Numerical Analysis – Homework 2

James Diffenderfer

February 2, 2017

Exercise 1. For $a > 0$ let

$$g(x) = \frac{x^3 + 3ax}{3x^2 + a}.$$

(a) Show that the fixed points of g are $0, \pm\sqrt{a}$.

(b) Show that g is locally convergent at $\sqrt{a}$.

(c) Show that g has a cubic rate of convergence at $\sqrt{a}$.

(d) Compute the asymptotic constant

$$M = \lim_{n \to \infty} \frac{|\sqrt{a} - x_{n+1}|}{|\sqrt{a} - x_n|^3}$$

where $x_n = g^n(x_0)$ and x_0 is chosen close enough to $\sqrt{a}$ so that $x_n \to \sqrt{a}$. Hint: Use the formulas for $g(x)$ and M directly and *not* the third derivative.

Proof of (a). Evaluating $g(x)$ at $x = 0, \pm\sqrt{a}$ we find that

$$g(0) = \frac{0}{a} = 0,$$

$$g\left(\sqrt{a}\right) = \frac{a^{3/2} + 3a^{3/2}}{3a + a} = \frac{4a^{3/2}}{4a} = \sqrt{a},$$

and

$$g\left(-\sqrt{a}\right) = \frac{-a^{3/2} - 3a^{3/2}}{3a + a} = \frac{-4a^{3/2}}{4a} = -\sqrt{a}.$$

Hence, $x = 0, \pm\sqrt{a}$ are fixed points of $g(x)$. □

Proof of (b). Observe that

$$g'(x) = \frac{(3x^2 + 3a)(3x^2 + a) - 6x(x^3 + 3ax)}{(3x^2 + a)^2} = \frac{3x^4 - 6ax^2 + 3a^2}{(3x^2 + a)^2} = \frac{3(x^2 - a)^2}{(3x^2 + a)^2}.$$

Note that, since $a > 0$, $3x^2 + a \neq 0$ for all x. Hence $g'(x) \in \mathcal{C}(\mathbb{R})$. Since $g'(\sqrt{a}) = 0$ and $g'(x) \in \mathcal{C}(\mathbb{R})$ there exists a $\delta > 0$ such that $|g'(x)| < 1$ for all $x \in \mathcal{B}(\sqrt{a}, \delta)$. Thus, by the contraction mapping theorem, $g(x)$ is locally convergent at $\sqrt{a}$. □

Proof of (c). Observe that

$$g''(x) = \frac{48ax(x^2 - a)}{(3x^2 + a)^3}.$$

Hence, $g'(\sqrt{a}) = g''(\sqrt{a}) = 0$. Since $g \in \mathcal{C}^3(\mathbb{R})$, Taylor's Theorem expanded about $\sqrt{a}$ implies that there exists a ξ between x_n and $\sqrt{a}$ such that

$$g(x_n) = g(\sqrt{a}) + \frac{g'''(\xi)}{6}(x_n - \sqrt{a})^3. \tag{1}$$

Since $x_{n+1} = g(x_n)$ and $g(\sqrt{a}) = \sqrt{a}$, (1) implies that

$$\frac{|x_{n+1} - \sqrt{a}|}{|x_n - \sqrt{a}|^3} = \frac{|g'''(\xi)|}{6}.$$

Finally, since $g'''(\sqrt{a}) = \frac{3}{2a}$ we have that

$$\lim_{n\to\infty} \frac{|x_{n+1} - \sqrt{a}|}{|x_n - \sqrt{a}|^3} = \frac{g'''(\sqrt{a})}{3!} = \frac{3}{12a} = \frac{1}{4a}.$$

□

Proof of (d). By considering the term $\sqrt{a} - x_{n+1}$ we find that

$$\sqrt{a} - x_{n+1} = \sqrt{a} - g(x_n) = \sqrt{a} - \frac{x_n^3 + 3ax_n}{3x_n^2 + a} = \frac{3x_n^2\sqrt{a} + a^{3/2} - x_n^3 - 3ax_n}{3x_n^2 + a} = \frac{(\sqrt{a} - x)^3}{3x_n^2 + a}.$$

