Theory of Hexaflexagons

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1 Pats and Their Formulas

Definition 1. An **unoriented leaf** is an equilateral triangle with two marked edges, called its **connecting edges**.

Definition 2. An **oriented leaf**, or simply a **leaf**, is a pair of an unoriented leaf and a unit vector $\vec{v}$ (called its **orientation**), perpendicular to the leaf.

Miriam's note 1. You can imagine $\vec{v}$ as a vector that points to the person who is looking at the leaf.

If a leaf is based on an equilateral triangle *ABC*, and *AB* and *BC* are its connecting edges, and the triple of vectors $<\overrightarrow{BA}, \overrightarrow{BC}, \vec{v}>$ is right-handed (i.e. oriented the same way as the standard three-dimensional system of coordinates – looking from *(0,0,1)*, the direction of rotation from *(1,0,0)* to *(0,1,0)* is counterclockwise), then we'll denote this leaf *ABC*. The same leaf with opposite orientation is denoted *CBA*.

Definition 3. **Right-handed splitting** of a leaf *ABC* is the transforming of *ABC* to a pile of two leaves CAB_1 and B_2CA connected at the edge *AC* such that $\overrightarrow{B_1B_2}$ has the same direction as $\vec{v}$. Similarly, **left-handed splitting** of a leaf *ABC* is the transforming of *ABC* to a pile of two leaves CAB_1 and B_2CA connected at the edge *AC* such that $\overrightarrow{B_1B_2}$ has the direction opposite to $\vec{v}$ (see Figure 1 below). Edges B_1C and AB_2 are called **descendants** of edges *AB* and *BC* respectively.

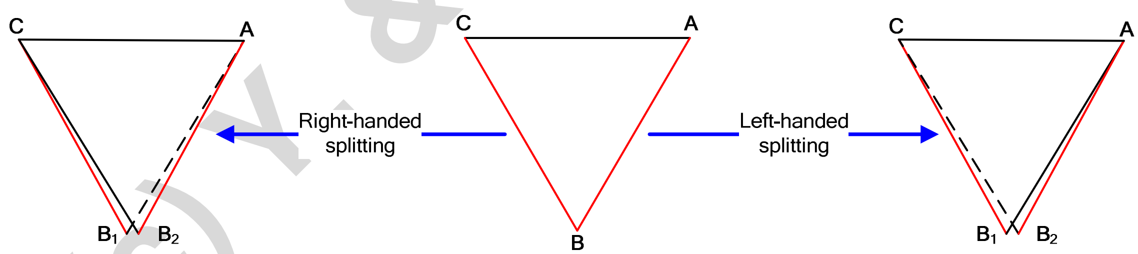

Figure 1. Right- and left-handed splitting ($\vec{v}$ is directed to the reader)

Fact 1. Handedness of splitting does not depend on leaf orientation.

Miriam's note 2. If you understood till now, you will probably understand the entire article – all the rest is much simpler! If not – the simple rule is: if the right connection is on the top – the splitting is right-handed; if the left connection is on the top – the splitting is left-handed.

Definition 4. A **right(left)-handed pat** with **protoleaf** *ABC* is a pile of connected leaves with the same orientation (called **pat orientation**) inductively defined as follows:

- A single leaf *ABC* is a right(left)-handed pat with protoleaf *ABC*; such a pat is called **thin**, or **singleton**.

- A pile produced from a right(left)-handed pat with protoleaf *ABC* by right(left)-handed splitting of one of its leaves is a right(left)-handed pat with protoleaf *ABC*; such a pat is called **thick**.

Definition 5. **Initial and final connecting edges** of a pat are descendants of connecting edges *BC* and *AB* of its protoleaf, respectively.

Fact 2. Any connecting edge of leaves of a pat, except for the initial and final connecting edges of the pat, connects two of its leaves, so the leaves form a sequence in which the initial connecting edge pertains to the first leaf of the sequence, the final edge pertains to the last leaf of the sequence, and each two leaves adjacent in the sequence have a common connecting edge – which means that the pat may be folded from a strip of equal equilateral triangles connected sequentially at their edges.

Definition 6. **Pat formula** of a pat is a sequence of natural numbers, each number corresponding to a connecting edge, the first and the last numbers corresponding to the initial and final connecting edges respectively, inductively defined as follows:

- In a singleton, *0* corresponds to both connecting edges, so the sequence is $<0,0>$.
- Suppose a pat *P* has pat formula $<x_1,\ldots,x_n>$, and pat *Q* is produced from pat *P* by splitting of a leaf *ABC*, where x_i and x_{i+1} correspond to edges *AB* and *BC* respectively. Then formula of *Q* is $<x_1,\ldots,x_i+1,1,x_{i+1}+1,\ldots,x_n>$, where x_i+1 corresponds to the edge AB_2, *1* – to *AC*, and $x_{i+1}+1$ – to B_1C.

Fact 3. Changing pat orientations swaps the initial and final connecting edges, and inverts the order of numbers in the pat formula.

Definition 7. **Pat order** is the number of leaves in the pat.

Fact 4. The sum of all numbers in the pat formula of a pat of order *n* is $3n-3$.

Proof. By induction. The formula of a pat of order *1* is $<0,0>$, and the sum is $3*1-3=0$. When a pat is produced from another pat by splitting, its order increases by *1*, and the sum of all numbers in the pat formula increases by *3*, by Definition 6.

2 Flexagon States and Their Formulas

Definition 8. A (general) **right(left)-handed flexagon state** is a cyclic sequence of six right(left)-handed pats with the same orientation, where each final connecting edge of a pat coincides with the initial connecting edge of the next pat; these six edges are called **pat connecting edges** of the flexagon state.

Fact 5. Alternatively **right(left)-handed flexagon state** can be defined inductively as follows:

- Six singletons connected cyclically as a hexagon is a right(left)-handed flexagon state (the **trivial flexagon state**)

- If F is a flexagon state, and G is produced from F by right(left)-handed splitting of one of its leaves, then G is a right(left)-handed flexagon state.

Definition 9. **Flexagon order** is the sum of orders of its six pats (i.e. the total number of leaves in the flexagon).

Definition 10. Let $< x_{i1}, \ldots, x_{i,n_i} >$, $1 \le i \le 6$, be the *6* pat formulas of a flexagon state. The **flexagon state formula** is

$< x_{6,n_6} + x_{11}, x_{12}, \ldots, x_{1,n_1-1}, x_{1,n_1} + x_{21}, x_{22}, \ldots, x_{6,n_6-1} >$ with *6* marked numbers $x_{6,n_6} + x_{11}, x_{1,n_1} + x_{21}, \ldots, x_{5,n_5} + x_{61}$, where each number corresponds to a connecting edge, and each marked number corresponds to a pat connecting edge.

We'll write pat formulas and flexagon state formulas as sequences of digits without commas, where the marked digits corresponding to pat connecting edges are **bold**.

Example 1. Let's take a straight strip with 36 leaves, plus a 37th leaf for gluing, as shown on Figure 2, and fold it several times as described below.

1. Fold together 1 and 2, 3 and 4, and so on. All folds (in this and following steps) must have the same handedness, e.g. always the upper leaf is connected to the right. You receive a strip consisting of 18 two-leaf pats (not counting the 37th leaf). We'll denote each pat with numbers of its leaves top to bottom. The resulting strip can be written as (1 2) (4 3) (5 6) (8 7) (see Figure 3). The formula of each pat is now 111.

2. Fold together (1 2) and (4 3), (5 6) and (8 7), and so on. You receive (2 1 4 3) (7 8 5 6) (10 9 12 11) (15 16 13 14) (18 17 20 19) (23 24 21 22) (26 25 28 27) (31 32 29 30) (34 33 35 36) (see Figure 4). The formula of each pat is now 21312.

