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Pitch-Set Equivalence and Inclusion

(A Comment on Forte's Theory of Set-Complexes)

JOHN CLOUGH

1.0 Introduction and synopsis

Forte's theory of set-complexes commands the attention of all who seek meaningful generalizations in the area of pitch-relations. *1 In its broad outlines — its formalizing and structuring of the notions of equivalence and inclusion — the theory offers a promising analytical framework. The present article attempts to provide a measure of the critical comment merited by such a theory. Questions will be raised concerning pitch set equivalence based on interval vector, in terms of its conceptual complexity, its contradiction in some cases of intuitive notions of pitch-set similarity, the conceptual obscurity of the inclusion relation based on it, an apparent defect in the mathematical substructure of the inclusion relation based on it, and, more speculatively, in terms of a possible illusion surrounding the concept of interval vector. An alternative, simpler, in fact altogether homely, equivalence relation, will be suggested, and an example of inclusion relations based on it will be given. Finally, the notion of exclusion, as derived from the equivalence relation suggested herein, will be explored briefly. Before turning directly to Forte's equivalence relation, we offer preliminary remarks, which, it is hoped, establish what is an acceptable line of critical discourse in these matters.

1.1 Musical vs. mathematical equivalence relations

Normally, in discussing unordered pitch sets, at least transpositional equivalence is recognized. Thus (0 1 3) and (1 2 4) are members of the same class of pitch sets. In many contexts, it is appropriate to recognize inversional equivalence as well. Thus (0 1 3), (0 2 3), (1 2 4), and (1 3 4), would be members of the same class of pitch sets. Now considering the matter casually, it seems possible that other equivalences may prove to be useful, and it would be foolish to bar from consideration a proposed equivalence relation, merely on the ground that it allowed for the equivalence of two sets not capable of superposition (which transpositions and inversions are).

If the matter is considered abstractly, a great many equivalences might suggest themselves. For example, consider a characteristic which we call the odd-even characteristic OE, as follows: OE of any set = the absolute value difference between the number of notes in one whole-tone scale and the number of notes in the other whole-tone scale. Thus, (OE) (0 2 4 5 7) = 1. (For (OE) read "OE of.") Now suppose we state that for any two sets A and B which contain the same number of pitch classes, A = B if and only if (OE) A = (OE) B. Among the consequences of the defined equivalence relation would be (0 2 5) ≡ (0 3 4), and (0 1 2 3 4) ≡ (0 2 4 5 7). From this we judge that, while OE may have some limited musical relevance, it is not useful as a general musical equivalence relation.

The point is simply that we reserve the right to evaluate a proposed relation at least partly, and certainly initially, in terms of our intuitive ideas of musical similitude. (This line of inquiry will be pursued further in Sections 2.0 through 2.4.) The same kind of criticism may be applied to the very important notion of pitch set inclusion. Again, normally, the assertion that (0 4 7 10) includes (0 4 10) is beyond question, and we wish to define "includes" to be able to assert that (0 4 7 10) includes (1 5 11) by transposition of (0 4 10) *2, and that (0 4 7 10) includes (0 4 6) by inversion and transposition of (0 4 10). However, the assertion that (0 4 7 10) includes (0 1 2) since (OE) (0 1 2) = (OE) (0 4 7) would obviously be musically insupportable.

Let it be emphasized that the foregoing is not in any sense a criticism of the Forte theory, but is merely intended to show how the music theoretical criticism of a proposed equivalence relation must differ from the mathematical view of it. Mathematically, it is interesting to explore a great many different kinds of relations between sets of numbers, but judgement must be exercised in determining which of these are musically of the greatest relevance. The following sections attempt to bring such judgement to bear upon the matter at hand.

2.0 Comparison of inversion-transposition and interval vector as musical concepts

It must be charged that a major fault of the Forte theory is this: It accounts complexly for what may be accounted for simply. The relation of pitch sets under IT† is a simple, venerable concept, of preeminent musical relevance. To treat it as a large subset of a complex relation, V equality, seems an overgeneralization. In most cases, if we choose at random two sets with equal V, the two sets are related by IT. To adduce V when IT demonstrates the equivalence of such sets, seems circuitous. Those cases in which the two randomly chosen sets have equal V but are unrelated by IT are not only relatively infrequent, but reflect a significantly lesser degree of structural similarity than those cases involving IT relations. This latter point we attempt to demonstrate in Sections 2.1, 2.20 and 2.21, while reserving a more formal resume of the above argument for section 2.3.

