Numerical Analysis – Homework 6

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Exercise 1. Consider the perturbed Euler method

$$
\begin{aligned}
u_0 &= \alpha + \delta_0 \\
u_{i+1} &= u_i + hf(u_i, t_i) + \delta_i
\end{aligned}
$$

with the δ_i representing round-off error. Show that for all i,

$$|Y(t_i) - u_i| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(b-a)} - 1\right) + |\delta_0| e^{L(b-a)},$$

where $\delta = \max|\delta_i|$ and $M = \max|Y''(t)|$, which we assume to be finite.

Proof. Note that I will often make use of the convention that $Y(t_i) = Y_i$. By Taylor's Theorem we have that there exists a $\xi_i \in [t_i, t_{n+1}]$ such that

$$Y_{i+1} = Y_i + Y'(t_i)h + \frac{h^2}{2}Y''(\xi_i) = Y_i + hf(Y_i, t_i) + \frac{h^2}{2}Y''(\xi_i).$$

Defining $e_i = Y_i - u_i$, we now have that

$$
\begin{aligned}
|e_{i+1}| &= \left| \left[Y_i + hf(Y_i, t_i) + \frac{h^2}{2}Y''(\xi_i) \right] - \left[u_i + hf(u_i, t_i) + \delta_i \right] \right| \\
&\leq |Y_i - u_i| + h\left|f(Y_i, t_i) - f(u_i, t_i)\right| + \frac{h^2}{2}\left|Y''(\xi_i)\right| + |\delta_i| \\
&\leq |Y_i - u_i| + hL\left|Y_i - u_i\right| + \frac{h^2M}{2} + \delta && \text{(Since } f \text{ is Lipschitz in y)} \\
&= (1 + hL)\left|e_i\right| + \frac{h^2M}{2} + \delta. && (1)
\end{aligned}
$$

Now recall the following lemma:

> *Lemma*: Suppose $s, t > 0$, $\{a_i\}_{i=0}^{k}$ with $a_0 \geq -\frac{t}{s}$ and $a_{i+1} \leq (1+s)a_i + t$. Then we have that $a_{i+1} \leq \left(e^{(1+i)s} - 1\right)\frac{t}{s} + a_0 e^{(1+i)s}$.

Accordingly, noting (1), we let $s = hL$, $t = \frac{h^2M}{2} + \delta$, and $a_0 = |\delta_0|$. Hence, we observe that (1) satisfies the hypotheses of the lemma and since

$$
\begin{aligned}
(1+i)s &= (1+i)hL = (t_{i+1} - a)L \\
\frac{t}{s} &= \frac{\frac{h^2M}{2} + \delta}{hL} = \frac{hM}{2L} + \frac{\delta}{hL} = \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)
\end{aligned}
$$

we conclude that

$$|e_{i+1}| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(t_{i+1}-a)} - 1\right) + |\delta_0| e^{L(t_{i+1}-a)}. \tag{2}$$

Since the function e^x is monotonically increasing, replacing t_{i+1} by b in (2) yields the desired inequality which holds for all i. $\square$

Exercise 2. With $\phi(w,t) = af(w+bh, t+ch)$, find the values of the parameters a, b, c so that the resulting one-step method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(w_i, t_i) \end{aligned}$$

has local truncation error $O(h^2)$.

Proof. Note that in this problem I will be using the notations $f(t_i) = f(Y(t_i), t_i)$, and $f_Y(t_i) = \nabla_Y f(Y(t), t)|_{t=t_i}$, and $f_t(t_i) = \nabla_t f(Y(t), t)|_{t=t_i}$ to prevent the derivation from becoming too cluttered. Recall that the truncation error is given by

$$\tau_i(h) = \frac{Y_{i+1} - (Y_i + h\phi(Y_i, t_i))}{h}. \tag{3}$$

By Taylor's Theorem we have that

$$\begin{aligned} Y_{i+1} - Y_i &= hY'(t_i) + \frac{h^2}{2}Y''(t_i) + O(h^3) \\ &= hf(t_i) + \frac{h^2}{2}\left(f_Y(t_i)f(t_i) + f_t(t_i)\right) + O(h^3) \end{aligned} \tag{4}$$

and

$$\begin{aligned} \frac{1}{a}\phi(Y_i, t_i) &= f(Y_i + bh, t_i + ch) \\ &= f(Y_i, t_i) + f_Y(t_i)bh + f_t(t_i)ch + O(h^2), \end{aligned}$$

which is equivalent to

$$\phi(Y_i, t_i) = af(t_i) + f_Y(t_i)abh + f_t(t_i)ach + O(h^2), \tag{5}$$

since a is a constant. Substituting (4) and (5) into (3) yields

$$\begin{aligned} \tau_i(h) &= \frac{hf(t_i) + \frac{h^2}{2}\left(f_Y(t_i)f(t_i) + f_t(t_i)\right) + O(h^3) - h\left(af(t_i) + f_Y(t_i)abh + f_t(t_i)ach + O(h^2)\right)}{h} \\ &= f(t_i) + \frac{h}{2}\left(f_Y(t_i)f(t_i) + f_t(t_i)\right) - \left(af(Y_i, t_i) + f_Y(t_i)abh + f_t(t_i)ach\right) + O(h^2) \\ &= (1-a)f(t_i) + h\left[f_Y(t_i)\left(\frac{1}{2}f(t_i) - ab\right) + f_t(t_i)\left(\frac{1}{2} - ac\right)\right] + O(h^2). \end{aligned} \tag{6}$$

