MATH266: UNSTABLE MOTIVIC HOMOTOPY THEORY
Fall 2024

Thomas Brazelton

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Contents

About these notes

These are notes for a topics class taught at Harvard in Fall 2024. The most recent version can be found on GitHub, where **I highly encourage you to submit edits and suggestions**:

https://github.com/tbrazel/math266-motivic

What isn't in these notes? A ton of core principles in motivic homotopy theory, for instance: purity, six functors formalism, the motivic complexes $\mathbb{Z}(q)$,... Despite the concise title, these notes are not intended to be a sprawling survey of all techniques in motivic homotopy theory, but rather a geodesic towards the motivic obstruction that allows us to talk about classification of torsors over smooth affine varieties. To that end, these notes shouldn't be viewed as a supplement to Morel's book or even a replacement for it. Rather it is a primer for people interested in the Asok–Fasel–et. al. program of research.

To that end, we treat the classification of torsors as *motivation* for constructing motivic spaces. It is important to note that this is very ahistorical, and is done only for pedagogical reasons. Although affine representability was one of the key early results explored by Morel, motivic spaces and spectra were developed in order to house theories such as algebraic K-theory, Bloch's higher Chow groups, and to grow new theories such as algebraic cobordism. We chose this route for a few reasons:

1. The Fall 2024 Thursday seminar is on the Wilson space hypothesis, so we're hoping this class will contrast nicely and provide some foundational background in techniques in motivic obstruction theory.
2. In order to tailor the course to a broader audience, we'd like to unify the class around a key question which has general appeal, so we've picked the classification of torsors over affine varieties using motivic methods as such a question. This has certain advantages, for instance we can pause in the sheaf topos and discuss classifying torsors there before building motivic spaces — this vista is useful to people across many fields.

Why did you spend so long on X if you're trying to get to affine representability? Perhaps the most mathematical mathematician, guided by pure force of will, can teach towards a singular goal, each word building towards this grand horizon. I haven't had the pleasure of meeting this person. The rest of us are tempted by the many sprawling inviting tangents that open their doors to us along the way, and what you're reading is my ambling path.

What background is assumed? We're assuming a strong handle of algebraic geometry and category theory, and a fair bit of familiarity with commutative algebra and homotopy theory. We'll see very quickly that ∞-categories (and/or model categories) are needed in order to develop the setting in which we wish to work. We've elected to take the ∞-categorical approach, since it streamlines many of the constructions and key ideas, at the cost of being a high technical investment; for this reason we've done our best to make the ∞-categorical machinery easy to black box. The reader should be aware that while the categorical language will be heavy in the first half of these notes, it will quickly fade into the background as we become familiar with the ambient setting we're working in and can set our focus towards computations.

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0.0 Introduction

0.0.1 Overview

What sorts of things about a ring R are still true when we move to the polynomial ring $R[t]$? In other words, what sorts of things about R can't be varied in a 1-parameter family?

Let's give a ton of examples! Don't stress if not all of the words are familiar, we'll break down what's happening here over the course of the semester, this is just motivation.

Example 0.0.1. Let R be a reduced ring. Then the inclusion $R \to R[t]$ induces an isomorphism after taking units[1]

$$R^\times \xrightarrow{\sim} (R[t])^\times. \tag{2}$$

Recall that the functor sending a commutative ring to its group of units is corepresented by $\mathbb{Z}[u, u^{-1}]$, so Equation 2 is equivalent to saying that the following map is a bijection

$$\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[u, u^{-1}], R) \to \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[u, u^{-1}], R[t]).$$

After taking Spec everywhere, this becomes

$$\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\,(R), \mathbb{G}_m) \to \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\,(R[t]), \mathbb{G}_m).$$

We therefore might rephrase Equation 2 as saying that the representable functor $\mathrm{Hom}_{\mathrm{Sch}}(-, \mathbb{G}_m)$ is $\mathbb{A}^1$-*invariant*, at least when we plug in something reduced.

Example 0.0.3. Let k be a field. Then the functor $k \to k[t_1, \ldots, t_n]$ induces an extension of scalars map

$$\begin{aligned} \mathrm{Mod}_k &\to \mathrm{Mod}_{k[t_1,\ldots,t_n]} \\ M &\mapsto M \otimes_k k[t_1, \ldots, t_n]. \end{aligned} \tag{4}$$

Serre's Problem: Is every finitely generated $k[t_1, \ldots, t_n]$-module free?

We can rephrase this in a few ways:

1. Is the functor

Recall finitely generated projective R-modules are the same as "algebraic vector bundles" over $\mathrm{Spec}\,(R)$, so we're asking whether every algebraic vector bundle on $\mathbb{A}^n_k$ is trivial.

Answer: Yes (Quillen–Suslin, 1974). Quillen actually proved more– for R a PID, he proved that the every finitely generated projective $R[t]$-module is extended from an R-module.[2] Lindel proved shortly thereafter that every finitely generated projective $A[t]$-module is extended from an A-algebra,

[1] If R is not reduced, say there is some $r \in R$ so that $r^2 = 0$, then $(1 + rt)(1 - rt) = 1$, so $1 + rt \in R[t]^\times$.

[2] Quillen's proof involves leveraging some previous work of Horrocks, flat descent for vector bundles, and a very clever technique he invented called *patching*. Suslin's proof, which appeared in the same year, is almost completely linear algebraic, leveraging the theory of *unimodular rows*.

where A is a smooth algebra containing a field k. We could read this as saying that the stack of algebraic vector bundles is $\mathbb{A}^1$-invariant over the category of smooth affine k-schemes.

More general: (Bass–Quillen conjecture) is it true that for every R regular Noetherian, the map

$$\mathrm{Mod}_R^{\text{f.g., proj}} \to \mathrm{Mod}_{R[t]}^{\text{f.g., proj}}$$
$$M \mapsto M \otimes_R R[t]$$

is essentially surjective? *Still open.*

Fundamental Theorem of Algebraic K-Theory (Quillen): For R regular Noetherian, we have that $R \to R[t]$ induces an equivalence[3][4]

$$K(R) \xrightarrow{\sim} K(R[t]).$$

In other words K-theory is $\mathbb{A}^1$-*invariant* for regular Noetherian rings (regular Noetherian schemes, more generally).

Example 0.0.5. If $X = \mathrm{Spec}(R)$ or more generally X is a scheme, then the map $\pi \colon X \times \mathbb{A}^1 \to X$ induces an isomorphism on Chow groups (see for instance [Ful98, 3.3])

$$\pi^* \colon \mathrm{CH}_*(X) \xrightarrow{\sim} \mathrm{CH}_{*+1}(X \times \mathbb{A}^1).$$

Example 0.0.6. Let $X = \mathrm{Spec}(R)$ where R is normal and Noetherian.[5] Then every line bundle on $X \times \mathbb{A}^1$ is extended from a line bundle on X, in other words $X \times \mathbb{A}^1 \to X$ induces an isomorphism

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X \times \mathbb{A}^1).$$

Example 0.0.7. We can also show $\mathbb{A}^1$-invariance for the Picard group over a PID. Let R be a PID, then it is a UFD, and we can show that $\mathrm{Pic}(R) = 0$, and therefore $\mathrm{Pic}(R[t_1, \ldots, t_n]) = 0$.

Example 0.0.8. [Aso21, 3.7.1.5] Check this doesn't hold for all rings, for example $R = k[x,y]/(y^2 - x^3)$.[6]

Definition 0.0.9. An *inner product space* over a ring R is a finitely generated productive R-module M and a symmetric bilinear form $\beta \colon M \times M \to R$ for which $m \mapsto \beta(-, m)$ defines an isomorphism $M \cong M^*$.

Theorem 0.0.10. (Harder's Theorem, VII.3.13 in Lam's book on Serre's problem) Let k be a field. Then every inner product space over $k[t]$ is extended from an inner product space over k.

Remark 0.0.11. The stable analogue of this has to do with $\mathbb{A}^1$-invariance of Hermitian K-theory [reference needed].

Algebraic vector bundles are GL_n-torsors (we will talk about torsors in more detail next week), so the Bass–Quillen conjecture is really asking about $\mathbb{A}^1$-invariance of torsors over affine schemes. We could ask an analogous question about G-torsors for any G. Here's an example result in this direction that we'll see later in the semester:

Theorem 0.0.12. [AHW20, 1.3] If k is a field, and G is an isotropic reductive group scheme, then G-torsors in the Nisnevich site are $\mathbb{A}^1$-invariant over any smooth affine k-scheme.

[3] So Bass–Quillen is really a question about *unstable* modules.

[4] The statement for K_0 is originally due to Grothendieck [Aso21, 5.6.1.3]. The statement for K_1 is due to Bass–Heller–Swan [Aso21, 5.8.2.1].

[5] We can get away with weaker assumptions on this, for example in [Aso21, 3.7.13] it is only assumed that R is a locally factorial Noetherian normal domain.

[6] More generally if A is reduced, Noetherian, and has a finite normalization, then its Picard group will be $\mathbb{A}^1$-invariant if and only if A is seminormal [Tra70, 3.6].

0.0.2 $\mathbb{A}^1$-homotopy theory

Recall from algebraic topology that $X \times [0,1] \to X$ is a weak homotopy equivalence, which implies that any cohomology theory is insensitive to taking a product with an interval, e.g. for $H^*(-,\mathbb{Z})$ integral cohomology we get

$$H^*(X,\mathbb{Z}) \xrightarrow{\sim} H^*(X \times [0,1], \mathbb{Z}).$$

In fact this type of homotopy invariance is an axiom of generalized Eilenberg–Steenrod cohomology theories.

Example 0.0.13. Let $k \subseteq \mathbb{C}$ be a subfield of the complex numbers. Then there is a *Betti realization* functor

$$\begin{aligned} \mathrm{Var}_k &\to \mathrm{Top} \\ X &\mapsto X(\mathbb{C}) \end{aligned}$$

sending a variety to its underlying analytic space. Note that

$$X \times \mathbb{A}^1_k \mapsto (X \times \mathbb{A}^1_k)(\mathbb{C}) = X(\mathbb{C}) \times \mathbb{C}.$$

Therefore any homotopy invariant functor out of spaces provides another example of an $\mathbb{A}^1$-invariant functor out of k-varieties, for example

$$\begin{aligned} X &\mapsto H^*(X(\mathbb{C}); \mathbb{Z}) \\ X &\mapsto \pi_*(X(\mathbb{C})). \end{aligned}$$

These observations lead to the following curiosity:

Motivational question: Can we build a homotopy theory of algebraic varieties in which the affine line $\mathbb{A}^1$ plays the role that the interval $[0,1]$ plays in classical topology?

This is what's known as $\mathbb{A}^1$-*homotopy theory* or *motivic homotopy theory*. This dates back to Morel and Voevodsky's seminal work in 1999, but many ideas date back to work of Karoubi–Villamayor, Jardine, Weibel in the 1980's, work of Brown, Gersten, Illusie and Joyal in the 1970's, and of Quillen and Grothendieck in the 1960's.

Chapter 1

Torsors

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1.1 Torsors

Assumption 1.1.1. For this lecture, every time we say "cover" you can assume we are working with Zariski covers, or even open covers of topological spaces, since the intuition will be the same and the results here will be mostly identical. If you know about other topologies, the statements here work for any site. We will go into sheaves and sites more next week, when we will remark that everything here works for other nice sites (étale, Nisnevich, flat, etc.).

Definition 1.1.2. Let G be a group. Then a G-set X is called a *torsor* if its G-action is simple and transitive. Equivalently, the map

$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\mapsto (x, gx) \end{aligned} \tag{1.3}$$

is a bijection.

Note there are two types of G-torsors — sets of the form G/e, and the empty set. Depending on the convention, we might want to exclude the case of the empty set by including $X \neq \varnothing$ in the definition.

Intuition 1.1.4. A G-torsor is like a group G which has remembered its multiplication but forgotten its identity. Any choice of basepoint $x \in X$ yields a canonical bijection $G \xrightarrow{\sim} X$ by sending $g \mapsto g \cdot x$.

Example 1.1.5. In a locally small category $\mathscr{C}$, given two objects $x, y \in \mathscr{C}$, the set of isomorphisms $\mathrm{Isom}_{\mathscr{C}}(x, y)$ is a left $\mathrm{Aut}_{\mathscr{C}}(x)$-torsor and a right $\mathrm{Aut}_{\mathscr{C}}(y)$-torsor.

Let's try to extend this definition to the setting where G isn't a single group, but a *sheaf of groups* $\mathcal{G}$ on a site. What is the appropriate analogue of a torsor in this setting? By abuse of notation we will also call this a *torsor*.

Definition 1.1.6. [Stacks, 03AH] Let $\mathcal{G}$ be a sheaf of groups on X, and let $\mathrm{Shv}_{\mathcal{G}}(X)$ denote the category of $\mathcal{G}$-sheaves, meaning sheaves of sets equipped with a $\mathcal{G}$-action, and equivariant morphisms between them. We define the category of $\mathcal{G}$*-torsors* $\mathrm{Tors}_{\mathcal{G}}(X) \subseteq \mathrm{Shv}_{\mathcal{G}}(X)$ to be the full subcategory on those $\mathcal{F}$ so that

1. if $\mathcal{F}(U)$ is non-empty then the action
$$\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$$
is simply transitive.[1]
2. there exists a covering $\{U_i \to X\}$ over which $\mathcal{F}(U_i) \neq \varnothing$.[2]

Terminology 1.1.7. The choice of topology comes into play in that second point. If $\mathcal{F} \in \mathrm{Shv}_{\mathcal{G}}(X)$, we say it is τ*-locally trivial* if $\mathcal{G}$ is a τ-sheaf of groups, $\mathcal{F}$ is a τ-sheaf of sets, and point (2) holds for any τ-cover.

Example 1.1.8. The sheaf $\mathcal{G}$, acting on itself by scaling, is called the *trivial* $\mathcal{G}$-torsor.

Example 1.1.9. For any group scheme G, we will refer to G*-torsors*, mean torsors the representable functor $\mathrm{Hom}(-, G)$.

Proposition 1.1.10. A $\mathcal{G}$-torsor $\mathcal{F}$ on X is trivial if and only if $\mathcal{F}(X) \neq \varnothing$, i.e. if it admits a global section.

Theorem 1.1.11. Every morphism in $\mathrm{Tors}_G(X)$ is an isomorphism.

So what are some examples of torsors, and why might we care to classify them?

Example 1.1.12. If L/k is a Galois field extension then $\mathrm{Spec}\,(L) \to \mathrm{Spec}\,(k)$ is a $\mathrm{Gal}(L/k)$-torsor in the étale topology.

- ▷ *the inverse Galois problem*: which groups G occur as Galois groups of number fields? This is asking to scratch the surface of understanding G-torsors over $\mathrm{Spec}\,(\mathbb{Q})$ for all finite groups G.

Example 1.1.13. The associated frame bundle of an algebraic vector bundle is a GL_n-torsor (say, in the Zariski topology).

Proof. Since GL_n is affine, every torsor is representable, hence GL_n-torsors are just principal GL_n-bundles, which are precisely algebraic vector bundles. □

- ▷ *Bass–Quillen conjecture*: this can be reframed as asking whether each GL_n-torsor over a regular Noetherian ring is trivial

[1] This means that in the category of $\mathcal{G}(U)$-sets, we have that $\mathcal{F}(U) \cong \mathcal{G}(U)$, but $\mathcal{F}(U)$ doesn't have a group structure — we might imagine that it has forgotten its identity element. Picking a basepoint $e \in \mathcal{F}(U)$ defines a group structure on $\mathcal{F}(U)$.

[2] In other words, we can find a cover in which to visualize $\mathcal{F}(U_i)$ as a group for each i.

▹ *Hartshorne's conjecture* concerns GL_2-torsors over $\mathbb{P}^n$ for $n \geq 7$

Example 1.1.14. A PGL_n-torsor is a Brauer–Severi variety (or a central simple algebra).

▹ the *period-index conjecture* then concerns the complexity of PGL_n-torsors.

Goal 1.1.15. Develop methods to classify torsors.

Let's do this, by first considering an alternative perspective on what a torsor is. We learned this from Alex Youcis' excellent note on torsors [You].

1.1.1 Sheaves and stacks

Recall if $\mathscr{F}$ is a sheaf of sets, this means for every cover $\{U_i \to U\}$, we have that the diagram is an equalizer

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j).$$

This first map is a monomorphism (injection) because it is an equalizer. This means that if $x, y \in \mathcal{F}(U)$ are equal in $\mathcal{F}(U_i)$ for each i, then they are equal in $\mathcal{F}(U)$. In other words, the map "reflects equality" — this is literally just what it means for something to be an injection.

Let's suppose now that $\mathcal{F}(U)$ is a *category* for every U. We'll define this concretely soon once we have more machinery, but for now let's just pretend that we know what this means — it means we can glue objects and morphisms along covers. Consider the analogous restriction functor:

$$\begin{aligned} \mathcal{F}(U) &\to \prod \mathcal{F}(U_i) \\ x &\mapsto (x|_{U_i})_i \, . \end{aligned}$$

Q: Does this map need to reflect isomorphisms?[3]

Example 1.1.16. Let $\mathcal{F}$ be the functor sending U to the category of line bundles over U. If the cover is picked appropriately small, then all line bundles are isomorphic to the trivial line bundle over U_i, but they need not be isomorphic globally as line bundles over U.

A: No, by the example above. This means that we can have $x, y \in \mathcal{F}(U)$ so that $x|_{U_i} \cong y|_{U_i}$ for each i, but we *do not have* that x and y are isomorphic in the category $\mathcal{F}(U)$. In other words, the following two notions are *different*.

▹ x and y agree *globally*, meaning x and y are isomorphic in $\mathscr{F}(U)$.
▹ x and y agree *locally*, meaning there exists an open cover $\{U_i \to U\}$ for which we have isomorphisms $x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ in $\mathcal{F}(U_i)$.[4]

Remark 1.1.17. This is a big difference between sheaves of sets (or 1-categories in general) and sheaves of categories (also called *stacks*). Equality is reflected along a cover for sheaves of sets, but isomorphism is not necessarily reflected along a cover.

Question 1.1.18. How many isomorphism classes of objects $y \in \mathcal{F}(U)$ are *locally isomorphic* to x along a cover, but not globally isomorphic?

We're going to build a sheaf that measures this! We'll call this sheaf $\mathrm{Aut}_{\mathcal{F}}(x)$, and it is defined by

$$U_i \mapsto \mathrm{Aut}_{\mathscr{F}(U_i)}(x). \tag{1.19}$$

[3] A functor which reflects isomorphisms is called *conservative*.
[4] The notation $x|_{U_i}$ is shorthand for the image of x under the restriction functor $\mathcal{F}(U) \to \mathcal{F}(U_i)$

Exercise 1.1.20. This is a priori just a presheaf of groups. Check it is actually a sheaf of groups.

Recall the following definition.

Definition 1.1.21. Let $\mathcal{G}$ denote a sheaf of groups over U and $\mathcal{U} = \{U_i \to U\}$ a cover.

1. We define a *Čech 1-cocycle* to be a collection of elements $g_{ij} \in \mathcal{G}(U_i \times_U U_j)$ for each i, j so that
$$g_{ij}g_{jk} = g_{ik}$$
on triple overlaps.[5]
2. We say two 1-cocycles (g_{ij}) and (γ_{ij}) are cohomologous if there are $\alpha_i \in \mathcal{G}(U_i)$ for each i so that
$$\alpha_i g_{ij} = \gamma_{ij}\alpha_j.$$
3. We define the *Cech cohomology* $\check{H}^1(U, \mathcal{G})$ to be the colimit of the Čech cohomology over covers, filtered with respect to refinement.

Theorem 1.1.22. There is a bijection between $H^1(U, \mathrm{Aut}_{\mathcal{F}}(x))$ and isomorphism classes of objects $y \in \mathscr{F}(U)$ which are locally isomorphic to x.

Proof. Let's first define a map. If y is locally isomorphic to x, then there is a cover $\{U_i\}$ of U and isomorphisms $\phi_i\colon\ x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ for each i. If the ϕ_i's agreed on overlaps then they would glue to a global isomorphism $\phi\colon x \xrightarrow{\sim} y$ because $\mathrm{Aut}(x)$ is a sheaf of groups, so it makes sense to look on overlaps to see what happens. Note that $\phi_i|_{U_{ij}}$ and $\phi_j|_{U_{ij}}$ will differ by an automorphism of $x|_{U_{ij}}$, call this g_{ij}:
$$g_{ij} := \left(\phi_i|_{U_{ij}}\right)^{-1}\left(\phi_j|_{U_{ij}}\right) :\ x|_{U_{ij}} \xrightarrow{\sim} x|_{U_{ij}}\,.$$
On triple overlaps, it is straightforward to verify that
$$g_{ij}g_{jk} = g_{ik}.$$
In other words we get a 1-cocycle! There was ambiguity here, since we *picked* isomorphisms $x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ as our starting data. The remaining thing to prove is that any other choice of local isomorphisms gives rise to a cohomologous 1-cocycle.

Suppose we instead picked some $\psi_i\colon x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ for each i, yielding $\gamma_{ij} = \psi_j^{-1}\psi_i$. Then ψ_i and ϕ_i differ by an automorphism of y which we call α_i:

$$\begin{tikzcd} x|_{U_i} \arrow[rr, "\phi_i"] \arrow[dr, "\alpha_i"'] & & y|_{U_i} \\ & x|_{U_i} \arrow[ur, "\psi_i"'] & \end{tikzcd}$$

Then on U_{ij} we have
$$g_{ij} = \phi_i^{-1}\phi_j = (\psi_i\alpha_i)^{-1}(\psi_j\alpha_j) = \alpha_i^{-1}\psi_i^{-1}\psi_j\alpha_j.$$
Hence
$$\alpha_i g_{ij} = \gamma_{ij}\alpha_j$$
And we get cohomologous 1-cocycles. ∎

Theorem 1.1.23. There is a bijection between isomorphism classes of $\mathcal{G}$-torsors and $\check{H}^1(U, \mathcal{G})$.

[5] See e.g. [Mil13, §11].

Sketch. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor, and pick $s_i \in \mathcal{F}(U_i)$ for each i. Then on the overlap $U_i \times_U U_j$, we have that s_i and s_j differ by a unique element $g_{ij} \in \mathcal{G}(U_i \times_U U_j)$. We run basically an identical argument. $\square$

So if $\mathcal{G} = \mathrm{Aut}_{\mathcal{F}}(x)$ then we have a bijection

$$\left\{ \begin{smallmatrix} \text{iso classes of } y \in \mathcal{F}(U) \\ \text{locally isomorphic to } x \end{smallmatrix} \right\} \leftrightarrow \check{H}^1(U, \mathrm{Aut}_{\mathcal{F}}(x)) \leftrightarrow \{\mathrm{Aut}_{\mathcal{F}}(x)\text{-torsors}\}.$$

These sorts of arguments are compatible with refinement of the cover, and since Čech and sheaf cohomology agree we see that G-torsors are in bijection with the first sheaf cohomology $H^1(U, G)$.

Exercise 1.1.24. Show that every sheaf of groups $\mathcal{G}$ is of the form $\mathrm{Aut}_{\mathcal{F}}(x)$ for some sheaf of categories $\mathcal{F}$.

Intuition 1.1.25. A $\mathcal{G}$-torsor is an object whose automorphisms locally look like $\mathcal{G}(U_i)$.

1.1.2 Interlude: representable G-torsors

When both the presheaf $\mathcal{G}$ and the sheaf of sets $\mathcal{F}$ are representable, we get a slightly different characterization.

Setup 1.1.26. Suppose $\mathscr{C} = \mathrm{Sch}_X$ is a site of schemes over X, and let $G \in \mathrm{Grp}(\mathrm{Sch}_X)$ be a group scheme over X. Suppose $\mathcal{G} = \mathrm{Hom}_X(-, G)$ is a representable sheaf of groups, and let $\mathcal{F} = \mathrm{Hom}_X(-, Y)$ for some $Y \in \mathrm{Sch}_X$, where Y comes equipped with a G-action.

In this setup, what does it mean in this case for $\mathcal{F}$ to be a $\mathcal{G}$-torsor?
Condition (1) from Definition 1.1.6 asks that

$$\mathrm{Hom}_X(U, G) \times \mathrm{Hom}_X(U, Y) \to \mathrm{Hom}_X(U, Y)$$

to be simply transitive for every $U \in \mathrm{Sch}_X$. This seems a bit tedious to check, but the following result gives us a cleaner characterization of it, which is a sheafy version of Equation 1.3.

Proposition 1.1.27. [Stacks, 0499] In Setup 1.1.26 the following two conditions are equivalent:

1. The map
$$G \times_X Y \to Y \times_X Y$$
$$(g, y) \mapsto (y, gy)$$
is an isomorphism of X-schemes.
2. For every $U \in \mathrm{Sch}_X$, the induced action
$$\mathrm{Hom}_X(U, G) \times \mathrm{Hom}_X(U, Y) \to \mathrm{Hom}_X(U, Y)$$
is simply transitive.

What about condition (2) from Definition 1.1.6? This condition translates, in Setup 1.1.26, to asking for a cover $\{U_i \to X\}$ for which $\mathrm{Hom}_X(U_i, Y) \neq \varnothing$ for each i. Pick any $s_i \in \mathrm{Hom}_X(U_i, Y)$, then it makes the diagram commute:

$$\begin{tikzcd} & Y \arrow[d] \\ U_i \arrow[ur, "s"] \arrow[r] & X. \end{tikzcd}$$

Then the pullback torsor $Y \times_X U_i \to U_i$ admits a section, which implies it is trivial by Proposition 1.1.10:

$$\begin{tikzcd} Y \times_X U_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ U_i \arrow[r] \arrow[u, dashed, bend left] & X. \end{tikzcd}$$

We summarize these observations in the following proposition.

Proposition 1.1.28. Let X be a scheme, let G be a group scheme over X, and let

$$f\colon Y \to X$$

be an X-scheme equipped with a G-action. Then Y is a G-torsor if and only if

1. The map $G \times_X Y \to Y \times_X Y$ is an isomorphism of X-schemes.
2. There exists a cover $\{U_i \to X\}$ for which $Y \times_X U_i \to U_i$ is isomorphic to the trivial G-torsor over U_i.

In this setting, we call $Y \to X$ a *principal G-bundle.*

Example 1.1.29. Let's double back to Example 1.1.12 and actually prove that $\operatorname{Spec} L \to \operatorname{Spec} k$ is a principal $\operatorname{Gal}(L/k)$-bundle in the étale site when L/k is a Galois field extension.

Proof. We check the conditions from Proposition 1.1.28. The first condition asks that

$$\left(\coprod_{g\in G} \operatorname{Spec}(k)\right) \times_{\operatorname{Spec}(k)} \operatorname{Spec}(L) \to \operatorname{Spec}(L) \times_{\operatorname{Spec} k} \operatorname{Spec}(L)$$

is an equivalence. Since everything in sight is affine, we can rephrase this as asking for the map

$$\prod_{g\in G} L \leftarrow L \otimes_k L$$

$$(x \cdot g(y))_{g\in G} \leftarrow\!\shortmid x \otimes y$$

to be an equivalence, which we recall is the *normal basis theorem* from Galois theory. The second condition from Proposition 1.1.28 asks us to find a cover of $\operatorname{Spec}(k)$ trivializing $\operatorname{Spec}(L) \to \operatorname{Spec}(k)$, but if we work in the étale site, then $\operatorname{Spec}(L) \to \operatorname{Spec}(k)$ is itself a cover, and it is clear that the product induces a section

$$\begin{tikzcd} \operatorname{Spec}(L \otimes_k L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec}(L) \arrow[d] \\ \operatorname{Spec}(L) \arrow[r] \arrow[u, dashed, bend left] & \operatorname{Spec}(k). \end{tikzcd}$$

Hence we have a trivialization, and hence a principal $\operatorname{Gal}(L/k)$-bundle (a $\operatorname{Gal}(L/k)$-torsor). $\square$

Often it's just enough to assume the sheaf of groups is representable to get all torsors are representable.

Proposition 1.1.30. Let G be an affine group scheme over X, and let $\tau \leq \mathrm{fppf}$. Then every G-torsor is representable. (see [Stacks, 0497], [You, 3.25])

Sketch. Every G-torsor is an algebraic space, and algebraic spaces which are locally affine are schemes. Since a G-torsor is locally isomorphic to G, which was assumed to be affine, then we conclude every torsor is actually a scheme. $\square$

Terminology 1.1.31. A representable fpqc-torsor for G is called a *principal homogeneous space.*

Remark 1.1.32. If $t \leq \tau$, then every t-cover is a τ-cover, hence if we can find a t-cover trivializing a G-torsor, then it also trivializes it in the τ-topology. hence

$$t \leq \tau \Rightarrow \{t\text{-torsors}\} \subseteq \{\tau\text{-torsors}\}.$$

So a very natural question is *how do we tell when a τ-torsor is also a t-torsor?* We'll discuss this next week.

1.1.3 Interlude: representability of torsors in topology

So it's a very reasonable goal to ask for any tools that could help us try to classify torsors. A natural idea, by analogy, is to look to homotopy theory, where we have a suite of tools for studying torsors.

Definition 1.1.33. Let X be a compact Hausdorff topological space and G a group. Then a *principal G-bundle* (or we might just say a G-torsor) is a fiber bundle $\pi\colon Y \to X$ so that G acts freely and transitively on the fibers.

In topology there is a *universal G-torsor*, which is denoted $EG \to BG$. This is universal in the sense that, given any map $f\colon X \to BG$, we can consider the fiber product

$$\begin{tikzcd} f^*EG \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EG \arrow[d] \\ X \arrow[r, "f"'] & BG. \end{tikzcd}$$

Then $f^*EG \to X$ is a principal G-bundle, and all principal G-bundles are obtained in this way. Not only that, but isomorphic principal G-bundles are given by homotopic classifying maps. In other words we have a bijection

$$\mathrm{Prin}_G(X) \leftrightarrow [X, BG].$$

So the data of a principal G-bundle is the data of a map $X \to BG$, and an isomorphism of principal G-bundles is equivalent to a homotopy between two maps $f, g\colon X \to BG$.

Example 1.1.34. We have that $B(\mathbb{Z}/2) = \mathbb{R}\mathrm{P}^\infty$ and $E\mathbb{Z}/2 = S^\infty$, so that $\mathbb{Z}/2$-torsors are real line bundles. Similarly $B\mathbb{C}^\times = \mathbb{C}\mathrm{P}^\infty$.

A big example comes from quotienting out by a compact subgroup:

Theorem 1.1.35. (Samelson, 1941) If $H \leq G$ is a compact subgroup of a Lie group, then $G \to G/H$ is a Serre fibration and principal H-bundle.

Corollary 1.1.36. We have fiber sequences

$$H \to G \to G/H$$
$$G/H \to BH \to BG.$$

Example 1.1.37. For the inclusions $O(n) \subseteq O(n+1)$ and $U(n) \subseteq U(n+1)$ we get fiber sequences

$$S^{2n+1} \to \mathrm{BU}(n) \to \mathrm{BU}(n+1)$$
$$S^n \to \mathrm{BO}(n) \to \mathrm{BO}(n+1).$$

This is how Bott periodicity is proved.

1.1.4 Why we like representability of torsors

This has a number of huge applications:

▷ Given any cohomology theory E^* and any class $c \in E^*(BG)$, if $f^*(c) \neq g^*(c)$ in $E^*(X)$, this means that f and g correspond to non-isomorphic torsors. This the basic idea of characteristic classes.

Example 1.1.38. If $G = \mathrm{GL}_n(\mathbb{C})$, then $\mathrm{BGL}_n(\mathbb{C}) = \mathrm{BU}(n) = \mathrm{Gr}_{\mathbb{C}}(n, \infty)$ is a Grassmannian of n-planes in $\mathbb{C}^\infty$. A map $f\colon X \to \mathrm{Gr}_{\mathbb{C}}(n,\infty)$ gives a complex n-dimensional vector space by pullback.

Theorem 1.1.39. (Pontryagin–Steenrod) There is a bijection

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X, \mathrm{BU}(n)]\,.$$

Since $H^*(\mathrm{BU}(n);\mathbb{Z}) = \mathbb{Z}[c_1,\ldots,c_n]$, if $E \to X$ is any rank n vector bundle classified by a map $f\colon X \to \mathrm{BU}(n)$, then its Chern classes are by definition $c_i(E) = f^*c_i$.

▷ We have access to *obstruction theory* – this lets us break down lifting problems into smaller manageable stages.

Example 1.1.40. If X is a complex n-dimensional manifold, and $E \to X$ is a rank n complex vector bundle, then it splits off a free summand if and only if $c_n(E) = 0$.

Example 1.1.41. If we fix $c_1,\ldots,c_n \in H^*(X;\mathbb{Z})$, we can ask how many isomorphism classes of complex rank n vector bundles on X have these given Chern classes. Since $c_i \in H^{2i}(X;\mathbb{Z}) = [X, K(\mathbb{Z},2i)]$ this is equivalent to asking how many lifts there are for

$$\begin{tikzcd} & \mathrm{BU}(n) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "c_1,\ldots,c_n"'] & \prod_i K(\mathbb{Z},2i). \end{tikzcd}$$

If X is a finite CW complex, there are only finitely many such lifts by basic obstruction theory.

▷ Suppose we have two groups G and K, and we want to study natural ways to create K-torsors out of G-torsors over any space. Then representability, combined with the Yoneda lemma, allows us to completely classify all the ways to do this.

Example 1.1.42. There is one and only one natural non-trivial function

$$\{\mathrm{GL}_n(\mathbb{R})\text{-torsors}\} \to \{(\mathbb{Z}/2)\text{-torsors}\}\,,$$

given by the nonzero class in $[\mathrm{BGL}_n(\mathbb{R}), B\mathbb{Z}/2] = \mathbb{Z}/2$. This is called the *determinant bundle* or the *first Stiefel–Whitney class.*

1.1.5 Motivation of what's to come

Question 1.1.43. By analogy to homotopy theory, we might ask, for a group scheme G, the following questions:

1. Is there an analogous universal space BG in algebraic geometry which classifies G-torsors?
2. If so, can we classify G-torsors over X via some "homotopy classes" of maps from X to BG?

The answer to both will be yes, but *not in the category of varieties.* We need more machinery than is available to us there.

