Numerical Analysis – Homework 3

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Exercise 1. For $a = x_0 < x_1 < \ldots < x_n = b$ and a function $f \in C[a,b]$, show that the interpolation problem has a unique solution

$$Q(x) = \sum_{j=0}^{n} c_j e^{jx} \tag{1}$$

with $Q(x_i) = f(x_i)$ for all i. *Hint*: Reduce to a usual polynomial interpolation problem.

Proof. Define $g : (0,\infty) \to \mathbb{R}$ by $g(z) = (f \circ \ln)(z)$. Since $f \in C[a,b]$ and $\ln \in C(0,\infty)$ we have that $g \in C[e^a, e^b]$. Now perform a change of variables by defining $z = e^x$. Since e^x is a strictly increasing function on $\mathbb{R}$ and $a = x_0 < \ldots < x_n = b$, by defining $z_i = e^{x_i}$, for $0 \le i \le n$, we have that $e^a = z_0 < z_1 < \ldots < z_n = e^b$. Since $g \in C[e^a, e^b]$ and $e^a = z_0 < z_1 < \ldots < z_n = e^b$, by Theorem 3.2 (in *Numerical Analysis* by Burden and Faires)

$$P(z) = \sum_{j=0}^{n} g(z_j) L_{n,j}(z),$$

is the unique polynomial of degree at most n satisfying $g(z_j) = P(z_j)$ for $0 \le j \le n$. Here the functions are $L_{n,j}(z) = \prod_{k \ne j} \frac{(z - z_k)}{(z_j - z_k)}, 0 \le j \le n$. So by defining

$$E_{n,j}(x) = \prod_{k \ne j} \frac{e^x - e^{x_k}}{e^{x_j} - e^{x_k}}$$

we now have that $E_{n,j}(x_i) = L_{n,j}(z_i) = \delta_{ij}$. Hence, letting

$$Q(x) = \sum_{j=0}^{n} f(x_j) E_{n,j}(x)$$

it follows that

$$Q(x) = \sum_{j=0}^{n} f(x_j) E_{n,j}(x) = \sum_{j=0}^{n} f(\ln(e^{x_j})) L_{n,j}(z) = \sum_{j=0}^{n} g(z_j) L_{n,j}(z) = P(z), \tag{2}$$

namely, $Q(x_i) = f(x_i)$, for $0 \le i \le n$. We prove uniqueness by way of contradiction. Accordingly, suppose that there exist $Q(x)$ and $R(x)$ satisfying (1) with $Q(x) \ne R(x)$. By our proof of existence, there exist corresponding polynomials $P(z)$ and $M(z)$ satisfying the equality in (2) for $Q(x)$ and $R(x)$, respectively. Thus, by the uniqueness of the interpolating polynomial from Theorem 3.2 we have that $Q(x) = P(z) = M(z) = R(x)$, a contradiction to $Q(x) \ne R(x)$. Thus, $Q(x)$ is unique. □

Exercise 2. Consider the inner product on $C[1,2]$,

$$\langle f, g\rangle = \int_1^2 f(x)g(x)e^{-x}dx.$$

(a) Starting with the basis $\{1, x, x^2\}$ for $\mathcal{P}_2[1,2]$ use Gram-Schmidt to determine the first three orthonormal polynomials on $[1,2]$ with respect to the inner product.

(b) Find the order 2 best least squares approximation for $f(x) = e^x$ on $[1,2]$ with respect to the given inner product.