3. Fold together (2 1 4 3) and (7 8 5 6), (15 16 13 14) and (18 17 20 19), (26 25 28 27) and (31 32 29 30), and glue the 37th leaf. You receive a flexagon with pats (3 4 1 2 7 8 5 6), (10 9 12 11), (15 16 13 14 19 20 17 18), (22 21 24 23), (27 28 25 26 31 32 29 30), (34 33 35 36) (see Figure 5).

The same folding is explained at http://library.thinkquest.org/J002441F/flexagon.htm, but with different numbering of the leaves. The formulas of the pats are:

313151313

21312

313151313

21312

313151313

21312

The formula of the entire flexagon is **5**131513131**5**131**5**1315131**5**131**5**1315131**5**131, where the first digit (**5**) corresponds to the edge between leaves 36 and 1, the second digit (1) – to the edge between leaves 1 and 2, and so on.

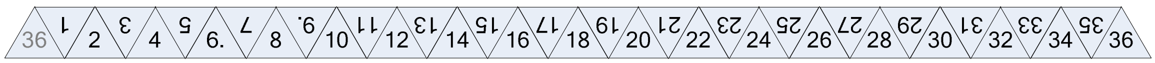

Figure 2. Dodecahexaflexagon strip before folding

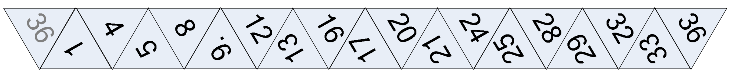

Figure 3. Dodecahexaflexagon strip after the first folding

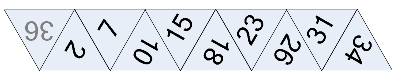

Figure 4. Dodecahexaflexagon strip after the second folding

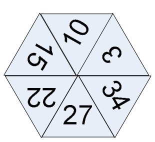

Figure 5. Fully folded dodecahexaflexagon

Definition 11. A flexagon state is called **regular** if it has rotational symmetry of order *3*.

Fact 6. The order of any regular flexagon state is divisible by *3*.

Fact 7. Starting from any point, the flexagon state formula of any regular flexagon state consists of *3* identical subsequences.

Definition 12. Any one of the three identical subsequences that comprise the flexagon state formula of a regular flexagon state (by Fact 7), seen as a cyclic sequence, is called the **compact flexagon state formula** of this regular flexagon state.

Example 2. The compact formula of the dodecahexaflexagon from Example 1 is **5**1315131**5**131.

3 Strips and Their Formulas

Definition 13. In a flexagon state let s_1,s_2,s_3 and s_4 be *4* consecutive leaves so that e_1 is the connecting edge common to s_1 and s_2; e_2 – the connecting edge common to s_2 and s_3; and e_3 – the connecting edge common to s_3 and s_4 (in other words, e_1,e_2 and e_3 are 3 consecutive connecting edges). The **binary symbol** of e_2 is *0* if e_1, e_2 and e_3 all have a common point, and 1 otherwise.

Definition 14. The **strip formula** of a flexagon state is a cyclic sequence of binary symbols of its connecting edges.

Fact 8. If b_{AB}, b_{BC} are binary symbols of connecting edges *AB, BC* of a leaf *ABC* of one of the pats of a flexagon state *F*, and flexagon state *G* is produced from *F* by splitting of *ABC* to B_1AC and ACB_2, then in the pat formula of *G* $b_{AB_1} = b_{AB} + 1 \bmod 2$, $b_{B_2C} = b_{BC} + 1 \bmod 2$, and $b_{AC}=1$.

Theorem 1. If *b* is the number corresponding to a connecting edge in the strip formula of a flexagon state, and *f* is the number corresponding to the same connecting edge in the flexagon state formula, then $b = f \bmod 2$.

Proof. By induction, using Fact 5. For the trivial flexagon state the statement is obvious. The inductive step follows from Definition 6, Definition 10 and Fact 8.

4 Tuckerman Map

In this chapter we'll introduce an alternative representation of the flexagon state formula and compact flexagon state formula, which for regular flexagon states surprisingly coincides with the well known Tuckerman traverse.

Definition 15. A **directed segment** is a combination of a straight segment and a direction along the segment. We'll show the direction of a directed segment using an arrow.

Definition 16. A **directed broken line** is a sequence of directed segments connected head to tail.

Definition 17. The **Tuckerman map of a pat** is a combination of (a) a directed broken line, where all segments of this line have integer lengths and (b) marked points lying on this broken line; where each segment corresponds to a connecting edge, ordered in the same order as connecting edges of the pat from the initial connecting edge to the final connecting edge, defined inductively as follows:

- The Tuckerman map of a singleton consists of one marked point and two zero-length segments starting and ending in the marked point, corresponding to the two connecting edges, with 120^0 angle between their directions. (It looks like a point, but to show the directions we'll draw it as shown on Figure 6 below.)

- If T is the Tuckerman map of a pat P, and pat Q is produced from P by the splitting of a leaf with connecting edges e_1 and e_2, where segments I_1 and I_2 correspond to e_1 and e_2, and the ending point of I_1 coincides with the starting point of I_2, then the Tuckerman map of Q is produced from T as follows (see Figure 7):
    - Prolong I_1 by 1 after its ending point and mark the new ending point. This line corresponds to the descendant of e_1.
    - Prolong I_2 by 1 before its starting point and mark the new starting point. This line corresponds to the descendant of e_2.
    - Connect the two new marked points with a new segment of length 1 corresponding to the new connecting edge.

If a new marked point coincides on the plane with an existing marked point, they are considered different points lying in different layers (like on Riemann surfaces).

Figure 6. Tuckerman map of a singleton

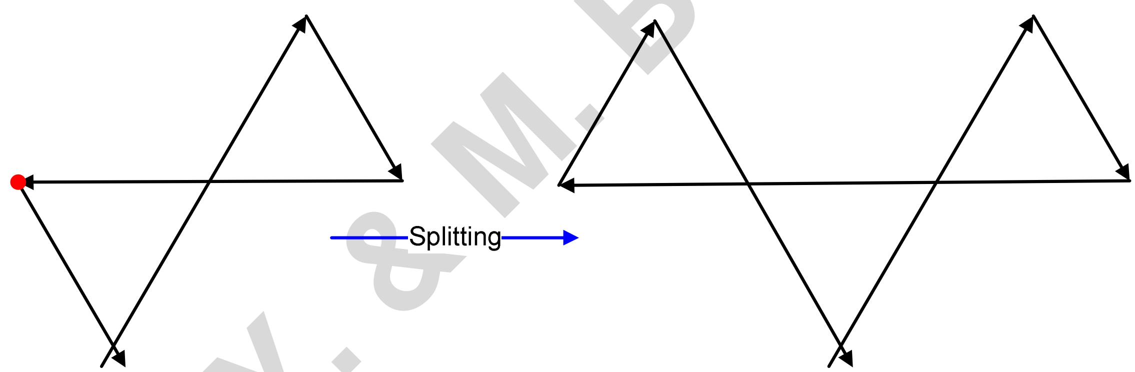

Figure 7. Effect of splitting on a Tuckerman map

Fact 9. The Tuckerman map of a pat consists of equal equilateral triangles with marked points in all vertices of these triangles.

Fact 10. A graph in which each node corresponds to a triangle in a Tuckerman map, and two nodes are connected if and only if the two triangles have a common vertex, is a tree.

Fact 11. It is possible to color the triangles of a Tuckerman map in two colors so that any two triangles with a common vertex are colored in different colors.

Fact 11 easily follows from Fact 10 and will be used in Chapter 7.

Fact 12. Flipping of a pat upside down inverts the order of numbers in its formula, and inverts the directions in its Tuckerman map.

Fact 13. The Tuckerman map of a pat of order *n* consists of *n-1* triangles and has *2n-1* marked points.