1.2 Torsors II: Simplicial methods

Definition 1.2.1. We denote by Δ the category whose objects are finite ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$, and whose morphisms $[n] \to [m]$ are order-preserving maps.

Notation 1.2.2. Let $\mathscr{C}$ be a category. Then we refer to a functor $\Delta \to \mathscr{C}$ as a *cosimplicial object* of $\mathscr{C}$, and a functor $\Delta^{\mathrm{op}} \to \mathscr{C}$ as a *simplicial object* of $\mathscr{C}$.

Intuition 1.2.3. We should think about a (co)simplicial object as a *data type* — it is a combinatorial gadget that is surprisingly convenient for bookkeeping and appears frequently in nature. We'll see quite a few examples, but let's first see how to compress this data.

There are a priori a lot of order-preserving functions $[n] \to [m]$, so we'd like a nice class of easy-to-manage morphisms in Δ so that any morphism factors into nice morphisms. These fall in two classes:

Terminology 1.2.4.
1. We have *coface* maps $d^i\colon [n] \to [n+1]$ for $0 \le i \le n+1$ which are defined by the property that they miss the element $i \in [n+1]$.
2. We have *codegeneracy* maps $s^j\colon [n] \to [n-1]$ for $0 \le j \le n-1$ defined by the property that they are surjective and that $s^j(j) = s_j(j+1)$.

Example 1.2.5. The coface maps $d^i\colon [2] \to [3]$ look like

$$\begin{tikzcd}
0\bullet \arrow[dr, "d^0"] & \bullet 0 & 0\bullet \arrow[r, "d^1"] & \bullet 0 & 0\bullet \arrow[r, "d^2"] & \bullet 0 & 0\bullet \arrow[r, "d^3"] & \bullet 0 \\
1\bullet \arrow[dr] & \bullet 1 & 1\bullet \arrow[dr] & \bullet 1 & 1\bullet \arrow[r] & \bullet 1 & 1\bullet \arrow[r] & \bullet 1 \\
2\bullet \arrow[dr] & \bullet 2 & 2\bullet \arrow[dr] & \bullet 2 & 2\bullet \arrow[dr] & \bullet 2 & 2\bullet \arrow[r] & \bullet 2 \\
& \bullet 3 & & \bullet 3 & & \bullet 3 & & \bullet 3
\end{tikzcd}$$

Example 1.2.6. The codegeneracy maps $[3] \to [2]$ look like

$$\begin{tikzcd}
0\bullet \arrow[r, "s^0"] & \bullet 0 & 0\bullet \arrow[r, "s^1"] & \bullet 0 & 0\bullet \arrow[r, "s^2"] & \bullet 0 \\
1\bullet \arrow[ur] & \bullet 1 & 1\bullet \arrow[r] & \bullet 1 & 1\bullet \arrow[r] & \bullet 1 \\
2\bullet \arrow[ur] & \bullet 2 & 2\bullet \arrow[ur] & \bullet 2 & 2\bullet \arrow[r] & \bullet 2 \\
3\bullet \arrow[ur] & & 3\bullet \arrow[ur] & & 3\bullet \arrow[ur] &
\end{tikzcd}$$

Terminology 1.2.7. Let $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathscr{C}$ denote a simplicial object.
1. We use the notation X_n to denote the object $X_\bullet([n])$ in $\mathscr{C}$.
2. We denote by $d_i\colon X_n \to X_{n-1}$ for $0 \le i \le n$ the image of the coface map d^i. We call d_i a *face map*
3. Similarly we denote by $s_i\colon X_n \to X_{n+1}$ for $0 \le i \le n$ the image of the codegeneracy map s^i, and call s_i a *degeneracy map*.

Exercise 1.2.8. Convince yourself that every map in Δ factors into face and degeneracy maps, and therefore any simplicial object $X_\bullet$ in $\mathscr{C}$ can be described by the data of X_n for each n, and its face and degeneracy maps.[6]

Definition 1.2.9. If $\mathscr{C}$ is a locally small category with all finite products (binary products and a terminal object), then a *group object* in $\mathscr{C}$ is the data of an object $G \in \mathscr{C}$ together with morphisms

$$\begin{aligned} m\colon G \times G &\to G \\ e\colon 1 &\to G \\ i\colon G &\to G \end{aligned}$$

called multiplication, identity, and inverse, such that the following diagrams commute.

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "e\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times e"'] \arrow[dr, "\mathrm{id}"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "i\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times i"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

Example 1.2.10. Let G be a group object in a category $\mathscr{C}$ as above. Then we can define a simplicial object

$$\begin{aligned} B_\bullet G\colon \Delta^{\mathrm{op}} &\to \mathscr{C} \\ [n] &\mapsto G^{\times n}, \end{aligned}$$

as:

$$\begin{tikzcd} * \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] & G \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] & G \times G \end{tikzcd}$$

Explicitly, $B_nG = G^{\times n}$, with face maps $d_i\colon G^{\times n} \to G^{\times(n-1)}$ given by

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & 1 \le i \le n-1 \\ (g_1, \ldots, g_{n-1}) & i = n, \end{cases}$$

and degeneracies $s_i\colon G^{\times n} \to G^{\times(n+1)}$ given by

$$s_i(g_1, \ldots, g_n) = \begin{cases} (e, g_1, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_n) & 1 \le i \le n. \end{cases}$$

Note that $d_1\colon G^{\times 2} \to G$ is precisely the group multiplication, while $s_0\colon * \to G$ is the identity on the group.

Exercise 1.2.11. Check the simplicial identities hold *precisely* because of the group axioms. In particular, observe that we didn't use anything special about spaces — if G is a group object in any category $\mathscr{C}$ we obtain an associated bar construction which is a simplicial object in $\mathscr{C}$ that we call $B_\bullet G$.

Example 1.2.12. (Important) For the formal categorical reason in Exercise 1.2.11, any group scheme G gives rise to a simplicial scheme $B_\bullet G \in \mathrm{Fun}\,(\Delta^{\mathrm{op}}, \mathrm{Sch})$. This dates back at least to work of Friedlander [Fri82, Example 1.2]. We'll use this object frequently.

Example 1.2.13. Simplicial objects appeared crucially in Deligne's work on resolution of singularities and mixed Hodge structures. Let X be a complete[7] singular variety. Then we can "replace" X

[6] Any time we define a simplicial object via this compressed data, we should check that the simplicial identities hold, although we'll mostly omit these arguments here.

[7] X is *complete* if it is proper over the base [Har77, p. 105]. This is a slightly more general notion than projective.

with a simplicial variety

$$\cdots \qquad X_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_0,$$

where each X_i is smooth and projective. There is a cohomological descent spectral sequence computing the cohomology of X in terms of the cohomology of the X_n's. This dates back to SGA 4. Additionally, this replacement is the core idea of *resolution of singularities*. Deligne uses these simplicial methods to endow singular varieties with Hodge structures, so-called *mixed Hodge structures* [Del74].

Example 1.2.14. Let $A_\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{Ab}$ be a simplicial abelian group. Then by taking the alternating sum of the face maps, we obtain a chain complex, purely by the simplicial identities. This assignment is functorial:

$$\mathrm{sAb} \to \mathrm{Ch}_{\geq 0}(\mathrm{Ab})$$
$$A_\bullet \mapsto \left(A_n, \partial = \sum_{i=0}^{n} (-1)^i d_i\right).$$

The associated chain complex is called the *Moore complex* of the simplicial abelian group. Two quick remarks about this:

1. This works if we replace Ab by any abelian category.
2. This process is invertible — this implies that simplicial objects and connective chain complexes in any abelian category are equivalent.[8] We will use this later, as we will want to construct certain simplicial objects in sheaves of abelian groups, and it will be more direct to construct them first as chain complexes, then pass through this equivalence.

Example 1.2.15 (Čech nerve $N_\bullet(U)$)**.** A simplicial object can conveniently encode the data of an open cover, its intersections, triple intersections, etc.[9] For this, given an open cover $\{U_i \to X\}$ of a variety X, denote by $U_{ij} := U_i \times_X U_j$ the double overlaps, by $U_{ijk} := U_i \times_X U_j \times_X U_k$ the triple overlaps, and so on. We define a simplicial object

$$\textstyle\coprod_i U_i \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \coprod_{i,j} U_{ij} \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \coprod_{i,j,k} U_{ijk} \qquad \cdots$$

The face maps are defined by omitting the jth index

$$d_j\colon U_{i_1\cdots i_n} \to U_{i_1\cdots \widehat{i_j}\cdots i_n},$$

and the degeneracy maps repeat the jth index

$$s_j\colon U_{i_1\cdots i_n} \to U_{i_1\cdots i_{j-1} i_j i_j i_{j+1}\cdots i_n}.$$

We call this the *Čech nerve* associated to the cover, and we denote it by $N(\mathcal{U})$.

1.2.1 More on simplicial sets

We denote by

$$\mathrm{sSet} := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$$

[8] There's a small lie here, the actual functor exhibiting the equivalence is not the Moore complex but the *normalized chains complex*, although the normalized chain complex maps to the Moore complex in a natural way, and this is a chain homotopy equivalence [GJ99, III.2.4], so they are essentially the same. This equivalence is the *Dold–Kan theorem*. For more detail see [GJ99, §III.2].

[9] Technically it should be an *augmented simplicial object* since we'd also like to remember the data of the maps $U_i \to X$.

the category of *simplicial sets*. Given a simplicial set

$$X_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$$
$$[n] \mapsto X_n,$$

we call X_n the set of *n-simplices*.

Example 1.2.16. Any set Y gives rise to a constant simplicial set $\underline{Y}$, given by sending $[n] \mapsto Y$, and every morphism in Δ to the identity on Y.

Example 1.2.17. We denote by Δ^n the simplicial set

$$\Delta^n := \mathrm{Hom}_{\Delta}(-, [n]) \colon \Delta^{\mathrm{op}} \to \mathrm{Set}.$$

By the Yoneda lemma, we have a natural bijection

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X_\bullet) \cong X_n$$

for any $X_\bullet \in \mathrm{sSet}$.

There is a functor called *geometric realization*, which "assembles" a simplicial set into a topological space: Let Δ^n_{top} denote the *topological* n-simplex[10]

$$|-| \colon \mathrm{sSet} \to \mathrm{Top}$$
$$X_\bullet \mapsto \amalg_{n \geq 0} X_n \times \Delta^n_{\mathrm{top}} / \sim,$$

where $\sim$ is

$$(x, d_i u) \sim (d_i x, u) \qquad x \in X_n,\ u \in \Delta^{n-1}_{\mathrm{top}}$$
$$(y, s_i v) \sim (s_i y, v) \qquad y \in X_{n-1},\ v \in \Delta^n_{\mathrm{top}}.$$

Remark 1.2.18. We discussed this universal space BG classifying principal G-bundles last week. We can define this as $BG := |B_\bullet G|$, i.e. it is the geometric realization of the bar construction for the group.[11]

Definition 1.2.19. We say two maps $f, g \colon X_\bullet \to Y_\bullet$ are *simplicially homotopic* if there is a map

$$H \colon \Delta^1 \times X_\bullet \to Y_\bullet$$

so that $H|_{\{0\} \times X_\bullet} = f$ and $H|_{\{1\} \times X_\bullet} = g$.

Remark 1.2.20. Unpacking this data, we can verify it is the same as asking for maps for every n:

$$H_i^n \colon X_n \to Y_{n+1} \qquad 0 \leq i \leq n,$$

so that $d_0 H_0^n = f_n$, and $d_{n+1} H_n^n = g_n$, and so that they satisfy the following relations with face and degeneracy maps:[12]

$$d_i H_j^n = \begin{cases} H_{j-1}^{n-1} d_i & i < j \\ d_i H_{j-1}^n & i = j \neq 0 \\ d_{j+1} H_{j+1} & = i = j+1,\ j \neq n \\ H_j^{n-1} d_{i-1} & i > j+1. \end{cases}$$

$$s_i H_j^N = \begin{cases} H_{j+1}^{n+1} s_i & i \leq j \\ H_j^{n+1} s_{i-1} & i > j. \end{cases}$$

This notion makes sense for maps between simplicial objects in any category, and it is called a *simplicial homotopy*.

[10] This is the locus $\Delta^n_{\mathrm{top}} := \{(x_0, \ldots, x_{n+1} \in \mathbb{R}^{n+1} \colon \sum x_i = 1,\ x_i \geq 0\}$.

[11] If G is a discrete group this is literally correct. If G is a topological group we have to modify the domain of geometric realization to be $|-| \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Top}) \to \mathrm{Top}$ in order to get the correct definition.

[12] This is in [Wei94, 8.3.11].

1. A priori we should be careful calling it a homotopy, since it doesn't require us to have any notion of a model structure or higher-categorical structure in order to state, however we will see this won't be a problem.
2. In the context of simplicial sets, this coincides with Definition 1.2.19, as we can check (see e.g. [Wei94, 8.3.12]).
3. It is not true (even in simplicial sets) that this is an equivalence relation, and indeed it isn't. Nevertheless it generates one, so when we say "up to simplicial homotopy" we often mean with respect to the transitive closure of this relation.

Theorem 1.2.21. Geometric realization preserves products.

Corollary 1.2.22. Simplicial homotopies become honest homotopies of spaces after geometric realization.

Remark 1.2.23. We can study simplicial sets up to homotopy or spaces up to homotopy, and in a sense that can be made precise, these are essentially the same theory.

Remark 1.2.24. Historically this connection between simplicial sets and spaces was part of a research program which used to be called "combinatorial homotopy theory," led by Kan in the 1950's.

1.2.2 Torsors via simplicial maps

The big takeaway of this entire section is the following:

Theorem 1.2.25. Let $\mathcal{U} = \{U_i \to X\}$ be a cover, and let G be a group scheme over X. To any map from the Čech nerve to the bar construction, we can assign a Čech 1-cocycle

$$\mathrm{Hom}_{\mathrm{Fun}(\Delta^{\mathrm{op}},\mathrm{Var}_k)}(N(\mathcal{U}), B_\bullet G) \to H^1(\mathcal{U}, G),$$

and this assignment is surjective.

Proof. We'll figure out some what's happening in lower degrees first and use this to see how to define maps in general.

Low-degree intuition: To specify a map, we need to know what happens on each level, so let's consider the diagram:

$$\begin{tikzcd}
N(\mathcal{U}) = & \coprod_i U_i \arrow[d] & \coprod_{i,j} U_{ij} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] \arrow[l, leftarrow] & \cdots \\
BG = & * \arrow[r] & G \arrow[l, shift left] \arrow[l, shift right] & \cdots
\end{tikzcd}$$

On 0-simplices, there is no data, and on 1-simplices, we are obtaining elements we will call $(g_{ij} \in \mathrm{Hom}(U_{ij}, G))$. For each i, the degeneracy maps require the following diagram to commute.

$$\begin{tikzcd}
U_i \arrow[r, "s_0"] \arrow[d] & U_{ii} \arrow[d, "g_{ii}"] \\
* \arrow[r, "e"'] & G,
\end{tikzcd}$$

which tells us that $g_{ii} = e$. That's about all we can learn from degeneracies. What happens on 2-cells? Let's consider the inner and outermost face maps $N_2(U) \to N_1(U)$:

$$\begin{tikzcd}
U_{jk} \arrow[d, "g_{jk}"'] & U_{ijk} \arrow[l, "d_0"'] \arrow[d] \\
G & G \times G \arrow[l, "d_0 = \mathrm{proj}_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{ij} \arrow[d, "g_{ij}"'] & U_{ijk} \arrow[l, "d_2"'] \arrow[d] \\
G & G \times G \arrow[l, "d_2 = \mathrm{proj}_1"]
\end{tikzcd}$$

We've decorated the face maps on the bar construction to recall whether they are projection or multiplication. By the universal property of the product, $U_{ijk} \to G \times G$ is determined by its post-composition with the projections, hence it makes sense to call this map (g_{ij}, g_{jk}). The d_1 relation gives us

$$\begin{tikzcd}
U_{ik} \arrow[d, "g_{ik}"] & U_{ijk} \arrow[l, "d_1"'] \arrow[d, "(g_{ij},g_{jk})"] \\
G & G \times G \arrow[l, "d_1=\mathrm{mult}"]
\end{tikzcd}$$

This implies that

$$g_{ik} = g_{ij} g_{jk}.$$

This is *exactly the 1-cocycle condition* from Definition 1.1.21. We claim that the remaining data of the map $N(\mathcal{U})_\bullet \to B_\bullet G$ is completely determined by this relation. In particular, every Čech 1-cocycle can be extracted from a map $N_\bullet(U) \to B_\bullet G$ by the above process.

In general: It is now clear how to define the map $N_n\mathcal{U} \to B_n G$ — its component on $U_{i_0 \cdots i_n}$ is given by the tuples of maps

$$\prod_{j=0}^{n-1} g_{i_j i_{j+1}} \colon U_{i_0 \cdots i_n} \to G^{\times n}.$$

This clearly commutes with degeneracies, and commutes with face maps by universal property of the product and by the 1-cocycle relation. We see in fact that this map is well-defined if and only if the 1-cocycle condition holds. ☐

Theorem 1.2.26. Two 1-cocycles for the cover $\mathcal{U}$ are cohomologous if and only if they are simplicially homotopic.

Proof sketch. Let $f, g \colon N(\mathcal{U})_\bullet \to B_\bullet G$ be two maps corresponding to 1-cocycles (f_{ij}) and (g_{ij}), respectively. Recall simplicial homotopy requires us to define maps $H_i^n \colon N(\mathcal{U})_n \to B_{n+1} G$ for $0 \le i \le n$ so that $d_0 H_0 = f_n$, $d_{n+1} H_n = g_n$, and various other relations hold. Again let's start in low degrees to gain some intuition and then prove the theorem directly.

Low-degree intuition: Let's see what H_0^0 looks like:

$$\begin{tikzcd}
\coprod_i U_i \arrow[d] \arrow[dr, "H_0^0"] & \coprod_{i,j} U_{ij} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] \arrow[dr, "{H_0^1, H_1^1}"] & \coprod_{i,j,k} U_{ijk} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \arrow[dr, "{H_0^2, H_1^2, H_2^2}"] & \cdots \\
* & G \arrow[l, shift left] \arrow[l, shift right] & G \times G \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \cdots
\end{tikzcd}$$

At level zero, the map H_0^0 specifies an element $\alpha_i \in G(U_i)$. Let's see what the degeneracy relations tell us at level one:

$$\begin{tikzcd}
\coprod_i U_i \arrow[dr, "H_0^0"'] & \coprod_{i,j} U_{ij} \arrow[l, shift right, "d^0"'] \arrow[l, shift left, "d^1"] \arrow[d] \arrow[dr, "{H_0^1, H_1^1}"] & & \\
& G & G \times G \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=2, "d_2"] & \cdots
\end{tikzcd}$$

Since H_i^1 is mapping into a product, it is determined by its projections, which are post-composing with d_2 and d_0, respectively. From the relations and the simplicial homotopy condition, we get

$$d_0 H_0^1 = f_{ij}$$
$$d_2 H_0^1 = H_0^0 d_1 = \alpha_i,$$

so H_0^1 is the tuple $(\alpha_i, f_{ij})\colon U_{ij} \to G \times G$.[13] Similarly we compute

$$d_0 H_1^1 = H_0^0 d_0 = \alpha_j$$
$$d_2 H_1^1 = g_{ij},$$

so $H_1^1 = (g_{ij}, \alpha_j)\colon U_{ij} \to G \times G$. The remaining relation states that

$$d_1 H_1^1 = d_1 H_0^1,$$

and since d_1 is the multiplication $G \times G \to G$, this tells us that $\alpha_i f_{ij} = g_{ij}\gamma_j$, which is the coboundary condition.

Exercise: Show that, as maps $U_{i_0 i_1 i_2} \to G \times G \times G$, we have

$$H_0^2 = (\alpha_{i_0}, f_{i_0 i_1}, f_{i_1 i_2})$$
$$H_1^2 = (g_{i_0 i_1}, \alpha_{i_1}, f_{i_1 i_2})$$
$$H_2^2 = (g_{i_0 i_1}, g_{i_1 i_2}, \alpha_{i_2})\,.$$

In general: We define

$$H_j^n \colon U_{i_0 i_1 \cdots i_n} \to G^{\times(n+1)}$$

by

$$\left(g_{i_0 i_1}, \ldots, g_{i_{j-1} i_j}, \alpha_{i_j}, f_{i_j i_{j+1}}, \ldots, f_{i_{n-1} i_n}\right).$$

We verify that the relevant relations hold if and only if the coboundary condition $\alpha_i f_{ij} = g_{ij}\alpha_j$ holds for every i, j. □

Motivation 1.2.27. Last week we asked for a universal space BG so that homotopy classes of maps $X \to BG$ classifies G-torsors, and we're getting close! We now have a fantastic candidate for BG, namely the bar construction $B_\bullet G$ above. However we (1) didn't have a notion of homotopy of maps to witness two torsors being equivalent, and (2) we were mapping from $N(\mathcal{U})$, *not* from X. To that end, let's write down what we're looking for.

Wishlist: We want some nice category $\mathscr{C}$ where both X and $B_\bullet G$ live (so our category should contain simplicial varieties), and we want our category to have a notion of *equivalence* with the following properties:

1. Two maps $N(\mathcal{U}) \to B_\bullet G$ are homotopic in $\mathscr{C}$ if and only if they classify cohomologous G-torsors. We want to make this notion of homotopy precise.
2. In $\mathscr{C}$, we have that $X \simeq N(\mathcal{U})$, i.e. a variety is equivalent to the Čech nerve of any cover over it, so that up to homotopy, we can classify torsors via maps $X \to B_\bullet G$.

Let's preview how this is going to go — the category Var_k of k-varieties doesn't support a nice homotopy theory, nor does simplicial varieties $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Var}_k)$. So what we could do instead is replace each variety by its representable presheaf (functor of points, for the algebraic geometers). The Yoneda embedding $\mathrm{Var}_k \to \mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{Set})$ then induces a functor

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Var}_k) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{Set}).$$

Doing some adjoint business, we get that this latter category is equivalent to

$$\mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})) = \mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{sSet}).$$

So altogether we get a Yoneda embedding

$$\mathrm{Var}_k \hookrightarrow \mathrm{Fun}(\mathrm{Var}_k^{\mathrm{op}}, \mathrm{sSet}).$$

This latter category is called the category of *simplicial presheaves*, or in higher category language it is just denoted $\mathrm{PSh}(\mathrm{Var}_k)$, and called the ∞-category of (∞-)presheaves. It comes with a notion of homotopy coming from simplicial homotopy theory. We'll see that this was not only a nice

[13] There's abuse of notation here with α_i.

well-behaved way to access homotopy theory starting from Var_k, it was actually the *universal* way to do this (see e.g. [Dug01]).

This category $\mathrm{PSh}(\mathrm{Var}_k)$ will take care of item (1) on our checklist! After we define what homotopy means there, homotopic maps in this category will give isomorphic torsors.

This doesn't take care of point (2), and we shouldn't really expect it to. The equivalence between a variety and the Čech nerve of the cover should depend on *what covers are permissible*. In other words, it should bake in the Grothendieck topology somehow. We'll see that we can get point (2) (while retaining a notion of homotopy compatible with that of presheaves) by passing to the category of *sheaves* in our site. We'll make all this precise soon.

1.3 Torsors III: Sites, sheaves, and Hilbert 90

We have seen that $H^1(X, G)$ classifies G-torsors for X, but we've been a little vague about Grothendieck topologies (i.e. what kinds of covers are we considering for our varieties). Today we'll make things more precise.

Definition 1.3.1. Let $\mathscr{C}$ be a category with pullbacks. Then a *Grothendieck topology* τ consists of collections of morphisms $\{U_i \to X\}_{i\in I}$ in $\mathscr{C}$ called *covers* or *coverings*, satisfying the following properties:

1. *Closure under pullbacks*: If $\{U_i \to X\}_{i\in I}$ is a covering and $f\colon Y \to X$ is any morphism, then the collection of base change morphisms $\{U_i \times_X Y \to Y\}_{i\in I}$ is a covering.
2. *Closure under refinement*: If $\{U_i \in X\}$ is a covering and $\{V_{ij} \to U_i\}_j$ is a covering for each i, then the composite $\{V_{ij} \to U_i \to X\}_{i,j}$ is a covering
3. *Isomorphisms*: Any isomorphism $f\colon Y \xrightarrow{\sim} X$ gives a one-element cover $\{Y \to X\}$.

A pair of a category and a topology $(\mathscr{C}, \tau)$ is called a *site*.

Example 1.3.2. A *Zariski cover* is a collection of open immersions $\{U_i \to U\}$ which are jointly surjective. This generates the *Zariski site* on Sch_S.

Example 1.3.3. An *étale cover* is a collection of étale morphisms, jointly surjective.

Example 1.3.4. A *Nisnevich cover* is a collection of étale morphisms $\{U_i \to U\}$ so that for each $x \in X$ there exists an i and a $y \in U_i$ so that $y \mapsto x$ induces an isomorphism on residue fields.

Note 1.3.5. If $R \to S$ is an étale ring extension, then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is an étale cover but not necessarily a Nisnevich cover. For example $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is étale if L/k is a finite separable extension, but not Nisnevich unless $k = L$.

Example 1.3.6. [Stacks, 021M] An *fppf cover* is a jointly surjective collection of morphisms $\left\{U_i \xrightarrow{f_i} U\right\}$ so that each f_i is flat and locally of finite presentation.

Example 1.3.7. [Stacks, 03NW] An *fpqc cover* is a jointly surjective collection of morphisms $\left\{U_i \xrightarrow{f_i} U\right\}$ so that $\amalg_i U_i \to X$ is faithfully flat and quasi-compact.

Example 1.3.8. A map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is an fpqc cover if and only if the ring map $R \to S$ is faithfully flat.

Terminology 1.3.9. If t and τ are two Grothendieck topologies, we write $t \leq \tau$ if every t-cover is a τ-cover. We say t is *coarser* than τ or that τ *refines* t in this case.

Proposition 1.3.10. We have that

$$\mathrm{Zar} \leq \mathrm{Nis} \leq \mathrm{et} \leq \mathrm{fppf} \leq \mathrm{fpqc}.$$

For more topologies see Belmans: topologies comparison.

Question 1.3.11. If $t \leq \tau$ then what is the difference between t-torsors and τ-torsors?

Let's break this into two parts

1.3.1 Torsors in finer topologies

Proposition 1.3.12. Let $\mathscr{C}$ be some fixed category of schemes, with topologies t and τ. Let $\mathcal{G}$ be a sheaf of groups in the τ-topology. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor in the t-topology. Then $\mathcal{F}$ is also a τ-torsor if $\mathcal{F}$ is a τ-sheaf.

Proof. The only thing that could fail is the sheaf condition in the definition. If $\mathcal{F}$ is a t-torsor, then there exists a t-cover $\{U_i \to U\}$ trivializing $\mathcal{F}$, and this is also a τ-cover since τ refines t. □

We want to make some guarantee that $\mathcal{G}$ will still be a sheaf in a finer topology. In our cases, we care about the case where $\mathcal{G}$ is representable, and a classical result guarantees this for us.

Theorem 1.3.13. (Grothendieck, [Stacks, 023Q]) Every representable presheaf is a sheaf in the fpqc topology (and hence in any coarser topology, e.g. Zariski, Nisnevich, étale, syntomic, fppf).

Corollary 1.3.14. Let G be an affine group scheme, and let $t \leq \tau \leq \mathrm{fppf}$. Then every t-torsor is a τ-torsor.

Proof. Let $\mathcal{F}$ be a G-torsor. By Proposition 1.1.30 it is representable, hence it is a sheaf in the τ topology as well by Theorem 1.3.13. Therefore by Proposition 1.3.12 it is also a τ-torsor. □

1.3.2 Torsors in coarser topologies

Now we're interested in the reverse question — when $t \leq \tau$, when is a τ-torsor a t-torsor? The only thing that could fail is condition (2) of Definition 1.1.6, so we get the following answer/definition.

Definition 1.3.15. Let $t \leq \tau$ be topologies on a fixed category of schemes $\mathscr{C}$, let $\mathcal{G}$ be a τ-sheaf of groups and let $\mathcal{F}$ be a $\mathcal{G}$-torsor. Then $\mathcal{F}$ is a t-torsor if and only if there exists a t-cover over which $\mathcal{F}$ is trivialized. We say $\mathcal{F}$ is *locally trivial in the t-topology.*

We can now leverage some tools from algebraic geometry to prove this.

Proposition 1.3.16. Every smooth morphism of schemes admits a section étale-locally.

Corollary 1.3.17. If G is a smooth group scheme, then there is an equivalence of categories between étale G-torsors and fppf-torsors.

Proof. See [Hal, Proposition 4] for a proof. □

We can summarize what we've learned in the following cheatsheet:

Cheatsheet: topologies

Let $t \leq \tau$ be two covers, we say that τ *refines* t, or that t is *coarser* than τ. What this means is that

1. Every t-cover is a τ-cover
2. Every τ-sheaf is a t-sheaf
3. If $\mathscr{F}$ is a t-torsor then it is a τ-torsor if it is a τ-sheaf
4. If $\mathscr{F}$ is a τ-torsor it is a t-torsor if and only if it is t-locally trivial.

1.3.3 Special algebraic groups

Definition 1.3.18. We say an algebraic group (a group object in varieties) over a field k is *linear* if it admits a faithful finite-dimensional representation (c.f. [Mil15, p. 72]).

Note 1.3.19. If an algebraic group is linear, it is automatically affine ([Mil15, 1.29]) and finite type. Also, all linear algebraic groups are closed subgroups of GL_n.

Proposition 1.3.20. Every affine group scheme of finite type over a field k is linear ([Mil15, 4.8]).

Definition 1.3.21. [Gro58, p. 5-11] A group scheme is called *special* if it is a linear algebraic group G with the property that every G-torsor[14] is locally trivial in the Zariski topology.

Theorem 1.3.22. [Gro58, Théorème 3] The group GL_n is special.

Corollary 1.3.23. [Mil13, 11.4] There is a natural bijection

$$H^1_{\mathrm{Zar}}(X, \mathrm{GL}_n) \leftrightarrow H^1_{\mathrm{Nis}}(X, \mathrm{GL}_n) \leftrightarrow H^1_{\mathrm{et}}(X, \mathrm{GL}_n) \leftrightarrow H^1_{\mathrm{fpqc}}(X, \mathrm{GL}_n).$$

Hence any of the groups above can be thought of as parametrizing isomorphism classes of *algebraic vector bundles* on X.

[14]Since G is affine, every G-torsor in $\tau \leq \mathrm{fppf}$ is automatically a principal G-bundle.

Chapter 2

Higher categories

Contents

2.1 Infinity categories

Example 2.1.1. If $\mathscr{C}$ is a small 1-category, it gives rise to a simplicial set $N_\bullet\mathscr{C}$, called the *nerve* of $\mathscr{C}$, with the following data:

- ▷ 0-simplices = objects of $\mathscr{C}$

▷ 1-simplices = morphisms in $\mathscr{C}$
▷ 2-simplices = pairs of composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathscr{C}$
⋮
▷ n-simplices = strings of n-composable morphisms

Here the degeneracy maps $(N\mathscr{C})_n \to (N\mathscr{C})_{n+1}$ insert an identity, while the face maps $(N\mathscr{C})_n \to (N\mathscr{C})_{n-1}$ compose maps. Observe that in $\mathscr{C}$, composition happens *strictly*, by which we mean there is no notion of homotopy between maps — if $x \xrightarrow{f} y \xrightarrow{g} z$ is a composite of maps, and $h\colon x \to z$, then either $h = g \circ f$, or it is not equal, and this is encoded by the data of a *unique* 2-cell:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z. \end{tikzcd}$$

We think about this 2-cell as a *witness* for the composition. Note that if no 2-cell exists filling the diagram above, this means that h is not equal to $g \circ f$. It is just some other map from x to z.

Example 2.1.2. If $\mathscr{C}$ is a one-object groupoid (e.g. a group G), then $N_\bullet\mathscr{C}$ is the bar construction of G, and its realization $B\mathscr{C} := |N_\bullet\mathscr{C}|$ is called the *classifying space* of the category. Some examples:

1. $BC_2 = \mathbb{R}\mathrm{P}^\infty$
2. $B\mathbb{N} = S^1$
3. $B\mathbb{Z} = S^1$
4. $B(\bullet \to \bullet) = \Delta^1_{\mathrm{top}}$
5. $B(\bullet \xrightarrow{\sim} \bullet) = S^\infty$
6. $B\mathrm{PBr}_n = \mathrm{Conf}_n(\mathbb{R}^2)$

Q: Given a simplicial set, when can you tell whether it arose as the nerve of a 1-category?

A: Given any diagram of the form $\bullet \to \bullet \to \bullet$, it has to fill in uniquely to a 2-cell. But we also need to fill in composites of three morphisms uniquely (to get a tetrahedron), and composites of four morphisms, and so on. To that end, let Λ^k_n be the simplicial set obtained from Δ^n by deleting the kth face. This is called a *horn*.

This isn't a definition of a horn — we might instead characterize the horns by their representable functors, i.e. $\mathrm{Hom}_{\mathrm{sSet}}(\Lambda^1_2, -)$ represents the set of "composable" edges $x \xrightarrow{f} y \xrightarrow{g} z$ in any simplicial set.

Proposition 2.1.3. A simplicial set $X_\bullet$ is the nerve of a 1-category if and only if it admits *unique inner horn filling*, meaning for every n and every $0 < k < n$, given any map $\Lambda^k_n \to X_\bullet$ it admits a unique lift:

$$\begin{tikzcd} \Lambda^k_n \arrow[r] \arrow[d] & X_\bullet \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Example 2.1.4. Let X be a topological space. Then it gives rise to a simplicial set called its *fundamental ∞-groupoid* $\Pi_\infty X$, with the data

▷ 0-simplices = points $x \in X$
▷ 1-simplices = paths x to y in X
▷ 2-simplices = homotopies between paths
▷ 3-simplices = homotopies between homotopies between paths

$\vdots$

Note that a 2-cell is no longer unique! There can be many homotopies between paths. In particular composition of paths isn't well-defined, in the sense that many paths can function naturally as a composite. We might define $g \circ f$ to be any path together with a 2-cell making the diagram commute:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed] & & z. \end{tikzcd}$$

In order to specify a composite now, we need to give the data not only of the 1-cell but also of the 2-cell! This is the vibe of higher-categorical composition. Note that horns don't fill uniquely here.

