Hyperbolic motion

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Here we are working in Minkowski space throughout.

1 Four-acceleration

The *four-velocity* of some object is given by

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\left(1, \frac{d\mathbf{x}}{dt}\right) \equiv \gamma\left(1, \mathbf{v}\right), \tag{1}$$

where $x^\mu = (t, \mathbf{x})$ is the *four-position* and τ is the *proper time* of the particle. We have defined $\gamma = \frac{dt}{d\tau}$, and it can be shown that $\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}}$. And we have defined $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ as usual. Let us compute $\frac{d\gamma}{dt}$, since we will need it later.

$$\frac{d\gamma}{dt} = \frac{d\gamma}{d\mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = \gamma^3 \mathbf{v} \cdot \mathbf{a}, \tag{2}$$

where we have defined $\mathbf{a} = \frac{d\mathbf{v}}{dt}$. This expression can be derived straight from the definition of γ in terms of $\mathbf{v}$. Now we want to define the *four-acceleration*

$$\begin{aligned}
a^\mu &= \frac{du^\mu}{d\tau} = \gamma \frac{du^\mu}{dt} \\
&= \gamma\left[\frac{d\gamma}{dt}\left(1, \mathbf{v}\right) + \gamma\left(0, \frac{d\mathbf{v}}{dt}\right)\right] \\
&= \gamma\left[\gamma^3 \mathbf{v} \cdot \mathbf{a}\left(1, \mathbf{v}\right) + \left(0, \gamma\mathbf{a}\right)\right] \\
&= \left(\gamma^4 \mathbf{v} \cdot \mathbf{a}, \gamma^2 \mathbf{a} + \gamma^4\left(\mathbf{v} \cdot \mathbf{a}\right)\mathbf{v}\right)
\end{aligned}$$

The contraction of the four-acceleration with itself is then

$$
\begin{aligned}
a^\mu a_\mu &= \eta_{\mu\nu} a^\mu a^\nu \\
&= -\gamma^8 \left(\mathbf{v}\cdot\mathbf{a}\right)^2 + \gamma^4 \mathbf{a}^2 + 2\gamma^6 \left(\mathbf{v}\cdot\mathbf{a}\right)^2 + \gamma^8 \mathbf{v}^2 \left(\mathbf{v}\cdot\mathbf{a}\right)^2 \\
&= \gamma^4 \mathbf{a}^2 + \gamma^6 \left(\mathbf{v}\cdot\mathbf{a}\right)^2 . \qquad (2)
\end{aligned}
$$

And since contractions are coordinate invariant the value that $a^\mu a_\mu$ takes is the same for all observers. Consider therefore the expression in the instantaneous rest frame of the object, where it feels an acceleration $\mathbf{a}_0$ and has velocity $\mathbf{v} = \mathbf{0}$. Then $\gamma = 1$ and the contraction becomes $\mathbf{a}_0^2$. This acceleration $\mathbf{a}_0$ in the instantaneous rest frame is called the *proper acceleration*. In some other reference frame where the instantaneous rest frame of the object has velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ the contraction is simply the expression in (2). This means that the proper acceleration $\mathbf{a}_0$ and the coordinate acceleration $\mathbf{a}$ are related through

$$|\mathbf{a}_0| = \sqrt{\gamma^4 \mathbf{a}^2 + \gamma^6 \left(\mathbf{v}\cdot\mathbf{a}\right)^2} \qquad (3)$$

2 Motion with constant proper acceleration

Consider now a scenario in which the acceleration at all times points in the same direction as the velocity - which is to say we have motion along a straight line. Then $\mathbf{v}\cdot\mathbf{a} = va$, where $v = |\mathbf{v}|$ and $a = |\mathbf{a}|$. So $a_0 = \gamma^3 a$. Here we want to assume that the proper acceleration a_0 is constant. And the coordinate acceleration $a = \frac{dv}{dt}$ is given by

$$a = \frac{a_0}{\gamma^3} \qquad (4)$$

From this we can already see that as the velocity, and hence also γ, increases the coordinate acceleration decreases. What we are interested in finding here are the exact expressions telling us how the accelerating object moves in some external inertial reference frame. Consider now

$$\frac{dv}{d\tau} = \frac{dt}{d\tau}\frac{dv}{dt} = \gamma a = \frac{a_0}{\gamma^2}. \qquad (5)$$

Keeping in mind the expression for γ this may seem like a relatively nasty differential equation. But we can transform it into a trivially simple one if we introduce the *rapidity* χ, which is defined through $v = \tanh\chi$. We can write

$$\frac{dv}{d\tau} = \frac{d\chi}{d\tau}\frac{dv}{d\chi}$$

$$= \frac{d\chi}{d\tau}\frac{1}{\cosh^2\chi},$$

since $\frac{d}{d\chi}(\tanh\chi) = \frac{1}{\cosh^2\chi}$. And

$$\frac{a_0}{\gamma^2} = a_0\left(1-\mathbf{v}^2\right) = a_0\left(1-\tanh^2\chi\right) = \frac{a_0}{\cosh^2\chi} \tag{6}$$

So (5) reduces to $\frac{d\chi}{d\tau} = a_0$, which of course has the trivial solution $\chi = a_0\tau$. Then $\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}} = \cosh\chi$. Then

$$dt = \gamma d\tau$$

$$\implies t = \int_0^\tau \gamma\left(\tau'\right)d\tau'$$

$$= \int_0^\tau \cosh\left(a_0\tau'\right)d\tau'$$

$$= \frac{1}{a_0}\sinh\left(a_0\tau\right)$$

where we assume $t=0$ when $\tau=0$. In order to find $x\left(\tau\right)$ we can use that

$$\frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt}$$

$$= \gamma v = \cosh\chi\tanh\chi$$

$$= \sinh\chi$$

$$= \sinh\left(a_0\tau\right).$$

Integrating this with respect to τ gives

$$x = \int_0^{\tau} \sinh(a_0 \tau')\, d\tau'$$

$$= \frac{1}{a_0} \left(\cosh(a_0 \tau) - 1 \right),$$

where we have assumed that $x(\tau = 0) = 0$. And since we have $\gamma(\tau)$ we can also find $a(\tau)$.

$$a = \frac{a_0}{\gamma^3} = \frac{a_0}{\cosh^3(a_0 \tau)} \tag{7}$$