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Supporting Online Material for

Generalized Voice-Leading Spaces

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SUPPORTING ONLINE MATERIAL

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1. Notation. C_4 is middle C, C_5 is an octave above middle C, and so on; B_3 is a semitone below C_4. $\mathbb{R}$ denotes the real numbers, $\mathbb{Z}$ the integers, $\mathbb{T}^n$ the *n*-torus, and $\mathcal{S}_n$ the symmetric group of order *n*. $\mathbf{T}_x$ denotes transposition by *x* semitones: $\mathbf{T}_x(p) = p + x$. $\mathbf{I}_y^x$ denotes the reflection that maps *x* to *y*: $\mathbf{I}_y^x(p) = (x + y) - p$.

We use subscripts to denote equivalence classes: $(x_1, \ldots, x_n)_{\mathcal{F}}$ denotes the equivalence class formed by applying the musical operations in $\mathcal{F}$ to the musical object $(x_1, \ldots, x_n)$. Thus (E_4, G_4, C_4), $(G_2, G_3, D_5, B_3) \in (C_4, E_4, G_4)_{\mathrm{OPTC}}$. For progressions, we use $(x_1, \ldots, x_n)_{\mathcal{F}} \rightarrow (y_1, \ldots, y_n)_{\mathcal{F}}$ to denote the equivalence class resulting from individual applications of $\mathcal{F}$, and $(x_1, \ldots, x_n) \xrightarrow{\mathcal{F}} (y_1, \ldots, y_n)$ to denote the equivalence class resulting from uniform applications of $\mathcal{F}$. The two notations can be combined: thus $(C_4, E_4, G_4)_{\mathrm{T}} \xrightarrow{\mathrm{OPI}} (C_4, F_4, A_4)_{\mathrm{T}}$ denotes the equivalence class of progressions related to $(C_4, E_4, G_4) \rightarrow (C_4, F_4, A_4)$ by individual T and uniform OPI (*S1*).

Note that there is a difference between the O and T operations: T operations send the pitches $(x_1, \ldots, x_n)$ to $(x_1 + c, \ldots, x_n + c)$, for some real number *c*, while O operations send $(x_1, \ldots, x_n)$ to $(x_1 + 12i_1, \ldots, x_n + 12i_n)$ for some collection of integers i_j. (These definitions reflect musical practice.) One could say that T acts "uniformly" on the notes in an object, while O acts "individually." In this paper, however, we use the terms "uniform" and "individual" only for progressions. Both O and T can apply to progressions either uniformly or individually.

The term "scalar transposition," used briefly in the main text, refers to transposition or translation along a scale. Given the ascending circular ordering of pitch-classes (C, D, E, F, G, A, B, [C]), scalar transposition by one ascending step sends (C, D, E) to (D, E, F), shifting every note upward by one scale tone. We can interpret a scale as defining a metric according to which adjacent scale tones are one "unit" apart; scalar transposition is just translation relative to this metric (*S2*).

2. Chords, set classes, voice leadings and chord-progressions. A "pitch class" is an equivalence class of one-note objects related by O. Pitch classes can be designated by letter names like "C" and "A" without an Arabic subscript identifying a particular octave. A "chord" is an equivalence class of objects related by OPC. Chords can be identified with unordered sets of pitch classes, and are often called "sets" in contemporary theory. A "chord type" (or "transpositional set class") is an equivalence class of objects related by OPTC. Terms like "major chord" and "dominant seventh chord" refer to chord types (*S3*). A "set class" is an equivalence class of objects related by OPTIC. Set classes have long been central to music theory: for example, Rameau used the term "perfect chord" to refer to the set class containing major and minor triads (*S4*).

The progression $(C_4, E_4, G_4) \rightarrow (C_4, F_4, A_4)$ defines three melodic voices that move C_4 to C_4, E_4 to F_4, and G_4 to A_4 (Fig. S2A). Theorists are often uninterested in the overall order of the voices. A voice leading in pitch space is an equivalence class of progressions under uniform permutation: $(C_4, E_4, G_4) \rightarrow (C_4, F_4, A_4)$ and $(E_4, G_4, C_4) \rightarrow (F_4, A_4, C_4)$ are instances of the same voice leading in pitch space, $(C_4, E_4, G_4) \xrightarrow{P} (C_4, F_4, A_4)$. Theorists also find it useful to abstract away from the particular octave in which voices appear. A voice leading in pitch-class space is an equivalence class of progressions under uniform permutation and octave shifts: $(C_4, E_4, G_4) \rightarrow (C_4, F_4, A_4)$ and $(E_2, C_5, G_3) \rightarrow (F_2, C_5, A_3)$ represent the same voice leading in pitch class space, $(C_4, E_4, G_4) \xrightarrow{OP} (C_4, F_4, A_4)$. Here, C moves to C by 0 semitones, E moves to F by 1 semitone, and G moves to A by 2 semitones.

A "chord-progression" is a progression with no implied mappings between its objects' elements. (Note the hyphen: chord-progressions are a particular subspecies of progression.) Chord-progressions can be modeled as equivalence classes under individual applications of OPC: thus $(C_4, E_4, G_4)\rightarrow(F_4, F_3, A_4, C_5)$ and $(E_4, C_4, G_5)\rightarrow(F_4, C_4, A_5)$ are instances of the chord-progression $(C_4, E_4, G_4)_{OPC}\rightarrow(F_4, A_4, C_5)_{OPC}$. Intuitively, chord-progressions represent sequences of chords considered as indivisible, harmonic "wholes," while voice leadings represent specific connections between the notes of successive chords. The difference between chord-progressions and voice leadings illustrates the difference between individual and uniform applications of the OPTIC operations.

Hugo Riemann classified triadic chord-progressions by uniform TI and individual OPC (*S5*). Thus he used a single term, *Quintschritt*, to label the progressions $(C_4, E_4, G_4)\rightarrow(G_4, B_4, D_5)$ and $(C_4, E\flat_4, G_4)\rightarrow(F_3, A\flat_3, C_4)$, even though the first progression ascends by fifth, while the second descends by fifth. We can generalize Riemann's terminology by saying that two progressions are *dualistically equivalent* if they are related by uniform TI and individual OPC.

3. Equivalence classes, quotient spaces, and music. A number of theorists (*S6–7*) have investigated the role of symmetries and equivalence classes in music theory. The present paper builds on this previous work in several ways: first, it describes a collection of five OPTIC transformations intrinsic to traditional musical discourse; second, it shows how to apply these transformations to progressions as well as individual objects; third, it identifies a particular class of quotient spaces in which natural geometrical structures correspond to familiar musical objects; fourth, it describes the specific geometry, topology, and musical interpretation of the resulting spaces; and fifth, it uses voice leading to reinterpret a variety of other music-theoretical terms, such as similarity, contour, and K-nets.

Progressions are atomic fragments of musical scores (Fig. S6). In the sixteen OPTI quotient spaces, there is an isomorphism between these fragments and a natural class of geometrical objects: for any atomic fragment of a musical score, such as that in

Figure S6A, there is a unique line segment (or pair of points) in the quotient, such as that in Figure S6C; conversely, for any line segment (or pair of points) in the quotient, there will be a corresponding equivalence class of atomic fragments.

This is not true of musical quotient spaces in general. For example, suppose one were to model equal-tempered pitch-classes as elements of the cyclic group $\mathbb{Z}_{12}$ and two-note chords as points in the discrete quotient space $((\mathbb{Z}_{12})^2 - \mathbb{Z}_{12})/\mathcal{S}_2$. (We use $((\mathbb{Z}_{12})^2 - \mathbb{Z}_{12})/\mathcal{S}_2$ because the music-theoretical tradition ignores "chords" with multiple copies of a single pitch class, such as {C, C}.) Now consider the two progressions shown in Fig S6A-B. In two-note OP space, $\mathbb{T}^2/\mathcal{S}_2$, these are represented by distinct paths (Fig S6C): one moves directly from {D, E} to {D♯, E}, while the other reflects off the space's singular boundary. However, the discrete space $((\mathbb{Z}_{12})^2 - \mathbb{Z}_{12})/\mathcal{S}_2$ does not distinguish these progressions, for three reasons. First, there are no line-segments in the discrete space. Second, chords such as {C, C} do not appear in $((\mathbb{Z}_{12})^2 - \mathbb{Z}_{12})/\mathcal{S}_2$, and hence there is no singular boundary to "bounce off." Thus, the line segment (D, E)→(E, D♯) is not represented even in $\mathbb{T}^2/\mathcal{S}_2 - \mathbb{T}^1$, the continuous noncompact Möbius strip representing two-note chords with two distinct pitch classes. Third, the path in Figure S6B intersects the singular boundary at a point with non-integral coordinates: thus, even if we were to use the space $(\mathbb{Z}_{12})^2/\mathcal{S}_2$, which has singular points, and even if we were to define a discrete analogue to the notion of "line segment," it would still be awkward to model this path.

Other musical quotient spaces create analogous complications (*S8*). In general, the most transparent isomorphism between familiar musical events and familiar geometrical concepts is obtained when pitches are modeled using real numbers, and when multisets such as {C, C, E, G} are included. Furthermore, as we will see in the next section, distances in the resulting spaces represent voice-leading size only when the operations that form the quotient are isometries of generic voice-leading metrics. Our suggestion is that the OPTI spaces are interesting precisely because they satisfy these conditions, thereby giving rise to geometrical structures with a particularly clear musical interpretation.

4. Proto-metrics and isometries. There are several ways to measure musical distance. For example one could consider the notes C_3 and G_4 to be close because the ratio of their fundamental frequencies can be expressed using small whole numbers. This is an *acoustic* conception of musical distance. Alternatively, one could consider the chords $(C, C\sharp, E, F\sharp)_{OP}$ and $(C, C\sharp, D\sharp, G)_{OP}$ to be close, or even identical, since each contains the same total collection of intervals: a semitone (C-C♯), a major second (E-F♯ and C♯-D♯), a minor third (C♯-E and C-D♯), and so on. This is an *intervallic* conception of musical distance (*S9*). Finally, it is important to distinguish perceptual models of musical distance, which focus on the experience of hearing music, from conceptual models, which attempt to describe more abstract cognitive structures musicians use to understand or compose music.

In this paper, we explore a conceptual model that measures distance melodically, using distance in log-frequency space (*S10–11*). Melodic distance is important for two reasons: first, because large melodic leaps can often be difficult to sing or play; and second, because short melodic distance facilitates auditory streaming, or the separation of the sound-stimulus into independent melodic lines (*S12–13*). As a result Western composers typically move from chord to chord in a way that minimizes the log-frequency distance traveled by each voice (*S2*, *S14–16*). What complicates matters is that Western polyphonic music involves multiple melodies at one time. Thus if one wants to measure the melodic distance between chords, it is necessary to determine whether the voice leading $(C_4, E_4, G_4) \xrightarrow{OP} (D\flat_4, F_4, A\flat_4)$, which moves three voices by one semitone each, is larger or smaller than $(C_4, E_4, G_4) \xrightarrow{OP} (C_4, E_4, A_4)$, which moves one voice by two semitones. The problem is that there is no obvious answer to this question: instead, there are a number of reasonable but subtly different ways to measure voice leading.

Let $A \rightarrow B \equiv (a_1, a_2, \ldots, a_n) \rightarrow (b_1, b_2, \ldots, b_n)$ be a progression of two n-note musical objects. We define this progression's *displacement multiset* as $\{|a_1 - b_1|, \ldots, |a_n - b_n|\}$, or $dm(A \rightarrow B)$. A proto-metric is a preorder over multisets of nonnegative reals satisfying what Tymoczko (*S2*) calls the "distribution constraint":

$$\{x_1 + c, x_2, \ldots, x_n\} \geq \{x_1, x_2 + c, \ldots, x_n\} \geq \{x_1, x_2, \ldots, x_n\}, \text{ for } x_1 > x_2, c > 0$$

A proto-metric allows comparison of some distances in $\mathbb{R}^n$, thereby endowing the space with more-than-topological, but less-than-geometrical structure. It is argued in (*S2*) that any music-theoretically reasonable method of measuring voice leading must be consistent with the distribution constraint, since otherwise voice leadings with "voice crossings" would be smaller than their naturally uncrossed alternatives. Furthermore, every existing music-theoretical method of measuring voice leading satisfies the constraint.

Uniform application of the OPTI symmetries preserves the displacement multiset $\{|a_1 - b_1|, \ldots, |a_n - b_n|\}$ and hence preserves the proto-metric: if $\mathcal{F}$ is any OPTI operation, then $dm(\mathcal{F}(A)\rightarrow\mathcal{F}(B)) = dm(A\rightarrow B)$; thus $A\rightarrow B$ is smaller than $C\rightarrow D$ if and only if $\mathcal{F}(A)\rightarrow\mathcal{F}(B)$ is smaller than $\mathcal{F}(C)\rightarrow\mathcal{F}(D)$. Consequently, quotients of the OPTI operations inherit the proto-geometrical structure of the parent space $\mathbb{R}^n$. For a large class of suitable metrics, this means that distance in the quotient space will correspond to the length of the shortest line-segment between two points, as measured using the quotient metric (*S17*). Musically, this means that we can use voice-leading size to measure distance between chord-types.

The C operation does not have this property, since it identifies the voice leading $C_4\rightarrow G_4$, whose displacement multiset is $\{7\}$, with the voice leading $(C_4, C_4, C_4)\rightarrow (G_4, G_4, G_4)$, whose displacement multiset is $\{7, 7, 7\}$. According to most music-theoretical metrics $\{7\}$ and $\{7, 7, 7\}$ do not have the same size: this is because it takes less musical "work" to move one voice by seven semitones than to move three voices by seven semitones. Consequently, the C operation is not an isometry of proto-metrics in general (*S18*).

This problem arises even in contexts that do not involve note duplication. Consider a chord A consisting of a large number of distinct pitches infinitesimally close to C_4, and a chord B consisting of a large number of distinct pitches infinitesimally close to G_4. According to standard metrics of voice leading size, the voice leadings taking the pitches of A to C_4 and the pitches of B to G_4 are infinitesimally small; the triangle inequality therefore requires that the voice leading $A\rightarrow B$ should be approximately the same size as the voice leading $C_4\rightarrow G_4$. But any voice leading from A to B must move a large number of voices by approximately seven semitones each, and (according to many standard metrics) is considerably larger than the voice leading $C_4 \rightarrow G_4$ (*S19*). Thus we must either abandon the triangle inequality or abandon the claim that $A \rightarrow B$ is significantly larger than $C_4 \rightarrow G_4$. Either way, we abandon the idea that voice-leading size corresponds to something like distance in the quotient space.

It follows that the C spaces are not ideal for studying voice leading, since we cannot use distance in the C spaces to explain the efficient voice-leading possibilities between chords. However, C spaces may usefully model similarity between chord types, as we will discuss in §6. In this context, violations of the triangle inequality do not necessarily present a serious music-theoretical obstacle.

5. Quotient spaces, fundamental domains, OPT, and OPTI. A quotient space is formed by identifying (or "gluing together") all points in a parent space related by some collection of operations $\mathcal{F}$. A fundamental domain for $\mathcal{F}$ is a region in the parent space satisfying two constraints: first, every point in the entire space is related by operations in $\mathcal{F}$ to some point in the region; and second, no two points in the interior of the region are related by operations in $\mathcal{F}$. (Typically, fundamental domains are also taken to be connected regions with a straightforward geometrical structure, but this is largely a matter of convenience.) The fundamental domain is analogous to a single "tile" of a piece of wallpaper.

The quotient space can be formed from the fundamental domain by gluing together all points on the fundamental domain's boundary that are related by operations in $\mathcal{F}$. For example, the (closed) upper half plane is a fundamental domain for 180° rotations in $\mathbb{R}^2$, since any point in the interior of the upper half-plane is related by 180° rotation to a point in the lower half plane (Figure S7). Since points on the positive x axis are related by 180° rotation to points on the negative x axis, they must be glued together to form the quotient space. The result of this identification is a cone, as shown in Figure S7.

We begin with $\mathbb{R}^n$, the space of n-note musical objects. The O operation transforms $\mathbb{R}^n$ into the n-torus $\mathbb{T}^n$. Musically, the most useful fundamental domain for the n-torus is an unusual one: those points $(x_1, x_2, \ldots, x_n)$ such that

$$\max(x_1, \ldots, x_n) \leq \min(x_1, \ldots, x_n) + 12$$
$$0 \leq \Sigma x_i \leq 12$$

where "max" and "min" represent the maximum and minimum of a collection of numbers, respectively. (See Table S2.) The first constraint ensures that the sequence of pitches spans less than an octave; the second determines its overall registral position. Intuitively, the resulting shape is a prism whose opposite faces are identified in a "twisted" fashion (Fig. S8). The base of this prism is the shadow of an n-dimensional hypercube orthogonally projected onto the $(n–1)$-dimensional plane containing points whose coordinates sum to 0: in two dimensions it is a line segment; in three, a hexagon; and in four, a rhombic dodecahedron. This fundamental domain is useful because transposition corresponds to motion along the prism's "height" dimension.

We can form a fundamental domain for the P operations by requiring that $x_1 \leq x_2 \leq \ldots \leq x_n$. This ensures that the object is in ascending order. We define a fundamental domain for the T operations by requiring that $\Sigma x_n = 0$, thereby fixing a default transposition for each object. A fundamental domain for the I operations is given by $x_2 – x_1 \leq x_n – x_{n–1}$. This selects between an object $(x_1, \ldots, x_n)$ and its inversion $(–x_1, \ldots, –x_n)$, choosing the one whose initial interval is smaller than its final interval. Fundamental domains for combinations of the OPTI operations often combine the relevant inequalities. However, the conjunction of O and T introduces additional constraints. (For example, fundamental domains for OT must satisfy an equation like $\min(x_1, \ldots, x_n) = x_1$.) This is because the action of OT orthogonally projects a twisted prism, producing a quotient of its cross-section (Fig. S8).

