

Dodecaflexagons

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Introduction

The original dodecaflexagon was discovered by Schwartz (2005), and variations have recently been described by Moseley (2007). Dodecaflexagons are made from 30° - 60° - 90° triangles, as are some other types of flexagon. Mitchell (2002, 2006) suggests calling 30° - 60° - 90° triangles bronze triangles, and flexagons made from them bronze flexagons. This interim report summarises the results of some recent investigations into dodecaflexagons.

Rings of Bronze Triangles

There are three possible flat regular rings of bronze triangles, one of four bronze triangles, one of 6 bronze triangles, and one of 12 bronze triangles. In some positions dodecaflexagons have the appearance of a flat ring of 12 bronze triangles. In some positions dodecaflexagons have 6-fold rotational symmetry and hence 6 identical sectors in which the pat structure is the same. There are also positions with 3-fold rotational symmetry (three sectors), and with 2-fold rotational symmetry (two sectors). There are also bronze flexagons that correspond to rings of four and 6 bronze triangles (Mitchell 2002, Pook 2007)

The Basic Bronze Flexagon with Six Sectors

The simplest dodecaflexagon is the basic bronze flexagon with six sectors. Moseley (2007) calls it the junior 12-gon flexagon. Mathematically, it can also be regarded as a stellated degenerate ring dodecagon flexagon (Pook 2007). Notation used here is from this viewpoint, and is partially described in Pook (2003). The net for the basic bronze flexagon with six sectors is shown in Figure 1. Transfer the numbers in brackets to the reverse face. Flexing using 6-fold 3-fold and 2-fold rotational symmetry is possible, and there are numerous possible configurations.

The six sectors make the flexagon difficult to handle when flexing using the 6-fold pinch flex: it is unstable and easily tangled. The dynamic behaviour is similar to that of the trihexaflexagon. As assembled the flexagon is in principal main position 1(3), which is flat (Figure 2(a)). The 3-cycle shown in the intermediate position map

(Figure 3) and the Tuckerman diagram (Figure 4) can be traversed.. Main positions 2(1) and 3(2) are skew (Figures 2(b) and 2(c).

B using a 3-fold pinch flex, as if the flexagon were a trihexaflexagon, 8 minor 3-cycles can be traversed. In the 8 minor cycles face numbers of main positions become mixed up, except in main position 1(2). Other main positions can be identified by a dual face marking scheme using both numbers and letters In the dual marked net (Figure 5). face numbers are the same as in Figure 1. Face letters are chosen so that all the leaves visible on a face of a main position have either the same number or the same letter. For example, in minor main position A(1) face A has six leaves marked 3A and six leaves marked 2AC, and face 1 has six leaves marked 1C and six leaves marked 1D.

There are two different types of minor 3-cycle, four of each type. In one type, pairs of leaves in the intermediate positions have an equilateral triangle outline (Figure 6(a)), and in the other type a kite outline (Figure 6(b)). Face letters, and some other markings, are not shown in these and subsequent photographs.. There are four types of minor cycle main position appearance (Figure 7). The four minor 3-cycles shown in the Tuckerman diagrams shown in Figure 8 can be traversed. For clarity, face letters for the other four minor 3-cycles are not shown on the net. Apart from face lettering, the Tuckerman diagrams for these other minor 3-cycles are the same as those shown in Figure 8.

In the cycles shown in Figure 8, main positions A(1) and B(1) have the appearance of an irregular ring of 12 bronze flexagons with a hexagonal outline (Figure 7(a)), main positions 3(A) and 3(B) have a propeller like outline made up of 9 bronze triangles (Figure 7(b)),and main positions C(3) and D(3) are slant bands of 12 bronze triangles so a push through flex is needed to traverse the minor 3-cycles. This is easy if the flexagon is made from origami paper. Main positions 1(C) and 1(D) are skew propellers made up of 9 bronze triangles (Figure 7(d), and main positions B(A) and D(C) have the same appearance as main position 1(2) (Figure 2(a)), but with a different pat structure.

There are a number of main positions which have 2-fold rotational symmetry, and these can be assembled into minor cycles in various ways, using both 2-fold flexing and irregular forms of 4-fold flexing. One possible minor 4-cycle is given as an example. In the minor main positions face numbers become mixed up so an appropriate dual marked net is used (Figure 9).