Proof. By induction. For a singleton the statement is obvious. Each splitting adds one triangle and two marked points.

Definition 18. The starting point of the segment corresponding to the initial connecting edge of a pat is called the **starting point** of the Tuckerman map of the pat.

Fact 14. The starting point of a Tuckerman map of a pat coincides with the ending point of the segment corresponding to the final connecting edge of the pat

Definition 19. The starting point of any directed segment of a Tuckerman map of a pat, except for the starting point of the map, is called a **turning point**.

Since Tuckerman map directed segments correspond to connecting edges, its turning points correspond to the leaves. If we mark leaves of the strip with sequential natural numbers as we did in Example 1, we'll mark the corresponding Tuckerman map turning points with the same numbers.

Definition 20. The **round traverse** of a pat is a directed broken line starting and ending in the starting point of the Tuckerman map, defined inductively as follows:

- The round traverse of a singleton consists of the same two zero-length segments as its Tuckerman map
- If pat Q with Tuckerman map U is produced from pat P with Tuckerman map T by splitting of a leaf corresponding to a turning point A in T, and U is T with added triangle ABC, then the round traverse of U is built from the round traverse of T as follows:
    - Go from the starting point to A
    - Add one of the directed segments AB or AC that does NOT have the same direction (i.e. does not lie on the same straight line) as the previous directed segment
    - Add directed segments BC,CA or CB,BA according to the choice made on the previous step
    - Continue from A to the ending point

Miriam's note 3. The round traverse is a different way to walk through all segments of a Tuckerman map. Every segment has length 1, and the next segment always has a different direction. When you get to a marked point – if this point is common to two triangles, go to the other triangle and don’t forget not to stay on the same straight line; otherwise, you don’t have choice anyway.

Theorem 2. The order of turning points along the round traverse of a pat corresponds to the order of the leaves – top to bottom or bottom to top depending on its handedness.

Definition 21. The **Tuckerman map of a flexagon state** is a union of Tuckerman maps of its pats where the same point serves as the starting points for all pats, and the directed segment corresponding to the initial connecting edge of a pat is a continuation of the segment corresponding to the same edge seen as the final connecting edge of another pat.

Example 3. Figure 8 shows the Tuckerman map of the same flexagon state as in Example 1. The thickness of the lines reflects the fact that each line is passed three times.

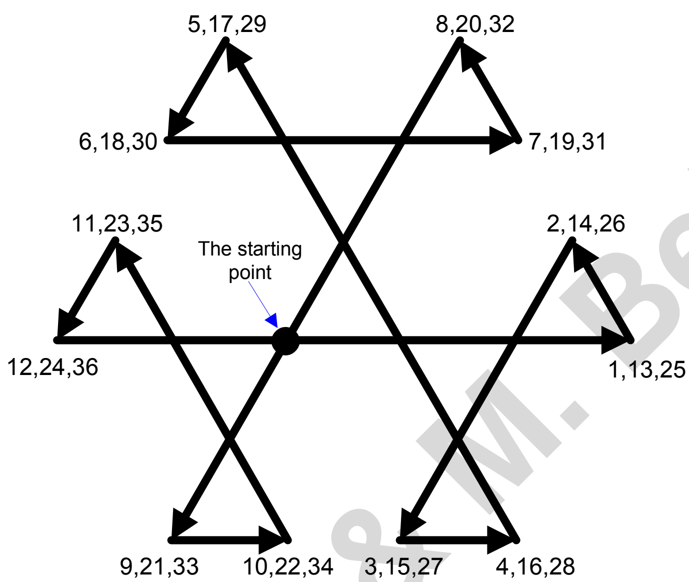

Figure 8. Tuckerman map of a dodecahexaflexagon state

Fact 15. The starting point of a Tuckerman map lies on exactly 6 directed segments corresponding to the pat connecting edges of the flexagon state.

Fact 16. The length of the directed segment of a Tuckerman map of a flexagon state corresponding to a connecting edge equals the number in the formula of this flexagon state corresponding to the same edge.

Fact 17. Let s_1, s_2 and s_3 be consecutive directed segments of a Tuckerman map. If s_2 has an even length, then s_1 and s_3 have the same direction; otherwise, the angle between their directions is 120^0.

Using Fact 16 and Fact 17, it is possible to convert the formula of a flexagon state to the Tuckerman map of the same state, and vice versa.

Fact 18. Flipping of a flexagon upside down inverts the order of numbers in its formula, and inverts the directions in its Tuckerman map (similarly to Fact 12 regarding pats).

Fact 19. Every directed segment of the Tuckerman map of a regular flexagon state is passed three times.

Definition 22. The **compact Tuckerman map of a regular flexagon state** is its Tuckerman map passed once instead of three times.

For a regular flexagon state of order *3n* (see Fact 6) it is sometimes more convenient to number them from *1* to *n*, and reuse each number three times. With this convention, each turning point receives only one number instead of three.

Example 4. Figure 9 shows the Tuckerman map of the same flexagon state as in Example 1**.** Using Theorem 2 it is possible to see that the order of leaves in the two types of pats are (10 9 12 11) and (3 4 1 2 7 8 5 6), so that we see 3 and 10 on one side of the flexagon, and 6 and 11 on the other side.

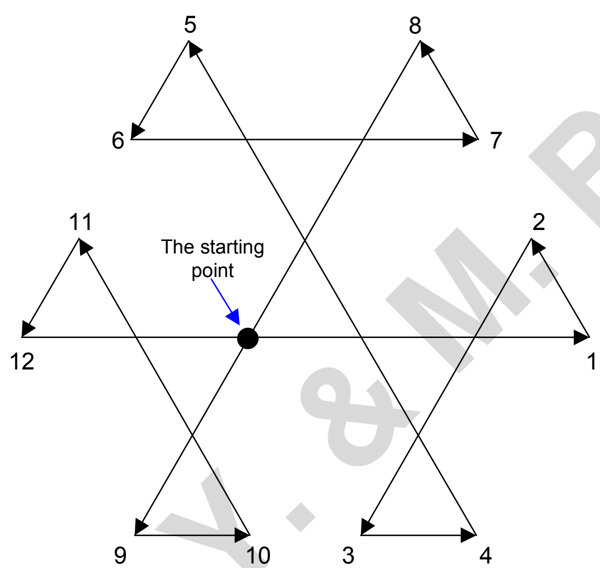

Figure 9. Compact Tuckerman map of a dodecahexaflexagon state

Fact 20. If F is a flexagon state, and G is produced from F by a pinch flex, then their Tuckerman maps have the same directed broken line, but different starting points at distance 1.

Fact 21. The Tuckerman map of a flexagon state of order n consists of n-6 triangles.

It easily follows from Fact 13.

Fact 22. The compact Tuckerman map of a regular flexagon state consists of n/3-2 triangles.

5 Pinch Flex

Definition 23. Two flexagon states F and G are called **pinch equivalent** if and only if it is possible to transform F to G by a series of flips, rotations, and pinch flexes.

Fact 23. Pinch equivalence is an equivalence relation.

Definition 24. The **pinch class** of a flexagon state F is a class of all flexagon states pinch equivalent to F.

Definition 25. A **pinch class Tuckerman map** is the same as a Tuckerman map but without segment directions and without starting point.

Fact 24. There as a canonical one-to-one mapping between pinch classes and pinch class Tuckerman maps.

Definition 26. A pinch class that contains a regular state is called a **regular pinch class**.

Fact 25, All states in a regular pinch class are regular.

Fact 26. There is a one-to-one correspondence between marked points of a regular pinch class Tuckerman map and states pertaining to this class.

