

# The Thrice Three-Fold Flexagon

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And thrice threefold the Gates; three folds were Brass,  
Three Iron, three of Adamantine Rock,  
Milton. *Paradise Lost*.

## Introduction

The discovery of the Thrice Three-Fold Flexagon originated in examination of the dynamic properties of some of the flexagons in Scott Sherman's Triangle Flexagon Bestiary. It appeared that there ought to be flexagons that, in some positions, had the appearance of a regular hexagon dissected into 6  $30^\circ$ - $30^\circ$ - $120^\circ$  isosceles triangles (Figure 1). There are, and the Thrice Three-Fold Flexagon is one of them. The name is based on Milton's description of the Gates of Hell. It was chosen because of the flexagon's structure in some positions, and also because of the bewildering complexity of its dynamic properties.

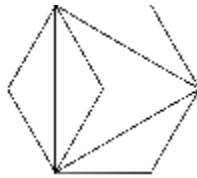


Figure 1. Regular hexagon dissected into 6  $30^\circ$ - $30^\circ$ - $120^\circ$  isosceles triangles.

The flexagon is a twisted band, and exists in two enantiomorphic forms. The torsion (number of half twists around the band) is 9. This high torsion makes the flexagon stable and easy to handle. The flexagon can be flexed to numerous positions, some of which are flat, and some have 2-fold or 3-fold rotational symmetry. Some flexing sequences are complicated, and cannot be described concisely. In practice it is difficult to flex between some positions. The nets shown below demonstrate some of the possible positions. Various markings are used to facilitate assembly and flexing. The Thrice Three-Fold Flexagon easily becomes mixed up, and it can be difficult to see how to return it to an initial position. It is therefore advisable to make one of the hinges easy to disconnect and reconnect in order that the flexagon can be unfolded and re-assembled. Also shown is the net for a triplex flexagon, which is a Thrice Three-Fold Flexagon with two of its faces deleted.

To assemble a net cut it out, and cut along any heavy dashed lines. Crease the lines between leaves to form hinges. Ensure that adjacent leaves superimpose neatly when folded together. Transfer markings in brackets on the upper face of a leaf to the reverse face, and delete them from the upper face. Join the three parts of the net at A-A and B-B using transparent adhesive tape. Follow additional instructions in the figure caption, and join the ends of the net. Leaves have to be bent during some flexes so the flexagon should be made using origami paper, as in the photographs,

### Triangle and Hexagon Positions

Triangle and hexagon positions of the Thrice Three Fold Flexagon are shown in Figures 2(a) and 2(d), and an appropriately marked net is shown in Figure 3.

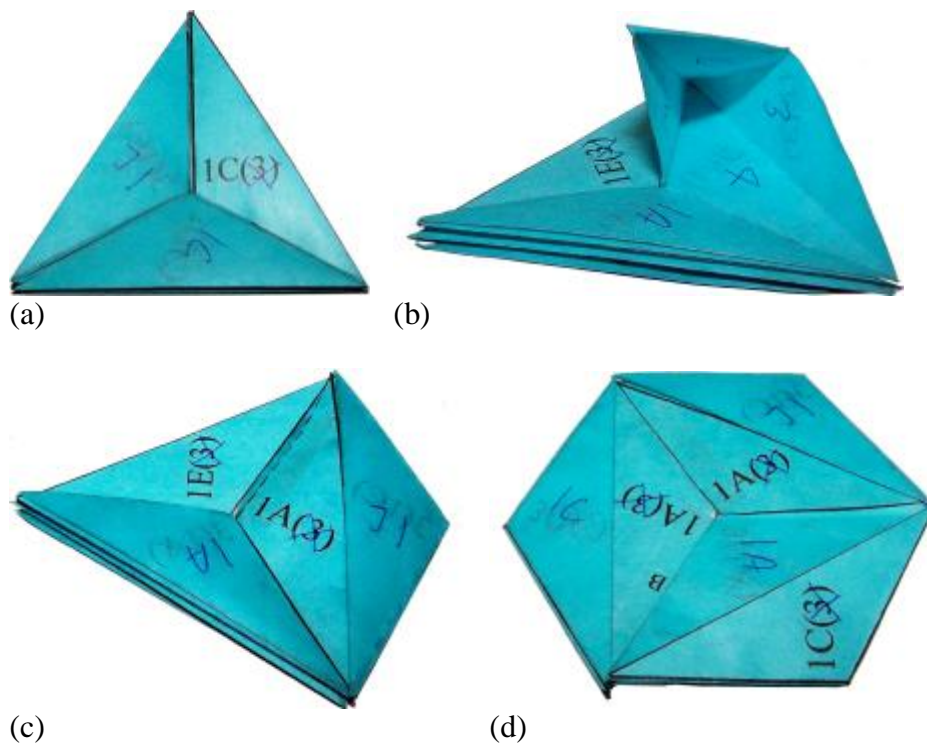


Figure 2. The Thrice Three-Fold Flexagon. (a) Triangle position. (b) Intermediate position. (c) Kite position. (d) Hexagon position.