The minor 4-cycle shown in the Tuckerman diagram (Figure 10) can be traversed as shown in Figure 11. All positions are flat, except where noted. As assembled the flexagon is in main position 1(3) (Figure 2(a)). Fold in two at the arrow heads to the

first minor intermediate position 1 (Figure 11(a)). Unfold at the parallel hinges to the first minor main position A(1) (Figure 11(b)). Fold in two at the perpendicular hinges to the second minor intermediate position A (Figure 11(c)). Unfold to the second minor main position B(A), which is skew (Figure 11(d)). Twist flex to the third minor main position C(1) by folding together the two pairs of leaves marked with asterisks, with the asterisks on the outside Figure (Figure 11(f)). The intermediate position is not clearly defined (Figure 11(e)). Flex to the fourth minor intermediate position, which is skew, by making valley folds at the hinges marked V, and mountain folds at adjacent hinges (Figure 11(g)). Then open into main position 1(3) to complete the minor 4-cycle.

A Linked Basic Bronze Flexagon with Six Sectors

This linked basic bronze flexagon with six sectors is formed by linking together two basic bronze flexagon with six sectors using a skew main position link. The net is shown in Figure 12. The dynamic properties are identical to those of the hexa-dodecaflexagon, or 12-gon flexagon, described by Schwartz (2005). She uses a straight strip net with additional faces that do not appear during flexing.

The dynamic properties of the flexagon are complex, and are not described in full. The six sectors make the flexagon difficult to handle when flexing using 6-fold rotational symmetry. It is unstable and very easily tangled. The two 3-cycles shown in the Tuckerman diagram (Figure 13) can be traversed using the 6-fold pinch flex. Main positions 3(1) and 2(4) are flat (similar to Figure 2(a)), and the other main positions are skew. In particular, main position 1(2) is skew (similar to Figure 2(b)), so a snap flex is needed when traversing from cycle A to cycle B, and vice versa. There are enough degrees of freedom for this to be accomplished without bending the leaves, but care is needed to avoid spurious positions in which face numbers are mixed up.

Various minor cycles can be traversed, as for the six sector version of the basic bronze flexagon. In particular, there are 8 minor cycles that can be traversed by using a 3-fold pinch flex, with a triangle intermediate position appearance (similar to Figure 6(a)). During flexing loose flaps consisting of a pair of pats appear. Schwartz calls these rogue triangles. All 8 of the minor 3-cycles are linked, as shown in the Tuckerman diagram (Figure 14), whereas in the six sector version of the basic bronze flexagon the four corresponding minor cycles are in two separate groups of two. The two groups of four minor 3-cycles, shown with straight arrows between main positions, can be traversed by using 3-fold pinch flexes. The curved lines, with arrowheads at both ends, are links between the two groups that can be traversed by turning the rogue triangles over in a flap flex. This type of link does not appear in the six sector version

of the basic bronze flexagon. In the linked basic bronze flexagon with six sectors the flap flex links provide an alternative, much easier, way of traversing between main positions 3(1) and 2(4) shown in Figure 13.

Concluding Remarks

Use of appropriate face marking schemes is the key to understanding the extraordinarily complex dynamic behaviour of dodecaflexagons. The comments in this article, together with other people's contributions, are only a small part of the overall picture. Random flexing leads to numerous positions with mixed up face numbers, some with lower degrees of rotational symmetry. Some of these could probably be formalised as resulting from flexes similar to the V-flex used for hexahexaflexagons (McLean 1979).

