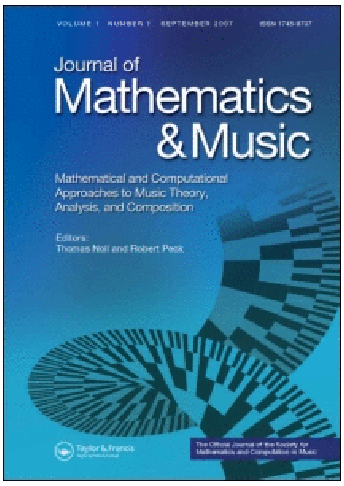


Journal of Mathematics and Music
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t741809807

Coherence and sameness in well-formed and pairwise well-formed scales

Norman Carey[a]
[a] CUNY Graduate Center, New York, NY, USA

To cite this Article Carey, Norman(2007) 'Coherence and sameness in well-formed and pairwise well-formed scales', Journal of Mathematics and Music, 1: 2, 79 — 98
To link to this Article: DOI: 10.1080/17459730701376743
URL: http://dx.doi.org/10.1080/17459730701376743

Journal of Mathematics and Music
Vol. 1, No. 2, July 2007, 79–98

Coherence and sameness in well-formed and pairwise well-formed scales

NORMAN CAREY*

CUNY Graduate Center, 365 Fifth Avenue, New York, NY 10016, USA

(*Received 29 December 2005; in final form 3 August 2006*)

A common theme running through many of the scale studies in recent years is a concern for the distribution of intervals and pitch classes. The question of good distribution becomes increasingly complex with the increase in parameters. Complexity increases when the cardinality, *N*, increases, and when the number of step sizes increases relative to cardinality. Complexity is also shown to be dependent upon the relative sizes of the step intervals. Two measures of scalar complexity are the properties known as difference and coherence. Difference rates a scale according to the number of distinct specific intervals it contains, whereas coherence concerns conflicts between generic and specific intervallic measures. There are two types of conflicts, 'ambiguity' and 'contradiction'. This paper demonstrates that well-formed scales have, as a class, the highest coherence rank—fewest numbers of ambiguities or contradictions—for scales of a given cardinality. They are, then, in this sense, 'minimally complex'. The paper concludes with a conjecture about pairwise well-formed scales, namely that these types are more complex than well-formed ones, but less so than all others.

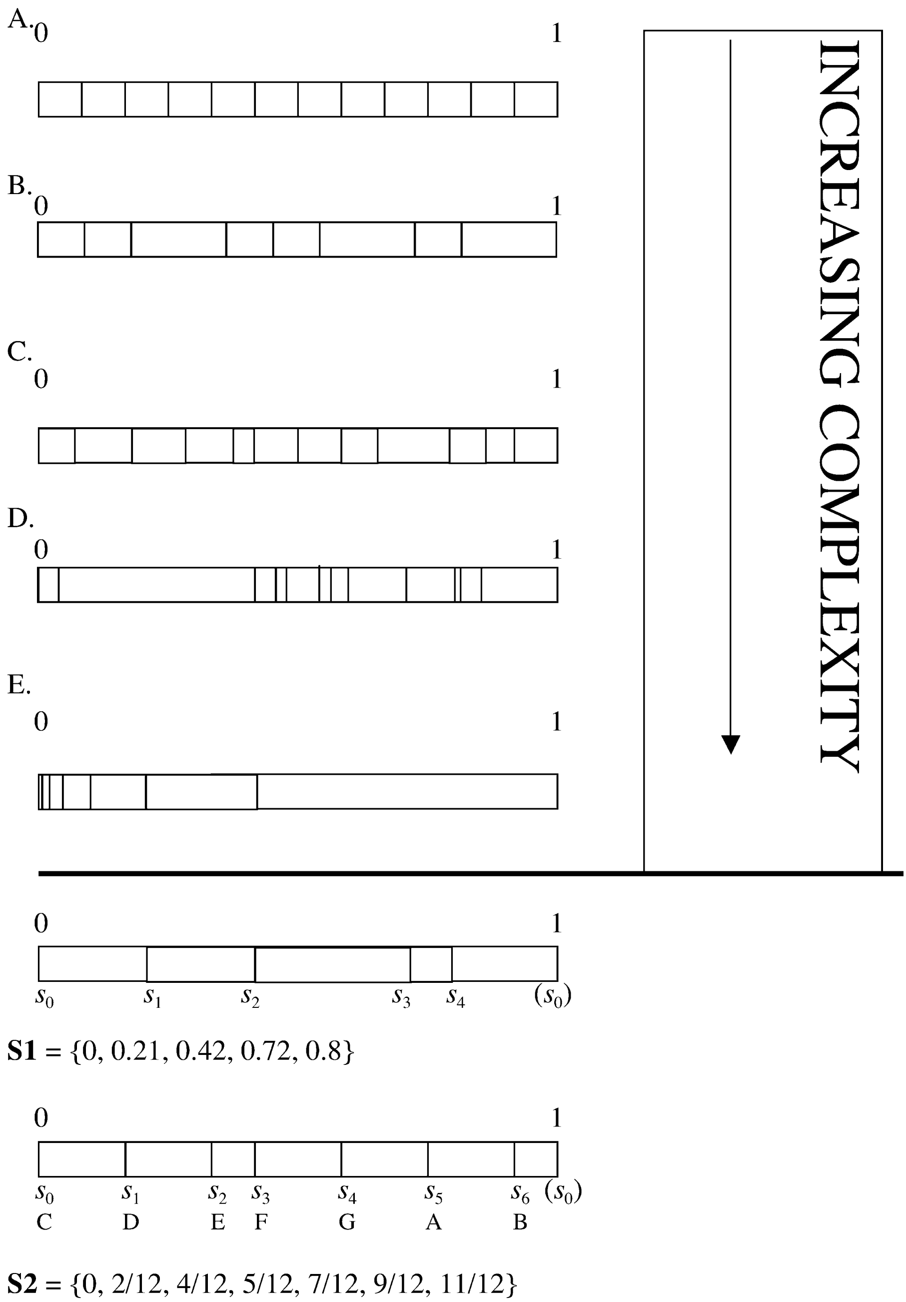
Keywords: Scale theory; Well-formed scales; Distribution modulo 1; Coherence; Fibonacci sequence

1. Models of musical scales

Musical scales may be represented as points lying within the unit interval. Let F be the frequency ratio of the octave, or more generally, the *interval of periodicity*. Let x be some arbitrary pitch, and assign it a frequency of 1 in some unit of measure. Then $\log_F(x) = 0$ and $\log_F(F) = 1$. Given any pitch of frequency ρ, we map ρ onto a number in the unit interval by taking the fractional part of $\log_F(\rho)$, that is, $\log_F(\rho)_{\mathrm{mod}1}$. On this basis, the sets of lines shown in Example 1 may be interpreted as musical scales in the unit interval. The last of these, S(2), corresponds to the usual 'Major' scale.

*Email: ncarey@gc.cuny.edu

Journal of Mathematics and Music
ISSN 1745-9737 print/ISSN 1745-9745 online
http://www.tandf.co.uk/journals
DOI: 10.1080/17459730701376743



Example 1. A–E: Sets of elements modulo 1.
S1: An arbitrary scale.
S2: The Major Scale in equal temperament.

