Harmonic Chains

- X a finite simplicial complex with an orientation.
- $(C_*(X), \partial_*)$ the simplicial chains complex over $\mathbb{R}$.
- $\Delta_* : C_*(X) \to C_*(X)$

$$\Delta_i = \partial_i^t \partial_i + \partial_{i+1} \partial_{i+1}^t$$

Def: The i-th harmonic space of X

is $\mathcal{H}_i(X) = \ker \Delta_i$

Recap of results:

Prop: (i) $\mathcal{H}_i(X) = \ker \partial_i \cap \ker \partial_{i+1}^t$

i-th simplicial homology

(ii) With $\pi_i : \mathcal{H}_i(X) \longrightarrow H_i(X)$
$$h \longmapsto [h]$$

is an isomorphism

Q: Within a homology class, how to identify the harmonic one?

Def: For $x \in C_i(X)$, its **energy** is defined by $\mathcal{E}(x) = \langle x, x \rangle = \|x\|^2$

If $x = \sum_{\sigma \in X_i} x_\sigma \sigma$ then $\mathcal{E}(x) = \sum_{\sigma \in X_i} |x_\sigma|^2$

Prop: Let $x \in \ker \partial_i$. The following are equivalent

(i) $h = \underset{y \in [x]}{\operatorname{argmin}} \, \mathcal{E}(y)$

(ii) $h = \pi_i^{-1}([x])$

Pf: $x \in \ker \partial_i$. $\quad x = h + \partial_{i+1} y$, $y \in C_{i+1}(X)$.

$h := \pi_i^{-1}([x])$

$$\mathcal{E}(x) = \underbrace{\langle h, h \rangle}_{\mathcal{E}(h)} + \underbrace{2\langle \partial_{i+1} y, h \rangle}_{=0} + \underbrace{\langle \partial_{i+1} y, \partial_{i+1} y \rangle}_{\mathcal{E}(\partial_{i+1} y)}.$$

Hodge decomp.

$$\langle \partial_{i+1} y, h \rangle = \langle y, \partial_{i+1}^t h \rangle \overset{\checkmark}{=} 0$$

$$\mathcal{E}(x) = \mathcal{E}(h) + \mathcal{E}(\partial_{i+1} y) \geq \mathcal{E}(h)$$

$\mathcal{E}(x) = \mathcal{E}(h)$ iff $\mathcal{E}(\partial_{i+1} y) = 0$ iff $x = h$.

<u>Examples</u>: (1) X is 1-dimensional aka a graph.

From Hodge decomposition $\mathcal{H}_0(X) \simeq H_0(X)$
$$\simeq k^{\pi_0(X)}$$

$$X = \bigsqcup_{C \in \pi_0(X)} C, \quad \mathbb{1}_C : X_0 \to k$$
$$x \mapsto \begin{cases} 1 \text{ if } x \in C \\ 0 \text{ else} \end{cases} \qquad \mathbb{1}_C = \sum_{\sigma \in C_0} \sigma$$

We have to prove that $\mathbb{1}_C$ is harmonic.

$f : X_0 \to k \in C_0(X)$ then for $v \in X_0$

$$\Delta_0(f)(v) = \sum_{w \sim v} \left(f(w) - f(v) \right)$$

"w is connected to v"

With this expression

$$\Delta_0(\mathbb{1}_C)(v) = \sum_{v \sim w} \underbrace{\mathbb{1}_C(w) - \mathbb{1}_C(v)}_{0} = 0$$

$\Rightarrow \mathbb{1}_C \in \mathcal{H}_0(X)$.

$(\mathbb{1}_C)_{C \in \pi_0(X)}$ is linearly independant in $C_0(X)$.

Since $\dim \mathcal{H}_0(X) = \# \pi_0(X)$

$(\mathbb{1}_C)_{C \in \pi_0(X)}$ is a basis of $\mathcal{H}_0(X)$.

We have proved that if X is 1-dim,

$$\mathcal{H}_0(X) = \{ f : X_0 \to \mathbb{R},\ f \text{ is constant on each c.c. of } X \}.$$

$$\big\downarrow$$

$$H_0(X)$$

$\mathbb{R}^{\#C}$ equipped $\|\cdot\|_2$.

$v \in \mathbb{R}^{\#C}$, $\Delta_0(v) = 0$ then

$$\|v\|_2^2 \geq \|\mathbb{1}_C\|_2^2 = \#C.$$

$x \in \ker \partial_i$, $h = \pi_i^{-1}([x])$.

$\exists y \in C_{i+1}(X)$, $x = h + \partial_{i+1} y$

$$\underset{y \in C_{i+1}(X)}{\cancel{\operatorname{argmin}}} \; \Sigma (x + \partial y)$$