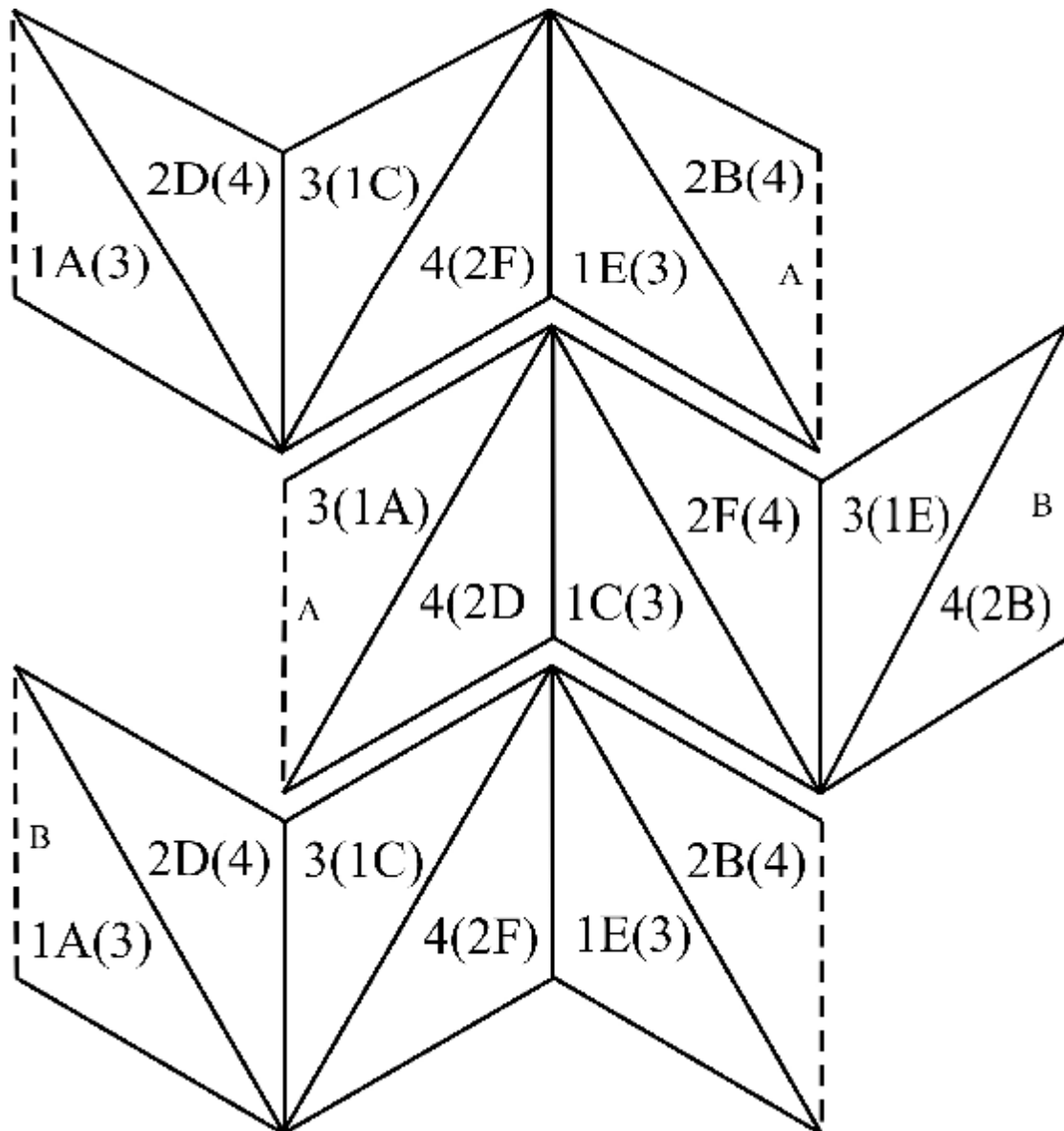


Figure 3. Net (triangle and hexagon positions) for the Thrice Three-Fold Flexagon. Fold each leaf numbered 3 onto a leaf numbered 4, and secure with a paper clip, next fold a leaf lettered C onto a leaf lettered E and in the same pat fold the leaf lettered D onto the leaf lettered F, then repeat with the other two pats.

