# Differential forms, Maxwell's equations and the generalized Stokes theorem

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This closely follows [1].

## 1 Differential forms

A *differential form* is a $(0, p)$ completely antisymmetric tensor. These objects are interesting because they very naturally lend themselves to being differentiated and integrated. As we will see this leads to a very elegant formulation of *Maxwell's equations* of electromagnetism, and the use of differential forms allows us to formulate the immensely beautiful *generalized Stokes theorem.* This post assumes familiarity with tensors.

An example of a 0-form is a scalar, while a dual vector $A_\mu$ is a one-form. These are both automatically antisymmetric. There is als othe antisymmetric electromagnetic field strength tensor $F_{\mu\nu}$, which is an example of a 2-form. Assume we have a $p$-form $A$ and a $a$-form $B$. we can produce a $(p+q)$-form from $A$ and $B$ using the *wedge product*, defined by

$$(A \wedge B)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\mu_{p+1} \ldots \mu_{p+q}]}, \tag{1}$$

where the square brackets define an antisymmetric tensor in the following way

$$A_{[\mu_1 \ldots \mu_n]} = \frac{1}{n!} \left( A_{\mu_1 \ldots \mu_n} - A_{\mu_n \ldots \mu_1} + \ldots \right), \tag{2}$$

where the ... here represents a sum over the permutations of the indices, but with alternating signs. The wedge product between two one-forms, for example, is given by

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu. \tag{3}$$

The *exterior derivative* $d$ lets us differentiate forms wile still returning a form - the exterior derivative of a $p$-form gives us a $(p+1)$-form. It is defined by

$$(dA)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]}. \tag{4}$$

It is well known that the partial derivative of a tensor, for example $\partial_\mu A_\nu$, is not itself a tensor in curved spacetime. But exterior derivatives, in contrast, are tensors. The non-tensorial part of the coordinate transformation of a partial derivative cancels out in the alterning positive and negative terms in the exterior derivative. This means the exterior derivative can be used in its current form even in curved spacetime. As a couple examples of the exterior derivative, the exterior derivatives of a 0-form and a 1-form are

$$(d\phi)_\mu = \partial_\mu \phi$$

$$(dA)_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

A fact about the exterior derivative which will have important consequences physically later is that, for any form $A$

$$d\,(dA) = 0. \tag{5}$$

The reason for this is very straight forward. Consider the one-form $A$ here. Then

$$[d\,(dA)]_{\mu\nu\rho} = 2 \cdot 3\partial_{[\mu}\partial_\nu A_{\rho]}$$

$$= 6\frac{1}{6}\left(\partial_\mu\partial_\nu A_\rho - \partial_\nu\partial_\mu A_\rho + \partial_\nu\partial_\rho A_\mu - \partial_\rho\partial_\nu A_\mu + \partial_\rho\partial_\mu A_\nu - \partial_\mu\partial_\rho A_\nu\right).$$

Partial derivatives commute, meaning that $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$. Then all the terms in the expression above cancel. So the reason for (5) is that partial derivatives commute. We can define yet another operator on differential forms - the *Hodge duality*. The *Hodge star operator* on an $n$-dimensional manifold is defined as a map from $p$-forms to $(n-p)$-forms in the following way

$$(\star A)_{\mu_1 \ldots \mu_{n-p}} = \frac{1}{p!}\epsilon^{\nu_1 \ldots \nu_p}{}_{\mu_1 \ldots \mu_{n-p}} A_{\nu_1 \ldots \nu_p}, \tag{6}$$

where $\epsilon_{\mu_1 \ldots \mu_n}$ is the *Levi-Civita tensor*, defined by

$$\epsilon_{\mu_1 \ldots \mu_n} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1 \ldots \mu_n}, \tag{7}$$

where $g$ is the determinant of the metric tensor on the manifold. And $\tilde{\epsilon}_{\mu_1 \ldots \mu_n}$ is the *Levi-Civita symbol*, defined to be $+1$ if $(\mu_1, ..., \mu_n)$ is an even permutation of $(1, 2, ..., n)$, $-1$ if $(\mu_1, ..., \mu_n)$ is an odd permutation of $(1, 2, ..., n)$ and 0 otherwise. The tensor

$$\epsilon^{\nu_1 \ldots \nu_p}_{\mu_1 \ldots \mu_{n-p}} \tag{8}$$

in (6) is just the Levi-Civita tensor with some of the indices raised by the metric. Consider now 3-dimensional Euclidean space, where $g = 1$. Let $U$ and $V$ be two one forms, corresponding to normal vectors. Consider the Hodge dual of the wedge product between these two

$$\star (U \wedge B)_i = \frac{1}{2} \epsilon_i^{jk} 2 U_{[j} V_{k]} = \epsilon_i^{jk} U_j V_k. \tag{9}$$

Here we have used that the Levi-Civita tensor is antisymmetric. We can then remove the antisymmetrizing square brackets from $U$ and $V$, since the Levi-Civita tensor automatically symmetrizes $U_j V_k$. Notice that $\epsilon_i^{jk} U_j V_k$ is of course the cross product between $U$ and $V$. This also shows why there is no cross product in higher (or lower) dimensional spaces - because in 3 dimensions there is this map from two dual vectors to a third dual vector.