Hence,

$$M = \lim_{n\to\infty} \frac{|\sqrt{a} - x_{n+1}|}{|\sqrt{a} - x_n|^3} = \lim_{n\to\infty} \frac{|\sqrt{a} - x_n|^3}{|\sqrt{a} - x_n|^3 \; |3x_n^2 + a|} = \lim_{n\to\infty} \frac{1}{|3x_n^2 + a|} = \frac{1}{4a}.$$

□

Exercise 2. Let $f(x) = \cos(x) + \sin^2(50x)$.

(a) Write a computer program to implement Newton's method for f.

(b) By doing numerical experiments, find out how close to the root $p = \frac{\pi}{2}$ an initial point x_0 needs to be in order to have the Newton iterates x_n converge to $p = \frac{\pi}{2}$.

(c) Explain your results.

Code, experiments, and results. The code is included on the final page of the homework assignment. I used C++ to implement Newton's method on a sampling of points from the interval $\left[\frac{\pi}{2} - .1, \frac{\pi}{2} + .1\right]$. The points sampled as initial guesses for Newton's method were $\frac{\pi}{2} - .1 + .0001n$, for integers $0 \le n \le 2000$. The program terminated if either $|x_k - \frac{\pi}{2}| < 1e-6$ or the number of steps exceeded 50. The program saved three sets of data points:

(1) Initial guesses x_0 which resulted in convergence to $\frac{\pi}{2}$ in less than 50 steps

(2) The number of steps resulting in convergence to $\frac{\pi}{2}$ for initial guesses x_0 corresponding to the data points from (1)

(3) Initial guesses x_0 which failed to converge to $\frac{\pi}{2}$ in less that 50 steps

Two plots were then created in Python using 'matplotlib'. The first was a graph of the data from (1) versus the data from (2), Figure 1, and the second graph was created using the data from (3) to illustrate the initial guesses from $\left[\frac{\pi}{2} - .1, \frac{\pi}{2} + .1\right]$ which failed to converge, Figure 2. Two intervals near $\frac{\pi}{2}$ which resulted in convergence in at most 10 steps were $[1.5431, 1.5709]$ and $[1.5941, 1.5982]$. This asymmetric nature of the basin of $\frac{\pi}{2}$ can be accounted for by observing the graph of $f(x)$ in Figure 3. The graph in Figure 3 shows that $f(x)$ has another root very close to $\frac{\pi}{2}$ which accounts for points to the left of $\frac{\pi}{2}$ failing to converge to $\frac{\pi}{2}$. By observing Figures 1 and 2 we see that most points outside of these intervals fail to converge to $\frac{\pi}{2}$ in less than 50 steps. Based on these experimental results, it seems that $B = \mathcal{B}\left(\frac{\pi}{2}, .0001\right)$ is the largest neighborhood about $\frac{\pi}{2}$ for which $x_0 \in B$ converge to $\frac{\pi}{2}$ using Newton's method.