6 V-Flex

Fact 27. The minimal flexagon state (**MinV**) in which a V-flex is possible has formula 1**11**1**121**1**0**, The only possible V-flex transforms it to **1111**1**0**1**12** (see Figure 10)

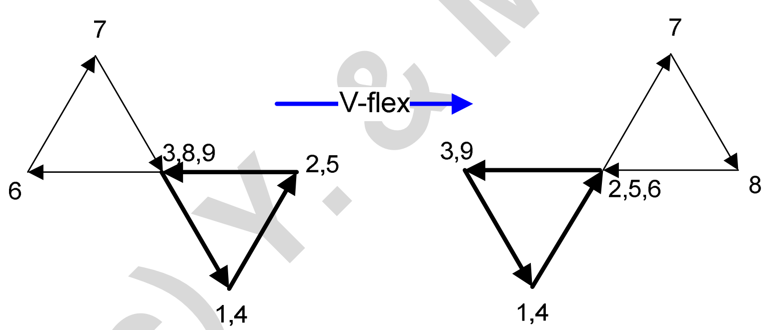

Figure 10. Minimal V-flexable flexagon

Definition 27. Pat Q is called an **extension** of pat P if it is possible to produce Q from P by a sequence of splittings.

Definition 28. Flexagon state G is called an **extension** of flexagon state F if each pat of G is an extension of the corresponding pat of F.

Fact 28. Any extension F of MinV can be V-flexed. Let’s call the resulting flexagon state G, and let T and U be Tuckerman maps of F and G respectively. Then U may be produced from T by the following transformations:

- Decrease by 2 the lengths of the descendant of edge (5,6)
- Increase by 2 the lengths of the descendant of edge (8,9)
- Place the starting point of the Tuckerman map in the intersection of descendants of edges (1,2), (4,5), (6,7) and descendant of edges (2,3), (5,6), (8,9).

Example 5. Let’s take a regular hexahexaflexagon folded from a straight strip. We’ll number its leaves $1_1 2_1 3_1 4_1 5_1 6_1 1_2 2_2 3_2 4_2 5_2 6_2 1_3 2_3 3_3 4_3 5_3 6_3$. Tuckerman maps of the initial and final states are shown on Figure 11.The directed segment corresponding to $(1_1 2_1)$ changes its length from 3 to 5; the directed segment corresponding to $(1_2 2_2)$ changes its length from 3 to 1. The corresponding formulae are 1**3**131**3**1**3**131**3**1**3**131**3** and 1**5**1313**11**1**3**131**3**1**3**13.

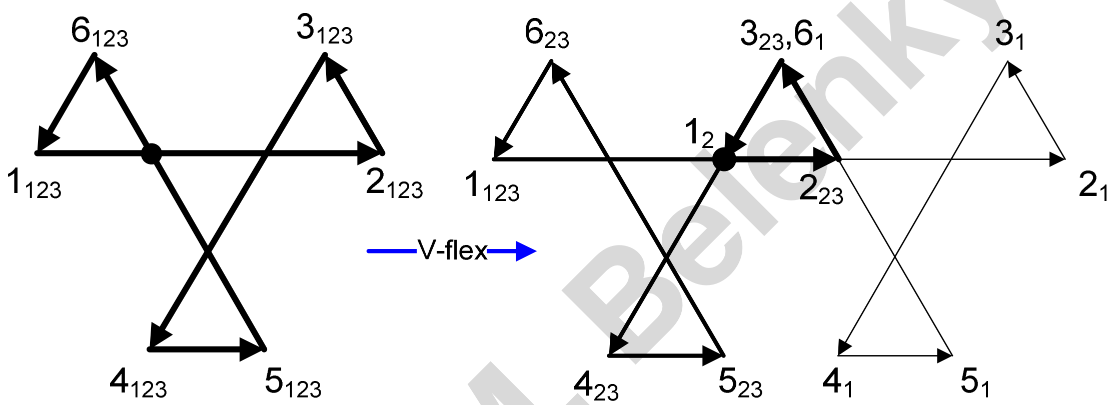

Figure 11. V-Flex of a hexahexaflexagon

Definition 29. Two flexagon states F and G are called **V-equivalent** if and only if it is possible to transform F to G by a series of flips, rotations, pinch flexes and V-flexes.

7 V-Equivalence of All Regular States Folded from a Straight Strip

Definition 30. A regular pinch class compact Tuckerman map is called a **straight compact Tuckerman map** if all segments have odd length; the regular pinch class is called in this case a **straight regular pinch class**, (It is called straight because the strip formula is all 1’s, i.e. the strip is straight.)

Fact 29. Any straight compact Tuckermap map of order n>9 can be produced from another straight compact Tuckerman map of order n-9 by addition of 3 triangles in the shape shown on Figure 12 (the **butterfly shape**).

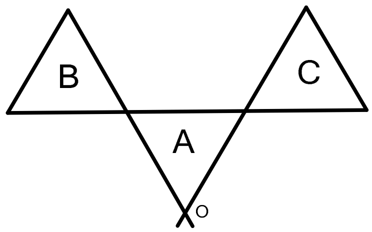

Figure 12. Straight Tuckermap map increment

Proof. Using Fact 11, let's color the triangles of the compact Tuckerman map in two colors, so that any two triangles with a common vertex are colored in different colors. Since all segments have odd length, it is easy to see that all turning points pertain to triangles of the same color. Let's call this color red, and the other one – blue. Since there are no turning points in blue triangles, each blue triangle is surrounded by 3 red triangles. Since the order is greater than 9, by Fact 22 the number of triangles in the map is greater than 9/3-2=1, so the map must include at least one red triangle and one blue triangle. Let A be the last blue triangle added to the Tuckerman map in the process of producing it from the trivial Tuckerman map by a series of splittings, and D – the red triangle having a common vertex with A that was added before D. Let B and C be the other two red triangles having a common vertex with A; they must have been added later than A. B and C cannot have a common vertex with any triangle other than A; otherwise this triangle would be a blue triangle added later than A, in contradiction with the choice of A. Triangles A, C and D form the butterfly shape, and it is easy to see that deleting of these triangles leaves the lengths of remaining segments odd.

The following statement follows easily from Fact 29 by induction.

Fact 30. Order of any straight regular pinch class is divisible by 9.

Fact 31. There exists a sequence of v-flexes, pinch flexes, rotations and flips (the **131313 sequence)** that converts the pinch class with compact formula 131313 to the pinch class with compact formula 313131 (see Figure 13).

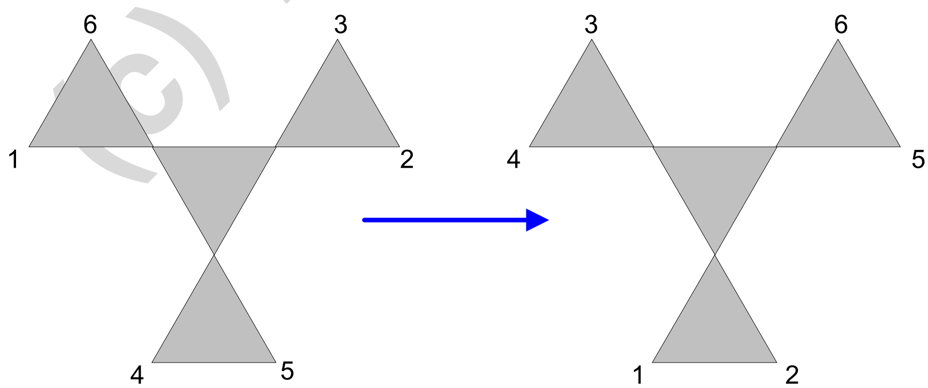

Figure 13. Action of 131313 Sequence

This sequence is described in Appendix A.

Now we are ready to prove the central theorem of this chapter.

Theorem 3. Using a sequence of v-flexes, pinch flexes, rotations and flips is possible to transform any straight regular flexagon state to any other straight regular flexagon state of the same order. The length of this sequence is linear in the order.