As assembled the flexagon is in a triangle position A(B), with leaves lettered A visible on one face, and leaves lettered B on the other. Ignoring face markings a triangle position has 3-fold rotational symmetry. By using the flap flex, in which the loose

flaps on a face are turned over, the 6-cycle of triangle positions shown in the Tuckerman diagram (Figure 4) can be traversed. Mixed up faces can be produced by turning over flaps individually. Nominally, the flexagon is also in main position 1(2) with leaves numbered 1 visible on one face, and leaves numbered 2 on the other. All 9 leaves numbered, say 1, can be seen by folding up the loose flaps on the appropriate face of a triangle position. The flexagon can be arranged so that all the leaves of faces 1 and 2 are visible, but this cannot be done with rotational symmetry. The resulting jumble cannot be photographed convincingly.

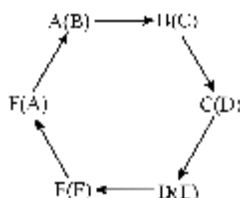


Figure 4. Tuckerman diagram for flap cycle of the thrice three-fold flexagon.

Faces 3 and 4 do not appear while traversing the 6-cycle shown in Figure 4 so, for this cycle, these faces can be deleted. The flexagon then becomes a triplex  $30^\circ$ - $30^\circ$ - $120^\circ$  isosceles triangle flexagon, the net for which is shown in Figure 5. This flexagon shows clearly the thrice 3-fold nature of a triangle position.

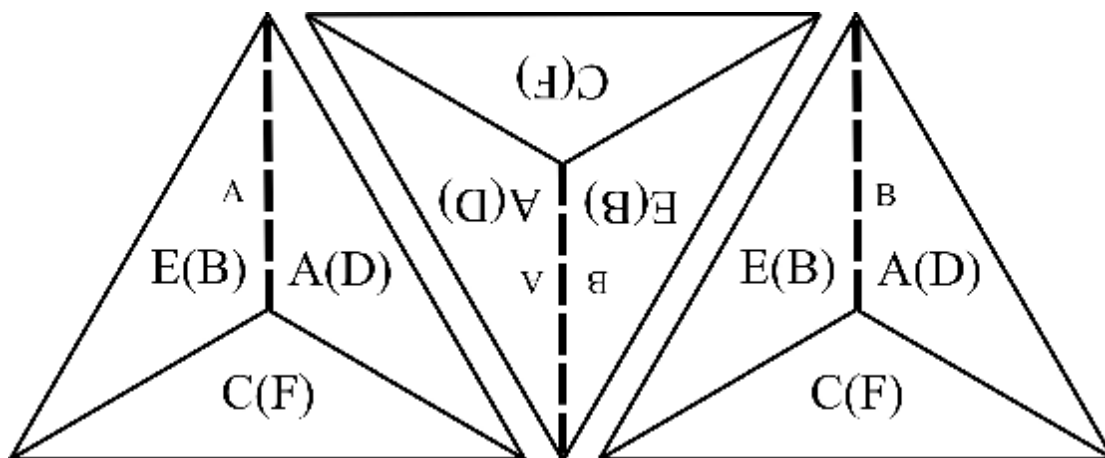


Figure 5. Net for a triplex  $30^\circ$ - $30^\circ$ - $120^\circ$  isosceles triangle flexagon. Fold a leaf lettered C onto a leaf lettered E and in the same pat fold the leaf lettered D onto the leaf lettered F, then repeat with the other two pats.

If one of the flaps on a face of a triangle position of the Thrice Three-Fold Flexagon is turned over, then the flexagon can be opened into the intermediate position shown in

Figure 2(b). Scott Sherman calls this a pocket flex. Completing the pocket flex by closing the pocket in the other direction leads to a kite position (Figure 2(c)). This can be done in 36 different ways. Repeating the pocket flex in two other places leads to a hexagon position which has 3-fold rotational symmetry, as shown in Figures 1 and 2(d). There are 12 different hexagon positions.

### Pyramid and Three Flap Positions

A pyramid position is an open triangular pyramid (Figure 6(b)). This is actually an irregular ring of 9 triangles, which has 3-fold rotational symmetry; there is no connection between the two triangles on the right hand side of the photograph. To flex a triangle position (Figure 2(a)) to a pyramid position fold up the three flaps on one face to reach the intermediate position shown in Figure 6(a). In this position, which has 3-fold rotational symmetry, the ends of the flaps coincide and some leaves are bent. Then open the flexagon into a pyramid position from underneath. There are 6 different pyramid positions.