DEFINITION 1. ***Scale**: A sequence* $\mathbf{S}$ *of elements* s_n *modulo 1. If N is the* ***cardinality*** *of* $\mathbf{S}$, *then* $\mathbf{S} = \{s_0 = 0 < s_1 < \ldots < s_{N-1} < 1\}$. *We say that the* s_n *are the* ***notes****, or* ***pitch classes*** *of* $\mathbf{S}$.

1.1. *A* ***mode*** *(or* ***rotation****) of* $\mathbf{S}$: *Let* s_n *be some member of* $\mathbf{S}$.
Then $\mathbf{S}_{\mathbf{MODE}(n)} = \{(s_0 - s_n), (s_1 - s_n), \ldots, (s_{N-1} - s_n)\}$, *is a mode of* ***S****, with elements reduced modulo 1 and reordered to begin on* $(s_n - s_n) = 0$.
1.2. *An* ***inversion*** *of* $\mathbf{S}$: *If* s_n *is some member of* $\mathbf{S}$.
Then $\mathbf{S}_{\mathbf{INV}(n)} = \{(s_n - s_0), (s_n - s_1), \ldots, (s_n - s_{N-1})\}$, *is an inversion of* ***S****, with elements reduced modulo 1 and reversed and reordered to begin on* $(s_n - s_n) = 0$.
$\mathbf{S}$ *is understood to be equivalent to its modes and inversions.*
1.3. *A scale* $\mathbf{S}$ *is* ***invariant under inversion*** *if, for elements* $s_n\ s_m \in \mathbf{S}$, $\mathbf{S}_{\mathbf{MODE}(n)} = \mathbf{S}_{\mathbf{INV}(m)}$.

We can define musical intervals on the ordered pairs of ***S****.*

DEFINITION 2. ***Interval****. An ordered pair* $(s_n, s_m) \in \mathbf{S} \times \mathbf{S}$.

2.1. ***Generic span*** *of an interval.* $\mathrm{SPAN}(s_n, s_m) = (m - n)_{\mathrm{mod}N}$.
2.2. ***Specific size*** *of an interval.* $\mathrm{SIZE}(s_n, s_m) = (s_m - s_n)_{\mathrm{mod}1}$.
2.3. ***Step****. An interval of span 1:* (s_n, s_{n+1}).

The function SPAN returns the *generic* span of the interval, which corresponds with the rough intervallic measure implied by the words 'third', 'fourth', and so on. The function SIZE returns the specific (logarithmic) size of the interval. The functions SPAN and SIZE serve to generalize the concepts of *generic* and *specific* interval, introduced by Clough and Myerson [1] (and implicit in Clough [2]). A *step interval*, or simply, a step, is a generic interval whose span is one.

2. Measures of scalar complexity

A common theme running through many of the scale studies in recent years is a concern for the distribution of intervals and pitch classes [3,4]. The question of good distribution becomes increasingly complex with the increase in parameters. We may invoke the literal translation of *scala* as 'ladder', and actually look at the scales in Example 1 as ladders by turning the page sideways. The ladder model reifies our notion of the generic step in the motion of ascending and descending, rung to rung. I would propose that we would feel safest in climbing ladder A, more and more challenged as we move over to E. On the other hand, the challenges might be fun. The question becomes, what is the best balance between boredom and complexity? A number of parameters come in to play. Complexity increases when the cardinality, *N*, increases, and when the number of step sizes increases relative to cardinality. When the step intervals vary greatly in size, this too adds complexity. Pitch-class sets that have served the role of musical scales are, as a general rule, relatively bland; more like A and B, less like D or E. In this paper, I use the properties of difference and coherence as a means to measure variations in scalar complexity. The concept of difference was introduced by Rahn [5]. An instance of difference exists when two intervals of the same span have different sizes. In equal temperament, the diatonic scale exhibits ten instances of difference among its scale steps: each of the minor seconds forms a difference with each of the five major seconds. There are twelve instances of difference among its thirds, six among its fourths, etc., for a total of 56 in all. The complement of "difference" is "sameness", to be discussed below.

DEFINITION 3. ***Difference***: *A* ***pair*** *of intervals* (s_a, s_b) *and* (s_c, s_d) *forms an instance of* ***difference*** *whenever* SPAN(s_a, s_b) = SPAN(s_c, s_d) *and* SIZE(s_a, s_b) ≠ SIZE(s_c, s_d).

DEFINITION 4. *The function* D(**S**) *returns the* ***number of differences in S***.

THEOREM 1. *The maximal number of differences for a scale* **S** *of cardinality N is the function* max D(N), *where* max $\mathrm{D}(N) = \dfrac{(N)(N-1)(N-1)}{2}$.

Proof: Carey [6], pp. 10–11. ■

Counting the total number of differences in a scale gives some intuition regarding a scale's complexity. Theorem 1 shows that the maximum number of differences for a scale of cardinality N is a function on N. Definitions 4 and 5 serve to introduce the *sameness quotient*.

DEFINITION 5. $SQ(\mathbf{S}) = 1 - \dfrac{D(\mathbf{S})}{\max \mathrm{D}(N)}$ *is the* **sameness quotient of *S***.

The higher the sameness quotient, the simpler the scale. For a given cardinality N, equal-interval scales are the simplest by this measure and those with maximal differences are the most complex. The sameness quotient of an equal-interval scale, **E** (such as the twelve-tone scale of western music) is unity, because D(**E**) = 0. (By definition, equal interval scales exhibit no differences.) Conversely, scales with a sameness quotient of zero are those with maximal differences.

Another way in which complexity arises in scales is when the spans and sizes of intervals are in conflict. Our intuition would lead us to expect that the largest step in a scale will be smaller than any third, all thirds smaller than any fourth, and so on. A scale is *coherent* when this is uniformly the case:

DEFINITION 6. ***Coherence*** *(of* **S***): The scale* **S** *is coherent if SPAN*(s_a, s_b) < SPAN(s_c, s_d) *implies* SIZE(s_a, s_b) < SIZE(s_c, s_d) *for all* **S** × **S**.

6.1. ***Coherence failure***: *The interval pair* ((s_a, s_b), (s_c, s_d)) *is an instance of coherence failure if* SPAN(s_a, s_b) < SPAN(s_c, s_d) *and* SIZE(s_a, s_b) ≥ SIZE(s_c, s_d).

Coherence may fail in one of two ways as indicated by '≥' in Definition 6.1:

6.2. ***Ambiguity***: *The interval pair* ((s_a, s_b), (s_c, s_d)) *is an ambiguity if* SPAN(s_a, s_b) ≠ SPAN(s_c, s_d) *and* SIZE(s_a, s_b) = SIZE(s_c, s_d).
6.3. ***Contradiction***: *The interval pair* ((s_a, s_b), (s_c, s_d)) *is a contradiction if* SPAN(s_a, s_b) < SPAN(s_c, s_d) *and* SIZE(s_a, s_b) > SIZE(s_c, s_d).

Coherence fails either due to *ambiguity*, in which two intervals of differing spans have the same size, or due to *contradiction* in which, for intervals A and B, A has a larger span, but B has a larger size. This definition of coherence accords with that provided in Balzano [7]. In Agmon [8], coherence signifies the lack of contradiction only. Further discussion is to be found in Agmon [9]. Coherence was described earlier in Rothenberg [10] with different terms. Rothenberg uses the term "strictly proper" to describe coherent scales, "proper" for scales with ambiguities but no contradictions, and "improper" for scales with contradictions.

