For $n \in \mathbb{N}$, let $F_n = (\varphi^n - \psi^n)/\sqrt{5}$.

Base case.

$$F_0 = 0 = \mathrm{Fib}(0),$$
$$F_1 = 1 = \mathrm{Fib}(1).$$

Induction step. Let $n \in \mathbb{N}$. Suppose $\mathrm{Fib}(k) = F_k$ holds for all $k \leq n + 1$.

Since both φ and ψ are solutions to the equation $x^2 - x - 1 = 0$, it follows that for all $n \in \mathbb{N}$,

$$\varphi^{n+2} = \varphi^{n+1} + \varphi,$$
$$\psi^{n+2} = \psi^{n+1} + \psi.$$

Then since F_n is a linear combination of φ^n and ψ^n for every $n \in \mathbb{N}$, we have

$$F_{n+2} = F_{n+1} + F_n = \mathrm{Fib}(n+1) + \mathrm{Fib}(n) = \mathrm{Fib}(n+2).$$

Hence, $\mathrm{Fib}(k) = F_k$ holds for all $k \leq (n + 1) + 1$.

Thus, we can conclude that $\mathrm{Fib}(n) = F_n$ for all $n \in \mathbb{N}$.

For all $n \in \mathbb{N}$, the number $\mathrm{Fib}(n)$ is an integer, and $\left|\varphi^n/\sqrt{5} - \mathrm{Fib}(n)\right| = |\psi|^n/\sqrt{5} \leq 1/\sqrt{5} < 1/2$. Therefore, $\mathrm{Fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$.