.4: Let $D_{12,8}$ = {0,1,2,4,7,8,9,11}. Then H_1={0,1,2,4} and H_2={7,8,9,11} are two hypertetrachords, and the gaps are 3 and 1. There is no hinge.

THEOREM 2.1: Suppose $D_{c,d}$ has hypertetrachordal structure.

(1) If d is odd then the multiplicity of one of the distinct clens of the step-intervals of $D_{c,d}$ must be odd, and the multiplicities of the rest must be even.

(2) If d is even then either there are precisely two distinct clens among the step-intervals of $D_{c,d}$ that have odd multiplicities or all the clens of the step-intervals have even multiplicities.

PROOF: (1) Every clen of a step-interval in one hypertetrachord must be matched with the clen of the corresponding step-interval in the other hypertetrachord. Only the gap has no match. (2) As above, every clen of a step-interval in one hypertetrachord must be matched with the clen of the corresponding step-interval in the other hypertetrachord. If the gaps are not the same then there are precisely two distinct clens of the step-intervals that have odd multiplicities; if they are the same then all the clens of the step-intervals have even multiplicities.∎

3. Maximally Even and Hyperdiatonic Sets

Clough and Douthett (1991) discussed a variety of inherent properties of the diatonic set. They explored chromatic universes other than U_{12} and found subsets within those universes that have properties that parallel those of the usual diatonic. One of the major theorems in that work gave ten equivalent conditions that define those subsets. All of those definitions assumed that the pcset was ME. We now state this theorem (without proof) after giving relevant definitions, and then we proceed to construct two more equivalent definitions that assume hypertetrachordal structure instead of the ME property.

For the sake of brevity, we shall provide few examples that pertain to definitions, theorems, and lemmas from Clough and Douthett (1991), to which the reader may refer for more elaboration.

DEFINITION 3.1 (1.6): The *spectrum of a dlen* of pcset $D_{c,d}$ is the set of clens corresponding to that particular dlen. We write $\langle I\rangle = \{i_1, i_2, \ldots i_n\}$ to indicate that the spectrum of dlen I is $\{i_1, i_2, \ldots i_n\}$. Thus, $i \in \langle I\rangle$ if and only if $0 \leq i \leq c\text{-}1$ and there exists an N, $0 \leq N \leq d\text{-}1$, such that $D_{N+I} - D_N \equiv i \pmod{c}$ where the subscripts are reduced modulo d.

DEFINITION 3.2 (1.7): $D_{c,d}$ is *maximally even (ME)* if and only if every spectrum consists of a single integer or two consecutive integers. We write $M(c,d)$ to represent the set of all ME sets with parameters c and d.[2]

A diatonic set is a ME set since $\langle 0\rangle = \{0\}$ (perfect unison), $\langle 1\rangle = \{1,2\}$ (minor and major seconds), $\langle 2\rangle = \{3,4\}$ (minor and major thirds), $\langle 3\rangle = \{5,6\}$ (perfect and augmented fourths), etc. The set of all diatonic sets is $M(12,7)$. Other familiar pcsets in U_{12} that are ME sets are the naked tritone, augmented triad, diminished seventh chord, anhemitonic (black key) pentatonic scale, whole-tone scale, and octatonic scale. In addition, using completely different approaches, Yasser (1932), Balzano (1980), Clough and Myerson (1985), Mathews, Pierce, and Roberts (1987), and Agmon (1989) have constructed subclasses of ME sets in universes other than U_{12}.[3]

DEFINITION 3.3: If a clen is in two or more interval spectra, we say it is an *ambiguous clen*.

DEFINITION 3.4 (2.2): Given $D_p, D_q, D_r, D_s \in D_{c,d}$, if $\mathrm{dlen}(D_p,D_q) \neq \mathrm{dlen}(D_r,D_s)$ and $\mathrm{clen}(D_p,D_q) = \mathrm{clen}(D_r,D_s)$, that is a case of *ambiguity.*[4]

DEFINITION 3.5 (2.4): A *tritone* $\{D_i, D_j\}$ is a two-element subset of $D_{c,d}$ such that $\mathrm{clen}(D_i,D_j) = c/2$. The tritone is defined only for c even.

In the ascending melodic minor scale, $<2> \cap <3> = \{3,4\} \cap \{4,5,6\} = \{4\}$. Thus, 4 is an ambiguous clen. This reflects the fact that there are major thirds and diminished fourths in this scale. A case of ambiguity involves a specific pair of intervals. In the ascending melodic minor scale, there is one diminished fourth that can be paired with any one of three major thirds; here we have three cases of ambiguity. Similarly, there are three more cases of ambiguity involving augmented fifths and minor sixths ($<4> \cap <5> = \{8\}$). Further, there are two augmented fourths and two diminished fifths ($<4> \cap <5> = \{6\}$). Each fourth can be paired with each fifth; thus, these create four more ambiguities. In total, the ascending melodic minor scale has three ambiguous clens (4,6, and 8) and ten ambiguities. The diatonic scale has only one ambiguous clen and one ambiguity: a single ambiguous tritone.

DEFINITION 3.6 (1.18): The *spectrum of a set $D_{c,d}$*, written $\mathrm{spec}(D_{c,d})$, is the set of all elements in the interval spectra $<I>$, $0 \leq I \leq d\text{-}1$, including their multiplicities; that is,

$$\mathrm{spec}(D_{c,d}) = \bigcup_{I=0}^{d-1} <I>.$$

Since the multiplicities are included, $\mathrm{spec}(D_{c,d})$ is a *multiset.*

If $D_{12,7} = \{0,2,4,5,7,9,11\}$ (the diatonic set), then $\mathrm{spec}(D_{12,7}) = \{0,1,2,3,4,5,6,6,7,8,9,10,11\}$. Note that the set has two 6's corresponding to the augmented fourth and diminished fifth.

DEFINITION 3.7 (1.19): We call the multiset $C_c=\{0,1,2, \ldots ,c\text{-}1\}$, with each element represented *once*, the *chromatic multiset.*

While U_c and C_c have the same elements, they are not identical. Consider the set $S = \{0,1,2,3,4,5,6,6,7,8,9,10,11\}$. If S is not a multiset then the second 6 is redundant, and $U_{12} = S$. But if S is a multiset then we cannot compare these two sets. In this case, the second 6 is not redundant. The multiset C_{12} is a proper subset of S since both sets are multisets and C_{12} contains only one 6. In this case S is the spectrum of the diatonic set.

There is but one more definition we must state before we formally give the ten equivalent definitions for hyperdiatonic sets.

DEFINITION 3.8 (2.6): *Min-c*, written $\underline{M}$ (c), is the set of all ME sets whose set spectra are the smallest multisets that properly contain C_c.

Note that $\underline{M}$ (12) = $M(12,7)$: the set of all diatonic sets. We now present a theorem that will lead to the formal definition of a hyperdiatonic set.

THEOREM 3.1 (2.2): Let $D_{c,d}$ be a ME set. Then the following conditions are equivalent.

(1) $D_{c,d}$ has precisely one ambiguity.
(2) $D_{c,d}$ has precisely one ambiguous clen.
(3) $D_{c,d}$ has precisely one ambiguous tritone.
(4) $D_{c,d}$ has precisely one tritone and $d \neq 2$.
(5) $c = 2(d - 1)$ and $(c,d)=1$ (i.e., c and d are coprime).
(6) $c = 2(d - 1)$ and d is odd.
(7) $c = 2(d - 1)$ and $c \equiv 0 \pmod 4$.
(8) $D_{c,d}$ has precisely two intervals of clen 1 and $c \equiv 0 \pmod 4$.
(9) $M(c,d) = \underline{M}(c)$ and $c \equiv 0 \pmod 4$.
(10) $|\mathrm{spec}(D_{c,d})| = c + 1$.

DEFINITION 3.9 (2.7): We say $D_{c,d}$ is a *hyperdiatonic set*[5] if it satisfies any one of the conditions in Theorem 3.1.[6]

We alert the reader that these ten conditions are equivalent only if $D_{c,d}$ is a ME set. Moreover, if $D_{c,d}$ is a ME set and meets any one of these ten conditions then $D_{c,d}$ satisfies all ten.

Before showing the connection between hypertetrachords and hyperdiatonic sets, it is necessary to present a few more definitions, lemmas, and theorems. In what follows, we use the *floor function* (also known as the *greatest integer function*) which is defined as follows:

$\lfloor x \rfloor$ = the greatest integer less than or equal to x.

DEFINITION 3.10 (1.8): Let c, d, m, and N be integers such that $0 \leq N < d \leq c$ and $0 \leq m \leq c - 1$. Then the *J-function* with these parameters is given by

$$J^m_{c,d}(N) = \left\lfloor \frac{cN + m}{d} \right\rfloor.$$

DEFINITION 3.11 (1.9): Let c, d, and m be as in the above definition. Then the *J-set* with these parameters is given by

$$J^m_{c,d} = \{J^m_{c,d}(0), J^m_{c,d}(1), J^m_{c,d}(2), \ldots, J^m_{c,d}(d-1)\}.$$

The superscript m is called the *mode index* or simply the *index.*

This algorithm is known as the *ME algorithm.* The reason for this name is made clear by the following theorem:

THEOREM 3.2 (1.1 and 1.5): $D_{c,d}$ is a ME set if and only if there exists an index m, $0 \leq m \leq c - 1$ such that $D_{c,d} = J_{c,d}^{m}$.