Proof. Instead of flexagon states, we'll speak about pinch classes. Let's call a pinch class with compact formula 1(2n+1)1313313…313 the **standard straight pinch class**, where the subsequence 313 is repeated n times (see Figure 14 for the corresponding compact Tuckerman map; marking of the turning points will be used later).

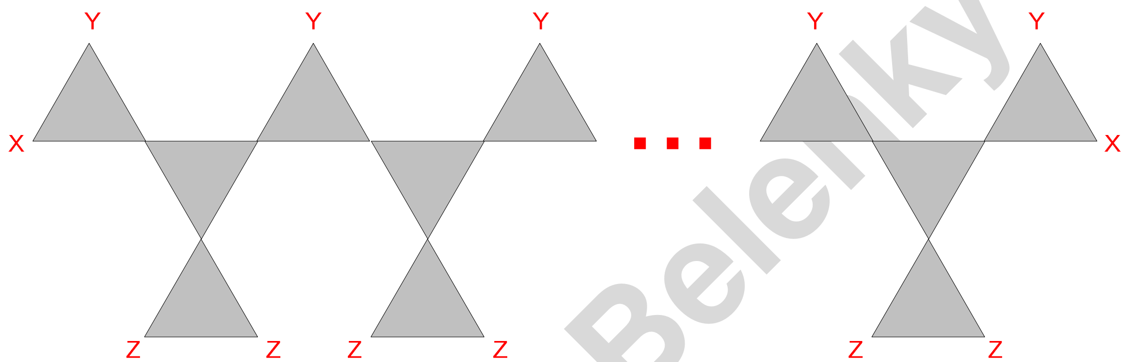

Figure 14. Standard straight compact Tuckerman map

We'll prove by induction on order that any straight pinch class can be transformed to the straight pinch class of the same order; the statement of the theorem follows from it.

For order 9 (the minimal possible order of a straight pinch class, according to Fact 30), there is only one pinch class, so the statement is true. Now let's suppose that the statement is true for order 9n. Let G be an arbitrary regular straight pinch class of order 9n+9, and T – its compact Tuckerman map. According to Fact 29, there exists a straight pinch class F of order 9n such that the compact Tuckerman map U of G can be produced from the compact Tuckerman map T of F by addition of a butterfly form W. Let's pretend that the four leaves corresponding to B are glued together; after this imaginary gluing G transforms to F. By induction assumption, it is possible to transform this pinch class to the standard straight pinch class. If we take the additional butterfly form B into account, the resulting compact Tuckerman map is the standard straight Tuckerman map with an additional butterfly form B attached to it at one of its turning points. On Figure 14 the turning points are marked with letters X,Y and Z. Let's analyze the 3 cases.

Case 1. B is attached at a turned point marked with X. In this case, we already reached a standard straight pinch clas.

Case 2. B is attached at a turned point marked with Y. See Figure 15 for a single 131313 sequence that transforms this state to a standard straight Tuckerman map. (The butterfly shape B is navy blue.)

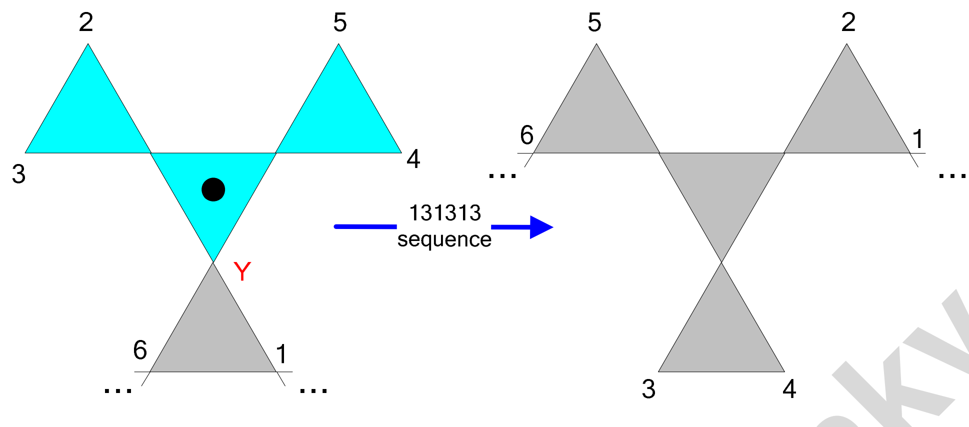

Figure 15. 131313 sequence for Case 2

Case 2. B is attached at a turned point marked with Z. See Figure 16Figure 20 for a series of four 131313 sequences that that transforms this state to a standard straight Tuckerman map. (The butterfly shape B on Figure 16 is navy blue.)

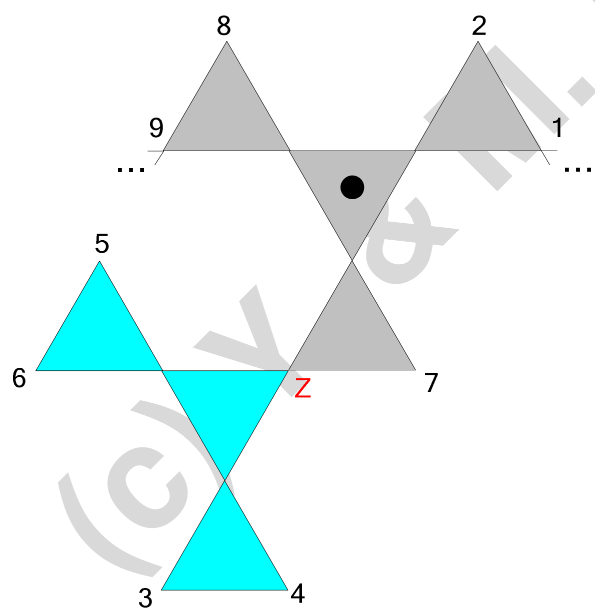

Figure 16. Initial state for Case 3. From this state apply the 131313 sequence to 12(3456)789

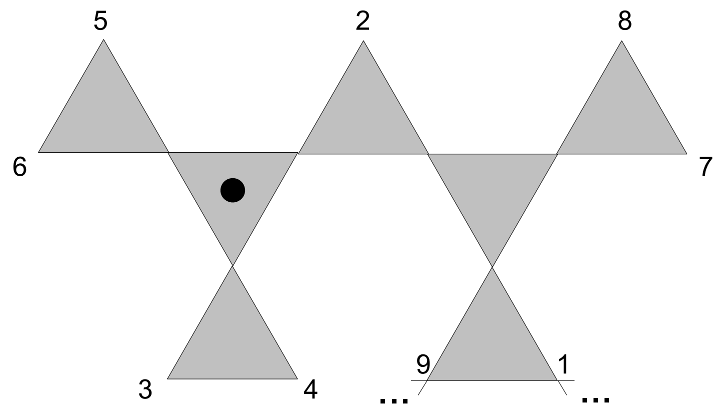

Figure 17. Second state for Case 3. From this state apply the 131313 sequence to (7891)23456

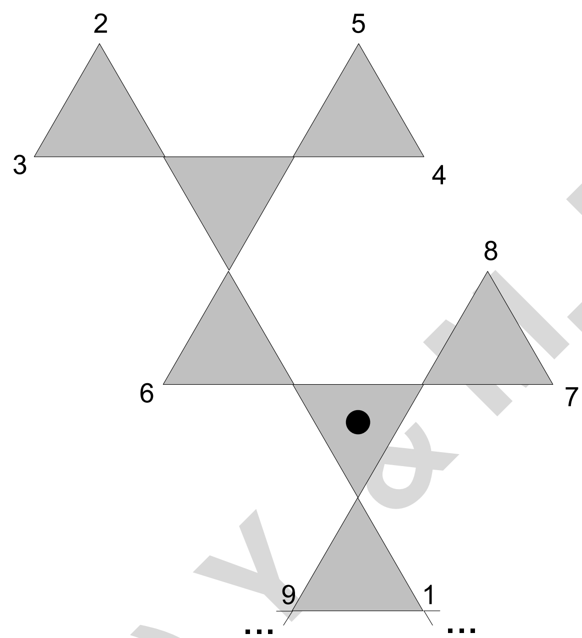

Figure 18. Third state for Case 3. From this state apply the 131313 sequence to 1(2345)6789