References

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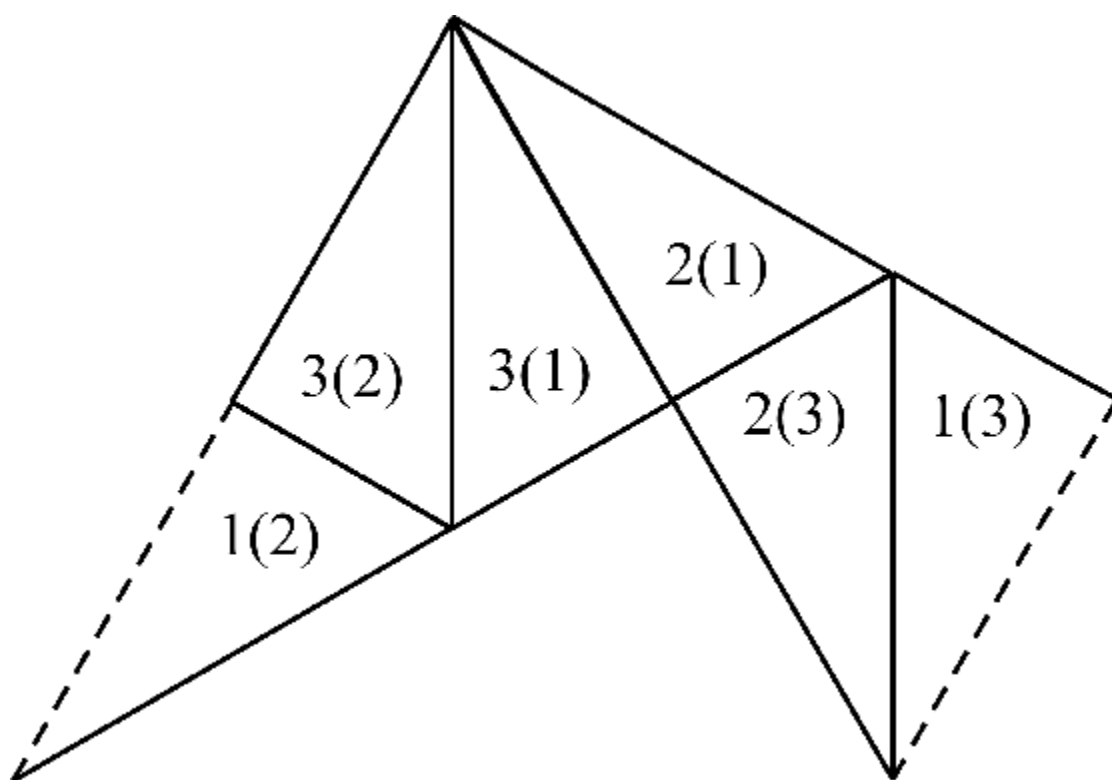


Figure 1. Net for the basic bronze flexagon. Six sector version. Three copies needed.
Fold together pairs of leaves numbered 2.

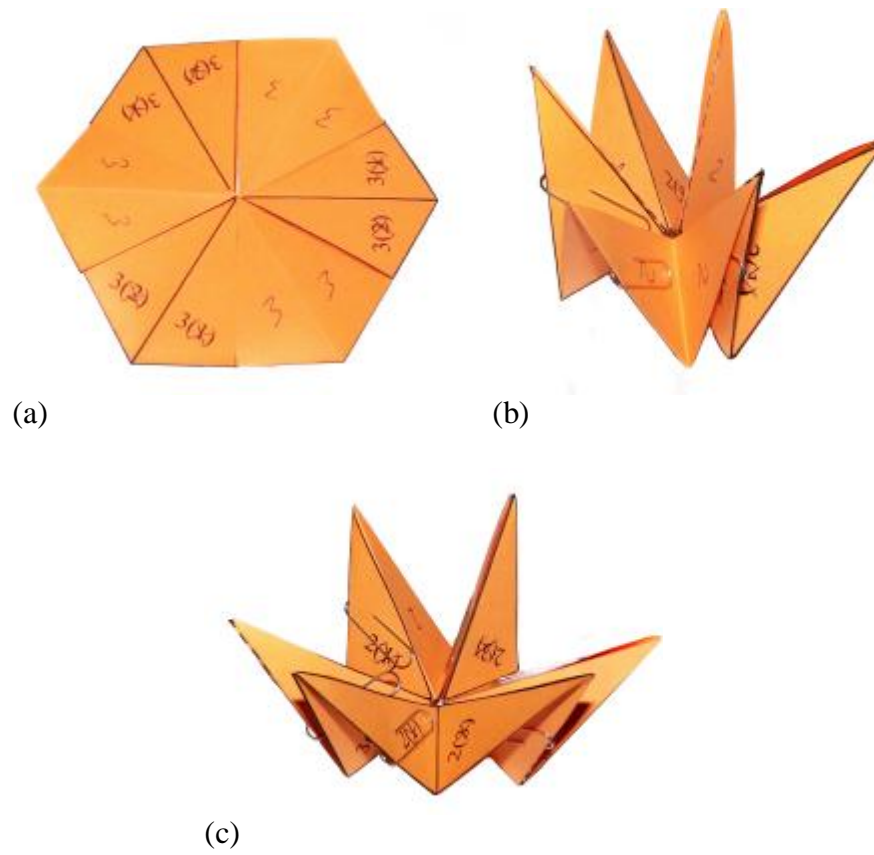


Figure 2. Basic bronze flexagon. Six sector version, 6-fold flexing (a) Principal main position. (b) Subsidiary 4-cycle main position. (c) Subsidiary 3-cycle main position.