Exercise 2.1.5. If you're familiar with the singular chains construction

$$|-|\colon \mathrm{sSet} \leftrightarrows \mathrm{Top} : \mathrm{Sing}(-),$$

convince yourself that $\mathrm{Sing}(-)$ is the same as $\Pi_\infty(-)$.

Definition 2.1.6. A *quasicategory* is any simplicial set with (not necessarily unique) inner horn filling. We denote by $\mathrm{qCat} \subseteq \mathrm{sSet}$ the full subcategory on the quasi-categories.

A natural question to ask is to what extent there is ambiguity in composition – how many choices do we have for horn filling? Do different choices *mean* different things? The following proposition answers this to some extent.

Proposition 2.1.7. (Joyal) If $\mathscr{C}$ is a quasi-category, then the map of simplicial sets

$$\mathrm{Fun}(\Delta^2, \mathscr{C}) \to \mathrm{Fun}(\Lambda^1_2, \mathscr{C})$$

has contractible fibers (i.e., the geometric realization of the fibers under this map are contractible spaces).[1]

Definition 2.1.8. A *Kan complex* is a quasi-category which also has *outer horn filling*, meaning we have a lift

$$\begin{tikzcd} \Lambda^k_n \arrow[r] \arrow[d] & X_\bullet \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

not only for $0 < k < n$, but also for $k = 0, n$. For $n = 2$, this means we are also allowed to fill the horns:

$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[rr] & & \bullet \end{tikzcd}$$

Exercise 2.1.9. Show a Kan complex is the nerve of a 1-groupoid if and only if its inner horn filling is unique.

The homotopy hypothesis: The functor

$$(\mathrm{Top}, \text{weak equiv}) \to (\mathrm{Kan}, \text{weak equiv})$$
$$X \mapsto \Pi_\infty(X)$$

[1] In fact $\mathscr{C}$ is a quasi-category if and only if this holds.

yields an equivalence of ∞-categories.[2] Hence we can think about spaces as Kan complexes without much loss of generality. We use $\mathcal{S}$ to denote the ∞-category of spaces.

Remark 2.1.10. (On other models of ∞-categories) There are a ton of different models of ∞-categories, we stick with quasi-categories since they have become the standard to some extent. The main advantage is that functors are much easier to describe – they are just maps of the underlying simplicial sets. Functor ∞-categories $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ between two quasi-categories are simply given by the internal hom in simplicial sets.

Proposition 2.1.11. If $\mathscr{C}$ is a quasi-category and S is any simplicial set, then the internal hom $\mathrm{Fun}(S, \mathscr{C})$ is also a quasi-category. We think about this as "S-shaped diagrams in $\mathscr{C}$."

2.1.1 Homotopy in an ∞-category

Definition 2.1.12. Let $\mathscr{C}$ be a quasi-category. We define its *homotopy category*, denoted $h\mathscr{C}$ to be the category freely generated by the 1-truncation $\tau_{\leq 1}\mathscr{C}$ (i.e. objects and edges), modulo the relations coming from 2-simplices.

Definition 2.1.13. We say that a morphism $f\colon x \to y$ in a quasi-category $\mathscr{C}$ is an *isomorphism/equivalence* if there exists some $g\colon y \to x$ so that $[gf] = \mathrm{id}_x$ and $[fg] = \mathrm{id}_y$ in $h\mathscr{C}$. Note that g is *not uniquely defined*, unlike in ordinary 1-category theory.

Remark 2.1.14. We should think about this less like isomorphisms in 1-categories, and more like homotopy equivalences in topology.

Example 2.1.15. A morphism $f\colon X \to Y$ between CW complexes is a (weak) homotopy equivalence if and only if $[f]$ is an isomorphism in $h\mathrm{Top}$. Hence the "isomorphisms" in the ∞-category of spaces are not homeomorphisms but rather homotopy equivalences.

Proposition 2.1.16. There is an adjunction[3]

$$h\colon \mathrm{qCat} \leftrightarrows \mathrm{Cat} : N.$$

Remark 2.1.17. We recall that the nerve is fully faithful. This is equivalent to the counit of the adjunction being a natural isomorphism:

$$hN\mathscr{C} \xrightarrow{\sim} \mathscr{C},$$

in other words any 1-category can be recovered as the homotopy category of its nerve.

Example 2.1.18. Let R be any ring. Then its category of chain complexes $\mathrm{Ch}(R)$ is naturally an ∞-category, and the notion of homotopy recovers the idea of chain homotopy equivalence. We have to be careful constructing this explicitly, refer to §13 of DAG for more info.

[2] We haven't defined what we mean by Top or Kan as an ∞-category, and it's a bit subtle. We want to incorporate the weak equivalences, so really we should take the hammock localization of Kan at the simplicial weak equivalences, then take its homotopy coherent nerve, but fibrantly replace $L^W\mathrm{Kan}$ first before taking N_Δ so that the resulting simplicial set is an honest quasi-category. An analogous procedure should be carried out with topological spaces, assuming we work with all spaces and not just CW complexes.

[3] If we consider the codomain of the nerve construction to be all simplicial sets, it still admits a left adjoint called the homotopy category, however it is not given by the formula in Definition 2.1.12. Technically there is a more general construction $h\colon \mathrm{sSet} \to \mathrm{Cat}$, which is left adjoint to N, and which agrees with Definition 2.1.12, which we should call $\tau_{\leq 1}$, when the simplicial set is a quasi-category. We'll only apply the homotopy category construction to quasi-categories here so this distinction won't matter.

2.1.2 Mapping spaces

We want to make precise the model of quasi-categories as $(\infty, 1)$-categories. The vibe of higher categories is that homs in 1-categories are 0-categories (sets). Homs in 2-categories are 1-categories, homs in 3-categories are 2-categories, etc. Hence homs in $(\infty, 1)$-categories should be $(\infty, 0)$-categories. From the models we're working in:

$$\begin{array}{rcl} (\infty,1)\text{-categories} & = & \text{quasi-categories} \\ (\infty,0)\text{-categories} & = & \text{Kan complexes,} \end{array}$$

hence we want to argue that, for any quasicategory $\mathscr{C}$, and any pair of objects (0-simplices) $x, y \in \mathscr{C}$, there is a mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$ which is a Kan complex.

Definition 2.1.19. For $x, y \in \mathscr{C}$, where $\mathscr{C}$ is a quasicategory, we denote by $\mathrm{Map}_{\mathscr{C}}(x, y)$ the pullback in simplicial sets:

$$\begin{tikzcd} \mathrm{Map}_{\mathscr{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathscr{C}) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ \{x,y\} \arrow[r] & \mathscr{C} \times \mathscr{C}. \end{tikzcd}$$

Here $\mathrm{Fun}(\Delta^1, \mathscr{C})$ denotes an internal hom from the interval Δ^1 to $\mathscr{C}$. The rightmost vertical map is what's called a *bifibration* (the proof that this map is a bifibration is [Lur09, 2.4.7.11]), which in particular means that $\mathrm{Map}_{\mathscr{C}}(x, y)$ is a Kan complex.

Intuition 2.1.20. The mapping space is intended to generalize the idea of the homotopy category, in the sense that

$$\pi_0 \mathrm{Map}_{\mathscr{C}}(x, y) = \mathrm{Hom}_{h\mathscr{C}}(x, y).$$

In particular its connected components correspond to homotopy classes of maps between x and y, but it remembers more information about *how* the homotopies were witnessed, encoded in the higher homotopy type of $\mathrm{Map}_{\mathscr{C}}(x, y)$.

Remark 2.1.21. Alternatively we may define $\mathrm{Map}_{\mathscr{C}}(x, y)$ as the simplicial set whose n-simplices are given by the set of all

$$z \colon \Delta^{n+1} \to \mathscr{C},$$

with the $\{n+1\}$-vertex mapping to y, and the vertices $\{0, \ldots, n\}$ mapping to x. Technically speaking this is the space of *right morphisms* but when $\mathscr{C}$ is an ∞-category this models the mapping space (it is canonically isomorphic in the homotopy category). As an exercise, verify that $\mathrm{Map}_{\mathscr{C}}(x, y)$ is indeed a Kan complex from the definition.

What is $\mathrm{Map}_{\mathscr{C}}(x, y)$ intended to capture? Its 0-simplices are maps from x to y *in the homotopy category* $h\mathscr{C}$. In other words, they are equivalence classes of zig-zags of morphisms in $\mathscr{C}$ from x to y, where maps going the wrong way are all invertible.

Notation 2.1.22. If $\mathscr{C}$ is an ∞-category and $x, y \in \mathscr{C}$, we denote by

$$[x, y] := \pi_0 \mathrm{Map}_{\mathscr{C}}(x, y).$$

We call this *homotopy classes of maps* from x to y.

Warning 2.1.23. It is not true that $[x, y]_{\mathscr{C}}$ is simply the edges from x to y in the quasi-category $\mathscr{C}$ modulo an equivalence relation, it is more subtle. In the presence of a model structure, we can replace x and y by equivalent objects Qx and Ry respectively, so that $[x, y] \cong [Qx, Ry]$, and this latter set can be literally identified with the 1-cells $Qx \to Ry$ in $\mathscr{C}$ modulo an explicit equivalence relation. We'll come back to this when we talk about sheaves.

Example 2.1.24. If $\mathscr{C}$ is a 1-category, we can view it trivially as an ∞-category via the nerve construction. In this case $\mathrm{Map}_{\mathscr{C}}(x,y) = \mathrm{Hom}_{\mathscr{C}}(x,y)$ is just a set (a discrete simplicial set). The homotopy category of $\mathscr{C}$ is just $\mathscr{C}$, because there are no equivalences which aren't isomorphisms.

2.1.3 Presentable ∞-categories

Modulo some set-theoretic technicalities, we can now be content with the existence of a model for infinity-categories. All notions of functors, colimits, adjunctions, etc. should now be understood in the higher categorical sense, i.e. up to higher coherence.

Definition 2.1.25. [Lur09, 5.4.2.1] An ∞-category is *accessible* if it is generated under κ-filtered colimits by a small category.

Example 2.1.26. The category $\mathcal{S}$ of spaces is accessible, since it admits all colimits and every space is built out of finite CW complexes.

Remark 2.1.27. By [Lur09, 5.4.3.6], a small ∞-cat is accessible if and only if it is idempotent complete.[4] Observe that the category of finitely generated free R-modules fail to contain retracts (projectives) so they're not idempotent complete and hence not accessible.

Definition 2.1.28. Given any ∞-category $\mathscr{C}$, we denote by $\mathrm{PSh}(\mathscr{C}) := \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S})$ the category of (∞-)*presheaves*.

Example 2.1.29. We can think about presheaves of simplicial sets $\mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{sSet})$ as the presheaf category $\mathrm{PSh}(\mathrm{Sch}_S)$. Note what lives in here:

1. Schemes all live in here via the Yoneda embedding $\mathrm{Sch}_S \hookrightarrow \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{sSet})$, by viewing $\mathrm{Set} \subseteq \mathrm{sSet}$ as discrete simplicial sets (no non-degenerate n-simplices for $n \geq 1$).
2. *Simplicial schemes* also live in here, by moving some adjoint stuff around:
$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sch}_S) \overset{y}{\hookrightarrow} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{Set})) \cong \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})) = \mathrm{PSh}(\mathrm{Sch}_S).$$
3. Spaces (viewed as simplicial sets by Π_∞) live in here as constant presheaves $\mathrm{sSet} \hookrightarrow \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathrm{sSet})$.

Thus we have a natural home for schemes and spaces, as well as these simplicial scheme data types we've been looking at.

Definition 2.1.30. We say an ∞-category $\mathscr{C}$ is *presentable* if it is accessible and admits all colimits (cocomplete).

Example 2.1.31. By the previous two examples, $\mathrm{PSh}(\mathscr{C})$ is presentable for any $\mathscr{C}$. This is the coYoneda lemma — that any presheaf is a colimit of representable ones.

Theorem 2.1.32. *(Adjoint functor theorem)* Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor between presentable categories. Then

- ▷ F admits a right adjoint if and only if F preserves all colimits
- ▷ F admits a left adjoint if and only if it preserves all limits and κ-filtered colimits

Really hard to write down functors explicitly in quasi-categories, since we are writing down a map of simplicial sets, which is a lot of data. AFT is nice because it lets us get functors without writing them explicitly, but they are still characterized by being adjoints.

[4] Idempotent complete has a number of definitions, in particular it implies that idempotent endomorphisms $f\colon X \to X$ (i.e. $f \circ f = f$) correspond bijectively to retracts of X, i.e. composites $Y \hookrightarrow X \to Y$. If $\mathscr{C}$ is idempotent complete then it is closed under retracts.

Notation 2.1.33. We denote by Pr^L the category of presentable ∞-categories and colimit-preserving functors between them. Note every functor in Pr^L is a left adjoint.

Theorem 2.1.34. Every presentable category is complete (admits all limits).

2.1.4 Localization

Definition 2.1.35. [Lur09, 5.2.7.2] A functor $f\colon \mathscr{C} \to \mathscr{D}$ is a *localization* if it admits a fully faithful right adjoint.

In many cases a localization is given by inverting a class of morphisms in $\mathscr{C}$. In particular let $S \subseteq \mathrm{mor}\mathscr{C}$ be a class of morphisms in $\mathscr{C}$, then we can try to *invert* S by cooking up a new category $\mathscr{C}[S^{-1}]$.

Example 2.1.36.

1. The *homotopy category* of spaces is obtained from the category of compactly generated weakly Hausdorff spaces by inverting all homotopy equivalences.
2. The *derived category* of a ring is obtained from the category of chain complexes by inverting the chain homotopy equivalences.
3. A group (as a one-object groupoid) is obtained from a monoid by freely inverting each morphism.

Definition 2.1.37. [Lur09, 5.5.4.1] Let $S \subseteq \mathrm{mor}\mathscr{C}$. We say $z \in \mathscr{C}$ is *S-local* if for every $s\colon x \to y$ in S, the induced map

$$\mathrm{Map}_{\mathscr{C}}(y, z) \to \mathrm{Map}_{\mathscr{C}}(x, z)$$

is an equivalence.

Remark 2.1.38. Let $\mathscr{C}_0 \subseteq \mathscr{C}$ be the full subcategory of S-local objects. If this admits a left adjoint, it makes sense to call that adjoint L_S, that is, S-localization, since it inverts every morphism in S. *This is where presentable categories give us an advantage.* In general arguing for the existence of a left adjoint isn't easy, however if $\mathscr{C}$ is presentable, then the adjoint functor theorem tells us that we just have to check the inclusion $\mathscr{C}_0 \subseteq \mathscr{C}$ preserves limits and filtered colimits.

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Proposition 2.1.39. [Lur09, 5.5.4.15] If $\mathscr{C}$ is presentable and $S \subseteq \mathrm{mor}\mathscr{C}$ is small, then the inclusion of the full subcategory of S-local objects admits a left adjoint.[5]

Example 2.1.40. In the next talk, our primary application of this machinery will be looking at the presheaf category $\mathrm{PSh}(\mathscr{C})$, which is presentable by Example 2.1.31. We can look at full subcategories of presheaves which satisfy a certain sheaf condition and argue this is a reflective subcategory hence we will have an adjoint we call *sheafification*.

Remark 2.1.41. Given a class of arrows $S \subseteq \mathrm{mor}\mathscr{C}$, we can always form $\mathscr{C}[S^{-1}]$ by adjoining formal inverses to S and considering all composites of morphisms in $\mathscr{C}$ and formal inverses (zig-zags). This is called *Dwyer–Kan localization* or *hammock localization*. This satisfies the correct universal property of localization, but we might encounter size issues. Bousfield localization is a particular example of Dwyer–Kan localization, but where we are able to guarantee that we don't encounter any size issues since the localization is a subcategory of the original category.

[5] The terminology for this is that $\mathscr{C}_0 \subseteq \mathscr{C}$ is a *reflective subcategory*.

Example 2.1.42. We define $\mathcal{S}$ to be the Dwyer–Kan localization of the category Top of compactly generated weakly Hausdorff spaces at the weak homotopy equivalences. This has the property that $h\mathcal{S} = \mathrm{Ho}(\mathrm{Top})$. See [Lur09, §1.2.16] for more information. As a model category we are invited to think about $\mathcal{S}$ as

1. simplicial sets with the classical model structure
2. Kan complexes with the classical model structure
3. topological spaces with the classical model structure

Hence we think about $\mathrm{PSh}(\mathscr{C}) = \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S})$ as simplicial presheaves, equipped with a *levelwise* notion of weak equivalence, coming from weak homotopy equivalence of simplicial sets.

2.2 Descent

Goal 2.2.1. Define the ∞-topos of Nisnevich sheaves $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k)$.

Assumption 2.2.2. We will work over a base scheme S which is qcqs and Noetherian.

2.2.1 Descent, higher categorically

Very roughly speaking, a *sheaf* is a presheaf that glues along covers. We're going to give a general definition, then show how it recovers what we know and remember.

Notation 2.2.3. Suppose $\mathcal{U} = \{U_i \to X\}_i$ is a cover in Sch_S, giving rise to a Čech nerve $\Delta^{\mathrm{op}} \to \mathrm{Sch}_S$. Then if $\mathscr{C}$ is any ∞-category and $F\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathscr{C}$ a presheaf, then we denote by $F(\mathcal{U}) \in \mathrm{Fun}(\Delta, \mathscr{C})$ the cosimplicial object given by applying F everywhere in the Čech nerve.[6]

Definition 2.2.4. Let $\mathscr{C}$ be an ∞-category with all limits, let (Sch_S, τ) be a site, and let $F\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathscr{C}$ be a presheaf valued in $\mathscr{C}$. Then we say F is a *τ-sheaf* if for every τ-cover $\mathcal{U} = \{U_i \to X\}$, we have that the induced map

$$F(X) \to \lim_{\Delta} F(\mathcal{U})$$

is an equivalence.

Example 2.2.5. If $\mathscr{C}$ is a 1-category, e.g. sets, then a higher categorical limit just recovers the notion of an ordinary limit, since there is no higher structure. In particular, the limit reduces to seeing that the pair of parallel morphisms $[0] \rightrightarrows [1]$, viewed as a subcategory of Δ, is *final* (see Example 2.3.7). In this case the sheaf condition reduces to asking whether the map

$$F(X) \to \lim \left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \right)$$

is an equivalence, i.e. it witnesses $F(X)$ as a 1-categorical limit (in particular, an equalizer).

Example 2.2.6. If $\mathscr{C}$ is an abelian category, then the equalizer of two maps is just the kernel of their difference, so we get the familiar sheaf condition that

$$0 \to F(X) \to \prod F(U_i) \to \prod_{i,j} F(U_{ij})$$

is left exact.

[6] If F is product-preserving this is immediate, if not we have to apply F at each level and then take products.

Example 2.2.7. If $\mathscr{C}$ is a 2-category, then $\Delta^{\mathrm{inj}}_{\leq 2} \subseteq \Delta$ is 2-final (reference needed), so we get that the sheaf condition becomes

$$F(X) \to \lim \left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \rightrightarrows \prod_{i,j,k} F(U_{ijk}) \right)$$

If $\mathscr{C} = \mathrm{Grpd}$ is the category of groupoids, viewed as an ∞-subcategory of qCat via the nerve construction, then this is precisely the stack condition! So this is what we meant when we said "sheaf of categories" earlier.

Remark 2.2.8. These limits are not 1-categorical limits, they are taking place in a higher categorical sense. The following example is worth thinking about as it makes this more concrete.

Example 2.2.9. Let R be a ring, and $\langle f, g\rangle = R$ two objects generating the unit ideal, giving rise to a two-object cover $\{\mathrm{Spec}\,(R_f) \to \mathrm{Spec}\,(R),\ \mathrm{Spec}\,(R_g) \to \mathrm{Spec}\,(R)\}$.

1. Argue that the stack condition for this particular cover truncates at the double overlaps, since there are no interesting triple overlaps.
2. See that $\mathcal{F}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpd}$ satisfies descent for this cover if and only if

$$\begin{tikzcd} \mathcal{F}(\mathrm{Spec}\,(R)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}(\mathrm{Spec}\,(R_f)) \arrow[d] \\ \mathcal{F}(\mathrm{Spec}\,(R_g)) \arrow[r] & \mathcal{F}(\mathrm{Spec}\,(R_{fg})) \end{tikzcd}$$

   is a *pullback of groupoids*.
3. As a particular example, show that $\mathrm{Mod}(-)$ satisfies descent for two-object Zariski covers. That is, $\mathrm{Mod}(R)$ is equivalent to the 2-categorical pullback, often called the *category of descent data* attached to the cover.

2.2.2 cd-structures

We saw in the previous example how a sheaf condition can simplify on covers with fewer objects. A natural question to ask would be whether descent along a small collection of covers implies descent along all covers. A formalism that often lets us deal with this is the idea of a *cd-structure*. We're also going to zoom in on presheaves of spaces.

Definition 2.2.10. A *cd-structure* is a collection of commutative squares in $\mathscr{C}$ closed under isomorphism:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D. \end{tikzcd}$$

Terminology 2.2.11. Given a cd-structure on Sch_S, we define its associated topology τ to be the coarsest topology for which $\{B \to D,\ C \to D\}$ is a τ-cover for every distinguished square.

Example 2.2.12. The *Zariski cd-structure* is defined by commutative squares of the form

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U \cup V. \end{tikzcd}$$

We will call these squares *distinguished Zariski squares.*

Theorem 2.2.13. Let $\mathcal{F} \in \mathrm{PSh}(\mathrm{Sch}_S^{\mathrm{op}})$ be a presheaf. Then $\mathcal{F}$ is a Zariski sheaf if and only if $\mathcal{F}(\varnothing) = *$ and $\mathcal{F}$ sends every distinguished Zariski square to a (homotopy) pullback square.

Example 2.2.14. The Nisnevich topology is generated by a cd-structure given by *distinguished Nisnevich squares*, of the form

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"'] & X, \end{tikzcd}$$

where i is an open immersion, p is étale, and p restricts to an isomorphism $p^{-1}(X - U) \to X - U$.

Example 2.2.15. *(Affine distinguished Nisnevich square).* Suppose $f : R \to S$ is a finite étale ring homomorphism, and $h \in R$ is some element for which $R/x \cong S/f(x)$ is a ring isomorphism. Then we have a distinguished Nisnevich square:

$$\begin{tikzcd} \mathrm{Spec}\,(S_{f(x)}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spec}\,(S) \arrow[d] \\ \mathrm{Spec}\,(R_f) \arrow[r] & \mathrm{Spec}\,(R). \end{tikzcd}$$

Proof. The right map is étale, the bottom is an open immersion, and the restriction of $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ to the complement of the open distinguished affine $D(f) \subseteq \mathrm{Spec}\,(R)$ is the hypyothesis that $R/x \cong S/f(x)$. □

Definition 2.2.16. We say $\mathcal{F} \in \mathrm{PSh}(\mathrm{Sm}_S)$ is a *Nisnevich sheaf* if $\mathcal{F}(\varnothing) = *$ and $\mathcal{F}$ sends every distinguished Nisnevich square to a pullback square.

Exercise 2.2.17. Show that $\mathrm{Mod}(-)$ is a Nisnevich sheaf on the site of affine schemes.

2.2.3 Sheafification and accessible localizations

Definition 2.2.18. [Lur09, 5.4.2.5] We say a functor between accessible ∞-categories is *accessible* if it is κ-continuous (i.e., preserves κ-limits by a regular cardinal κ).

Definition 2.2.19. If $\mathscr{C} \subseteq \mathscr{D}$, then we say a localization $L \colon \mathscr{D} \to \mathscr{C}$ is *accessible* if and only if the composite $\mathscr{D} \xrightarrow{L} \mathscr{C} \hookrightarrow \mathscr{D}$ is accessible. If $\mathscr{D}$ is an accessible category, this is equivalent to the statement that $\mathscr{C} \subseteq \mathscr{D}$ is an accessible subcategory [Lur09, 5.5.4.2].

Definition 2.2.20. [Lur09, 6.1.0.4] If $\mathscr{X}$ is an ∞-category, we say it is an ∞*-topos* if there exists a small category $\mathscr{C}$ and an accessible left exact localization functor $\mathrm{PSh}(\mathscr{C}) \to \mathscr{X}$.

This is some higher categorical analogue of the fact from topos theory that every (Grothendieck) topos is the category of sheaves of sets on a site.

Proposition 2.2.21. [Lur09, 6.2.2.7] If $\mathscr{C}$ is a (small) ∞-category with a Grothendieck topology, then $\mathrm{Shv}_\tau(\mathscr{C})$ is an accessible left exact localization of $P(\mathscr{C})$, in particular it is an ∞-topos.

Proof. The precise statement of [Lur09, 6.2.2.7] is that $\mathrm{Shv}_\tau(\mathscr{C}) \subseteq \mathrm{PSh}(\mathscr{C})$ is a so-called *topological* localization (defined in [Lur09, 6.2.1.4]). In [Lur09, 6.2.1.6] it is proved that every topological localization of a presentable ∞-category is accessible and left exact. □

Terminology 2.2.22. We refer to the localization functor

$$L_\tau \colon \mathrm{PSh}(\mathscr{C}) \leftrightarrows \mathrm{Shv}_\tau : i$$

as *sheafification.*

Proposition 2.2.23. [Lur09, 6.2.2.17] If $\mathscr{C}$ is a small ∞-category there is a bijection between Grothendieck topologies on $\mathscr{C}$ and (equivalence classes of) topological localizations of $\mathrm{PSh}(\mathscr{C})$.

2.2.4 About the sheaf topos

Corollary 2.2.24. Some consequences:

1. The *sheafification* functor $L_\tau \colon \mathrm{PSh}(\mathscr{C}) \to \mathrm{Shv}_\tau(\mathscr{C})$ preserves all colimits (being a left adjoint) and all small limits (being left exact).
2. The inclusion functor $i \colon \mathrm{Shv}_\tau(\mathscr{C}) \hookrightarrow \mathrm{PSh}(\mathscr{C})$ preserves all limits and all filtered colimits, by the adjoint functor theorem.

Explicitly, by this second point, we have that *limits and filtered colimits of sheaves can be computed in the presheaf category.* This is a crucial fact.

Example 2.2.25. (Examples of τ-sheaves):

1. Any τ-sheaf of *sets* is a τ-sheaf of discrete spaces. So we have
$$\mathrm{Sh}_\tau(\mathscr{C}; \mathrm{Set}) \subseteq \mathrm{Sh}_\tau(\mathscr{C}).$$
2. If τ is subcanonical, then the representable presheaf
$$h_X := \mathrm{Hom}_{\mathscr{C}}(-, X) \colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$$
is a sheaf of sets, and hence a sheaf of spaces.
3. Consider any presheaf of groupoids
$$\mathscr{F} \colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Grpd}.$$
We denote by $B\mathscr{F}$ the composite $\mathscr{C}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathrm{Grpd} \xrightarrow{N} \mathrm{sSet}$. Then $\mathscr{F}$ is a τ-stack if and only if $B\mathscr{F}$ is a τ-sheaf [Hol08, 3.9].
4. If $Y \in \mathcal{S}$ is any space, we may sheafify the constant presheaf valued at Y in order to obtain a sheaf $\underline{Y}$.

Terminology 2.2.26. From the canonical equivalence

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Set})) \cong \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{sSet}),$$

simplicial objects in set-valued presheaves naturally give rise to ∞-categorical presheaves. Consider an object on the left, of the form $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{PSh}(\mathscr{C}; \mathrm{Set})$. If this functor factors through $\mathrm{Sh}_\tau(\mathscr{C}; \mathrm{Set}) \subseteq \mathrm{PSh}(\mathscr{C}; \mathrm{Set})$, that is if X_n is a sheaf of sets for each n, we call this a *simplicial object in sheaves.*

Warning 2.2.27. (c.f. [Lur09, 7.1.3.1]) In the literature, we can find objects $X_\bullet$ of the form in Terminology 2.2.26 referred to as “simplicial sheaves.” This is overloaded terminology, and suggests that ∞-categorical sheaves are just those presheaves of simplicial sets which are levelwise sheaves of sets, an erroneous claim that can be found throughout the literature.

2.2.5 Slice categories

Recall if $\mathscr{C}$ is any (∞-)category and $x \in \mathscr{C}$, we have the under and over categories $\mathscr{C}_{x/}$ and $\mathscr{C}_{/x}$, respectively. If $* \in \mathscr{C}$ is a terminal object, then it becomes both initial and terminal in $\mathscr{C}_{*/}$, that is, it is a zero object. We call a category with a zero object *pointed.*

Proposition 2.2.28. We have that $\Delta^0 \in \mathcal{S}$, viewed as a constant sheaf, is terminal in the sheaf topos $\mathrm{Shv}_\tau(\mathscr{C})$ for any site.

Proof. The inclusion $\mathrm{Sh}_\tau(\mathscr{C}) \subseteq \mathrm{PSh}(\mathscr{C})$ will preserve terminal objects, being a limit over an empty diagram, so it suffices to observe that the constant presheaf Δ^0 is already a sheaf. $\square$

Proposition 2.2.29. Let $h_S \in \mathrm{PSh}(\mathrm{Sm}_S)$ denote the representable sheaf attached to the base. Then the map $h_S \xrightarrow{!} \Delta^0$ of presheaves is a *local equivalence*, i.e. it sheafifies to an equivalence.

Proof. It suffices to observe that the presheaves are identical, which is true because for any $U \in \mathrm{Sm}_S$, we have that $h_S(U) = \mathrm{Hom}_S(U, S)$, which is a one-object set consisting of the structure map (since S is terminal in Sm_S). $\square$

Notation 2.2.30. We denote by $\mathrm{Shv}_\tau(\mathscr{C})_* := \mathrm{Shv}_\tau(\mathscr{C})_{\Delta^0/}$ the pointed slice category. An object here is a map of sheaves $\Delta^0 \to F$, which we observe is equivalent to picking a basepoint in $F(U)$ for every $U \in \mathscr{C}$. Hence we can think of these as sheaves of *pointed* spaces.

2.2.6 Connectivity

Definition 2.2.31. Let $(X, x) \in \mathrm{Shv}_\tau(\mathscr{C})_*$ and $n \geq 0$. Then we denote by $\pi_n(X, x)$ the nth *homotopy sheaf*, defined to be the τ-sheafification of the presheaf of sets

$$\begin{aligned} \mathscr{C}^{\mathrm{op}} &\to \mathrm{Set} \\ U &\mapsto \pi_n(X(U), x). \end{aligned}$$

Remark 2.2.32. There is a more intrinsic definition of π_n, leveraging that an ∞-topos is cotensored over spaces,[7] and considering the map $X^{S^n} \to X$ in the slice topos $\mathrm{Shv}_{/X}$, and defining π_n to be its 0-truncation.

Proposition 2.2.33. We have that

1. π_0 is a sheaf of sets
2. π_1 is a sheaf of groups
3. π_n is a sheaf of abelian groups for $n \geq 2$.

Notation 2.2.34. If $\mathscr{C}$ is any pointed ∞-category admitting limits, and $X \in \mathscr{C}$, we denote by ΩX its *loop space*, defined as the pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & X. \end{tikzcd}$$

The loop space interacts with the homotopy groups in a topos in the way we might expect from topology:

Proposition 2.2.35. If $\mathscr{C}$ is an ∞-topos and $(X, x) \in \mathscr{C}_*$, then

$$\pi_n(X, x) = \pi_0(\Omega^n(X, x)).$$

[7]This means that for any $F \in \mathrm{Shv}_\tau(\mathrm{Sch}_S)$ and any $X \in \mathrm{sSet}$, we have a natural object $F^X \in \mathrm{Shv}_\tau(\mathrm{Sch}_S)$ with natural equivalences of mapping spaces

$$\mathrm{Map}_{\mathrm{Shv}}(G, F^X) \cong \mathrm{Map}_{\mathrm{sSet}}(X, \mathrm{Map}_{\mathrm{Shv}}(G, F)).$$

Proof. (reference needed) □

Example 2.2.36. Let $\mathscr{F}\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Set}$ be any sheaf of sets, groups, abelian groups, etc. Then

$$\pi_n(\mathscr{F}) = \begin{cases} \mathscr{F} & n = 0 \\ * & n > 0. \end{cases}$$

Proof. It suffices to observe that the presheaf $\pi_n\mathscr{F}$ is identical to the constant presheaf sheaf Δ^0 since $\mathscr{F}(U)$ has no higher homotopy for each $U \in \mathrm{Sm}_S$. □

Definition 2.2.37. [Lur09, 6.5.1.10] Let $\mathscr{C}$ be an ∞-topos, and take $X \in \mathscr{C}$.

1. We say X is *n-connective* if $\pi_k(X, x) = *$ for every $k < n$ and for every basepoint x.
2. We say that X is *connected* if it is 1-connective, meaning $\tau_{\leq 0}X = *$.
3. We say X is *n-truncated* if $\pi_k(X, x) = *$ for all $k > n$ and for every basepoint x.

Proposition 2.2.38. [Lur09, 5.5.6.18] Let $\mathscr{C}$ denote a presentable category and $\tau_{\leq k}\mathscr{C} \subseteq \mathscr{C}$ the full subcategory spanned by k-truncated objects. Then the inclusion admits an accessible left adjoint, in other words there is a truncation functor $\tau_{\leq k} : \mathscr{C} \to \tau_{\leq k}\mathscr{C}$ which is a localization.

In particular, the collection of n-truncated spaces is closed under limits.

Definition 2.2.39. If $(\mathscr{C}, \tau)$ is any site, then the category $\tau_{\leq 0}\mathrm{Sh}_\tau(\mathscr{C}) =: \mathrm{Sh}_\tau(\mathscr{C})_{\leq 0}$ is the category of sheaves of sets $\mathrm{Sh}_\tau(\mathscr{C}; \mathrm{Set})$.

Notation 2.2.40. We write $\mathrm{Ab}(\mathrm{Sh}_\tau(\mathscr{C})_{\leq 0})$ for the sheaves of abelian groups.

2.2.7 Whitehead's theorem

Theorem 2.2.41. (Whitehead's theorem) Let $f\colon X \to Y$ be a map of CW complexes. Then f is a homotopy equivalence if and only if $f\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for each $i \geq 0$.

A way to say this is that "∞-connective morphisms are homotopy equivalences." The ∞-categorical analogue of this is the notion of *hypercompleteness*.

Definition 2.2.42. An ∞-topos is *hypercomplete* if and only if every object is ∞-connective.