As discussed above, the diatonic sets are ME sets and, hence, can be represented as J-sets. The diatonic sets cycle by fifths as the indices of their J-set representations increase incrementally; that is, $J_{12,7}^{0}$ = D♭ major, $J_{12,7}^{1}$ = A♭ major, $J_{12,7}^{2}$ = E♭ major, etc.

THEOREM 3.3 (1.7 and 1.8): For a given c and d, all ME sets are I-symmetrical and equivalent under transposition.

LEMMA 3.1 (1.16): For any $D_{c,d}$ and associated dlen I, the number of intervals in $\langle I \rangle$ with clen k equals the number of intervals in $\langle d - I \rangle$ with clen $c - k$.

COROLLARY 3.1: If $D_{c,d}$ has a single ambiguity then that ambiguity is a tritone.

PROOF: If an ambiguity with clen k exists then an ambiguity with clen $c - k$ also exists (Lemma 3.1). Hence, if there is only one ambiguity then $k = c - k$ implying $k = c/2$.∎

LEMMA 3.2 (1.9): For $D_{c,d}$ and any associated dlen I, if $\langle I \rangle = \{k_I\}$ or $\{k_I, k_I + 1\}$, then $k_I = \lfloor cI/d \rfloor$.

LEMMA 3.3: For $D_{c,d}$, suppose the clens of step-intervals are all the same or come in two consecutive integer sizes. Let $c = dq + r$ where q and r are non-negative integers and $0 \leq r < d$. Then there are r step-intervals whose clens are $\lfloor c/d \rfloor + 1$ and $d - r$ step-intervals whose clens are $\lfloor c/d \rfloor$.

PROOF: From Lemma 3.2 we know that the clen of any interval in $\langle 1 \rangle$ is $\lfloor c/d \rfloor$ or $\lfloor c/d \rfloor + 1$. Let h be the number of step-intervals whose clen is $\lfloor c/d \rfloor + 1$. Then there will be $d - h$ step-intervals whose clens are $\lfloor c/d \rfloor$. Now we sum up all the step-intervals

$$h\left(\left\lfloor \frac{c}{d} \right\rfloor + 1\right) + (d - h)\left\lfloor \frac{c}{d} \right\rfloor = c$$

$$h = c - d\left\lfloor \frac{c}{d} \right\rfloor.$$

We apply the division algorithm and let $c = dq + r$. Then

$$h = c - d\left\lfloor \frac{c}{d} \right\rfloor = dq + r - d\left\lfloor \frac{dq + r}{d} \right\rfloor = r.$$

Thus, there are r step-intervals whose clens are $\lfloor c/d \rfloor + 1$ and hence $d - r$ step-intervals whose clens are $\lfloor c/d \rfloor$.∎

DEFINITION 3.12: The *I–span* of a pcset $D_{c,d} = \{D_0, \ldots, D_{d-1}\}$ is defined as follows:

$$\text{span}_I\,(D_{c,d}) = cI - \max <I>$$

where max$<I>$ is the largest integer in $<I>$ and $1 \leq I \leq d - 1$. Thus, for any fixed n, $0 \leq n \leq d - 1$,

$$\text{span}_I(D_{c,d}) \leq \sum_{k=1}^{d-1} \text{clen}(D_{n+(k-1)I}, D_{n+kI}),$$

where the subscripts are reduced modulo d. Moreover, equality holds for at least one n. When equality holds, we say the sequence

$$I_{\text{span}}(D_{c,d}) = (\text{clen}(D_n, D_{n+I}), \text{clen}(D_{n+I}, D_{n+2I}), \ldots \text{clen}(D_{n+(d-2)I}, D_{n+(d-1)I}))$$

is a I–spanning sequence of $D_{c,d}$. Whence, $\text{span}_I\,(D_{c,d})$ is the sum of the clens in $I_{\text{span}}(D_{c,d})$.[7]

In the context of this paper, we will be concerned only with the case $I = 1$. Thus, we will refer to $\text{span}_1(D_{c,d})$ and $1_{\text{span}}\,(D_{c,d})$ simply as the *span* and a *spanning sequence* of $D_{c,d}$, respectively.

Note that if H_1 and H_2 are a pair of hypertetrachords of a pcset, $\text{span}_1(H_1) = \text{span}_1(H_2)$. Moreover, the spanning sequence is the same for both hypertetrachords; whence, $1_{\text{span}}(H_1) = 1_{\text{span}}(H_2)$.

Now we present two other equivalent definitions for a hyperdiatonic set that depend on hypertetrachordal structure and step-interval sizes rather than maximal evenness.

THEOREM 3.4: Let $d \geq = 4$. The following statements are equivalent.

(1) $D_{c,d}$ is a hyperdiatonic set.

(2) The following conditions hold for $D_{c,d}$.

(a) $D_{c,d}$ has a hypertetrachordal structure.

(b) $D_{c,d}$ has precisely one ambiguity.

(c) Step-intervals come in two consecutive integer sizes.

(3) The following conditions hold for $D_{c,d}$.

(a) $D_{c,d}$ has a hypertetrachordal structure.

(b) $D_{c,d}$ has precisely one tritone.

(c) Step-intervals come in two consecutive integer sizes.

PROOF: We will show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). If $D_{c,d}$ is a hyperdiatonic set then (b) and (c) are automatically satisfied (Theorem 3.1). It is clear that any pcset that is a transposition or inversion of a pcset with hypertetrachordal structure will, itself, have hypertetrachordal structure. Further, for a given c and d, all ME sets are equivalent under transposition (Theorem 3.3). Hence, we must only show that for a given $d \geq 4$, any particular hyperdiatonic set has hypertetrachordal structure. Consider the hyperdiatonic set with index 0, $J^0_{c,d}$ where $c = 2(d - 1)$.

$$\begin{aligned}\text{clen}(J^0_{c,d}(0), J^0_{c,d}(1)) &= J^0_{c,d}(1) - J^0_{c,d}(0)\\ &= \left\lfloor \frac{c}{d} \right\rfloor - \left\lfloor \frac{0}{d} \right\rfloor\\ &= \left\lfloor \frac{2(d-1)}{d} \right\rfloor - 0\\ &= 1\end{aligned}$$

Similarly

$$\text{clen}\left(J^0_{c,d}\left(\frac{d-1}{2}\right), J^0_{c,d}\left(\frac{d+1}{2}\right)\right) = 1$$

Since hyperdiatonic sets are ME sets and there are precisely two intervals of clen 1 (Theorem 3.1), all other step-intervals must have clen 2. Thus, the subsets $H_1 = \{J^0_{c,d}(0), J^0_{c,d}(1), \ldots, J^0_{c,d}((d-1)/2)\}$ and $H_2 = \{J^0_{c,d}((d-1)/2), J^0_{c,d}((d+1)/2), \ldots, J^0_{c,d}(d-1)\}$ are two step-related hypertetrachords in $J^0_{c,d}$.

(2) $\Rightarrow$ (3). Clearly (a) and (c) are satisfied. Since there is precisely one ambiguity, that ambiguity must be a tritone (Corollary 3.1). We only need to show there are no unambiguous tritones.

Assume d is even. If hypertetrachordal structure exists, then there are two gaps. If these gaps are equal then the clens between the corresponding pcs of the hypertetrachords are tritones. Further, the same number of pcs occur on both sides of each tritone. This implies there are no ambiguous tritones (contradiction). If the gaps are not equal then their clens must differ by one. Hence, the sum of their clens must be odd. Since the spans of the hypertetrachords are equal (i.e., span(H_1) = span(H_2)), the chromatic cardinality must be odd (contradiction). It follows that d is odd. If d is odd then the number of pcs on either side of a tritone must be different. Hence, there are no unambiguous tritones.

(3) $\Rightarrow$ (1). Clearly c must be even; else tritone would not be defined. If d is even then the gaps must be equal; else c would be odd. But then the intervals between corresponding pcs in the hypertetrachords must all be tritones. Since $d \geq 4$ there would be more than one tritone (contradiction). Thus, d must be odd, and $D_{c,d}$ has a hinge.

Since transposition does not effect hypertetrachordal structure, we may assume without loss of generality that

$$D_{c,d}=\{D_0, D_1, D_2, \ldots, D_{(d-1)}\}$$

where $D_0 = 0$ and $D_{k+(d-1)/2} = D_{(d-1)/2} + \mathrm{D}_k$ for $0 \leq k \leq (d-1)/2$. Then the hypertetrachords are

$$H_1=\{0, D_1, D_2, \ldots, D_{(d-1)/2}\} \text{ and } H_2 = \{D_{(d-1)/2}, D_{(d+1)/2}, D_{(d+3)/2} \ldots, D_{d-1}\},$$

and the gap is clen(D_{d-1}, D_0). The gap must be even since c is even and

$$\mathrm{clen}(D_{d-1}, D_0) = c - \mathrm{span}_1(H_1) - \mathrm{span}_1(H_2) = c - 2D_{(d-1)/2}$$

Thus, there are two cases: either the gap is $\lfloor c/d \rfloor$ or $\lfloor c/d \rfloor + 1$ (Lemma 3.3).