The OP space $\mathbb{T}^n/\mathcal{S}_n$ has a fundamental domain that is a prism whose base is an $(n–1)$-simplex. As shown in (*S2*), the space $\mathbb{T}^n/\mathcal{S}_n$ is formed from the fundamental domain by identifying points on the base with those on the opposite face (Figure S9).

(The rectangular boundaries of the fundamental domain in Figure S9A are singular orbifold points, a matter we will disregard in the following discussion.) This identification involves a “twist”—a rigid transformation cyclically permuting the simplex’s vertices. OPT space $\mathbb{T}^{n-1}/\mathcal{S}_n$ is the orthogonal projection of this space along the direction of transposition, and is therefore the (n–1)-simplex modulo the “twist.” This space can be visualized as a cone over a quotient of the (n–2)-sphere, since the simplex is homeomorphic to a ball, which is itself a cone over the sphere. (Note, however, that this “cone” will contain additional orbifold points with complicated topology: the vertex, the base, and for chords with a nonprime number of notes, points on other layers as well.) When n is prime, the group generated by cyclic permutation has no fixed points and the base of $\mathbb{T}^{n-1}/\mathcal{S}_n$ is a lens space (*S20*). Mathematically, transposition induces a foliation of chord space ($\mathbb{T}^n/\mathcal{S}_n$), with chord-type space ($\mathbb{T}^{n-1}/\mathcal{S}_n$) being the “leaf space” of the foliation.

Musically, the OPT spaces can be understood as a series of similar “layers,” each of which contains the n-note chord-types whose smallest interval is some particular size. The base of the cone contains chord-types whose smallest interval is size 0—i.e. chords with note duplications. (These are all orbifold points acting like mirrors: images of line-segments in the parent space appear in the quotient to “bounce off” the base, much as a ball bounces of the edge of the pool table [*S2*, *S21–22*].) The vertex is an orbifold point representing the perfectly even chord-type that divides the octave into n equal pieces. Figure S10 represents the equal-tempered layers of the cone in Figure 2A: these layers are topological circles, and contain chords whose smallest interval is 0, 1, 2, 3, and 4 semitones respectively. The same pattern can be seen in Figure S5, which shows the individual layers of $\mathbb{T}^3/\mathcal{S}_4$ containing chords whose smallest intervals are 0, 1, 2, and 3 semitones. Since four is a composite number, there are additional orbifold points within each layer: for example, the point 0167 in Figure S5B is singular.

The OPTI spaces are quotients of these OPT spaces by the reflection that identifies inversionally-related chord types. Since inversionally-related chord types share the same smallest interval, the resulting space can again be visualized as a cone over a quotient of the sphere. To form Figure 2B out of 2A, take the quotient of the cone by the reflection that fixes 048 (the vertex), 000 (the "kink" in the base) and 006 (the point on the base antipodal to the kink), as shown in Figure S9. This produces the triangle in Figure 2B and S9E—a "cone" over the line segment from 000 to 006. Similarly, to form Figure S5E out of Figure S5A identify each point with its geometrical inversion through the center of the square. Three-note OPTI space was first described by Callender (*S22*) while four-note OPTI space was partially described by Cohn (*S23*, fig. S4F), and described more completely in unpublished work by Quinn. Interested readers can explore these spaces further by downloading "ChordGeometries," a free computer program written by Dmitri Tymoczko (*S24*).

6. Voice leading, orbifold points, and conical OPT space. The description of OPT and OPTI spaces as cones allows for a concise reformulation of one of the central conclusions in (*S2*), that "nearly even" chords can be linked to their transpositions by efficient voice leading. In conical OPT space, the "unevenness" of a chord corresponds to its distance from the vertex. Nearly even chords can be linked to their transpositions by efficient voice leading because short line segments near the vertex of a cone can self-intersect. These self-intersecting line segments represent progressions combining harmonic consistency (since they link transpositionally related chords) with efficient voice leading (since they are short), two cardinal desiderata of traditional Western music (*S2*). Thus a fact of central importance to Western music reduces to a familiar feature of the geometry of cones.

In fact, there is a more general relation between voice leading and orbifold points. To explore this, we will temporarily adopt the Euclidean voice-leading metric—mathematically very convenient, because the voice leading $A \rightarrow \mathbf{T}_x(B) \equiv (a_0, \ldots, a_{n-1}) \rightarrow (b_0 + x, \ldots, b_{n-1} + x)$ is minimized when A and $\mathbf{T}_x(B)$ sum to the same value. Furthermore, from the standpoint of the distribution constraint, the Euclidean metric is nicely intermediate between the extremal cases of L^1 and L^∞ (*S25*). Thus, results that are exactly true in the Euclidean case will be approximately true for other metrics satisfying the distribution constraint, with the accuracy of the approximation controlled by the particular metric in question.

We begin with a simple theorem of Euclidean geometry. Let A be any vector and let B^i be a collection of n vectors that add to zero. The sum of the squared Euclidean distances $|B^i - A|$ is equal to

$$\Sigma|B^i - A|^2 = \Sigma|B^i|^2 - 2\Sigma(B^i \bullet A) + n|A|^2 = \Sigma|B^i|^2 + n|A|^2$$

since $\Sigma B^i = 0$. When the B^i are all the same length, $|B|$, we have $|A|^2 + |B|^2 = \Sigma|A - B^i|^2/n$. We now examine three cases where this fact has an interesting music-theoretical interpretation. In each case, the lengths $|A|^2$ and $|B|^2$ will represent intrinsic properties of chords A and B (such as their "evenness" or "spread") while $\Sigma|A - B^i|^2/n$ will represent something about the voice-leading possibilities between them.

1. Unevenness and the size of the minimal voice-leadings between chord types. Define the "unevenness" of a chord as the size of the minimal voice leading to the nearest perfectly even chord. Let us translate the origin of OP space so that it lies at a perfectly even chord $(0, 12/n, \ldots, 12(n-1)/n)$, considering two chords A and B whose coordinates x_i $(0 \leq i \leq n - 1)$ sum to zero and obey the inequalities $-12/n \leq x_{i+1 \,(\mathrm{mod}\, n)} - x_i < 12 - 12/n$. It follows that (1) the quantity $|A|^2 + |B|^2$ represents the sum of the squared "unevennesses" of A and B; and (2) the voice leading $A \rightarrow B$ will have no crossings, since A and B each lie within the region bounded by chords with pitch-class duplications. Now consider the n voice leadings

$$A \rightarrow \sigma^i(B) \equiv (a_0, \ldots, a_{n-1}) \rightarrow (b_{0+i\,(\mathrm{mod}\, n)}, \ldots, b_{n-1+i\,(\mathrm{mod}\, n)})$$

The n chords on the right of the voice leading have coordinates summing to 0, are the same distance from the origin, and are related by transposition; they are also represented by n vectors whose vector sum is zero. The voice leadings $A \rightarrow \sigma^i(B)$ have no crossings and, for nonsingular A and B, are not individually T-related. (When both chords are in nondescending order spanning less than an octave, we obtain them by repeatedly moving the lowest note of the second chord up by octave, and transposing the entire chord down by $12/n$, as in Figure S11B.) Consequently, the quantity $|A|^2 + |B|^2 = \Sigma|A - \sigma^i(B)|^2/n$

represents both the sum of the chords' squared "evennesses" and the average squared size of the crossing-free OPT voice-leading classes between chord-types A_{OPT} and B_{OPT} (Figure S11A-C). For very even chords, such as the major and minor triads of the classical tradition, there will be multiple small crossing-free voice-leading possibilities between their transpositions. (Recall from §4 that crossing-free voice leadings are desirable for a number of reasons, not least because there is always a minimal voice leading between two chords that is crossing free.) This gives composers a wealth of contrapuntal options to choose from.

2."Spread" and the size of crossed voice-leadings between chords. A similar result relates the size of voice leadings with crossings to the "spread" of two chords—that is, their distance from the perfectly *clustered* chord-type $\{0, \ldots, 0\}_{\mathrm{T}}$. Consider chords A and B whose coordinates x_i sum to zero. If the origin is the perfectly clustered chord, then $|A|^2 + |B|^2$ represents the sum of the squared "spreads" of A and B. The n voice leadings $A \rightarrow \sigma^i(B)$ are now obtained by circularly permuting the notes of the second chord without any transposition (Fig. S11D). The average squared size of these voice leadings will be equal to the sum of the chords' "spread." Thus a very clustered chord can be connected to some transposition of another very clustered chord by many different efficient voice leadings. The result can be generalized to all permutations of chord B.

3. Inversions. Choose an inversionally symmetrical sequence of pitch classes such that $(x_1, \ldots, x_n) = (c - x_n, \ldots, c - x_1)$, and translate the origin of OT so that it lies at this point; thus, all chords B and $-B$ are inversionally related. The quantities $|A|^2$ and $|B|^2$ represent the squared distance to the inversionally symmetrical chord at the origin. Our result now says that the sum of these squared distances is equal to the average squared size of the voice leadings $A \rightarrow B$ and $A \rightarrow -B$. The closer A and B are to the inversionally symmetrical chord midway between B and $-B$, the smaller these two voice leadings will be.

In each of case, we find a similar relationship between voice leading and distance from orbifold points. If we assume the Euclidean metric, we can express this relationship in quantitative terms: the sum of the squared distances between two chords and some particular orbifold point is equal to the average of the squared size of some musically interesting collection of voice leadings between them. For an arbitrary metric obeying the distribution constraint, there is no equivalently elegant quantitative result, but similar relationships obtain approximately. Western composers have exploited these facts to write progressions connecting structurally similar chords by efficient voice leading (*S2*).

7. Similarity among chord types. Over the last twenty years, a number of theorists have attempted to model musical conceptions of "similarity" (or inverse distance) between chord types. These models have been based on the intervallic conception of distance described in §4 (*S26–29*), Fourier-transform-based extensions to these models (*S9*), the prevalence of shared subsets (*S30–33*), or meta-analyses of models in these categories (*S34–35*). Following Roeder (*S36*) and Straus (*S37*), we suggest an approach based on voice leading: specifically, we propose modeling the similarity between equivalence classes as the size of the smallest voice leading between their elements. Conceiving of similarity in this way has a number of advantages:

1. As described in the text, this approach is consistent with the flexibility inherent in terms such as "scale fragment," "triad," and "major triad." Furthermore, composers often do vary musical material in accordance with this notion of similarity (Fig. 3).
2. Since this conception of similarity is consistent with aggregate physical distance on a keyboard instrument, it is plausible that composers would be sensitive to it.
3. This approach permits different chords to have different degrees of "self similarity," here measured by the size of the smallest nontrivial voice leading from a chord-type to itself. Chord-types with a high degree of self-similarity have played a prominent role in Western music (*S2*).
4. This approach generalizes naturally to continuous spaces, in such a way that chord-types differing by imperceptible distances are judged highly similar. Furthermore, the approach provides similarity measurements that are independent of the underlying chromatic universe. Other similarity metrics (*S26–S35*) lack these properties.
5. Unlike intervallic similarity metrics (*S9*, *S26–S29*), this approach distinguishes "Z-related" (or nontrivially homometric) chords such as {C, C♯, E, F♯} and {C, C♯, D♯, G} which are often thought to be dissimilar.
6. There exist a range of OPTI quotient spaces modeling different degrees of musical abstraction. Thus, unlike other metrics (*S9*, *S26– S29*), the

approach itself does not require one particular set of symmetries—such as, for example, the identification of inversionally-related chords.

7. The approach generalizes naturally to measures of voice-leading similarity, as will be discussed shortly.

We do not assert that voice-leading-based similarity metrics represent the only coherent approach to the problem. However, we do suggest that the seven considerations adduced above provide good reason to explore them.

Note that when modeling judgments of chord-type similarity it may be useful to impose C equivalence: thus we may want to consider {C, C, E} and {C, E, E} to be identical, and to consider{B, C♯, G} to be highly similar to{C, F♯, G}. (This is because the two chords can be linked by the voice leading (B, C♯, G, G)→(C, C, F♯, G), which is both a nonbijective voice leading from $\{B, C\sharp, G\}_C$ to $\{C, F\sharp, G\}_C$ and a bijective voice leading from {B, C♯, G, G} to {C, C, F♯, G}.) The fact that "distances" in the resulting C-spaces do not obey the triangle inequality should not necessarily be a cause for alarm, as there is no reason to expect psychological similarity judgments to be metrically well-behaved (*S38*).

Infinite-dimensional C space is of course quite complex. Fortunately, in most practical situations, the goal is to model relatively coarse-grained similarity judgments: for example, to explain straightforward processes of musical variation such as that in Figure 3B. In these contexts, it is often sufficient to work with a finite-dimensional OPT or OPTI space. Furthermore, choosing a specific metric of voice leading size is often unnecessary: the distribution constraint alone ensures that $(E, F, A)_{OPT}$ is as close to $(G, G\sharp, B)_{OPT}$ as any other equal-tempered transpositional set class. Thus Schoenberg's process of melodic variation in Figure 3 is a series of minimal changes in equal-tempered OPTIC space—no matter what measure of voice-leading size we prefer.

We can also use voice leading to measure the similarity of musical progressions: one compares two-element progressions by parallel-transporting them so that they start at the same point in the relevant quotient space; the distance between their endpoints represents their relatedness. Figure S12A identifies two pairs of voice leadings in Brahms's Op. 116, no. 5, while Figure S12B models these voice leadings in $\mathbb{T}^2/\mathcal{S}_2$. If we parallel transport the vectors in the most direct way (S12B), they nearly though not exactly coincide. This reflects the intuitive sense that the gestures are closely, but not exactly, related. However, the vectors can also be parallel transported so that they exactly coincide (S12D), revealing a non-obvious symmetry in Brahms's piece. The symmetry is hard to spot in the musical notation, though it is obvious in the geometrical representation. It is also fairly easy to hear: X1 and Y2 contain contrary motion where both voices move by semitone, while X2 and Y1 contain contrary motion where one voice moves by one semitone and the other moves by two semitones. Since the two pairs of vectors are related by reflection, the figure also illustrates the Möbius strip's nonorientability.

8. Contour. T-space is divided into regions whose points share the same contour, or ordinal ranking of their elements in pitch space. Figure S13 illustrates the three-dimensional case, showing that the six regions are described by inequalities of the form $x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)}$ where σ is a permutation. Music theorists have used these inequalities to define equivalence classes, treating objects as contour-equivalent if they belong to the same region (Fig. S13B) (*S39*). Elizabeth West Marvin has defined a "similarity metric" for contours, equivalent (to within a linear function) to Kendall's tau rank correlation coefficient (*S40–41*). Geometrically, Marvin's similarity metric counts the number of half-spaces $x_i < x_j$ containing both contours (Fig. S13C), divided by the total number of sharable half-spaces (*S42*). The similarity between contours is a linearly decreasing function of the smallest number of hyperplanes that must be traversed in moving between their respective contour regions (Figure S13D).

Traditional models of musical contour sometimes deliver counterintuitive results. For example, the passages in Figures S14A-B are contour-equivalent, since they both begin with their lowest note, move to their highest note, and end with a note between the first two. However, the resemblance between S14A and C is in some ways more striking than that between A and B: the lowest two notes of A and C are quite close together, which makes the difference in contour seem relatively unimportant. By contrast, since the middle note of B is close to its upper note, its contour intuitively seems dissimilar to that of A, even though they both exemplify the sequence low-high-middle. One can capture these intuitions by individuating contours more finely, representing them as normalized vectors in T space. Figure S14D shows that the vector representing A is closer to that of C than that of B.

These ideas refine Quinn's earlier extensions to Marvin's theory of contours (*S43*). Quinn was concerned with two problems: "averaging" a series of contours to produce a composite representing the group's general properties, and extending Marvin's similarity metric to these averaged contours. We can average a set of traditional contours by associating each region with the unit vector ($\sigma(1)$, …, $\sigma(n)$) at its center (Fig. S14E). These vectors can then be averaged using their (normalized) vector sum. (Alternatively if we wish to individuate contours more finely, as unit vectors, we can average them directly.) If we define an inner product on T-space (for instance, by using the Euclidean norm to measure voice-leading size), then we can take the distance between these vectors to be the angle between them (*S44*). Somewhat less elegant alternatives are available for the other L^p norms.

It is also possible to represent sequences of notes as points in T-space, measuring the similarity between them using a voice-leading metric. Thus, as illustrated in S15F, our geometrical perspective provides a range of possibilities lying between traditional pitch-set theory and traditional contour theory, depending on whether we represent contours as points, rays, or regions of T-space. These represent progressive degrees of abstraction from the musical surface.

9. Larger equivalence classes: lines, planes, and K-nets. Points in the OPTI spaces represent musical objects, while line segments represent progressions. The question naturally arises whether other geometrical entities—such as lines and planes—might correspond to interesting music-theoretical ideas. Here we show that lines and planes define larger equivalence classes known in the recent music-theoretical literature as "K-nets." This geometrical reinterpretation suggests a new and more general way to understand these somewhat controversial objects.

A K-net (*S45–47*) is a twofold partition of a chord, written $\{a_1\} + \{a_2\}$. Pitch classes in the same partition are interpreted as being related by transposition, while those in opposite partitions are interpreted as being related by inversion (Fig. S15A-B) (*S48*). A K-net thus determines a family of voice leadings that move the two partitions by exact contrary motion (*S49*), leaving invariant the graph of transpositional and inversional relations between the resulting notes (Figure S15C-D). For this reason, chords that can be connected by such voice leadings are said to be "strongly isographic." Strong isography is useful for describing music in which exact contrary motion produces chords of diverse harmonic character (Figure S15E-G).