The presence of coherence failures, ambiguities and/or contradictions, is not only not detrimental to a scale, but possibly quite beneficial. Many of the sets shown in Example 1 contain coherence failures. The only claim being made here is that they invariably add complexity to a pitch structure and that there might be some cognitive limit on how much complexity a scale can tolerate. In equal-tempered tuning, the major scale contains the ambiguous interval of the tritone. The tritone, whose specific size is exactly half that of the octave, can be found in two generic guises, as an augmented fourth (F to B), or as a diminished fifth (B to F.) The span in the first case is 3, and of the second, 4. The presence of this single ambiguity enriches the diatonic scale, adds some complexity, but also opens up opportunities for enharmonic relationships.

THEOREM 2. *The maximal number of coherence failures for a scale* **S** *of cardinality* N *is the function* maxF(N), *where*

$$\mathrm{maxF}(N) = \frac{N(N-1)(N-2)(3N-5)}{24}.$$

Proof: Carey [6], pp. 14–17. ■

For the sequel, we assume that $N \geq 3$, in order to ensure that max F(N) returns only non-zero values. Not much is lost by eliminating 1- and 2-note scales from our discussion. Definitions 7 and 8 establish the *coherence quotient*, in parallel to the sameness quotient. An algorithm for F(**S**) is given in Carey [6], p. 20.

DEFINITION 7. F(**S**) *returns the* ***number of coherence failures in S***.

DEFINITION 8. $CQ(\mathbf{S}) = 1 - \dfrac{F(\mathbf{S})}{\mathrm{maxF}(N)}$ *is the* ***coherence quotient of S***.

The scale illustrated in Example 1A has no coherence failures and no differences. Thus, both the coherence quotient and the sameness quotient of this set is unity. By contrast the scale illustrated in Example 1E has maximal coherence failure and maximal difference for a 7-note set. Thus, both the coherence quotient and the sameness quotient of this set is zero.

3. Generated scales

One class of scales that has received considerable attention, not only recently, but also historically, is the class of generated scales.

DEFINITION 9. ***Generated scale***: *A scale,* $S_{(\theta)(N)}$, *such that, for positive real number* θ *and integer* N, $S_{(\theta)(N)} = \{n\theta_{\mathrm{mod}\ 1} | n \in \mathbf{Z}_N\}$.

Generated scales are well modelled by the kinds of sets described by the so-called 'three-gap theorem', formerly known as the Steinhaus conjecture. See Halton [11], Sós [12], and Carey [13], pp. 115–120. If θ is irrational, then the number of such sets $S_{(\theta)(N)}$ is unlimited. (Given integers n and m, if $n \neq m$ then $n\theta_{\text{mod } 1} \neq m\theta_{\text{mod } 1}$.) If θ is rational $= p/q$, then the largest set $S_{(\theta)(N)}$ is determined by $N = q$. (For integers n and k, $n\left(\frac{p}{q}\right)_{\text{mod}1} = (kq+n)\left(\frac{p}{q}\right)_{\text{mod}1}$.) Thus, $S_{\left(\frac{p}{q}\right)(q)}$ contains exactly one step size, which models the equal-interval scale of cardinality q. Let $\theta = 7/12$, and consider the sets $S_{\left(\frac{7}{12}\right)(N)} = \{n(7/12) \mid n \in \mathbf{Z}_N\}$. First, consider $N = 5$. Then $S_{\left(\frac{7}{12}\right)(5)}$ contains five elements, 0, 2/12, 4/12, 7/12, 9/12. (Multiplying these values by 1200 gives the cents values for the anhemitonic pentatonic scale.) There are two step sizes here, 2/12 and 3/12. Now let $N = 6$. The set $S_{\left(\frac{7}{12}\right)(6)}$ contains the five elements above and adds a new element, 11/12. This six-element set corresponds to the diatonic or 'Guidonian' hexachord. It contains three step sizes, the two previous ones, 2/12 and 3/12, and now also 1/12. Note that of these three, the largest (3/12) is the sum of the smaller two (1/12 and 2/12). The proof of the three-gap theorem demonstrates that this is always the case when the set yields three step sizes: the largest step size, call it x, is the sum of the smaller two step sizes, y and z. Note, then, that such sets necessarily contain at least one ambiguity: the step or 'second' of size x is equal to the 'third' of size $y+z$. Carl Dahlhaus [14] comments on this phenomenon in the case of the diatonic hexachord: 'The pentatonic and heptatonic scales are systems. In comparison, the hexachord is a mere auxiliary construction. As a system it would be self-contradictory. The procedure of filling in the *d-f* but not the *a-c* third with an intermediate degree would, if conceived as a principle of the system, lead to the absurd consequence that the listener would have to alternate between the idea of the minor third as a "step" and as a "leap".'

Continuing with our example, when $N = 7$, the seven elements correspond with Dahlhaus's heptatonic system, that is, with the values of the diatonic scale: 0, 2/12, 4/12, 6/12, 7/12, 9/12, 11/12. Again there are just two step sizes, 1/12 and 2/12. As Dahlhaus suggests, musical scales that are generated and that contain two step sizes have been found to have greater musical utility than generated scales with three step sizes. Furthermore, the inherent ambiguity and greater number of 'differences' in these sets may explain their lack of favour.

When θ is irrational, a set with two step sizes is the result of choosing an $N\,(>1)$ that corresponds with the denominator of a convergent or intermediate convergent of the continued fractions expansion of θ. Presently we will ascribe the term 'well-formed' to sets with N so restricted. Well-formed scales were originally described in Carey and Clampitt [15]. When N is not such a convergent, the resulting set has three step sizes. In the case of a rational θ the case is the same except for values of N between the penultimate (intermediate) convergent and q. In our example, the penultimate intermediate convergent of 7/12 is 4/7. That is, when $N = 8$, 9, 10, or 11, N is not an intermediate convergent of 7/12. Assuming the results of irrational θ the resulting sets $S_{\left(\frac{7}{12}\right)(N)}$ would be expected to have three step sizes. In fact, because 7/12 is rational, these sets have only two step sizes. Nonetheless, except for the last of those, all exhibit ambiguities. When $N = 11$, or, more generally, when $N = q-1$, the generated set

$S_{\left(\frac{p}{q}\right)(q-1)}$ contains no ambiguities. I refer to this case as the *Wooldridge Anomaly* after Marc Wooldridge, who first pointed this out [16].

We see, then, that not all generated scales have proven to be of equal musical utility. Generated scales that are also *well-formed* scales appear to be the most privileged in this regard (see Carey and Clampitt [15]). Based upon the above arguments, it should be clear that in generated sets, coherence implies well-formedness. The converse is not true: there are well-formed sets with ambiguities and/or contradictions; however, their number is restricted by well-formedness, as we shall see.

Definition 10. ***Well-formed scale***: *A generated scale $S_{(\theta)(N)}$ in which all intervals of size θ are of constant span K. (Using continued fractions it is possible to show that*

$$\frac{K}{N} \approx \theta.$$

See Appendix for continued fraction conventions.)

10.1. ***Non-degenerate well-formed scale***: *A well-formed scale in which there is exactly one interval of span K that is not of size θ.*
10.2. ***Degenerate well-formed scale***: *A well-formed scale in which all intervals of span K are of size θ. An equal-interval scale. Also: any scale* **S** *in which* $\mathrm{D}(\mathbf{S}) = 0$.

