Numerical Analysis – HW 1

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January 20, 2017

Exercise 1. Let $g(x) = 2^{-x-1}$.

(a) Show that $g([0,1]) \subset [0,1]$.

(b) Show that g is a contraction on $[0,1]$.

(c) Compute the smallest possible contraction constant k for g on $[0,1]$ and estimate how many iterates n it will take for $g^n(0)$ to be within 10^{-4} of the fixed point.

Proof of (a). Since $g'(x) = -\ln(2)2^{-x-1} < 0$ for all x, it follows that g is decreasing on $[0,1]$. This fact combined with $g(0) = \frac{1}{2}$ and $g(1) = \frac{1}{4}$ yields that $g([0,1]) \subseteq [\frac{1}{4}, \frac{1}{2}] \subset [0,1]$. □

Proof of (b). Since $g''(x) = (\ln(2))^2 \, 2^{-x-1} > 0$ for all x, it follows that g is increasing on $[0,1]$. Combining this information with $g'(0) = -\frac{\ln 2}{2}$ and $g(1) = -\frac{\ln 2}{4}$ yields that $g([0,1]) \subseteq [-\frac{\ln 2}{2}, -\frac{\ln 2}{4}] \subset (-1,0)$. So $|g'(x)| < 1$ for all $x \in [0,1]$ implying that g is a contraction on $[0,1]$ by the Contraction Mapping Theorem. □

Proof of (c). From part (b), the smallest possible contraction constant for g on $[0,1]$ is $k = -\frac{\ln 2}{2}$. By the Contraction Mapping Theorem we have that there is a unique fixed point $p \in [0,1]$ such that

$$|g^n(x) - p| \leq \frac{k^n}{1-k}|g(x) - x|,$$

for all $x \in [0,1]$. Taking $x = 0$ we find that

$$|g^n(0) - p| \leq \frac{k^n}{1-k}|g(0)| = \frac{\left(\frac{\ln 2}{2}\right)^n}{2\left(1 - \frac{\ln 2}{2}\right)}.$$

Based on this estimate, $|g^8(0) - p| \leq 1.6 \times 10^{-4}$ and $|g^9(0) - p| \leq 5.52 \times 10^{-5}$ so $|g^n(0) - p| \leq 10^{-4}$ provided for $n \geq 9$. □

Exercise 2. Let $f(x) = \ln(1-x)$.

(a) Estimate the error in using the second order Taylor polynomial T_2 expanded about zero for f to compute $\ln(.9)$.

(b) Compute $T_2(.1)$.

(c) Compute the actual error in using $T_2(.1)$ to compute $\ln(.9)$.

(d) Compare the actual error with your error estimate.

Proof of (a). By Taylor's theorem, there exists ξ between 0 and .1 such that $f(.1) - T_2(.1) = \frac{1}{6}(.1)^3 f'''(\xi)$. Since $f'''(x) = -\frac{2}{(1-x)^3}$, we have that $|f'''(x)| \in [2, \frac{2}{.729}]$ for $x \in [0,.1]$. Hence,

$$|f(.1) - T_2(.1)| = \frac{1}{6}(.1)^3 |f'''(\xi)| \leq \frac{1}{3}\frac{(.1)^3}{(.9)^3} = \frac{1}{3^7} \approx .00045724.$$

□

Computation in (b). Since $f'(x) = -1/(1-x)$ and $f''(x) = -1/(1-x)^2$ we have that the second order Taylor polynomial expanded about zero is

$$T_2(x) = \ln 1 - x - \frac{1}{2}x^2 = -x - \frac{1}{2}x^2.$$

Hence, $T_2(.1) = -(.1) - \frac{1}{2}(.1)^2 = -.1 - .005 = -.105$. $\square$

Computation in (c). *Actual Error* $= |\ln(.9) - T_2(.1)| \approx |-.10536051565 + .105| = .00036051565$. $\square$

Comparison in (d). $|$*Error Estimate* $-$ *Actual Error*$| = |.00036051565 - .00045724| = .00009672435$. $\square$

Exercise 3. Assume that $g \in C[a,b]$, $g(a) > a$ and $g(b) < b$. Show that g has a fixed point in $[a,b]$.

Proof. Let $f(x) = g(x) - x$. By our hypothesis we have that

$$g(a) > a \Leftrightarrow g(a) - a > 0 \Leftrightarrow f(a) > 0,$$

and

$$g(b) < b \Leftrightarrow g(b) - b < 0 \Leftrightarrow f(b) < 0.$$

Since $f \in C[a,b]$, by the Intermediate Value Theorem there exists a point $p \in [a,b]$ such that $f(p) = 0$, or equivalently $g(p) = p$. $\square$

Exercise 4. Write a computer program that implements the midpoint method and use it to compute *all* the roots of $f(x) = x^3 - 6.1x^2 + 10.8x - 5.8$ to within an accuracy of 10^{-5}.

Solution. Code and output on final page of homework. The assignment was coded using C and compiled using the command: "gcc -o midpoint_test midpoint.c". $\square$

Exercise 5. Assume $g \in C^2[a,b]$ with $g([a,b]) \subset [a,b]$ and a fixed point $p \in (a,b)$. Assume that $g'(p) = 0$. Show using the Taylor theorem with remainder expanded about p that for any $x \in [a,b]$ with $x \neq p$

$$\frac{|g(x) - p|}{|x - p|^2} \leq M,$$

where $M = \max\{|g''(z)| : z \in [a,b]\}/2$.

Proof. Fix $x \in [a,b] \setminus \{p\}$. By Taylor's Theorem, there exists ξ between x and p such that

$$g(x) = g(p) + g'(p)(x-p) + \frac{1}{2}g''(\xi)(x-p)^2.$$

Since, by our hypotheses, $g(p) = p$, $g'(p) = 0$, and $x \neq p$ we now have that

$$\frac{g(x) - p}{(x-p)^2} = \frac{1}{2}g''(\xi).$$

Taking the absolute value of both sides yields

$$\frac{|g(x) - p|}{|x-p|^2} = \frac{1}{2}|g''(\xi)| \leq M.$$

Since x was taken to be arbitrary, the inequality holds for all $x \in [a,b] \setminus \{p\}$. $\square$