Two K-nets $\{a_1\} + \{a_2\}$ and $\{b_1\} + \{b_2\}$ are "positively isographic" if a_1 is transpositionally related to b_1 and a_2 is transpositionally related to b_2 (Figure S16A). Similarly, they are "negatively isographic" if a_1 is inversionally related to b_1 and a_2 is inversionally related to b_2. Positive and negative isography are useful for describing music in which diverse chords are formed by superposing subsets that are transpositionally or inversionally related (Fig. S16B). Note that "strong isography" is primarily a contrapuntal notion, identifying pairs of chords that can be linked by a certain kind of voice leading. "Positive" and "negative" isography are primarily harmonic, referring to features of chord-structure that can be manifest even in music that does not articulate distinct melodic voices.

Any K-net defines a line in OP space (a K-net line), containing strongly isographic chords that can be reached by exact contrary motion of the two partitions (Figure S17A). Two chords are related by $<\mathbf{T}_x>$ ("hyper $\mathbf{T}_x$") or $<\mathbf{I}_y^x>$ ("hyper $\mathbf{I}_y^x$") if they lie on K-net lines related by $\mathbf{T}_x$ or $\mathbf{I}_y^x$ respectively (*S50–51*). $<\mathbf{T}_x>$-related chords are positively isographic and project onto a line in OPT space containing all the chord-types that can be formed by superimposing transpositions of the K-net's two parts; $<\mathbf{I}_y^x>$-related chords project onto inversionally related lines in OPT space. (Chords project onto a line in OPTI space only if they are either positively or negatively isographic.) Figure S17 shows a K-net line in three-note OP space along with its projection in OPT space. The figure shows that in twelve-note chromatic space, semitonal contrary motion reaches only half of the positively isographic chord types. This is because, for example, moving the two subsets {C} and {E, G} by semitonal contrary motion produces {C♯} and {E♭, G♭}, thereby decreasing the distance between the singleton and the lowest note of the minor third by two (*S52*). Consequently, in equal-tempered chromatic universes of even cardinality, chords can be positively isographic even though no two of their (equal-tempered) transpositions are strongly isographic. Remarkably, the traditional definition of "positive isography" reflects this, even though the theory of K-nets was developed without reference to geometry.

The geometrical perspective also suggests a natural generalization of K-nets. A collection of chords or voice-leadings can be said to be strongly *L*-isographic if they lie on a single line *L* in chord space (OP space). Such "generalized K-nets" can be used to describe musical situations in which the same contrapuntal schema, not necessarily purely contrary, links chords of different types. For example, Figure S18 shows a famous classical-music pattern (the "omnibus progression") that relates strongly *L*-isographic chords. (The example cannot be analyzed using K-nets since the inner voices remain stationary.) Here, the line *L* represents voice leadings in which the top voice moves up by semitone and the bottom voice moves down by semitone (*S53*). Intuitively speaking, a collection of chords is "strongly *L*-isographic" if repeated application of the same contrapuntal schema will produce all the chords in that collection. Chords relate by $<\mathbf{T}_x>_L$ and are "positively *L*-isographic" if they lie on lines *L* and $\mathbf{T}_x(L)$; they relate by $<\mathbf{I}_y^x>_L$ and are "negatively *L*-isographic" if they lie on lines *L* and $\mathbf{I}_y^x(L)$.

It is also possible to generalize to higher dimensions. A *pair* of linearly independent voice leadings $\alpha \rightarrow \beta$, $\alpha \rightarrow \gamma$ defines a plane *P* in OP chord space (Figure S19). Chords relate by $<\mathbf{T}_x>_P$ or $<\mathbf{I}_y^x>_P$ if they lie on planes *P* and *P′* relating by $\mathbf{T}_x$ or $\mathbf{I}_y^x$, respectively. (Again, chords are positively *P*-isographic if they relate by $<\mathbf{T}_x>_P$ and are negatively isographic if they relate by $<\mathbf{I}_y^x>_P$.) Philip Stoecker's term "strong axial isography" (*S54*) describes pairs of chords lying on one of these planes in three-note chord space (Figure S19B-C); other theorists have generalized Stoecker's idea to four-note chords (*S55*). One natural use of these ideas is to describe geometrical subspaces containing chords that can be partitioned into subsets of the same type. For example, Figure S5A illustrates the real projective plane formed by four-note chord-types of the form $\{0, 0, a, b\}_{\mathrm{OPT}}$. (These are chord-types containing at least one "doubled" note.) Chords belonging to these types are all strongly *P*-isographic, where *P* is a plane defined by voice leadings $(0, 0, a, b) \rightarrow (0, 0, a + \varepsilon, b)$ and $(0, 0, a, b) \rightarrow (0, 0, a, b + \varepsilon)$. Such spaces can be useful in analyzing music where composers create chords by superimposing sonorities of three fixed types.

K-nets are a notoriously difficult and even controversial topic (*S56*). There are, perhaps, four reasons for this. First, previous discussion of K-nets used algebraic language to describe objects that are more easily understood geometrically: lines and planes in OP, OPT, and OPTI space. Second, previous discussions treat a special case of a much more general phenomenon, considering only some of the lines and planes in the relevant quotient spaces. Third, theorists have investigated K-nets in discrete 12-note musical space, where the underlying relation between strong and positive isography is obscured. And fourth, traditional theorists labeled the “hyper” transpositions and inversions in ways that obscure their relation to ordinary transposition and inversion; as a result, comparisons between these two forms of “transposition” are sometimes problematic (*S50, S51, S56*). We hope that our geometrical reinterpretation of these music-theoretical constructions leads to greater understanding of both their utility and their limitations.

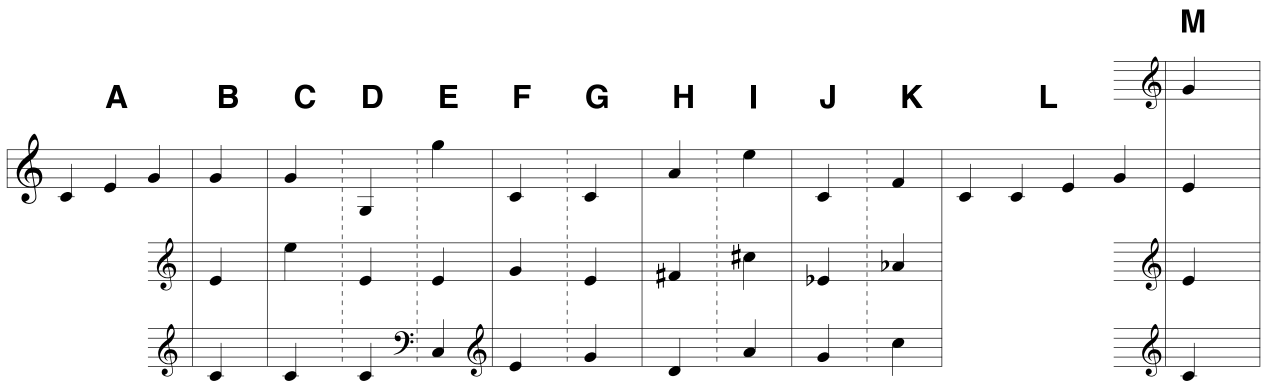


Figure S1. A musical object is a series of pitches, ordered in time or by instrument. (Instruments are labeled arbitrarily.) **(A)** The object (C_4, E_4, G_4) ordered in time. (**B)** The object (C_4, E_4, G_4) ordered by instrument (bottom to top). **(C–E)** Applying O operations to (C_4, E_4, G_4) yields (C_4, E_5, G_4) (**C**), (C_4, E_4, G_3) (**D**), (C_3, E_4, G_5) (**E**), and so on. **(F-G)** Applying P operations to (C_4, E_4, G_4) yields (E_4, G_4, C_4) (**F**), (G_4, E_4, C_4) (**G**). **(H-I)** Applying T operations to (C_4, E_4, G_4) yields (D_4, $F\sharp_4$, A_4) (**H**), (A_4, $C\sharp_4$, E_5) (**I**). **(J-K)** Applying I operations to (C_4, E_4, G_4) yields (G_4, $E\flat_4$, C_4) (**J**), (C_5, $A\flat_4$, F_4) (**K**). **(L-M)** Applying C operations to (C_4, E_4, G_4) yields (C_4, C_4, E_4, G_4) (**L**), (C_4, E_4, E_4, G_4) (**M**).

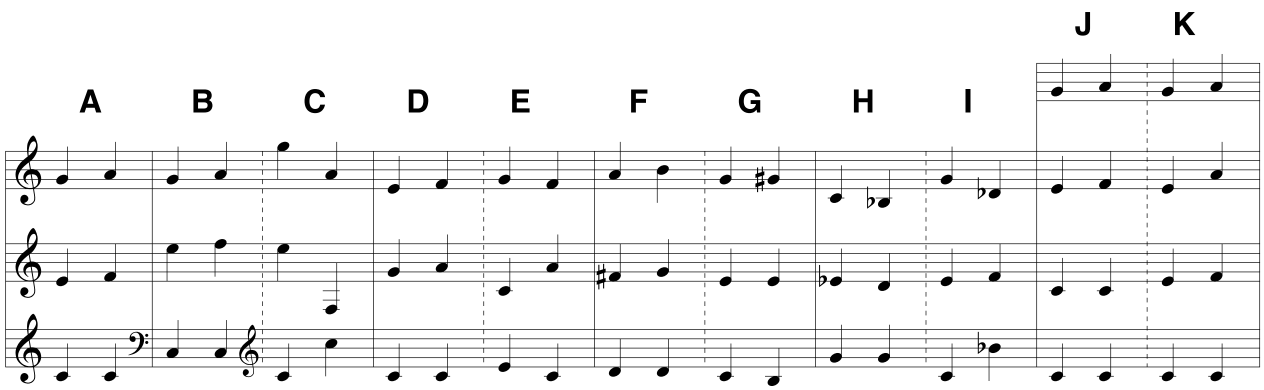


Figure S2. A progression is a sequence of musical objects. **(A)** The progression $(C_4, E_4, G_4) \rightarrow (C_4, F_4, A_4)$, ordering instruments from bottom to top. **(B)** $(C_3, E_5, G_4) \rightarrow (C_3, F_5, A_4)$ relates to (A) by uniform octave shifts. **(C)** $(C_4, E_5, G_5) \rightarrow (C_5, F_3, A_4)$ relates to (A) by individual octave shifts. **(D)** $(C_4, G_4, E_4) \rightarrow (C_4, A_4, F_4)$ relates to (A) by uniform permutation. **(E)** $(E_4, C_4, G_4) \rightarrow (C_4, A_4, F_4)$ relates to (A) by individual permutation. **(F)** $(D_4, F\sharp_4, A_4) \rightarrow (D_4, G_4, B_4)$ relates to (A) by uniform transposition. **(G)** $(C_4, E_4, G_4) \rightarrow (B_3, E_4, G\sharp_4)$ relates to (A) by individual transposition. **(H)** $(G_4, E\flat_4, C_4) \rightarrow (G_4, D_4, B\flat_3)$ relates to (A) by uniform inversion. **(I)** $(C_4, E_4, G_4) \rightarrow (B\flat_4, F_4, D\flat_4)$ relates to (A) by individual inversion. **(J)** $(C_4, C_4, E_4, G_4) \rightarrow (C_4, C_4, F_4, A_4)$ relates to (A) by uniform cardinality change. **(I)** $(C_4, E_4, E_4, G_4) \rightarrow (C_4, F_4, A_4, A_4)$ relates to (A) by individual cardinality change. Note that uniform shifts typically preserve the identity of musical voices while individual shifts typically do not.

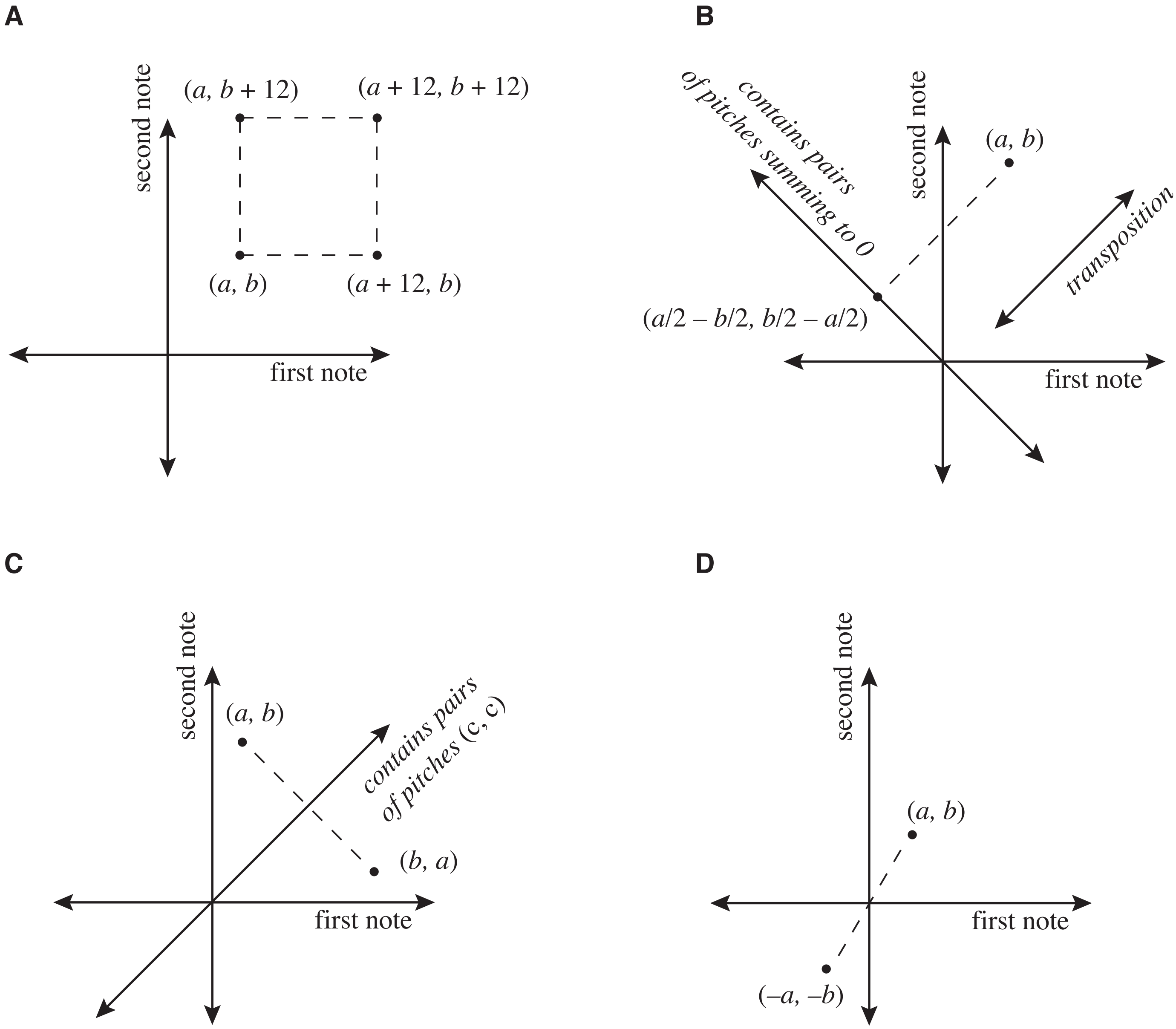
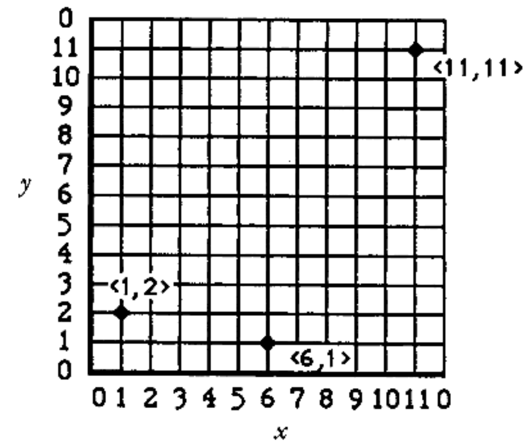


Figure S3. Each of the four OPTI operations has a natural geometrical interpretation. **A.** O identifies all points (a, b), $(a + 12, b)$ and $(a, b + 12)$. **B.** T identifies points with their (Euclidean) orthogonal projections onto the hyperplane containing chords whose pitches sum to 0. **C.** P identifies points with their reflections in the subspaces containing chords with "duplicate" pitches: in two dimensions, this is the line (x, x). In three dimensions, these are the planes (x, x, y), (x, y, x), and (x, y, y). **D.** Finally, I operates by central inversion, sending (x, y) to $(-x, -y)$.