A given generator θ engenders a hierarchy of well-formed scales.

10.3. H_θ *the set (or hierarchy) of well-formed scales generated by θ.*

For example, $\mathrm{H}_{\frac{7}{12}}$ contains the black-key pentatonic scale and the white-key diatonic scale, as embedded in the twelve-tone chromatic scale. As noted in Theorem 3, the cardinalities of the well-formed scales in a given hierarchy are determined by the series of denominators of the full and intermediate convergents in the continued fraction expansion of θ.

Theorem 3. *Let $S_{(\theta)(N)}$ be a scale of cardinality N generated by θ. $S_{(\theta)(N)} \in H_\theta$ if and only if N is the denominator of a full or an intermediate convergent in the continued fraction of θ.*

Proof: Carey [13], pp. 125–126. ■

Corollary. *We may construct the elements of well-formed scale $S_{(\theta)(N)}$ in order, given θ and N: Let $1 < t \leq t_k$. Then, by Theorem 3, N is the denominator of at least an intermediate convergent of θ, and so $N = b_k^t$ with $1 \leq t \leq t_k$. If k is odd, then let $g = b_{k-1}$. If k is even, then let $g = N - b_{k-1}$. Then scale $S_{(\theta)(N)} = \{((ng)_{mod\ N}\theta)_{mod\ 1} | n \in Z_N\}$ is well-formed. The elements of $S_{(\theta)(N)}$ are given by,* $s_0 = ((0g)_{\mathrm{mod}N}\theta)_{\mathrm{mod}1} < s_1 = ((1g)_{\mathrm{mod}N}\theta)_{\mathrm{mod}1} < \ldots < s_{(N-1)} = (((N-1)g)_{\mathrm{mod}N}\theta)_{\mathrm{mod}1}$.

Proof: Carey [13], pp. 123–124 (see Appendix for continued fraction conventions). ■

Clough and Myerson [1,17] define *Myhill's Property* as obtaining in scales where for every non-zero span there exist intervals of two distinct sizes:

DEFINITION 11. ***Myhill's Property***: *A scale has Myhill's Property if there exist exactly two interval sizes for each non-zero span.*

THEOREM 4. *A scale has Myhill's Property if and only if it is non-degenerate well-formed.*

Proof: Carey and Clampitt [18], pp. 66–71. See also Carey [13], pp. 126–128. ■

DEFINITION 12. ***Multiplicity***: *The number of different intervals of a given span and size.*

Example: The multiplicity of minor thirds in the diatonic scale is four.

DEFINITION 13. ***The class of well-formed scales***, *WF(N, g). The class of N-note well-formed scales whose larger step interval has multiplicity g.*

For example, the diatonic scale in equal temperament, Pythagorean tuning, and mean-tone temperament belong to distinctly different hierarchies, but all are members of WF(7, 5). The value g in Definition 13 derives from Theorem 3. If N is the denominator of a full or intermediate convergent of θ, then g (or $N-g$) is the denominator of previous full or intermediate convergent. Thus, well-formed scales that belong to WF(N, g) are generated by values of θ that have identical continued fraction expansions up until the partial quotient associated with the convergent whose denominator is N.

4. Difference and coherence in well-formed scale s

In Carey [6], I show that a well-formed scale of cardinality N has significantly lower maxima for both differences and coherence failures. (The table in Example 2 illustrates.) The following definitions apply to non-degenerate well-formed scales:

DEFINITION 14a. μ: *The size of the larger step interval in a well-formed scale.*

DEFINITION 14b. ν: *The size of the smaller step interval in a well-formed scale.*

As well as describing the two step-interval sizes in a well-formed scale as larger and smaller, we also note that, with $N>2$, they are never equal in multiplicity, as a consequence of Theorem 3 and its corollary. Furthermore, the rare step interval never appears twice in succession.

Following Blackwood [19], I use the letter R to represent the ratio between the sizes of the larger and smaller step intervals:

DEFINITION 15. $R=\frac{\mu}{\nu}$. *The ratio of the larger step interval to the smaller.*

The ratio R parameterizes elements in WF(N,g):

N	MaxD (N)	WF Diff	SQ (WF)	PWWF Diff	SQ (PWWF)	MaxF (N)	MAX WF F	CQ (Max WF F)	Max PWWF F	CQ (Max PWWF F)
3	6	4	0.33333	6	0.00000	1	1	0.00000	1	0.00000
4	18	10	0.44444			7	5	0.28571		
5	40	20	0.50000	30	0.25000	25	15	0.40000	17	0.32000
6	75	35	0.53333			65	35	0.46154		
7	126	56	0.55556	84	0.33333	140	70	0.50000	82	0.41429
8	196	84	0.57143			266	126	0.52632		
9	288	120	0.58333	180	0.37500	462	210	0.54545	250	0.45887
10	405	165	0.59259			750	330	0.56000		
11	550	220	0.60000	330	0.40000	1155	495	0.57143	595	0.48485
12	726	286	0.60606			1705	715	0.58065		
13	936	364	0.61111	546	0.41667	2431	1001	0.58824	1211	0.50185
14	1183	455	0.61538			3367	1365	0.59459		
15	1470	560	0.61905	840	0.42857	4550	1820	0.60000	2212	0.51385
16	1800	680	0.62222			6020	2380	0.60465		
17	2176	816	0.62500	1224	0.43750	7820	3060	0.60870	3732	0.52276
18	2601	969	0.62745			9996	3876	0.61224		
19	3078	1140	0.62963	1710	0.44444	12597	4845	0.61538	5925	0.52965
20	3610	1330	0.63158			15675	5985	0.61818		
21	4200	1540	0.63333	2310	0.45000	19285	7315	0.62069	8965	0.53513
22	4851	1771	0.63492			23485	8855	0.62295		
23	5566	2024	0.63636	3036	0.45455	28336	10626	0.62500	13046	0.53960
24	6348	2300	0.63768			33902	12650	0.62687		
25	7200	2600	0.63889	3900	0.45833	40250	14950	0.62857	18382	0.54330

Example 2. Results for N from 3 to 25.

Definition 16. ***The well-formed scale,*** wfs(N, g, R). *A well-formed scale with parameters N, g, and R.* wfs(N, g, R) $\in$ WF(N, g).

The number of differences in a well-formed scale is entirely dependent upon N. Theorem 5 and a corollary provide the number of differences in a well-formed scale and its sameness quotient:

Theorem 5. *The number of differences in a well-formed scale*

$$D(\mathrm{wfs}(N, g, R)) = \frac{(N-1)(N)(N+1)}{6}.$$

Corollary. *The sameness quotient of a well-formed scale: By Theorem 1, the maximum number of differences for a scale of cardinality N is* $\frac{(N)(N-1)(N-1)}{2}$*. Then*

$$SQ(\mathrm{wfs}(N, g, R)) = 1 - \frac{D(\mathrm{wfs}(N, g, R))}{\max\mathrm{D}(N)} = 1 - \frac{\frac{(N-1)(N)(N+1)}{6}}{\frac{(N)(N-1)(N-1)}{2}} = \frac{2(N-2)}{3(N-1)}.$$

As N increases, the sameness quotient approaches 2/3.