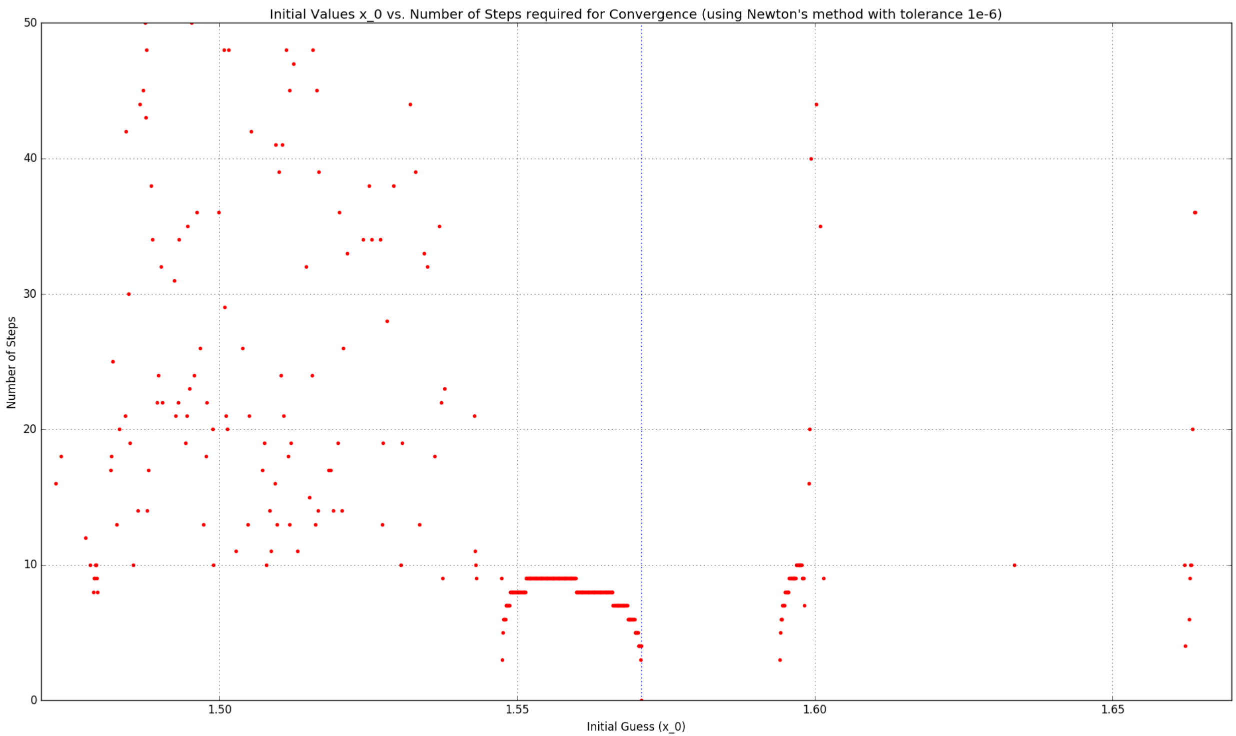

Figure 1: Plot of initial guesses x_0 from $\left[\frac{\pi}{2} - .1, \frac{\pi}{2} + .1\right]$ which converged to $\frac{\pi}{2}$ using Newton's method in less than 50 steps verses the number of steps required for convergence.