# 2 Maxwell's equations

## 2.1 Formulating Maxwell's equations in tensorial form

The standard form in which one first encounters Maxwell's equations is the following

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = 4\pi \mathbf{J} \tag{10}$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho \tag{11}$$

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \tag{12}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{13}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field vectors. $\mathbf{J}$ is the current vector and $\rho$ is the charge density. (10) tells us that a time-varying electric field or an electric current can generate a magnetic field. (11) tells us that electric charges generate an electric field. (12) tells us that a time-varying magnetic field induces an electric field, and finally (13) tells us that there are no magnetic monopoles - no magnetic analogues of electric charges. These equations are

*Lorentz invariant*, meaning they take the same form in all coordinate systems, but that is not manifest when written in this form. Let us rewrite Maxwell's equations in tensor notation as

$$\epsilon^{ijk}\partial_j B_k - \partial_0 E^i = 4\pi J^i \tag{14}$$

$$\partial_i E^i = 4\pi J^0 \tag{15}$$

$$\epsilon^{ijk}\partial_j E_k + \partial_0 B^i = 0 \tag{16}$$

$$\partial_i B^i = 0. \tag{17}$$

Here we have collected the charge density and current into the current *four-vector* $J^\mu = (\rho, \mathbf{J})$. Now let us define the electromagnetic field tensor with lower indices as (seemingly somewhat arbitrarily)

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

We can form the field tensor with upper indices by contracting with the inverse of the Minkowski metric (for the moment we are assuming flat spacetime)

$$F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}. \tag{18}$$

Now consider $F^{0i}$, where Latin indices denote spatial indices.

$$F^{0i} = \eta^{0\rho}\eta^{i\sigma}F_{\rho\sigma} = \eta^{00}\eta^{ij}\delta^\rho_0\delta^\sigma_j F_{\rho\sigma}$$

$$= -\eta^{ij}F_{0j} = E^i$$

and

$$F^{ij} = \eta^{i\rho}\eta^{j\sigma}F_{\rho\sigma} = \eta^{ik}\eta^{jl}F_{kl}$$

$$= \eta^{ik}\eta^{jl}\epsilon^{m}_{kl}B_l$$

$$= \epsilon^{ijk}B_k$$

Then (14) and (15) become

$$\partial_j F^{ij} - \partial_0 F^{0i} = 4\pi J^i$$

$$\partial_i F^{0i} = 4\pi J^0$$

Because of the antisymmetry of $F^{\mu\nu}$ we can combine these two into

$$\partial_\mu F^{\nu\mu} = 4\pi J^\nu \tag{19}$$

It can also be shown, although somewhat tediously, that (16) and (17) can be combined to form

$$\partial_{[\mu}F_{\nu\lambda]} = 0 \tag{20}$$

(19) and (20) are tensorial equations, so if they hold in one frame they also hold in every other frame. These are the covariant Maxwell's equations.

## 2.2 Maxwell's equations in the formalism of differential forms

This form of the Maxwell's equations is important in its own right, but here we are interested in another formulation - namely one using the language of differential forms. Notice first how equation (20) is the definition of the exterior derivative of the 2-form $F_{\mu\nu}$ up to an irrelevant constant. So we can right away write

$$dF = 0 \tag{21}$$

We know that the electromagnetic field strength tensor can be written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{22}$$

where $A_\mu$ is the four-potential. (22) can of course be written as $F = dA$. Then $dF = 0$ arises very naturally, since $d(dA) = 0$ for all $A$. The discussion of the gauge invariance of our physical system is central in physics, and in electromagnetism we have the following gauge transformation

$$A_\mu \to A_\mu + ie\partial_\mu \phi, \tag{23}$$

with $A_\mu$ being the four-potential, $e$ the elementary charge and $\phi$ a scalar potential. This means that the electromagnetic field tensor is invariant under such transformations. This gauge transformation of the four-potential can be expressed as $A \to A + ied\phi$ using differential forms. Then the requirement that $F_{\mu\nu}$ is invariant under the gauge transformation again arises naturally, because $F = dA \to dA + d(d\phi)$, and the last term is always zero.

(19) can be written as

$$d\left(\star F\right) = 4\pi\left(\star J\right) \tag{24}$$

through yet another tedious derivation which we skip here.

# 3 The generalized Stokes theorem

An incredibly powerful application of differential geometry is the *generalized Stokes theorem*

$$\int_M d\omega = \int_{\partial M} \omega \tag{25}$$

where $M$ is some manifold, $\partial M$ is the boundary of the manifold, $\omega$ is a differential form and $d$ is the exterior derivative. From this deceptively simple equation we can derive a number of famous results from calculus.