Definition 2.2.43. [Lur09, 7.2.11] An ∞-topos has *homotopy dimension* $\leq n$ if every $(n-1)$-connected object X receives a map from the terminal object $* \to X$.

Theorem 2.2.44. If an ∞-topos has finite homotopy dimension then it is hypercomplete.

Warning 2.2.45. The étale topos $\mathrm{Shv}_{\mathrm{et}}(\mathrm{Sm}_S)$ need not be hypercomplete, even for nice S (reference needed).

Theorem 2.2.46. (Voevodsky) If S is qcqs and Noetherian, then the homotopy dimension of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$ is bounded above by $\dim(S)$. In particular this implies it is hypercomplete.

Truncation and connectivity fit into fiber sequences

$$\tau_{>n} \to \mathrm{id} \to \tau_{\leq n} \tag{2.47}$$

which allow us to form Postnikov towers out of objects of ∞-topoi:

$$\begin{tikzcd} & \tau_{\leq n}X \arrow[d] \\ & \vdots \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \tau_{\leq 0}X \end{tikzcd}$$

Hypercompleteness implies *Postnikov completeness* (reference needed) meaning that the induced map

$$X \to \lim_n \tau_{\leq n}X$$

is an equivalence. That is, the Postnikov towers converge and we can make sense of obstruction theory.

We'll see soon that this is a valuable perspective, e.g. for classifying torsors via homotopy theory. However the obstruction theory won't be very useful in the sheaf topos setting. This is one of the advantages we gain by passing to motivic spaces.

Upshot 2.2.48. In a hypercomplete topos, equivalences can be checked on homotopy sheaves.

Corollary 2.2.49. Let $F \to G$ be a map in the sheaf topos $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S)$ (or any hypercomplete topos) where S is qcqs and Noetherian. Then the following are equivalent:

1. The map $F \to G$ is an equivalence in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$
2. The induced maps $\pi_n(F) \to \pi_n(G)$ are isomorphisms for every $n \geq 0$.

So this gives us a way to check equivalence on homotopy groups.

Notation 2.2.50. For $F, G \in \mathrm{Shv}_\tau(\mathrm{Sch}_S)$, we denote by

$$[F, G]_\tau := \pi_0 \mathrm{Map}_{\mathrm{Shv}_\tau(\mathrm{Sch}_S)}(F, G).$$

Remark 2.2.51. If F, G are presheaves, then their mapping space in presheaves and the mapping space of their associated sheafifications are quite different. We notice that if G is already a sheaf, then the unit $G \to iL_{\mathrm{Nis}}G$ is an equivalence, inducing an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{PSh}(\mathrm{Sch}_S)}(F, G) \cong \mathrm{Map}_{\mathrm{PSh}(\mathrm{Sch}_S)}(F, iL_{\mathrm{Nis}}G) \cong \mathrm{Map}_{\mathrm{Shv}_\tau(\mathrm{Sch}_S)}(L_{\mathrm{Nis}}F, L_{\mathrm{Nis}}G).$$

We think of this as a higher categorical analogue of the universal property of sheafification: that a map from a presheaf to a sheaf factors uniquely through its sheafification.

2.2.8 Equivalences of sheaves and presheaves

Recall that if $f\colon F \to G$ is a map in $\mathrm{PSh}(\mathrm{Sch}_S)$, then it is an equivalence if and only if $F(U) \to G(U)$ is an equivalence of spaces for each $U \in \mathrm{Sch}_S$, also called a *sectionwise equivalence*. If $f\colon F \to G$ is an equivalence, then so is the associated map after sheafification. The converse doesn't hold though, since different presheaves can admit the same sheafification. To that end, we introduce a new notion.

Definition 2.2.52. Let $F, G \in \mathrm{PSh}(\mathrm{Sch}_S)$ be presheaves, and let $f\colon F \to G$ be a map of presheaves. We say that f is a *local equivalence* if

$$L_{\mathrm{Nis}}(f)\colon L_{\mathrm{Nis}}F \to L_{\mathrm{Nis}}G$$

is an equivalence of sheaves.

Example 2.2.53. The unit of the sheafification adjunction $L_{\text{Nis}} \dashv i$ has components
$$F \to iL_{\text{Nis}}F,$$
which are all local equivalences.

Proof. Since i is fully faithful, we have that the counit $L_{\text{Nis}}i \to \text{id}$ is a natural equivalence. We'd like to argue that
$$L_{\text{Nis}}F \to L_{\text{Nis}}iL_{\text{Nis}}F$$
is an equivalence. Since i is fully faithful, it reflects equivalences, so it suffices to check that
$$iL_{\text{Nis}}F \to iL_{\text{Nis}}iL_{\text{Nis}}F$$
is a natural equivalence, which follows from the counit being a natural equivalence. □

Notation 2.2.54. Let X be a scheme and $x \in X$. We denote by $\text{Hen}_{X,x}$ the category of maps $f\colon (Y, y) \to (X, x)$ where $f\colon Y \to X$ is étale, $f(y) = x$, and f induces an isomorphism $k(x) \xrightarrow{\sim} k(y)$.

Definition 2.2.55. Let X be a scheme and $x \in X$. We denote by
$$X^h_x := \lim_{(Y,y)\in\text{Hen}_{X,x}} Y = \lim_{(Y,y)\in\text{Hen}_{X,x}} Y_y$$
See [Stacks, 04GV], or [Bac, 2.22].

Notation 2.2.56. Denote by $\text{Sm}_S \subseteq \text{Sch}_S$ the full subcategory of smooth S-schemes. Here S is still qcqs and Noetherian as assumed. We will restrict our attention here as it will be needed for upcoming results.

Definition 2.2.57. Let $f\colon F \to G$ be a map of presheaves in $\text{PSh}(\text{Sm}_S)$. Then we say f is a *stalkwise (Nisnevich) weak equivalence* if for every $X \in \text{Sm}_S$ and every $x \in X$, the induced map
$$\text{colim}_{\text{Hen}_{X,x}} F(Y) \to \text{colim}_{\text{Hen}_{X,x}} G(Y)$$
is an equivalence of spaces.

Warning 2.2.58. This is *not* the same as saying that $F(X^h_x) \to G(X^h_x)$ is an equivalence.

Theorem 2.2.59. (Voevodsky) Let S be qcqs and Noetherian, and let $f\colon F \to G$ in $\text{PSh}(\text{Sm}_S)$. Then f is a local equivalence if and only if it is a stalkwise equivalence.

Proof. (todo — hard) □

As a particular case of Theorem 2.2.59 applied to Example 2.2.53, we recover the following familiar result.

Corollary 2.2.60. A presheaf and its Nisnevich sheafification admit the same stalks in the Nisnevich site.

2.2.9 Torsors revisited

Proposition 2.2.61. Let S be qcqs and Noetherian, and let τ be a topology on Sch_S. Then given any τ-cover $\mathcal{U} := \{U_i \to X\}$, we can look at the induced map from the Čech nerve (viewed as a simplicial object of representable presheaves) to X viewed as a discrete representable simplicial presheaf. This map
$$N_\bullet(\mathcal{U}) \to X$$
is a local equivalence of presheaves.

Remark 2.2.62. (How to prove this)

1. One way to prove this is by identifying the sheaf topos, as a simplicial model category, with the Bousfield localization of the category of simplicial presheaves, endowed with the projective model structure, at the class of hypercovers in the topology. We insist on S being qcqs and Noetherian so we don't have to stress about the difference between covers and hypercovers here, although there is a more general statement over any base. This follows a body of work by Jardine, Bousfield and Kan, Dugger, Hollander and Isaksen.
2. Another direction is by formal nonsense of ∞-topos theory. For instance if $X \in \mathrm{Sm}_S$, we can apply [Lur18, A.5.3.1] to the Čech nerve of a cover after pulling it back to the slice topos $\mathrm{Shv}_\tau(\mathrm{Sm}_S)_{/X}$.

What about classifying spaces $B_\bullet\mathcal{G} \in \mathrm{PSh}(\mathrm{Sch}_S)$? We would like to understand them in the sheaf topos.

Notation 2.2.63. If τ is a topology, we denote by $B_\tau\mathcal{G} := L_\tau B_\bullet\mathcal{G}$ the sheafification of the bar construction in the τ topology.

Example 2.2.64. For any $\mathcal{G}$, we have a stack of $\mathcal{G}$-torsors, which we denote by
$$\mathrm{Tors}_\tau(\mathcal{G})\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Grpd}.$$
This gives rise to a sheaf $B\mathrm{Tors}_\tau(\mathcal{G})$ by post-composing with the nerve (Example 2.2.25).

Theorem 2.2.65. [AHW18, 2.3.2] There is a morphism of simplicial presheaves $B_\bullet\mathcal{G} \to B\mathrm{Tors}_\tau(\mathcal{G})$, defined on sections by sending the unique vertex of $B_\bullet\mathcal{G}(U)$ to the trivial $\mathcal{G}$-torsor over U. This map is a local equivalence.

Corollary 2.2.66. For any τ-sheaf of groups $\mathcal{G}$, there is a natural isomorphism of sheaves of sets
$$[-, B_\tau\mathcal{G}]_\tau = \pi_0\mathrm{Map}_{\mathrm{Shv}_\tau(\mathrm{Sch}_S)}(-, B_\tau\mathcal{G}) \cong H^1_\tau(-, \mathcal{G}).$$

This is the result we've been hoping for. It tells us we can classify torsors in the sheaf topos $\mathrm{Sh}_\tau(\mathrm{Sch}_S)$, and it is a completely topos-theoretic fact. In particular we gain access to all the homotopically flavored tools available to us in an ∞-topos. For instance, we could attempt to leverage Postnikov towers to deal with obstruction theory for the classifying sheaf $B_\tau\mathcal{G}$. The following result indicates that this tower won't contain any interesting information.

Example 2.2.67. If $\mathcal{G}$ is a sheaf of discrete groups, we have that
$$\pi_n B_\tau\mathcal{G} = \begin{cases} \mathcal{G} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. By Theorem 2.2.65 we have that $B_\tau\mathcal{G} \simeq B\mathrm{Tors}_\tau\mathcal{G}$, and the latter is a sheaf whose sections are exactly the nerves of discrete groupoids, and hence $K(\pi, 1)$'s. □

2.3 Bonus: more category theory

2.3.1 Cofinality

Definition 2.3.1. We say a subcategory $I \subseteq J$ is *cofinal* if for any functor $F\colon J \to \mathscr{C}$, the induced map on colimits
$$\mathrm{colim}_I F \to \mathrm{colim}_J F \tag{2.2}$$

is an isomorphism.[8] In other words, in order to compute a J-shaped colimit, it suffices to restrict to the subdiagram $I \subseteq J$.

Remark 2.3.3. This definition makes sense in 1-category theory as well as it does in ∞-category theory, however we remark that the two notions are different, so let's differentiate between the two, continuing the story of Definition 2.3.1:

- $I \subseteq J$ is 1-*cofinal* if for any 1-category $\mathscr{C}$ and 1-functor $F: J \to \mathscr{C}$, the induced map Equation 2.2 is an isomorphism.
- $I \subseteq J$ is *cofinal* (∞-cofinal if we want to be really pedantic) if for any ∞-category and ∞-functor $F: J \to \mathscr{C}$, the induced map Equation 2.2 is an equivalence.

Note this latter definition extends to the case where I and J are themselves ∞-categories, or even just simplicial sets.

Example 2.3.4. The subcategories $2\mathbb{N} \subseteq \mathbb{N}$ and $2\mathbb{N}+1 \subseteq \mathbb{N}$ are both 1-cofinal and ∞-cofinal.

It is straightforward to check when a subdiagram is 1-cofinal:

Proposition 2.3.5. [Stacks, 04E6] $I \subseteq J$ is 1-cofinal if

- every $j \in J$ has some $i \in I$ with a morphism $j \to i$
- for every $j \in J$ and pair of objects $i, i' \in I$, there is a zig-zag of morphisms between $i, i' \in I$ and maps from j into the zig-zag making the diagram commute:

$$\begin{tikzcd} & & j \arrow[dl] \arrow[d] \arrow[dr] & & \\ \cdots \arrow[r] & i_n & i_{n+1} \arrow[l] \arrow[r] & i_{n+2} & \cdots \arrow[l] \end{tikzcd}$$

Notation 2.3.6. Let $\Delta^{\text{inj}} \subseteq \Delta$ be the subcategory of injective maps, and let $\Delta_{\leq n} \subseteq \Delta$ denote the full subcategory of objects $[k]$ for $k \leq n$. For example $\Delta^{\text{inj}}_{\leq 1}$ is just a the category with two parallel arrows, i.e. the "(co)equalizer" category:

$$\Delta^{\text{inj}}_{\leq 1} := \bullet \rightrightarrows \bullet$$

Example 2.3.7. We have that $\Delta^{\text{inj, op}}_{\leq 1} \subseteq \Delta^{\text{op}}$ is 1-cofinal (c.f. [Rie14, 8.3.8]).

Proposition 2.3.8. The inclusion $\Delta^{\text{inj,op}}_{\leq n} \subseteq \Delta^{\text{op}}$ is n-cofinal for any $1 \leq n \leq \infty$.

It turns out by a souped-up extension of Quillen Theorem A, originally due to Joyal, we have a necessary condition for cofinality in the ∞-categorical setting.

Theorem 2.3.9. (Joyal, Quillen) Let $f: I \to J$ be a functor of 1-categories which is (∞-)cofinal. Then the induced map on classifying spaces

$$BI \to BJ$$

is a weak homotopy equivalence [Lur09, 4.1.3.1, 4.1.3.3].

Example 2.3.10. The same subcategory $\Delta^{\text{inj, op}}_{\leq 1} \subseteq \Delta^{\text{op}}$ is *not* ∞-cofinal.

Proof. The classifying space of the coequalizer diagram is S^1, however since $[0] \in \Delta$ is terminal, it is initial in Δ^{op}, hence $B\Delta^{\text{op}} \simeq *$ is contractible. $\square$

[8] We dually say $I \subseteq J$ is *final* if the natural map $\lim_I F \to \lim_J F$ is an isomorphism for any F.

Remark 2.3.11. Removing this injectivity hypothesis is also interesting, since we include the opposite of the map $[1] \to [0]$ — the category $\Delta^{\mathrm{op}}_{\leq 1}$ is the *split coequalizer* category:

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[from=1-1, to=1-2, bend left] \arrow[from=1-1, to=1-2, bend right] \end{tikzcd}$$

Proposition 2.3.12. Each of the composites

$$\Delta^{\mathrm{inj,\,op}}_{\leq 1} \subseteq \Delta^{\mathrm{op}}_{\leq 1} \subseteq \Delta^{\mathrm{op}}$$

is 1-cofinal, however none of these inclusions are ∞-cofinal.

Proof. The universal property of the 1-categorical colimit for both $\Delta^{\mathrm{inj,\,op}}_{\leq 1}$ and $\Delta^{\mathrm{op}}_{\leq 1}$ agree, so this is direct. The other inclusion now follows by Example 2.3.7. $\square$

In general we cannot truncate in order to obtain an ∞-cofinal diagram. We can, however, restrict only to face maps and throw out degeneracies:

Lemma 2.3.13. [Lur09, 6.5.3.7] The inclusion $\Delta^{\mathrm{inj,op}} \subseteq \Delta$ is cofinal.

Definition 2.3.14. An ∞-category $\mathscr{C}$ is *sifted* if the diagonal map $\mathscr{C} \to \mathscr{C} \times \mathscr{C}$ is cofinal [Lur09, 5.5.8.1].

Proposition 2.3.15. [Lur09, 5.5.8.11] Sifted colimits valued in Set commute with finite products.

Remark 2.3.16. In the 1-categorical setting, the converse of Proposition 2.3.15 holds, meaning we can take this to be the definition of sifted colimits.

Example 2.3.17.

1. The category Δ^{op} is sifted
2. Any filtered category is sifted

The examples above are the only interesting examples, in the following more precise sense.

Proposition 2.3.18. A category $\mathscr{C}$ admits all sifted colimits if and only if it admits all filtered colimits and it admits geometric realizations (meaning Δ^{op}-indexed colimits).

Corollary 2.3.19. A 1-category admits all sifted colimits if and only if it admits all filtered colimits and it has coequalizers.

Proof. Δ^{op}-indexed colimits are just coequalizers in 1-categories by Example 2.3.7. $\square$

Moreover, we may add in *finite coproducts* to capture all colimits.

Proposition 2.3.20. Let $\mathcal{C}$ be a category, $\mathcal{D} \subset \mathcal{C}$, a full subcategory, and X an object. Then, the following are equivalent.

(a) X is a geometric realization of coproducts of elements in $\mathcal{D}$.
(b) X is a sifted colimit of finite coproducts of elements in $\mathcal{D}$.
(c) X is a colimit of object in $\mathcal{D}$.

Proof. The implication (a) $\implies$ (b) is essentially Proposition 2.3.18 (as coproducts are filtered colimits of finite coproducts) and the implication (b) $\implies$ (c) is the fact that colimits of colimits are colimits. The implication (c) $\implies$ (a) follows from the Bousfield-Kan formula for a limit of $p\colon K \to \mathcal{C}$:

$$\operatorname{colim}_K p \xleftarrow{\sim} \operatorname{colim}\left(\coprod_{x \in K_0} p(x) \leftleftarrows \coprod_{\alpha \in K_1} p(\alpha(0)) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; \coprod_{\alpha \in K_2} p(\alpha(0)) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\; \vdots \right)$$

see e.g. [Sha23, Cor 12.5]. $\square$

2.3.2 Essential smallness

Being small is not a property of categories that is invariant under equivalence, so it is more meaningful to ask whether a category is *essentially small* (whether it is equivalent to a small category). This is equivalent to a category admitting a small skeleton, although the axiom of choice is required in order to pick representatives for each isomorphism class of object.

Theorem 2.3.21. The category of finite type S-schemes is essentially small.

Proof sketch. (see MO251044 for details) If $S = \operatorname{Spec}(A)$ is affine, then the category $\mathrm{Alg}_A^{\mathrm{f.t.}}$ of finite type A-algebras is essentially small, since all the objects are isomorphic to algebras of the form $A[x_1, \dots, x_n]/(f_1, \dots, f_r)$, of which there are a set. We can glue finite type S-schemes over affines, and then bootstrap to the more general case of the base S not being affine by gluing finite type schemes over all affine subschemes of S. $\square$

Remark 2.3.22. The category of all S-schemes is not essentially small.

2.4 Bonus: on localizing invariants

2.4.1 Idempotent-completion (for 1-categories)

Suppose we are given a strict retract diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & X \arrow[d, "r"] \\ & Y. \end{tikzcd}$$

Then we have that $f := i \circ r \colon X \to X$ is an idempotent endomorphism of X, since $f \circ f = f$. This establishes a correspondence between isomorphism classes of retractive X-objects and idempotent endomorphisms of X:

$$\begin{aligned} \operatorname{ob}\left(\mathscr{C}_{X//X}\right)^{\simeq} &\to \left\{ f \in \operatorname{End}_{\mathscr{C}}(X) \colon f^2 = f \right\} \\ (Y, i, r) &\mapsto \left[X \xrightarrow{i \circ r} X \right]. \end{aligned} \tag{2.1}$$

Proposition 2.4.2. The map in Equation 2.1 is always injective.

Proof. Suppose we have another idempotent

$$\begin{tikzcd} Z \arrow[r, "\iota"] \arrow[dr, "\mathrm{id}"'] & X \arrow[d, "\rho"] \\ & Z, \end{tikzcd}$$

so that $\iota\rho = ir$. Then we claim Z and Y are isomorphic as retractive X-objects. Indeed we see that

$$Y \xrightarrow{i} X \xrightarrow{\rho} Z$$

admits an inverse $Z \xrightarrow{\iota} X \xrightarrow{r} Y$, since $(\rho i)(r\iota) = \rho(ir)\iota = \rho\iota\rho\iota = \mathrm{id}$, and vice versa. In particular the diagram commutes

$$\begin{tikzcd}
 & Z \arrow[d, "\iota"] \arrow[dr, "\mathrm{id}"] & \\
Y \arrow[ur, "\rho\circ i", "\sim"'] \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & X \arrow[r, "\rho"] \arrow[d, "r"] & Z \\
 & Y \arrow[ur, "\rho\circ i"', "\sim"] &
\end{tikzcd}$$

Which exhibits Y and Z as isomorphic in the category $\mathscr{C}_{X//X}$ of retractive X-objects. $\square$

Proposition 2.4.3. If $\mathscr{C}$ admits equalizers, then Y can be recovered from the associated idempotent as the equalizer:

$$Y = \mathrm{eq}(\mathrm{id}, i \circ r \colon X \rightrightarrows X).$$

Proof. We claim this rectangle is a pullback:

$$\begin{tikzcd}
Y \arrow[rr, "i"] \arrow[d, "i"'] & & X \arrow[d, "\mathrm{id}"] \\
X \arrow[r, "r"'] & Y \arrow[r, "i"'] & X
\end{tikzcd}$$

Indeed suppose $h \colon Z \to X$ satisfies $h = irh$. Then the diagram commutes:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "hr"] \arrow[drrr, bend left, "h"] \arrow[ddr, bend right, "h"'] & & & \\
 & Y \arrow[rr, "i"] \arrow[d, "i"'] & & X \arrow[d, "\mathrm{id}"] \\
 & X \arrow[r, "r"'] & Y \arrow[r, "i"'] & X.
\end{tikzcd}$$

$\square$

Definition 2.4.4. We say that a 1-category $\mathscr{C}$ is *idempotent complete* if for every object $X \in \mathscr{C}$, the correspondence between retractive objects and idempotent morphisms in Equation 2.1 is a bijection.

Example 2.4.5. If $\mathscr{C}$ admits equalizers, it is idempotent complete.

Proposition 2.4.6. Let $\mathscr{C} \subseteq \mathscr{D}$ be a full subcategory which is idempotent complete. Then it is closed under retracts in $\mathscr{D}$.

Proof. Let $d \xrightarrow{i} c \xrightarrow{r} d$ be a retract of c which lies in $\mathscr{D}$. Then $i \circ r \colon c \to c$ lies in $\mathscr{C}$ since it is a full subcategory, and since it is idempotent complete, we obtain that $(d, i, r) \in \mathscr{C}_{c//c}$, hence $d \in \mathscr{C}$. $\square$

Example 2.4.7. The category of free finitely generated R-modules is not idempotent complete.

Proof. Let P be a finitely generated projective R-module. Then it is a summand in a free R-module, i.e. $P \oplus Q \cong R^n$, and the combination of the inclusion and projection off the direct sum yields the identity:

$$P \hookrightarrow R^n \to P.$$

Hence P is a retract of R^n. Considering the category of finitely generated free R-modules as a full subcategory of Mod_R, we then observe it isn't closed under retracts, and hence isn't idempotent complete. $\square$

2.5 Idempotent completion, ∞-categorically

The main distinction between idempotents in 1-categories versus ∞-categories is that being an idempotent in a 1-category is a *property*, whereas in an ∞-category it is a *structure* (witnessing idempotence requires additional data). This has various consequences, in particular that the admission of finite (co)limits is no longer sufficient to guarantee idempotent completeness.

Example 2.5.1. [Lur09, 4.4.5.1] Let $C_\bullet(R)$ denote the category of bounded chain complexes of finitely generated free R-modules. Then $C_\bullet(R)$ is a stable ∞-category, but is only idempotent complete if every finitely generated projective R-module is stably free (if $K_0(R) = \mathbb{Z}$).

Definition 2.5.2. [Lur09, 4.4.5.3] The simplicial set Idem is defined by the property that it has exactly one non-degenerate simplex in each dimension, and any face of any non-degenerate simplex is non-degenerate.

So if $\mathrm{Idem} \to \mathscr{C}$ is a map of simplicial sets, it picks out a morphism f in degree 1, and in degree 2 witnesses the composite

$$\begin{tikzcd} & \bullet \arrow[dr, "f"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "f"'] & & \bullet. \end{tikzcd}$$

The higher data witnesses higher coherence. So we say an *idempotent* in an ∞-category $\mathscr{C}$ is a map of simplicial sets $\mathrm{Idem} \to \mathscr{C}$ [Lur09, 4.4.5.4].

Definition 2.5.3. An ∞-category $\mathscr{C}$ is *idempotent complete* if any idempotent $F\colon \mathrm{Idem} \to \mathscr{C}$ admits a colimit.

Not every ∞-category is idempotent complete, but any category can be completed to one which is. This process is called *idempotent completion* — we say $f\colon \mathscr{C} \to \mathscr{D}$ is an *idempotent completion* if ([Lur09, 5.1.4.1])

1. $\mathscr{D}$ is idempotent complete
2. f is fully faithful
3. every object in $\mathscr{D}$ is a retract of something in the image of $\mathscr{C}$.

Proposition 2.5.4. [Lur09, 5.1.4.2, 5.4.2.4] Every ∞-category admits an idempotent completion, given by

$$\mathrm{Idem}(\mathscr{C}) := \mathrm{Ind}(\mathscr{C})^{\omega}.$$

Proof. Assume $\mathscr{C}$ is small without loss of generality (changing universes). Consider the Yoneda embedding $y\colon \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$, and let $\mathscr{C}'$ denote the closure of $y(\mathscr{C})$ under retracts. More explicitly, if κ is a regular cardinal, then we take $\mathrm{Ind}_\kappa(\mathscr{C})^\kappa$. □

Let $\mathrm{Cat}_\infty^\vee \subseteq \mathrm{Cat}_\infty$ be the full subcategory of idempotent-complete categories. Then this inclusion admits a left adjoint, given by idempotent completion [Lur09, 5.4.2.18].

Proposition 2.5.5. A small ∞-category is accessible if and only if it is idempotent complete [Lur09, 5.4.3.?].

We denote by

$$\mathrm{Cat}_\infty^{\mathrm{perf}} := \mathrm{Cat}_\infty^{\mathrm{st}} \cap \mathrm{Cat}_\infty^\vee,$$

the category of idempotent complete stable ∞-categories (and exact functors). Then idempotent completion descends to an adjunction

$$\mathrm{Idem}\colon \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Cat}_\infty^{\mathrm{perf}}.$$

2.5.1 Presentably symmetric monoidal categories

Definition 2.5.6. A *presentably symmetric monoidal category* is any object in $\mathrm{CAlg}(\mathrm{Pr}^L, \otimes)$. This is the same as a symmetric monoidal ∞-category in which the tensor product

$$\mathscr{C} \times \mathscr{C} \xrightarrow{\otimes} \mathscr{C}$$

preserves colimits in each variable.

Proposition 2.5.7. If $\mathscr{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ is presentably symmetric monoidal, and $S \subseteq \mathscr{C}$ is a set of objects, we can universally invert tensoring with each $x \in \mathscr{C}$, giving a new presentably symmetric monoidal category $\mathscr{C}[S^{-1}]$ with the obvious universal property. See Robalo 2.1

Notation 2.5.8. We denote by $\mathrm{Pr}^L_{\mathrm{st}} \subseteq \mathrm{Pr}^L$ the full subcategory of stable categories.

Proposition 2.5.9. *(Properties of* $\mathrm{Pr}^L_{\mathrm{st}}$*)*

1. The category $\mathrm{Pr}^L_{\mathrm{st}}$ is symmetric monoidal, with unit Sp the category of spectra
2. The Eilenberg–Watts theorem gives an equivalence
$$\mathrm{Sp} \cong \mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp})$$
$$X \mapsto X \otimes -.$$
3. If $\mathscr{C} \in \mathrm{Pr}^L_{\mathrm{st}}$ is dualizable, then its trace
$$\mathrm{Sp} \to \mathscr{C}^\vee \otimes \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}^\vee \to \mathrm{Sp}$$
is a spectrum (by Eilenberg–Watts), which is $\mathrm{THH}(\mathscr{C})$.

Notation 2.5.10. We denote by $\mathrm{Cat}^{\mathrm{perf}} \subseteq \mathrm{Cat}^{\mathrm{ex}}$ the full subcategory of idempotent-complete categories. This is part of an adjunction

$$\mathrm{Idem}(-)\colon \mathrm{Cat}^{\mathrm{ex}} \leftrightarrows \mathrm{Cat}^{\mathrm{perf}} \colon \text{inclusion}.$$

Given a cardinal ω, taking ind-completion or compact objects gives an equivalence

$$\mathrm{Ind}\colon \mathrm{Cat}^{\mathrm{perf}}_\infty \overset{\sim}{\leftrightarrows} \mathrm{Pr}^L_{\mathrm{st},\omega} \colon (-)^\omega$$

In particular the unit $\mathrm{id} \to \mathrm{Ind}(-)^\omega$ is an equivalence because this is exactly idempotent completion. The counit being an equivalence is precisely that the category is accessible (generated by its compact objects under accessible colimits).

Definition 2.5.11. An exact functor $F\colon \mathscr{C} \to \mathscr{D}$ of presentable stable categories is called *strongly continuous* if either of the equivalent conditions hold:

1. The right adjoint $F \dashv G$ is continuous
2. The right adjoint to F admits a further right adjoint.

Denote by $\mathrm{Fun}^{LL}(\mathscr{C}, \mathscr{D})$ the category of strongly continuous functors. We denote by $\mathrm{Pr}^{LL}_{\mathrm{st}} \subseteq \mathrm{Pr}^L_{\mathrm{st}}$ the subcategory on the same objects but with only strongly continuous morphisms.

Definition 2.5.12. We denote by $\mathrm{Cat}^{\mathrm{dual}} \subseteq \mathrm{Pr}^L_{\mathrm{st}}$ the (not full) subcategory of dualizable categories and strongly continuous functors. Or equivalently $\mathrm{Cat}_{\mathrm{dual}} \subseteq \mathrm{Pr}^{LL}_{\mathrm{st}}$ the full subcategory on dualizable categories.

Theorem 2.5.13. (Efimov) A presentable stable ∞-category is dualizable if and only if it is flat in $\mathrm{Pr}^L_{\mathrm{st}}$, meaning $\mathscr{C} \otimes -$ preserves fully faithful functors.

Remark 2.5.14. By [Lur09, 5.4.3.6], a small ∞-cat is accessible if and only if it is idempotent complete.[9] so finitely generated free R-modules fail to contain retracts (projectives) so they're not idempotent complete and hence not accessible.

[9] Idempotent complete has a number of definitions, in particular it implies that idempotent endomorphisms $f\colon X \to X$ (i.e. $f \circ f = f$) correspond bijectively to retracts of X, i.e. composites $Y \hookrightarrow X \to Y$. If $\mathscr{C}$ is idempotent complete then it is closed under retracts (todo: check this).

2.5.2 Exact sequences in $\mathrm{Cat}_\infty^{\mathrm{perf}}$

Definition 2.5.15. [BGT13, 5.8] A sequence

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

in $\mathrm{Pr}^L_{\mathrm{st}}$ is *exact* if

1. the composite is zero
2. $\mathcal{A} \to \mathcal{B}$ is fully faithful
3. the map $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence.

Chapter 3

Motivic spaces

Contents

3.1 Motivic spaces

We are interested in studying $\mathbb{A}^1$-*invariance* of presheaves. There are a few ways we might impose this, the first being the most naive — we can study presheaves F for which

1. any projection $X \times \mathbb{A}^1 \to X$ induces an equivalence $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$
2. any vector bundle torsor $E \to X$ induces an equivalence $F(X) \xrightarrow{\sim} F(E)$
3. any algebraic vector bundle *torsor* $Y \to X$ induces an equivalence $F(X) \to F(Y)$.

We have a strengthening of conditions here, but we notice that in the context of sheaves they all become equivalent, so the definition is only really important for presheaves. We will work with the first one as it is classically what people work with, and the last one since it will simplify some proofs. Let's first define vector bundle torsors explicitly.

Definition 3.1.1. [Wei89, 4.2] A *vector bundle torsor* is an affine map $Y \to X$ which is a torsor for an algebraic vector bundle $E \to X$. Explicitly, $Y \to X$ is a Zariski locally trivial affine morphism with fibers isomorphic to affine space.

Example 3.1.2. If $X = \mathrm{Spec}(A)$ is an affine scheme, then algebraic vector bundle torsors over X are the same as vector bundles over X.[1]

Notation 3.1.3. Let Sm_S denote the category of smooth S-schemes of finite type over S. The finite type assumption is needed in order to guarantee that Sm_S is essentially small (see Theorem 2.3.21).

Definition 3.1.4. Let $F \in \mathrm{PSh}(\mathrm{Sm}_S)$ be a presheaf.

1. We say F is $\mathbb{A}^1$-*invariant* if for every $X \in \mathrm{Sm}_S$, the projection map $X \times \mathbb{A}^1 \to X$ induces an equivalence
$$F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1).$$
2. We say F is *strongly homotopy invariant* if for every vector bundle torsor $Y \to X$, the restriction map
$$F(X) \xrightarrow{\sim} F(Y)$$
is an equivalence.

We denote by $\mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_S)$ and $\mathrm{PSh}_{\mathrm{htp}}(\mathrm{Sm}_S)$ the full subcategories of $\mathrm{PSh}(\mathrm{Sm}_S)$ spanned by the $\mathbb{A}^1$-invariant and strongly homotopy invariant presheaves, respectively.

Remark 3.1.5. It is clear that we have containments
$$\mathrm{PSh}_{\mathrm{htp}}(\mathrm{Sm}_S) \subseteq \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_S) \subseteq \mathrm{PSh}(\mathrm{Sm}_S),$$
but neither of the reverse inclusions hold.

Proposition 3.1.6. Both of $\mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_S)$ and $\mathrm{PSh}_{\mathrm{htp}}(\mathrm{Sm}_S)$ are accessible subcategories of $\mathrm{PSh}(\mathrm{Sm}_S)$, and therefore their inclusions admit left adjoints.

Proof. By Proposition 2.1.39 it suffices to check they are defined by being local with respect to a *set* of maps. Note that Sm_S has a small skeleton, hence we may pick a form a set S containing a projection $X \times \mathbb{A}^1 \to X$ for each isomorphism class of smooth schemes X. Now note that a presheaf F is $\mathbb{A}^1$-invariant if and only if it is S-local, meaning that
$$\mathrm{Map}_{\mathrm{PSh}(\mathrm{Sm}_S)}(h_X, F) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{PSh}(\mathrm{Sm}_S)}(h_{X \times \mathbb{A}^1}, F)$$
is an equivalence. We have used here the Yoneda lemma, and we conclude by noting that the Yoneda embedding preserves finite products. This proves the statement for $\mathbb{A}^1$-invariant presheaves, and an analogous argument works for $\mathrm{PSh}_{\mathrm{htp}}(\mathrm{Sm}_S)$. $\square$

[1] Question - can we find an illuminating example of a non-affine X for which vector bundle torsors and vector bundles don't agree?