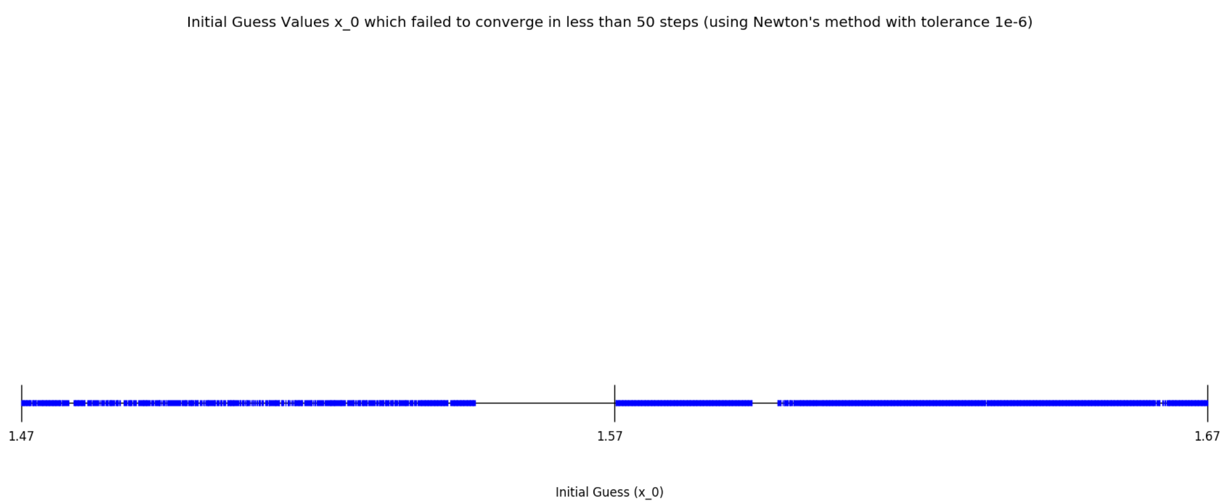

Figure 2: Plot of initial guesses x_0 from $\left[\frac{\pi}{2} - .1, \frac{\pi}{2} + .1\right]$ which failed to convergence to $\frac{\pi}{2}$ in less than 50 steps using Newton's method with tolerance $1e-6$.

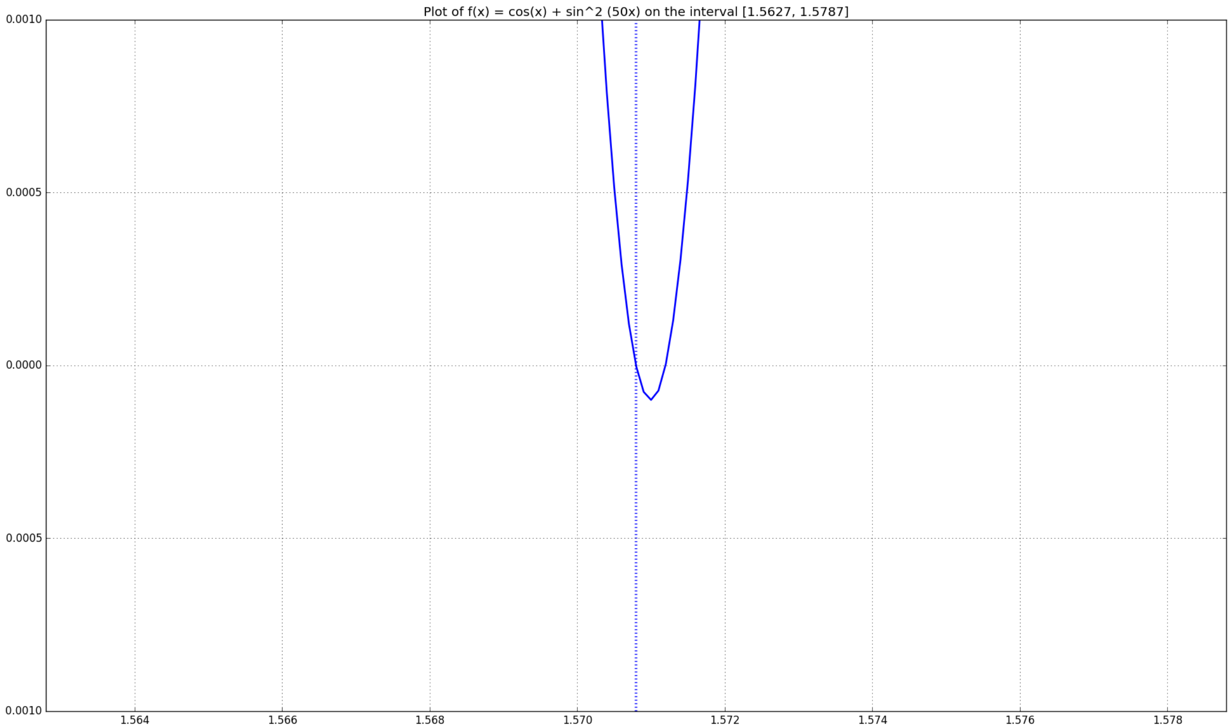

Figure 3: Plot of $f(x) = cos(x) + sin^2(50x)$ on the interval $[1.5627, 1.5787]$. The vertical dashed line intersecting the graph of f at $y = 0$ is $x = \frac{\pi}{2}$.

$\square$

Exercise 3. Let

$$G(x,y) = \left(\frac{.25}{1+(x+y)^2}, \frac{.25}{1+(x-y)^2} \right).$$

Find a convex region B with $G(B) \subset B$ and $\|DG\|_\infty \le k < 1$ for some k for all $(x,y) \in B$. Conclude then that G has a unique fixed point in B.