2.1 Structural differences in some sets with equal V

For any two sets A and B, if A = (IT) B, then (V) A = (V) B, but the converse does not hold: (V) A = (V) B does not imply A = (IT) B. Therefore, as stated above, and as Forte acknowledges, under the equivalence relation: A ≡ B if and only if (V) A = (V) B, many sets unrelated by IT are held equivalent, for example (0 1 4 5 7) and (0 1 2 5 8). As stated in section 1.1

† In the remainder of the article "inversion and/or transposition" will be abbreviated IT. "Interval vector" will be abbreviated V.

of this article, this fact is not in itself damaging. However, suppressing our initial intuitive reaction that such sets are "obviously different", we are bound to observe that their relationship is conceptually more complex than IT.

While there is no difficulty in comprehending aurally the relationship between (0 1 2 3 5 6) and (1 2 3 4 6 7) (transposition), and the same may be said of (0 1 2 3 5 6) and (0 1 3 4 5 6) (inversion), the aural relation between (0 1 2 3 5 6) and (0 1 2 3 4 7) strikes us as being of a different order. The latter hexachord is the inverted and transposed complement of the former, hence the two have equal V; but they are unrelated by IT. It would seem that there is no real objection to maintaining hexachord complementation and hexachord equivalence as separate (though not mutually exclusive) relationships. Other examples of differently "shaped" pairs of sets with equal V are: (0 1 3 4 8) and (0 1 2 5 9)*3; (0 2 5 6) and (0 1 3 7). In the latter case (all-interval tetrachords) and, by extension, in all cases where two sets with equal V are unrelated by IT, a large degree of similarity surely exists, but one may question whether a more useful theory of pitch set relations might not be founded upon equivalence under IT only, with V as a separate measure of similarity for non-equivalent sets. Under such a theory, the designation "maximally similar" would be reserved for two non-equivalent sets with equal V (such as the two all-interval tetrachords cited above.) It is believed that a re-structuring of the Forte theory along these lines would preserve the best aspects of the theory, would be more generally applicable, more precise, and more in accordance with our intuitive notions.

2.20 Minimum span

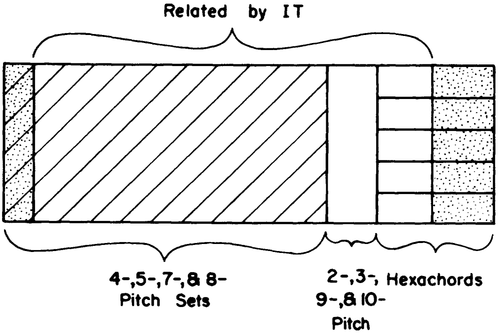
Here and in Section 2.21 we give, in amplification of the previous section, two simple measures which formalize and underline structural differences between sets with equal V but unrelated by IT. Let the minimum span of a set be defined as the smallest interval within which all its elements may be gathered. Thus, minimum span of (0 1 3 5 6) = 6, and minimum span of (0 1 3 5 10) = 7 ((0 1 3 5 10) transposed = (0 2 3 5 7)). Now V of (0 1 3 5 6) = V of (0 1 2 4 7), yet minimum span (0 1 3 5 6) = 6 and minimum span of (0 1 2 4 7) = 7. It is, of course, true that for any two sets A and B related by IT, (min. span) A = (min. span) B.

2.21 Cluster characteristic

Understanding "cluster" to be a group of notes much as (0 1) or (0 1 2) or (0 1 2 3) or (0 10 11), etc., in which the minimum span of the group equals the number of notes in the group minus one, let the cluster characteristic CC of a set be defined (rather informally, we fear) as the number of 2-note clusters not contained in larger clusters, the number of 3-note clusters not contained in larger, etc., etc. (CC) (0 1 2 4 9 10) = one two-note cluster and one three-note cluster. In the majority of cases sets with equal V but unrelated by IT have different CC. One example: (CC) (0 1 2 4 5 6) = two 3-note clusters; (CC) (0 4 5 6 7 8) = one 5-note cluster. Obviously, for any two sets A and B related by IT, (CC) A = (CC) B.