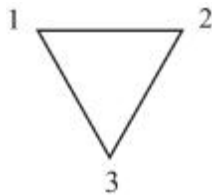


Figure 3. Intermediate position map for basic bronze flexagon, 6-fold flexing.

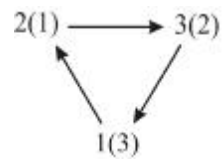


Figure 4. Tuckerman diagram for basic bronze flexagon, 6-fold flexing.

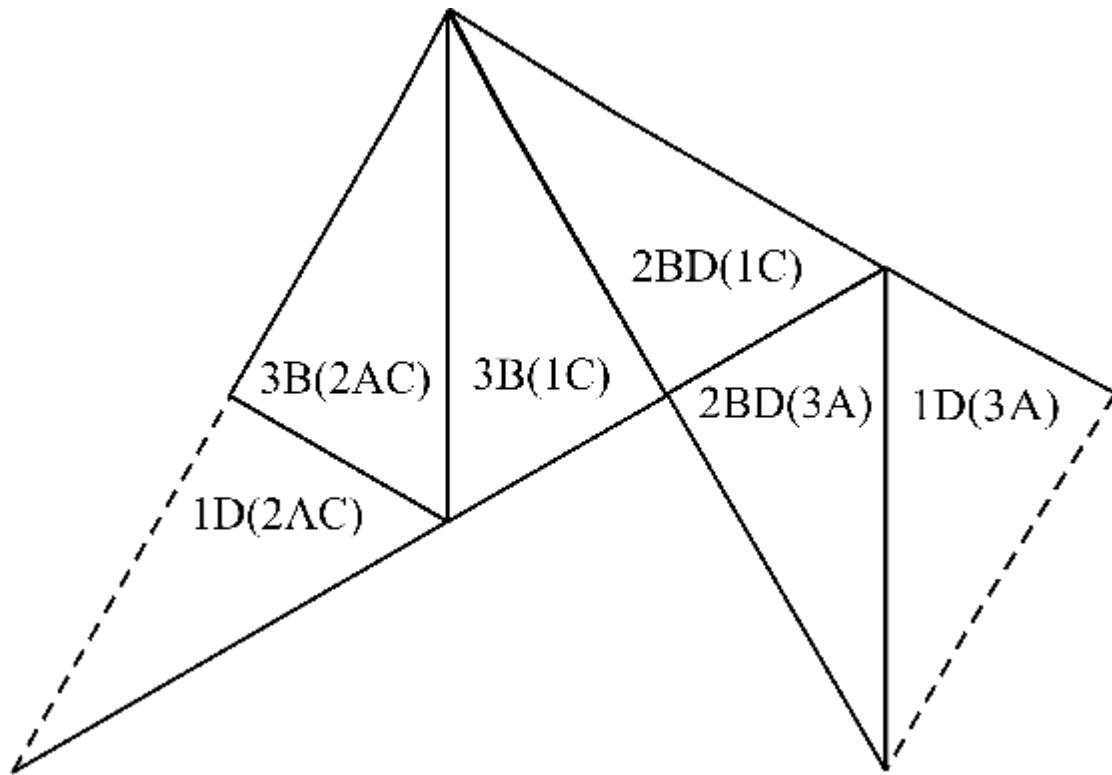


Figure 5. Dual marked net for the basic bronze flexagon. Six sector version, 3-fold flexing. Three copies need. Fold together pairs of leaves numbered 2.