A



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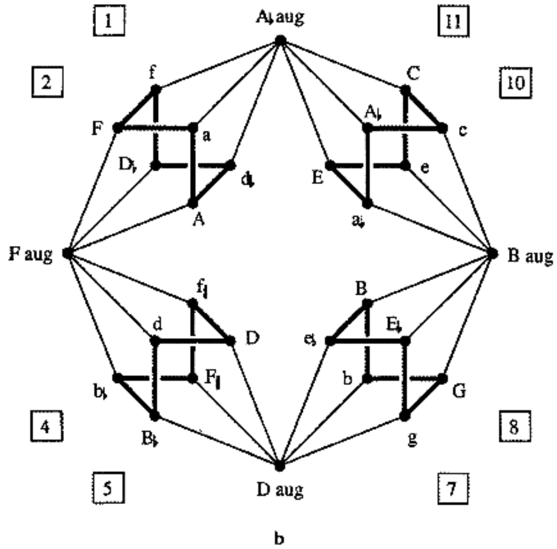

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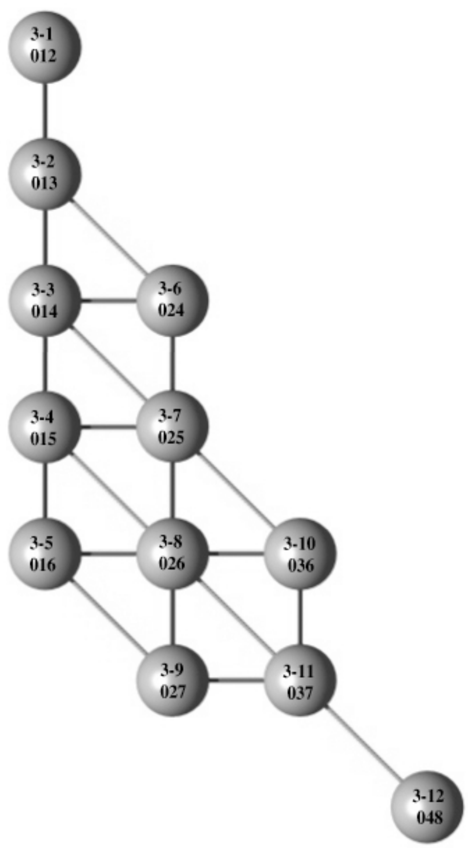


Figure S4. Recent geometrical models of musical structure. All are portions of the quotient spaces described in this report. **A.** John Roeder's "ordered interval space" *(S57)* represents three-note OT equivalence classes in $\mathbb{T}^2$, using the coordinates (0, *x*, *x* + *y*). **B.** Jack Douthett and Peter Steinbach's "Cube Dance" *(S58)* is a graph appearing in $\mathbb{T}^3/\mathcal{S}_3$. **C.** Joe Straus's graph of single-semitone voice leadings among three-note set classes *(S37)* is a graph appearing in $\mathbb{T}^2/(\mathcal{S}_3 \times \mathbb{Z}_2)$. (*continued on next page*)

D **E**

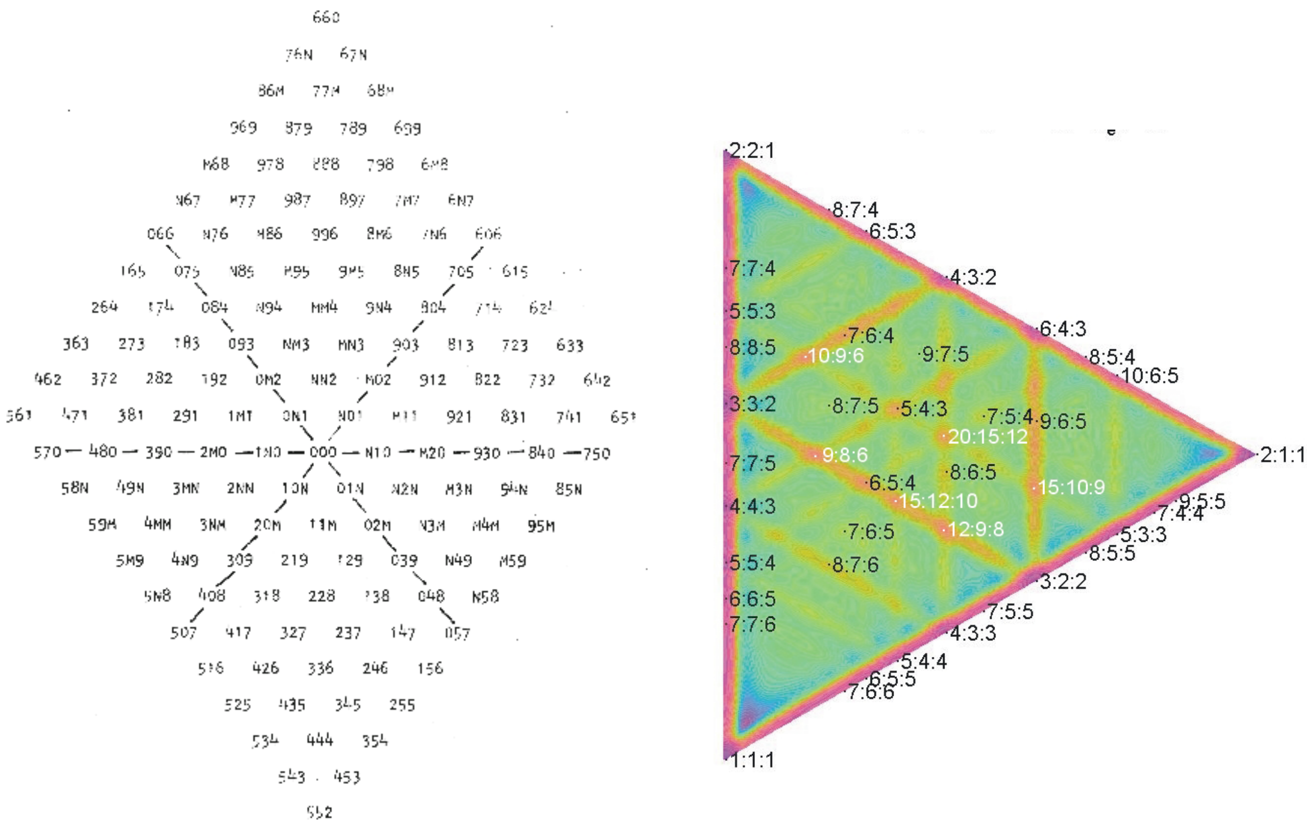


Figure S4 (*continued*). **D**. Roiter's unpublished (1983) model of 3-note OT space, showing its simplicial coordinate system. **E.** Erlich's unpublished (2000) continuous model of 3-note OTI space, here coordinatized with frequency ratios. Colors represent the consonance of the chords, as judged according to Erlich's "harmonic entropy" model. (*continued on next page*)

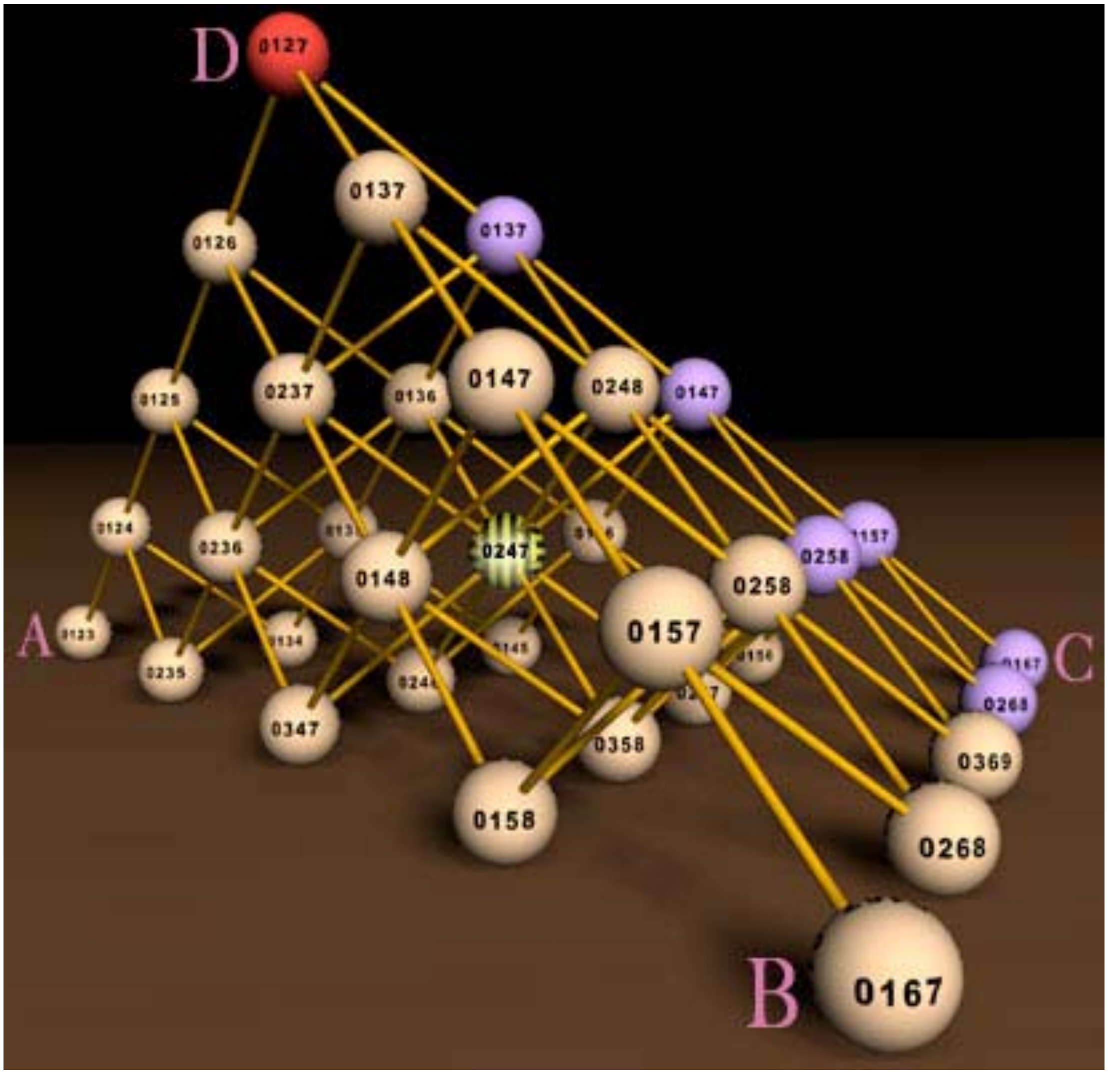


Figure S4 (*continued*). **F.** Richard Cohn's graph of four-note OPTI equivalence classes (*S23)* is a graph in a fundamental domain for $\mathbb{T}^3/(\mathcal{S}_4 \times \mathbb{Z}_2)$. The 29 points drawn here, six of which are drawn twice, correspond to the 29 points in Figure S5F-H. Six points are drawn twice also in Figure S5, four in part F and two in part G; but the choice of which six points to repeat has been made differently in the two figures.

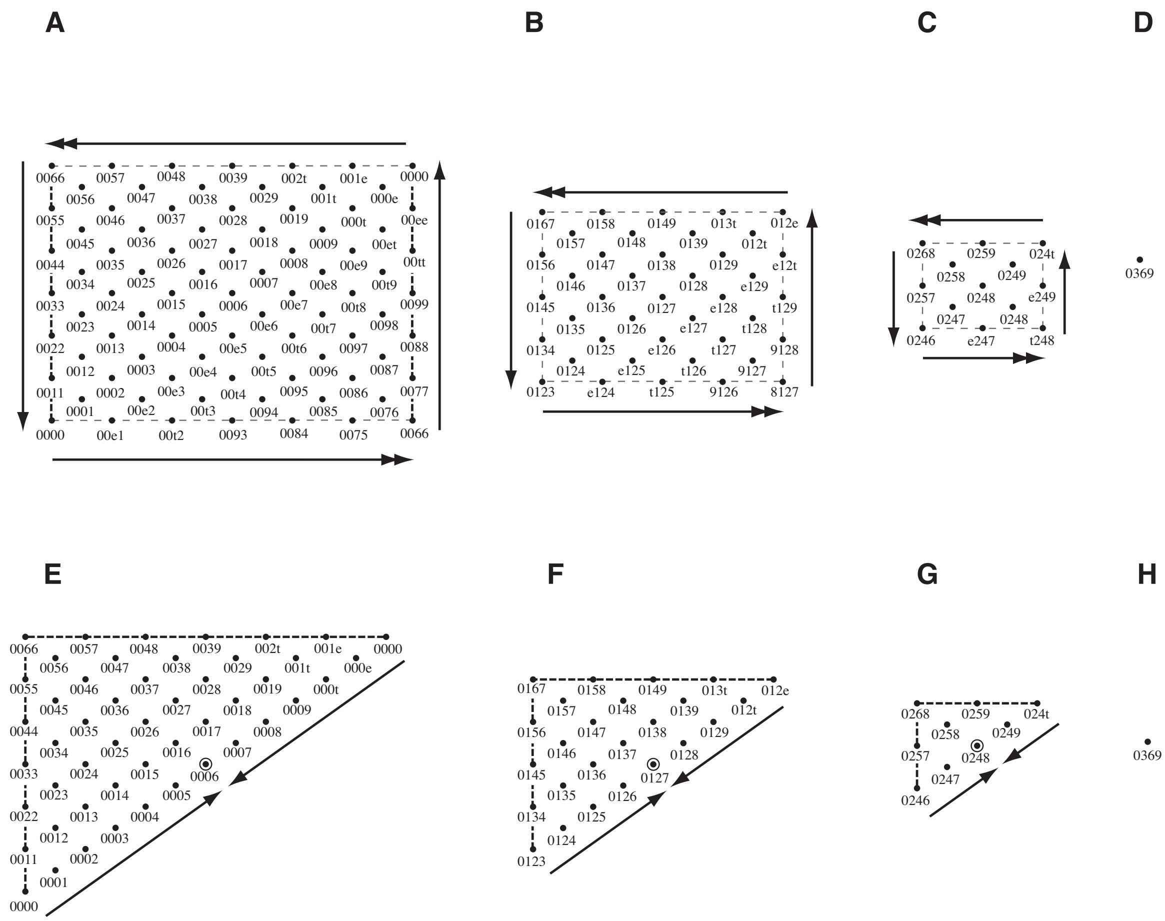
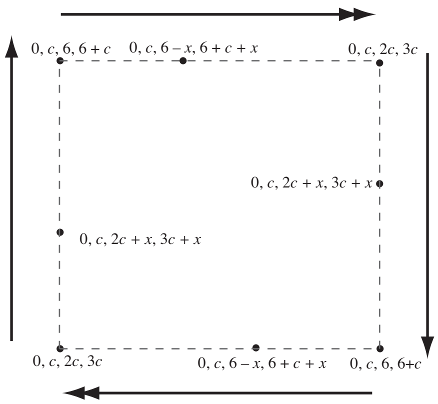
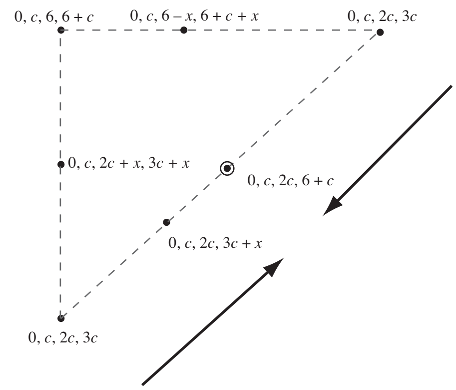


Figure S5. A-D. The space of four-note chord types, $\mathbb{T}^3/\mathcal{S}_4$, can be visualized as a cone whose base is a real projective plane (A). However, this "cone" contains additional orbifold points. (Here "t" = 10 and "e" = 11.) Every point is an OPT equivalence class. The left and right edges of (A) are singular and act like mirrors. The cone consists of a series of geometrically similar "layers." **E-H**. The space of four-note set-classes (OPTI equivalence classes) is $\mathbb{T}^3/(\mathcal{S}_4 \times \mathbb{Z}_2)$, whose layers are quotients of A-D by central inversion. (*continued on next page*)

I



J



K

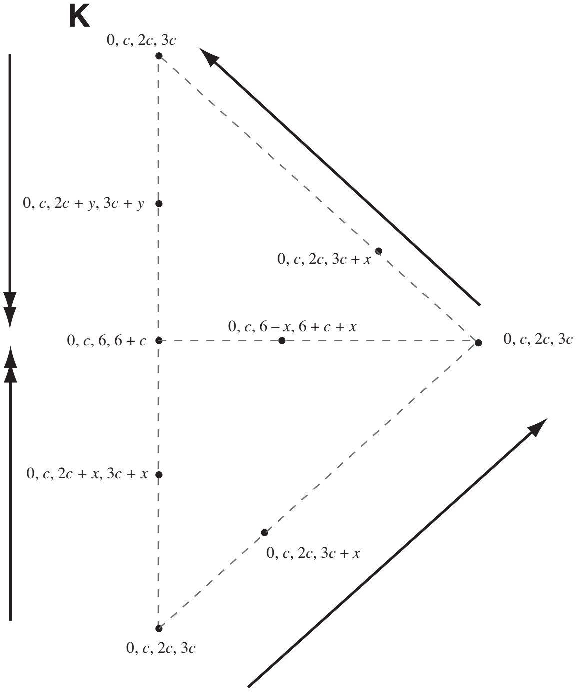


Figure S5 (*continued*). **I-J.** A schematic representation of the layers in A-H. The structure of each layer is determined by a parameter, *c*, representing the smallest interval in the chord. Figures S5A–D correspond to *c* values of 0, 1, 2, and 3, respectively. Note that these coordinates represent OPT and OPTI equivalence classes. Thus, {0, 2, 4} and {0, 2, 10} are both instances of {0, 2, 4}. **K.** In A-D, it is clear that the base of the cone is the real projective plane. However, this representation conceals the relation between line segments and voice leadings: for instance, there is no voice leading corresponding to the line segment (e, 1, 2, 7)→(0, 1, 3, 7) in Figure S5B. (This is analogous to the fact that, on Figure S10, there is no voice leading that follows an inner contour through its “kink.”) For a representation in which line segments faithfully represent voice leadings, we can depict the base as a triangle, as in Figure S5K. Here, however, it is somewhat less obvious that the base of the cone is a real projective plane. (*continued on next page*)