2.3 Universes of equivalences under V and under IT

Consider the statement: Set A is equivalent to set B. For how many different pairs of sets (allowing that two "different" pairs of sets may have one set in common) is the statement true when the condition for equivalence is (1) V equality, and (2) relation by IT? Computation shows that for V equality there are 50, 098 different equivalent pairs, of which 40, 882 are related by IT *8. Of the remaining 9, 216 pairs, 6, 048 are complementary hexachords (either literally complementary or complementary by IT) and 3, 168 are pairs of 4-, 5-, 7-, and 8-pitch sets. Diagrammatically, the kinds of set pairs equivalent under V may be represented thus:



Bearing in mind that (a) sets related by IT are structurally more similar than sets merely related by V equality, and (b) a majority of sets with equal V but unrelated by IT are hexachords related by complementation, it seems better to establish equivalence classes solely on the basis of IT, while maintaining complementation and V as separate, related, concepts, each to be applied as the occasion demands. The 3, 168 equivalences involving neither IT relation nor hexachord complementation, which are thus severed, as it were, by Occam's razor, are precisely those sets which seem particularly to violate principles of conceptual simplicity and economical description. For example, given (0 4 6 7) the problem of constructing (not recalling) a set with equal V but unrelated by IT, is anything but routine.

2.4 Inclusion under V equivalence

That (0 2 4 5 7) includes (0 4 7) is obvious, and generally we would define "includes" so that (0 2 4 5 7) would include (1 5 8) ((1 5 8) = (0 4 7) transposed), and (0 3 7) ((0 3 7) = (0 4 7) inverted). But the sense in which (0 3 5 6 7 8) may be said to include (0 1 3 5 6) is obscure:

(V) (0 3 5 6 7) = (V) (0 1 3 5 6)
and (0 3 5 6 7 8) includes (0 3 5 6 7)
therefore (0 3 5 6 7 8) includes (0 1 3 5 6 *4

The following is equally obscure:

(V) (0 1 2 3 5 6) = (V) (0 1 2 3 4 7)
and (0 1 2 3 4 7) includes (0 4 7)
therefore (0 1 2 3 5 6) includes (0 4 7)

As the above indicates, inclusion relations based on V pose the following problem: In some cases it may not easily be demonstrable (except by table look-up) whether one set does or does not include another. With equivalence under IT only, routine inspection of two sets discloses the presence or absence of an inclusion relation *5.

2.5 Transitive property of the inclusion relation

A most serious fault is that Forte's inclusion relation does not have the transitive property claimed for it *6. The transitive property requires that for three sets A, B and C, if A includes B, and B includes C, then A must include C. Now under V-type inclusion, and using Forte's set reference numbers:

(5-32) includes (4-15)
and (4-15) includes (3-2)
but (5-32) does not include (3-2)

And there are other such cases. This inconsistency would seem to cast a shadow over set theoretic operations based on V and hence on the idea of the set-complex itself. Inclusion relations based on equivalence under IT have the transitive property in all cases.

3.0 A possible illusion surrounding V

Here is offered a speculation as to why V, while it may be a useful measure of similarity, fails to define a musically meaningful equivalence relation in all cases. If A and B are 3-pitch sets such that (V) A = (V) B, then A and B are related by IT in all cases. Therefore we do not dispute that, for 3-pitch sets, when (V) A = (V) B, A is equivalent to B. But what is really perceived as equivalent about A and B? Is it their interval content or their dyad content? The importance of this seemingly trivial distinction is crucial. If equal dyad content is the test of 3-pitch-set equivalence, that is, if we perceive the (0 2) in (0 2 3) more as a subset of two elements separated by 2 semitones than as a distance of 2 semitones, then the notion suggests itself that equal dyad content and equal 3-pitch-set content are both necessary conditions for 4-pitch-set equivalence, and that equal dyad, 3-pitch-set and 4-pitch-set content are all necessary conditions for 5-pitch-set equivalence, etc. Example: (V) (0 1 3 7) = (V) (0 2 5 6), but the former includes (0 1 3) and (0 3 7), which the latter excludes; the latter includes (0 2 5) and (0 3 4), which the former excludes.

What is suggested here is that the perception of an interval as a distance between two elements of a set is less meaningful than the perception of it as a structured subset. Or, in general, that a pitch set should be defined by nothing more or less than the sum of all its structured subsets. All of which adds up to the simple, unoriginal, proposition that for any two sets A and B,

A ≡ B if and only if
A and B are related by IT

4.0 Inclusion relations based upon equivalence under IT

Here we tabulate, without comment, and simply to display the application of equivalence under IT to the inclusion relation, the proper subsets of (0 1 4 5 8):

(0 1 4 5 8) includes	(0 1) (0 1 4) (0 1 4 5) (0 3) (0 1 5) (0 1 4 8) (0 4) (0 3 7) (0 1 5 8) (0 5) (0 4 8) (0 3 4 7)

The above tabulation is based on the idea that, for a set A and a smaller set B, A includes B if and only if B is related by IT to some subset of A. For example, the elements 3 and 7 do not appear in (0 1 4 5 8), but (0 1 4 5 8) includes (0 3 7) since (0 3 7) is related by IT to (1 5 8).