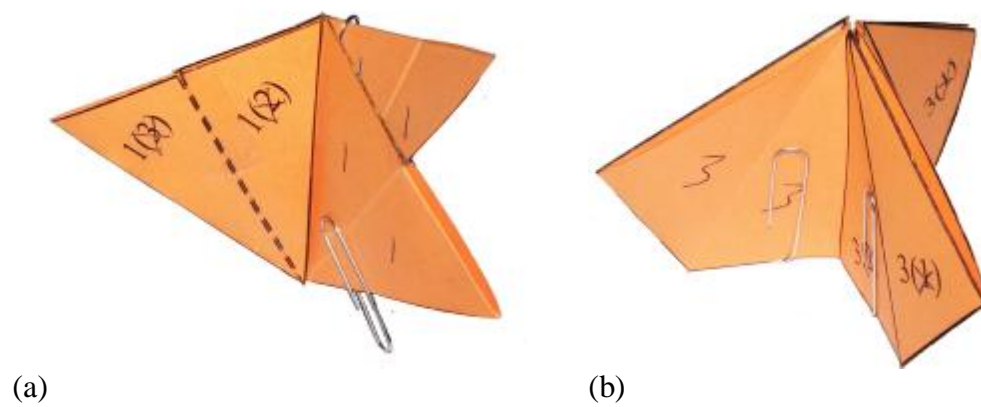


Figure 6. Basic bronze flexagon. Six sector version, 3-fold flexing. Minor cycle intermediate positions. (a) Triangle. (b) Kite.

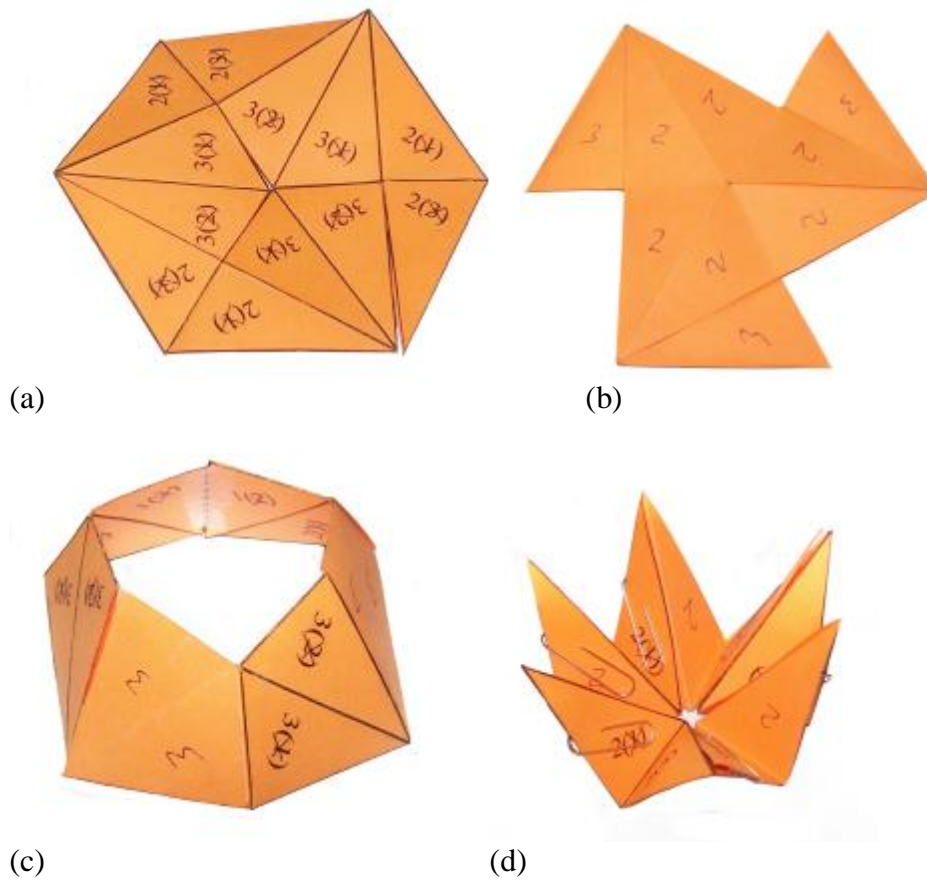


Figure 7. Basic bronze flexagon. Six sector version, 3-fold flexing. Minor cycle main positions. (a) Hexagonal. (b) Propeller. (c) Slant. (d) Skew.

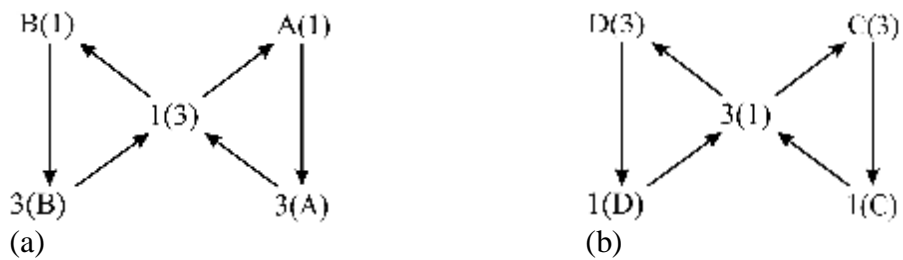


Figure 8. Tuckerman diagram for minor cycles of the basic bronze flexagon. Six sector version, 3-fold flexing (a) Triangle intermediate positions. (b) Kite intermediate positions.