Terminology 3.1.7. In light of the previous remark, we might also call presheaves or sheaves $\mathbb{A}^1$-*local* instead of $\mathbb{A}^1$-invariant. This is the terminology used in [MV99].

Example 3.1.8. *(Not every representable is $\mathbb{A}^1$-invariant)* Representable presheaves need not be $\mathbb{A}^1$-invariant. For example:
- ▷ $\mathbb{G}_m$ is $\mathbb{A}^1$ invariant assuming the base is reduced. This is because it represents units, which are $\mathbb{A}^1$-invariant.
- ▷ $\mathbb{A}^1$ is not $\mathbb{A}^1$-invariant, since it represents global sections.

Notation 3.1.9. We denote by
$$L_{\mathbb{A}^1}\colon \mathrm{PSh}(\mathrm{Sm}_S) \to \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_S),$$
$$L_{\mathrm{htp}}\colon \mathrm{PSh}(\mathrm{Sm}_S) \to \mathrm{PSh}_{\mathrm{htp}}(\mathrm{Sm}_S)$$
the associated localizations which are left adjoint to the inclusions.

3.1.1 Singular chains

Here we develop an explicit formula for $L_{\mathbb{A}^1}$ which will help us do computations.

Notation 3.1.10. We denote by Δ^n the *algebraic n-simplex*
$$\Delta^n := \mathrm{Spec}\,(\mathbb{Z}[t_0, \ldots, t_n]/(\sum t_i - 1)).$$
These give a cosimplicial scheme $\Delta^\bullet \in \mathrm{Fun}(\mathbb{\Delta}, \mathrm{Sch})$.

Definition 3.1.11. We define the *singular chains* construction
$$\mathrm{Sing}\colon \mathrm{PSh}(\mathrm{Sm}_S) \to \mathrm{PSh}(\mathrm{Sm}_S)$$
by the formula
$$\mathrm{Sing}(F)(X) = \mathrm{colim}_{\mathbb{\Delta}^{\mathrm{op}}} F(X \times \Delta^n)$$

Proposition 3.1.12. We have that $\mathrm{Sing}(F)$ is $\mathbb{A}^1$-invariant for any F.

Proof sketch. We want to prove for any $X \in \mathrm{Sch}_S$ that the projection map $\pi\colon X \times \mathbb{A}^1 \to X$ induces an equivalence
$$\pi^*\colon (\mathrm{Sing}F)(X \times \mathbb{A}^1) \to (\mathrm{Sing}F)(X).$$
Let $z\colon X \to X \times \mathbb{A}^1$ denote the zero section. Then we claim z^* exhibits a simplicial homotopy equivalence with π^*. Since $\pi z = \mathrm{id}$, it is clear that $z^*\pi^* = \mathrm{id}$, so it suffices to exhibit a simplicial homotopy $\pi^* z^* \simeq \mathrm{id}$. Since any functor will preserve simplicial homotopies, and geometric realizations will send simplicial homotopies to honest homotopies, it suffices to exhibit a homotopy of cosimplicial simplicial varieties
$$\mathrm{id}, z \circ \pi\colon X \times \mathbb{A}^1 \times \Delta^\bullet \to X \times \mathbb{A}^1 \times \Delta^\bullet,$$
meaning maps whose opposites satisfy the simplicial homotopy identities. (todo: add explicit simplicial homotopy) □

Proposition 3.1.13. If F is already $\mathbb{A}^1$-invariant, then the natural map
$$F \to \mathrm{Sing}(F)$$
is an equivalence of presheaves.

Proof. Immediate since F is $\mathbb{A}^1$-invariant, so we can turn $F(X \times \Delta^\bullet)$ into a constant diagram. □

Proposition 3.1.14. There is a natural equivalence of functors $L_{\mathbb{A}^1} \simeq \mathrm{Sing}$.

Proof. We note that the essential image of Sing is precisely $\mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k)$ by Proposition 3.1.13. By that same result, the natural transformation

$$\eta\colon \mathrm{id} \to \mathrm{Sing}$$

induces an equivalence between the maps $\mathrm{Sing}(\eta_{(-)})$ and $\eta_{\mathrm{Sing}(-)}$ on its components. By [Lur09, 5.2.7.4], this implies that $\mathrm{Sing}(-)$ is a left adjoint, with right adjoint given by the fully faithful inclusion of its essential image $\mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k) \subseteq \mathrm{PSh}(\mathrm{Sm}_k)$. By uniqueness of adjoints, this implies $\mathrm{Sing} \simeq L_{\mathbb{A}^1}$. □

Corollary 3.1.15. $L_{\mathbb{A}^1}$ preserves finite products.

Proof. This is not immediate from the definition, but it follows after identifying $L_{\mathbb{A}^1}$ with $\mathrm{Sing}(-)$. Since Δ^{op} is sifted (c.f. [Lur09, 5.5.4.8]), colimits indexed over Δ^{op} commute with products, hence the result follows. □

Proposition 3.1.16. For any $X \in \mathrm{Sm}_S$, we have that $X \times \mathbb{A}^n_S \to X$ is an equivalence after $\mathbb{A}^n$-localization.

Proof. For $n = 1$ this is by definition of the localization, and for higher n this follows from $L_{\mathbb{A}^1}$ preserving finite products. □

Remark 3.1.17. An analogous formula holds for L_{htp} — if we let $\mathrm{VBT}_{/X}$ denote the category of vector bundle torsors over X, then $\mathrm{VBT}_{/X}$ is cosifted, and we obtain an identification

$$(L_{\mathrm{htp}}F)(X) = \mathrm{colim}_{Y \in \mathrm{VBT}_{/X}} F(Y), \tag{3.18}$$

with the same properties that it preserves finite products and is locally cartesian [Hoy17, 3.5]

3.1.2 The category of motivic spaces

Definition 3.1.19. We define the category of *motivic spaces* $\mathrm{Spc}(k)$ as

$$\mathrm{Spc}(k) = \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \cap \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k) \subseteq \mathrm{PSh}(\mathrm{Sm}_k),$$

that is, the full subcategory of presheaves which are both Nisnevich sheaves and are $\mathbb{A}^1$-invariant.

Remark 3.1.20. Since every affine bundle torsor is locally trivialized, once we impose the sheaf condition, the properties of being $\mathbb{A}^1$-invariant and strongly homotopy invariant can be checked locally, and hence agree:

$$\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S) \cap \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_S) = \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S) \cap \mathrm{PSh}_{\mathrm{htp}}(\mathrm{Sm}_S).$$

In other words we could equivalently define motivic spaces via $\mathrm{PSh}_{\mathbb{A}^1}$, or via $\mathrm{PSh}_{\mathrm{htp}}$. The former is more common, although the latter has some nice advantages (see [Hoy17, p. 204] for a discussion, and [Hoy17, 3.13] for the equality above).

Problem: Nisnevich sheafifying an $\mathbb{A}^1$-invariant presheaf needs not preserve $\mathbb{A}^1$-invariance, and $\mathbb{A}^1$-localizing a sheaf may break the sheaf condition.

Example 3.1.21. [MV99, 3.2.7] Let $U_0 = \mathbb{A}^1 - 0$ and $U_1 = \mathbb{A}^1 - 1$, and let $U_{01} = U_0 \cap U_1 \subseteq \mathbb{A}^1$. Since both U_0 and U_1 are $\mathbb{A}^1$-invariant, we claim that U_{01} is as well. Pick a closed embedding of U_{01} in $\mathbb{A}^n$ for some n, and consider the non-smooth scheme

$$Y := (U_0 \times \mathbb{A}^n) \amalg_{U_{01}} (U_1 \times \mathbb{A}^n) \in \mathrm{Sch}_S.$$

Then for any connected $X \in \mathrm{Sm}_S$, we have that

$$\mathrm{Hom}_{\mathrm{Sch}_S}(X, Y) = \mathrm{Hom}_{\mathrm{Sm}_S}(X, U_0 \times \mathbb{A}^n) \coprod_{\mathrm{Hom}_{\mathrm{Sm}_S}(X, U_{01})} \mathrm{Hom}_{\mathrm{Sm}_S}(X, U_1 \times \mathbb{A}^n).$$

Since $L_{\mathbb{A}^1}$ preserves pushouts, we have that

$$L_{\mathbb{A}^1} h_Y = L_{\mathbb{A}^1}(U_0 \times \mathbb{A}^n) \amalg_{L_{\mathbb{A}^1}(U_{01})} L_{\mathbb{A}^1}(U_1 \times \mathbb{A}^n).$$

By Proposition 3.1.16 we can contract away the $\mathbb{A}^n$'s, and we invoke that U_0, U_1, and U_{01} were already $\mathbb{A}^1$-local to get that the above is equivalent to

$$\simeq L_{\mathbb{A}^1}(U_0) \coprod_{L_{\mathbb{A}^1}(U_{01})} L_{\mathbb{A}^1}(U_1) = U_0 \coprod_{U_{01}} A_1.$$

If this were a sheaf, it would agree with its sheafification, and since sheafification preserves pushouts, we would have that it is equal to the *pushout of sheaves*

$$U_0 \amalg_{U_{01}} U_1 = \mathbb{A}^1,$$

which is the representable sheaf given by $\mathbb{A}^1$ by Zariski descent. But note that $\mathbb{A}^1$ is not $\mathbb{A}^1$-invariant, so we get that $L_{\mathbb{A}^1} h_Y$ cannot be a sheaf.

Proposition 3.1.22. The category $\mathrm{Spc}(S) \subseteq \mathrm{PSh}(\mathrm{Sm}_S)$ is an accessible localization, hence the inclusion admits a left adjoint.

We call this adjoint *motivic localization*, and we can describe it explicitly as the infinite composition of both functors.

Definition 3.1.23. We define $L_{\mathrm{mot}} \colon \mathrm{PSh}(\mathrm{Sm}_S) \to \mathrm{Spc}(S)$ by the formula

$$L_{\mathrm{mot}} := \operatorname{colim}\left(L_{\mathrm{Nis}} \to L_{\mathbb{A}^1} L_{\mathrm{Nis}} \to L_{\mathrm{Nis}} L_{\mathbb{A}^1} L_{\mathrm{Nis}} \to \cdots\right)$$

where this colimit is computed in the presheaf category.

Remark 3.1.24. By cofinality (Example 2.3.4), this is canonically equivalent to the following two colimits:

$$\operatorname{colim}\left(L_{\mathrm{Nis}} \circ L_{\mathbb{A}^1} \to (L_{\mathrm{Nis}} \circ L_{\mathbb{A}^1})^{\circ 2} \to \cdots\right)$$
$$\operatorname{colim}\left(L_{\mathbb{A}^1} \circ L_{\mathrm{Nis}} \to (L_{\mathbb{A}^1} \circ L_{\mathrm{Nis}})^{\circ 2} \to \cdots\right).$$

The former is a colimit computed in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$, which is closed under filtered colimits by Proposition 3.1.6, hence the resulting object is a sheaf. The latter is a colimit computed in $\mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k)$, which is also closed under filtered colimits (again by Proposition 3.1.6), hence it is $\mathbb{A}^1$-invariant.

3.1.3 Properties of motivic localization

We know that L_{mot} preserves colimits, being a left adjoint. What other properties does it admit?

Proposition 3.1.25. [Hoy14, C.6] We have that L_{mot} preserves finite products.

Proof. This follows by both L_{Nis} and $L_{\mathbb{A}^1}$ preserving finite products, together with transfinite composition preserving products. ◻

Proposition 3.1.26. [Hoy17, 3.15] L_{mot} is locally cartesian, meaning that for a cospan $X \to Y \leftarrow Z$ in $\mathrm{PSh}(\mathrm{Sm}_S)$, if both X and Y are motivic spaces, then the natural map

$$L_{\mathrm{mot}}(X \times_Y Z) \to X \times_Y L_{\mathrm{mot}}(Z)$$

is an equivalence. Note the pullback on the domain is in the category of presheaves, while the pullback on the right is in the category of motivic spaces.

Corollary 3.1.27. Colimits in $\mathrm{Spc}(S)$ are universal (pullback-stable).

Proof. Let X and Y be some motivic spaces, and let $j\colon I \to \mathrm{Spc}(S)$ be a diagram, whose colimit maps to Y. We claim that

$$\mathrm{colim}_{i\in I}\,(X \times_Y j(i)) \to X \times_Y \mathrm{colim}_{i\in I} j(i)$$

is an equivalence in L_{mot}. This is true in the ambient presheaf category since it is a topos, but this doesn't immediately imply the result for motivic spaces, since pullbacks of presheaves do not in general yield pullbacks of motivic spaces. However by applying L_{mot} to both sides, and using the fact that it is locally cartesian, the result follows. □

Proposition 3.1.28. We have that L_{mot} is *not left exact*. In particular $\mathrm{Spc}(k)$ is not an ∞-topos.

We will come back to prove Proposition 3.1.28 later when we have a little more machinery.

3.1.4 Motivic equivalences

Definition 3.1.29. We say that $f\colon F \to G$ in $\mathrm{PSh}(\mathrm{Sm}_S)$ is a *motivic equivalence* if $L_{\mathrm{mot}} f$ is an equivalence in $\mathrm{Spc}(S)$.

Proposition 3.1.30. We have that $\Delta^0 \cong S$ in $\mathrm{Spc}(S)$, and moreover these are terminal, hence we will often denote them by $*$.

Proof. This follows from $\mathrm{Spc}(S) \subseteq \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$ being an accessible subcategory (hence preserving terminal objects) together with Proposition 2.2.29. □

Proposition 3.1.31. The map to the terminal object in schemes $\mathbb{A}^n_S \to S$ is a motivic equivalence.

Proof. It is clear that $\mathbb{A}^1_S \to S$ is a motivic equivalence essentially by definition. The more general statement follows from both the Yoneda embedding and motivic localization preserving finite products. □

The following proposition is also essentially by definition.

Proposition 3.1.32. For any $X \in \mathrm{Sm}_S$, the projection map $X \times \mathbb{A}^n_S \to X$ is a motivic equivalence.

Proposition 3.1.33. For any $F \in \mathrm{PSh}(\mathrm{Sm}_S)$, the projection map $F \times \mathbb{A}^n \to F$ is a motivic equivalence.

Proof. By the co-Yoneda lemma, any presheaf is a colimit of representable presheaves. Since L_{mot} is locally cartesian, colimits are pullback stable, and hence product-stable, meaning that

$$F \times \mathbb{A}^n \cong (\mathrm{colim}_U h_U) \times \mathbb{A}^n.$$

Since colimits are universal in any ∞-topos (in particular in the presheaf category), we have that we can distribute the colimit over the product. Then the result follows by Proposition 3.1.32. □

3.2 Jouanolou devices and motivic equivalences

Theorem 3.2.1. (Jouanolou–Thomason trick) Suppose that S is qcqs and further suppose either S is affine or it is noetherian, regular, and separated. Then for any quasi-projective $X \in \mathrm{Sch}_S$, there exists an affine bundle $Y \to X$, where Y is affine.

Remark 3.2.2. The above result probably holds under weaker assumptions – we're citing [Hoy17, 2.20], pulling our assumptions on S from [Hoy17, 2.8] and letting G be trivial. In most cases we apply this, S will be a field, and X will be smooth, and we can apply this trick by e.g. [AF14b, 3.3.3].

Example 3.2.3. The easiest example is $\mathbb{P}^1_k$ over a field k. There is a map

$$\operatorname{Spec}\left(\frac{k[x,y,z,w]}{x+w-1, xw-yz}\right) \to \mathbb{P}^1$$

$$(x,y,z,w) \mapsto \begin{cases} [x:y] & (x,y) \neq (0,0) \\ [z:w] & (z,w) \neq (0,0). \end{cases}$$

We claim this is well-defined, exhibiting an affine torsor over $\mathbb{P}^1$.

Example 3.2.4. In a similar vein, the map

$$\mathrm{SL}_2 \to \mathbb{A}^2 \smallsetminus 0$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (b,d)$$

is an affine vector bundle torsor over $\mathbb{A}^2 \smallsetminus 0$. More generally

$$\mathrm{SL}_n \to \mathbb{A}^n \smallsetminus \{0\}$$

is an affine vector bundle torsor, by sending an invertible determinant one matrix to its last column.

Proposition 3.2.5. Any affine vector bundle torsor $E \to B$ is a motivic equivalence.

Proof. It suffices to verify on a local cover where the bundle is trivialized, at which point it follows by Proposition 3.1.32. □

3.2.1 Checking motivic equivalences on affines

Proposition 3.2.6. The category $\mathrm{Spc}(S)$ is generated under sifted colimits by (the motivic localization of the representable presheaves attached to) affine S-schemes in Sm_S [Hoy17, 3.16].

Proof. We do this in two steps:

1. First we argue that $\mathrm{Spc}(S)$ is generated under sifted colimits by Sm_S. Let $F \in \mathrm{Spc}(S)$ be any motivic space, then, considered as a presheaf, it is a colimit of representable presheaves. What is furthermore true is that it is a sifted colimit of finite coproducts of representable presheaves (Proposition 2.3.20). While it is not true that $h_U \amalg h_V$ is $h_{U \amalg V}$ in the category of presheaves, it *is true* in the category of sheaves (and hence in the category of motivic spaces), by descent. Hence any motivic localization is in fact a sifted colimit of the motivic localization of representable sheaves.
2. Second we argue that $\mathrm{Sm}_S \subseteq \mathrm{Spc}(S)$ is generated under sifted colimits by affine S-schemes in Sm_S. Given any $X \in \mathrm{Sm}_S$, we can write it as an Δ^{op}-indexed colimit in the category $\mathrm{Spc}(S)$ over the Čech nerve of a Nisnevich cover. Each fiber product appearing in the cover is equivalent to a smooth affine scheme by Jouanolou's trick, and we can get rid of finite coproducts by the same trick. Hence X can be written as a Δ^{op}-indexed colimit over smooth affine S-schemes.

As sifted colimits are combinations of filtered colimits and geometric realizations (Proposition 2.3.18), the result follows. □

This gives us a powerful way to check motivic equivalences between presheaves.

Proposition 3.2.7. [Hoy17, 3.16] Let $f\colon F \to G$ be a morphism in $\mathrm{PSh}(\mathrm{Sm}_S)$. If
$$F(U) \to G(U)$$
is an equivalence for every affine $U \in \mathrm{Sm}_S$, then f is a motivic equivalence.

Proof. By Yoneda it suffices to argue that
$$\mathrm{Map}_{\mathrm{Spc}(S)}(-, L_{\mathrm{mot}}F) \to \mathrm{Map}_{\mathrm{Spc}(S)}(-, L_{\mathrm{mot}}G)$$
is a natural equivalence. Since $\mathrm{Map}(-,-)$ sends colimits to limits in the first variable, by Proposition 3.2.6 it suffices to argue that
$$\mathrm{Map}_{\mathrm{Spc}(S)}(L_{\mathrm{mot}}h_U, L_{\mathrm{mot}}F) \to \mathrm{Map}_{\mathrm{Spc}(S)}(L_{\mathrm{mot}}h_U, L_{\mathrm{mot}}G)$$
is an equivalence for each affine $U \in \mathrm{Sm}_S$. By adjunction, this is equivalent to asking that $L_{\mathrm{mot}}F \to L_{\mathrm{mot}}G$ is a sectionwise equivalence of presheaves when restricted to affines.

We first argue $L_{\mathrm{htp}}F \to L_{\mathrm{htp}}G$ is a sectionwise equivalence. Since every $X \in \mathrm{Sm}_S$ admits an affine bundle torsor $U \to X$ which is itself affine by the Jouanolou trick, we have that $(L_{\mathrm{htp}}F)(X) \xrightarrow{\sim} (L_{\mathrm{htp}}F)(U)$ is an equivalence, so it suffices to check that $L_{\mathrm{htp}}F \to L_{\mathrm{htp}}G$ is a sectionwise equivalence on affines. Since every vector bundle torsor over an affine is affine[2], we can leverage the formula for L_{htp} (Equation 3.18) in order to write
$$(L_{\mathrm{htp}}F)(U) = \mathrm{colim}_{V \in \mathrm{VBT}_{/U}} F(V).$$
Since $F(V) \to G(V)$ is an equivalence for each V above, it is clear that $(L_{\mathrm{htp}}F)(U) \to (L_{\mathrm{htp}}G)(U)$ is an equivalence for each affine U, hence $L_{\mathrm{htp}}F \to L_{\mathrm{htp}}G$ is a sectionwise equivalence.

Finally, since a sectionwise equivalence of presheaves is a local equivalence, we have that
$$L_{\mathrm{Nis}}L_{\mathrm{htp}}F \to L_{\mathrm{Nis}}L_{\mathrm{htp}}G$$
is an equivalence of presheaves. Passing to the colimit it is now clear that $F \to G$ is a motivic equivalence. □

3.3 Pointed motivic spaces

Notation 3.3.1. By abuse of notation, if $X \in \mathrm{Sm}_S$, then we also denote by $X \in \mathrm{Spc}(S)$ the motivic space $L_{\mathrm{mot}}h_X$.

Notation 3.3.2. We denote by $\mathrm{Spc}(S)_*$ the category of pointed motivic spaces. This has a zero object, which we denote by $*$. This comes with an adjunction
$$(-)_+ \colon \mathrm{Spc}(S) \leftrightarrows \mathrm{Spc}(S)_* : U,$$
where X_+ is the coproduct $X \amalg S$, pointed at that copy of S, and the right adjoint forgets the basepoint.

Notation 3.3.3. If $Y \to X$ is a map of (pointed) motivic spaces, we denote by X/Y the cofiber, computed as the pushout
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/Y. \end{tikzcd}$$

[2] Over an affine scheme, vector bundle torsors are just vector bundles. If $\mathscr{F}$ is a vector bundle (considered as a sheaf of modules) over $X = \mathrm{Spec}(A)$, then the underlying scheme is $\underline{\mathrm{Spec}}_X(\mathrm{Sym}(\mathcal{F}^*))$ which is affine over X, and hence is itself affine over the base.

We denote by $X \vee Y$ the coproduct in the category $\mathrm{Spc}(S)_*$ of pointed motivic spaces, and by

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

the *smash product.*

Notation 3.3.4. Since $\mathrm{Spc}(S)_*$ has a zero object, we can denote by ΣX the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

for any motivic space X.

Proposition 3.3.5. If X is a pointed motivic space, then there is a canonical equivalence

$$\Sigma X \simeq S^1 \wedge X.$$

Proof. Since L_{mot} preserves coproducts and $\mathrm{PSh}(\mathrm{Sm}_S)$ is cocomplete, it suffices to compute the pushout at the level of simplicial presheaves. In this setting, the coproduct is computed levelwise, in which case it is clear that $\Sigma X(U) = S^1 \wedge X(U)$ for any $U \in \mathrm{Sm}_S$. Hence as pointed presheaves, we have that $\Sigma X \simeq S^1 \wedge X$. Finally, L_{mot} preserves the smash product construction since it preserves finite products, as well as coproducts and cofibers. $\square$

Proof 2. We can first show it for $X = S^0$, then leverage universality of colimits and [DH21, 2.26] to argue in general. $\square$

By descent, Nisnevich covers of schemes give rise to colimits of motivic spaces. One of the most immediate examples is the following:

Example 3.3.6. The following diagram of (pointed) motivic spaces is a pushout:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "z"] \arrow[d, "z^{-1}"'] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1. \end{tikzcd}$$

Since $\mathbb{A}^1 \xrightarrow{\sim} *$ is trivial, and since all our colimits are ∞-colimits, we can replace the above diagram with the weakly equivalent span $* \leftarrow \mathbb{G}_m \to *$ and compute the same colimit. This implies that the natural map

$$\Sigma \mathbb{G}_m \xrightarrow{\sim} \mathbb{P}^1 \tag{3.7}$$

is a motivic equivalence.

3.3.1 Motivic spheres

Terminology 3.3.8. We have *two kinds of spheres* in motivic homotopy theory: ones coming from algebraic geometry (the multiplicative group scheme $\mathbb{G}_m$ or the projective line $\mathbb{P}^1$) and ones coming from algebraic topology (the constant presheaf at S^1 or at S^n for any n). So we get bigradings on the spheres. There are competing conventions in the literature for motivic grading, but the increasingly standardized convention is to write

$$S^{1,1} := \mathbb{G}_m$$
$$S^{1,0} = S^1$$
$$S^{2,1} = \mathbb{P}^1.$$

Example 3.3.9. We have that $S^1 \simeq \mathbb{A}^1/\{0,1\}$.

Proof. We can consider the cofiber diagram

$$\begin{tikzcd} S^0 \arrow[r, "{(0,1)}"] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ * \arrow[r] & S^1. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

□

Exercise 3.3.10. The diagram

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \Sigma(X \wedge Y) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout.

Proposition 3.3.11. There is a motivic equivalence

$$\mathbb{A}^n \setminus \{0\} \simeq (S^1)^{\wedge(n-1)} \wedge (\mathbb{G}_m)^{\wedge n} = S^{2n-1,n}.$$

Corollary 3.3.12. We have that

$$\frac{\mathbb{A}^n}{\mathbb{A}^n \setminus \{0\}} \simeq S^{2n,n}.$$

Proposition 3.3.13. We have that

$$\frac{\mathbb{P}^n}{\mathbb{P}^n - 0} \simeq S^{n,n}.$$

Proof. By covering $\mathbb{P}^n$ with $\mathbb{A}^n$ and $\mathbb{P}^n \setminus \{0\}$, we get a pushout diagram

$$\begin{tikzcd} \mathbb{A}^n \setminus \{0\} \arrow[r] \arrow[d] & \mathbb{P}^n \setminus \{0\} \arrow[d] \\ \mathbb{A}^n \arrow[r] & \mathbb{P}^n. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The cofibers of the vertical maps are equivalent, yielding

$$S^{2n,n} \xrightarrow{\sim} \frac{\mathbb{P}^n}{\mathbb{P}^n - 0}.$$

□

3.4 $\mathbb{A}^1$-homotopy classes of maps

Given two simplicial presheaves $F, G \in \mathrm{PSh}(\mathrm{Sm}_S)$, by abuse of notation we denote by $[F,G]_{\mathbb{A}^1}$ homotopy classes of maps of their associated motivic localizations

$$[F,G]_{\mathbb{A}^1} := \pi_0 \mathrm{Map}_{\mathrm{Spc}(S)}(L_{\mathrm{mot}}F, L_{\mathrm{mot}}G).$$

A goal is to understand how to compute this. There are some easy cases first:

Example 3.4.1. Let X be a representable sheaf and $\mathcal{Y}$ be a motivic space. Then via the Yoneda lemma, we have

$$\mathrm{Map}_{\mathrm{Spc}(S)}(L_{\mathrm{mot}}h_X, \mathcal{Y}) \cong \mathrm{Map}_{\mathrm{PSh}(\mathrm{Sm}_S)}(h_X, \mathcal{Y}) \cong \mathcal{Y}(X).$$

Hence $[X, \mathcal{Y}]_{\mathbb{A}^1} = \pi_0 \mathcal{Y}(X)$.

Example 3.4.2. We have a bijection
$$[X, \mathbb{G}_m]_{\mathbb{A}^1} = \mathbb{G}_m(X)$$
for any $X \in \mathrm{Sm}_S$.

Example 3.4.3. If X and Y are both varieties, then
$$[X, Y]_{\mathbb{A}^1} = \pi_0 \left(L_{\mathrm{mot}} h_Y\right)(X).$$
This is much harder to access in general, since we don't have control over the sections of the presheaf $L_{\mathrm{mot}} h_Y$.

3.4.1 Connected components

Given a motivic space X, we can consider the presheaf of sets
$$\begin{aligned} \mathrm{Sm}_S^{\mathrm{op}} &\to \mathrm{Set} \\ U &\mapsto [U, X]_{\mathbb{A}^1}. \end{aligned}$$
Its sheafification is called the *sheaf of connected components*, and we denote it by
$$\pi_0^{\mathbb{A}^1}(X) \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k).$$

Note 3.4.4. We have that $\pi_0^{\mathbb{A}^1}(X) = \pi_0(X)$ is the same as the homotopy sheaf in the sheaf topos, provided X is a motivic space. We have to be a bit careful what we mean here — if F is an arbitrary presheaf, then $\pi_0(F)$ will mean $\pi_0(L_{\mathrm{Nis}} F)$ by convention, while $\pi_0^{\mathbb{A}^1} F$ will mean $\pi_0(L_{\mathrm{mot}} F)$.

Proposition 3.4.5. We have that $\pi_0^{\mathbb{A}^1}(\mathrm{SL}_n) \simeq *$.

Proof sketch. It suffices to argue the inclusion of any element is homotopic to the identity. First suppose M is an elementary matrix. Then there is a map
$$\mathbb{A}^1 \to \mathrm{SL}_n,$$
sending 0 to the identity and 1 to M. Since SL_n is generated by elementary matrices, the result follows. □

Definition 3.4.6. If (X, x) is a pointed motivic space, we have that $\pi_n^{\mathbb{A}^1}(X, x)$ is defined to be the Nisnevich presheaf of
$$U \mapsto [\Sigma^n U_+, X]_{\mathrm{Spc}(S)_*}.$$

Proposition 3.4.7. If (X, x) is a pointed motivic space, then $\pi_n^{\mathbb{A}^1}(X, x) = \pi_n(X, x)$.

Proof. We show the presheaves are identical before sheafifying. By adjunction we have that
$$[\Sigma^n U_+, X]_{\mathrm{Spc}(S)_*} \cong [U_+, \Omega^n X]_{\mathrm{Spc}(S)_*} = [U, \Omega^n X]_{\mathrm{Spc}(S)}.$$
The latter is the presheaf attached to $\pi_n(X, x)$ by Proposition 2.2.35. □

Proposition 3.4.8. Let $f\colon F \to G$ be a map of simplicial presheaves. Then it is a motivic equivalence if and only if
$$\pi_n^{\mathbb{A}^1}(f)\colon \pi_n^{\mathbb{A}^1}(F, x) \to \pi_n^{\mathbb{A}^1}(G, f(x))$$
is an equivalence for all $n \geq 0$ and all basepoints $x \in F$.

Proof. The forward direction is clear, since f being a motivic equivalence would induce a natural isomorphism $[-, X]_{\mathbb{A}^1} \cong [-, Y]_{\mathbb{A}^1}$, hence the associated homotopy (pre)sheaves would be identical.

For the backwards direction, this condition on homotopy groups unwinds to tell us that $L_{\mathrm{mot}}F \to L_{\mathrm{mot}}G$ is an equivalence of sheaves by Corollary 2.2.49, which in particular implies it is an equivalence of motivic spaces. □

Terminology 3.4.9. We say that a motivic space X is $\mathbb{A}^1$-*n-connected* if $\pi_i^{\mathbb{A}^1}(X, x)$ is trivial for all $i \le n$ and all basepoints. Some special cases:

$$\begin{array}{lcl} \mathbb{A}^1\text{-}\textit{connected} & = & \mathbb{A}^1\text{-0-connected} \\ \mathbb{A}^1\text{-}\textit{simply connected} & = & \mathbb{A}^1\text{-1-connected} \\ \mathbb{A}^1\text{-}\textit{contractible} & = & \mathbb{A}^1\text{-}\infty\text{-connected.} \end{array}$$

Example 3.4.10. We have that $S^{i+j,j}$ is $\mathbb{A}^1$-$(i-1)$-connected (see [AO19, 2.4.5]). In particular it depends only on the number of copies of $S^{1,0}$ and not on the number of copies of $\mathbb{G}_m$.

Example 3.4.11. As a particular example, when $i = d-1$ and $j = d$, we obtain that punctured affine space $\mathbb{A}^d \smallsetminus \{0\} = S^{2d-1,d}$ is $\mathbb{A}^1$-$(d-2)$-connected.

3.5 Strong and strict $\mathbb{A}^1$-invariance

Goal 3.5.1. We have seen that in order to check a map is a motivic equivalence, it suffices to look at homotopy sheaves. We'd therefore like to better understand when homotopy sheaves are equivalent. It turns out homotopy sheaves (for $n \ge 2$) are a prototypical example of a particularly nice class of sheaves of abelian groups called *strictly invariant* sheaves. These have nice properties that make them easier to work with.

Note that $\pi_n^{\mathbb{A}^1}(X)$ is a Nisnevich sheaf of sets, but it is not necessarily a motivic space. We can ask to what extent it is $\mathbb{A}^1$-invariant.

It was conjectured by Morel that $\pi_0^{\mathbb{A}^1}(X)$ was always $\mathbb{A}^1$-invariant, but a counterexample was found by Ayoub (todo). Nevertheless we can ask for $\pi_i^{\mathbb{A}^1}(X)$ for $i \ge 1$. A more general question is to understand conditions that tell us a sheaf of groups is invariant.

Let $\mathcal{G}$ denote a Nisnevich sheaf of groups. Since $\mathcal{G}(X) = H^0_{\mathrm{Nis}}(X, \mathcal{G})$, the condition that $\mathcal{G}$ is $\mathbb{A}^1$-invariant is equivalent to asking whether

$$H^0_{\mathrm{Nis}}(X; \mathcal{G}) \to H^0_{\mathrm{Nis}}(X \times \mathbb{A}^1; \mathcal{G})$$

is $\mathbb{A}^1$-invariant.

Definition 3.5.2. Let $\mathcal{G}$ be a Nisnevich sheaf of groups over S. Then we say

1. $\mathcal{G}$ is $\mathbb{A}^1$-*invariant* if $H^0_{\mathrm{Nis}}(X; \mathcal{G}) \to H^0_{\mathrm{Nis}}(X \times \mathbb{A}^1; \mathcal{G})$ is an equivalence for every $X \in \mathrm{Sm}_S$
2. $\mathcal{G}$ is *strongly* $\mathbb{A}^1$-*invariant* if
$$H^i_{\mathrm{Nis}}(X; \mathcal{G}) \to H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1; \mathcal{G})$$
is an equivalence for $i = 0, 1$ and for all X
3. if $\mathcal{G}$ is a sheaf of abelian groups, we say it is *strictly* $\mathbb{A}^1$-*invariant* if
$$H^i_{\mathrm{Nis}}(X; \mathcal{G}) \to H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1; \mathcal{G})$$
is an equivalence for all i and for all X.