Case 1: Assume the gap is $\lfloor c/d \rfloor$. Since the gap is even and the spans of the hypertetrachords are equal, the span of each hypertetrachord is $(c - \lfloor c/d \rfloor)/2$. Now the hypertetrachords can be written

$$H_1 = \left\{0, D_1, \ldots, \frac{1}{2}\left(c - \left\lfloor \frac{c}{d} \right\rfloor\right)\right\}$$
$$\text{and } H_2 = \left\{\frac{1}{2}\left(c - \left\lfloor \frac{c}{d} \right\rfloor\right), \frac{1}{2}\left(c - \left\lfloor \frac{c}{d} \right\rfloor\right) = D_1, \ldots, c - \left\lfloor \frac{c}{d} \right\rfloor\right\}.$$

Since the spans of the hypertetrachords must be less then $c/2$, the single tritone must have one pc in each hypertetrachord. It follows that there exist i and j such that $0 \leq i,j \leq (d-1)/2$ and

$$\text{clen}\left(D_i, \frac{1}{2}\left(c - \left\lfloor\frac{c}{d}\right\rfloor\right) + D_j\right) = \frac{c}{2}$$

$$D_j - D_i = \frac{1}{2}\left\lfloor\frac{c}{d}\right\rfloor.$$

But then $D_j - D_i > 0$ implying that this difference is at least as large as a step-interval. Hence,

$$\frac{1}{2}\left\lfloor\frac{c}{d}\right\rfloor = D_j - D_i \geq \left\lfloor\frac{c}{d}\right\rfloor$$

(Lemma 3.3). By dividing the left and right side of the above inequality by $\lfloor c/d \rfloor$ we are led to the contradiction $1/2 \geq 1$.

Case 2: Assume the gap is $\lfloor c/d \rfloor + 1$. Since the gap is even, the span of each hypertetrachord must be $(c - \lfloor c/d \rfloor - 1)/2$; whence, the hypertetrachords can be written

$$H_1 = \left\{0, D_1, \ldots, \frac{1}{2}\left(c - \left\lfloor\frac{c}{d}\right\rfloor - 1\right)\right\} \text{ and}$$

$$H_2 = \left\{\frac{1}{2}\left(c - \left\lfloor\frac{c}{d}\right\rfloor - 1\right), \frac{1}{2}\left(c - \left\lfloor\frac{c}{d}\right\rfloor - 1\right) + D_1, \ldots, c - \left\lfloor\frac{c}{d}\right\rfloor - 1\right\}.$$

Thus, there exist i and j, $0 \leq i, j \leq (d - 1)/2$, such that

$$\text{clen}(D_i, \frac{1}{2}\left(c - \left\lfloor\frac{c}{d}\right\rfloor - 1\right) + D_j) = \frac{c}{2}$$

$$D_j - D_i = \left(\left\lfloor\frac{c}{d}\right\rfloor + 1\right).$$

As in case 1, $Dj - D_i \geq \lfloor c/d \rfloor$. But now

$$\frac{1}{2}\left(\left\lfloor\frac{c}{d}\right\rfloor + 1\right) = D_j - D_i \geq \left\lfloor\frac{c}{d}\right\rfloor$$

implying that $\lfloor c/d \rfloor \leq 1$. But since $c \geq d$, $\lfloor c/d \rfloor = 1$. Thus, $d > c/2$. Further, by the pigeonhole principle, $d \leq c/2+1$; else there would be more than one tritone. It follows that

$$d = \frac{c}{2} + 1$$

$$c = d + (d - 2)$$

implying that there are two step-intervals with clen 1 and $d - 2$ intervals with clen 2 (Lemma 3.3). Since there are precisely two step-intervals with clen 1, each hypertetrachord must have one, and they must be between corresponding pcs in the hypertetrachords. Thus, for some i, $0 \leq i \leq (d - 1)/2$, $\text{clen}(D_i, D_{i+1}) = 1$ and $\text{clen}(D_{(d-1)/2} + D_i, D_{(d-1)/2} + D_{i+1}) = 1$

where $D_{(d-1)/2} = c/2-1 = d - 2$, and all other step-intervals have clen 2. But this is precisely the hyperdiatonic set with index $2D_i$; that is, $J^{2D_i}_{2(d-1),d}$ (Theorems 3.1 and 3.2).∎

Theorem 3.4 provides two more equivalent definitions for hyperdiatonic sets.

Our next goal is to construct a more general class of sets that admits not only hyperdiatonic sets but also sets we will call *hypergramas* (to be defined formally later), which are generalizations of the Indian *gramas.* However, before this can be done, it is necessary to develop a few handy mathematical tools that will allow us to achieve this goal.

4. Consecutive Integer and Multiplicity Sequences

The step-intervals of the diatonic set have clens 1 and 2 (1 and 2 half-steps). Thus, the distinct step-interval sizes are consecutive integers. The sequence representing the sizes of the steps is (12). Such a sequence will be called a *consecutive integer sequence (CIS).* Since a CIS consists of step-intervals, this sequence is < 1 > written as a sequence instead of a set. Note that it is only possible to write < 1 > as a CIS when < 1 > consists of consecutive integers.

In the diatonic set, there are two step-intervals with clen 1 and five with clen 2. The sequence reflecting the corresponding multiplicities of these clens can be written (25), and we call this sequence the *multiplicity sequence (MS)* of the pcset. For the harmonic minor set, the MS and CIS are (331) and (123) respectively. Note that the elements in the MSs always sum up to the diatonic cardinality 7. Further, the dot product between these sequences is the chromatic cardinality 12. In general, there are two equations relating these sequences. If $A = (a_0, a_1, \ldots, a_n)$ and $B = (b, b+1, \ldots, b+n)$ are the respective MS and CIS of a pcset $D_{c,d}$ then

$$\sum_{k=0}^{n} a_k = d \text{ and } A \bullet B = \sum_{k=0}^{n} a_k(b+k) = c. \tag{1}$$

For now, let us consider only the MSs.

DEFINITION 4.1: Let d be a positive integer. Then the set

$$\mathcal{F}(d) = \left\{(a_0 a_1 \ldots a_n) \mid n \geq 0,\ a_k \text{ is a positive integer, and } \sum_{k=0}^{n} = d\right\}$$

is called the *d-family of MSs.*

Consider the elements of $\mathcal{F}(5)$ (Figure 4.1). There are $2^4 = 16$ such sequences. In general, there will be 2^{d-1} sequences. Now, consider the sequence (2111) and the following system of *reduction.*

1 element sequences	(5)
2 element sequences	(14),(23),(32),(41)
3 element sequences	(113),(131),(122),(212),(221)(311)
4 element sequences	(1112),(1121),(1211),(2111)
5 element sequences	(11111)

Figure 4.1: Sequences in $\mathcal{F}(5)$.

$$(2111) \rightarrow (2-1, 1+1, 1+1, 1-1) = (122)$$

Figure 4.2: First step in the reduction of (2111).

$$(2111) \rightarrow (122) \rightarrow (41)$$

Figure 4.3: Complete reduction of (2111).

$$(11111) \rightarrow (212) \rightarrow (131) \rightarrow (5)$$

Figure 4.4: Complete reduction of (11111).

$P_1(5)$	$P_2(5)$	$P_3(5)$	$P_4(5)$	$P_5(5)$
(14)	(23)	(32)	(41)	(5)
(221)	(311)	(113)	(122)	(131)
(1112)	(1121)	(1211)	(2111)	(212)
				(11111)

Figure 4.5: Partitioning of $\mathcal{F}(5)$.

$P_1(7)$	$P_2(7)$	$P_3(7)$	$P_4(7)$	$P_5(7)$	$P_6(7)$	$P_7(7)$
(16)	(25)	(34)	(43)	(52)	(61)	(7)
(241)	(331)	(115)	(511)	(133)	(142)	(151)
(322)	(412)	(421)	(124)	(214)	(223)	(232)
(1132)	(1141)	(1231)	(1321)	(1411)	(2311)	(313)
(1213)	(1222)	(1312)	(2131)	(2221)	(3121)	(1123)
(4111)	(2113)	(2122)	(2212)	(3112)	(1114)	(3211)
(12211)	(13111)	(11113)	(31111)	(11131)	(11221)	(21112)
(21121)	(21211)	(22111)	(11122)	(11212)	(12112)	(12121)
(111112)	(111121)	(111211)	(112111)	(121111)	(211111)	(11311)
						(1111111)

Figure 4.6: Partitioning of $\mathcal{F}(7)$.

First subtract 1 from the left and right boundary numbers of the sequence and add 1 to the numbers to the immediate right and left of these boundary numbers. If either of the resultant boundary numbers is 0, shorten the sequence by discarding the 0 (Figure 4.2). Then continue this process until a sequence has only 1 or 2 numbers (Figure 4.3). The sequence (11111) reduces to (5) (Figure 4.4). Hence, this reduction process maps $\mathcal{F}(5)$ onto the subset of itself that contains all 1 or 2 element sequences. Now the sequences in $\mathcal{F}(5)$ can be partitioned into sets that include the reductions of their constituent sequences (Figure 4.5). For a given partition P_k, the index subscript, k, is the first element of the shortest sequence in that partition.[8]

The number $d = 7$ is of particular interest to us since heptatonic sets exist in both Indian and western cultures (as well as other cultures). For this number, there are $2^6 = 64$ possible MSs in $\mathcal{F}(7)$ which can be partitioned into seven subsets (Figure 4.6).