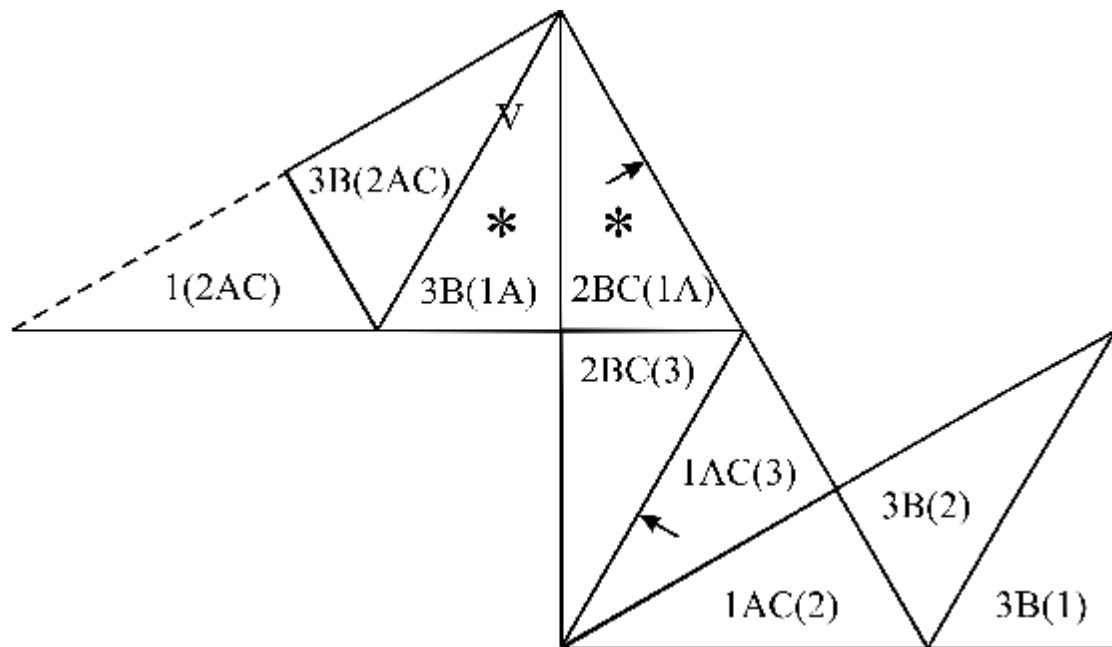


Figure 9. Dual marked net for the basic bronze flexagon. Six sector version, 2-fold flexing. Two copies need. Fold together pairs of leaves numbered 2. (10 Jan 07 A 12) (XVII p. 14)

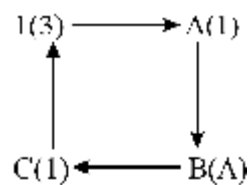


Figure 10. Tuckerman diagram for a minor cycle of the basic bronze flexagon. Six sector version, 2-fold flexing.