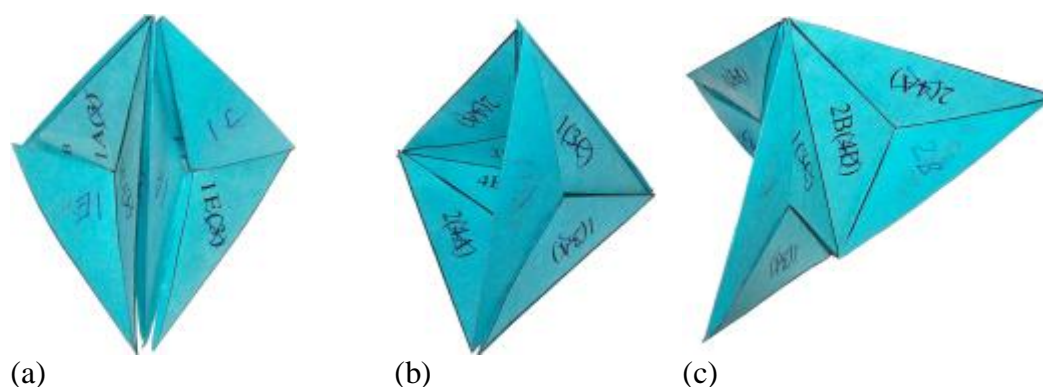


Figure 6. The thrice three-fold flexagon. (a) 3-fold intermediate position. (b) Pyramid position. (c) Three flap position.

Alternatively, the net shown in Figure 7 can be used to construct a pyramid position. To flex this pyramid position to a three flap position (Figure 6(c)), which also has 3-fold rotational symmetry, fold together pairs of leaves inside the pyramid that have the same letter. The three flap position can either be returned to the original pyramid position, or opened into another pyramid position. With difficulty, a pyramid position can be flexed to main position 1(2) by using a 3-fold pinch flex. Hence, theoretically, a pseudo 4-cycle can be traversed, as shown schematically in the Tuckerman diagram (Figure 8), but this is very difficult. Pyramid 1 has leaves numbered 1 on the outside of the pyramid, and Pyramid 2 leaves numbered 2. In the three flap position leaves numbered 1 and 2 are visible.