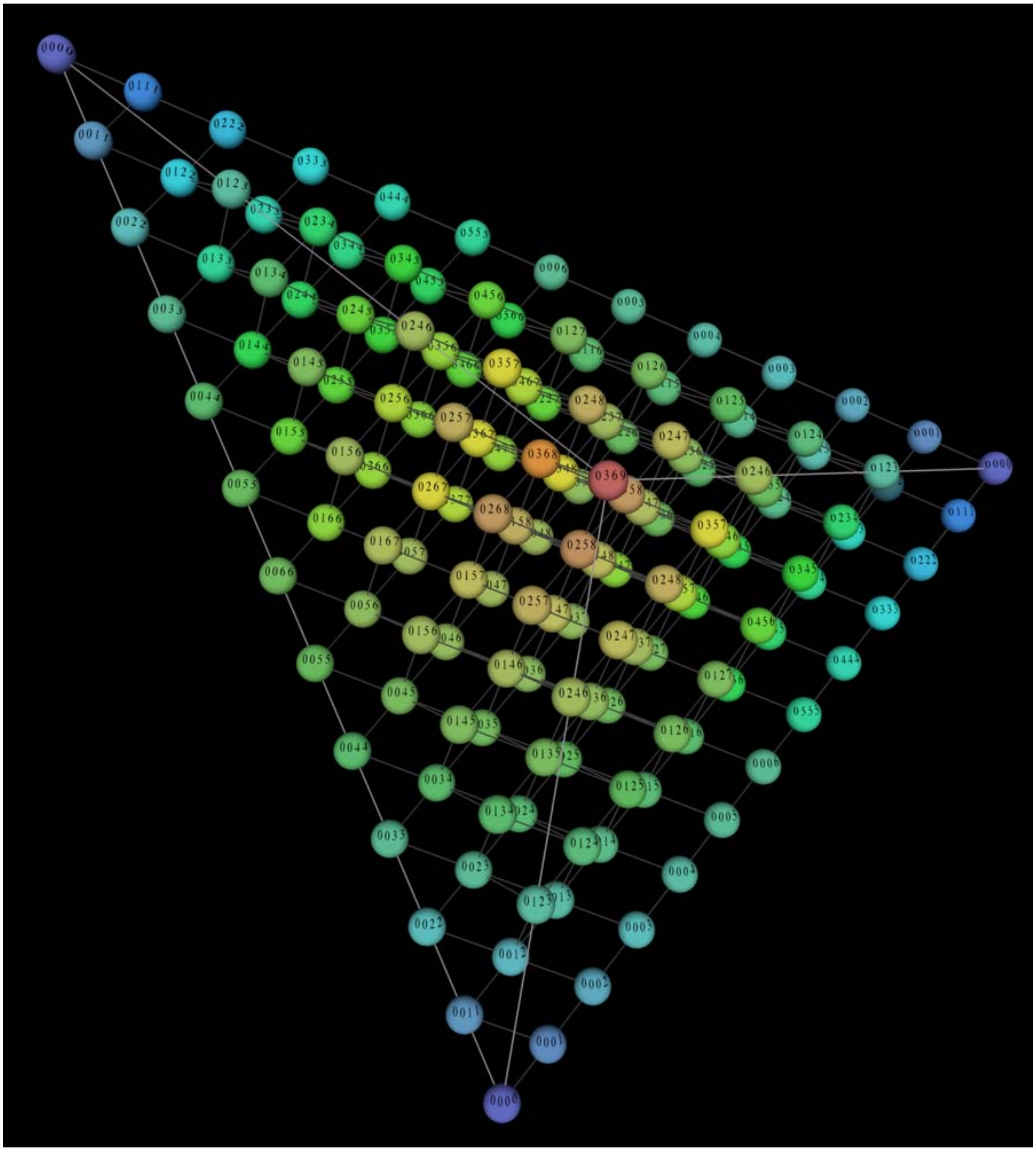


Figure S5 (*continued*). **L.** $\mathbb{T}^3/\mathcal{S}_4$, as seen from above. Figure S5K presents a schematic view of each of the layers. Color indicates how evenly a chord divides the octave, with the red chord (0369) being perfectly even, and the deep blue chords (0000) being perfectly clustered. (*continued on next page*)

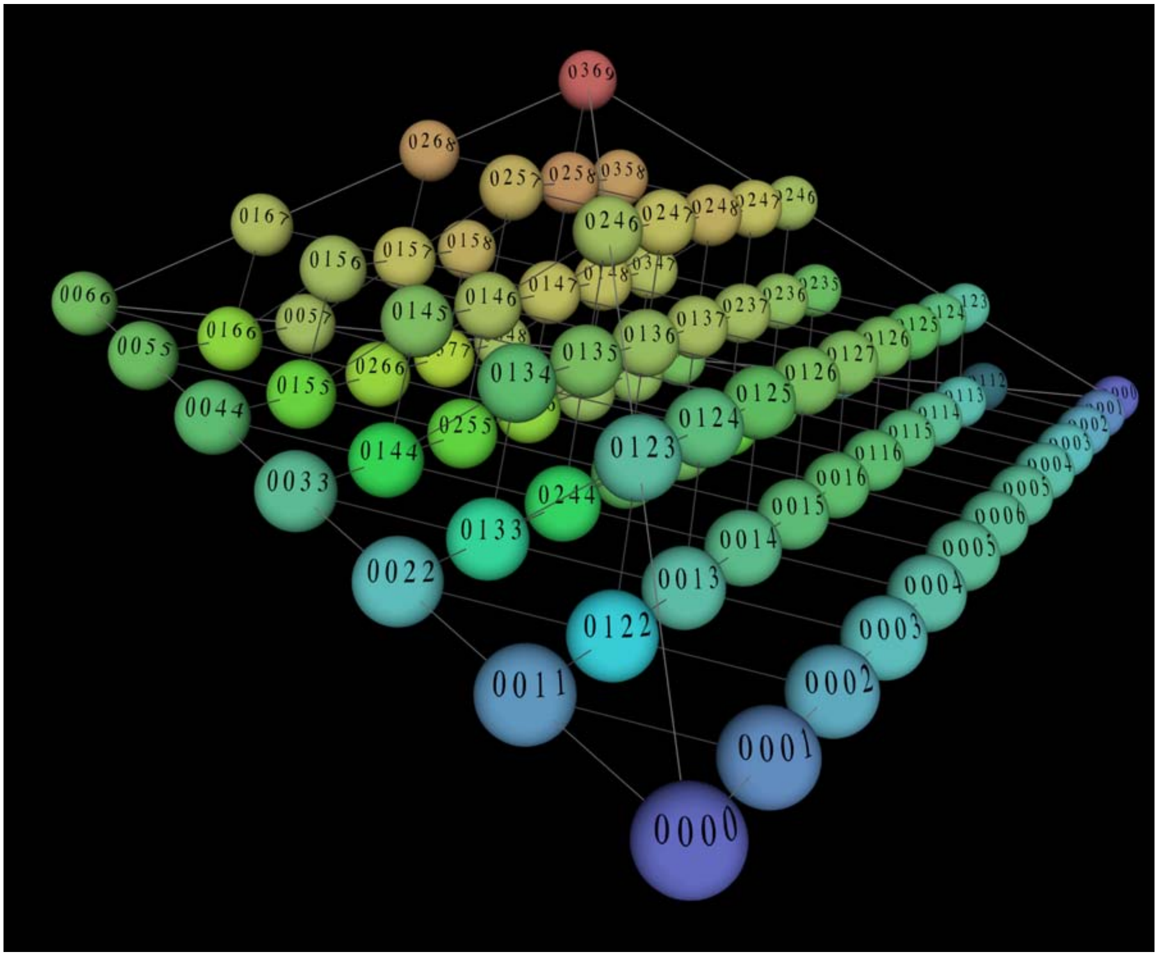


Figure S5 (*continued*). **M.** $\mathbb{T}^3/(\mathcal{S}_4 \times \mathbb{Z}_2)$, seen from a side view. The layers correspond to Figure S5E-H. This and the preceding image were made with Dmitri Tymoczko's "ChordGeometries" program.

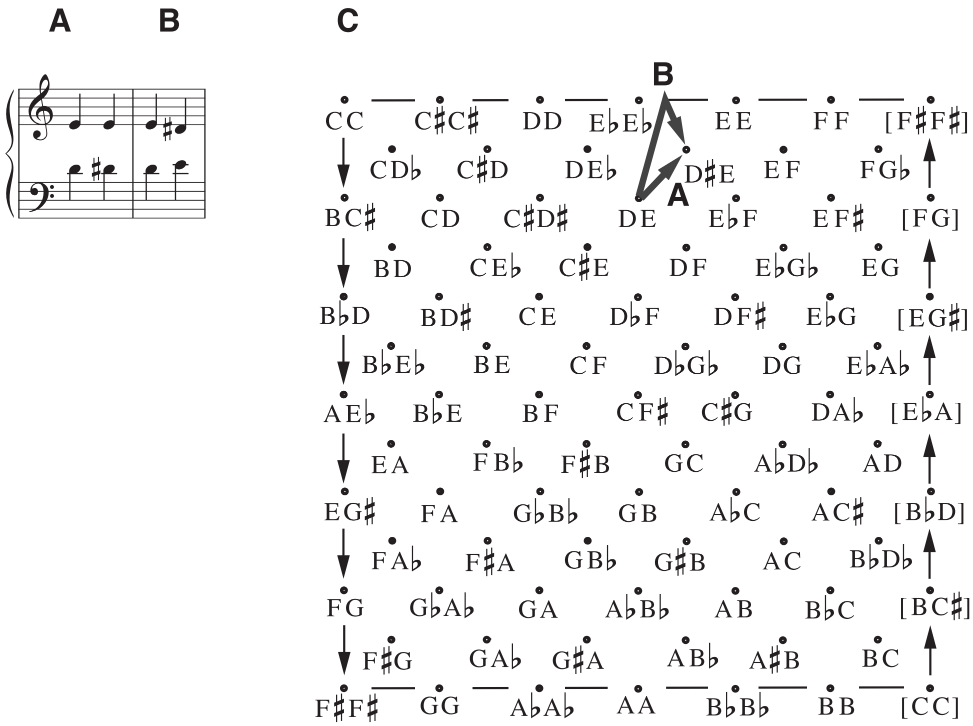


Figure S6. A–B. The progressions (D, E)→(D♯, E) and (D, E)→(E, D♯). **C.** The line segments in $\mathbb{T}^2/\mathcal{S}_2$ corresponding to these progressions.

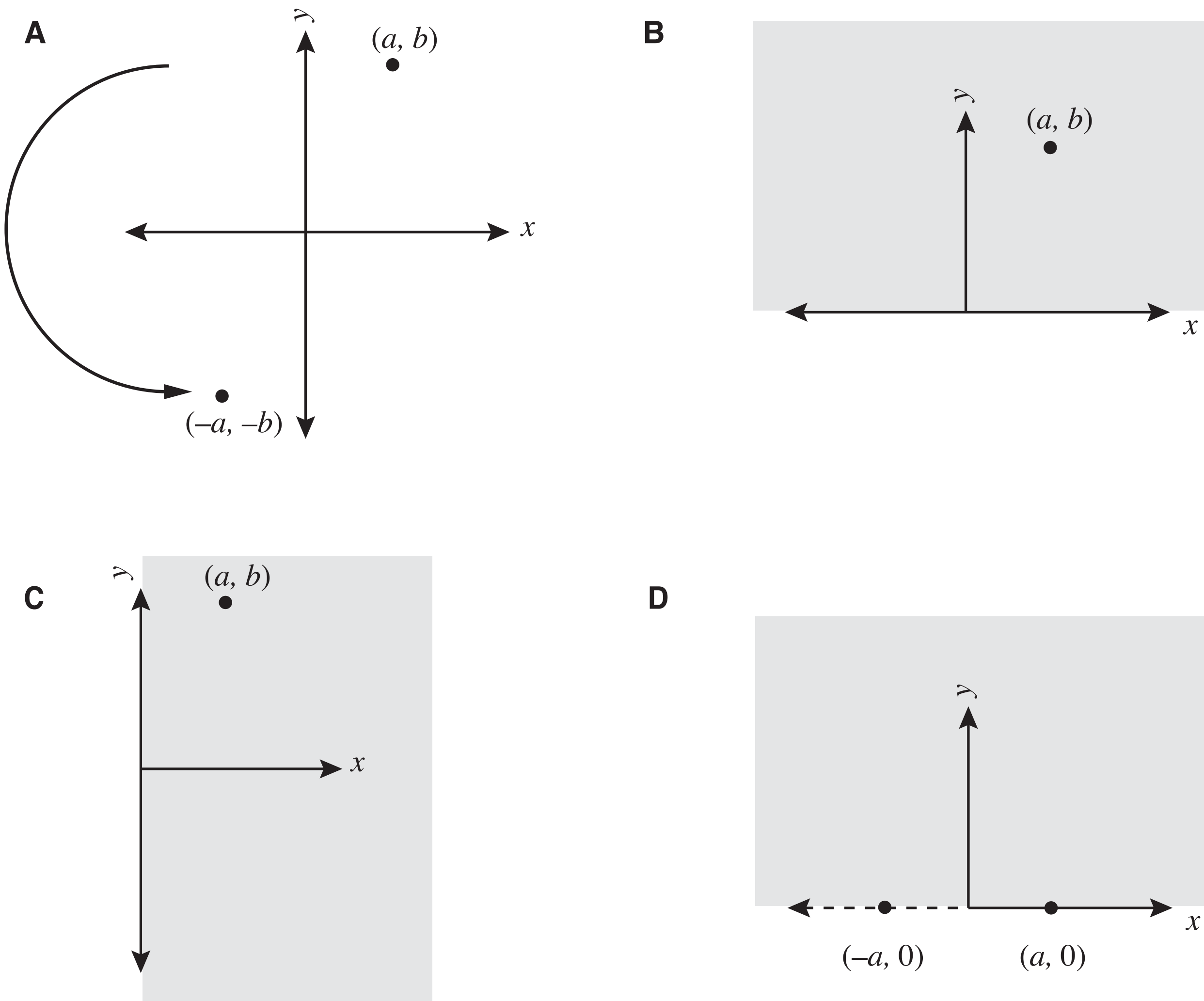


Figure S7. Fundamental domains and quotient spaces. **A.** Every point in the upper half plane is related by 180° rotation to a point in the lower half-plane. **B.** The upper half plane (including the *x* axis) is therefore a fundamental domain for the rotation. This fundamental domain is a region of $\mathbb{R}^2$, and does not possess any unusual topological or geometrical properties. **C.** Fundamental domains are not in general unique. Thus the right half-plane is also a fundamental domain for 180° rotation. **D.** To transform a fundamental domain into the quotient space, it is necessary to identify boundary points related by the symmetry operation. Here, this requires gluing the positive *x* axis to the negative *x* axis so as to attach (a, 0) to (–a, 0). The result is a cone. Though there are many fundamental domains, they all produce the same quotient space upon identification of the appropriate boundary points.

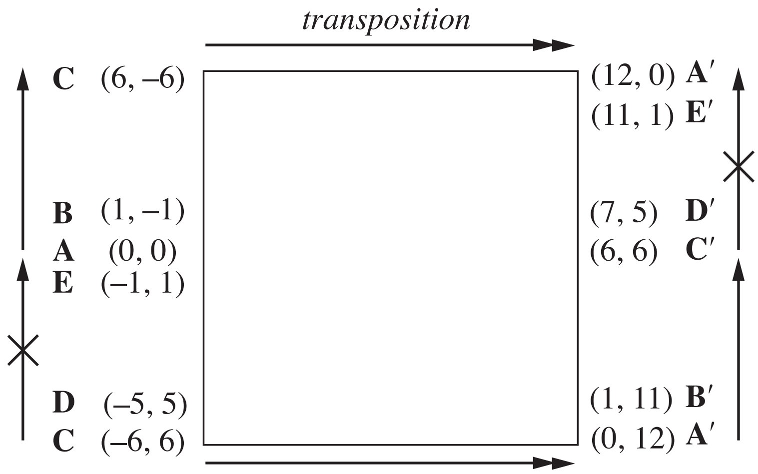


Figure S8. A fundamental domain for the 2-torus $\mathbb{T}^2$. The top edge is glued to the bottom, forming a cylinder. The left edge is then glued to the right so as to match the appropriate chords. This is an unusual fundamental domain for the 2-torus, but it has the advantage of clearly representing transposition, shown here as horizontal motion.