The number of ambiguities, and the number of total coherence failures can be determined as functions requiring the three parameters N, g, and R:

Theorem 6. *Coherence, Ambiguity and Contradictions in well-formed scales. For proofs, see Carey* [6], *pp. 22–31.*

6.1. *If* $g = N-1$ *then wfs(N, g, R) is coherent.*
6.2. *If* $1 \leq R < 2$ *then wfs(N, g, R) is coherent.*
6.3. *If wfs(N, g, R) has ambiguities, then* $R \in Z$ *and* $2 \leq R < \frac{N-1}{g} + 1$.

The number of ambiguities in such cases is:

$$\frac{((N-1) - g(R-1))((N) - g(R-1))((N+1) - g(R-1))}{6}$$

6.4. *If* $R > 2$ *and* $g < N-1$, *then wfs(N, g, R) contains contradictions.*
For real number x, let[x] *mean 'the greatest integer* ***less than or equal to x****'. Let* $\lfloor x \rfloor\downarrow$ (*the 'sub-floor' function*) *mean 'the greatest integer* ***strictly less than x****'.* (*For all x,* $\lfloor x \rfloor\downarrow < x$.) *Let* $J = \min\left([R-1], \left\lfloor \frac{N-1}{g} \right\rfloor \downarrow\right)$.

6.4.1. *The total number of coherence failures in* wfs(*N, g, R*) *is*

$$\mathrm{F}(\mathrm{wfs}(N, g, R)) = \sum_{k=1}^{J} \frac{(N-1-gk)(N-gk)(N+1-gk)}{6}.$$

6.4.2. *The number of contradictions in wfs(N, g, R): same as the value given in 6.4.1 above, but with J redefined as:* $J = \min\left(\lfloor R - 1 \rfloor\downarrow, \left\lfloor \frac{N-1}{g} \right\rfloor \downarrow\right)$.

Note that the cases shown in Theorems 6.4.1 and 6.4.2 will only differ when R is an integer. When R is an integer, the value of J is, potentially, one less than if it is not.

I will refer to the situation described in Theorem 6.1 as 'the special case of well-formed coherence'. To paraphrase, whenever the smaller interval is of multiplicity $N-1$, the well-formed scale is coherent, whatever the value of R may be. Theorem 6.2 is, then, 'the general case of well-formed coherence'. It sets $R<2$ as an upper limit for coherent well-formed scales. Theorem 6.3 first asserts that a well-formed scale has ambiguities only when R is an integer, and then, when such is the case, gives the number of ambiguities. Theorem 6.4 says that, except for the special case shown as Theorem 6.1, a well-formed scale with R greater than 2 will contain contradictions. Theorem 6.4.1 gives the total number of coherence failures; ambiguities plus contradictions. A slight retooling of Theorem 6.4.1 yields a way to count the number of contradictions only, shown as Theorem 6.4.2.

6.5. *If* $R \geq N-1$ *and* $g=1$, *then* wfs(*N, g, R*) *has the greatest number of coherence failures for a well-formed scale of cardinality N. Under these conditions,* $J=N-2$, *and the result of Theorem 6.4.1 becomes* $\frac{(N-2)(N-1)(N)(N+1)}{24}$. *By Theorem 2,*

$$\mathrm{maxF}(N)\frac{N(N-1)(N-2)(3N-5)}{24}.$$

Then

$$\mathrm{CQ}(\mathrm{wfs}(N, 1, R(\geq N-1)) = \left(1 - \frac{F(\mathrm{wfs}(N, 1, N-1))}{\mathrm{maxF}(N)}\right) = \left(1 - \frac{\frac{(N-2)(N-1)(N)(N+1)}{24}}{\frac{N(N-1)(N-2)(3N-5)}{24}}\right)$$

$$= \frac{2(N-3)}{3N-5}.$$

Contradictions and ambiguities proliferate in well-formed scales only when the two step intervals are significantly different in size, and when the smaller steps outnumber the larger ones. As a class, however, well-formed scales are significantly freer from coherence failures than the average. For a given N, the well-formed scales that have the most coherence failures are those in WF(N,1) in which $R \geq N-1$. In such scales, there is exactly one larger step interval that is at least $N-1$ times larger than the smaller one. (A chromatic cluster of seven pitch classes in the 12-tone scale is a case in point.)

Theorem 6.5 shows that as N increases the coherence quotients of even the least coherent well-formed scales rise towards but never exceeds 2/3.

A more comprehensive understanding of coherence in well-formed scales is possible through the perspective of continued fractions. We saw in Theorem 3 that in order for a scale to be well-formed, it must be generated by some value θ, and its cardinality, N, must be the denominator of a convergent or intermediate convergent of θ. These full and intermediate convergents may be themselves expressed as continued fractions, namely the finite initial segments of the continued fraction of θ. The ratio R may be recovered as the remainder of the continued fraction of θ, with a slight modification at the point of intersection:

THEOREM 7. *Let θ be a number in the unit interval. Then* $\theta = [t_0 = 0, t_1, t_2, \ldots, t_k \ldots]$. *Given well-formed scale* wfs(N, g, R) $\in H_\theta$, if $N = b_k^t$ where $\frac{a_k^t}{b_k^t} = [t_0, t_1, ..., t_{k-1}, t]$, *then* $g = (b_{k-1})$, *and* $R = [(t_k + 1 - t), t_{k+1}, t_{k+2}, \ldots]$.

Proof: Carey [13]. ■

THEOREM 8. *If N is the denominator of a full convergent of θ, then* wfs(N, g, R) *is coherent, that is,* CQ(wfs(N,g,R)) $= 1$.

Proof: $\frac{p_k}{q_k} = [t_0, t_1, \ldots, t_k]$ and so, by Theorem 7, R $= [(t_k + 1 - t_k),\ t_{k+1},\ t_{k+2}, \ldots.] = [1, t_{k+1}, t_{k+2}, \ldots.]$.

Thus, $1 < R < 2$, and, by Theorem 6.2, wfs(N, g, R) is coherent. ■

The value R is greatly significant in determining the coherence profile of a well-formed scale. Note that R, by virtue of the value shown in Theorem 7, is always irrational if the continued fraction of θ is non-terminating (that is, if θ is irrational). R is always rational when θ is rational, and R is always an integer for the intermediate convergents and the full convergent determined by the final partial quotient. Concordant with the special status of convergents as best approximations of the value of the continued fraction, all scales determined by full convergents are coherent.

Theorem 6.3 implies that ambiguity occurs relatively rarely in well-formed scales. For this to be the case, it is necessary but not sufficient for the real number R to take an integer value. The case occurs only in finite hierarchies, that is, when θ is rational, and in scales associated with the final term of the continued fraction. Only in these cases will the continued fraction of R contain only one term, hence only then will R be an integer. Contradictions are more common in well-formed scales than ambiguities, and will arise, generally, when N is the denominator of an intermediate convergent of θ.

Example 3 presents the finite hierarchy of well-formed scales $H_{\frac{179}{306}}$, The value $\frac{179}{306}$ is a very close approximation to the pure fifth of Pythagorean tuning (the difference being less than a hundredth of a cent).