Consider first a 0-form $\omega$. Then $d\omega = \partial_\mu \omega \; dx^\mu$. The generalized Stokes theorem then states that

$$\int_M \partial_\mu \omega \; dx^\mu = \int_{\partial M} \omega \tag{26}$$

Let $M$ be the $x$-axis. Then $\int_a^b \frac{d\omega}{dx} \; dx = \omega(b) - \omega(a)$. This is of course the *fundamental theorem of calculus.*

## 3.1 Stokes theorem from vector calculus

Consider now a 1-form $\omega_\mu dx^\mu$. Assume also that we work in 3-dimensional Euclidean space. We have

$$
\begin{aligned}
d\omega &= 2\partial_{[\mu}\omega_{\nu]}dx^\mu \otimes dx^\nu \\
&= \left(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu\right) dx^\mu \otimes dx^\nu \\
&= \frac{1}{2}\left(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu\right) dx^\mu \wedge dx^\nu \\
&= \left(\partial_y\omega_z - \partial_z\omega_y\right) dy \wedge dz + \left(\partial_z\omega_x - \partial_x\omega_x\right) dz \wedge dx + \left(\partial_x\omega_y - \partial_y\omega_x\right) dx \wedge dy,
\end{aligned}
$$

where we switched from direct product to wedge products by including the combinatorical factor $\frac{1}{2!}$. We are allowed to do this change because the multiplying factor is already antisymmetric. We recognize this (or we can at least verify after the fact ;)) as the first, second and third components of $\nabla \times \vec{\omega}$. Each of the terms contain a 2-form like $dx \wedge dy$, meaning that we are integrating over a surface. So from the generalized Stokes theorem we have

$$\int_S (\nabla \times \vec{\omega}) \cdot \hat{n} dS = \int_C \vec{\omega} \cdot d\vec{r}, \tag{27}$$

where $S$ is some surface and $C$ is the boundary to that surface. This is the normal *Stokes theorem* familiar from vector calculus.

## 3.2 Gauss' theorem/the divergence theorem

We will now consider a 2-form $\omega = \omega_{\mu\nu}dx^\mu \otimes dx^\nu$, whose components $\omega_{\mu\nu}$ can be represented in matrix form as the following

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \omega_x & -\omega_y \\ -\omega_x & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix}. \tag{28}$$

The antisymmetry of the 2-form means there are only 3 independent components in $\omega_{\mu\nu}$, and we have named these components in this specific way for future convenience. Then

$$
\begin{aligned}
d\omega &= 3\partial_{[\mu}\omega_{\nu\rho]} dx^\mu \otimes dx^\nu \otimes dx^\rho \\
&= \frac{1}{2}\left(\partial_\mu\omega_{\nu\rho} - \partial_\nu\omega_{\mu\rho} + \partial_\nu\omega_{\rho\mu} - \partial_\rho\omega_{\nu\mu} + \partial_\rho\omega_{\mu\nu} - \partial_\mu\omega_{\rho\nu}\right) dx^\mu \otimes dx^\nu \otimes dx^\rho \\
&= \left(\partial_\mu\omega_{\nu\rho} + \partial_\nu\omega_{\rho\mu} + \partial_\rho\omega_{\mu\nu}\right) dx^\mu \otimes dx^\nu \otimes dx^\rho \\
&= \frac{1}{6}\left(\partial_\mu\omega_{\nu\rho} + \partial_\nu\omega_{\rho\mu} + \partial_\rho\omega_{\mu\nu}\right) dx^\mu \wedge dx^\nu \wedge dx^\rho \\
&= \left(\partial_x\omega_{23} + \partial_y\omega_{31} + \partial_z\omega_{12}\right) dx \wedge dy \wedge dz,
\end{aligned}
$$

where in the third step we used the antisymmetry of $\omega_{\mu\nu}$. In the fourth step we again replaced the direct products with wedge products, remembering the combinatorical factor of $\frac{1}{6!}$. The fifth step just involves lots of tedious algebra and collecting like terms, while remembering the antisymmetry of the wedge product and $\omega_{\mu\nu}$. Now we can use (28) to write $d\omega$ as

$$
d\omega = (\partial_x\omega_x + \partial_y\omega_y + \partial_z\omega_z) dx \wedge dy \wedge dz = \nabla \cdot \vec{\omega}\, dx \wedge dy \wedge dz \tag{29}
$$

While for the 2-form itself we have

$$
\begin{aligned}
\omega &= \omega_{\mu\nu} dx^\mu \otimes dx^\nu \\
&= \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu \\
&= \omega_{23} dy \wedge dz + \omega_{31} dz \wedge dx + \omega_{12} dx \wedge dy \\
&= \omega_x dy \wedge dz + \omega_y dz \wedge dx + \omega_z dx \wedge dy \\
&= \vec{\omega} \cdot \hat{n} dS,
\end{aligned}
$$

So we have

$$
\int_V \left(\nabla \cdot \vec{\omega}\right)\, dxdydz = \int_S \left(\vec{\omega} \cdot \hat{n}\right) dS, \tag{30}
$$

which is Gauss' theorem, also called the divergence theorem. Here we have just written the volume element as $dxdydz$. This is just a simplification of the more precise $dx \wedge dy \wedge dz$.

## References

[1] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 2019.