Example 3.5.3. We have that $\mathbb{G}_m$ is strongly invariant, since both units and the Picard group are $\mathbb{A}^1$-invariant over a base field.

Theorem 3.5.4. (Morel) If X is a motivic space over a field k, then $\pi_1^{\mathbb{A}^1}(X)$ is strongly $\mathbb{A}^1$-invariant.

Theorem 3.5.5. (Morel) If k is a perfect field, and $\mathcal{A}$ is a sheaf of abelian groups on Sm_k, then it is strong if and only if it is strict.

The proof of these theorems is hard — to see them worked out in detail we refer the reader to an amazing recent survey paper of Bachmann [Bac24].

Assumption 3.5.6. From here on out, we'll assume k is a perfect field, so that we can access these results.

Corollary 3.5.7. If $X \in \mathrm{Spc}(k)_*$, then $\pi_n^{\mathbb{A}^1}(X)$ is strictly invariant for $n \geq 2$ and strongly invariant for $n = 1$.

Proof. We apply Theorem 3.5.4 to $\Omega^{n-1}X$ and get that

$$\pi_n(X) = \pi_1(\Omega^{n-1}X)$$

is strongly invariant, which is strictly invariant for $n > 1$ by Theorem 3.5.5. □

3.5.1 Unramified sheaves

Let $\mathcal{F}$ be a sheaf of sets. We say it *ramifies* if, for some X, the map

$$\mathcal{F}(X) \to \bigcap_{x \in X^{(1)}} \mathcal{F}(\operatorname{Spec} \mathcal{O}_{X,x})$$

has nontrivial kernel (here the intersection takes place in $\mathcal{F}(\kappa(X))$). This is very related to the idea of *purity* for torsors.

Definition 3.5.8. [Mor12, 2.1] An *unramified presheaf of sets* on Sm_k is a presheaf $\mathcal{F}$ so that

1. If X has irreducible components $\{X_\alpha\}$, then
$$\mathcal{F}(X) \to \prod_\alpha \mathcal{F}(X_\alpha)$$
is a bijection
2. If $U \subseteq X$ is open and dense, then the restriction map
$$\mathcal{F}(X) \to \mathcal{F}(U)$$
is injective
3. The map
$$\mathcal{F}(X) \to \bigcap_{x \in X^{(1)}} \mathcal{F}(\operatorname{Spec} \mathcal{O}_{X,x})$$
is an injection.

Example 3.5.9. Any unramified presheaf of sets is automatically a Zariski sheaf.

Theorem 3.5.10. (Morel) Any strictly $\mathbb{A}^1$-invariant sheaf is unramified.

Example 3.5.11. Some other examples of unramified sheaves:

1. Homotopy modules with transfers
2. Rost cycle modules
3. Unramified étale cohomology (ref needed)

Unramified sheaves are determined in a precise sense by their values on fields, which we will now explain (and eventually reach in Theorem 3.5.17.

3.5.2 Unramified sheaves from theories on fields

Definition 3.5.12. A *homotopy sheaf* is a strongly (=strictly, therefore unramified) $\mathbb{A}^1$-invariant sheaf of abelian groups. We denote by $\mathrm{HI}(k) \subseteq \mathrm{Ab}_{\mathrm{Nis}}(k)$ the full subcategory of homotopy sheaves.

Theorem 3.5.13. [Mor12, 6.24] The subcategory $\mathrm{HI}(k) \subseteq \mathrm{Ab}_{\mathrm{Nis}}(k)$ is abelian, with exact inclusion.

We will see that elements in $\mathrm{HI}(k)$ are completely determined by their value on certain fields over k. Let's fix some notation.

Notation 3.5.14.

1. We let $\mathcal{F}_k \subseteq \mathrm{Alg}_k$ be the full subcategory of fields F so that F/k has finite transcendence degree [Mor12, p. vi].
2. We let $\mathcal{E}_k \subseteq \mathcal{F}_k$ be the full subcategory of those fields F which are further assumed to be finitely generated over k [Dég07, p. 43]. These are precisely the *algebraic function fields* over k, equivalently given as finite field extensions of $k(t_1, \ldots, t_n)$ for some n.

Remark 3.5.15. Since every $F \in \mathcal{F}_k$ is a (filtered) colimit over its finitely generated subfields, we conclude that $\mathcal{F}_k$ is generated by $\mathcal{E}_k$ under filtered colimits. We will be mostly concerned with continuous functors out of $\mathcal{F}_k$, which by definition are then determined by their values on $\mathcal{E}_k$.

We'll work with unramified sheaves of abelian groups here (c.f. [Fel21, §1.5]), but the more general definitions are for sheaves of pointed sets.

Definition 3.5.16. [Mor12, 2.6,2.9] An *unramified $\mathcal{F}_k$-datum* is the data of

D1 A continuous functor $M\colon \mathcal{F}_k \to \mathrm{Ab}$.

D2 For every $F \in \mathcal{F}_k$ and discrete valuation v on F, a subgroup
$$M(\mathcal{O}_v) \subseteq M(F),$$

D3 For every $F \in \mathcal{F}_k$ and valuation v on F a specialization map
$$s_v\colon M(\mathcal{O}_v) \to M(\kappa(v)).$$

This data is subject to the axioms

A1 If $(E, v) \subseteq (F, w)$ is a separable extension of discretely valued fields (so that $w_{|E} = v$), and v has ramificiation index 1 on E[3] then the square commutes

$$\begin{tikzcd} M(\mathcal{O}_w) \arrow[r, dashed] \arrow[d, hook] & M(\mathcal{O}_w) \arrow[d, hook] \\ M(E) \arrow[r] & M(F). \end{tikzcd}$$

If the field extension $\kappa(v) \to \kappa(w)$ is an isomorphism then this square is cartesian.

A2 If X is irreducible with function field F, and $m \in M(F)$, then m lies in all but a finite number of $S(\mathcal{O}_{X,x})$, where $x \in X^{(1)}$.

A3(i) If $(E, w) \subseteq (F, v)$ is an extension in $\mathcal{F}_k$, then the diagram commutes:

$$\begin{tikzcd} M(\mathcal{O}_v) \arrow[r, dashed] \arrow[d] & M(\mathcal{O}_w) \arrow[d] \\ M(\kappa(v)) \arrow[r] & M(\kappa(w)). \end{tikzcd}$$

[3]This means that $\pi_v = u\pi_w$ for some unit u, c.f. [Stacks, 09E4].

A3(ii) If $E \subseteq F$ is an extension in $\mathcal{F}_k$ and v a discrete valuation on F restricting to zero on E, and $j\colon E \hookrightarrow \kappa(v)$ the induced field extension, then the diagram commutes

$$\begin{tikzcd} & M(\kappa(v)) \\ M(E) \arrow[ur, "M(j)"] \arrow[r, dashed] \arrow[dr] & M(\mathcal{O}_v) \arrow[u, "s_v"'] \arrow[d, hook] \\ & M(F). \end{tikzcd}$$

A4(i) If $X \in \mathrm{Sm}_k$ is local of dimension two with closed point z, and $y_0 \in X^{(1)}$ has smooth reduced scheme $\bar{y}_0$, then the diagram commutes

$$\begin{tikzcd} \bigcap_{y\in X^{(1)}} M(\mathcal{O}_y) \arrow[r, dashed] \arrow[d, hook] & M(\mathcal{O}_{\bar{y}_0,z}) \arrow[d, hook] \\ M(\mathcal{O}_{y_0}) \arrow[r, "s_{y_0}"'] & M(\kappa(y_0)). \end{tikzcd}$$

A4(ii) The composite

$$\bigcap_{y\in X^{(1)}} M(\mathcal{O}_y) \to M(\mathcal{O}_{\bar{y}_0,z}) \to M(\kappa(z))$$

doesn't depend on the choice of y_0.

We say that M is *strongly unramified* if it satisfies some further axioms (see [Mor12, §2]).

Theorem 3.5.17. [Mor12, 2.27] By restricting $\mathcal{F} \in \mathrm{Ab}_{\mathrm{Nis}}(k)$ to its values on $\mathcal{F}_k$, we obtain the following results.

1. There is an equivalence of categories between unramified sheaves of abelian groups and unramified $\mathcal{F}_k$-data.
2. There is an equivalence of categories between $\mathrm{HI}(k)$ and *strongly* unramified $\mathcal{F}_k$ data.

This lets us take theories on fields and bootstrap them up to presheaves of abelian groups! Suppose that M is an unramified sheaf, and $X \in \mathrm{Sm}_k$ is an irreducible smooth scheme. Then we can define

$$M(X) := \bigcap_{x\in X^{(1)}} M(\mathcal{O}_x) \subseteq M(F).$$

If X is smooth with irreducible components X_α, we define

$$M(X) := \prod_{\alpha\in X^{(0)}} M(X_\alpha).$$

This defines M on objects. Now if $f\colon X \to Y$ is any morphism in Sm_k we can factor it as

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & Z \arrow[d, two heads] \\ & Y, \end{tikzcd}$$

where the first map is a closed immersion and the latter is a smooth projection. Factoring the closed immersion as a sequence of closed immersions, each of which is codimension one, we can define $M(X) \to M(Z)$ [Mor12, 2.13]. To define $M(Z) \to M(Y)$, we leverage that $Z \to Y$ is smooth and define it in terms of the induced map on function fields [Mor12, p. 17].

Motivation 3.5.18. We'll leverage this equivalence of categories to construct certain unramified sheaves (Milnor K-theory, Milnor–Witt K-theory, Witt theory, etc.) by defining them on fields. We'll then be able to have a strong handle on cohomology with coefficients in these elements in

$\mathrm{HI}(k)$, and we will better understand how to construct Eilenberg–MacLane spaces for them in the category of motivic spaces.

As an immediate application of this equivalence of categories, we obtain the following.

Theorem 3.5.19. Let $f\colon F \to G$ be a morphism of strictly invariant sheaves on Sm_k, where k is a perfect field. Then f is an isomorphism if and only if

$$f(L)\colon F(L) \to G(L)$$

is an equivalence for every finitely generated (separable) field extension L/k ([Mor12, 2.3, 2.8], c.f. [Hoy15, 2.7]).

Proof. By the equivalence above, we can restrict to $\mathcal{F}_k$, and continuity allows us to further restrict to finite separable field extensions. □

Corollary 3.5.20. Let $f\colon X \to Y$ be a map of motivic spaces in $\mathrm{Spc}(k)$, for k perfect. Then it is an equivalence if and only if it induces an isomorphism on π_0, and for $n \geq 1$ and every finitely generated field extension L/k we have that

$$\pi_n(X,x)(L) \to \pi_n(Y,y)(L)$$

is an equivalence.

3.6 Milnor K-theory

By Theorem 3.5.17, we can define unramified sheaves of abelian groups via their values on fields. In the following few sections we provide some examples.

Definition 3.6.1. *Milnor K-theory* of a field F is defined to be the graded algebra K_*^M generated by symbols $\{a\} \in K_1^M$ for $a \in F^\times$ subject to the relations

(M1) $\{a\} \cdot \{1-a\} = 0$

(M2) $\{ab\} = \{a\} + \{b\}$.

Notation 3.6.2. We denote by

$$\{a_1, \ldots, a_n\} := \{a_1\} \cdots \{a_n\} \in K_n^M(F).$$

Proposition 3.6.3. For any field F we have that

$$K_{-n}^M(F) = 0$$
$$K_0^M(F) = \mathbb{Z}$$
$$K_1^M(F) = F^\times.$$

Notation 3.6.4. We denote by $K_*^M(F) = \oplus_{n\geq 0} K_n^M(F)$. This is a graded abelian group, with multiplication coming from the multiplication of symbols.

3.6.1 Basic symbol algebra in K_*^M

Much of this can be found in [GS17, §7.1].

An immediate corollary of bilinearity of symbols is the following:

Proposition 3.6.5. In $K_1^M(F)$ we have that $\{1\} = 0$.

Proof. By **(M2)**, we have that $\{1 \cdot 1\} = \{1\} + \{1\}$, hence $0 = \{1\}$ by subtracting a copy of $\{1\}$ from either side. $\square$

Proposition 3.6.6. In $K_2^M(F)$ we have that

1. $-\{x, y\} = \left\{x^{-1}, y\right\}$.
2. More generally for any $i, j \in \mathbb{Z}$ we have
$$\left\{x^i, y^j\right\} = (i+j)\{x, y\}.$$

Proposition 3.6.7. [Mil69, 1.1] Multiplication on $K_*^M(F)$ is graded commutative, meaning if $\alpha \in K_m^M(F)$ and $\beta \in K_n^M(F)$, we have that
$$\alpha\beta = (-1)^{mn}\beta\alpha.$$

Proposition 3.6.8. [Mil69, 1.1, 1.2] We have that
$$\{x, -x\} = 0$$
$$\{x, x\} = \{x, -1\}.$$
In particular $2\{x, x\} = 0$ for any x.

Proof. Note for $x \neq 1$, we have that
$$-x = \frac{1-x}{1-x^{-1}}.$$
Hence
$$\{x, -x\} = \{x, 1-x\} - \left\{x, 1-x^{-1}\right\} = -\left\{x, 1-x^{-1}\right\} = \left\{x^{-1}, 1-x^{-1}\right\} = 0.$$
This last equality uses Proposition 3.6.6. The argument that $\{x, x\} = \{x, -1\}$ is similar. The last statement follows from observing that
$$2\{x, x\} = 2\{x, -1\} = \{x, 1\} = 0,$$
since $\{1\} = 0$. $\square$

3.6.2 Computations

Example 3.6.9. [Wei13, III.6.1, III.7.2] We have that $K_n^M(\mathbb{F}_q) = 0$ for $n \geq 2$.

Proof. We will show that it vanishes for $n = 2$. Pick x to generate $\mathbb{F}_q^\times$, then any element in $K_2^M(\mathbb{F}_q)$ is of the form $\left\{x^i, x^j\right\}$. By Proposition 3.6.6, this is equal to $(i+j)\{x, x\}$ so it will suffice to verify that $\{x, x\} = \{x, -1\}$ vanishes. This has order dividing two. We want to show it is killed by an odd number as well, and we'll be done. If $q = 2^m$, then $x^{2^m-1} = 1$, and hence
$$0 = \{1, x\} = \left\{x^{2^m-1}, x\right\} = (2^m - 1)\{x, x\},$$
which concludes the proof. If q has odd exponential characteristic, we can find two non-squares in $\mathbb{F}_q$ which sum to 1 (c.f. [GS17, 1.3.6]). This gives us $x^k + x^\ell = 1$ for k, ℓ odd. Hence we get
$$0 = \left\{x^k, x^\ell\right\} = (k+\ell)\{x, x\},$$
and we are done. $\square$

Example 3.6.10. [Wei13, III.7.2] If $F = \bar{F}$ is algebraically closed, then $K_n^M(F)$ is uniquely divisible for $n \geq 1$. In particular $K_n^M(F)/\ell = 0$ for any $n \geq 1$ and any $\ell \neq 0$.

These come equipped with subrings
$$K_*^M(\mathcal{O}_v) \subseteq K_*^M(F)$$

for every discrete valuation v on F, and specialization maps, which we omit (see Milnor's original paper or [Wei13, 7.3]. These satisfy the axioms for being an unramified $\mathcal{F}_k$-datum, proving the following.

Proposition 3.6.11. Milnor K-theory gives rise to an unramified sheaf of abelian groups $\mathbf{K}_n^{\mathrm{M}}$ for every n.

We will see later that the sheaf cohomology of $\mathbf{K}_n^{\mathrm{M}}$ computes the Chow groups of a scheme.

Proposition 3.6.12. There is a *symbol map* for any field F valued in Quillen K-theory

$$K_n^M(F) \to K_n(F),$$

which is an isomorphism for $n \leq 2$. (For $n = 0, 1$ this is easy, for $n = 2$ this is *Matsumoto's theorem*).

3.7 Milnor–Witt K-theory

We refer the reader to [Dég23; Car23] for more in-depth discussions of what's found here.

Definition 3.7.1. *Milnor–Witt K-theory* of a field F is defined to be the graded algebra $K_*^{\mathrm{MW}}(F)$ defined by symbols $[a] \in K_1^{\mathrm{MW}}(F)$ for $a \in F^\times$ and $\eta \in K_{-1}^{\mathrm{MW}}(F)$, modulo the relations:

(MW1) $[a][1-a] = 0$ for $a \neq 0, 1$
(MW2) $[ab] = [a] + [b] + \eta[a][b]$
(MW3) $\eta[a] = [a]\eta$
(MW4) $\eta(2 + \eta[-1]) = 0$.

It will benefit us to have some notation for various special elements in Milnor–Witt K-theory.

Notation 3.7.2. *(Special elements in $K_*^{\mathrm{MW}}(F)$)*

1. For any $a \in F$ we denote by
$$\langle a \rangle := 1 + \eta[a] \in K_0^{\mathrm{MW}}(F).$$
2. We denote by $h := 1 + \langle -1 \rangle = 2 + \eta[-1]$ the *hyperbolic element.*
3. We denote by ϵ the element
$$\epsilon = -\langle -1 \rangle = -(1 + \eta[-1]) \in K_0^{\mathrm{MW}}(F).$$

Remark 3.7.3. Observe that relation MW4 can take either of the following equivalent forms

(MW4) $\eta h = 0$
(MW4) $\epsilon\eta = \eta$.

3.7.1 Symbol algebra in Milnor–Witt K-theory

We will develop some basic properties, starting in lower degrees and going to higher degrees.

Proposition 3.7.4. *(Properties in K_0^{MW})*

1. $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for any $a, b \in F^\times$.
2. $1 = \langle 1 \rangle$ is the multiplicative unit.
3. $\eta[1] = 0$.
4. $\epsilon^2 = 1$

Proof.

1. We see that
$$\langle a\rangle \langle b\rangle = (1+\eta[a])(1+\eta[b]) = 1+\eta([a]+[b])+\eta^2[a][b]$$
$$\overset{(\mathrm{MW2})}{=} 1+\eta([ab]-\eta[a][b])+\eta^2[a][b] = 1+\eta[ab] = \langle ab\rangle .$$
2. By definition, we have that $\eta[1] = \langle 1\rangle - 1$. Multiplying $[1]$ into (MW4) we get that
$$\begin{aligned} 0 &= \eta[1](1+\langle -1\rangle) = (\langle 1\rangle - 1)(1+\langle -1\rangle) \\ &= \langle 1\rangle - 1 + \langle -1\rangle - \langle -1\rangle \\ &= \langle 1\rangle - 1. \end{aligned}$$
Hence $1 = \langle 1\rangle$.
3. Since $\eta[1] = \langle 1\rangle - 1$, we get that $\eta[1] = 0$ by the previous result.
4. By 1, it is clear that $\epsilon^2 = \langle 1\rangle$, which is equal to 1 by 2.

□

In degree one, we have that $[1] = 0$, completely analogous to Milnor K-theory, and various other properties:

Proposition 3.7.5. *(Properties in K_1^{MW})* Some basic properties to record are that $[1]$ vanishes, analogous to Milnor K-theory, as well as the following commutativity relations:
1. $[1] = 0$.
2. $\eta[a][b] = \eta[b][a]$
3. $[a]\langle b\rangle = \langle b\rangle [a]$.
4. $\epsilon[a] = [a]\epsilon$

Some further relations let us expand various degree one elements, and are useful in further computation:

4. $[a^2] = (1+\langle a\rangle)[a]$
5. $[a] = -\langle a\rangle [a^{-1}]$
6. $[ab] = [a] + \langle a\rangle [b]$
7. $[a/b] = [a] - \langle a/b\rangle [b]$

Proof.
1. By MW2 we get
$$[1\cdot 1] = [1]+[1]+\eta[1][1].$$
Since $\eta[1] = 0$ by Proposition 3.7.4, we conclude that $[1] = [1]+[1]$, from which the result follows.
2. Since $ab = ba$ in F, we have that $[ab] = [ba]$. Expanding each of these using MW2 gives the desired result.
3. By applying item 2, we see that
$$[a]\langle b\rangle = [a](1+\eta[b]) = [a]+\eta[a][b] = [a]+\eta[b][a] = \langle b\rangle [a].$$
4. This follows from Proposition 3.7.5(3) with $b = -1$ and a negative sign.
5. Applying MW2 we get
$$[a^2] = 2[a]+\eta[a][a] = (2+\eta[a])[a] = (1+\langle a\rangle)[a].$$
6. Applying MW2 to a, a^{-1} we get
$$0 = [1] = [a^{-1}]+[a]+\eta[a^{-1}][a] = [a^{-1}] + \left\langle a^{-1}\right\rangle [a],$$
Subtracting $[a^{-1}]$ from both sides then replacing a with a^{-1} yields the desired result.
7. We see that
$$[ab] = [a]+[b]+\eta[a][b] = [a]+\langle a\rangle [b].$$

8. This is a previous case of the previous result:
$$[a] = \left[\frac{a}{b}b\right] = \left[\frac{a}{b}\right] + \langle a/b\rangle [b].$$
Rearranging, the result follows.

□

In degree two, we see the value of ϵ, that it measures the failure of commutativity:

Proposition 3.7.6. *(Properties in K_2^{MW})*

1. $[a][-a] = 0$
2. $[a][a] = \epsilon[a][-1] = \epsilon[-1][a]$
3. $[a][-1] = [-1][a]$.
4. $[a][b] = \epsilon[b][a]$.

Proof.

1. We do the same trick as in Proposition 3.6.8, and write $-a = \frac{1-a}{1-a^{-1}}$. By item 7, we get
$$[-a] = [\frac{1-a}{1-a^{-1}}] = [1-a] - \langle -a\rangle [1-a^{-1}].$$
Scaling through by $[a]$ we get
$$[a][-a] = -[a]\langle -a\rangle [1-a^{-1}].$$
So this reduces to showing that
$$[a]\langle -a\rangle [1-a^{-1}] = 0.$$
We can commute $[a]$ and $\langle a\rangle$ by Proposition 3.7.5(3), then expand $[a]$ using Proposition 3.7.5(5) to get
$$\langle -a\rangle [a][1-a^{-1}] = -\langle -a\rangle \langle a\rangle [a^{-1}][1-a^{-1}].$$
The latter two terms multiply to zero by MW1.
2. By Proposition 3.7.5(6), we have
$$[-a] = [-1] + \langle -1\rangle [a]. \tag{3.7}$$
Multiplying Equation 3.7 by $[a]$ on the left and applying 1 we get
$$0 = [a][-1] + [a]\langle -1\rangle [a] = [a][-1] + \langle -1\rangle [a][a].$$
Rearranging, we get
$$\langle -1\rangle [a][a] = -[a][-1],$$
and multiplying through by $\langle -1\rangle$ gives the desired result.
Multiplying Equation 3.7 by $[a]$ on the right instead, we get
$$0 = [-1][a] + \langle -1\rangle [a][a],$$
which gives us a similar equality.
3. By Proposition 3.7.6(2) we have $\epsilon[a][-1] = \epsilon[-1][a]$. Multiplying both sides by ϵ and using that $\epsilon^2 = 1$ (Proposition 3.7.4(4)) gives the desired result.
4. We can write
$$\begin{aligned}0 = [ab][-ab] &= ([a] + \langle a\rangle [b])([-a] + \langle -a\rangle [b])\\ &= \langle -a\rangle [a][b] + \langle a\rangle [b][-a] + \langle -1\rangle [b][b].\end{aligned}$$
Scaling through by $\langle a\rangle$ we get
$$0 = -\epsilon[a][b] + [b][-a] + \langle a\rangle \langle -1\rangle [b][b]$$
Since $[-a] = [a] + \langle a\rangle [-1]$ by Proposition 3.7.5(6), and $\langle -1\rangle [b][b] - [b][-1]$ by Proposi-

tion 3.7.6(2), we can plug these into the above equation to get

$$\begin{aligned} 0 &= -\epsilon[a][b] + [b]\left([a] + \langle a\rangle [-1]\right) - \langle a\rangle [b][-1] \\ &= -\epsilon[a][b] + [b][a] + \langle a\rangle [b][-1] - \langle a\rangle [b][-1] \\ &= -\epsilon[a][b] + [b][a]. \end{aligned}$$

□

3.7.2 Comparison to Milnor K-theory and Grothendieck–Witt

Proposition 3.7.8. There is a surjective homomorphism of graded algebras

$$\begin{aligned} K_*^{\mathrm{MW}}(F) &\to K_*^M(F) \\ [a] &\mapsto \{a\} \\ \eta &\mapsto 0. \end{aligned}$$

That is, Milnor K-theory is obtained from Milnor–Witt K-theory by killing η.

Proof. Relations M1 and M2 are just relations MW1 and MW2 after modding out by η, so the map is well-defined, and it is clearly surjective since $\{u_1, \ldots, u_n\}$ is hit by $[u_1, \ldots, u_n]$. □

Definition 3.7.9. Let F be a field of characteristic $\neq 2$. We define the *Grothendieck–Witt ring* of F, denoted $\mathrm{GW}(F)$ to be the ring of isomorphism classes of non-degenerate symmetric bilinear forms, group completed.

Proposition 3.7.10. $\mathrm{GW}(F)$ is generated by rank one forms

$$\begin{aligned} \langle a\rangle : k \times k &\to k \\ (x, y) &\mapsto axy, \end{aligned}$$

modulo the relations

(GW1) $\langle ab^2\rangle = \langle a\rangle$ for any $a, b \in F^\times$
(GW2) $\langle a\rangle + \langle b\rangle = \langle ab(a+b)\rangle + \langle a+b\rangle$
(GW3) $\langle a\rangle \langle b\rangle = \langle ab\rangle$.

Exercise 3.7.11. Show that

$$\langle 1\rangle + \langle -1\rangle = \langle a\rangle + \langle -a\rangle$$

for any $a \in F^\times$. This is sometimes taken as a relation, but it is implied by (GW1) and (GW2).

Notation 3.7.12. We often save space and write

$$\begin{aligned} \{a_1, \ldots, a_n\} &:= \{a_1\} \cdots \{a_n\} \in K_n^{\mathrm{MW}}(F) \\ [a_1, \ldots, a_n] &:= [a_1] \cdots [a_n] \in K_n^M(F) \\ \langle a_1, \ldots, a_n\rangle &:= \langle a_1\rangle + \ldots + \langle a_n\rangle \in \mathrm{GW}(F). \end{aligned}$$

Proposition 3.7.13. There is a ring isomorphism

$$\begin{aligned} \mathrm{GW}(F) &\to K_0^{\mathrm{MW}}(F) \\ \langle a\rangle &\mapsto \langle a\rangle. \end{aligned}$$

Proof. We first check this is well-defined, in that the map respects the relations for the Grothendieck–Witt ring. Clearly **(GW3)** holds by Proposition 3.7.4(1). Since $\langle a\rangle$ is multiplicative, checking **(GW1)** reduces to showing that $\langle b^2\rangle = 1$ in K_0^{MW}, which reduces to checking that $\eta[b^2] = 0$. By MW2, and Proposition 3.7.5(4) we have that

$$[b^2] = 2[b] + \eta[b][b] = 2[b] + \eta[-1][b] = (2 + \eta[-1])[b].$$

Multiplying by η and applying MW4 gives zero. Finally we want to check **(GW2)**, and by multiplicativity we can assume that $b = 1 - a$, from which we get

$$\begin{aligned}
\langle a \rangle + \langle 1-a \rangle &= 1 + \eta[a] + 1 + \eta[1-a] \\
&= 2 + \eta\left([a] + [1-a]\right) \\
&\overset{\text{(MW2)}}{=} 2 + \eta\left([a(1-a)] - \eta[a][1-a]\right) \\
&\overset{\text{(MW1)}}{=} 2 + \eta[a(1-a)] \\
&\overset{[1]=0}{=} 1 + \eta[a(1-a)] + 1 + \eta[1] \\
&= \langle a(1-a) \rangle + \langle 1 \rangle\,.
\end{aligned}$$

It is clear by construction that this map is injective, so it suffices to see it is surjective by verifying that every element in $K_0^{\mathrm{MW}}(F)$ is a sum of elements in the image of the homomorphism produced above. Since $1 = \langle 1 \rangle$, we have that $\eta[a] = \langle a \rangle - \langle 1 \rangle$ is in the image of the homomorphism above. Iterated application of (MW2) and (MW3) yields the desired result. $\square$

Via the equivalence of categories between unramified sheaves of groups and unramified data for fields, we get sheaves corresponding to each of the invariants above.

Proposition 3.7.14. [Mor12, p. 71] $\mathbf{K}_n^{\mathrm{MW}}$ is unramified and strongly $\mathbb{A}^1$-invariant.

Warning 3.7.15. This is a bit of a lie. Unlike the case of Milnor K-theory, the residue homomorphisms for Milnor–Witt K-theory depend on a choice of uniformizing parameter, so we have to be careful about twists here.

3.8 The Milnor conjecture

Definition 3.8.1. We define the *Witt ring* to be the quotient $W(F) := \mathrm{GW}(F)/h$.

Definition 3.8.2. We define the *fundamental ideal* $I(F)$ to be the kernel of the modulo two rank homomorphism $W(F) \to \mathbb{Z}/2$. We denote by I^j the powers of the fundamental ideal.

Proposition 3.8.3. We obtain unramified sheaves of groups

$$\mathbf{GW},\ \mathbf{W},\ \mathbf{I}^j,$$

associated to each of these $\mathcal{F}_k$-data.

Proposition 3.8.4. There is a pullback square of abelian groups for any field F

$$\begin{tikzcd}
\mathrm{GW}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d] \\
W(F) \arrow[r] & \mathbb{Z}/2.
\end{tikzcd}$$

This extends to a pullback of unramified sheaves of groups.

Proposition 3.8.5.

1. The fundamental ideal is equivalently the kernel of the rank homomorphism $\mathrm{GW}(F) \to \mathbb{Z}(F)$, which we denote by $\hat{I}(F)$. Explicitly, there is an isomorphism

$$\begin{tikzcd}
\hat{I}(F) \arrow[r] \arrow[d, "\cong"'] & \mathrm{GW}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d] \\
I(F) \arrow[r] & W(F) \arrow[r] & \mathbb{Z}/2.
\end{tikzcd}$$

2. $\hat{I}(F)$ is generated by elements of the form $\langle 1\rangle - \langle a\rangle$ for $a \in F^\times$
3. The isomorphism $\hat{I}(F) \to I(F)$ is given by sending
$$\begin{aligned} \hat{I}(F) &\to I(F) \\ \langle 1\rangle - \langle a\rangle &\mapsto \langle 1, -a\rangle . \end{aligned}$$

Definition 3.8.6. For $a \in F^\times$ we denote by $\langle\langle a\rangle\rangle := \langle 1\rangle - \langle a\rangle$ the *Pfister form* attached to a.

Remark 3.8.7. (On notation) Authors in the literature use $\langle\langle a\rangle\rangle$ to denote $\langle 1\rangle - \langle a\rangle \in \mathrm{GW}(F)$ as we have done, or to denote $\langle 1, -a\rangle \in W(F)$ (see e.g. [EKM08, p. 24]. We should be careful about the context when using this notation, and we also warn the reader that a different sign convention is used in [Lam05, X.1.1].

Example 3.8.8. *(Examples of fundamental ideals)*

1. Every Pfister form is hyperbolic over an algebraically closed field, or even just a quadratically closed field. Hence $I(F) = 0$ if $F = \bar{F}$.
2. $I(\mathbb{R}) \cong \mathbb{Z}$, generated by $\langle\langle -1\rangle\rangle$. We can check that
$$\langle\langle -1\rangle\rangle^n = 2^{n-1}\langle\langle -1\rangle\rangle ,$$
therefore $I(\mathbb{R}) \cong I^2(\mathbb{R}) \cong \cdots \cong I^n(\mathbb{R}) \cong \mathbb{Z}$, and we have that
$$I^n(\mathbb{R})/I^{n+1}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}.$$
3. This is a more general fact for any field that
$$I(F)/I^2(F) \cong F^\times/\left(F^\times\right)^2 .$$
This is a result of Pfister (c.f. [Lam05, II.2.3]).
4. For any F we have that
$$I^2(F)/I^3(F) \cong \mathrm{Br}(F)[2]$$
is the 2-torsion in the Brauer group.

Proposition 3.8.9. Given two quadratic forms $\alpha, \beta \in \mathrm{GW}(F)$, they agree if and only if they agree modulo $I^n(F)$ for every n, and therefore if and only if they agree in the associated graded $I^n(F)/I^{n+1}(F)$ for every n.

Proof idea. It suffices to argue that
$$\bigcap_{n=0}^{\infty} I^n(F) = 0,$$
since we can show that if $\alpha - \beta \in \bigcap_{n=0}^{\infty} I^n(F)$, then $\alpha = \beta$. The question of whether this intersection is zero was first raised by Milnor [Mil69, 4.4], and proven shortly thereafter by Aarason and Pfister as a consequence of their Hauptsatz [Lam05, X.5.1]. □

Proposition 3.8.10. (Milnor) There is a homomorphism
$$\begin{aligned} K_n^M(F)/2 &\to I^n(F)/I^{n+1}(F) \\ \{a_1, \dots, a_n\} &\mapsto \prod_{i=1}^{n} \langle\langle a_i\rangle\rangle . \end{aligned} \tag{3.11}$$

This homomorphism was constructed and shown to be surjective by Milnor [Mil69, 4.1], who conjectured it was bijective for all n [Mil69, 4.3].

Theorem 3.8.12. (Milnor Conjecture, Orlov–Vishik–Voevodsky) The homomorphism Equation 3.11 is an isomorphism. Moreover, we obtain an isomorphism of unramified sheaves of groups
$$\mathbf{K}_n^{\mathrm{M}}(F)/2 \xrightarrow{\sim} \mathbf{I}^n/\mathbf{I}^{n+1}.$$

Motivation 3.8.13. We'll double back and compute cohomology in these theories. We first want to show these cohomology groups are *representable* by Eilenberg–MacLane spaces. Formal consequences of representability will motivate the construction of complexes which resolve these sheaves and let us carry out computations more directly.

Corollary 3.8.14. There is a pullback diagram of unramified sheaves

$$\begin{tikzcd} \mathbf{K}_n^{\mathrm{MW}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{K}_n^{\mathrm{M}} \arrow[d] \\ \mathbf{I}^n \arrow[r] & \mathbf{K}_n^{\mathrm{M}}/2. \end{tikzcd}$$

Remark 3.8.15.