The column labelled t_k shows the partial quotients of the continued fraction of $\frac{179}{306}$, and the next column, labelled N, shows the denominators of the full and intermediate

k	t_k	N	g	μ_1	ν_1	R	Differences	Ambiguities	Contradictions
1.	1	**2**	1	701.9 (P5)	498.0 (P4)	**1.409**	1	0	0
		***3**	2	498.0 (P4)	203.9 (M2)	*2.442*	4	0	0
2.	2	**5**	2	294.1 (m3)	203.9 (M2)	**1.442**	20	0	0
		7	5	203.9 (M2)	90.2 (m2)	2.260	56	1	0
3.	2	**12**	5	113.7 (A1)	90.2 (m2)	**1.260**	286	0	0
		17	12	90.2 (m2)	23.5 (d2)	3.833	816	0	20
		29	12	66.7	23.5 (d2)	2.833	4,060	0	816
4.	3	**41**	12	43.1	23.5 (d2)	**1.833**	11,480	0	0
5.	1	**53**	41	23.5	19.6	**1.120**	24,804	0	0
		94	53	19.6	3.9	5.0	138,415	0	11,480
		147	53	15.6	3.9	4.0	529,396	0	149,895
		200	53	11.7	3.9	3.0	1,333,300	138,415	529,396
		253	53	7.8	3.9	2.0	2,699,004	1,333,300	0
6.	5	**306**	253	3.9	3.9	**1.0**	0	0	0

Bold:	**Coherent**	**Determined by full convergents**
Plain:	Not coherent	Determined by intermediate convergents
Italics:	**Special case:*	*Coherent, but determined by Intermediate convergent: smaller step appears once.*

Example 3. The hierarchy of well-formed scales H_θ with $\theta = \frac{179}{306} = [0,1,1,2,2,3,1,5] = 0.58496732\ldots$

convergents. Denominators of full convergents are in bold. The table shows that each scale associated with a full convergent is coherent, and so both the number of ambiguities and contradictions are zero. In general, it may be noted how relatively free this hierarchy is of ambiguities and contradictions, even as differences increase.

COROLLARY 8.1. *Let* $\phi = \frac{\sqrt{5}-1}{2} = [0, \bar{1}] = .618033989$. *For all* wfs($N$, g, R) $\in H_{\phi}$, CQ(wfs(N, g, R)) $= 1$ *and* $R = 1/\phi$.

COROLLARY 8.2. *Let* $\zeta = \frac{15-\sqrt{5}}{22} = [0, 1, 1, 2, \bar{1}] = 0.580178728....$ *For all* wfs(N, g, R) $\in H_{\zeta}$, CQ(wfs(N, g, R)) $= 1$, *and, if* $N \geq 7$, $R = 1/\phi$.

Corollaries 8.1 and 8.2 concern the two hierarchies of well-formed scales which contain *only* coherent scales. The first is H_{ϕ}, where ϕ stands for the golden number, $\frac{\sqrt{5}-1}{2}$, The non-zero partial quotients of its continued fraction consist entirely of ones, and so it yields no intermediate convergents. A hierarchy based upon the standard octave and a generator of about 741 cents embodies this proportion. The cardinalities of the scales in this hierarchy are all of the members of the Fibonacci Series, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc. This is also the only hierarchy in which R is constant in every scale. The continued fraction of ϕ is $[0, \bar{1}]$ (the line over the final one meaning to repeat without termination). Therefore $R = 1/\phi$ in every scale of this hierarchy, and, as $1/\phi$ is between 1 and 2, every scale in the hierarchy is coherent. The other hierarchy in which *all* of the scales are coherent is generated by $\frac{15-\sqrt{5}}{22}$. This is a most remarkable value, because the interval that generates the scale is approximately equal to 696.21¢. Recall that the perfect fifth of quarter-comma mean-tone tuning ($\sqrt[4]{5}$ as a frequency ratio) is equal to 696.58¢, which is far less than one cent different from the above value. This is Kornerup's tuning [20], which yields the cardinalities 1, 2, 3, 5, 7, 12, 19, 31, 50, 81, etc. These are also the cardinalities in Yasser's system [21]. The continued fraction of $\frac{15-\sqrt{5}}{22}$ is $[0, 1, 1, 2, \bar{1}]$. This continued fraction produces only one intermediate convergent; however, the three-note scale determined by that convergent is coherent, as it satisfies the conditions for the special case. The seven-note scale in this hierarchy is wfs(7, 5, $1/\phi$). This is a diatonic scale in a mean-tone type tuning with $1/R = \phi$. In fact, $1/R = \phi$ for the all of the scales in the remainder of the hierarchy as well. There are then, essentially two hierarchies of well-formed scales that contain only coherent scales. It is rather astounding that one of them corresponds so closely to mean-tone tuning.

5. Trivalent scales

If well-formed scales represent the best distribution of two differently-sized scale steps in the unit interval, we may be interested in generalizing this problem for three step sizes.

Generalizing to sets with four or more step sizes is considerably more difficult. I will end the paper with some conjectures that lead towards these kinds of generalizations. Sets with three step sizes are already richly diverse in structure. The most interesting scales with three step sizes might be those that instantiate a generalized form of Myhill's property. David Clampitt [22] uses the term *trivalence* to describe such sets.

DEFINITION 17. ***Trivalence***. *A scale has trivalence if there are exactly three distinct interval sizes for each non-zero span.*

The diatonic in Just Intonation is trivalent. Clampitt further proposes the category, *pairwise well-formed scale*:

DEFINITION 18. ***Pairwise well-formed scale***. *A scale* **S** *of odd cardinality whose step intervals come in three sizes,* A, B, *and* C, *which yields a non-degenerate well-formed scale when any pair of step sizes is taken to be equivalent. That is, taking* A *equivalent to* B, *together with* C, *and taking* A *equivalent to* C, *together with* B, *and* B *equivalent to* C, *together with* A, *in each case yields the step pattern of a non-degenerate well-formed scale.*

The diatonic scale under Just Intonation is not only trivalent, but also an example of a pairwise well-formed scale. The scale displays the pattern ABCABAC, where A corresponds to the major tone, 9/8, B to the minor tone, 10/9, and C to the half step, 16/15. If A and B are taken to be equivalent, the pattern becomes AACAAAC. If B and C are taken to be equivalent, the pattern becomes ABBABAB. If A and C are taken to be equivalent, the pattern becomes ABAABAA. In all three cases, the resultant pattern is that of some well-formed scale.

18.1. ***Singular pairwise well-formed scale***. *A pairwise well-formed scale with the pattern,* A B A C A B A. *For the sequel, this scale is excluded from our consideration of PWWF scales.*

An instance of this scale is the so-called 'gypsy' scale, or 'Hungarian minor', G-A-B^b-C$^\#$-D-E^b-F$^\#$-G. Interval A here would be the half-step, interval B the augmented second, and interval C the whole step. This scale has a number of interesting properties, but, as its name implies, is an outlier with respect to the class of PWWF scales. It is excluded from the discussion that follows.

18.2. ***Non-singular pairwise well-formed scale***. *A pairwise well-formed scale that contains a step interval of odd multiplicity* (A)*, and two steps of equal multiplicity* (B *and* C).

Consider again the diatonic in Just Intonation. It is a non-singular PWWF scale with step pattern ABCABAC. Step interval A is of odd multiplicity (3), and the multiplicities of step intervals B and C are equal (2 each). The characterization of step multiplicities in Definition 18.2 is implicit in Clampitt's dissertation: His Theorem 3.2 proves that '[a]ll pairwise well-formed scales are of odd cardinality', and his Theorem 3.6 shows that those of the non-singular type 'have two step intervals of the same multiplicity' (Clampitt [22], pp. 93, 101). Taking these two together yields the description found in Definition 18.2: If two step multiplicities are equal, and if the cardinality of the scale is odd, then the multiplicity of the third interval must be odd.