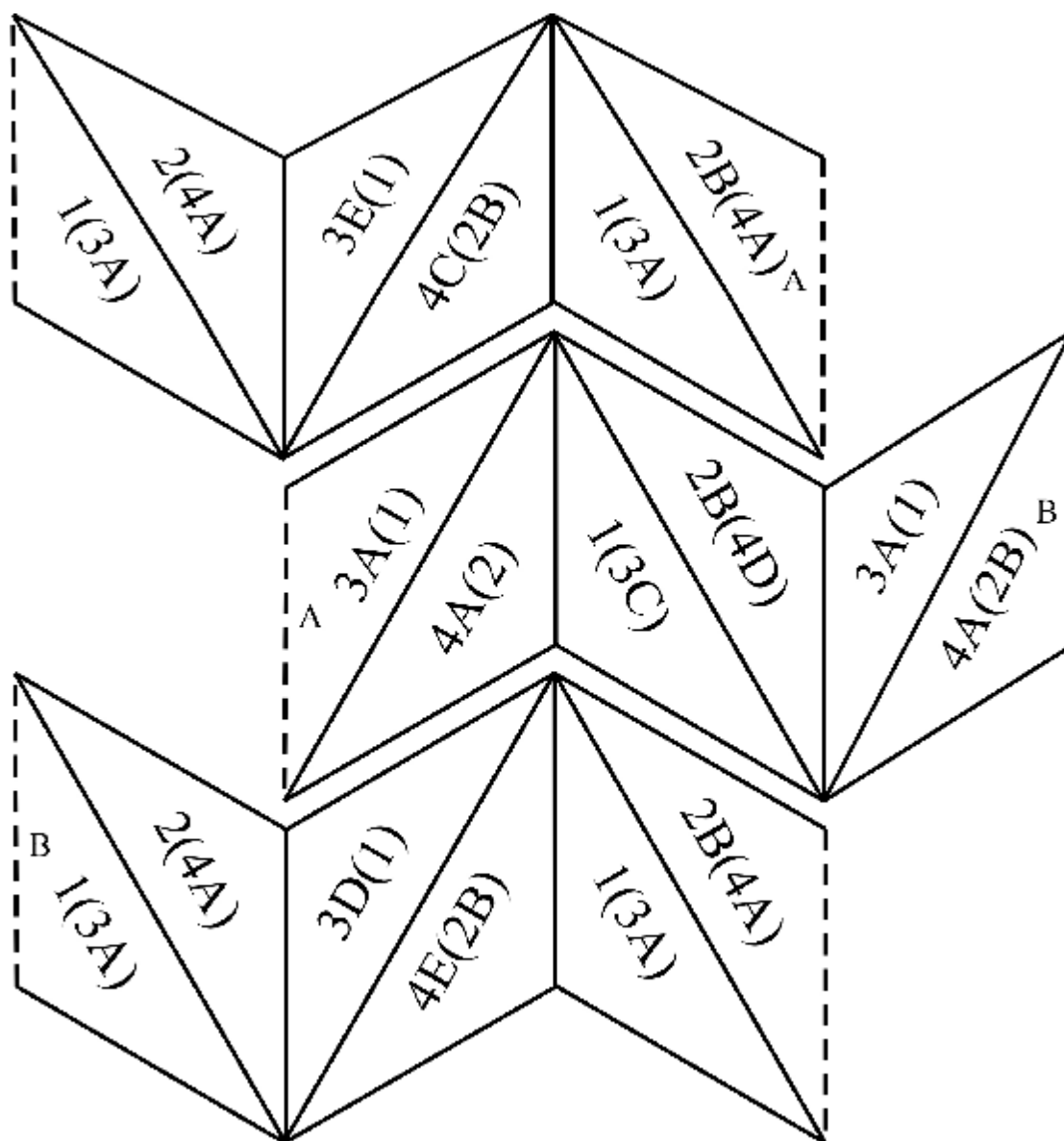


Figure 7. Net (pyramid and three flap positions) for the Thrice Three-Fold Flexagon. Fold pairs of leaves together in the order A and B. Ensure that leaves numbered 3 and 4 are on the inside of the pyramid.

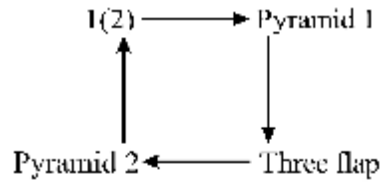


Figure 8. Schematic Tuckerman diagram for a pseudo 4-cycle of the Thrice Three-Fold Flexagon.

### Trapezium and Rhombus Positions

A trapezium position is shown in Figure 9(a); the pat structure has 2-fold rotational symmetry about a line in its plane. A trapezium position can be reached by a twist flex from a triangle position (Figure 2(a)), but this is difficult. To carry out the twist flex hold the middle pair of leaves in a pat, and pull and twist the opposite vertex. There are 9 different trapezium positions.

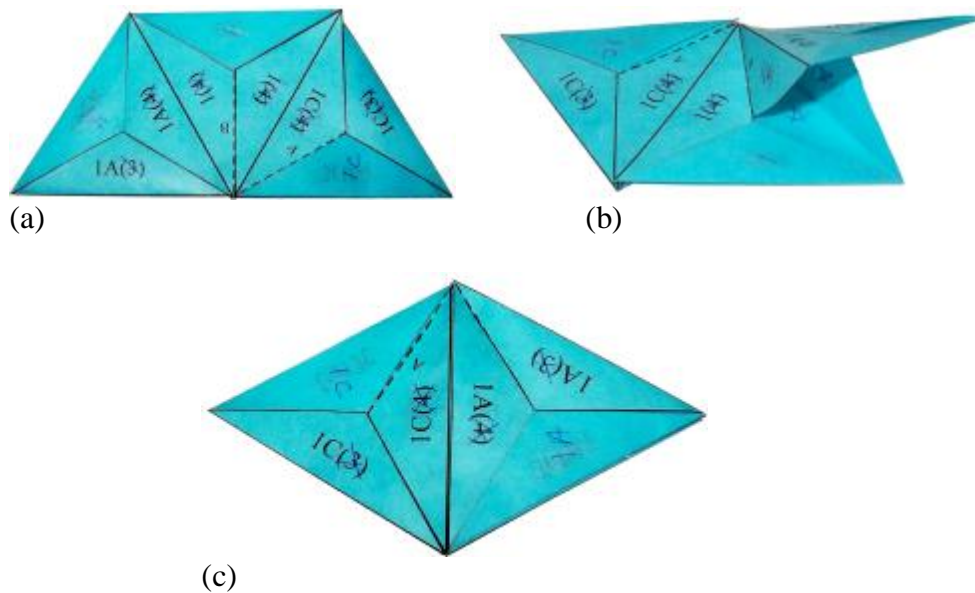


Figure 9. The Thrice Three-Fold Flexagon. (a) Trapezium position. (b) Intermediate position. (c) Rhombus position.

