Numerical Analysis – Homework 7

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Exercise 1. The modified Euler's method is

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + \frac{h}{2}\left[f(w_i,t_i) + f(w_i + hf(w_i,t_i), t_{i+1})\right]. \end{aligned}$$

Apply this method to the IVP,

$$\begin{aligned} y' &= \lambda y, \quad \lambda < 0 \\ y(0) &= 1 \end{aligned}$$

and find the conditions on λ and h which ensure $w_i \to 0$ as $i \to \infty$.

Proof. From the update provided by modified Euler's method, we have that

$$\begin{aligned} w_n &= w_{n-1} + \frac{h}{2}\left[\lambda w_{n-1} + \lambda w_{n-1} + \lambda^2 h w_{n-1}\right] \\ &= w_{n-1}\left[1 + h\lambda + \frac{h^2\lambda^2}{2}\right]. \end{aligned}$$

Applying this formula recursively yields

$$w_n = \alpha\left[1 + h\lambda + \frac{h^2\lambda^2}{2}\right]^n.$$

Hence, it follows that $w_n \to 0$ provided that $\left|1 + h\lambda + \frac{h^2\lambda^2}{2}\right| < 1$. Since $\lambda < 0$ and the polynomial $1 - x + \frac{x^2}{2}$ has no real roots, it suffices to determine conditions for h such that $1 + h\lambda + \frac{h^2\lambda^2}{2} < 1$. Accordingly,

$$\begin{aligned} & 1 + h\lambda + \frac{h^2\lambda^2}{2} < 1 & \\ \iff \quad & \frac{h^2\lambda^2}{2} < -h\lambda & \\ \implies \quad & \frac{h\lambda}{2} > -1 & \text{(Since } h > 0 \text{ and } \lambda < 0\text{)} \\ \implies \quad & h < -\frac{2}{\lambda}. & \text{(Since } \lambda < 0\text{)} \end{aligned}$$

Since $h = 0$ results in $w_n = \alpha$ we also require that $h > 0$. Thus, we conclude that $w_n \to 0$ as $n \to \infty$ for all h satisfying $0 < h < -\frac{2}{\lambda}$. □

Exercise 2. Show that the second column, $R_{k,2}$, of Romberg integration is a composite Simpson's rule.

Proof. Define $h_k = \frac{b-a}{2^{k-1}}$ and recall that Romberg integration is defined by

$$R_{k,j} = \frac{4R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1},$$

where $R_{k,1} = \frac{h_k}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{2^{k-1}-1} f(a + ih_k)\right]$. Since

$$\begin{aligned}
4R_{k,1} &= 4 \cdot \frac{h_k}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{2^{k-1}-1} f(a + ih_k)\right] \\
&= h_k\left[2f(a) + 2f(b) + 4\sum_{i=1}^{2^{k-1}-1} f(a + ih_k)\right] \\
&= h_k\left[2f(a) + 2f(b) + 4\sum_{i=1}^{2^{k-2}-1} f(a + (2i)h_k) + 4\sum_{i=1}^{2^{k-2}} f(a + (2i+1)h_k)\right]
\end{aligned}$$

and

$$\begin{aligned}
R_{k-1,1} &= \frac{h_{k-1}}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{2^{k-2}-1} f(a + ih_{k-1})\right] \\
&= h_k\left[f(a) + f(b) + 2\sum_{i=1}^{2^{k-2}-1} f(a + (2i)h_k)\right]
\end{aligned}$$

it follows that

$$\begin{aligned}
4R_{k,1} - R_{k-1,1} = \quad & h_k\left[2f(a) + 2f(b) + 4\sum_{i=1}^{2^{k-2}-1} f(a + (2i)h_k) + 4\sum_{i=1}^{2^{k-2}} f(a + (2i+1)h_k)\right] \\
& -h_k\left[f(a) + f(b) + 2\sum_{i=1}^{2^{k-2}-1} f(a + (2i)h_k)\right] \\
= \quad & h_k\left[f(a) + f(b) + 2\sum_{i=1}^{2^{k-2}-1} f(a + (2i)h_k) + 4\sum_{i=1}^{2^{k-2}} f(a + (2i+1)h_k)\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
R_{k,2} &= \frac{4R_{k,1} - R_{k-1,1}}{3} \\
&= \frac{h_k}{3}\left[f(a) + f(b) + 2\sum_{i=1}^{2^{k-2}-1} f(a + (2i)h_k) + 4\sum_{i=1}^{2^{k-2}} f(a + (2i+1)h_k)\right],
\end{aligned}$$

for $k \geq 2$, which is Composite Simpson's Rule with $k-2$ composite intervals. $\square$

Exercise 3. Assume $N(h)$ is the computed approximation for M for each $h > 0$ and

$$M = N(h) + c_1 h + c_2 h^2 + c_3 h^3 + \cdots . \tag{1}$$

Use the values $N(h)$, $N(h/3)$, and $N(h/9)$ to produce a $O(h^3)$ approximation to M.