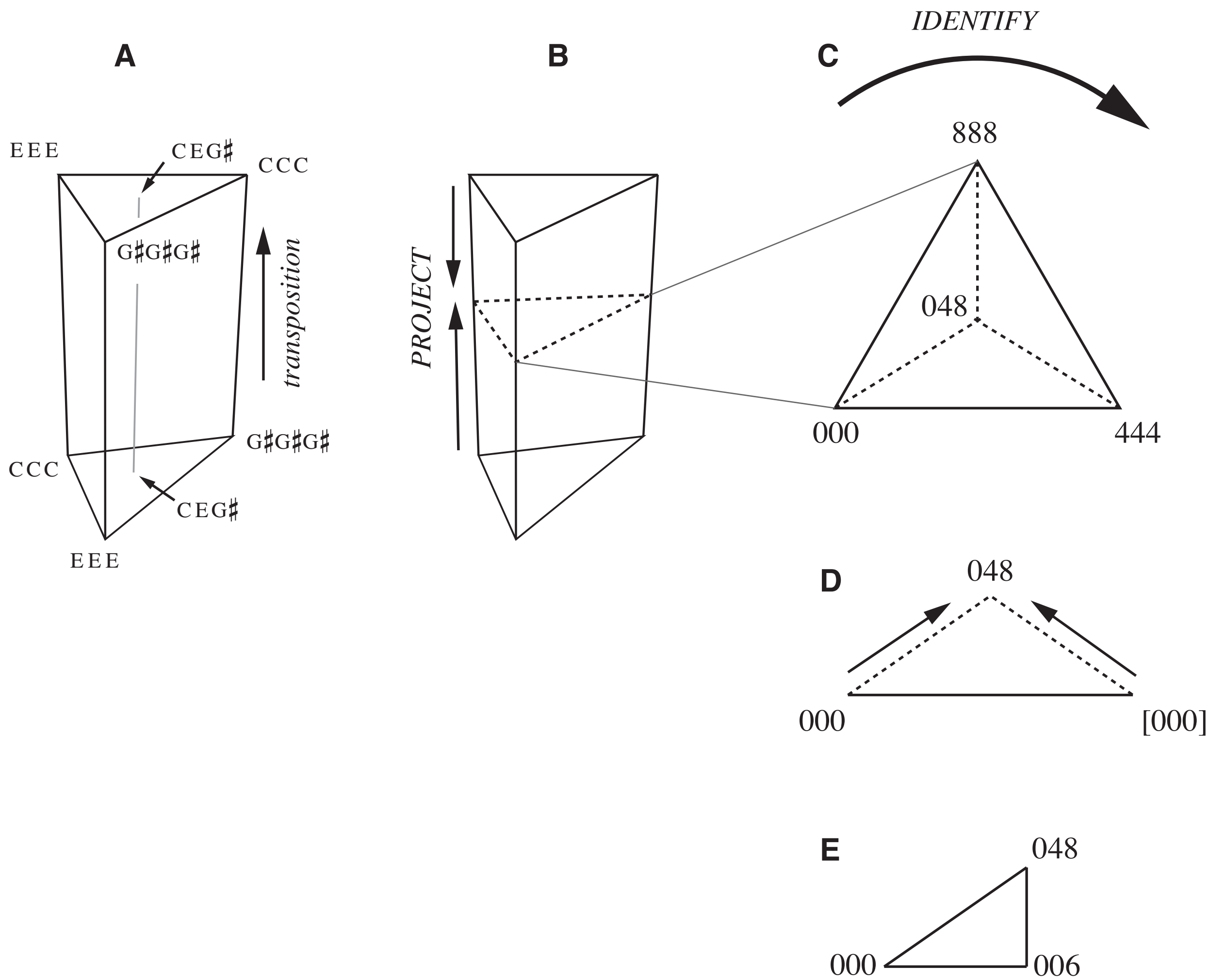


Figure S9. A. Three-note chord space $\mathbb{T}^3/\mathcal{S}_3$ is a prism whose opposite faces are identified by way of a 120° "twist," matching CCC to CCC, EEE to EEE, and G♯G♯G♯ to G♯G♯G♯. Transposition is represented by vertical motion. To identify transpositionally related points, take a cross section (**B**) of the prism and identify all points related by 120° rotation (**C**). This produces a cone (**D**). To identify inversionally related chord types, take the further quotient by reflection, as shown in **E**.

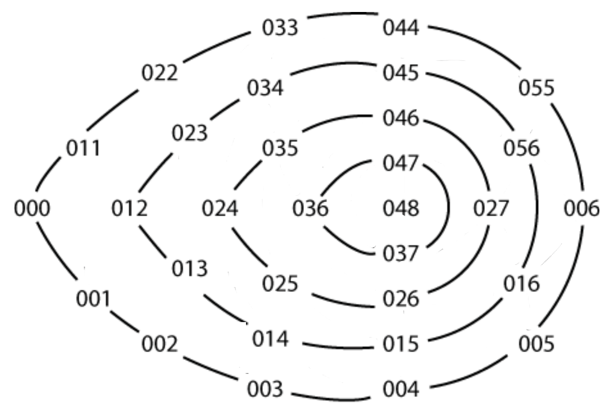


Figure S10. Three-note OPT space ($\mathbb{T}^2/\mathcal{S}_3$) can be understood as a series of similar "layers." Here, the outermost layer contains chord-types whose smallest interval is 0; the next layer contains chord-types whose smallest interval is 1, and so on.

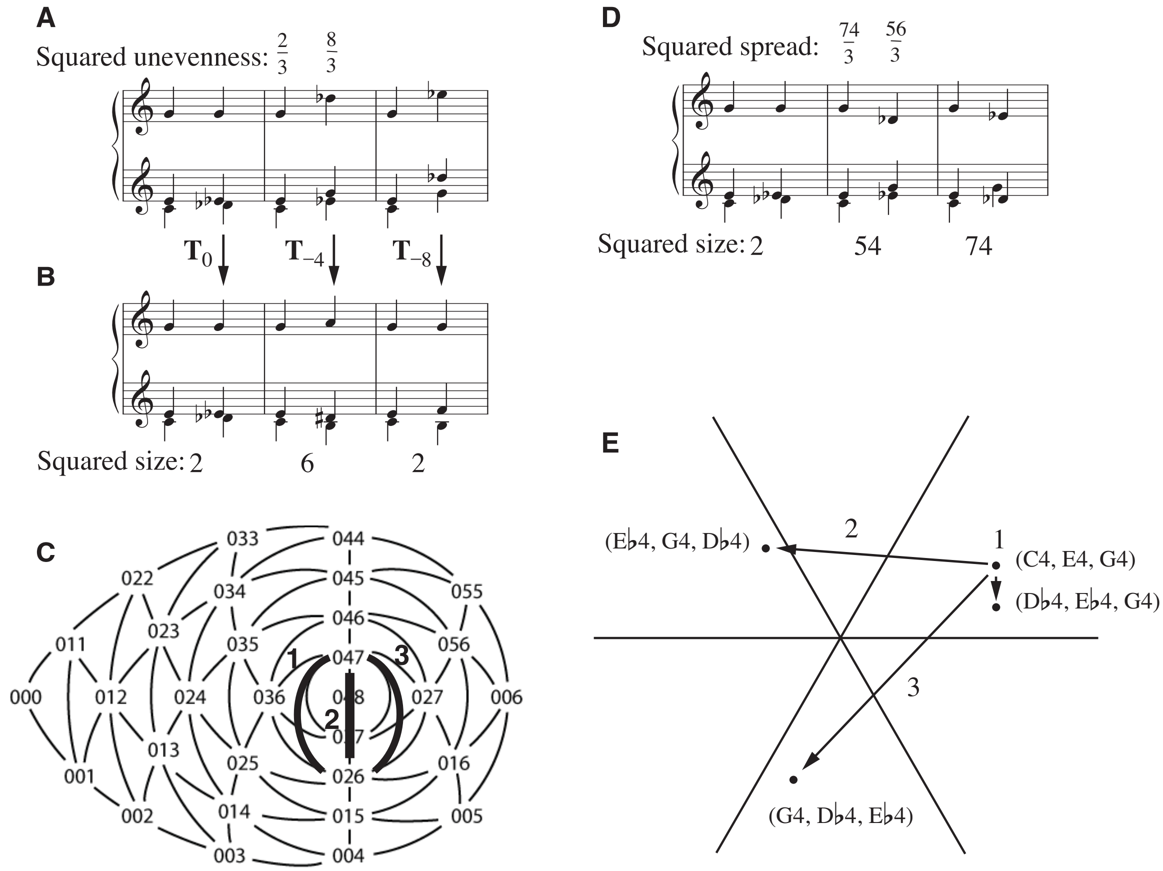


Figure S11. A. Beginning with a voice leading that does not touch the boundary of OPT space, transform the second chord by successively transposing its bottom note up by octave. **B.** Then consider the smallest voice leadings individually T-related to these. If one is using the Euclidean metric, the resulting voice leadings will connect chords summing to the same value, and their average squared size will be equal to the sum of the squared "unevenness" of the two original chords (here 10/3). ("Unevenness" is measured by the size of smallest voice leading from a chord to the nearest chord dividing the octave perfectly evenly.) **C.** In three-note OPT space, the three voice leadings in B represent three different line segments between the same two points, none touching the space's singular boundary. Their average squared length is equal to the sum of their endpoints' squared distances to the cone's vertex. **D.** Beginning with any voice leading in pitch space, circularly permute the pitches of second chord. Consider the smallest voice leadings individually T-related to these. (In the example shown, these voice leadings are already as small as possible.) Using the Euclidean metric, the average squared size of these voice leadings (here 130/3) will be equal to the sum of the squared "spread" of the two original chords. ("Spread" is measured by the size of smallest voice leading from the original chord to the nearest chord with just a single pitch class.) **E.** The three voice leadings in **D**, shown in T-space.

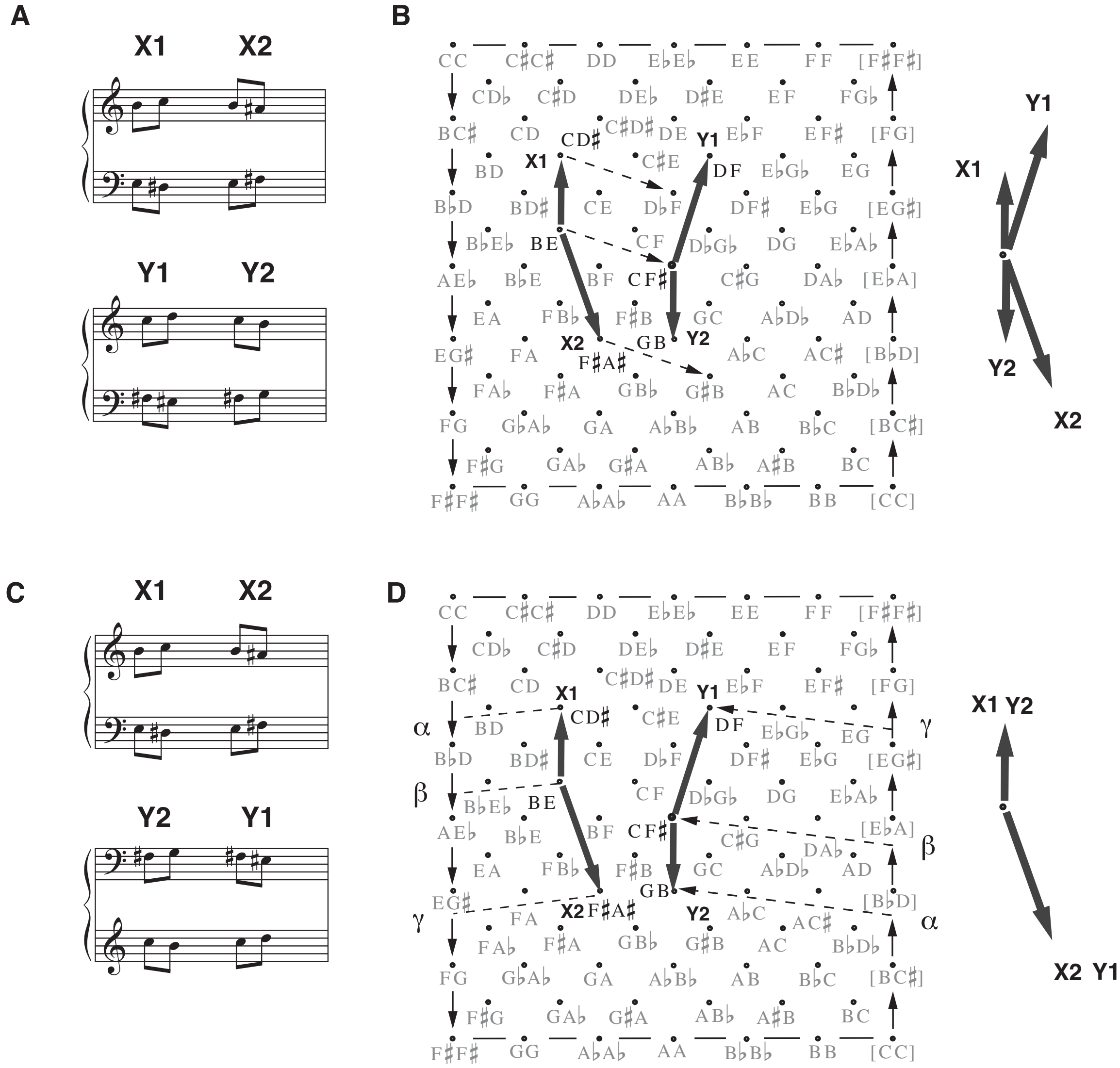


Figure S12. A. Four voice-leadings from the opening of Brahms's Op. 116, no. 5: X1 is similar to Y1 and X2 is similar to Y2. **B.** Graphing the voice leadings in the Möbius strip representing two-note chords ($\mathbb{T}^2/\mathcal{S}_2$) shows that X1 and X2 can be parallel-transported so as to nearly coincide with Y1 and Y2, respectively. **C.** X1 and X2 are also highly similar to Y2 and Y1, as can be seen by switching the two hands. **D.** In $\mathbb{T}^2/\mathcal{S}_2$, X1 and X2 can be parallel transported to exactly coincide with Y2 and Y1, respectively. This can be seen from the fact that the two pairs of arrows are related by reflection.

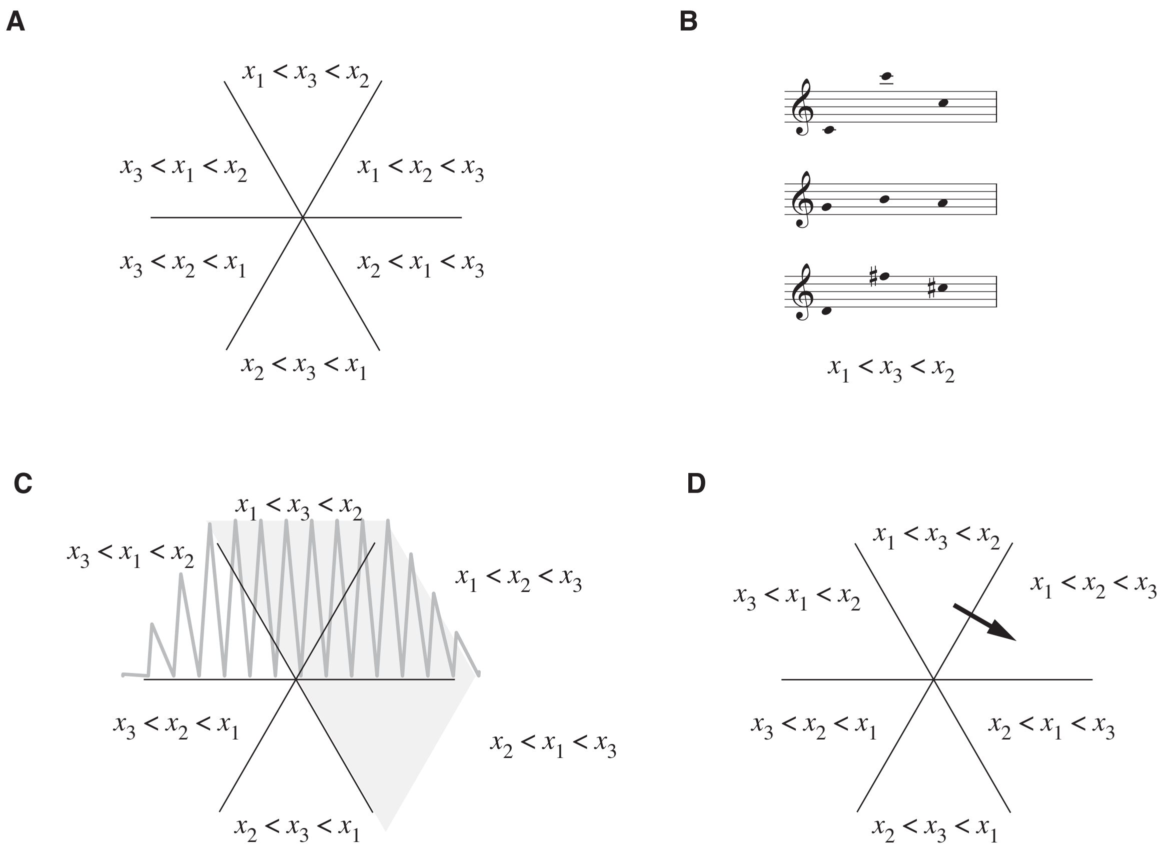


Figure S13. A. T-space is divided into regions representing objects sharing the same contour. **B.** Three musical objects sharing the contour $x_1 < x_3 < x_2$. **C.** Elizabeth West Marvin's "contour similarity index" is equal to the number of half planes of the form $x_i < x_j$ containing the two contours, divided by the maximum number of potential matches. Here, the similarity of $x_1 < x_3 < x_2$ and $x_1 < x_2 < x_3$ is equal to 2/3. **D.** The similarity of two contours is a linearly decreasing function of the number of hyperplanes that need to be crossed in moving between their respective regions.