Note that the partitioning of d-families depends only on d. But how do these sequences relate to c? We begin to explore this connection with the following definition.

DEFINITION 4.2: Let $c \geq d$ be positive integers. Then the set

$$\mathcal{F}(c,d) = \{(A,B) \mid A \in \mathcal{F}(d), B \text{ is a CIS, and } A \bullet B = c\}$$

is called the *(c,d)-family of sequence pairs.*

Thus, the sequence pairs in $\mathcal{F}(c,d)$ are precisely the pairs that satisfy equation (1) above. Recall CIS members must be positive integers; as a consequence, there are only three sequence pairs in $\mathcal{F}(12,7)$ (Figure 4.7). Note that the set of MSs in these pairs is a subset of P_2 (7) (Figure 4.6).

A	B
(412)	(123)
(331)	(123)
(25)	(12)

Figure 4.7: Sequence pairs in $\mathcal{F}(12,7)$.

Partition	$P_1(7)$	$P_2(7)$	$P_3(7)$	$P_4(7)$	$P_5(7)$	$P_6(7)$	$P_7(7)$
Size	9	9	9	9	9	9	10

Figure 4.8: Cardinalities of the partitions in $\mathcal{F}(7)$.

Partition	$P_1(9)$	$P_2(9)$	$P_3(9)$	$P_4(9)$	$P_5(9)$	$P_6(9)$	$P_7(9)$	$P_8(9)$	$P_9(9)$
Size	28	28	29	28	28	29	28	28	30

Figure 4.9: Cardinalities of the partitions in $\mathcal{F}(9)$.

This is not a coincidence! In general, one can determine the appropriate partition of MSs for a given c and d by using the division algorithm $c = qd + r$. If $r \neq 0$ then the doubleton $(d - r, r)$ is in the partition; hence, $P_{d-r}(d)$ is the desired partition. Otherwise, the singleton (d) is in the partition, and the correct partition will, again, be $P_{d-r}(d) = P_d(d)$. In the above case, the appropriate sequences are those in the partition $P_{7-5}(7) = P_2(7)$. The same partition would also be appropriate for $c = 19,26,33,40, \ldots$. Another way of putting it is that A is in $P_{7-5}(7) = P_2(7)$ if and only if there exists a CIS B such that $A \bullet B \equiv 5 (\text{mod } 7)$. More generally, for an integer r, $0 \leq r < d$, a sequence A is in $P_{d-r}(d)$ if and only if there exists a CIS B such that

$$A \bullet B \equiv r (\text{mod } d).$$

Thus, the following is an equivalent definition of a (c,d)-family of sequence pairs.

DEFINITION 4.2a: Let $c \geq d$ be positive integers, and let r be the smallest non-negative integer such that $r \equiv c (\text{mod } d)$. Then the set

$$\mathcal{F}(c,d) = \{(A,B) \mid A \in P_{d-r}(d), B \text{ is a CIS, and } A \bullet B = c\}$$

is called the *(c,d)-family of sequence pairs.*

Since the step-intervals must have a clen of at least 1, only the three sequence pairs in Figure 4.7 have musical application. Each pair of sequences can be associated with more than one pcset. The pair ((25),(12)) is associated with the diatonic set, ascending melodic minor set, and whole-tone-plus-one set. Note that if $c = 22$ and $d = 7$ (as in the *gramas*), then $22 = 3 \cdot 7 + 1$. Hence, the MSs in $P_{7-1}(7) = P_6(7)$ are the applicable ones, as we demonstrate in the following section.

This type of partitioning appears to divide the set of MSs into partitions of approximately equal sizes. For example, if $d = 7$ then the cardinalities of the partitions range from 9 to 10 (Figure 4.8), and for $d = 9$ the cardinalities range from 28 to 30 (Figure 4.9).

5. Hyperscales and Their Axioms

Finally, we investigate similarities between the diatonic sets and the Indian *gramas.* We develop two sets of *axioms* which allow us to find pcsets in other chromatic universes that share these similarities. Our principal requirement for a set of axioms is that the axioms generate all the diatonic sets when $c = 12$ and all the *gramas* when $c = 22$ while, at the same time, eliminating all other pcsets in these universes. After developing these axioms, we find that, as a bonus, the hyperdiatonic sets also satisfy them. Then we devise a method for constructing hyperdiatonic sets based on their hypertetrachords and mimic this construction for the *gramas*. As a result, we discover a new class of pcsets that also satisfy our sets of axioms.[9]

Some explanation of methodology is in order. In this section (as in some of our previous work) we confess to an experimental process of contriving and progressively "tweaking" sets of conditions until they capture exclusively the pcsets in question. We exhibit certain stages of that process in our exposition, believing that they may interest some readers. If the resulting sets of conditions—which connect musical scales of two different cultures—are seen to be interesting, provoke surprise, convey insight, or stimulate further inquiry, then we shall feel rewarded for our efforts.

We start by supplying necessary details about the heptatonic *gramas* and then state a couple of theorems that connect the diatonic sets and the *gramas.* The chromatic universe of this ancient Indian system consists of 22 microtones or *srutis.* There are four principal heptatonic sets: the (i) *sa-grama,* (ii) *ma-grama,* (iii) *sa-grama doubly altered,* and (iv) *ma-grama doubly altered* (Figure 5.1). Note that (i) and (iv) are related by inversion, as are (ii) and (iii). We will consider pcset inversions as being distinct whereas pcset transpositions will be considered equivalent. We call the class of all transpositions of a given pcset the *transposition-class*

Grama	Pcset	Step-Interval Sequence
sa-grama	$J^{0,2}_{22,12,7} = \{0,3,5,9,12,14,18\}$	$(\underline{324}\,\underline{324}\,4)$
ma-grama	$J^{6,0}_{22,12,7} = \{0,2,6,9,11,15,18\}$	$(\underline{243}\,\underline{243}\,4)$
sa-grama doubly altered	$J^{0,4}_{22,12,7} = \{0,3,7,9,12,16,18\}$	$(\underline{342}\,\underline{342}\,4)$
ma-grama doubly altered	$J^{2,0}_{22,12,7} = \{0,2,5,9,11,14,18\}$	$(\underline{234}\,\underline{234}\,4)$

Figure 5.1: Ancient Indian *gramas* and their step interval sequences.

A	*B*	Tetrachord Structure	Single Tritone
(331)	(123)	no (Thm 2.1)	–
(412)	(123)	yes	no
(25)	(12)	yes	yes (diatonic sets)

Figure 5.2: Sequence pairs in $\mathcal{F}(12,7)$ and their implications.

A	*B*	Tetrachord Structure	Single Tritone
(211111)	(123456)	no (Thm 2.1)	–
(11221)	(12345)	no (Thm 2.1)	–
(12112)	(12345)	no (Thm 2.1)	–
(1114)	(1234)	no (Thm 2.1)	–
(3121)	(2345)	no (Thm 2.1)	–
(2311)	(2345)	no (Thm 2.1)	–
(223)	(234)	yes	yes (*gramas*)
(142)	(234)	yes	no
(61)	(34)	yes	no

Figure 5.3: Sequence pairs in $\mathcal{F}(22,7)$ and their implications.

(SC_T) of that set. Thus, for a given pcset, its SC_T and SC are the same if and only if the pcset is I-symmetrical. The Indian system also includes *singly altered sa-grama* and *singly altered ma-grama,* but, for our purposes, these are not distinct from *ma-grama* and *sa-grama doubly altered,* respectively, since they are equivalent under transposition.[10] (For further details, see Rowell 1992, ch. 7 and Clough et al. 1993.) We will call the union of the SC_T's of these four pcsets the *grama family* or simply the *gramas.*

THEOREM 5.1: $D_{12,7}$ is a diatonic set if and only if the following conditions hold.

(1) $D_{12,7}$ has a tetrachordal structure.
(2) $D_{12,7}$ has precisely one tritone.
(3) Step-intervals come in (any number of) consecutive integer sizes.

PROOF: The direct implication is a result of Theorem 3.4. Clearly the converse can be shown via a simple computer program, but we wish to illustrate the power of Theorem 2.1 and the sequence pairs discussed above by employing them to show these three conditions imply the pcset is diatonic.

As discussed in section 4, there are only three sequence pairs in $\mathcal{T}(12,7)$ (Figure 4.7). Pcsets related to the sequence pairs ((412),(123)) and ((25),(12)) are the only ones that satisfy conditions (1) and (3) (Figure 5.2). We use step-interval sequences to find the pcsets. There are only four: (1131132), (1311312), and (3113112) are associated with ((412),(123)), and (2122122) is associated with ((25),(12)). Finally, we inspect these four to discover that only the step-interval sequence associated with the diatonic set has precisely one tritone.∎

We find that the same general properties are true of the gramas.

THEOREM 5.2: $D_{22,7}$ is a *grama* if and only if the following conditions hold.

(1) $D_{22,7}$ has a tetrachordal structure.
(2) $D_{22,7}$ has precisely one tritone.
(3) Step-intervals come in consecutive integer sizes.

PROOF: It is a simple matter to verify the direct implication. We proceed to show the converse.