It is much easier to assemble the flexagon into a trapezium position using the net shown in Figure 10. As assembled using this net, the flexagon is in trapezium position 1(2). By using a pocket flex the flexagon can be opened into an intermediate position (Figure 9(b)) and then closed into a rhombus position (Figure 9(c)). This can be done

in four different ways. A second pocket flex leads to a triangle position, but with a different pat structure to that of the triangle position shown in Figure 2(a). There are two of these, triangle positions A(B) and C(D).

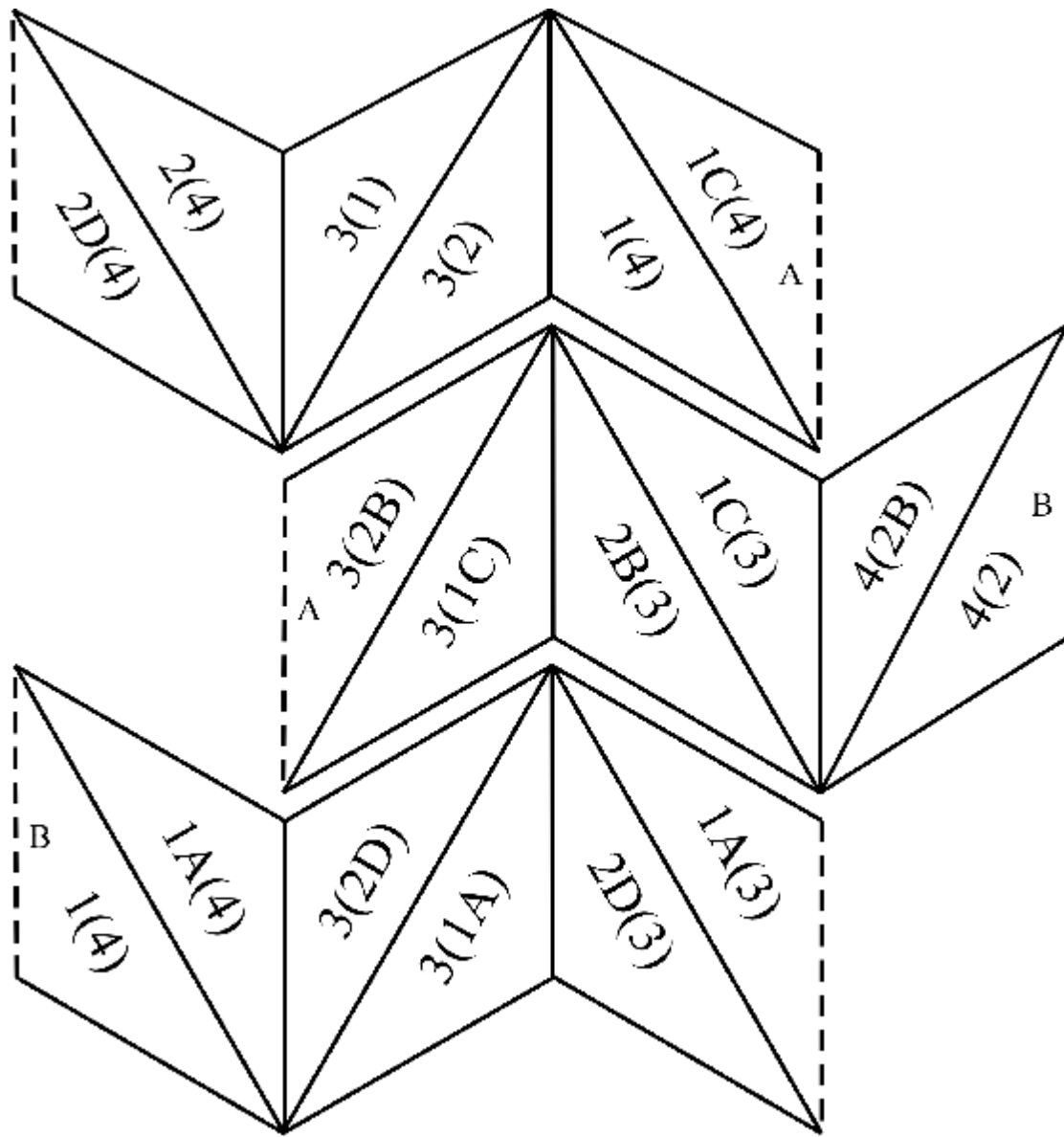


Figure 10. Net (trapezium, rhombus and triangle positions) for the Thrice Three-Fold Flexagon. Fold pairs of leaves numbered 3 together, and interleave so that leaves numbered 4 are opposite each other.