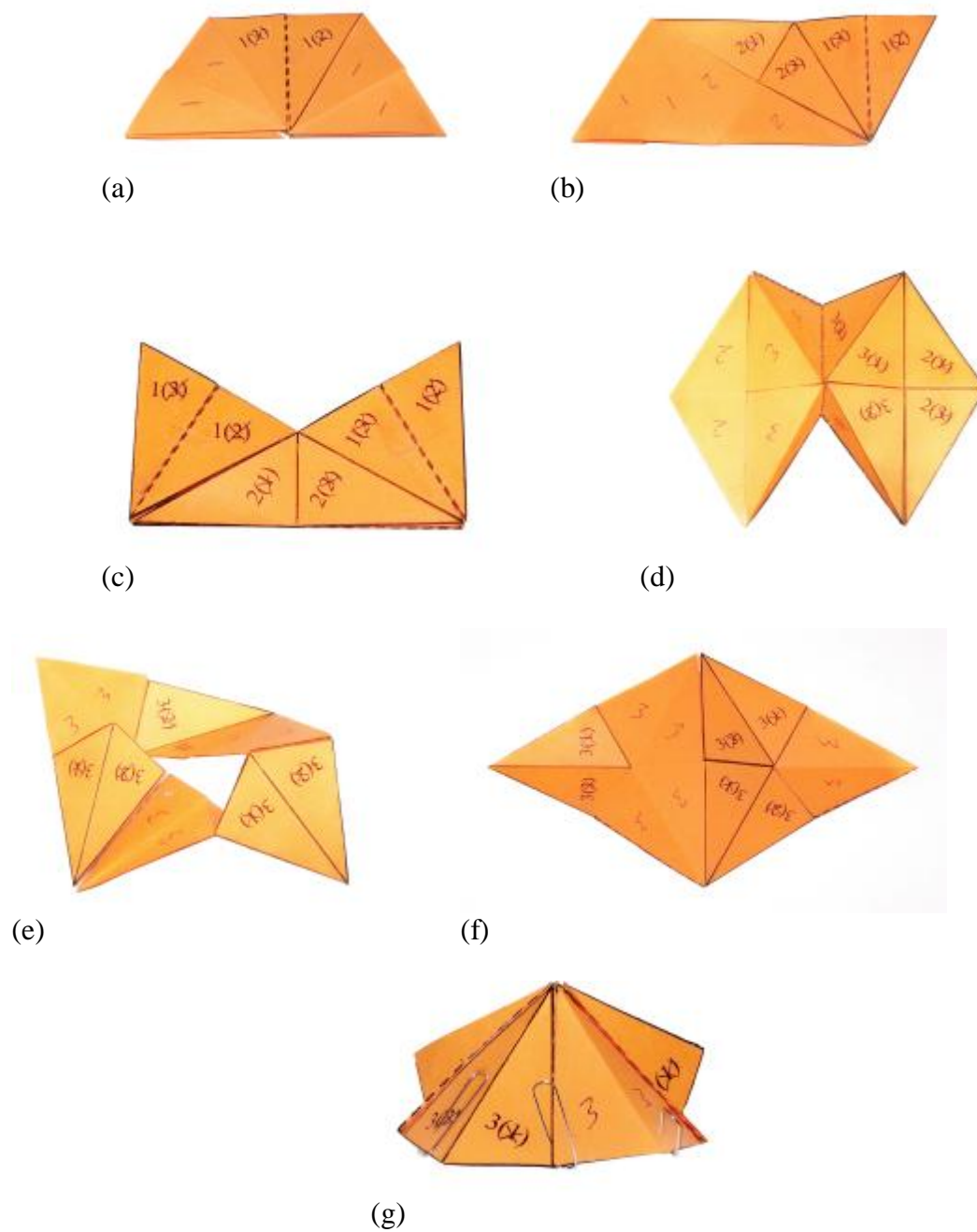


Figure 11. Basic bronze flexagon. Six sector version, 2-fold flexing. Flexing sequence for a minor cycle. (a) First minor intermediate position. (b) First minor main position. (c) Second minor intermediate position. (d) Second minor main position. (e) Third minor intermediate position. (f) Third minor main position. (g) Fourth minor intermediate position.