Proof of (a). Using Gram-Schmidt, the first orthonormal polynomial on $[1,2]$ with respect to the given inner product is

$$q_0(x) = \frac{e}{\sqrt{e-1}} \approx 2.073706473.$$

Then $q_1(x) = \dfrac{p_1(x)}{\langle p_1(x), p_1(x)\rangle^{1/2}}$, where $p_1(x) = x - \langle x, q_0(x)\rangle q_0(x)$. Since

$$\begin{aligned}
\langle x, q_0(x)\rangle &= \frac{e}{\sqrt{e-1}}\int_1^2 xe^{-x}dx \\
&= \frac{e}{\sqrt{e-1}}\left(\left[-xe^{-x}\right]_1^2 - \int_1^2 -e^{-x}dx\right) \\
&= \frac{e}{\sqrt{e-1}}\left(-2e^{-2} + e^{-1} - e^{-2} + e^{-1}\right) \\
&= \frac{e}{\sqrt{e-1}}\left(-3e^{-2} + 2e^{-1}\right) \\
&\approx 0.683810998,
\end{aligned}$$

we have that $p_1(x) = x - 1.418023293$. Hence, letting $k = 1.418023293$, we have

$$\begin{aligned}
\langle p_1(x), p_1(x)\rangle &= \int_1^2 \left(x^2 - 2kx + k^2\right)e^{-x}dx \\
&= \int_1^2 x^2e^{-x}dx - 2k\int_1^2 xe^{-x}dx + k^2\int_1^2 e^{-x}dx \\
&= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_1^2 - 2k[-3e^{-2} + 2e^{-1}] + k^2\left[-e^{-x}\right]_1^2 \\
&= -10e^{-2} + 5e^{-1} - 2k[-3e^{-2} + 2e^{-1}] + k^2[e^{-1} - e^{-2}] \\
&\approx 0.018446892.
\end{aligned}$$

Thus,

$$q_1(x) = \frac{x - 1.418023293}{\sqrt{0.018446892}} = 7.362721909x - 10.44051117.$$

Next, $q_2(x) = \dfrac{p_2(x)}{\langle p_2(x), p_2(x)\rangle^{1/2}}$, where $p_2(x) = x^2 - \langle x^2, q_1(x)\rangle q_1(x) - \langle x^2, q_0(x)\rangle q_0(x)$. Using a computer we find that

$$\langle x^2, q_0(x)\rangle \approx 1.0079133634 \quad \text{and} \quad \langle x^2, q_1(x)\rangle \approx 0.3983825436.$$

Hence,

$$\langle x^2, q_0(x)\rangle q_0(x) \approx 2.09011647 \quad \text{and} \quad \langle x^2, q_1(x)\rangle q_1(x) \approx 2.93317988197x - 4.15931739645.$$

Thus, $p_2(x) = x^2 - 2.93317988197x + 2.06920092645$. Since

$$\langle p_2(x), p_2(x)\rangle \approx 0.001238094453$$

we conclude that

$$q_2(x) = \frac{x^2 - 2.93317988197x + 2.06920092645}{0.03518656637} = 28.419937x^2 - 83.360787x + 58.80656.$$

$\square$

Proof of (b). Define $P : C[1,2] \to \mathcal{P}_2[1,2]$ by

$$Pf = \sum_{j=0}^{2} \langle f, q_j(x) \rangle q_j(x),$$

for all $f \in C[1,2]$. By our theorem we have that $w = Pe^x$ minimizes the value $\|w - e^x\|$ over all $w \in \mathcal{P}_2[1,2]$. Since

$$\langle e^x, q_0(x) \rangle = 2.073706473, \quad \langle e^x, q_1(x) \rangle = 0.6035716935, \quad \text{and} \quad \langle e^x, q_2(x) \rangle = 0.078565833$$

we have that

$$\begin{aligned} Pe^x &= 0.078565833\ q_2(x) + 0.6035716935\ q_1(x) + 2.073706473\ q_0(x) \\ &= (2.23284x^2 - 6.54931x + 4.62019) + (4.44393x - 6.3016) + 4.30025853 \\ &= 2.23284x^2 - 2.10538x + 2.61885. \end{aligned}$$

For a comparison of this polynomial to e^x, consider the plot of both functions in Figure 1. $\square$