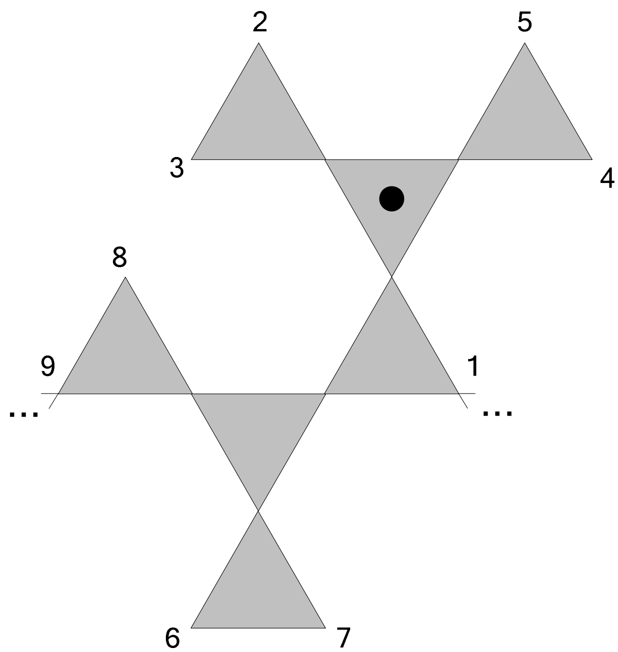

Figure 19. Fourth state for Case 3. From this state apply the 131313 sequence to 12345(6789)

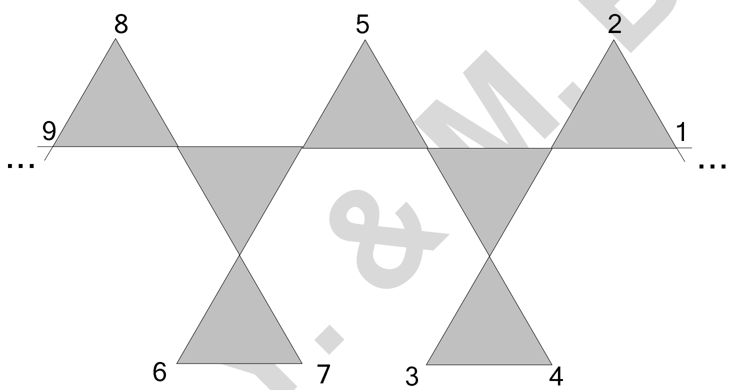

Figure 20. Final state for Case 4

From this proof follows that the sequence of flexes for order 9n+9 consists of the sequence of flexes for order 9n and not more than four 131313 sequences; it proves linearity in the order.

8 The 131313 Sequence

The following figures describe the 131313 sequence. Each figure includes the Tuckerman map, the flexagon state formula, and the view of the flexagon from both sides. The segment that is shortened by the next V-flex is marked in red; the segment that is lengthened by the next V-flex is marked in green. The blue arrow from the starting point defines the upper side of the flexagon.