Solution. Define $g_1(x,y) = \dfrac{.25}{1+(x+y)^2}$ and $g_2(x,y) = \dfrac{.25}{1+(x-y)^2}$. Observe that for all $r \in \mathbb{R}$,

$$1 \le 1 + r^2 \quad \Leftrightarrow \quad \frac{1}{1+r^2} \le 1 \quad \Rightarrow \quad g_1(x,y) \le \frac{1}{4} \text{ and } g_2(x,y) \le \frac{1}{4}.$$

Hence, by taking $B = \left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{4}\right]$, B is convex and $G(B) \subset B$. Next, we have that

$$\nabla g_1 = \left[-\frac{(x+y)}{2(1+(x+y)^2)^2}, -\frac{(x+y)}{2(1+(x+y)^2)^2} \right] \quad \text{and} \quad \nabla g_2 = \left[-\frac{(x-y)}{2(1+(x-y)^2)^2}, \frac{(x-y)}{2(1+(x-y)^2)^2} \right].$$

Hence,

$$\|\nabla g_1\|_1 = \frac{|x+y|}{(1+(x+y)^2)^2} \quad \text{and} \quad \|\nabla g_2\|_1 = \frac{|x-y|}{(1+(x-y)^2)^2}.$$

Now for $(x,y) \in B$, it follows that $|x+y| \le \frac{1}{2}$, $|x-y| \le \frac{1}{4}$, $(1+(x+y)^2)^2 > 1$, and $(1+(x-y)^2)^2 > 1$. Thus, $\|\nabla g_1\|_1 \le \frac{1}{2}$ and $\|\nabla g_2\|_1 \le \frac{1}{4}$. We now have that

$$\|DG\|_\infty = \max_{1 \le n \le 2} \|\nabla g_n\|_1 \le \frac{1}{2}$$

which, by the Generalized Contraction Mapping Theorem, yields that G has a unique fixed point $p \in B$. $\square$

Exercise 4. Let $f(x) = \log(x)$, $x_0 = 1$, $x_1 = 1.75$, $x_2 = 2$. Find the *total* error bound for the degree two interpolating polynomial $P_2(x)$ with these nodes and f on the interval $[1,2]$, i.e. find a K with

$$\max_{t \in [1,2]} |f(t) - P_2(t)| \le K.$$

Solution. The total error bound for the degree two interpolating polynomial $P_2(x)$ on $[1,2]$ is given by

$$\max_{t \in [1,2]} |f(t) - P_2(t)| \le \frac{1}{3!} \left\{ \max_{x \in [1,2]} |f'''(x)| \right\} \left\{ \max_{t \in [1,2]} |w(t)| \right\},$$

where $w(t) = (t-1)(t-1.75)(t-2)$. Since $f'''(x) = 2x^{-3}$ is decreasing on $[1,2]$, we have that

$$\max_{x \in [1,2]} f'''(x) = f'''(1) = 2.$$

Next, observe that

$$w'(t) = (t-1.75)(t-2) + (t-1)(t-2) + (t-1)(t-1.75) = 3t^2 - 9.5t + 7.25.$$

Solving $w'(t) = 0$ using the quadratic formula yields the roots

$$t_1 = \frac{9.5 - \sqrt{3.25}}{6} \approx 1.282871 \quad \text{and} \quad t_2 = \frac{9.5 + \sqrt{3.25}}{6} \approx 1.8837959.$$

Since $w(1) = w(2) = 0$, $|w(t_1)| \approx .094759452$ and $|w(t_2)| \approx .013740933$ we conclude that

$$\max_{t \in [1,2]} |w(t)| = |w(t_1)| = .094759452.$$

Thus,

$$\max_{t \in [1,2]} |f(t) - P_2(t)| \le \frac{1}{6}(2)(.094759452) \approx .031586484.$$

$\square$