There are nine sequence pairs in $\mathcal{T}(22,7)$ (Figure 5.3). Only pcsets related to the last three pairs satisfy conditions (1) and (3). We leave it to the reader to find all possible step-interval sequences associated with these pairs (there are eight in all). Of these, only the step-interval sequences of the *gramas* contain precisely one tritone!∎

A	B	**Tetrachord Structure**	**Single Tritone**
(12211)	(12345)	no (Thm 2.1)	–
(21121)	(12345)	no (Thm 2.1)	–
(4111)	(2345)	no (Thm 2.1)	–
(1132)	(1234)	no (Thm 2.1)	–
(1213)	(1234)	no (Thm 2.1)	–
(241)	(234)	yes	yes
(322)	(234)	yes	no
(16)	(34)	yes	no

Figure 5.4: Sequence pairs in $\mathcal{F}(20,7)$ and their implications.

Pcset	**Step-Interval Sequence**
$J^{0,0}_{20,15,7} = \{\underline{0},2,5,8,\underline{10},13,16\}$	$(\underline{233}\underline{2\ 33}4\ )$
$J^{5,0}_{20,15,7} = \{0,\underline{3},5,8,11,\underline{13},16\}$	$(\underline{3\ 23}\underline{32\ 3}4)$
$J^{10,0}_{20,15,7} = \{0,3,\underline{6},8,11,14,\underline{16}\}$	$(\underline{33\ 2}\underline{332}\ 4)$

Figure 5.5: Pcsets with $c = 20$ and $d = 7$ that satisfy the three conditions in Theorems 5.1 and 5.2.

We give one more example (which will be useful later) to illustrate this technique.

THEOREM 5.3: For $c = 20$ and $d = 7$, there are three distinct pcsets (up to transposition) that satisfy the three conditions in Theorems 5.1 and 5.2.

PROOF: There are eight sequence pairs in $\mathcal{F}(20,7)$ (Figure 5.4). Only pcsets related to the last three pairs satisfy conditions (1) and (3), and of these, there are only three pcsets satisfying condition (2) (Figure 5.5). The underlined subsequences are the hypertetrachord spanning sequences (see Definition 3.12). In the pcsets, the underlined pcs form the single tritones and the associated step-interval sequences are spaced so the corresponding half-octaves can easily be observed. For each of these pcsets, the fourth pc is the hinge and the last clen in the associated step-interval sequence is the gap.∎

Theorems 5.1, 5.2 and 5.3 all assume $d = 7$. Might it be possible to generalize these theorems even more? Suppose the assumption that $d = 7$ is dropped and hypertetrachordal structure is required instead of the more restrictive tetrachordal structure. Then one problem pcset comes up immediately: the naked tritone, which (in the trivial sense) satisfies all three conditions. But if we eliminate this case then the following is true.

THEOREM 5.4: Let $d \geq 3$. Then $D_{12,d}$ is a diatonic set if and only if the following conditions hold.

(1) $D_{12,d}$ has a hypertetrachordal structure.
(2) $D_{12,d}$ has precisely one tritone.
(3) Step-intervals come in consecutive integer sizes.

PROOF: ($\Rightarrow$) Clear!
($\Leftarrow$) All sequence pairs in the families $\mathcal{F}(12,d)$, $3 \leq d \leq 7$, are checked in the same way we checked the sequence pairs in $\mathcal{F}(12,7)$ in Theorem 5.1. For $d > 7$ the pigeonhole principle would require that $D_{12,d}$ have more than one tritone.∎

Now, is the same more general approach also true for the *gramas*? Unfortunately, the answer is "No!"

EXAMPLE 5.1: Consider $D_{22,9} = \{0,1,4,7,10,12,13,16,19\}$ associated with the sequence pair ((216),(123)) in $\mathcal{F}(22,9)$. This pcset satisfies the following three conditions.

(1) $D_{22,d}$ has a hypertetrachordal structure.
(2) $D_{22,d}$ has precisely one tritone.
(3) Step-intervals come in consecutive integer sizes.

This is one of several *non-grama* sets that satisfy these conditions. Thus, if we wish to eliminate all *non-gramas*, we must strengthen our conditions.

Recall that the two equivalent definitions of hyperdiatonic (and, hence, diatonic) sets that include hypertetrachords (Theorem 3.4) differ in that one requires a single tritone and the other, a single ambiguity. In the context of this theorem (step-intervals come in *two* consecutive integer sizes) the two are mutually implicative. But in a larger context (step-intervals come in any number of consecutive integer sizes) this is not true. So, we will explore "a single tritone" verses "a single ambiguity."

Note that for $D_{22,9}$ above, $< 1 > \cap < 2 > = \{1,2,3\} \cap \{3,4,5,6\} = \{3\}$. With the exception of an ambiguous tritone all ambiguities come in pairs (Lemma 3.1 and Corollary 3.1). Hence, $D_{22,9}$ has more than one ambiguity. This is also true of the other *non-gramas* that satisfy the three conditions above. So, in our conditions, we replace "a single tritone" with "a single ambiguity," and we get the following parallel theorems.

THEOREM 5.5: $D_{12,d}$ is a diatonic set if and only if the following conditions hold.

(1) $D_{12,d}$ has a hypertetrachordal structure.
(2) $D_{12,d}$ has precisely one ambiguity.
(3) Step-intervals come in consecutive integer sizes.

THEOREM 5.6: $D_{22,d}$ is a *grama* if and only if the following conditions hold.

(1) $D_{22,d}$ has a hypertetrachordal structure.
(2) $D_{22,d}$ has precisely one ambiguity.
(3) Step-intervals come in consecutive integer sizes.

We leave it to the reader to check these theorems. With some effort, they can be checked without the aid of a computer (as we did in our original calculations) by employing Theorem 2.1 and the sequence pairs, but it is clearly easier for the computer to search for the qualifying pcsets once it has been "taught" how to use Theorem 2.1 and the sequence pairs (or some other appropriate set of tools). Without these tools, even the computer would have some difficulty with Theorem 5.6 since, as calculated from David Reiner's (1985) formulas based on Polya's Enumeration Theorem and Burnside's Lemma, there are 96,908 distinct SC's containing a total of $2^{22} - 1 = 4{,}194{,}303$ non-empty subsets of U_{22}.

In Theorems 5.5 and 5.6, the number 7 arises not by assumption as in Theorems 5.1, 5.2, and 5.3, but it arises as a result of conditions completely independent of the diatonic cardinality. This brings us to our first set of axioms.

FIRST SET OF HYPERSCALE AXIOMS

(1A) $D_{c,d}$ has a hypertetrachordal structure.
(2A) $D_{c,d}$ has precisely one ambiguity.
(3A) Step-intervals come in consecutive integer sizes.

For $c = 12$ and $c = 22$, the diatonic sets and the *gramas* are the only pcsets that satisfy this set of axioms. It is surprising to discover that a set of three simple properties, independent of diatonic cardinality, can generate the diatonic sets when $c = 12$ and the *gramas* when $c = 22$, and, at the same time, eliminate *all* other pcsets in their respective universes (Theorems 5.5 and 5.6). Moreover, Theorem 3.4 implies that all hyperdiatonic sets satisfy the axioms as well. In addition, pcsets in Figure 5.5 satisfy this set of axioms since the single tritones in these pcsets are also single ambiguities (this fact is easy to verify). This implies that when

$c \equiv 0 (\mathrm{mod}\ 4)$, hyperdiatonic sets need not be the only pcsets satisfying these axioms.

Clough and Douthett (1991) discussed informally the concept of *iterated* ME sets and their role in the infrastructure of Western music. In their later collaboration with Ramanathan and Rowell (Clough et al. 1993), this concept was extended (informally) to the Indian *gramas*. We now give formal definition to this concept, preparing to show its relationship to hypertetrachords and to construct a second set of axioms that replaces property (3A) with an iterated ME property.

DEFINITION 5.1: Suppose $c > d_1 > d_2 > \ldots > d_n$, $0 \le m_1 \le c - 1$, and $0 \le m_i + 1 \le d_i - 1$ for $1 \le i \le n - 1$ and $0 \le N \le d_n - 1$. Then the *nth-order J-function* with these parameters is defined as follows:

$$J_{c,d_1,d_2,\ldots,d_n}^{m_1,m_2,\ldots,m_n}(N) = J_{c,d_1}^{m_1}\left(J_{d_1,d_2}^{m_2}\left(\ldots J_{d_{n-1},d_n}^{m_n}(N)\ldots\right)\right).$$

DEFINITION 5.2: Let c, d_i, and m_i be as above. Then the *nth-order ME set* with these parameters is given by

$$J_{c,d_1,d_2,\ldots,d_n}^{m_1,m_2,\ldots,m_n} = \left\{J_{c,d_1,d_2,\ldots,d_n}^{m_1,m_2,\ldots,m_n}(0), J_{c,d_1,d_2,\ldots,d_n}^{m_1,m_2,\ldots,m_n}(1), \ldots J_{c,d_1,d_2,\ldots,d_n}^{m_1,m_2,\ldots,m_n}(d_n - 1)\right\}$$

The triads and seventh chords embedded in the diatonic set are second-order ME sets. The ME set $J_{12,7}^{5}$ is the C major set (C = 0) and the second-order ME sets $J_{12,7,3}^{5,0}, J_{12,7,3}^{5,1}, J_{12,7,3}^{5,2}, \ldots$ are the triads C major, A minor, F major, etc. Similarly, the second-order ME sets $J_{12,7,4}^{5,0}, J_{12,7,4}^{5,1}, J_{12,7,4}^{5,2}$, ... are the chords D minor 7th, F major 7th, A minor 7th, etc. Moreover, the harmonic triads imbedded in an octatonic set are second-order ME sets. In particular, the collection $\{J_{12,8,3}^{0,k}\}_{k=0}^{7}$ is the collection of harmonic triads imbedded in the octatonic set $J_{12,8}^{0}$.