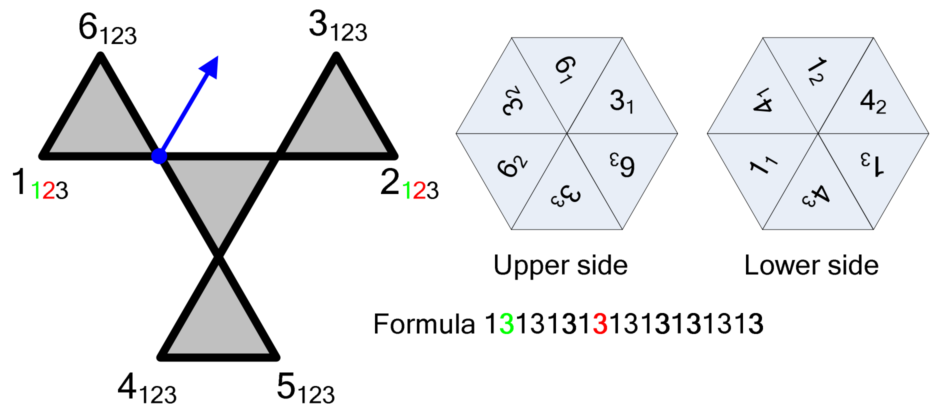

Figure 21. 131313 sequence – the initial state.V-flex

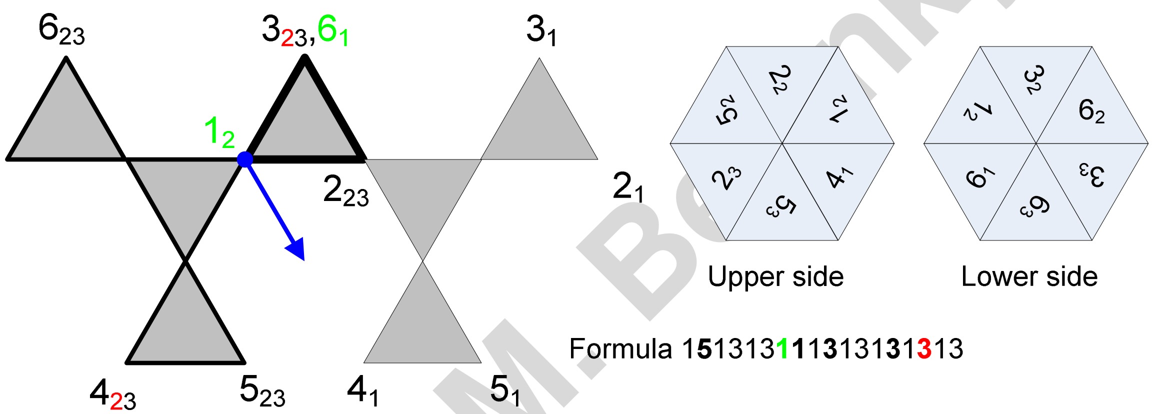

Figure 22. 131313 sequence – state No.2.V-flex

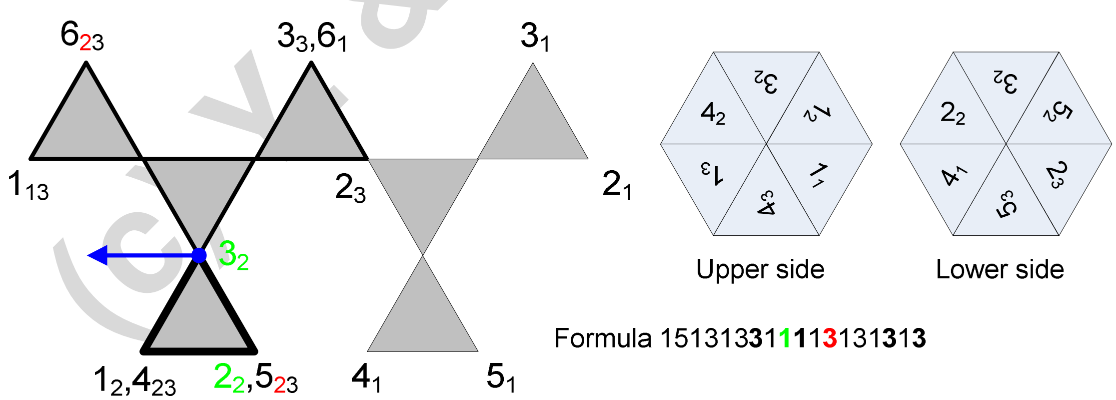

Figure 23. 131313 sequence – state No.3.V-flex

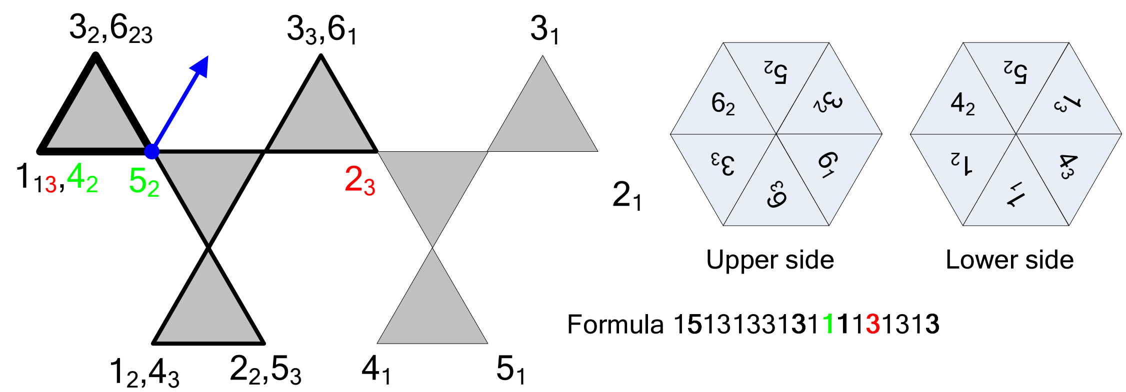

Figure 24. 131313 sequence – state No.4.V-flex

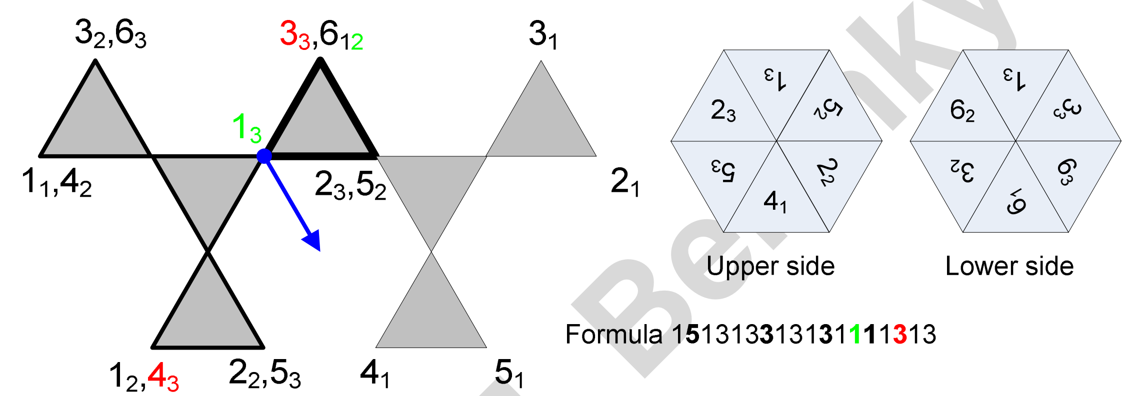

Figure 25. 131313 sequence – state No.5.V-flex

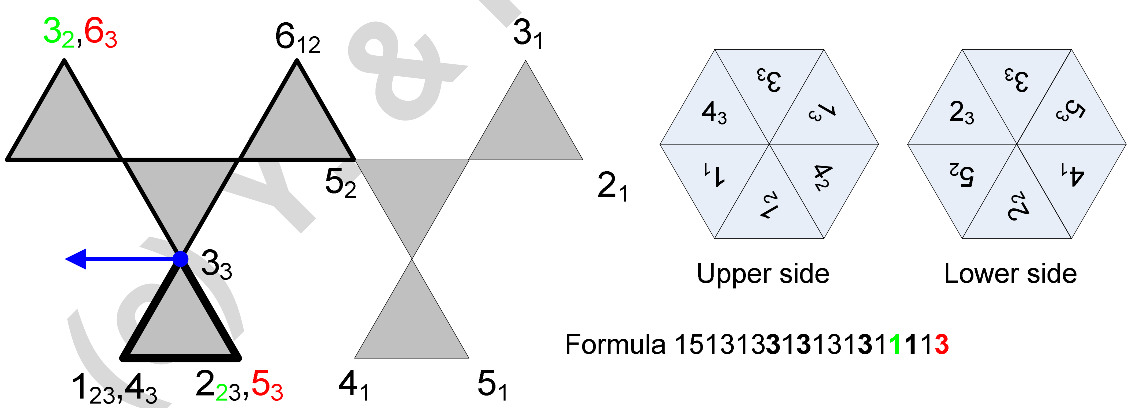

Figure 26. 131313 sequence – state No.6.V-flex

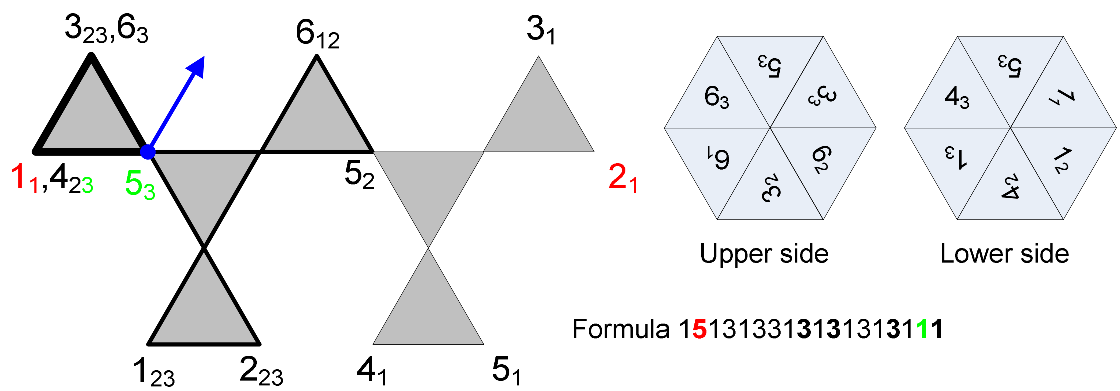