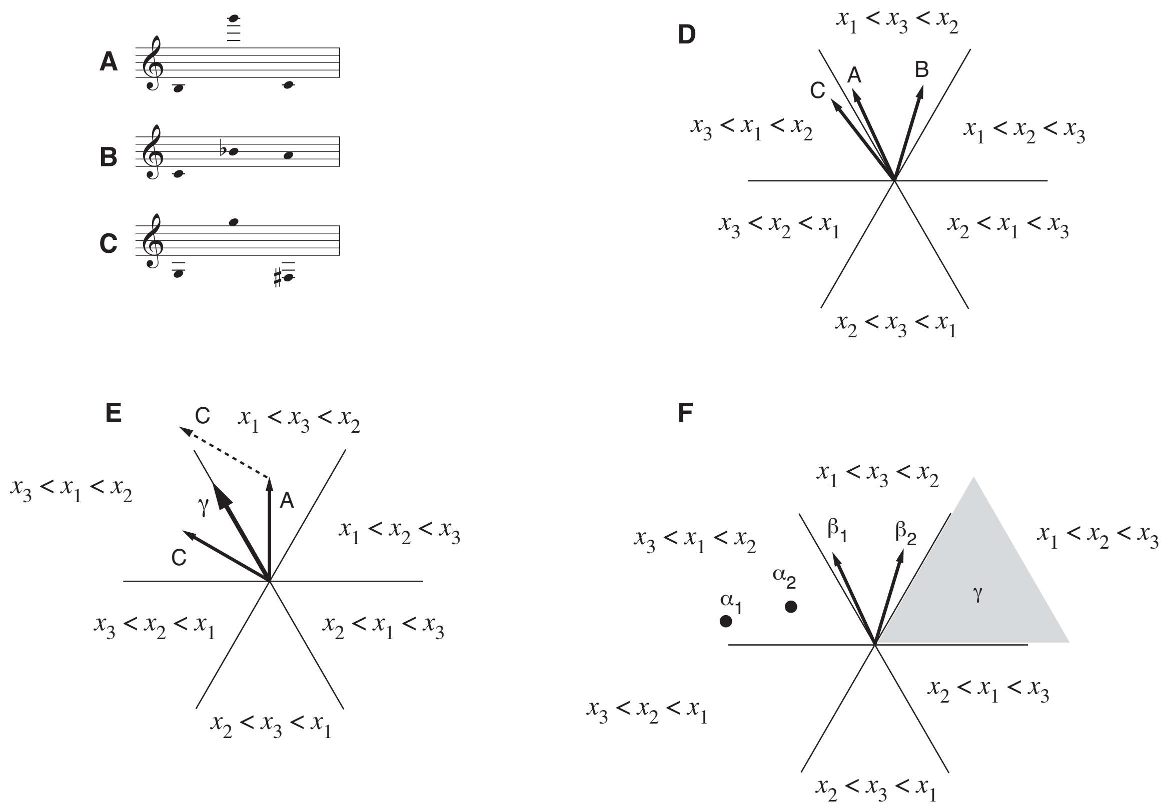


Figure S14. A-C. Contour A is in some ways more similar to C than B, even though A and B both represent the sequence low-high-middle, while C represents middle-high-low. **D.** We can individuate contours more finely by representing them as line segments of unit length in T-space. In this representation, A is closer to C than B. **E.** A contour region can be represented by the vector ($\sigma(1)$, $\sigma(2)$, …, $\sigma(n)$) at its center. These vectors can be averaged in the standard way. Here, γ represents the average of the contours A ($x_1 < x_3 < x_2$) and C ($x_3 < x_1 < x_2$). **F.** The geometrical perspective provides a range of theoretical abstractions. Contours can be represented as points (α_1, α_2), rays (β_1, β_2) or regions (γ). Measures of voice-leading size can be used to determine the distance between these objects.

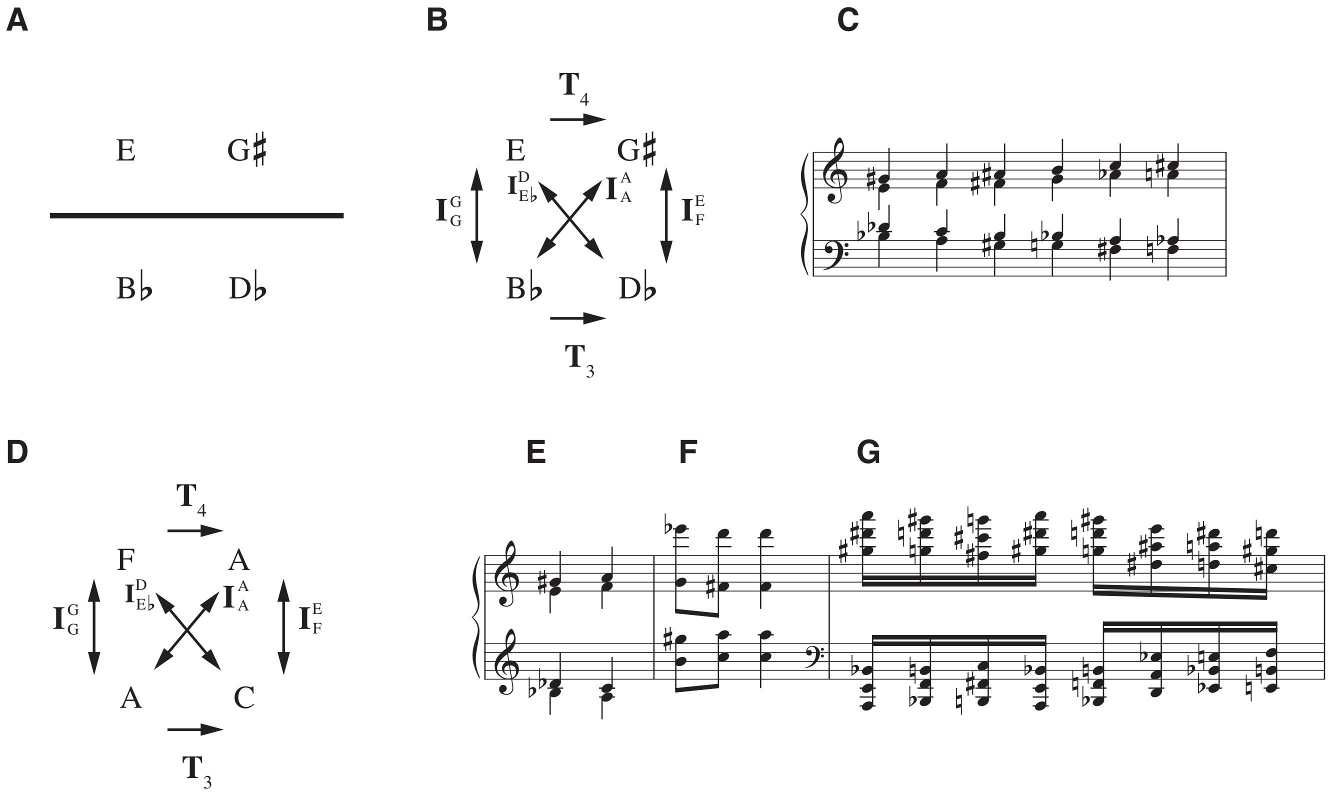


Figure S15. A. A K-net is a chord partitioned into two parts. Here, the K-net {B♭, D♭} + {E, G♯} partitions {B♭, D♭, E, G♯} into a major and minor third. **B.** Notes in the same partition are interpreted as being related by transposition, while notes in opposite partitions are interpreted as being related by inversion. **C-D.** A K-net defines a collection of voice leadings in which the two parts move semitonally in contrary motion. These voice leadings produce chords of many different types, all of which share the same graph of transpositional and inversional relationships (D). **E-G.** Exact contrary motion in Strauss's *Till Eulenspiegel,* Mozart's Symphony no. 40, movement 1, and Messiaen's *Vingt Regards,* no. 20.

A

A C♯ E	F A C	D F♯ A	F A C
F	F♯	G♯	G♯

B

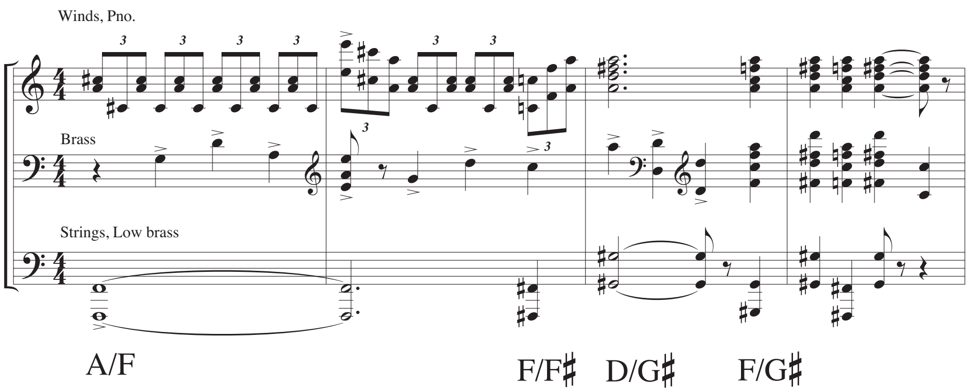


Figure S16. Chords in **A** are all strongly isographic because they can be partitioned into a major triad and a single note. **B.** Movement 3 of Stravinsky's *Symphony of Psalms,* m. 46-51. The passage contains the four chords in (A).

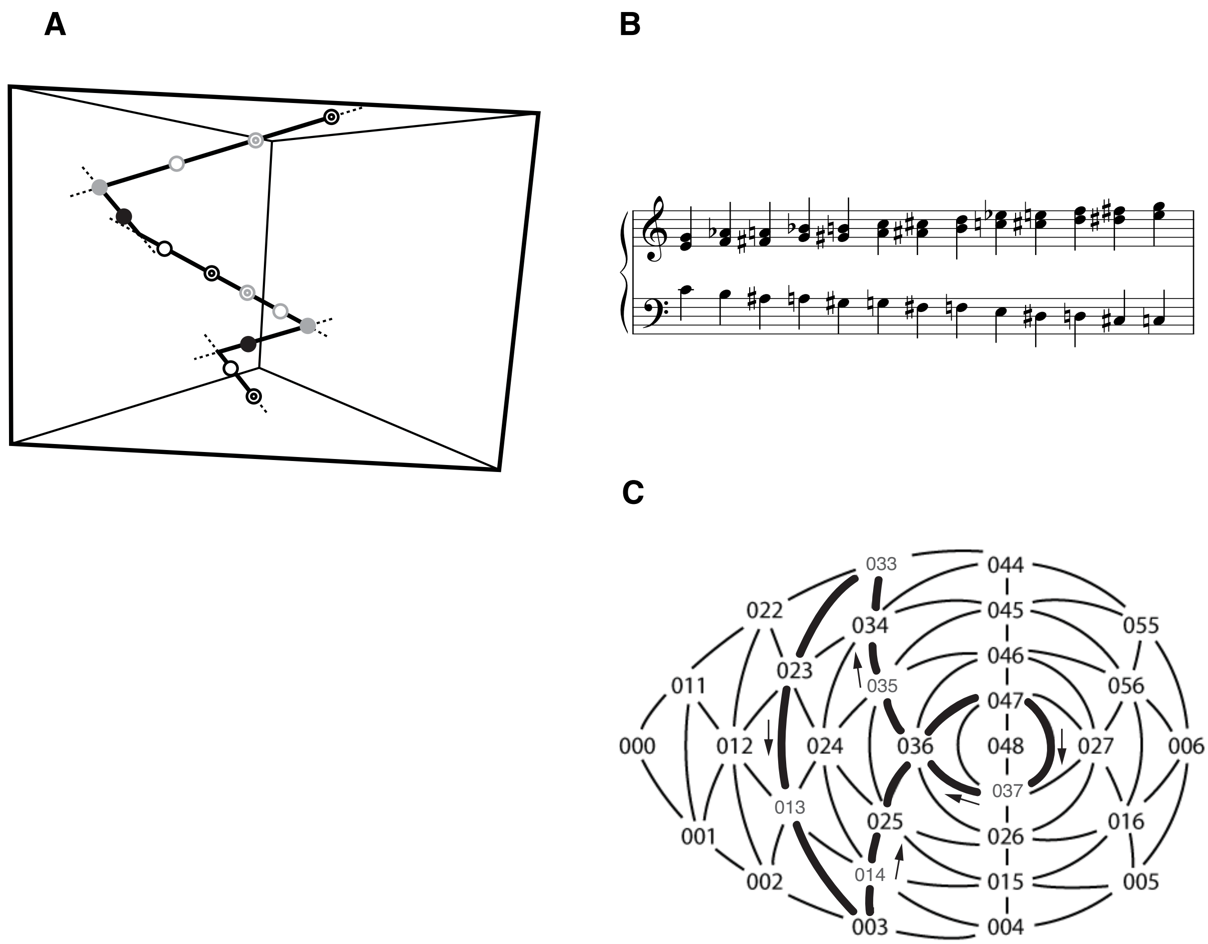
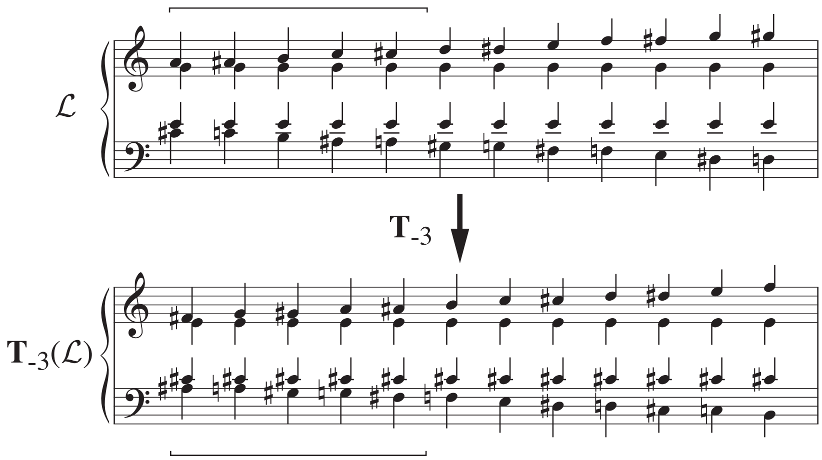


Figure S17. A. A K-net defines a line in OP space. Here, the line contains the equal-tempered chords shown in (**B**). (Major triads and diminished triads are represented by the grey and black "target" patterns, respectively.) The line reflects off the space's mirror boundaries, and hence appears to contain "kinks." **C.** The image of this K-net line in OPT space contains all three-note chord-types with a minor third. The sequence in (B) skips every other equal-tempered chord-type on this line, thus avoiding chord-types such as 037 altogether.

A



B

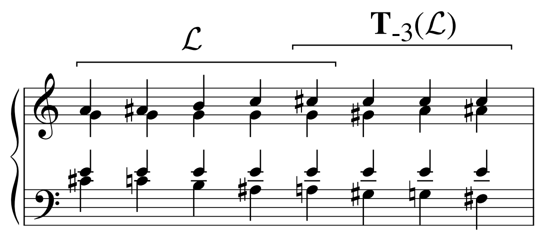


Figure S18. A. The "omnibus" pattern defines a line $\mathcal{L}$ in four-note chord space. Chords on this line contain the notes {E, G} and have two other notes related by inversion around B. Transposing these chords downward by three semitones produces the line $\mathbf{T}_{-3}(\mathcal{L})$. Chords on the top staff are therefore positively $\mathcal{L}$-isographic to those on the bottom, and relate by the "hyper transposition" $<\mathbf{T}_{-3}>_{\mathcal{L}}$. **B**. Classical composers often used the fact that both lines contain an A^7 chord to shift from one to the other.

A

B

voice leading α→γ

voice leading α→β

CE+G	BE+G	B♭E+G	AE+G	A♭E+G	GE+G	G♭E+G	FE+G	EE+G	E♭E+G	DE+G	D♭E+G
D♭F+G♭	CF+G♭	BF+G♭	B♭F+G♭	AF+G♭	A♭F+G♭	GF+G♭	G♭F+G♭	FF+G♭	EF+G♭	E♭F+G♭	DF+G♭
DF♯+F	D♭F♯+F	CF♯+F	BF♯+F	B♭F♯+F	AF♯+F	A♭F♯+F	GF♯+F	G♭F♯+F	FF♯+F	EF♯+F	E♭F♯+F
E♭G+E	DG+E	D♭G+E	CG+E	BG+E	B♭G+E	AG+E	A♭G+E	GG+E	G♭G+E	FG+E	EG+E
EG♯+E♭	E♭G♯+E♭	DG♯+E♭	D♭G♯+E♭	CG♯+E♭	BG♯+E♭	B♭G♯+E♭	AG♯+E♭	A♭G♯+E♭	GG♯+E♭	G♭G♯+E♭	FG♯+E♭
FA+D	EA+D	E♭A+D	DA+D	D♭A+D	CA+D	BA+D	B♭A+D	AA+D	A♭A+D	GA+D	G♭A+D
G♭B♭+D♭	FB♭+D♭	EB♭+D♭	E♭B♭+D♭	DB♭+D♭	D♭B♭+D♭	CB♭+D♭	BB♭+D♭	B♭B♭+D♭	AB♭+D♭	A♭B♭+D♭	GB♭+D♭
GB+C	G♭B+C	FB+C	EB+C	E♭B+C	DB+C	D♭B+C	CB+C	BB+C	B♭B+C	AB+C	A♭B+C
A♭C+B	GC+B	G♭C+B	FC+B	EC+B	E♭C+B	DC+B	D♭C+B	CC+B	BC+B	B♭C+B	AC+B
AC♯+B♭	A♭C♯+B♭	GC♯+B♭	G♭C♯+B♭	FC♯+B♭	EC♯+B♭	E♭C♯+B♭	DC♯+B♭	D♭C♯+B♭	CC♯+B♭	BC♯+B♭	B♭C♯+B♭
B♭D+A	AD+A	A♭D+A	GD+A	G♭D+A	FD+A	ED+A	E♭D+A	DD+A	D♭D+A	CD+A	BD+A
BD♯+A♭	B♭D♯+A♭	AD♯+A♭	A♭D♯+A♭	GD♯+A♭	G♭D♯+A♭	FD♯+A♭	ED♯+A♭	E♭D♯+A♭	DD♯+A♭	D♭D♯+A♭	CD♯+A♭

Figure S19. A. The voice leadings α→β and α→γ define a plane in three-note chord space (**B**, shown also in **C**). Chords on this plane are said to be "axially isographic." (*continued on next page*)

C

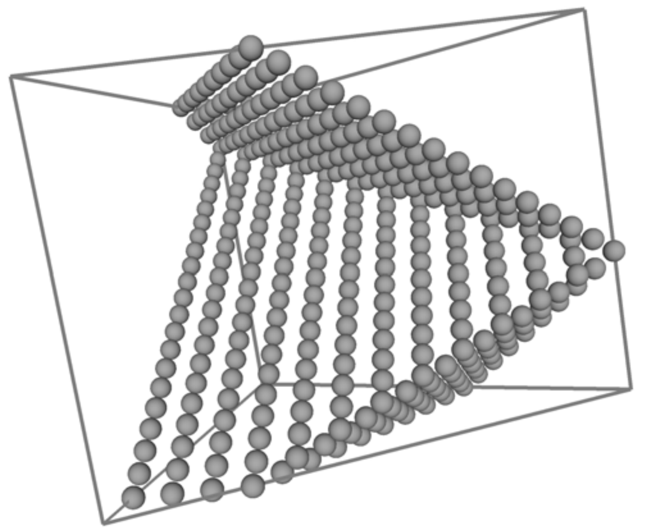


Figure S19 (*continued*). **C.** The chords in B define a plane in three-note OP space ($\mathbb{T}^3/\mathcal{S}_3$). Since the plane reflects of the space's mirror boundaries, it appears to have kinks.