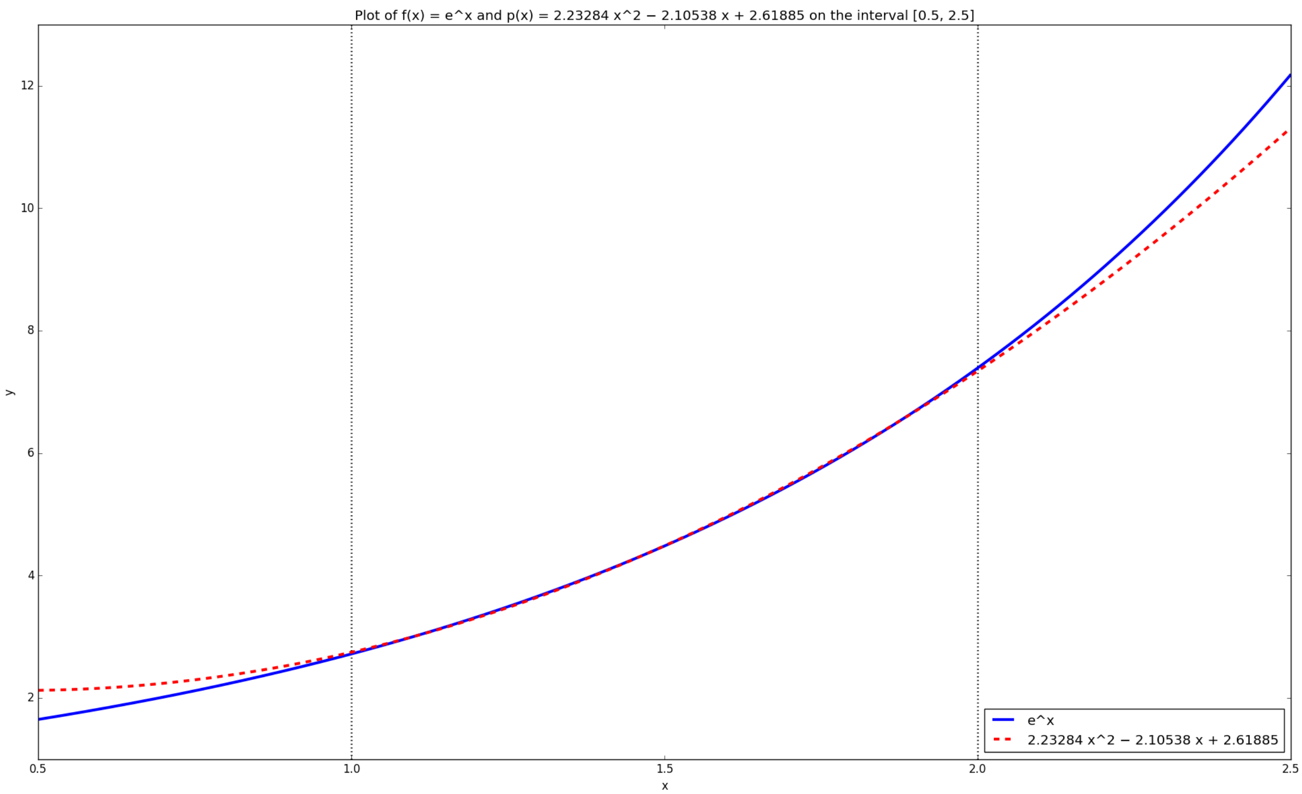

Figure 1: Plot of $f(x) = e^x$ (solid line) and $p(x) = 2.23284x^2 - 2.10538x + 2.61885$ (dashed line), the polynomial constructed in 2 (a).

Exercise 3. Let $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$.

(a) Write a program which uses Newton divided differences to find the interpolating polynomial p to f with respect to $n = 10$ uniformly spaced nodes.

(b) Evaluate $p(.4835)$ and compute the absolute error $|p(.4835) - f(.4835)|$.