5.0 Exclusion

A remarkable fact is that a set may be defined more economically by exclusion than by inclusion. There exists for any set A a relatively small collection of excluded sets E_1, E_2, . . ., E_n (with no two E's equivalent under IT and with any two E's mutually exclusive under IT), such that for a set B, of fewer elements than A and excluded by A, B includes at least one E. For example:

for A = (0 1 4 5 8) the collection of E sets is

(0 2) (0 6)

for A = (0 2 4 6 8 10) the collection of E sets is

(0 1) (0 3) (0 5)

for A = (0 2 4 5 7 9 11) the collection of E sets is

(0 1 2) (0 1 6 7) *7
(0 1 4) (0 2 6 8)
(0 4 8) (0 3 6 9)

for A = (0 1 2 3 4 5 7) the collection of E sets is

(0 4 8) (0 2 5 8)
(0 1 5 6)(0 2 6 8)
(0 1 6 7)(0 3 5 8)
(0 1 5 8)(0 3 6 9)

In other words, a set that cannot be found in A contains at least one E set (and conversely, any set which contains an E set is excluded by A). The E sets for any set A might be called "basic excluded sets" of A. In a sense, a set is more neatly characterized by its relatively few types of "enemies" (basic excluded sets) than by its relatively many "friends" (included sets). In a later article the author hopes to develop more formally and in greater depth the notion of basic excluded sets and its applications in analysis.

references

1 Allen Forte, A Theory of Set-Complexes for Music. In: The Journal of Music Theory, VIII/2(1964). Familiarity with Forte's theory is assumed in the present article. The interval vector concept, defined in section 2.1 of Forte, will not be redefined here. As in Forte, pitch sets will be represented numerically, with 0 = any fixed pitch-class.

2 More precisely, what we wish to be able to assert is that the general set of which (0 4 7 10) is one form (of 24 possible transpositional and inversional forms) includes the general set of which (1 5 11) is one form (of 24 possible forms). A similar qualification applies to the next statement in the text.

3 The existence of such sets was first pointed out by Lewin. See David Lewin, The Intervallic Content of a Collection of Notes. . . .. In: The Journal of Music Theory, IV/1(1960). However, Lewin's statement "these are the only possibilities" is not quite correct. As inspection of Forte's Z-forms and computation shows, the sets (0 1 2 5 9) and (0 1 3 4 8) also qualify as "exceptional".

4 In Forte, the sense in which one set does not include another is unclear, since the meaning of an assertion such as "(10-1) and (6-35) are incomparable" is not satisfactorily defined. On page 160 it is translated as "there is no subset of (10-1) such that its interval vector is the same as the interval vector of (6-35)". However, the expression "subset of (10-1)" is unclear. It cannot mean, as it literally seems to, one of the 10-pitch sets in the equivalence class (10-1), for none of these can have the same V as a 6-pitch set. We therefore take it to mean "subset of any pitch set in the set of pitch sets labelled (10-1)". This clarification is by no means trivial, for in some cases, V-type inclusion may be proved by only some pitch sets within a class of sets, as shown in the present example.

5 There are fundamental errors in Forte section 8.0. For example, it is not true that "all three-note sets (with the exception of 3-12) are contained in each seven-note set". Nor is it true, as stated, that any (every) eight-note set includes any (every) four-note set. Two cases, of many: (7-35) $\not\supset$ (3-3), (8-21) $\not\supset$ (4-28).

6 Forte, p. 160.

7 These six E sets plus (0 3 4), the inversion of (0 1 4), are analogous to Boatwright's "chromatic formulae". See Howard Boatwright, Introduction to The Theory of Music (New York, 1956), p. 149.

8 These figures are obtained by tabulating the numbers of equivalent pairs of pitch sets for each interval set, and making the corresponding totals. For example, the 48 different pitch sets represented by the interval set (1 1 1 1 1 1) (set 4-15 in Forte) form 47+46+45. . .+1 = 1128 different equivalent pairs if the condition for equivalence is V equality. However, equivalence based on IT divides the 48 sets into two groups of 24, and the number of equivalent pairs under IT is therefore 2 (23+22+21. . .+1) = 552.