Proof. Observing that

$$M = N(h/3) + c_1 \frac{h}{3} + c_2 \frac{h^2}{3^2} + c_3 \frac{h^3}{3^3} + \cdots \tag{2}$$

we have that 3 (1) - (2) yields

$$2M = 3N(h/3) - N(h) - \frac{2}{3} c_2 h^2 - \frac{8}{9} c_3 h^3 + \cdots \iff M = \frac{3N(h/3) - N(h)}{2} - \frac{1}{3} c_2 h^2 - \frac{4}{9} c_3 h^3 + \cdots .$$

Letting $N_2(h) = \frac{3N(h/3) - N(h)}{2}$ we have that

$$M = N_2(h) - \frac{1}{3} c_2 h^2 - \frac{4}{9} c_3 h^3 + \cdots . \tag{3}$$

Next, observing that

$$M = N(h/9) + c_1 \frac{h}{9} + c_2 \frac{h^2}{9^2} + c_3 \frac{h^3}{9^3} + \cdots \tag{4}$$

we have that 3 (4) - (2) yields

$$2M = 3N(h/9) - N(h/3) - \frac{2}{27} c_2 h^2 - \frac{8}{243} c_3 h^3 + \cdots$$

or, equivalently,

$$M = \frac{3N(h/9) - N(h/3)}{2} - \frac{1}{27} c_2 h^2 - \frac{4}{243} c_3 h^3 + \cdots .$$

Since $N_2(h/3) = \frac{3N(h/9) - N(h/3)}{2}$ we have that

$$M = N_2(h/3) - \frac{1}{27} c_2 h^2 - \frac{4}{243} c_3 h^3 + \cdots . \tag{5}$$

Now taking 9 (5) - (3) yields

$$8M = 9N_2(h/3) - N_2(h) + \frac{11}{27} c_3 h^3 + \cdots .$$

Letting $N_3(h) = \frac{9N_2(h/3) - N_2(h)}{8}$ we conclude that

$$M = N_3(h) + O(h^3).$$

$\square$

Exercise 4. Taylor's formula yields the following:

$$f'(x_0) = \frac{1}{h}\left(f(x_0+h) - f(x_0)\right) - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

Use this with extrapolation to derive a $O(h^3)$ formula for $f'(x_0)$.

Proof. First, define $N(h) = \frac{1}{h}\left(f(x_0+h) - f(x_0)\right)$, $c_1 = \frac{f''(x_0)}{2}$, and $c_2 = \frac{f'''(x_0)}{6}$. Then

$$M = N(h/2) - c_1 h - c_2 h^2 + O(h^3) \tag{6}$$

and

$$M = N(h/2) - c_1\frac{h}{2} - c_2\frac{h^2}{4} + O(h^3). \tag{7}$$

Hence, letting $N_2(h) = 2N(h/2) - N(h)$, taking 2 (7) - (6) yields

$$M = N_2(h) + c_2\frac{h^2}{2}O(h^3). \tag{8}$$

Next, observing that

$$M = N(h/4) - c_1\frac{h}{4} - c_2\frac{h^2}{16} + O(h^3). \tag{9}$$

we have that 2 (9) - (7) yields

$$M = N_2(h/2) + c_2\frac{h^2}{8}O(h^3) \tag{10}$$

Now taking 4 (10) - (8) yields

$$3M = 4N_2(h/2) - N_2(h)O(h^3).$$

Letting $N_3(h) = \frac{4N_2(h/2) - N_2(h)}{3}$ we have that

$$M = N_3(h) + O(h^3).$$

Since

$$N_2(h) = \frac{4}{h}\left[f\left(x_0 + \frac{h}{2}\right) - f(x_0)\right] - \frac{1}{h}\left[f(x_0+h) - f(x_0)\right] = -\frac{1}{h}\left[f(x_0+h) - 4f\left(x_0+\frac{h}{2}\right) + 3f(x_0)\right]$$
$$N_2(h/2) = -\frac{2}{h}\left[f\left(x_0+\frac{h}{2}\right) - 4f\left(x_0+\frac{h}{4}\right) + 3f(x_0)\right]$$

we conclude that

$$N_3(h) = \frac{1}{3h}\left[f(x_0+h) - 12f\left(x_0+\frac{h}{2}\right) + 32f\left(x_0+\frac{h}{4}\right) - 21f(x_0)\right]$$

is a $O(h^3)$ formula for $f'(x_0)$. □

Exercise 5. Let $f(x) = \ln(x)$.