1. Note that when $n = 0$, the pullback diagram in Corollary 3.8.14 recovers that of Proposition 3.8.4.
2. The map $\mathbf{K}_n^{\mathrm{MW}} \to \mathbf{I}^n$ is given by sending $[a]$ to $\langle\langle a \rangle\rangle$.

3.9 Eilenberg–MacLane spaces

We denote by $B_{\mathrm{mot}}G := L_{\mathrm{mot}}BG$, and we have seen the following:

Proposition 3.9.1. If X is a scheme, then

$$\pi_0 \mathrm{Map}_{\mathrm{Spc}(k)} \left(L_{\mathrm{mot}} h_X, B_{\mathrm{mot}} G \right) \cong H^1_{\mathrm{Nis}}(X, G).$$

We wonder whether higher Nisnevich cohomology is represented by Eilenberg–MacLane spaces? The answer is yes!

Theorem 3.9.2. (Dold–Kan) If $\mathrm{Ab}_{\mathrm{Nis}}(k)$ denotes the category of Nisnevich sheaves of abelian groups on X, there is an equivalence of categories

$$\mathrm{Ch}_{\geq 0}(\mathrm{Ab}_{\mathrm{Nis}}(k)) \cong \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}_{\mathrm{Nis}}(k)).$$

Given a chain complex of abelian sheaves, we can view it as an object on the right hand side. Forgetting the levelwise sheaf structure, we can view it as a simplicial presheaf of abelian groups, and therefore a presheaf of simplicial sets:

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}_{\mathrm{Nis}}(k)) \subseteq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{Ab})) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Fun}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{Set})) \cong \mathrm{PSh}(\mathrm{Sm}_k).$$

We'll note by DK this composite:

$$\mathrm{DK}\colon \mathrm{Ch}_{\geq 0}\left(\mathrm{Ab}_{\mathrm{Nis}}(k)\right) \to \mathrm{PSh}(\mathrm{Sm}_k).$$

Proposition 3.9.3. If $\mathcal{A} \in \mathrm{Ch}_{\geq 0}(\mathrm{Ab}_{\mathrm{Nis}}(k))$, then there is an isomorphism

$$H_n(\mathcal{A}) \cong \pi_n(L_{\mathrm{Nis}}\mathrm{DK}(\mathcal{A})).$$

Proof. This follows from the more general statement that the homology of a chain complex agrees with the simplicial homotopy groups of the associated simplicial abelian group produced by the Dold–Kan correspondence (c.f. [GJ99, III.2.5]). □

Definition 3.9.4. For any $A \in \mathrm{Ab}_{\mathrm{Nis}}(k)$ we denote by

$$K(A, n) \in \mathrm{PSh}(\mathrm{Sm}_k)$$

the space $K(A, n) := \mathrm{DK}(A[n])$ given by applying the Dold–Kan construction to the chain complex with A concentrated in degree n.

Proposition 3.9.5. *(Properties of $K(A, n)$)*

1. We have that $K(A, n)$ is already an object in the sheaf topos $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$
2. We have that

$$\pi_i K(A, n) = \begin{cases} A & i = n \\ 0 & \text{else} \end{cases}$$

There is a natural identification

$$\pi_0 \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Nis}}}(-, K(A, n)) \cong H^n_{\mathrm{Nis}}(-, A).$$

Proof references. The classical way to approach this is using explicit model structures on simplicial presheaves and chain complexes of sheaves of abelian groups, and arguing that the construction of an Eilenberg–MacLane object preserves fibrancy, which is implies that $K(A, n)$ is a sheaf in the local model structure. See [Mor12, Chapter 6] or [MV99, pp.56—59] for this approach. The high-level perspective on Eilenberg–MacLane objects in a general ∞-topos is in [Lur09, §7.2.2], from which these properties are formal. □

3.9.1 Strong and strict invariance revisited

With EM spaces in hand in the sheaf topos, we can reframe our definitions of strong and strict invariance.

Proposition 3.9.6. Let $\mathcal{G}$ be a sheaf of groups and $\mathcal{A}$ a sheaf of abelian groups. Then

1. $\mathcal{G}$ is strongly invariant if and only if $B_{\mathrm{Nis}}\mathcal{G}$ is $\mathbb{A}^1$-local
2. $\mathcal{A}$ is strictly invariant if and only if $K(\mathcal{A}, n)$ is $\mathbb{A}^1$-local for every $n \geq 0$.

Proof. The backwards direction is immediate by representability of cohomology. The forwards direction needs a nontrivial argument, see [Bac24, 1.5]. For the forwards direction, we want to see that

$$(B_{\mathrm{Nis}}G)(X) \to (B_{\mathrm{Nis}}G)(X \times \mathbb{A}^1)$$

is an equivalence of spaces. Each space is 1-truncated and we have an isomorphism on π_0 since $H^0(-, G)$ is $\mathbb{A}^1$ invariant. So we just want to show an isomorphism on π_1 for each choice of basepoint $x \in X$. Since the presheaf

$$(\mathrm{Sm}_k)_{/X} \to \mathrm{Grp}$$
$$Y \mapsto \pi_1\left((B_{\mathrm{Nis}}G)(Y), x\right)$$

is a Nisnevich sheaf (equivalent to $\Omega_x(B_{\mathrm{Nis}}G)_{|X}$), we can check it is $\mathbb{A}^1$-invariant locally. Locally x is a trivial torsor, in which case the presheaf above is $H^0 -, G)$ which we assumed $\mathbb{A}^1$-invariant. An analogous argument works for the forward direction of $K(\mathcal{A}, n)$. □

We now have a list of nice properties that homotopy sheaves satisfy. For $\mathcal{A} \in \mathrm{HI}(k)$ we have that

1. $\mathcal{A}$ is determined by its underlying $\mathcal{F}_k$-datum
2. $K(\mathcal{A}, n)$ is a motivic space for each $n \geq 0$
3. $H^n_{\mathrm{Nis}}(-, \mathcal{A})$ is $\mathbb{A}^1$-invariant for any $n \geq 0$
4. Isomorphisms between $\mathcal{A} \to \mathcal{B}$ in $\mathrm{HI}(k)$ can be checked on separable field extensions of the base
5. $\mathcal{A}$ lives in an abelian category $\mathrm{HI}(k)$

Proposition 3.9.7. For any sheaf of abelian groups, we have that

$$\Omega K(A, n) \cong K(A, n-1).$$

Proof. The statement in the sheaf topos is a formal consequence of the fact that $\pi_i(\Omega X) = \pi_{i+1}X$. $\square$

If A is strictly invariant, we will prove that this is an equivalence of motivic spaces as well. It's not immediately obvious that this is the case — we first need to know that computing loops in the sheaf topos agrees with computing loops in motivic spaces. The following argument lets us prove this.

Remark 3.9.8. If $B \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-local, and ΩB is $\mathbb{A}^1$-local, then

$$\Omega B = \lim_{\mathrm{Spc}(k)} (* \to B \leftarrow *) = \lim_{\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)} (* \to B \leftarrow *).$$

Corollary 3.9.9. If A is strictly $\mathbb{A}^1$-invariant, we have an identification in $\mathrm{Spc}(k)$ of the form

$$\Omega^{1,0}K(A,n) \simeq K(A,n-1).$$

3.9.2 Cofiber sequences

We say that

$$X \to Y \to C$$

is a *cofiber sequence* in $\mathrm{Spc}(k)_*$ if the following diagram is a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & C. \end{tikzcd}$$

Since adding a disjoint basepoint is a left adjoint, it doesn't matter if X and Y are pointed or not, only C. In this setting, we have the following result.

Proposition 3.9.10. For G any strongly invariant sheaf of groups and any cofiber sequence $X \to Y \to C$, we obtain a long exact sequence

$$0 \to H^0_{\mathrm{Nis}}(C;G) \to H^0_{\mathrm{Nis}}(Y,G) \to H^0_{\mathrm{Nis}}(X;G)$$
$$\to H^1_{\mathrm{Nis}}(C;G) \to H^1_{\mathrm{Nis}}(Y,G) \to H^1_{\mathrm{Nis}}(X;G)$$

if G is a sheaf of abelian groups (hence strictly invariant), this continues to H^2 and so on.

Proposition 3.9.11. If $X \to Y \to C$ is a cofiber sequence in $\mathrm{Spc}(k)$, and $B \in \mathrm{Spc}(k)$ is arbitrary, then

$$X \times B \to Y \times B \to C \wedge B_+$$

is a cofiber sequence.

Proof. This is a direct consequence of universality of colimits in $\mathrm{Spc}(k)$. Alternatively, we can use that $B \times -$ is a left adjoint, to see that

$$\begin{tikzcd} X \times B \arrow[r] \arrow[d] & Y \times B \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C \times B \end{tikzcd}$$

is a pushout. The induced maps on the cofibers of the horizontal arrows will be an equivalence, and the bottom is clearly $C \wedge B_+$. $\square$

Example 3.9.12. For any motivic space B, we have a cofiber sequence

$$\mathbb{G}_m \times B \to B \to \mathbb{P}^1 \wedge B_+.$$

This follows from Example 3.3.6 and Proposition 3.9.11.

3.10 Contraction

Definition 3.10.1. If F is a sheaf of pointed sets, we define the *contraction* of F to be the sheafification of the presheaf F_{-1}, where F_{-1} is defined to be the kernel in the short exact sequence

$$0 \to F_{-1}(U) \to F(U \times \mathbb{G}_m) \to F(U) \to 0,$$

and the latter map is $\mathrm{id} \times 1 \colon U \to U \times \mathbb{G}_m$. See [ABH23, 2.1.10].

Remark 3.10.2. Since $F(U \times \mathbb{G}_m) \to F(U)$ is split by the inclusion of units, contraction is equivalently defined as the cokernel of the projection off of $\mathbb{G}_m$:

$$0 \to F(U) \to F(U \times \mathbb{G}_m) \to F_{-1}(U) \to 0.$$

Proposition 3.10.3. For any $U \in \mathrm{Sm}_k$ and any strongly $\mathbb{A}^1$-invariant sheaf of groups G, we have that $H^0(\mathbb{P}^1 \wedge U_+, G) = 0$.

Proof. This follows by connectivity – smashing with $\mathbb{P}^1$ (and hence S^1) makes $\mathbb{P}^1 \wedge U_+$ connected, so every map into $G = K(G, 0)$ is trivial. □

Remark 3.10.4. If G is strongly $\mathbb{A}^1$-invariant, we have our cofiber sequence

$$\mathbb{G}_m \times U \to U \to \mathbb{P}^1 \wedge U_+.$$

Let's look at the long exact sequence with coefficients in G, together with the vanishing of $H^0(\mathbb{P}^1 \wedge U_+, G)$ by Proposition 3.10.3. Then there is an induced map out of the contraction

$$\begin{tikzcd} 0 \arrow[r] & G(U) \arrow[r, hook] & G(U \times \mathbb{G}_m) \arrow[r] \arrow[d, two heads] & H^1(\mathbb{P}^1 \wedge U_+) \arrow[r] & \cdots \\ & & G_{-1}(U) \arrow[ur, dashed] & & \end{tikzcd}$$

Lemma 3.10.5. [Mor12, 2.34] Let G be strongly $\mathbb{A}^1$-invariant and $U \in \mathrm{Sm}_k$. Then the canonical map:

$$G_{-1}(U) \to H^1(\mathbb{P}^1 \wedge U_+, G).$$

is a bijection (an isomorphism if G is abelian).

Proof. The map

$$H^1(U, G) \to H^1(\mathbb{G}_m \times U, G)$$

is split by evaluation at one and hence injective, so the sequence

$$0 \to G(U) \to G(U \times \mathbb{G}_m) \to H^1(\mathbb{P}^1 \wedge U_+) \to 0$$

is exact, from which the result follows. □

Corollary 3.10.6. For any strongly $\mathbb{A}^1$-invariant sheaf of abelian groups, we have that

$$\Omega^{2,1} K(G, 1) = K(G_{-1}, 0) = G_{-1}.$$

Proof. We just verified there is a canonical bijection of presheaves

$$G_{-1}(-) \cong H^1(\mathbb{P}^1 \wedge (-)_+, G) = \pi_0 \mathrm{Map}(\Sigma^{2,1}(-)_+, BG) = \pi_0 \mathrm{Map}((-)_+, \Omega^{2,1} BG).$$

□

This suggests that $\mathbb{A}^1$-invariance of a sheaf is related to $\mathbb{A}^1$-invariance of its contraction.

Proposition 3.10.7. (Properties of contraction), c.f. [Bac24, §4.1]

1. If F is a sheaf of (abelian) groups, so is F_{-1}

2. If F is $\mathbb{A}^1$-invariant, so is F_{-1}.

Proof. The first statement is clear from its definition as a presheaf. The second statement follows from Corollary 3.10.6 — if G is strongly invariant, then $K(G_{-1}, 0) = G_{-1}$ is a motivic space, hence $\mathbb{A}^1$-invariant. □

Lemma 3.10.8. If G is a strongly $\mathbb{A}^1$-invariant sheaf of groups, then

$$\Omega_{\mathbb{G}_m} K(G, 1) \simeq K(G_{-1}, 1).$$

Proof sketch. Easy to check that $\pi_1 K(G_{-1}, 1) = G_{-1}$. What's less obvious is $\Omega_{\mathbb{G}_m} K(G, 1)$ is connected. We've learned we can check connectivity on fields, so this reduces to asking that

$$[\mathrm{Spec}\,(F)_+, \Omega_{\mathbb{G}_m} K(G, 1)]_{\mathbb{A}^1}$$

is trivial for each F/k finitely generated. By adjunction and representability of cohomology, this is asking whether $H^1(\mathbb{G}_m(F), G)$ is trivial. We can pass to the Zariski site by some formal nice properties of strongly invariant sheaves of groups (c.f. [Mor12, 2.24]), and then we can conclude by an argument involving 1-cocycles (see e.g. [Bac24, 4.2]). □

Corollary 3.10.9.

1. If G is a strongly $\mathbb{A}^1$-invariant sheaf of groups, then so is G_{-1}.
2. If A is a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups. Then there is a canonical equivalence

$$\Omega_{\mathbb{G}_m} K(A, n) \xrightarrow{\sim} K(A_{-1}, n).$$

Proof.

1. Since G was strongly $\mathbb{A}^1$-invariant, we have that $B_{\mathrm{Nis}} G = K(G, 1)$ is $\mathbb{A}^1$-local by Proposition 3.9.6. Since $\Omega_{\mathbb{G}_m}$ preserves $\mathbb{A}^1$-invariant objects, we have that $\Omega_{\mathbb{G}_m} B_{\mathrm{Nis}} G$ is $\mathbb{A}^1$-local as well. Hence the result follows.
2. It is clear by adjunction that

$$\pi_n \Omega^{1,1} K(A, n) = A_{-1},$$

and the higher homotopy groups vanish. To check that $\pi_i \Omega^{1,1} K(A, n) = 0$ for $i < n$, it suffices to see that $H^i(\mathbb{G}_m(F), A) = 0$ for all $i < n$. This is clear for $i \geq 2$ since $\mathbb{G}_m$ has Krull dimension one, and the case $i = 1$ follows by the proof of Lemma 3.10.8.

□

Remark 3.10.10. It is clear from Corollary 3.10.9(1) that if A is strictly $\mathbb{A}^1$-invariant then so is A_{-1} by Theorem 3.5.5. However the argument in Corollary 3.10.9(2) also allows us to prove that contraction preserves strict invariance without reference to the "strong=strict" theorem.

Corollary 3.10.11. We have that

$$\Omega^{2,1} K(A, n) \cong K(A_{-1}, n-1).$$

This is a higher-dimensional generalization of Corollary 3.10.6.

3.10.1 Examples of contraction

We want to compute contractions of these sheaves we've developed, being Milnor/Milnor–Witt K-theory, etc.. To do this, we introduce the so-called *Milnor exact sequence*.

Theorem 3.10.12. [Mor12, 3.24] For any field F, we have a split short exact sequence of abelian groups (actually of $K_*^{\mathrm{MW}}(F)$-modules):

$$0 \to K_n^{\mathrm{MW}}(F) \to K_n^{\mathrm{MW}}(F(t)) \xrightarrow{\sum \partial^p_{(p)}} \oplus_p K_{n-1}^{\mathrm{MW}}(F[t]/p) \to 0,$$

where p runs over monic irreducible polynomials $p(t) \in F[t]$. Phrased differently, this is a short exact sequence

$$0 \to K_n^{\mathrm{MW}}(F) \to K_n^{\mathrm{MW}}(F(t)) \to \oplus_{x\in(\mathbb{A}^1)^{(1)}} K_{n-1}^{\mathrm{MW}}(\kappa(x)) \to 0.$$

Corollary 3.10.13. We have that

$$\left(\mathbf{K}_n^{\mathrm{MW}}\right)_{-1} \cong \mathbf{K}_{n-1}^{\mathrm{MW}}.$$

Proof. It suffices to check on fields. Take another exact sequence almost identical to the Milnor one, but without the valuation at zero.

$$\begin{tikzcd}
K_n^{\mathrm{MW}}(F \times \mathbb{G}_m) \arrow[r] & K_n^{\mathrm{MW}}(F(t)) \arrow[r] & \oplus_{x\neq 0} K_{n-1}^{\mathrm{MW}}(\kappa(x)) \\
K_n^{\mathrm{MW}}(F) \arrow[r] \arrow[u] & K_n^{\mathrm{MW}}(F(t)) \arrow[r] \arrow[u, equal] & \oplus_x K_{n-1}^{\mathrm{MW}}(\kappa(x))
\end{tikzcd}$$

Since all these sequences are split exact, we get

$$K_n^{\mathrm{MW}}(F \times \mathbb{G}_m) = K_n^{\mathrm{MW}}(F) \oplus K_{n-1}^{\mathrm{MW}}(F),$$

where the latter is the target of the residue map at zero. Since the contraction $F_{-1}(U)$ is alternatively described as a summand in $F_{-1}(\mathbb{G}_m \times U)$ complementary to $F_{-1}(U)$, the result follows. □

Corollary 3.10.14. We have that

$$(\mathbf{K}_n^{\mathrm{M}})_{-1} \cong \mathbf{K}_{n-1}^{\mathrm{M}} \text{ and } (\mathbf{I}_n)_{-1} \cong \mathbf{I}_{n-1}.$$

Proof. A nearly identical argument to Corollary 3.10.13 works. Alternatively, we can leverage some techniques of dévissage and compactly supported cohomology to prove it, as in [AF14c, 2.9]. □

Note 3.10.15. There should(?) be a more direct algebraic argument that $\left(\mathbf{K}_n^{\mathrm{M}}\right)_{-1} \cong \mathbf{K}_{n-1}^{\mathrm{M}}$ which doesn't make reference to the Milnor exact sequence, and just leverages the symbol algebra of Milnor K-theory.

3.10.2 Contraction is exact

Contraction yields a functor from homotopy sheaves to itself:

$$(-)_{-1} \colon \mathrm{HI}(k) \to \mathrm{HI}(k).$$

Proposition 3.10.16. For any $A \in \mathrm{HI}(k)$ and any function field $E \in \mathcal{E}_k$ we have that $H^1(\mathbb{G}_m \times \operatorname{Spec}(E); A) = 0$ (see [Dég07, 4.10]).

Proposition 3.10.17. As an endofunctor on $\mathrm{HI}(k)$, contraction is exact.

Proof. Suppose that

$$A \to B \to C$$

is an exact sequence in $\mathrm{HI}(k)$. Since $H^1(\mathbb{G}_m \times -, A) = 0$ by Proposition 3.10.16, the sequence

$$0 \to A(\mathbb{G}_m \times -) \to B(\mathbb{G}_m \times -) \to C(\mathbb{G}_m \times -) \to 0$$

is exact as well. Finally since $A_{-1}(U) \oplus A(U) \cong A(\mathbb{G}_m \times U)$ and this splitting is natural in A and U, the result follows. □

Corollary 3.10.18. Exactness of contraction implies that $(M/n)_{-1} = M_{-1}/n$ for any $M \in \mathrm{HI}(k)$ and any $n \in \mathbb{Z}$. It also implies quotients pass through contraction — in particular the isomorphism

$$\mathbf{K}_n^{\mathrm{M}}/2 \cong \mathbf{I}^n/\mathbf{I}^{n+1}$$

are compatible with contraction.

3.11 Homotopy modules

We now want to take these properties that $\mathbf{K}_n^{\mathrm{MW}}$, $\mathbf{K}_n^{\mathrm{M}}$, and $\mathbf{I}^n$ all satisfy and axiomatize them. First we establish further properties about homotopy sheaves.

3.11.1 Monoidal structure on $\mathrm{HI}(k)$

We claim that $\mathrm{HI}(k)$ admits a closed symmetric monoidal structure. The origin of this perhaps makes sense with reference to other objects — it is the unique symmetric monoidal structure making the forgetful map from the category of "sheaves with transfers" symmetric monoidal [Dég11, Lemme 1.8], alternatively it is the unique symmetric monoidal structure induced from that on the $\mathbb{A}^1$-derived category c.f. [Fel21, 1.5.1.20]. We won't define it explicitly, but just comment that it exists, and explore some key properties:

Theorem 3.11.1. The category $\mathrm{HI}(k)$ admits a closed symmetric monoidal structure, which we denote by $\otimes_{\mathrm{HI}}$. The unit for the monoidal structure is given by the constant sheaf $\mathbb{Z}$.

We'll come back and add a proof for this later. The slick proofs use a bit more machinery than we've developed so far, see e.g. [Fel21, 4.2.1.20].

Since the monoidal structure is closed, we have an internal hom object $\underline{\mathrm{Hom}}_{\mathrm{HI}(k)}(-,-)$. Since contraction was a loop space construction, we might expect it to be a right adjoint, and indeed this is true!

Proposition 3.11.2. For any $M \in \mathrm{HI}(k)$, we have an isomorphism

$$M_{-1} \cong \underline{\mathrm{Hom}}_{\mathrm{HI}(k)}\left(\mathbf{K}_1^{\mathrm{MW}}, M\right),$$

which is natural in M.

There is a $\mathbf{K}_0^{\mathrm{MW}}$-module structure on any M_{-1} coming from its action on $\mathbf{K}_1^{\mathrm{MW}}$:

$$\mathbf{K}_0^{\mathrm{MW}} \times \mathbf{K}_1^{\mathrm{MW}} \to \mathbf{K}_1^{\mathrm{MW}}$$
$$(\langle a \rangle, \beta) \mapsto \langle a \rangle \cdot \beta.$$

So multiplication by $\langle a \rangle$ induces an endomorphism of $\mathbf{K}_1^{\mathrm{MW}}$, which therefore induces an endomorphism

$$M_{-1} \cong \underline{\mathrm{Hom}}_{\mathrm{HI}}(\mathbf{K}_1^{\mathrm{MW}}, M) \xrightarrow{\langle a \rangle \cdot -} \underline{\mathrm{Hom}}_{\mathrm{HI}}(\mathbf{K}_1^{\mathrm{MW}}, M) \cong M_{-1}.$$

Corollary 3.11.3. There is an adjunction

$$\mathbf{K}_1^{\mathrm{MW}} \otimes_{\mathrm{HI}} - \colon \mathrm{HI}(k) \rightleftarrows \mathrm{HI}(k) : (-)_{-1}.$$

Since contraction is exact, the identification above states that $\mathrm{Hom}(\mathbf{K}_1^{\mathrm{MW}}, -)$ is exact, implying that $\mathbf{K}_1^{\mathrm{MW}}$ is projective as an object in the abelian category $\mathrm{HI}(k)$. This implies it is flat (since $\mathrm{HI}(k)$ is a module category), and it is moreover faithfully flat.

Proposition 3.11.4. [Dég11, 1.15] We have that $\mathbf{K}_1^{\mathrm{MW}} \otimes_{\mathrm{HI}} -$ is fully faithful.

A key property of the tensor product is the following

Proposition 3.11.5. We have that
$$\mathbf{K}_m^{\mathrm{MW}} \otimes_{\mathrm{HI}(k)} \mathbf{K}_n^{\mathrm{MW}} \cong \mathbf{K}_{m+n}^{\mathrm{MW}}.$$

We had these isomorphisms $\mathbf{I}^{n-1} \xrightarrow{\sim} \mathbf{I}^n_{-1}$. So now under adjunction it seems natural to look at their mates
$$\mathbf{K}_1^{\mathrm{MW}} \otimes_{\mathrm{HI}} \mathbf{I}^{n-1} \to \mathbf{I}^n.$$
In particular we want to take this analogy kind of seriously:
$$\mathbf{K}_1^{\mathrm{MW}} \leftrightsquigarrow \Sigma$$
$$(-)_{-1} \leftrightsquigarrow \Omega$$
$$\text{a sequence of homotopy sheaves } \left\{\mathbf{K}_n^{\mathrm{MW}} \otimes_{\mathrm{HI}} M\right\}_{n\geq 0} \leftrightsquigarrow \text{a suspension spectrum } \left\{\Sigma^n X\right\}_{n\geq 0}$$
$$\text{a sequence where the mates are equivalences} \leftrightsquigarrow \text{a spectrum}$$
In particular $\mathbf{I}^*$, $\mathbf{K}_*^{\mathrm{MW}}$ and $\mathbf{K}_*^{\mathrm{M}}$ would all be *spectra* in the above analogy. This idea of spectra admits a name, called *homotopy modules*. Let's make this definition precise.

3.11.2 The definition of homotopy modules

Definition 3.11.6. (c.f. [Fel21, 3.4.1.2]) A *homotopy module* is a pair (M_*, ω_*) where M_* is a $\mathbb{Z}$-graded strictly invariant sheaf of graded abelian groups (i.e. $M_n \in \mathrm{HI}(k)$ for each n), and
$$\omega_n \colon M_{n-1} \xrightarrow{\sim} (M_n)_{-1}$$
is an isomorphism called a *desuspension*. A morphism of homotopy sheaves is a levelwise homomorphism compatible with desuspensions. This gives a category $\mathrm{HM}(k)$, we call the category of *homotopy modules*.

Remark 3.11.7. Alternatively, we can describe $\mathrm{HM}(k)$ as the localization $\mathrm{HM}(k) = \mathrm{HI}(k)\left[\mathbf{K}_1^{\mathrm{MW}} \otimes_{\mathrm{HI}} -\right]$. It is also abelian, and we will prove it is symmetric monoidal.

Let's note some natural functors. We have
$$\omega^\infty \colon \mathrm{HM}(k) \to \mathrm{HI}(k)$$
$$M_* \mapsto M_0,$$
which just picks out the zero space. Let's try to build a map back the other way — in trying to do so, we bump into a puzzle: given M, how do we cook up an N for which $M \cong N_{-1}$? With our new understanding of the monoidal structure on $\mathrm{HI}(k)$, this becomes easier to answer.

Corollary 3.11.8. For any $M, N \in \mathrm{HI}(k)$, if
$$f \colon \mathbf{K}_1^{\mathrm{MW}} \otimes_{\mathrm{HI}} M \xrightarrow{\sim} N$$
is an equivalence, then the mate
$$M \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{HI}}(\mathbf{K}_1^{\mathrm{MW}}, N) = N_{-1}$$
is an isomorphism as well.

Proof. It is not true in general that the mate of an isomorphism needs to be an isomorphism. It is however true in this case since $\mathbf{K}_1^{\mathrm{MW}}$ is fully faithful. The mate is defined by
$$M \xrightarrow{\eta_M} \underline{\mathrm{Hom}}_{\mathrm{HI}}(\mathbf{K}_1^{\mathrm{MW}}, \mathbf{K}_1^{\mathrm{MW}} \otimes_{\mathrm{HI}} M) \xrightarrow{\underline{\mathrm{Hom}}_{\mathrm{HI}}(\mathbf{K}_1^{\mathrm{MW}}, f)} \underline{\mathrm{Hom}}_{\mathrm{HI}}(\mathbf{K}_1^{\mathrm{MW}}, N) = N_{-1}.$$
The latter map is an isomorphism by hypothesis, and the first is because a left adjoint is fully faithful if and only if the unit of the adjunction is a natural isomorphism. $\square$

Corollary 3.11.9. If $M \in \mathrm{HI}(k)$, then M is a contraction of $M \otimes_{\mathrm{HI}} \mathbf{K}_1^{\mathrm{MW}}$.

Proposition 3.11.10. There is a functor

$$\sigma^\infty \colon \mathrm{HI}(k) \to \mathrm{HM}(k),$$

defined by the property that

$$(\sigma^\infty M)_n = \begin{cases} M \otimes_{\mathrm{HI}} \mathbf{K}_n^{\mathrm{MW}} & n \geq 0 \\ M_{-j} & n = -j < 0 \end{cases}$$

Proof. The gluing maps

$$(\sigma_\infty M)_n \to ((\sigma^\infty M)_n)_{-1}$$

are defined to be mates to the natural equivalences

$$\mathbf{K}_1^{\mathrm{MW}} \otimes \mathbf{K}_n^{\mathrm{MW}} \otimes M \xrightarrow{\sim} \mathbf{K}_{n+1}^{\mathrm{MW}} \otimes M.$$

□

3.11.3 Monoidal structure on homotopy modules

Definition 3.11.11. An object $X \in \mathscr{C}$ in a symmetric monoidal 1-category (resp. ∞-category) is called *cyclic* if some permutation of $X^{\otimes n}$ is equal to (resp. homotopic to) the identity.

We'll phrase this in two settings. The ∞-categorical one will come in use later.

Theorem 3.11.12.

1. If $\mathscr{C}$ is a closed symmetric monoidal 1-category and $X \in \mathscr{C}$ is cyclic, then the localization $\mathscr{C}[X^{-1}]$ exists, defined as the colimit
$$\mathscr{C} \xrightarrow{X \otimes -} \mathscr{C} \xrightarrow{X \otimes -} \cdots$$
It is symmetric monoidal, and the localization $\mathscr{C} \to \mathscr{C}[X^{-1}]$ is strong monoidal.
2. If $\mathscr{C}$ is a presentably symmetric monoidal category, then if X is cyclic, the localization $\mathscr{C}[X^{-1}]$ is still presentably symmetric monoidal, with monoidal localization functor [Rob15, §2].

Proposition 3.11.13. We have that $\mathbf{K}_1^{\mathrm{MW}} \in \mathrm{HI}(k)$ is cyclic.

Proof. We claim the permutation (1 2 3) going from

$$\mathbf{K}_3^{\mathrm{MW}} \cong (\mathbf{K}_1^{\mathrm{MW}})^{\otimes 3} \to (\mathbf{K}_1^{\mathrm{MW}})^{\otimes 3} \cong \mathbf{K}_3^{\mathrm{MW}}$$

is equal to the identity, proving $\mathbf{K}_1^{\mathrm{MW}}$ is cyclic. Since $\mathbf{K}_n^{\mathrm{MW}}$ is generated by symbols $[u_1, \ldots, u_n]$ for $n \geq 1$, we are reduced to checking that $[c,a,b] = [a,b,c]$ for any $a, b, c \in F^\times$. We use that ϵ commutes with $[u]$ for any u by Proposition 3.7.5(4), and that $\epsilon^2 = 1$ by Proposition 3.7.4(4). This is now an immediate check by what we've done:

$$[c,a,b] := [c][a][b] = \epsilon[a][c][b] = \epsilon[a]\epsilon[b][c] = \epsilon^2[a][b][c] = [a,b,c].$$

□

Combining Proposition 3.11.13 and Theorem 3.11.12, we immediately obtain the following.

Corollary 3.11.14. The category of homotopy modules $\mathrm{HM}(k)$ is symmetric monoidal, and

$$\sigma^\infty \colon \mathrm{HI}(k) \to \mathrm{HM}(k)$$

is strong symmetric monoidal.

Proposition 3.11.15. We have that σ^∞ is fully faithful and ω^∞ is exact.

Proof. It is clear σ^∞ is fully faithful by construction, since the data of a map $\sigma^\infty M \to \sigma^\infty N$ is the data of a map $M \to N$ in $\mathrm{HI}(k)$. Meanwhile, exactness in $\mathrm{HM}(k)$ means exactness at each level, which implies exactness at level zero, hence ω^∞ is exact. $\square$

Let's also remark something interesting — if M_* is a homotopy module, then the mates to the structure isomorphisms are all of the form

$$\mathbf{K}_1^{\mathrm{MW}} \otimes_{\mathrm{HI}} M_* \to M_{*+1},$$

and by iterating this we get

$$\mathbf{K}_n^{\mathrm{MW}} \otimes_{\mathrm{HI}} M_* \to M_{*+n}.$$

Remark 3.11.16. Every homotopy module is a graded $\mathbf{K}_*^{\mathrm{MW}}$-module.

3.12 Gersten complexes

If F is a homotopy sheaf and $X \in \mathrm{Sm}_k$, we want to *resolve* $F(X) = H^0(X, F)$ by its values on codimension one, two, three, points. By this we mean its values on the function fields of those points.

The goal is to write down a complex that looks like

$$\bigoplus_{x \in X^{(0)}} F(\kappa(x)) \to \bigoplus_{x \in X^{(1)}} F_{-1}(\kappa(x)) \to \bigoplus_{x \in X^{(2)}} F_{-2}(\kappa(x)) \to \cdots$$

The primary difficulty is defining the differentials and showing that they actually yield a complex. This is one of the main complicating factors in the literature. Following the slogan that homotopy sheaves are easier once they are contractions, we might approach this when a homotopy sheaf is a 1-fold or 2-fold contraction and leverage the nice properties it inherits to try to construct this complex. This is the focus of a large chunk of Morel's book – building these complexes for M_{-1} and for M_{-2} when M is a homotopy sheaf.

For the purposes of this notes, we will work with a smaller class of homotopy sheaves, namely those which are infinite contractions — i.e. homotopy modules!

Theorem 3.12.1. If M_* is a homotopy module, then for any n, we can define well-defined differentials so that

$$C^*(X, M_n) := \left(\bigoplus_{x \in X^{(0)}} M_n(\kappa(x)) \to \bigoplus_{x \in X^{(1)}} M_{n-1}(\kappa(x)) \to \cdots \right)$$

is a complex, natural in X, called the *Rost-Schmid complex* [Mor12, 5.31]. This is an acyclic resolution, so the jth cohomology group of this complex is *precisely* the sheaf cohomology

$$H^j C^*(X, M_n) = H^j_{\mathrm{Nis}}(X, M_n).$$

Remark 3.12.2.