## Irregular Pyramid Positions

The flexagon can also be flexed by using an asymmetric 4-fold pinch flex, starting from a triangle position of the type shown in Figure 2(a). Six of the pats are pinched together in three pairs, with the remaining three pats left in a flat triangle, to reach the intermediate position shown in Figure 11(a). This can be done in 18 different ways. To complete the flex the intermediate position is opened up into an irregular open pyramid with a flap attached (Figure 11(b)). Apart from the flap, and ignoring pat structure, this open pyramid has 4-fold rotational symmetry. The flap can then be closed against the pyramid to give a slightly different pyramid position (Figure 11(c)). Ignoring pat structure, this pyramid position has 2-fold reflectional symmetry.

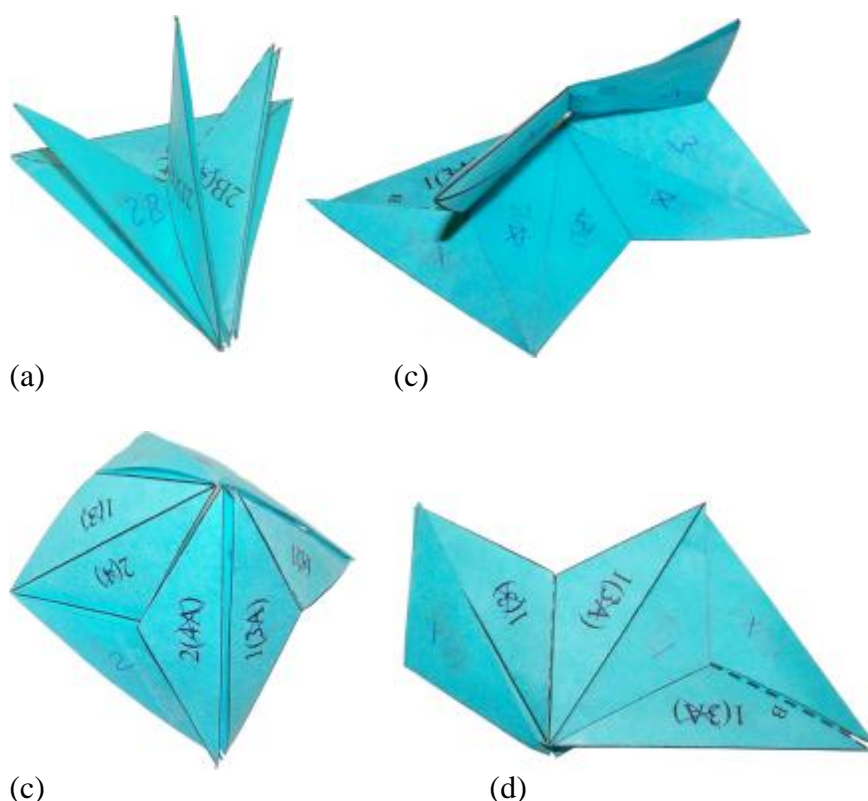


Figure 11. The Thrice Three Fold Flexagon. (a) Asymmetric 4-fold intermediate position. (b) Irregular pyramid position, flap open. (c) Irregular pyramid position, flap closed. (d) A flat position.

The asymmetric 4-fold pinch flex is difficult. It is much easier to assemble the flexagon into an irregular open pyramid by using the net shown in Figure 12. Either form of the pyramid can be flattened in four different ways, one of which is shown in

Figure 11(d). These flat positions can be opened into various asymmetric positions, but it does not appear to be possible to traverse any cycles which include an asymmetric 4-fold pinch flex