Figure 27. 131313 sequence – state No.7.V-flex

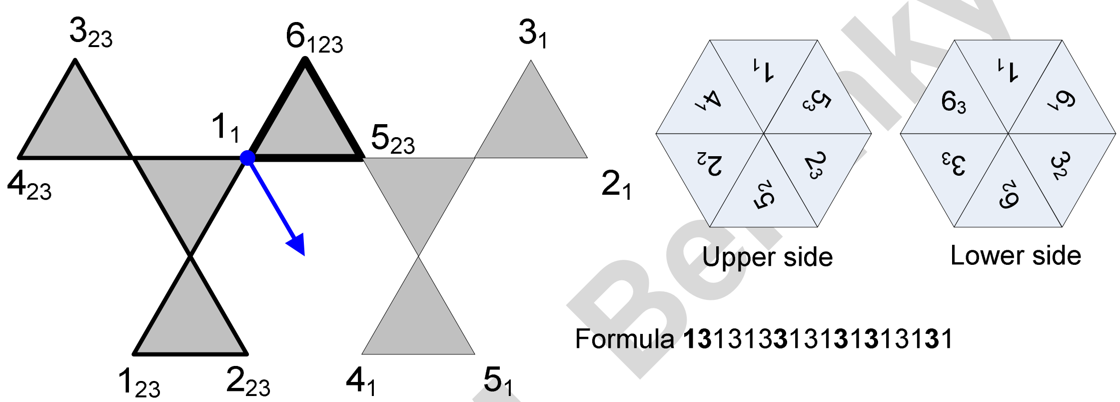

Figure 28. 131313 sequence – state No.8. Pinch flex and flip

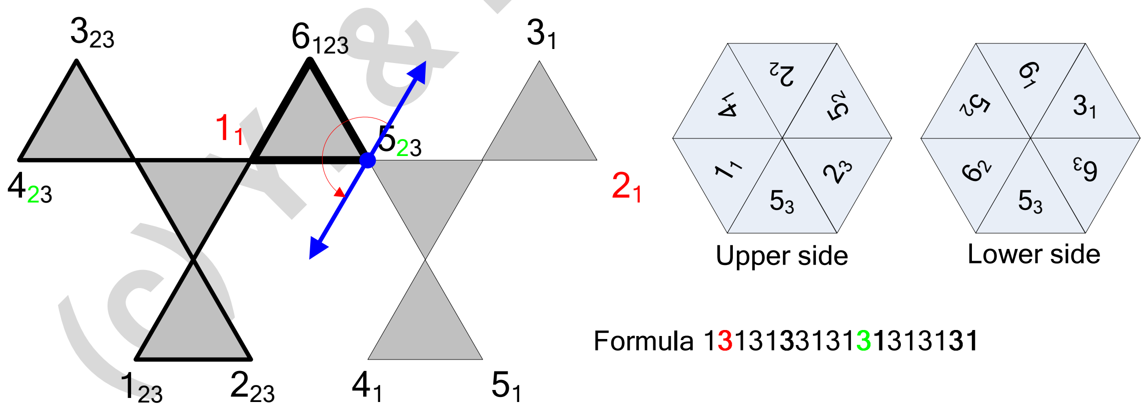

Figure 29. 131313 sequence – state No.9. V-flex and flip

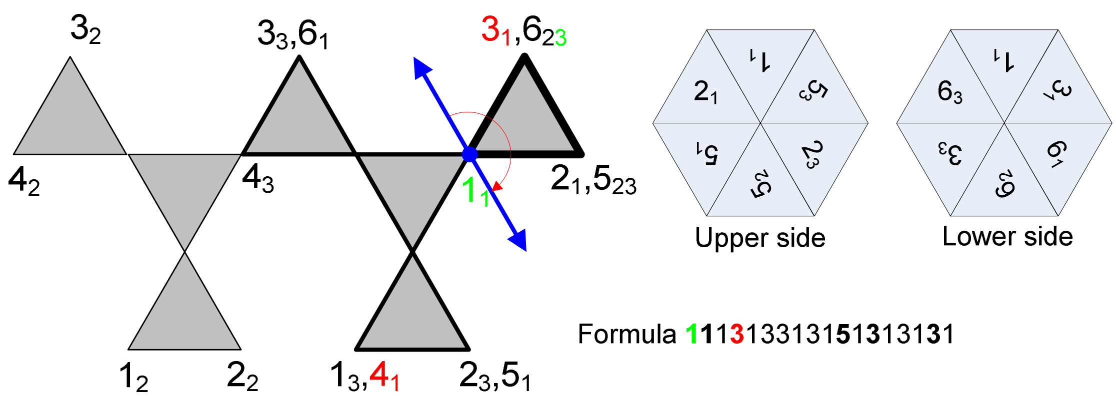

Figure 30. 131313 sequence – state No.10. V-flex

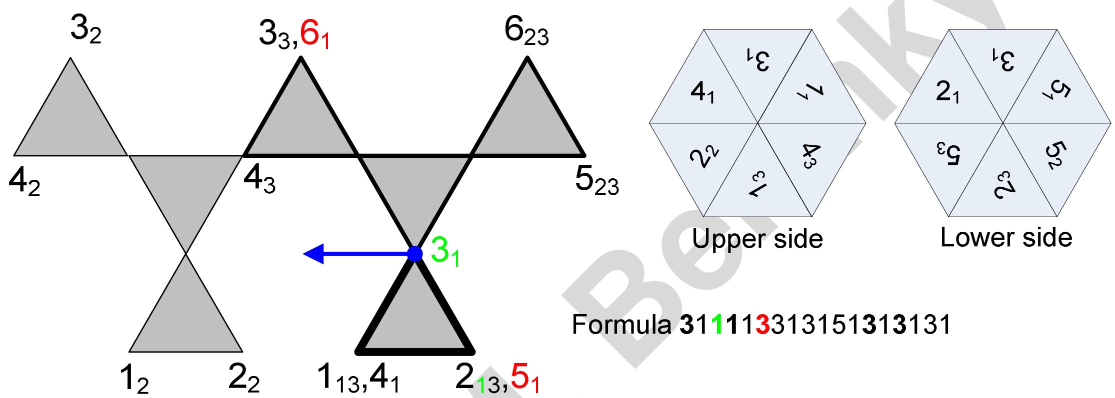

Figure 31. 131313 sequence – state No.11. V-flex

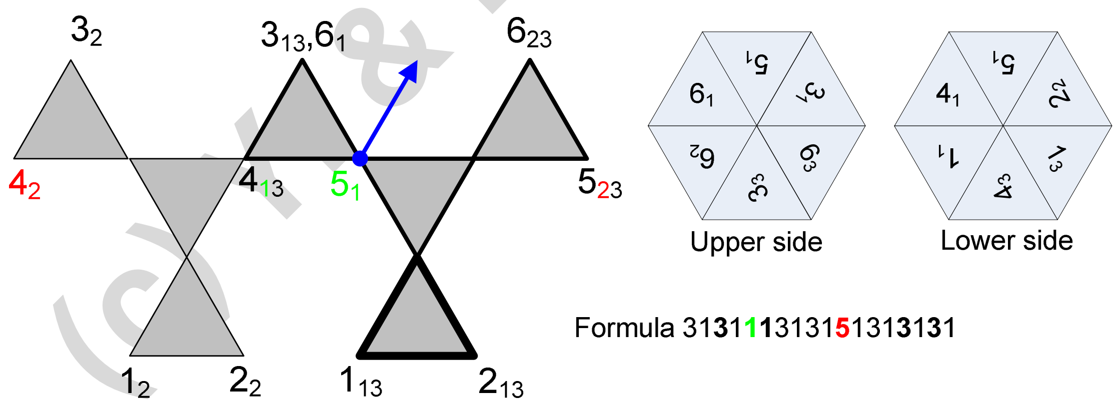

Figure 32. 131313 sequence – state No.12. V-flex

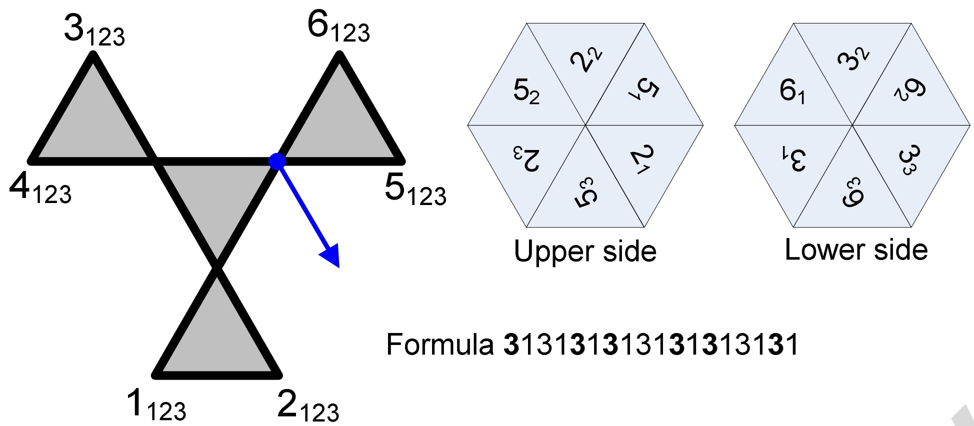

Figure 33. 131313 sequence – final state