Term	Symmetry	Definition	Example
Single objects			
pitch class	O	An octave-free note-type	G3, G4, G5
chord of pitches, pitch set	P	An unordered collection of pitches	(C4, E4, G4), (E4, C4, G4)
chord, set	OPC	An unordered set of pitch classes	(C4, E4, G4), (E4, G4, G5, C3)
transpositional set class	OPTC	A class of T-related chords	(C4, E4, G4), (F♯3, D2, A5, D6)
TI set class	OPTIC	A class of T or I related chords	(C4, E♭4, G4), (F♯3, D2, A5, D6)
Progressions I: Intervals (one-note progressions)			
pitch-class interval (I)	individual O	An ordered pair of pitch classes	C4→E4, C2→E2, C4→E5
pitch-class interval (II)	individual O, uniform T	Pitch class intervals (I) related by T (e.g. "ascending major third")	C4→E4, D4→F♯2
pitch-class interval class	individual O, uniform TI	Pitch class intervals (I) related by T or I	C4→E4, F♯2→D5
path in pitch-class space	uniform OT	A specific path between pitch classes (e.g. "eleven ascending semitones")	C4→B3, C5→B4, G4→F♯4 but not C4→B4
Progressions II: Chord progressions (individual OPC)			
chord progression	individual OPC	A succession of pitch class sets (e.g. "C major moving to F major")	(C4, E4, G4)→(C4, F4, A4) (C2, E3, C4, G4)→(F2, A3, C5)
chord progression type, transpositional chord-progression class	individual OPC, uniform T	Chord progressions related uniformly by transposition. (e.g. "major chords descending by fifth")	(C4, E4, G4)→(C4, F4, A4) (D2, F♯3, D4, A4)→(G2, B3, D5)
Schritt, *Wechsel* (Oettingen, Riemann)	individual OPC, uniform TI	Chord progressions related uniformly by TI (e.g. Riemann's "*Gegenquintschritt*")	(C4, E♭4, G4)→(B♭4, D4, G4) (D2, F♯3, D4, A4)→(G2, B3, D5)
progression of set classes	individual OPTIC	A succession of set classes (e.g. "an 013 followed by an 015")	(D2, F♯3, D4, A4)→(F2, A3, C5) (C4, E♭4, G4)→(B4, D♯4, G♯4))

Table S1, page 1.

Progressions III: Voice Leadings (uniform P)			
voice leading between pitch sets	uniform P	A bijective function between two (unordered) chords of pitches	(C4, E4, G4)→(C4, F4, A4) (E4, G4, C4)→(F4, A4, C4)
path-neutral voice leading between pitch- class sets (Lewin, Straus)	individual O, uniform P	A bijective function between two chords of pitch classes	(C4, E4, G4)→(C4, F4, A4) (E5, C2, G3)→(F2, C6, A4)
path-specific voice leading between pitch-class sets (Tymoczko)	uniform OP	A bijective function from the pitch classes in one set to paths in pitch-class space.	(C4, E4, G4)→(C4, F4, A4) (G2, E4, C5)→(A2, F4, C5)
OPT voice-leading class	uniform OP, individual T	Path-specific voice leadings related individually by transposition	(C4, E4, G4)→(C4, F4, A4) (E4, C5, G5)→(E4, B4, G♯5)
OPTI voice-leading class	uniform OPI, individual T	Path-specific voice leadings related by individual transposition or uniform inversion	(C4, E4, G4)→(C4, F4, A4) (C4, E♭4, G4)→(B3, D♯4, G♯4)
Progressions IV: General categories of voice leading			
transpositional voice leading (Straus)	any path-neutral voice leading related to the identity by individual T	A voice leading that moves every pitch class by the same pitch-class interval (I) (*see above*).	(C4, E4, G4)→(E♭2, G4, B♭5)
inversional voice leading (Straus)	any path-neutral voice leading related to the identity by uniform I	A voice leading that moves each pitch class to its inversion around some fixed point in pitch-class space.	(C4, E4, G4)→(E4, C4, A3)

Table S1, page 2.

Symmetry	Space	Fundamental Domain
O	$\mathbb{T}^n$	$\max(x_1, x_2, \ldots, x_n) \le \min(x_1, x_2, \ldots, x_n) + 12$ $0 \le \Sigma x_i \le 12$
P	$\mathbb{R}^n/\mathcal{S}_n$	$x_1 \le x_2 \le \ldots \le x_n$
T	$\mathbb{R}^{n\text{-}1}$	$\Sigma x_i = 0$
I	$\mathbb{R}^n/\mathbb{Z}_2$	$x_2 - x_1 \le x_n - x_{n\text{-}1}$
OP	$\mathbb{T}^n/\mathcal{S}_n$	$x_1 \le x_2 \le \ldots \le x_n \le x_1 + 12$ $0 \le \Sigma x_i \le 12$
OT	$\mathbb{T}^{n\text{-}1}$	$\min(x_1, x_2, \ldots, x_n) = x_1$ $\max(x_1, x_2, \ldots, x_n) \le x_1 + 12$ $\Sigma x_i = 0$
OI	$\mathbb{T}^n/\mathbb{Z}_2$	$\max(x_1, x_2, \ldots, x_n) \le \min(x_1, x_2, \ldots, x_n) + 12$ $0 \le \Sigma x_i \le 12$ $x_2 - x_1 \le x_n - x_{n\text{-}1}$
PT	$\mathbb{R}^{n\text{-}1}/\mathcal{S}_n$	$x_1 \le x_2 \le \ldots \le x_n$ $\Sigma x_i = 0$
PI	$\mathbb{R}^n/(\mathcal{S}_n \times \mathbb{Z}_2)$	$x_1 \le x_2 \le \ldots \le x_n$ $x_2 - x_1 \le x_n - x_{n\text{-}1}$
TI	$\mathbb{R}^{n\text{-}1}/\mathbb{Z}_2$	$\Sigma x_i = 0$ $x_2 - x_1 \le x_n - x_{n\text{-}1}$
OPT	$\mathbb{T}^{n\text{-}1}/\mathcal{S}_n$	$x_n \le x_1 + 12$ $\Sigma x_i = 0$ $x_1 + 12 - x_n \le x_{i+1} - x_i,\ 1 \le i < n$
OPI	$\mathbb{T}^n/(\mathcal{S}_n \times \mathbb{Z}_2)$	$x_1 \le x_2 \le \ldots \le x_n \le x_1 + 12$ $0 \le \Sigma x_i \le 12$ $x_2 - x_1 \le x_n - x_{n\text{-}1}$
OTI	$\mathbb{T}^{n\text{-}1}/\mathbb{Z}_2$	$\min(x_1, x_2, \ldots, x_n) = x_1$ $\max(x_1, x_2, \ldots, x_n) \le x_1 + 12$ $\Sigma x_i = 0$ $x_2 - x_1 \le x_n - x_{n\text{-}1}$
PTI	$\mathbb{R}^{n\text{-}1}/(\mathcal{S}_n \times \mathbb{Z}_2)$	$x_1 \le x_2 \le \ldots \le x_n$ $\Sigma x_i = 0$ $x_2 - x_1 \le x_n - x_{n\text{-}1}$
OPTI	$\mathbb{T}^{n\text{-}1}/(\mathcal{S}_n \times \mathbb{Z}_2)$	$x_n \le x_1 + 12$ $\Sigma x_i = 0$ $x_1 + 12 - x_n \le x_{i+1} - x_i,\ 1 \le i < n - 1$ $x_2 - x_1 \le x_n - x_{n\text{-}1}$

Table S2. Fundamental domains for the OPTI quotient spaces.

Theorist	Name of structure	Description
Werckmeister (1698) *(S59)*	LR chain	Graph in 3-note OP space
Heinichen (1728) *(S59)*	Circle of fifths	Graph in 7-note OP space
Vial (1767) *(S59)*	Chart of the regions	Graph in 7-note OP space
Hauptmann (1853) *(S60)*	Diatonic circle of thirds	Graph in 3-note OP space
Euler (1739), Oettingen (1866), Riemann (1873)*(S61)*	The *Tonnetz*	The geometrical dual of graphs in $\mathbb{R}^3$ and 3-note OP space
Roiter (1983) *(S62)*		OT spaces
Roeder (1984) *(S57)*	Ordered interval space	Graphs in OT space; fundamental domains for OPT and OPTI spaces using an unusual projection.
Lewin (1990) *(S46)* Klumpenhouwer (1991) *(S45)*	K-nets	Lines in OP, OPT, and OPTI space
Cohn (1996) *(S63)*	Hexatonic Cycle	Graph in 3-note OP space
Douthett/Steinbach (1998) *(S58)*	Cube dance, Power towers	Graphs in 3- and 4-note OP space
Morris (1998) *(S64)*		Graphs in OP and OPT spaces
Erlich (2000) *(S62)*		Three-note OTI space
Stoecker (2002) *(S54)*	axial isography	Planes in three-note OPT space
Callender (2002) *(S22)* (published 2004)		3-note T, PT, PTI, and OPTI space (complete)
Straus (2003) *(S37)*		Graphs in 3- and 4-note OPTI space
Cohn (2003) *(S23)*		4-note OPT and OPTI space (incomplete)
Quinn (2003) *(S62)*		4-note OPT and OPTI space (complete)
Tymoczko (2004) *(S65)*	Scale lattice	Graph in 7-note OP space, also found in other dimensions
Tymoczko (2006) *(S2)*		All OP spaces

Table S3. Previous implicitly or explicitly geometrical models of musical structure. All can be interpreted as depicting portions of the quotient spaces described in this paper.

REFERENCES AND NOTES

S1. This equivalence class consists of all three-voice progressions that either move the root, third, and fifth of major triad by x, $x + 1$, and $x + 2$ semitones, respectively, or the root, third, and fifth of a minor triad by $x - 2$, $x - 1$, and x semitones, respectively, where x is any real number.

S2. D. Tymoczko, *Science* **313**, 72 (2006).

S3. This definition of "chord type" is a technical term of art that corresponds reasonably well with ordinary usage. In some circumstances, however, musicians use "chord type" to refer to other equivalence classes—such as PT equivalence classes.

S4. J. P. Rameau, *Treatise on Harmony* (Dover, New York, 1971).

S5. See H. Klumpenhouwer, *Music Theory Online* **0.9** (1994).

S6. D. Lewin, *Generalized Musical Intervals and Transformations* (Yale, New Haven, 1987).

S7. G. Mazzola, *The Topos of Music* (Birkhäuser, Boston, 2002).

S8. If one were to follow Allen Forte in identifying homometric or Z-related points, such as $(\text{C}, \text{C}\sharp, \text{E}, \text{F}\sharp)_{\text{OPTI}}$ and $(\text{C}, \text{C}\sharp, \text{D}\sharp, \text{G})_{\text{OPTI}}$ in four-note OPTI space ($\mathbb{T}^3/[\mathcal{S}_4 \times \mathbb{Z}_2]$), there would not be a well-defined coordinate system at the point of identification. Thus, quotient spaces formed using this equivalence relation will be significantly less straightforward than those considered in this paper: in particular, distance cannot be interpreted as representing voice leading. For more on these spaces, see reference 9.

S9. I. Quinn, *Perspectives of New Music* **44.2,** 114 (2006).

S10. Perceptual models of voice-leading distance need to be quite complex, as there are subtle interactions between displacement size and the number of moving voices, the direction of motion, and so on. See reference 11.

S11. C. Callender, and N. Rogers, *Proceedings of the International Conference of Music Perception and Cognition* **9**, 1686 (2006).

S12. A. Bregman, *Auditory Scene Analysis* (Cambridge, MIT Press, 1990).

S13. D. Huron, *Music Perception* **19**, 1 (2001).

S14. C. Masson, *Nouveau Traité des Règles pour la Composition de la Musique* (Da Capo, New York, 1967).

S15. O. Hostinský, *Die Lehre von den musikalischen Klängen* (H. Dominicus, Prague, 1879).

S16. A. Schoenberg, *Theory of Harmony* (University of California, Los Angeles, 1983).

S17. Note that the distribution constraint is a weakened version of the triangle inequality: it asserts that all voice leadings such as B in Figure S6C, which form two legs of a triangle by "bouncing off" a singularity, must be at least as large as the voice leading forming the third leg of the triangle (e.g. A in the same figure). However, not all

triangles can be translated so that they are of this form. Consequently, the distribution constraint is consistent with, but does not imply, the triangle inequality.

S18. The C operation is an isometry of the L^∞ norm. Hence, if one is willing to use this metric, then distance in C spaces can be taken to represent voice-leading size.

S19. The claim that $A \rightarrow B$ is significantly larger than $C_4 \rightarrow G_4$ depends on the stipulation that the motion from *A* to *B* consists of a number of conceptually distinct voices. In many cases (for instance, a string section playing in unison), such slight variations in pitch would not be conceived (or perceived) as distinct, and should instead be represented as a single, fused voice moving from C_4 to G_4.

S20. A. Hatcher, *Algebraic Topology* (Cambridge, New York, 2002).

S21. W. Thurston, *The Geometry and Topology of Three Manifolds,* available at http://www.msri.org/publications/books/gt3m/.

S22. C. Callender, *Music Theory Online* **10.3** (2004).

S23. R. Cohn, *Music Theory Online* **9.4** (2003).

S24. http://music.princeton.edu/~dmitri/ChordGeometries.html

S25. R. Hall and D. Tymoczko, paper presented to the tenth annual Bridges Conference, Donostia, Spain, July 24-27, 2007.

S26. R. Morris, *Perspectives of New Music* **18**, 445 (1979-80).

S27. C. Lord, *Journal of Music Theory* **25,** 91 (1981).

S28. E. Isaacson, *Journal of Music Theory* **34**, 1 (1990).

S29. D. Scott, and E. Isaacson, *Perspectives of New Music* **36.2**, 107 (1998).

S30. J. Rahn, *Perspectives of New Music* **18**, 483 (1979-80).

S31. D. Lewin, *Perspectives of New Music* **18**, 498 (1979-80).

S32. M. Castrén, thesis, Sibelius Academy (1994).

S33. M. Buchler, *Journal of Music Theory* **45**, 263 (2000).

S34. T. Demske, *Music Theory Online* **1.2**, (1995).

S35. I. Quinn, *Perspectives of New Music* **39**, 108 (2001).

S36. J. Roeder, *Perspectives of New Music* **25**, 362 (1987).

S37. J. Straus, *Music Theory Spectrum* **25**, 305 (2003).

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S40. E. Marvin, thesis, Eastman School (1988).

S41. I. Shmulevich, *Journal of New Music Research* **33.1**, 17 (2004).

S42. W. Cook and L. Seiford, *American Statistician* **37.4**, 307 (1983).

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S44. Quinn's original proposal represents a continuous variant of Marvin's "half-space counting" approach, such that the half-spaces are given more or less weight depending on how deeply the vector points into them.

S45. H. Klumpenhouwer, thesis, Harvard University (1991).
S46. D. Lewin, *Music Theory Spectrum* **12**, 83 (1990).
S47. D. Lewin, *Journal of Music Theory* **38**, 79 (1994).
S48. Any two pitch classes are related by some transposition and some inversion; the K-net description, by partitioning the chord, asserts that for a particular pair of pitch classes either the transposition or the inversion is more significant musically.
S49. Two notes move by "exact contrary motion" if they move by the same distance in opposite directions. In the voice leadings under discussion, any pair of notes in opposite partitions move by exact contrary motion.
S50. This notation for hyper-transpositions and inversions is nonstandard. Traditional theorists use $\langle\mathbf{T}_{2x}\rangle$ to describe the relation between K-net lines related by $\mathbf{T}_x$, and $\langle\mathbf{I}_{2x}\rangle$ to describe the relation between K-net lines related by $\mathbf{I}_x$. Thus, according to traditional terminology, $\langle\mathbf{T}_2\rangle$ relates the K-net {C, E} + {G} to {D♭, F} + {A♭}, even though the chords themselves (as well as their K-net lines) relate by single-semitone transposition. See reference 51.
S51. D. Tymoczko, *Music Theory Online* **13.3** (2007).
S52. This voice leading consequently skips the equal-tempered set-class in which the singleton is three semitones below the lowest note of the minor third.
S53. Of course, voice leadings are defined only up to uniform permutations of their voices. Thus the voice leading could also be expressed in pitch space such that the inner two voices moved while the outer voices remained stationary.
S54. P. Stoecker, *Music Theory Spectrum* **24**, 231 (2002).
S55. J. Ianni and L. Shuster, paper presented to the Mathematics and Computation in Music Conference, Berlin, May 18-20, 2007.
S56. M. Buchler, *Music Theory Online* **13.2** (2007).
S57. J. Roeder, thesis, Yale University (1984).
S58. J. Douthett, P. Steinbach, *Journal of Music Theory* **42**, 241 (1998).
S59. See J. Lester, *Compositional Theory in the Eighteenth Century* (Cambridge: Harvard, 1992).
S60. M. Hauptmann, *Die Natur der Harmonik und der Metrik* (Leipzig: Breitkopf, 1853).
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