Hence, by taking $a = 1$, $b = \frac{1}{2}f(t_i)$, and $c = \frac{1}{2}$, the equation in (6) simplifies to

$$\tau_i(h) = O(h^2),$$

yielding the desired result. $\square$

Exercise 3. Consider

$$y' = \frac{2}{t}y + t^2e^t; \quad t \in [1,2]; \quad y(1) = 0, \tag{7}$$

which has solution $Y(t) = t^2(e^t - e)$.

(a) Using the (unperturbed) Euler method error estimate, find the value of h required to ensure that the Euler method solution w_i of (7) satisfies $|Y(t_i) - w_i| \leq 0.1$, for all i.

(b) Write a program to implement Euler's method and run it with the h value computed in (a) (or more precisely, a value h near it with $1/h = n \in \mathbb{N}$) and confirm that $|Y(t_i) - w_i| \leq 0.1$, for all i.

(c) Derive the Taylor method of order 3 for (7).

(d) Write a program to implement this Taylor method and, by experimentation, find a value of h which ensures that $|Y(t_i) - v_i| \leq 0.1$, for all i, where v_i is the solution computed via Taylor's method.

Proof for (a). In order to determine the required value of h we need to determine $M = \max_{t\in[1,2]} |Y''(t)|$ and the Lipschitz constant L. Accordingly, computing the derivatives we find that

$$\begin{aligned}
Y' &= f(Y,t) = \frac{2}{t}Y + t^2e^t \\
Y'' &= \frac{2}{t}Y' - \frac{2}{t^2}y + e^t(t^2+2t) = \frac{2}{t^2}Y + t^2e^t + 4te^t
\end{aligned}$$

Subsituting the solution $Y(t) = t^2(e^t - e)$ into the formula for Y'' yields

$$Y'' = \frac{2}{t^2}(e^t - e) + t^2e^t + 4te^t.$$

Hence $Y''(t)$ is increasing for $t \in [1,2]$ so we have that

$$M = \max_{t\in[1,2]} |Y''(t)| = |Y''(2)| = 2(e^2 - e) + 4e^2 + 8e^2 = 2e(7e - 1).$$

Next, observing that

$$\begin{aligned}
|f(y,t) - f(w,t)| &= \left| \left(\frac{2}{t}y + t^2e^t\right) - \left(\frac{2}{t}w + t^2e^t\right) \right| \\
&= \frac{2}{|t|}|y - w|,
\end{aligned}$$

we conclude that $|f(y,t) - f(w,t)| \leq 2|y-w|$ for all y, w and $t \in [1,2]$. Hence, $L = 2$ is the Lipschitz constant. Using the formula for the error bound on Euler's method we have

$$|Y(t_i) - w_i| \leq \left(e^{(b-a)L} - 1\right)\frac{hM}{2L} = (e^2 - 1)\frac{2e(7e-1)h}{4} \leq 0.1.$$

Solving for h we obtain the inequality

$$h \leq \frac{4}{2e(7e-1)(e^2-1)} \approx 0.0006387808944 \quad \Longrightarrow \quad 1566 \leq \frac{1}{h}.$$

$\square$

Proof for (c). Computing derivatives we find that

$$\begin{aligned}
Y' &= f(Y,t) = \frac{2}{t}Y + t^2e^t \\
Y'' &= \frac{2}{t^2}Y + t^2e^t + 4te^t \\
Y''' &= e^t(t^2 + 6t + 6).
\end{aligned}$$

The update for the 3rd order Taylor Method is given by

$$
\begin{aligned}
w_{i+1} &= w_i + hf(w_i, t_i) + \frac{h^2}{2}\nabla_t f(w_i, t_i) + \frac{h^3}{6}\nabla_t^2 f(w_i, t_i) \\
&= w_i + h\left(\frac{2}{t_i}w_i + t_i^2 e^{t_i}\right) + h^2\left(\frac{1}{t_i^2}w_i + \frac{1}{2}t_i^2 e^{t_i} + 2t_i e_i^t\right) + h^3 e_i^t\left(\frac{t_i^2}{6} + t_i + 1\right).
\end{aligned}
$$

□

Numerical Results. The code for this assignment was written using python and is included on the final page of the assignment. Based on the work from part (a), the value of h was set to $\frac{1}{1566}$ and yielded the error result

$$||\texttt{Y} - \texttt{w}||_{\texttt{inf}} = 0.0234241022462,$$

as desired. For the third order Taylor's method, experimentation with various values of h yielded that the largest value of h for which $\|Y - w\|_\infty \leq 0.1$ was $h = 0.2$. In this instance,

$$||\texttt{Y} - \texttt{w}||_{\texttt{inf}} = 0.0579913386802.$$

□