The *gramas* and hyperdiatonic sets share many properties, among which are that all have precisely one tritone and precisely one ambiguity. Such pcsets can be ME sets only if $c \equiv 0 (\mathrm{mod}\ 4)$ (Theorem 3.1). Thus, the *gramas* are not ME sets. So, we look for the next best thing—second-order ME sets. And are the *gramas* second-order ME sets? Remarkably, yes! Their parent sets have cardinality 12 (perhaps coincidentally, but nevertheless the same cardinality as the Western chromatic universe). We can represent the *gramas* in Figure 5.1 as $J_{22,12,7}^{0,2}$, $J_{22,12,7}^{6,0}$, $J_{22,12,7}^{0,4}$, and $J_{22,12,7}^{2,0}$, respectively. Assuming transpositional equivalence, there are only two other types of pcsets in this cardinal hierarchy: e.g, $J_{22,12,7}^{0,0}$, and $J_{22,12,7}^{0,1}$. However, these two pcsets do not have tetrachordal structure as do the *gramas*. Next, using the above observation, we develop a set of axioms in which the consecutive step-interval property is replaced by an iterated ME property.

Recall that a hyperdiatonic set is a ME set with precisely one ambiguity. Theorem 3.1 shows that such pcsets exist if and only if $c \equiv 0 (\mathrm{mod}\ 4)$. For the universe that includes the *gramas*, $c = 22 \equiv 2(\mathrm{mod}\ 4)$. It follows that if a subset of U_{22} has precisely one ambiguity, it cannot be a (first-order) ME set. The *gramas* are such pcsets. Recall, however, that the *gramas are* second-order ME sets. This leads us to the following theorem.

THEOREM 5.7: A pcset $D_{22,d}$ is a *grama* if and only if the following conditions hold.

(1) $D_{22,d}$ has a hypertetrachordal structure.
(2) $D_{22,d}$ has precisely one ambiguity.
(3) $D_{22,d}$ is a second-order ME set.

(Again, we leave it to the reader to verify this theorem.) This theorem and Theorem 3.4 suggest another set of axioms.

SECOND SET OF HYPERSCALE AXIOMS

(1B) $D_{c,d}$ has a hypertetrachordal structure.
(2B) $D_{c,d}$ has precisely one ambiguity.
(3B) $D_{c,d}$ is a minimally iterated ME set within the class of sets that satisfy (1B) and (2B).

Unlike the first set, this set of axioms implies that for cases $c \equiv 0(\mathrm{mod}\ 4)$, one need merely identify the hyperdiatonic sets in $\underline{M}(\mathrm{c})$. One need not check further since the hyperdiatonic sets are first-order ME sets and are the only such pcsets that satisfy Axiom 2B. As Theorems 3.1 and 3.4 illustrate, all hyperdiatonic sets (ME sets with precisely one ambiguity) have hypertetrachordal structure. Thus, for hyperdiatonic sets, Axiom 1B is redundant. However, if hyperdiatonic sets cannot be found, Axiom 3B relaxes the first-order ME condition and allows a search through second-order ME sets. In doing so, hypertetrachordal structure is no longer implied; so Axiom 1B becomes relevant. For the chromatic universe of 22 divisions to the octave, there are no first-order ME sets with precisely one ambiguity. We must then consider all second-order ME sets. We find that, for c=22, the *gramas* are the only second-order ME sets satisfying Axioms 1B and 2B. If we had found no second-order ME sets satisfying both these axioms, we would then search through third-order ME sets, and so on.

DEFINITION 5.3: A *hyperscale* is a pcset that satisfies one or both of the sets of hyperscale axioms above.

The two sets of hyperscale axioms are not equivalent. The pcsets in Figure 5.5 satisfy the first set of axioms but not Axiom 3B in the second

Hyperdiatonic Set	Step-Interval Sequence
$J^0_{20,11} = \{\underline{0},1,3,5,7,9,\underline{10},12,14,16,18\}$	$(\underline{12222}\underline{1\ 2222}2\ )$
$J^2_{20,11} = \{0,\underline{2},3,5,7,9,11,\underline{12},14,16,18\}$	$(\underline{2\ 1222}\underline{21\ 222}2)$
$J^4_{20,11} = \{0,2,\underline{4},5,7,9,11,13,\underline{14},16,18\}$	$(\underline{22\ 122}\underline{221\ 22}2)$
$J^6_{20,11} = \{0,2,4,\underline{6},7,9,11,13,15,\underline{16},18\}$	$(\underline{222\ 12}\underline{2221\ 2}2)$
$J^8_{20,11} = \{0,2,4,6,\underline{8},9,11,13,15,17,\underline{18}\}$	$(\underline{2222\ 1}\underline{22221}\ 2)$

Figure 5.6: Hyperdiatonic sets in the universe U_{20}.

set since U_{20} is a universe that contains hyperdiatonic sets (i.e., *first-order* ME sets that satisfy axioms 1B and 2B). Conversely, the second-order ME set $J^{0,0}_{6,5,4} = \{0,1,2,3\}$ satisfies the criterion in the second set of axioms but not Axiom 3A in the first set. This suggests two kinds of hyperscales, but we will not explore that distinction here. Instead we will focus on constructing two classes of pcsets whose members satisfy both sets of axioms.

For the first class of pcsets to be constructed, assume $n \geq 0$. Choose n 2's and one 1, and put them together in any order to form the spanning sequence of the first hypertetrachord. Repeat this sequence and append a clen of 2 on the end to form the complete step-interval sequence of the set. The resultant sequence is that of a set in a universe of size $c = 2(2n + 1) + 2 = 4(n + 1)$; in fact, the set is a hyperdiatonic set. Thus, all pcsets constructed in this way are I-symmetrical and equivalent under transposition (Theorem 3.3). It follows that these sets comprise the SC $\underline{M}$(c) precisely when $c \equiv 0 (\text{mod } 4)$ and $c \geq 4$ (Theorem 3.1).

EXAMPLE 5.2: Suppose $c = 20$. Then $n = 4$. The spanning sequence of the hypertetrachord consists of four 2's and one 1 in any order. Thus, the span of the hypertetrachord is $4 \cdot 2 + 1 = 9$. We repeat this sequence and append clen 2 to the end of this sequence to make the step-interval sequence of a hyperdiatonic set. Figure 5.6 lists the hyperdiatonic sets that begin with 0 and correspond to the step-interval sequences listed to the right of the pcset. The underlined subsequences are the spanning sequences of the hypertetrachords. The underlined pcs in the pcsets are the single ambiguities, and the associated step-interval sequences are spaced to reflect these ambiguities. Note that the mode indices of the J-sets are multiples of 2. We can write these pcsets as $J^{2k-2}_{20,11}$ where $1 \leq k \leq 5$.

Although we have pointed out that the pcsets generated above are hyperdiatonic (and certainly intuition suggests they are), this should be shown formally. From their construction, it is clear that these pcsets satisfy Axioms 1A, 1B, and 3A. If we can show that the pcsets constructed in this way are, indeed, hyperdiatonic sets then the rest of the axioms will be automatically satisfied (Theorem 3.4).

For the hyperdiatonic set $J^{2k-2}_{4n,2n+1}$, $1 \leq k \leq n$, we define the following function:

$$f^k_n(N) = J^{2k-2}_{4n,2n+1}(\mathrm{N}) - J^{2k-2}_{4n,2n+1}(N\text{-}1)$$

where $1 \leq N \leq 2n + 1$. Whence, $f^k_n(N)$ is a step-interval of $J^{2k-2}_{4n,2n+1}$, and the step-interval sequence of $J^{2k-2}_{4n,2n+1}$ is

$$(f^k_n(1), f^k_n(2), \ldots f^k_n(2n + 1)).$$

To show that these step-interval sequences are the same as the ones generated by the hypertetrachords above, we must show

$$f^k_n(N) = \begin{cases} 2 & \text{if } 1 \leq N \leq k - 1 \\ 1 & \text{if } N = k \\ 2 & \text{if } k + 1 \leq N \leq n \\ 2 & \text{if } n + 1 \leq N \leq n + k - 1 \\ 1 & \text{if } N = n + k \\ 2 & \text{if } n + k + 1 \leq N \leq 2n \\ 2 & \text{if } N = 2n + 1. \end{cases}$$

If this is true, then

$$1_{\text{span}}(H_1) = (f^k_n(1), f^k_n(2), \ldots f^k_n(n)) \text{ and}$$
$$1_{\text{span}}(H_2) = (f^k_n(n + 1), f^k_n(n + 2), \ldots f^k_n(2n)),$$

and $f^k_n(2n + 1)$ is the gap. Once these seven cases are shown, we will know the corresponding pcsets are hyperdiatonic, and, hence, satisfy all the axioms. We will show only one and leave it to the reader to check the rest.[11]

Suppose $n + 1 \leq N \leq n + k - 1$, and assume $1 \leq k \leq n$. Then

$$\begin{aligned} f^k_n(N) &= J^{2k-2}_{4n,2n+1}(\mathrm{N}) - J^{2k-2}_{4n,2n+1}(N\text{-}1) \\ &= \left\lfloor \frac{4Nn + 2k - 2}{2n + 1} \right\rfloor - \left\lfloor \frac{4(N - 1)n + 2k - 2}{2n + 1} \right\rfloor \\ &= \left(2N + \left\lfloor \frac{2(k - N - 1)}{2n + 1} \right\rfloor\right) - \left(2N - 2 + \left\lfloor \frac{2(k - N)}{2n + 1} \right\rfloor\right) \\ &= (2N - 1) - (2N - 3) \\ &= 2. \end{aligned}$$

The definition of a *hypergrama* set is inherent in its construction. This construction parallels the previous construction of the hyperdiatonic sets.