We can define classes of pairwise well-formed scales similarly to the definition of well-formed scale classes:

DEFINITION 19. PWWF(N, G)*: The class of (non-singular) pairwise well-formed scales of cardinality N whose 'A' step interval appears G times. Consequently,* G *is always odd, and the multiplicities of both 'B' and 'C' are* $(N-G)/2$.

Example 4 shows how to convert the pattern of a given well-formed scale class into that of a pairwise well-formed one. In any well-formed scale of odd cardinality, one step type is of even multiplicity, called 'B' in the example. The step of odd multiplicity, A, maps into the 'A'-type in the PWWF scale (indicated by white arrows). Meanwhile, the step of even multiplicity, B, alternately maps into 'B'- and 'C'- types in the PWWF scale (indicated by dark arrows).

THEOREM 9. *The set of interval multiplicities in a pairwise well-formed scale is, in some order,* $\{(N-2,1,1),(N-2,1,1),(N-4,2,2),(N-4,2,2),\ldots(1,\frac{N-1}{2},\frac{N-1}{2}),(1,\frac{N-1}{2},\frac{N-1}{2})\}$. *This follows from the definition of a non-singular pairwise well-formed scale and Clampitt's Theorems 3.3 and 3.6 ([22], pp. 94–95, 101–104).*

In the diatonic under just intonation, the steps come in multiplicities of (3, 2, 2): There are three major tones, 9/8, two minor tones, 10/9, and two half steps, 16/15. The multiplicities of the thirds are (1,3,3): There is one syntonic minor third, 40/27, three just minor thirds, 6/5, and three just major thirds, 5/4. The multiplicities of the fourths are (5,1,1): There are five perfect fourths, 4/3, one syntonic fourth, 27/20, and one augmented fourth, 45/32. These multiplicities are duplicated for sevenths, sixths, and fifths, respectively. Thus, the set of multiplicities in the just scale is $\{(5,1,1),(5,1,1),(3,2,2),(3,2,2),(1,3,3),(1,3,3)\}$.

Since a pairwise well-formed scale is of odd cardinality, it always contains an even number of (non-zero) generic intervals. This information allows us to calculate the number of differences in a pairwise well-formed scale of cardinality N.

THEOREM 10. $\mathrm{D}(\mathrm{PWWF}(N,G)) = \dfrac{(N-1)(N)(N+1)}{4}$. *This follows directly from Theorem 9. Multiplying and adding the multiplicities gives*

$$2\left((2(N-2)+1^2)+(2(N-4)+2^2)+\ldots+\left(2+\left(\frac{N-1}{2}\right)^2\right)\right)$$

$$=2\left(\sum_{i=1}^{\frac{N-1}{2}} 2Ni-3i^2\right)=\frac{(N-1)(N)(N+1)}{4}.\mathrm{QED}.$$

COROLLARY 10.1. *The sameness quotient of a pairwise well-formed scale: By Theorem 1, the maximum number of differences for a scale of cardinality N is*

$$\frac{(N)(N-1)(N-1)}{2}.$$

Then

$$\mathrm{SQ}(\mathrm{PWWF}(N,G)) = 1-\frac{\mathrm{D}(\mathrm{PWWF}(N,G))}{\mathrm{maxD}(N)} = 1-\frac{\frac{(N-1)(N)(N+1)}{4}}{\frac{(N)(N-1)(N-1)}{2}} = \frac{N-3}{2(N-1)}.$$

As N increases, the sameness quotient of pairwise well-formed scales rises asymptotically towards 1/2.

A.
WF(7,5)/WF(7,2)

A	A	B	A	A	A	B
⇩	⇩	⬇	⇩	⇩	⇩	⬇

PWWF(7,5)

A	A	B	A	A	A	C

B.
WF(7,3)/WF(7,4)

A	B	B	A	B	A	B
⇩	⬇	⬇	⇩	⬇	⇩	⬇

WF(7,3)

A	B	C	A	B	A	C

C.
WF(11,5)/WF(11/6)

A	A	B	A	A	B	A	A	B	A	B
⇩	⇩	⬇	⇩	⇩	⬇	⇩	⇩	⬇	⇩	⬇

PWWF(11,5)

A	A	B	A	A	C	A	A	B	A	C

D.
WF(9,1)/WF(9,8)

A	B	B	B	B	B	B	B	B
⇩	⬇	⬇	⬇	⬇	⬇	⬇	⬇	⬇

PWWF(9,1)

A	B	C	B	C	B	C	B	C

E.
The Singular Pairwise Well-formec Scale:

A	B	A	C	A	B	A

Example 4. Well-formed scales transformed into pairwise well-formed scales.

COROLLARY 10.2. $\dfrac{\mathrm{D}(\mathrm{PWWF}(N,G))}{\mathrm{D}(\mathrm{WF}(N,g))} = \dfrac{3}{2}$. *By Theorem 10 and Theorem 5. The number of differences in a pairwise well-formed N-note scale is* 3/2 *greater than the number of differences of an N-note well-formed scale.*

This last result reflects the fact that pairwise well-formed scales are trivalent, and well-formed ones have Myhill's Property. Thus, a pairwise well-formed scale of arbitrary cardinality N has three interval sizes in each generic class while the corresponding well-formed scale has two.

Having determined significant facts about coherence and sameness in well-formed and pairwise well-formed scales, I would like to conclude with four conjectures, which may form the basis for further research:

CONJECTURE 1. *The pairwise well-formed scale class that has the most potential for coherence failures is* PWWF(N, $N-2$), *when the interval (*A*) of multiplicity $N-2$ is the smallest step. Note that this implies that steps* B *and* C *each occur only once. Such a scale has a pattern of the following type: 5 notes:* ABAAC; *7 notes:* AABAAAC; *9 notes:* AAABAAAAC, *etc. The scale will attain maximal failure only if*

$$\text{Size(B)} \geq (\tfrac{N-1}{2})(\text{Size(A)}),$$

and if

$$\text{Size(C)} \geq \text{Size(B)} + (N-2)\text{Size(A)}.$$

A seven-note PWWF scale in 16-note equal temperament exhibiting maximal failure has the pattern AABAAAC, *with* Size(A) = 1, Size (B) = 3, Size(C) = 8 (0 1 2 5 6 7 8 (16)).

CONJECTURE 2. *The number of coherence failures in such a set (maximal for a pairwise well-formed scale of cardinality N is*

$$\frac{(N-1)(N+1)(5N^2-12N+3)}{96}.$$

Consequently, the coherence quotient of a pairwise well-formed scale with maximal failure is

$$1-\frac{\frac{(N-1)(N+1)(5N^2-12N+3)}{96}}{\frac{N(N-1)(N-2)(3N-5)}{24}} = 1-\frac{(N+1)(5N^2-12N+3)}{4N(N-2)(3N-5)}, \quad \text{or} \quad \frac{7N^3-37N^2+49N-3}{12N^3-44N^2+40N}.$$

An upper bound for this value is $\frac{7}{12}$. *For well-formed and pairwise well-formed scales that are of the same cardinality and that have maximal coherence failure, the well-formed scale will have the higher coherence quotient. In the cases of both well-formed and pairwise well-formed scales with maximal coherence failures, the greater the cardinality, the* ***higher*** *the coherence quotient.*