(a) Find the linear Taylor polynomial $T_1(x)$ of $f(x)$ expanded about $x_0 = \frac{3}{2}$ and find the maximum error $|T_1(x) - f(x)|$ on $[1, 2]$.

(b) Find the linear minimax approximation $p_*^{(1)}(x)$ to $f(x)$ on $[1, 2]$ and find the maximum error $|p_*^{(1)}(x) - f(x)|$ on $[1, 2]$.

Solution for (a). $T_1(x) = \ln(3/2) + \frac{2}{3}x - 1$. Define $g(x) = T_1(x) - \ln(x) = \ln(3/2) + \frac{2}{3}x - 1 - \ln(x)$. Then

$$g'(x) = \frac{2}{3} - \frac{1}{x} \iff g'(x) = 0 \text{ at } x = \frac{3}{2}.$$

Then

$$\begin{aligned} g(1) &= \frac{2}{3} + \ln(3/2) - 1 - 0 \approx 0.07213177477 \\ g(3/2) &= 1 + \ln(3/2) - 1 - \ln(3/2) = 0 \\ g(2) &= \frac{4}{3} + \ln(3/2) - 1 - \ln(2) \approx 0.04565126088 \end{aligned}$$

Thus, $\max_{x\in[1,2]} |T_1(x) - \ln(x)| \approx 0.07213177477$. □

Solution for (b). Let $p_*^{(1)}(x) = a_0 + a_1 x$ and define $E(x) := p_*^{(1)}(x) - \ln(x)$. By Chebychev's Equioscillation Theorem, $p_*^{(1)}(x)$ is characterized by at least 3 points, say x_0, x_1, and x_2. Since $\ln(x)$ is a concave function, we have that

$$E(x_0) = M, \quad E(x_1) = -M, \quad \text{and} \quad E(x_2) = M,$$

where $M = \min_{p\in\mathcal{P}_n} \|f - p\|_\infty$. Letting $x_0 = 1$, $x_2 = 2$, and x_1 a point between x_0 and x_2 where $E(x_1) < 0$ we have the following system of equations:

$$M = E(1) = a_0 + a_1 \tag{11}$$

$$-M = E(x_1) = a_0 + a_1 x_1 - \ln(x_1) \tag{12}$$

$$M = E(2) = a_0 + 2a_1 - \ln(2) \tag{13}$$

$$0 = E'(x_1) = a_1 - \frac{1}{x_1}, \tag{14}$$

where (14) follows since $E(x)$ attains a minimum value at $x = x_1$. From (14) we have that

$$x_1 = \frac{1}{a_1},$$

which, when substituted in (12) yields

$$\begin{aligned} -M &= a_0 + 1 - \ln\left(\frac{1}{a_1}\right) = a_0 + 1 - \ln(1) + \ln(a_1) = a_0 + 1 + \ln(a_1) \\ \iff \quad a_0 &= -M - 1 - \ln(a_1). \end{aligned}$$

Substituting this value for a_0 into (11) and (13) gives

$$\begin{aligned} M &= -M - 1 - \ln(a_1) + a_1 \\ \iff \quad 2M &= a_1 - 1 - \ln(a_1), \end{aligned} \tag{15}$$

$$\begin{aligned} M &= -M - 1 - \ln(a_1) + 2a_1 - \ln 2 \\ \iff \quad 2M &= 2a_1 - 1 - \ln(a_1) - \ln 2, \end{aligned} \tag{16}$$

respectively. Now (16) - (15) yields

$$a_1 = \ln 2.$$

Substituting this value for a_1 into (15) yields that the maximum error is

$$\begin{aligned} M &= \frac{1}{2}\left(-1 - \ln(a_1) - \ln(2)\right) = \frac{1}{2}\left(-\ln(e) - \ln(a_1) - \ln 2\right) = \frac{1}{2}\ln\left(\frac{2}{ea_1}\right) \\ \iff \qquad M &= \frac{1}{2}\ln\left(\frac{2}{e\ln 2}\right) \approx 0.02983005057. \end{aligned}$$

Finally, substituting the values for a_1 and M into (11) yields

$$a_0 = M - a_1 \qquad \Longrightarrow \qquad a_0 = \frac{1}{2}\ln\left(\frac{2}{e\ln 2}\right) - \ln 2.$$

Hence, we conclude that

$$p_*^{(1)}(x) = x\ln 2 + \frac{1}{2}\ln\left(\frac{2}{e\ln 2}\right) - \ln 2.$$

$\square$