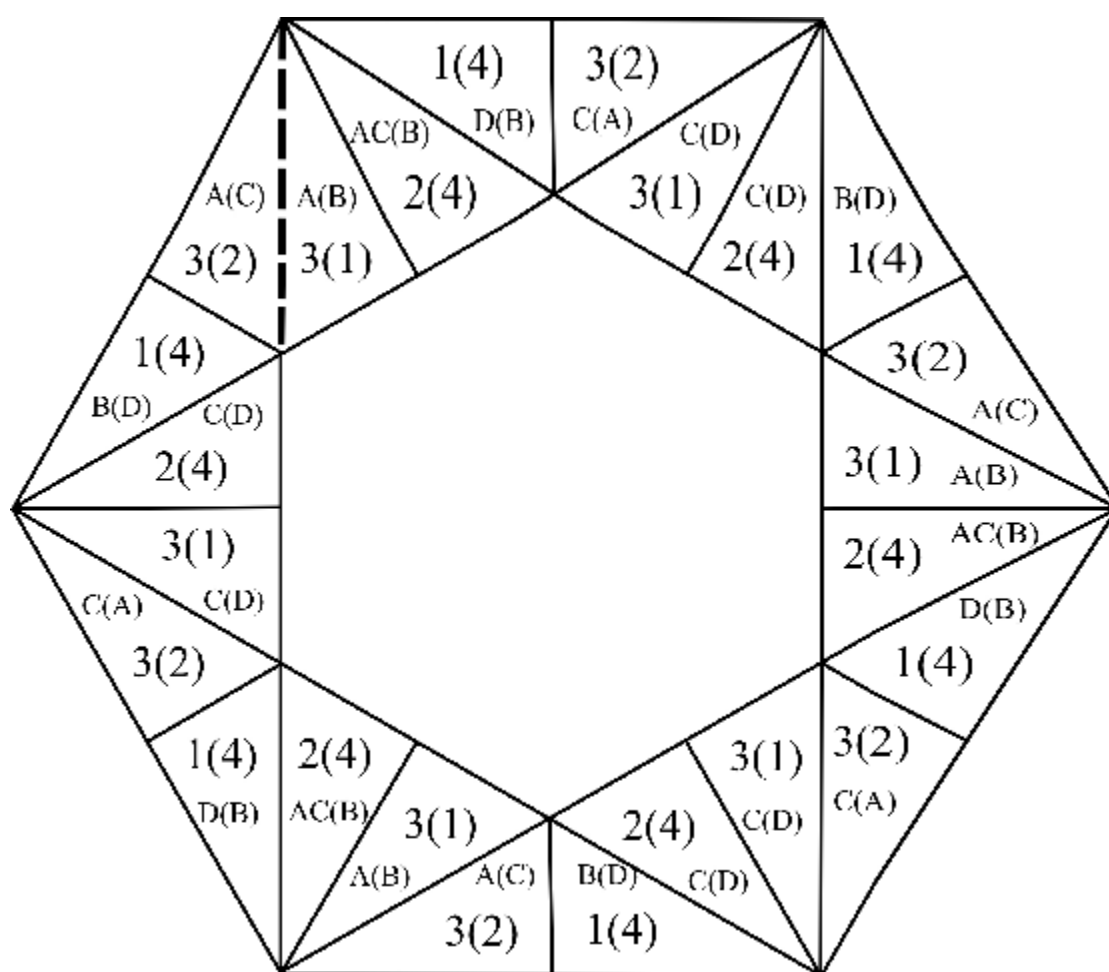


Figure 12. Net for a skew linked bronze flexagon. One copy needed. Cut along the dashed line, fold together pairs of leaves numbered 4 and 2, and rejoin.

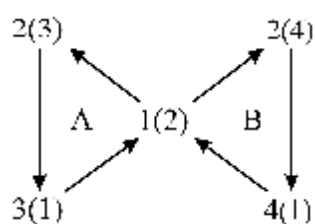


Figure 13. Tuckerman diagram for a skew linked bronze flexagon.

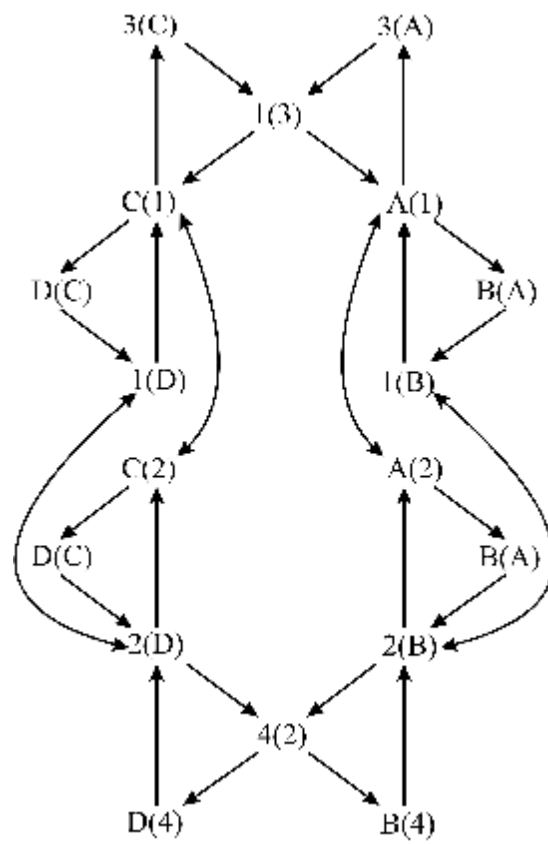


Figure 14. Minor cycle Tuckerman diagram for a skew linked bronze flexagon.