Code, experiments, and results. The code is included on the final page of the homework. The program was coded using C++ and Figure 2 was generated using the 'matplotlib' package in python. Instead of computing the coefficients of the interpolating polynomial I used a recursion to compute values in my program. The output from the program I wrote is listed below:

`p(0.4835) = 0.852867` and `|p(0.4835) - f(0.4835)| = 0.0423448.`

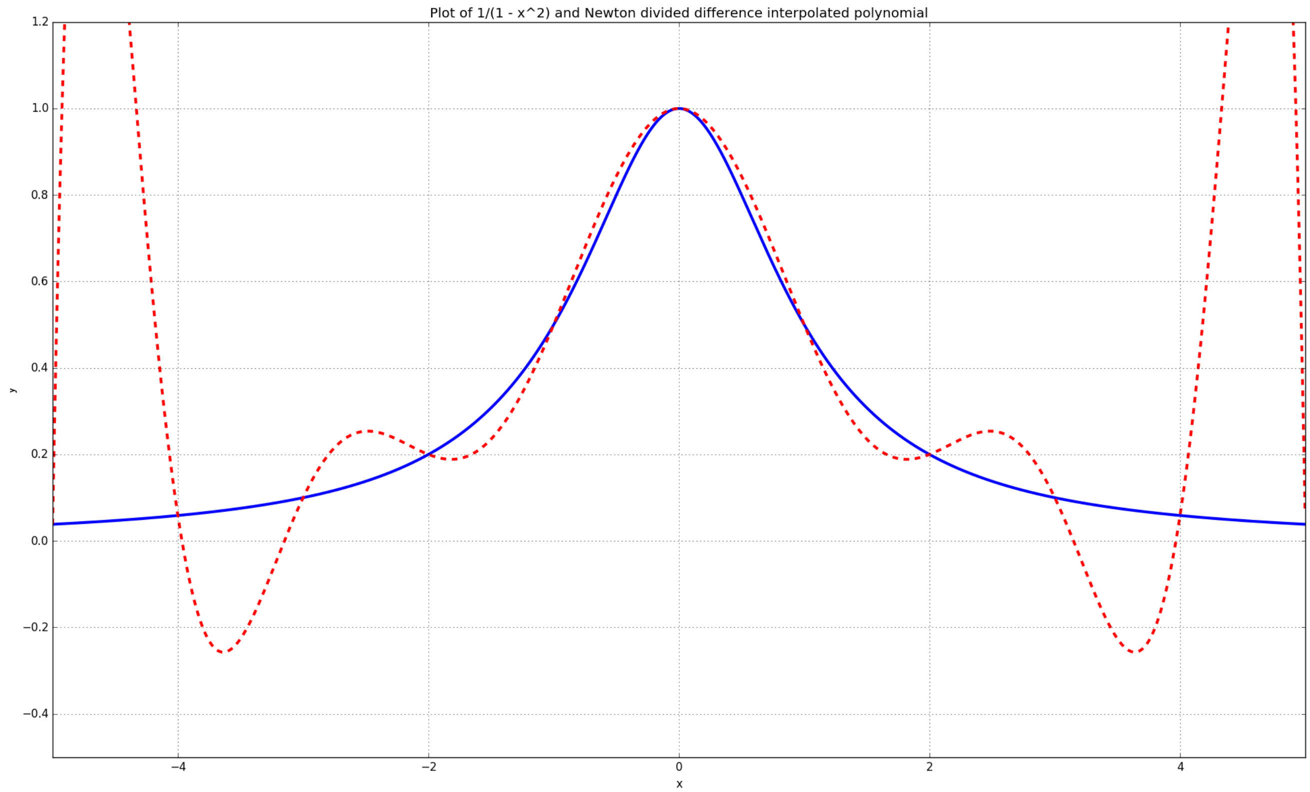

Figure 2: Plot of $f(x) = \frac{1}{1+x^2}$ (solid line) and $p(x)$, the interpolated polynomial generated by Newton divided differences (dashed line).