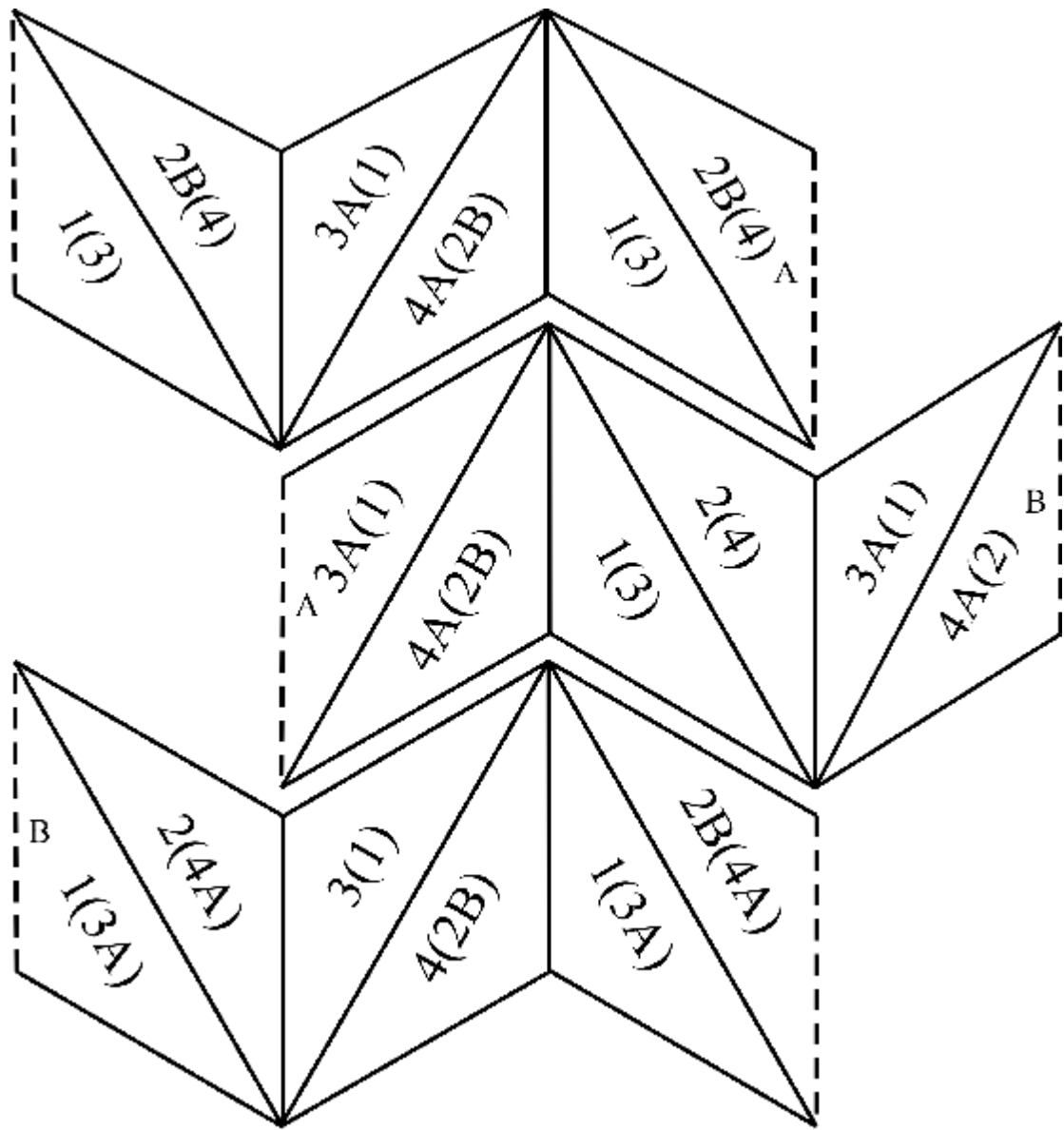


Figure 12. Net for the Thrice Three-Fold Flexagon (irregular pyramid and flat positions). Fold pairs of leaves together in the order A and B. Ensure that leaves numbered 3 and 4 are on the inside of the pyramid.

## **Further Remarks**

The Thrice Three-Fold Flexagon is an entertaining and at times frustrating flexagon, some of whose subtleties have still to be explored. It is an odd flexagon in that there are positions that have the appearance of a ring of an odd number of identical polygons. Most flexagons are even, with even numbers of polygons in rings.

The traditional symmetrical pinch flex, with pats pinched together in pairs, is not possible with odd flexagons and, in general, this results in a limited number of flexing options. However, 9 is not a prime number and in the Thrice Three-Fold Flexagon this results in a relatively large number of flexing options. In particular, 3 is a factor of 9 and there are positions and flexes with 3-fold rotational symmetry. There is no equivalent behaviour in odd flexagons with five and 7 polygons in rings because these are prime numbers.

## **Disclaimer**

The author is not responsible for sleepless nights, or other annoyances, that might arise from manipulation of the Thrice Three-Fold (Gates of Hell) Flexagon