DEFINITION 5.4: Let $n \geq 0$. Choose n 4's, one 2, and one 3, and put them together in any order to form the spanning sequence of the first hypertetrachord. Repeat this sequence and append a 4 to the end to form the complete step-interval sequence of a set. Any pcset whose step-interval sequence can be constructed in this way is called a *hypergrama.*

In this case, the size of the universe is $c = 2(4n + 3+ 2) + 4 = 8(n + 1) + 6$. Thus, for a universe that supports hypergrama sets, $c \geq 14$ and $c \equiv 6 \pmod 8$. Note that if $n = 1$ (i.e., $c = 22$), the pcsets generated in this way are the *gramas* (Figure 5.1). Unlike hyperdiatonic sets, for a given c, hypergramas are not necessarily equivalent under transposition. But then, for a given $c \equiv 6 \pmod 8$, how many hypergrama SC_Ts are there? We can answer this by observing the spanning sequences of the hypertetrachords. If this sequence begins with 2 then there are $n + 1$ places in this sequence to place the 3. Each of these sequences generates a different hypergrama step-interval sequence. These are associated with $n + 1$ distinct SC_Ts. (e.g., the *ma-grama* and *ma-grama doubly altered* in Figure 5.1). Similarly (assuming transpositional but not inversional equivalence), if the spanning sequence of the hypertetrachord begins with 3 then there are $n + 1$ places in the sequence to place the 2, and each of these sequences generates still another hypergrama step-interval sequence (e.g., the *sa-grama* and *sa-grama doubly altered*). But if the spanning sequence of the hypertetrachord begins with a 4 then the generated hypergrama step-interval sequence will not be distinct from one of the above. This can be seen if we take this step-interval sequence and rotate the 4's in the front to the back until a 2 or a 3 becomes the first element. This sequence will be identical to one of the hypergrama step-interval sequences generated by a hypertetrachord spanning sequence that begins with a 2 or a 3. It follows that there are $2(n + 1) = (c - 6)/4$ hypergrama SC_Ts. Moreover, a hypergrama associated with each hypertetrachord spanning sequence that begins with a 2 are equivalent under inversion to a hypergrama associated with a sequence that begins with a 3, and conversely (e.g., the *ma-grama* and *sa-grama doubly altered* and the *sa-grama* and *ma-grama doubly altered*). Whence, the hypergramas comprise $n + 1 = (c - 6)/8$ SCs. The following example illustrates the presence of the single ambiguity in the hypergramas which we will later prove in a general setting.

EXAMPLE 5.3: Suppose $c = 30$. Then $n = 2$. The hypergrama step-interval sequence will begin with a subsequence two 4's, one 3, and one 2 in any order which is repeated and has a clen of 4 appended at the end. The step-interval sequences for the first three pcsets in Figure 5.7 begin with

Hypergrama	Step-Interval Sequence
$J^{0,2}_{30,16,9} = \{0,\underline{3},5,9,13,16,\underline{18},22,26\}$	$(\underline{3\ 244}\underline{32\ 44}4)$
$J^{0,4}_{30,16,9} = \{0,3,\underline{7},9,13,16,20,\underline{22},26\}$	$(\underline{34\ 24}\underline{342\ 4}4)$
$J^{0,6}_{30,16,9} = \{0,3,7,\underline{11},13,16,20,24,\underline{26}\}$	$(\underline{344\ 2}\underline{3442}\ 4)$
$J^{2,0}_{30,16,9} = \{\underline{0},2,5,9,13,\underline{15},18,22,26\}$	$(\underline{2344}\underline{2\ 344}4\ )$
$J^{6,0}_{30,16,9} = \{\underline{0},2,6,9,13,\underline{15},19,22,26\}$	$(\underline{2434}\underline{2\ 434}4\ )$
$J^{10,0}_{30,16,9} = \{\underline{0},2,6,10,13,\underline{15},19,23,26\}$	$(\underline{2443}\underline{2\ 443}4\ )$

Figure 5.7: Hypergramas in the universe U_{30}.

a 3. Note that the J-sets corresponding to these three sequences are $J^{0,2k}_{30,16,9}$ where $k = 1,2,3$ respectively. Similarly, the step-interval sequences for the last three pcsets in Figure 5.7 begin with a 2, and these pcsets can be represented as $J^{4k\text{-}2,0}_{30,16,9}$ where k=1,2,3 respectively. The hypertetrachords and ambiguities are indicated as in Example 5.2 and Figure 5.5 and 5.6.

It is more difficult to show that the hypergramas satisfy our axioms than it was to show that the hyperdiatonic sets satisfied them since many tools that exist for the hyperdiatonic sets do not apply to the hypergramas. As in the case of the hyperdiatonic sets, it is clear by our construction that the hypergramas satisfy axioms 1A, 1B, and 3A. But now we must show not only that these pcsets are second-order ME sets (which will show they satisfy Axiom 3B), but we must also show they satisfy Axioms 2A and 2B.

It is not difficult to see that if a pcset is a nth-order ME set then so are its transpositions. (Simply transpose the entire hierarchy of sets.) So, to show the hypergramas are second-order ME sets, we need to show that the pcsets $J^{0,2k}_{8n+6,4n+4,2n+3}$, $1 \leq k \leq n$ correspond to the step-interval sequences generated by the hypertetrachord spanning sequences that begin with a 3 and $J^{4k\text{-}2,0}_{8n+6,4n+4,2n+3}$, $1 \leq k \leq n$ correspond to the to the step-interval sequences generated by the hypertetrachord spanning sequences that begin with a 2. We will demonstrate the technique needed to show this and leave it to the reader it follow through.

Suppose the hypertetrachord spanning sequence begins with a 3. Define the function

$$g^k_n(N) = J^{0,2k}_{8n+6,4n+4,2n+3}(N) - J^{0,2k}_{8n+6,4n+4,2n+3}(N - 1)$$

where $1 \leq N \leq 2n + 3$ and $1 \leq k \leq n$. Then $g^k_n(N)$ is a step-interval $J^{0,2k}_{8n+6,4n+4,2n+3}$, and the step-interval sequence of $J^{0,2k}_{8n+6,4n+4,2n+3}$, $1 \leq k \leq n$, is

$$(g_n^k(1), g_n^k(2), \ldots g_n^k(2n+3)).$$

To show that these sequences are the same as those generated by the hypertetrachord construction, we must show

$$g_n^k(N) = \begin{cases} 3 & \text{if } N = 1 \\ 4 & \text{if } 2 \leq N \leq k \\ 2 & \text{if } N = k+1 \\ 4 & \text{if } k+2 \leq N \leq n+1 \\ 3 & \text{if } N = n+2 \\ 4 & \text{if } n+3 \leq N \leq n+k+1 \\ 2 & \text{if } N = n+k+2 \\ 4 & \text{if } n+k+3 \leq N < 2n+2 \\ 4 & \text{if } N = 2n+3. \end{cases}$$

If this is true then

$$1_{\text{span}}(H_1) = (g_n^k(1) \ldots g_n^k(n+1)) \text{ and } 1_{\text{span}}(H_2) = (g_n^k(n+2) \ldots g_n^k(2n+2)),$$

and the gap is $g_n^k(2n+3)$. We will show only one case.

Suppose $2 \leq N \leq k$ and $1 \leq k \leq n$. Then

$$\begin{aligned} g_n^k(N) &= J^{0,2k}_{8n+6,4n+4,2n+3}(N) - J^{0,2k}_{8n+6,4n+4,2n+3}(N-1) \\ &= J^0_{8n+6,4n+4}\left(2N + \left\lfloor \frac{2(k-N)}{2n+3} \right\rfloor\right) - J^0_{8n+6,4n+4}\left(2N - 2 + \left\lfloor \frac{2(k-N+1)}{2n+3} \right\rfloor\right) \\ &= J^0_{8n+6,4n+4}(2N) - J^0_{8n+6,4n+4}(2N-2) \\ &= \left(4N + \left\lfloor \frac{-N}{n+1} \right\rfloor\right) - \left(4n - 4 + \left\lfloor \frac{1-N}{n+1} \right\rfloor\right) \\ &= (4N-1) - (4N-5) \\ &= 4. \end{aligned}$$

The reader is invited to check the rest of these cases as well as the nine additional cases required for the hypergramas whose hypertetrachord spanning sequences begin with a 3. This will verify that the hypergramas are second-order ME sets.