CONJECTURE 3. *A consequence of Myhill's Property is that the difference between the larger and smaller steps is the same as the difference between the larger and smaller intervals of any given span. For example, in the diatonic under Pythagorean tuning, the difference between the steps is equal to the apotome,* $\frac{2187}{2048}$. *But the difference between the perfect and diminished fifth in this scale is equal to the same value. This property might generalize to sets with three or more intervals sizes per span as follows: Let n represent the number of distinct step sizes per span. If the set of* (n) ($n-1$)/2 *(positive) differences between the n step sizes is the same for each span, the set has* ***strong n-valence****. Some sets, such as* 5-22 (01478), *have trivalence without strong trivalence. I conjecture that iff a set of odd cardinality has strong trivalence it is pairwise well-formed.*

CONJECTURE 4. *For n-valent scales with n > 3 the situation becomes quite complex. Nevertheless, I conjecture that strong n-valent scales have higher sameness and coherence quotients than other scales of the same cardinality. This conjecture has important implications for scale theory. First, it seems likely that n-valence becomes increasingly rare as n increases. Second, this property promotes a kind of 'best distribution' of intervals analogous to the distributions found in well-formed and pairwise well-formed scales. The following six-note scale in* 25*-note equal temperament has strong four-valence:* (0 3 7 12 15 21 (25)). *The steps form the pattern,* 3, 4, 5, 3, 6, 4, *consisting of four values,* 3, 4, 5, *and* 6. *The thirds in this scale are of one of the four sizes,* 7, 8, 9, 10. *The fourths are of sizes* 11, 12, 13, 14. *Note that the scale is coherent.*

6. Conclusion

This study has revealed that well-formed and pairwise well-formed scales are endowed with properties that limit them from becoming over-weighted with differences or coherence failures. This enables these sets to be more readily accepted as scales: the symmetries that are their constant features help to regulate the specific interval sizes in each generic class. Myhill's Property and strong trivalence are the operative properties. These properties do not seem to generalize to scales that have four or more step sizes. Thus, not only for cognitive reasons, but also for structural ones, we are not likely to find musically useful scales with four or more step sizes.

References

[1] Clough, J. and Myerson, G., 1985, Variety and multiplicity in diatonic systems. *Journal of Music Theory*, **29**, 249–270.

[2] Clough, J., 1979, Aspects of diatonic sets. *Journal of Music Theory*, **23**, 45–61.

[3] Clough, J. and Douthett, J., 1991, Maximally even sets. *Journal of Music Theory*, **35**, 93–173.

[4] Clough, J., Engebretsen, N. and Kochavi, J., 1999, Scales, sets, and interval cycles: a taxonomy. *Music Theory Spectrum*, **21**, 74–104.

[5] Rahn, J., 1981, Coordination of interval sizes in seven-tone collections. *Journal of Music Theory*, **35**, 33–60.

[6] Carey, N., 2002, On coherence and sameness, and the evaluation of scale candidacy claims. *Journal of Music Theory*, **46**, 1–56.

[7] Balzano, G., 1980, The group-theoretic description of 12-fold and microtonal pitch systems. *Computer Music Journal*, **4**(4), 66–84.

[8] Agmon, E., 1989, A mathematical model of the diatonic system. J*ournal of Music Theory*, **33**, 1–25.

[9] Agmon, E., 1996, Coherent tone systems: A study in the theory of diatonicism. *Journal of Music Theory*, **40**, 39–59.

[10] Rothenberg, D., 1978, A model for pattern perception with musical applications. Part I: Pitch structures as order preserving maps. *Mathematical Systems Theory*, **11**, 199–234. Part II: The information content of pitch structures. *Mathematical Systems Theory*, **11**, 353–372. Part III: The graph embedding of pitch structures *Mathematical Systems Theory*, **12**, 73–101.

[11] Halton, J.H., 1965, The distribution of the sequence $\{n\zeta\}$ ($n = 0, 1, 2, \ldots$). *Proceedings of the Cambridge Philosophical Society*, **61**, 665–670.

[12] Sós, V.T., 1958, On the distrubution mod 1 of the sequence $n\alpha$. *Annals of the University of Science, Budapest Eötvös, Sect Math*, **I**, 127–134.

[13] Carey, N., 1998, Distribution modulo 1 and musical scales. PhD dissertation, University of Rochester.

[14] Dahlhaus, C., 1990, *Studies on the Origin of Harmonic Tonality* (translated by Robert Gjerdingen from the original: *Untersuchungen über die Entstehung der Harmonischen Tonalität* (Kassel: Bärenreiter, 1967, 1988)). (Princeton: Princeton University Press).

[15] Carey, N. and Clampitt, D., 1989, Aspects of well-formed scales. *Music Theory Spectrum*, **11**(2), 187–206.

[16] Wooldridge, M., 1992, Rhythmic implications of diatonic theory: A study of Scott Joplin's Ragtime piano works. PhD dissertation, State University of New York at Buffalo.

[17] Clough, J. and Myerson, G., 1986, Musical scales and the generalized circle of fifths. *American Mathematical Monthly*, **93**, 695–701.
[18] Carey, N. and Clampitt, D., 1996, Self-similar pitch structures, their duals, and rhythmic analogues. *Perspectives of new music*, **34**(2), 62–87.
[19] Blackwood, E., 1985, *The Structure of Recognizable Diatonic Tunings* (Princeton: Princeton University Press).
[20] Kornerup, T., 1930, *Das Goldene Tonsystem als Fundament der theoretischen Akustik* (Copenhagen: Aschehoug Forlag).
[21] Yasser, J., 1975 [1932], *A Theory of Evolving Tonality. American Library of Musicology* (Reprint: New York: Da Capo Press).
[22] Clampitt, D., 1997, Pairwise well-formed scales: Structural and transformational properties. PhD dissertation, State University of New York at Buffalo.

Appendix: Continued fraction conventions

C.1. Let θ be a real number >0.

C.2. $\theta = [t_0, t_1, t_2, t_3, \ldots, t_k, \ldots]$.The bracket notation indicates a continued fraction. That is,

$$\theta = [t_0, t_1, t_2, \ldots] = \cfrac{1}{t_0 + \cfrac{1}{t_1 + \cfrac{1}{t_2 + \cdots}}}.$$

The t_i are the *terms*, or *partial quotients* of the continued fraction.

C.3. If θ is rational, then the number of terms in the continued fraction is finite, infinite if θ is irrational.

C.4. The fraction $\frac{a_k}{b_k} = [t_0, t_1, \ldots, t_k]$ is a *full convergent* of the continued fraction of θ.

C.5. Let $a_{-1} = b_{-2} = 1$. Let $a_{-2} = b_{-1} = 0$.

C.6. Then $\frac{t_k a_{k-1} + a_{k-2}}{t_k b_{k-1} + b_{k-2}}$ is the convergent $\frac{a_k}{b_k}$ (=$[t_0, t_1, \ldots, t_k]$).

C.7. If $t_k > 1$, then let t be an integer, $1 < t < t_k$.

C.8. The fraction $\frac{a_k^t}{b_k^t} = [t_0, t_1, \ldots, t_{k-1}, t]$ is an *intermediate convergent* of the continued fraction of θ. (By extension, $\frac{a_k}{b_k} = \frac{a_k^{t_k}}{b_k^{t_k}}$.)

C.9. Then $\frac{t(a_{k-1}) + a_{k-2}}{t(b_{k-1}) + b_{k-2}}$ is the intermediate convergent $\frac{a_k^t}{b_k^t}$.