$\square$

Exercise 4. If $f \in C^{2n+2}[a,b]$ and $x_0, \ldots, x_n$ are distinct points in $[a,b]$ and H_{2n+1} is the corresponding Hermite polynomial, show that for $t \in [a,b]$

$$f(t) = H_{2n+1}(t) + \frac{(t-x_0)^2 \cdots (t-x_n)^2}{(2n+2)!} f^{(2n+2)}(\eta),$$

for some $\eta \in [a,b]$.

Proof. Begin by defining $E(x) = f(x) - H_{2n+1}(x)$, $w(x) = (x-x_0)^2 \cdots (x-x_n)^2$, and

$$G(x) = E(x) - \frac{w(x)}{w(t)} E(t).$$

Based on these choices, we observe that

(i) $E(x_i) = f(x_i) - H_{2n+1}(x_i) = 0$, for $0 \le i \le n$.

(ii) $G(x_i) = E(x_i) - \frac{w(x_i)}{w(t)} E(t) = 0 - 0 = 0$, for $0 \le i \le n$.

(iii) $G(t) = E(t) - \frac{w(t)}{w(t)} E(t) = 0$.

By definition $G \in C^{(2n+2)}[a,b]$. From (ii) and (iii) G has $n+2$ distinct zeros in $[a,b]$. By Rolle's Theorem, there exist $\xi_i \in (a,b)$, for $0 \leq i \leq n$, such that

(a) $G'(\xi_i) = 0$, for $0 \leq i \leq n$

(b) $\xi_i \neq x_j$, for $0 \leq i, j \leq n$

(c) $\xi_i \neq t$, for $0 \leq i \leq n$.

Since $G'(x_i) = E'(x_i) - \frac{w'(x_i)}{w(t)^2} E(t) = 0$, for $0 \leq i \leq n$, $G'(x)$ has $2n+2$ distinct roots in (a,b), namely $\{x_0, x_1, \ldots, x_n, \xi_0, \xi_1, \ldots, \xi_n\}$. Thus, Generalized Rolle's Theorem yields that there exists a $\eta \in (a,b)$ with $G^{(2n+2)}(\eta) = 0$. Computing derivatives with respect to x we find that

$$w^{(2n+2)}(x) = (2n+2)! \quad \text{and} \quad E^{(2n+2)}(x) = f^{(2n+2)}(x),$$

where the second equality holds since $H_{2n+1}^{(2n+2)}(x) = 0$. Hence,

$$\begin{aligned} G^{(2n+2)}(x) &= f^{(2n+2)}(x) - \frac{w^{(2n+2)}(x)}{w(t)} E(t) \\ &= f^{(2n+2)}(x) - \frac{(2n+2)!}{w(t)} E(t). \end{aligned}$$

Substituting $x = \eta$ we get that

$$E(t) = \frac{f^{(2n+2)}(\eta)}{(2n+2)!} w(t),$$

the desired result. $\square$

Exercise 5. Fix n. For the basic Lagrange polynomials

$$L_k(x) = \prod_{i \neq j} \frac{x - x_j}{x_i - x_j},$$

for $k = 0, \ldots, n$. Show that

$$\sum_{j=0}^{n} L_j(x) = 1,$$

for all x.

Proof. Define the degree n polynomial $p(x) = 1 - \sum_{j=0}^{n} L_j(x)$. Since $L_j(x_i) = \delta_{ij}$ we have that

$$p(x_j) = 1 - L_j(x_j) = 1 - 1 = 0,$$

for $0 \leq j \leq n$. Hence, $p(x)$ has $n+1$ distinct roots. However, since $\deg p \leq n$ it follows that $p(x) \equiv 0$. Thus, we conclude that $\sum_{j=0}^{n} L_j(x) = 1$. $\square$