Now we must show that each of these hypergramas has precisely one ambiguity. Let $a_I = \min < \mathrm{I} >$ and $b_I = \max < \mathrm{I} >$. We can see from our construction that, for $1 \leq I \leq n+1$, $a_I \geq 2 \cdot 1 + 3 \cdot 1 + 4(I-2) = 4I - 3$

(one step-interval with clen 2, one with clen 3, and I - 2 with clen 4) and $b_I \leq 4I$ (I step-intervals with clen 4). Thus, $b_I < a_{I+1}$ for all I, $1 \leq I \leq n$. This implies $< I_1 > \cap < I_2 > = \varnothing$ for all I_1 and I_2 such that $1 \leq I_1 < I_2 \leq n + 1$. Then by Lemma 3.1 $< I_1 > \cap < I_2 > = \varnothing$ for all I_1 and I_2 such that $n + 2 \leq I_1 < I_2 \leq 2n + 2$. It follows that the only ambiguous clens are in $<n+1> \cap < n + 2 >$. But

$$\begin{aligned} <n + 1> &= \{2 \cdot 1 + 3 \cdot 1 + 4(n - 1), 2 \cdot 1 + 4n, 3 \cdot 1 + 4n\} \\ &= \{1 + 4n, 2 + 4n, 3 + 4n\}. \end{aligned}$$

By Lemma 3.1, $< n + 2 > = \{3 + 4n, 4 + 4n, 5 + 4n\}$. Thus, $< n + 1 > \cap < n + 2 > = \{3 + 4n\}$; so, the only ambiguous clen is $3 + 4n$. Moreover, we can see from our construction that there is only one interval with clen $3 + 4n$ (composed of one step-interval with clen 3 and n with clen 4) associated with $n + 1$. It follows a hypergrama has precisely one ambiguity and, hence, satisfies Axioms 2A and 2B. (Note that $3 + 4n = c/2$. Thus, the single ambiguity is a tritone which satisfies Corollary 3.1.) Whence, hypergramas are, indeed, hyperscales that satisfy both sets of axioms.

We can find the parent ME set of a hypergrama by breaking down all the 4's in its step-interval sequence into pairs of 2's, one of the 3's into 1+2, and the other 3 into 2 + 1 (the choice of which 3 divides into 1 + 2 and which into 2 + 1 is critical). The resultant sequence will correspond to a ME set $J^m_{c,(c+2)/2}$ for some index m, $0 \leq m \leq c - 1$. Note that if d is the diatonic cardinality and e is the cardinality of the parent set then, strikingly, $e = 2(d - 1)$ and $c = 2(e - 1)$. (Recall that for the hyperdiatonic set, $c = 2(d - 1)$.) Moreover, if a universe that supports hypergramas has chromatic cardinality $c \equiv 6 \pmod 8$, the cardinality of the parent set of a hypergrama is $e \equiv 0 \pmod 4$. Since a hypergrama is maximally even within its parent and $e = 2(d - 1)$, there is a "hyperdiatonic substructure" within this universe (Theorem 3.1); that is, hypergramas are not hyperdiatonic sets within their chromatic universe, but they *are* hyperdiatonic sets with respect to their parent set. This can easily be seen if we call clen 2 "small" and clen 3 and 4 "large." Then the distribution of "small's" and "large's" in a hypergrama is the same as the distribution of clen 1's and clen 2's in a hyperdiatonic set. This substructure is inherent in our construction, and it is not hard to see the diatonic substructure" of the *gramas* in Figure 5.1. In such cases, the parent set can be thought of as a "ME temperament."

We conclude with several questions. From the above, we know that a hyperdiatonic set satisfies both sets of axioms, and when $c \equiv 0 \pmod 4$, a pcset is a hyperdiatonic set if and only if it satisfies the second set of axioms. Moreover, we know a hypergrama also satisfies both sets of axioms, but, when $c \equiv 6 \pmod 8$, are these the only pcsets satisfying one or both sets of axioms? A computer search establishes that for $c =$ 14,22,30, they are, but for larger values of c, we do not know!

Also, we have not explored classes of hyperscales in which $c \equiv 2$ (mod 8). For these chromatic cardinalities, our program reveals that the smallest universe that has pcsets satisfying both sets of axioms has cardinality $c = 26$, but, in general, it is not known what classes of pcsets are generated by these sets of Hyperscale Axioms, or how they can be constructed.

We have also pointed out that these sets of axioms are not equivalent. What can be said about hyperscales that satisfy one set of axioms but not the other, and what properties must a hyperscale have to satisfy both sets of axioms? Although, in general, we have not investigated these questions (except as they relate to hyperdiatonic sets and hypergramas), we do know something about the frequency of occurrence of such hyperscales. Our computer search suggests that pcsets satisfying either set of axioms are very rare. For chromatic cardinalities $8 \leq c \leq 30$, there are only 22 SC's whose members satisfy the first set of axioms, and 22 SC's whose members satisfy the second set (recall that earlier calculations for $c = 22$ established that, for this universe alone, there are 96,908 SC's). Of these SC's, 14 have members that satisfy both sets of axioms. Thus, for $8 \leq c \leq 30$, there are only $22 + 22\text{-}14 = 30$ SC's whose members satisfy one or both sets of axioms. Of the SC's whose members satisfy both sets, all but 2 are hyperdiatonic or hypergrama SC's.

We have not considered pcsets in universes whose chromatic cardinalities are odd because the single ambiguity (a tritone) appears to be such a powerful condition. Are there sets of axioms that allow us to discard this condition so that we can consider pcsets in universes that have odd chromatic cardinalities while, at the same time, satisfying our principal requirement (that they generate precisely the diatonic sets when $c =$ 12 and the *gramas* when $c = 22$)?

These are all questions that we feel warrant further investigation.

NOTES

A portion of this paper, in an earlier version by Clough and Douthett, was presented at the International Conference on Music and Mathematics, Bucharest, May 1994. The contents of a handout from that presentation (which correspond in part to examples in the present paper) appear in the Romanian journal *Muzica* (1995): 100–109.

1. Readers who wish for background on the history of TC in music theory will recall that ancient Greek theory describes tetrachords of various intervallic structure; similar (i.e., transpositionally related) tetrachords may be joined to make larger systems. For a survey of later theoretical precedents (from Guido of Arezzo to the 1980's), see Cohn (1987, ch. 1).
2. Block and Douthett (1993) have devised a measurement that compares the *evenness* of pcsets in the same cardinal family (e.g., the augmented triad is more even than the harmonic triad which, in turn, is more even than the chromatic trichord). This measurement is consistent with Definition 3.2 and can provide the reader with an intuitive sense of *maximally even.*
3. Applications of maximally even sets are not restricted to musical structures. Douthett and Krantz (1996) have shown that this formalism can be useful in describing the structure of one-dimensional lattices of up- and down-spins that minimize configurational energy in a physical model known as the *Ising model.* If the white keys are thought of as up-spins and the black keys as down-spins then the piano keyboard would represent such a lattice.
4. This definition was first given by Jay Rahn (1991).
5. Our terminology here is slightly different from that in Clough and Douthett (1991). The *hyperdiatonic sets* of this paper are the *diatonic sets* of the earlier paper. In this paper, we use the term *diatonic set* and *usual diatonic set* interchangeably; either represents that which we call the *usual diatonic set* in the earlier paper. Thus, in the present paper, the *diatonic set* or the *usual diatonic set* is a special case of the *hyperdiatonic set.* We regret any confusion this causes, but the shift in terminology seems necessary for parallelism with the natural definitions for tetrachord and hypertetrachord, the former being a special case of the latter. Moreover, this change in terminology better suits our development of *hyperscales* in the last section of this paper.
6. This class of sets was first isolated by Agmon (1989), who used the term *diatonic system* for the set of parameters (including cardinalities and generating intervals) which yields that which we call a hyperdiatonic set.
7. This definition yields an alternative definition for maximally even sets: THEOREM: A pcset $D_{c,d}$ is a ME set if and only if
$$\text{span}_I(D_{c,d}) = \lfloor cI(d-1)/d \rfloor$$
for every I, $1 \le I \le d - 1$.
8. Note that the reduction algorithm does not necessarily form a single path within a given partition. For example, the sequences (1121) and (311) in $P_2(5)$ reduce as follows:
$$(1121) \rightarrow (23) \leftarrow (311).$$
From a graph–theoretic standpoint, the reduction algorithm acting on a partition induces a rooted tree whose root is a sequence of either one or two integers.
9. Theorems 5.1 and 5.2 in the present paper correspond to "irreducible feature set"

E1 in Clough et al. (1993, 53), generalized later in this section as the "first set of hyperscale axioms." Irreducible feature set E3 from the earlier paper is suggestive of the "second set of hyperscale axioms."

10. It was Lewis Rowell who, at the Annual Meeting of the Society of Music Theory, Austin, 1989, first suggested to us that maximally even properties might be lurking in this ancient Indian system.
11. Using Proposition 4 from Agmon (1989, 18) and/or the observations by Clough and Douthett (1991, 141) regarding tetrachordal structure, there may be a simpler more direct approach to showing that these sets are hyperdiatonic sets. However, the approach we use here is more general, and this generality will be needed in the construction of another class of hyperscales to be discussed later.

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