1. We are being vague about differentials and twists — these need to be carefully considered although for the sake of time we are omitting them.
2. A Rost-Schmid complex exists for a strongly invariant sheaf of abelian groups, and its construction is a key input in proving the strongly=strictly theorem. See [Bac24, §6] for details.

Definition 3.12.3. The full subcategory

$$\mathrm{Mod}_{\mathbf{K}_*^{\mathrm{M}}} \subseteq \mathrm{HM}(k)$$

of modules over Milnor K-theory (or modules over $\mathbf{K}_*^{\mathrm{MW}}$ on which η acts trivially) is called the category of *homotopy modules with (Voevodsky) transfers*. There is an equivalence of categories between these and so-called *Rost cycle modules*.

Remark 3.12.4.

1. It is easier to define differentials for homotopy modules with transfers since we don't have to stress about twists.
2. Rost cycle modules were used famously in the proof of the Milnor conjecture. Moreover over a perfect field, the category of Rost cycle modules is equivalent to the heart of the category of Voevodsky motives in the homotopy t-structure (see Deglise modules de cycles et motifes mixtes)

Theorem 3.12.5. *(Properties of the Rost–Schmid complex)*

1. For X smooth, the Rost–Schmid complex provides an acyclic resolution in the Zariski or Nisnevich sites
2. The projection $X \times \mathbb{A}^1 \to X$ induces a quasi-isomorphism
$$C^*(X, M_n) \to C^*(X \times \mathbb{A}^1, M_n).$$
This is [Mor12, 5.38], and it is a bit out of order here – one proves this first for a strongly invariant sheaf of abelian groups, and leverages this to conclude that strongly implies strictly.

Theorem 3.12.6. (Rost ?) For any smooth X, we have that $H^n(X, \mathbf{K}_n^{\mathrm{M}}) = \mathrm{CH}^n(X)$.

Proof. At the tail end of the Rost–Schmid complex. we have
$$\cdots \to \bigoplus_{x \in X^{(n-1)}} K_1^M(\kappa(x)) \to \bigoplus_{x \in X^{(n)}} K_0^M(\kappa(x)) \to 0,$$
since a contraction of K_0^M is trivial. Note that $K_1^M(\kappa(x)) = \kappa(x)^\times$ and $K_0^M(\kappa(x)) = \mathbb{Z}$. This differential can precisely be identified with the divisor function in the definition of Chow groups. The result follows immediately. □

Corollary 3.12.7. We have that
$$H^n(X, \mathbf{K}_n^{\mathrm{M}}) = H^n(X, \mathbf{K}_n^{Q}) = \mathrm{CH}^n(X).$$
This is the so-called *Gersten conjecture*.

Definition 3.12.8. The sheaf cohomology groups
$$H^n(X, \mathbf{K}_n^{\mathrm{MW}}) = \widetilde{\mathrm{CH}}^n(X)$$
are called *Chow–Witt groups*. Note that negative Milnor–Witt K-theory sheaves are not identically zero like Milnor K-theory, so the Rost–Schmid complex doesn't truncate. Nevertheless, the differential going from $\mathbf{K}_1^{\mathrm{MW}}$ to $\mathbf{K}_0^{\mathrm{MW}}$ is still an analogue of a divisor of a rational function, however equipped with some orientation data.

Remark 3.12.9. The multiplicative structure on cycle modules induce a multiplicative structure on the Rost–Schmid complexes, and therefore a multiplicative structure on the total cohomology.

3.12.1 Vanishing of I^j-cohomology

Proposition 3.12.10. If $\mathbf{I}^n \neq 0$ for some n, then $\mathbf{I}^n \neq \mathbf{I}^{n+1}$.

Proof. It suffices to check on fields, from which it was originally proven by Arason and Pfister [AP71, Korollar 2]. □

Corollary 3.12.11. If $\mathbf{I}^n/\mathbf{I}^{n+1} = 0$ then $\mathbf{I}^n = 0$.

Theorem 3.12.12. Let X be a smooth scheme of dimension d over an algebraically closed field k. Then $\mathbf{I}^n$ is identically zero over X for $n > d$.

Proof. It suffices to check on the residue field of X, and suffices to check that $\mathbf{I}^n/\mathbf{I}^{n+1} = 0$ by Corollary 3.12.11. By the Milnor conjecture, we have an isomorphism

$$\mathbf{I}^n/\mathbf{I}^{n+1} \cong \mathbf{K}_n^{\mathrm{M}}/2,$$

so we have that $I^n(k(X))/I^{n+1}(k(X)) = H^n_{\mathrm{Gal}}(k(X), \mu_2^{\otimes n})$, which vanishes above the dimension of X. □

Remark 3.12.13. *(Vanishing of I^j-cohomology)*

1. Over an arbitrary field k, the same vanishing result will hold for $n \geq d+r+1$, where $r = \mathrm{cd}_2(k)$ is the 2-cohomological dimension of the field k [AF14a, 5.1].
2. An even strongly vanishing statement is proven in [AF14a, 5.2], using a Bloch–Ogus spectral sequence argument.

3.12.2 Contraction yoga

Proposition 3.12.14. If $M \to N$ is a map of homotopy sheaves, which is an isomorphism after $k$fold contraction, then

$$H^n(X, M) \cong H^n(X, N)$$

for $n > k$.

Proof. The map induces a map of Rost complexes and see that they are identical starting at the kth slot. □

We can look at some maps of sheaves we currently have and see how far we have to contract them to get an isomorphism. Recall we have a short exact sequence

$$0 \to \mathbf{I}^{j+1} \to \mathbf{K}_j^{\mathrm{MW}} \to \mathbf{K}_j^{\mathrm{M}} \to 0.$$

Since $(\mathbf{K}_j^{\mathrm{M}})_{-k} = 0$ if $k > j$, we have that $\mathbf{I}^{j+1} \to \mathbf{K}_j^{\mathrm{MW}}$ is an isomorphism after $j+1$ contractions. This gives us

Proposition 3.12.15. We have that

$$H^n(X, \mathbf{I}^{j+1}) \cong H^n(X, \mathbf{K}_j^{\mathrm{MW}})$$

for $n > j+1$.

Proposition 3.12.16. We have that

$$H^n(X, \mathbf{K}_j^{\mathrm{M}}) = 0$$

for $j < n$.

What happens to $\mathbf{I}^j$ and $\mathbf{K}_j^{\mathrm{MW}}$ when they are contracted below zero?

Proposition 3.12.17. We have that

$$\mathbf{K}_{-n}^{\mathrm{MW}} \cong \mathbf{W}$$

is the Witt for each $n \geq 1$.

Proof. We first make the claim that $\mathbf{K}_{-1}^{\mathrm{MW}}$ is generated by elements of the form $\eta\langle a\rangle$. In other words, the map

$$\begin{aligned} \mathbf{K}_0^{\mathrm{MW}} &\to \mathbf{K}_{-1}^{\mathrm{MW}} \\ \langle a\rangle &\mapsto \eta\langle a\rangle \end{aligned}$$

is onto. This is analogous to the argument we did before (now we remove an η^{-1}, and get everything else in $\mathrm{GW}(k)$).

We first show the claim for $n = 1$. We claim that the map factors

$$\begin{tikzcd} \mathbf{K}_0^{\mathrm{MW}} \arrow[r, "\cong"] \arrow[d, "\eta"'] & \mathbf{GW} \arrow[d, "\mathrm{mod}\ h"] \\ \mathbf{K}_{-1}^{\mathrm{MW}} \arrow[r, dashed] & \mathbf{W}, \end{tikzcd}$$

where the bottom map sends $\eta\langle a\rangle$ to $\langle a\rangle$. We have to check this is well-defined, i.e. that if we have $\alpha, \beta \in \mathrm{GW}(k)$, and $\eta\alpha = \eta\beta$ is it true that $\alpha \equiv \beta \pmod{h}$. This is clear since η will kill any hyerbolic elements. From this it is also clear it is bijective. To see it for $n > 1$, we observe that multiplication by η induces an isomorphism $\mathbf{K}_{-n}^{\mathrm{MW}} \xrightarrow{\sim} \mathbf{K}_{-n-1}^{\mathrm{MW}}$. $\square$

Corollary 3.12.18. We have that $\mathbf{I}_{-1} = \mathbf{W}$.

Corollary 3.12.19. For $j < n$ we have that

$$H^n(X, \mathbf{I}^{j+1}) \cong H^n(X, \mathbf{K}_j^{\mathrm{MW}}).$$

3.13 The fundamental group of $\mathbb{P}^1$

Our goal is to compute $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. Recall we have

$$\mathrm{HI}(k) \subseteq \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k; \mathrm{Set}_*).$$

It makes sense for any F to ask for some *universal* approximation to it, lying in the category $\mathrm{HI}(k)$. Let G be a strongly invariant sheaf of groups, and S a sheaf of pointed sets. Then there is a bijection

$$[\Sigma S, BG]_{\mathrm{Spc}(k)_*} \cong [S, G]_{\mathrm{Shv}(\mathrm{Sm}_k;\mathrm{Set}_*)},$$

given by taking $\pi_1^{\mathbb{A}^1}$. In particular this gives the following definition.

Definition 3.13.1. [Mor12, 7.23] If S is a Nisnevich sheaf of pointed sets, we denote by

$$F_{\mathbb{A}^1}(S) := \pi_1^{\mathbb{A}^1}(\Sigma S)$$

the *free strongly invariant sheaf* of groups on S. This has the universal property that for any strongly invariant sheaf of groups G, we have

$$\begin{tikzcd} S \arrow[r] \arrow[d] & G \\ F_{\mathbb{A}^1}(S) \arrow[ur, dashed] & \end{tikzcd}$$

Definition 3.13.2. If S is a sheaf of pointed sets, we can alternatively consider the free strongly invariant sheaf of *abelian groups* on S. We denote this by[4]

$$F_{\mathbb{A}^1}^{\mathrm{ab}}(S),$$

and it is the universal element in $\mathrm{HI}(k)$ receiving a morphism of sheaves of pointed sets from S. We don't have an obvious way to compute this yet, unless $F_{\mathbb{A}^1}(S)$ already happens to be abelian.

[4] The notation here is nonstandard.

Proposition 3.13.3. For any $n \geq 2$, we have that
$$F_{\mathbb{A}^1}^{\mathrm{ab}}(S) \cong \pi_n(\Sigma^n S),$$
compatibly with suspension homomorphisms.

Proof. Let $n \geq 2$, and let G be any strictly invariant sheaf of abelian groups. Then by adjunction, we have a string of equivalences
$$[S, G]_* \cong [S, \Omega^n K(G,n)] \cong [\Sigma^n S, K(G,n)]\,.$$
Since $K(G,n)$ is n-connected, any map factors through its n-truncation $\tau_{\leq n}\Sigma^n S$, which is equal to $K(\pi_n(\Sigma^n S), n)$ since $\Sigma^n S$ is n-connected. Therefore by taking π_n we get a bijection
$$[\Sigma^n S, K(G,n)] \cong [K(\pi_n \Sigma^n S, n), K(G,n)] = \mathrm{Hom}_{\mathrm{Shv(Grp)}}\left(\pi_n(\Sigma^n S), G\right).$$
$\square$

The independence on n can be thought of as a preliminary version of the S^1-Freudenthal suspension theorem.

Remark 3.13.4. There is a natural map
$$F_{\mathbb{A}^1}(S) \to F_{\mathbb{A}^1}^{\mathrm{ab}}(S),$$
which is not necessarily an isomorphism.

Example 3.13.5. We can consider $\mathbb{G}_m$ as a sheaf of pointed sets. Then by definition, we have that
$$F_{\mathbb{A}^1}(\mathbb{G}_m) = \pi_1^{\mathbb{A}^1}(\mathbb{P}^1).$$
This is the sheaf of groups we want to compute.

Notation 3.13.6. For $n \geq 1$, we denote by
$$F_{\mathbb{A}^1}(n) := F_{\mathbb{A}^1}\left(\mathbb{G}_m^{\wedge n}\right)$$
the free strongly invariant sheaf of groups on the smash product $\mathbb{G}_m^{\wedge n}$.

We have that $\mathbb{G}_m$, as a motivic space, represents units of global sections. Since smash products are computed sectionwise, we have that $\mathbb{G}_m^{\wedge n}$ represents the n-fold smash product of $\mathcal{O}_X(X)^\times$ for any X. This is a sheaf of pointed sets, with no higher homotopy. Hence it gives rise to a map
$$\left(\mathcal{O}_X(X)^\times\right)^{\wedge n} \to (k(X)^\times)^{\wedge n} \to K_n^{\mathrm{MW}}(k(X))$$
$$(a_1, \ldots, a_n) \mapsto [a_1, \ldots, a_n]\,.$$
By the yoga of $\mathcal{F}_k$-data, this bootstraps up to a *symbol map*, which we denote by
$$\sigma_n \colon \mathbb{G}_m^{\wedge n} \to \mathbf{K}_n^{\mathrm{MW}}.$$
This factors through the universal strongly invariant presheaf of *abelian* groups:
$$\begin{tikzcd}
\mathbb{G}_m^{\wedge n} \arrow[r, "\sigma_n"] \arrow[d] & \mathbf{K}_n^{\mathrm{MW}} \\
F_{\mathbb{A}^1}^{\mathrm{ab}}(n) \arrow[ur, dashed] &
\end{tikzcd}$$

Theorem 3.13.7. [Mor12, 3.37] For $n \geq 1$, the induced symbol map
$$F_{\mathbb{A}^1}^{\mathrm{ab}}(n) \to \mathbf{K}_n^{\mathrm{MW}}$$
is an equivalence. In other words, $\mathbf{K}_n^{\mathrm{MW}}$ is the free strongly invariant sheaf of *abelian* groups on $\mathbb{G}_m^{\wedge n}$.

Proof intuition. Let $M \in \mathrm{HI}(k)$. Then any map of pointed sheaves of sets $\mathbb{G}_m^{\wedge n} \to M$ can be viewed as a map of pointed motivic spaces, and we see by adjunction:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Spc}(k)_*}(S^{n,n}, M) &= \mathrm{Hom}_{\mathrm{Spc}(k)_*}\left(S^0, \Omega^{n,n} M\right) \\ &= \mathrm{Hom}_{\mathrm{Spc}(k)_*}\left(S^0, M_{-n}\right) \\ &= M_{-n}(k). \end{aligned}$$

Since $M_{-n} = \underline{\mathrm{Hom}}_{\mathrm{HI}}(\mathbf{K}_n^{\mathrm{MW}}, M)$, we might expect that this gives rise to a unique map $\mathbf{K}_n^{\mathrm{MW}} \to M$. This ends up being true, and the difficulty is just in verifying that this is indeed a morphism of homotopy sheaves. In particular one must construct it from the symbol map and verify it is well-defined with respect to the relations for Milnor–Witt K-theory. ☐

Proposition 3.13.8. We have that

$$F_{\mathbb{A}^1}(2) \cong F_{\mathbb{A}^1}^{\mathrm{ab}}(2) \cong \mathbf{K}_2^{\mathrm{MW}}.$$

Proof. Recall by Proposition 3.3.11 we have

$$\mathrm{SL}_2 \simeq \mathbb{A}^2 \smallsetminus 0 \simeq S^{3,2} \simeq \Sigma\left(\mathbb{G}_m \wedge \mathbb{G}_m\right).$$

Since SL_2 is a group object, its π_1 is abelian by an Eckmann–Hilton argument, hence the result follows. ☐

3.13.1 The fundamental group of $\mathbb{P}^1$

Since $\mathbb{A}^2 \smallsetminus 0 \to \mathbb{P}^1$ is the total space of a $\mathbb{G}_m$-torsor, we get an $\mathbb{A}^1$-fiber sequence

$$\mathbb{G}_m \to \mathbb{A}^2 \smallsetminus 0 \to \mathbb{P}^1,$$

which deloops to

$$\mathbb{A}^2 \smallsetminus 0 \to \mathbb{P}^1 \to B\mathbb{G}_m.$$

Since both $B\mathbb{G}_m$ and $\mathbb{A}^2 \smallsetminus 0$ are connected, the long exact sequence on homotopy becomes a short exact sequence of the form [Mor12, (7.4)]

$$0 \to \mathbf{K}_2^{\mathrm{MW}} \to F_{\mathbb{A}^1}(1) \to \mathbb{G}_m \to 0.$$

This turns out to be a central extension, but $F_{\mathbb{A}^1}(1)$ is not commutative [Mor12, 7.29].

3.14 The Brouwer degree

We are interested in leveraging our computation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ to say something about $\mathrm{End}_{\mathrm{Spc}(k)_*}(\mathbb{P}^1)$. By convention, $\mathbb{P}^1$ is pointed at ∞. Given any pointed endomorphism of motivic spaces $\mathbb{P}^1 \to \mathbb{P}^1$, we can look at the induced map on $\pi_1^{\mathbb{A}^1}$, and we get an assignment

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]_{\mathrm{Spc}(k)_*} \to \mathrm{End}(F_{\mathbb{A}^1}(1)). \tag{3.1}$$

We are going to argue that this is a bijection.

Proposition 3.14.2. There is a bijection

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]_{\mathbb{A}^1} \cong \left[\mathbb{P}^1, B\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)\right]_*.$$

Proof. We have that

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]_* = \left[S^1, \Omega^{1,1}\mathbb{P}^1\right]_* = \pi_1(\Omega^{1,1}\mathbb{P}^1) = \left[S^1, B\pi_1(\Omega^{1,1}\mathbb{P}^1)\right]_*.$$

Since $\pi_1(\mathbb{P}^1)_{-1}$ is strongly invariant, the above is isomorphic to

$$\left[S^1, \Omega^{1,1} B\pi_1(\mathbb{P}^1)\right]_* \cong \left[\mathbb{P}^1, B\pi_1(\mathbb{P}^1)\right]_*.$$

☐

Corollary 3.14.3. The assignment Equation 3.1 is a bijection.

Proof. Since $BF_{\mathbb{A}^1}(1)$ is 1-truncated, we have that any map $\mathbb{P}^1 \to BF_{\mathbb{A}^1}(1)$ factors through $L_{\mathrm{mot}}\tau_{\leq 1}\mathbb{P}^1$. Since $\mathbb{P}^1$ is connected,[5] we have that $\tau_{\leq 1}\mathbb{P}^1$ is equal to $BF_{\mathbb{A}^1}(1)$, which is $\mathbb{A}^1$-local. □

Proposition 3.14.4. The contraction of $\mathbb{G}_m$ is $\mathbb{Z}$.

Proof. We first check that

$$\mathrm{Hom}_{\mathrm{Sm}_k}(\mathbb{G}_m \times U, \mathbb{G}_m) = \mathrm{Hom}_{\mathrm{Sm}_k}(U[t, t^{-1}], \mathbb{G}_m).$$

If U is reduced and irreducible, the units in $U[t, t^{-1}]$ are all of the form ut^n for some $u \in U^\times$ and $n \in \mathbb{Z}$. Hence we get an isomorphism

$$\mathbb{G}_m(\mathbb{G}_m \times U) \cong \mathbb{G}_m(U) \times \mathbb{Z}.$$

In the short exact sequence defining contraction, this gives us that $(\mathbb{G}_m)_{-1}$ is the constant presheaf $\mathbb{Z}$. □

Proposition 3.14.5. The short exact sequence, obtained from the contraction above

$$0 \to \mathbf{K}_1^{\mathrm{MW}} \to F_{\mathbb{A}^1}(1)_{-1} \to \mathbb{Z} \to 0$$

is canonically split exact, giving an isomorphism

$$F_{\mathbb{A}^1}(1)_{-1} \cong \mathbf{K}_1^{\mathrm{MW}} \oplus \mathbb{Z}.$$

Proof. It suffices to see that $\mathrm{Ext}(\mathbb{Z}, \mathbf{K}_1^{\mathrm{MW}}) = 0$, which is clear since $\mathbb{Z}$ is projective. □

Proposition 3.14.6. We have that $\mathrm{End}(F_{\mathbb{A}^1}(1))$ is a group, and moreover is an associative commutative ring.

Proposition 3.14.7. There is a canonical isomorphism

$$\mathrm{End}(F_{\mathbb{A}^1}(1)) \cong F_{\mathbb{A}^1}(1)_{-1}(k).$$

Proof. Since $F_{\mathbb{A}^1}(1)$ is by definition the free pointed sheaf of abelian groups on $\mathbb{G}_m$, there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Shv(Grp)}}\left(F_{\mathbb{A}^1}(1), F_{\mathbb{A}^1}(1)\right) = \mathrm{Hom}_{\mathrm{Shv}_*}\left(\mathbb{G}_m, F_{\mathbb{A}^1}(1)\right).$$

This is the contraction $F_{\mathbb{A}^1}(1)_{-1}$. □

Since $F_{\mathbb{A}^1}(1)_{-1}(k) = \mathrm{End}(F_{\mathbb{A}^1}(1))$ is the group we want to compute, we can take global sections $H^0_{\mathrm{Nis}}(\mathrm{Spec}\,(k), -)$ in the above SES to get

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]_* = \mathrm{End}(F_{\mathbb{A}^1}(1)) \cong K_1^{\mathrm{MW}}(k) \oplus \mathbb{Z}. \tag{3.8}$$

Any endomorphism of $\mathrm{End}(F_{\mathbb{A}^1}(1))$ induces an endomorphism of the contraction $F_{\mathbb{A}^1}(1)_{-1}$ by functoriality, which restricts to an endomorphism of $\mathbf{K}_1^{\mathrm{MW}}$. All endomorphisms of $\mathbf{K}_1^{\mathrm{MW}}$ are given by multiplication by some element in $K^M W_0 = \mathrm{GW}(k)$. This gives a natural homomorphism called the $\mathbb{A}^1$-*Brouwer degree*:

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]_* = \mathrm{End}(F_{\mathbb{A}^1}(1)) \xrightarrow{\deg^{\mathbb{A}^1}} \mathrm{End}(\mathbf{K}_1^{\mathrm{MW}}) = \mathrm{GW}(k).$$

Some facts are true about this:

Proposition 3.14.9.

1. The $\mathbb{A}^1$-Brouwer degree is an epimorphism

[5]

2. The following diagram commutes

$$\begin{tikzcd} & & & \mathrm{GW}(k) \arrow[d, "\mathrm{rank}"] & \\ 0 \arrow[r] & K_1^{\mathrm{MW}}(k) \arrow[r] & {[\mathbb{P}^1,\mathbb{P}^1]_*} \arrow[ur, "\deg^{\mathbb{A}^1}"] \arrow[r] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

Proof. todo ☐

Hence we think about $\mathbb{Z}$ as recording *rank*.

Question 3.14.10. What is the kernel of the degree homomorphism

$$\deg^{\mathbb{A}^1} \colon [\mathbb{P}^1,\mathbb{P}^1]_* \to \mathrm{GW}(k).$$

To answer this, we can consider the following commutative diagram, which is just rewriting the one before:

$$\begin{tikzcd} 0 \arrow[r] & ? \arrow[r] \arrow[d] & {[\mathbb{P}^1,\mathbb{P}^1]_*} \arrow[r] \arrow[d, equal] & \mathrm{GW}(k) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_1^{\mathrm{MW}} \arrow[r] & {[\mathbb{P}^1,\mathbb{P}^1]_*} \arrow[r] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

By the snake lemma, the cokernel of the map $? \to K_1^{\mathrm{MW}}(k)$ is the kernel of $\mathrm{GW}(k) \to \mathbb{Z}$, which is $I(k)$.

Proposition 3.14.11. The connecting homomorphism $K_1^{\mathrm{MW}}(k) \to I(k)$ is given by the right exact sequence

$$K_1^{\mathrm{MW}}(k) \xrightarrow{h} K_1^{\mathrm{MW}}(k) \twoheadrightarrow I(k) \to 0,$$

where $[u] \mapsto \langle u \rangle - \langle 1 \rangle$.

Proof. todo ☐

Proposition 3.14.12. The kernel of $K_1^{\mathrm{MW}}(k) \to I(k)$ is $(k^\times)^2$.

Proof. Recall the pullback diagram from Corollary 3.8.14:

$$\begin{tikzcd} K_1^{\mathrm{MW}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & K_1^M(k) \arrow[d] \\ I(k) \arrow[r] & K_1^M(k)/2. \end{tikzcd}$$

The fiber of the left vertical map is isomorphic to the fiber of the rightmost vertical map, which is $2K_1^M(k)$. Since addition on $K_1^M(k) = k^\times$ is multiplication of field units, the result follows. ☐

Altogether we get the following result.

Theorem 3.14.13. [Mor12, 7.36] We have a short exact sequence of abelian groups

$$0 \to (k^\times)^2 \to [\mathbb{P}^1,\mathbb{P}^1]_* \xrightarrow{\deg^{\mathbb{A}^1}} \mathrm{GW}(k) \to 0. \tag{3.14}$$

Remark 3.14.15. *(On Theorem 3.14.13)*

1. Morel writes $(k^\times)/\pm 1$ instead of $(k^\times)^2$ in his book, which I believe is an error?

2. The short exact sequence above can alternatively be repackaged as a pullback diagram (the sequence is obtained by taking horizontal fibers):

$$\begin{tikzcd} \left[\mathbb{P}^1, \mathbb{P}^1\right]_* \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{GW}(k) \arrow[d] \\ k^\times \arrow[r] & k^\times / \left(k^\times\right)^2. \end{tikzcd}$$

   This is the form that Theorem 3.14.13 takes in various forms in the literature, where people write

$$\left[\mathbb{P}^1, \mathbb{P}^1\right]_* \cong \mathrm{GW}(k) \times_{k^\times/(k^\times)^2} k^\times.$$

3. The ring structure on $\mathrm{GW}(k) \times_{k^\times/(k^\times)^2} k^\times$ is spelled out explicitly in [Caz09].
4. The units will go away after stabilizing, and we will see that $\mathrm{GW}(k)$ is the home for the *stable* $\mathbb{A}^1$-Brouwer degree, in other words it is the $(0,0)$th stable stem motivically.

The additive structure comes from the isomorphism $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$. We have a pinch map of simplicial sets

$$S^1 \to S^1 \vee S^1.$$

Smashing with $\mathbb{G}_m$, we use that smash products distribute over wedge products to get a comultiplication

$$\mathbb{P}^1 \to \mathbb{P}^1 \vee \mathbb{P}^1.$$

This cogroup structure induces an addition on $\left[\mathbb{P}^1, \mathbb{P}^1\right]_{\mathbb{A}^1}$ which corresponds to addition in $\mathrm{GW}(k)$.

3.14.1 Explicit computations

Given an endomorphism $f/g\colon \mathbb{P}^1 \to \mathbb{P}^1$, explicit formulas exist for computing the associated symmetric bilinear form in $\mathrm{GW}(k)$ [Caz12]. We spell these out now.

By convention $\mathbb{P}^1$ is pointed at ∞ as a motivic space, however we could pick any basepoint and get analogous formulas.

Note 3.14.16. We say a morphism of schemes $f/g\colon \mathbb{P}^1 \to \mathbb{P}^1$ is *pointed* if it sends ∞ to ∞. Explicitly this implies that $\deg(g) < \deg(f)$. WLOG we can assume f is monic, and we can dehomogenize and write

$$\begin{aligned} f(x) &= x^n + a_{n-1}x^{n-1} + \ldots + a_0 \\ g(x) &= b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_0. \end{aligned}$$

Then we require f and g to not be simultaneously zero in order to give a well-defined morphism. Taking affine space in the coefficients

$$\mathbb{A}^{2n}_k = \operatorname{Spec} k[a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}],$$

the resultant is a hypersurface in here. So the data of f/g of leading degree n is the same as a rational point in the hypersurface complement.

Definition 3.14.17. Let $f/g\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a pointed morphism of degree $\deg(f) = n$. We introduce auxiliary variables X and Y, and consider the quantity

$$\frac{f(X)g(Y) - f(Y)g(X)}{X - Y} = \sum_{1 \le i,j \le n} c_{ij} X^{i-1} Y^{j-1}.$$

This is called the *Bézoutian*, and we denote by $\mathrm{Béz}(f/g)$ the $n \times n$ matrix $(c_{ij})_{i,j}$ over k.

Theorem 3.14.18. (Cazanave) The Bézoutian is a symmetric bilinear matrix, equal to the $\mathbb{A}^1$-degree of f/g, and the bijection

$$\begin{aligned}\left[\mathbb{P}^1, \mathbb{P}^1\right]_* &\to \mathrm{GW}(k) \times_{k^\times/(k^\times)^2} k^\times \\ (f/g) &\mapsto \left(\mathrm{Béz}(f/g), (-1)^{\frac{n(n-1)}{2}} \det \mathrm{Béz}(f/g)\right)\end{aligned}$$

is an isomorphism of abelian groups.

Example 3.14.19. Consider $f(x) = x^2$ and $g(x) = 1$. Then we have that

$$\frac{f(X)g(Y) - f(Y)g(X)}{X - Y} = \frac{X^2 - Y^2}{X - Y} = X + Y = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Hence

$$\deg^{\mathbb{A}^1}(x^2/1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = h.$$

Corollary 3.14.20. An analogous argument shows that

$$\deg^{\mathbb{A}^1}\left(\frac{x^n}{1}\right) = \begin{cases} \frac{n}{2} h & n \text{ even} \\ \frac{n-1}{2} h + \langle 1 \rangle & n \text{ odd.} \end{cases}$$

3.15 Naive homotopies

We localized at $\mathbb{A}^1$ in order to build the category of motivic spaces, but we can talk briefly about a more naive way to go about this.

Definition 3.15.1. Let $X, Y \in \mathrm{Sm}_k$, and take two maps $f, g \colon X \to Y$ between them in the category of schemes. A *naive* $\mathbb{A}^1$*-homotopy* is a morphism $H \colon X \times \mathbb{A}^1 \to Y$ so that the diagram commutes

$$\begin{tikzcd} X \arrow[dr, "0"] \arrow[drr, bend left, "f"] & & \\ & X \times \mathbb{A}^1 \arrow[r, "H"] & Y \\ X \arrow[ur, "1"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

Example 3.15.2. If $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine k-algebras, then a map

$$X \times_k \mathbb{A}^1_k \to Y$$

is the same data as a k-algebra homomorphism $B[t] \leftarrow A$. A homotopy between $f, g \colon A \to B$ is a map that restricts to f and g by setting $t = 0$ and $t = 1$, respectively.

We define a relation $\sim_N$ on $\mathrm{Hom}_{\mathrm{Sm}_k}(X, Y)$, where we identify two maps if there exists a naive homotopy between them.

Note 3.15.3. $\sim_N$ is not an equivalence relation (it isn't transitive). However it admits a transitive closure, giving rise to an equivalence relation, which we also denote by $\sim_N$ by abuse of notation. We use the notation

$$[X, Y]_N := \mathrm{Hom}_{\mathrm{Sm}_k}(X, Y)/\sim_N.$$

We call this *naive* $\mathbb{A}^1$*-homotopy classes of maps.*

Proposition 3.15.4. If $X, Y \in \mathrm{Sm}_k$ we have that

$$[X, Y]_N = \pi_0\left(L_{\mathbb{A}^1} h_Y\right)(X).$$

Proof. We recall $L_{\mathbb{A}^1}h_Y$ was the sifted colimit

$$L_{\mathbb{A}^1}h_Y = \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} h_Y(\mathbb{A}^n \times -).$$

Since h_Y is a discrete sheaf, it is equivalent to its own π_0, so we can pass π_0 through the colimit. Using that we are now valued in a 1-category we can use the cofinality of $\Delta^{\mathrm{op,inj}}_{\leq 1} \subseteq \Delta^{\mathrm{op}}$ Example 2.3.7 to rewrite the diagram as

$$\begin{aligned}\pi_0\left(L_{\mathbb{A}^1}h_Y\right)(X) &= \operatorname{colim}\left(\pi_0\left(L_{\mathbb{A}^1}h_Y\right)(X\times\mathbb{A}^1) \rightrightarrows \pi_0\left(L_{\mathbb{A}^1}h_Y\right)(X)\right)\\ &= \operatorname{colim}\left(\operatorname{Hom}_{\mathrm{Sm}_k}(X\times\mathbb{A}^1, Y) \rightrightarrows \operatorname{Hom}_{\mathrm{Sm}_k}(X,Y)\right),\end{aligned}$$

so it is the coequalizer of evaluation at 0 and 1, which is precisely the quotient of $\operatorname{Hom}_{\mathrm{Sm}_k}(X,Y)$ by the equivalence relation generated by naive homotopy. □

Remark 3.15.5. There is an analogous notion of a pointed naive homotopy which is the definition one would expect, and a pointed version of Proposition 3.15.4 can be formulated. See e.g. [Bar+23, Appendix A].

There is a natural map

$$L_{\mathbb{A}^1}h_Y \to L_{\mathrm{mot}}h_Y,$$

which induces a map

$$[X,Y]_N = \pi_0\left(L_{\mathbb{A}^1}h_Y\right)(X) \to \pi_0\left(L_{\mathrm{mot}}h_Y\right)(X) = [X,Y]_{\mathbb{A}^1}.$$

Here the latter equality is by Example 3.4.3.

We remark that if $L_{\mathbb{A}^1}h_Y$ is already a Nisnevich sheaf, then we get that these two classes agree, however this rarely happens.

Definition 3.15.6. [MV99, §3.2.4] We say that $Y \in \mathrm{Sm}_k$ is $\mathbb{A}^1$-*rigid* if h_Y is $\mathbb{A}^1$-invariant.

Example 3.15.7. $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid, as is any smooth curve of genus $g \geq 1$ [MV99, §3.2.4].

Proposition 3.15.8. If Y is $\mathbb{A}^1$-rigid, then for any $X \in \mathrm{Sm}_k$, we get that

$$[X,Y]_N \cong [X,Y]_{\mathbb{A}^1}.$$

In general this is rare. Recently, it has been observed that it is much more common for representable presheaves (and motivic spaces) to be $\mathbb{A}^1$-invariant on affines.

Definition 3.15.9. We say that a simplicial presheaf $F \in \mathrm{PSh}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-*naive* if the induced map

$$(L_{\mathbb{A}^1}F)(U) \to (L_{\mathrm{mot}}F)(U)$$

is an equivalence for every affine $U \in \mathrm{Sm}_k^{\mathrm{aff}}$.

Proposition 3.15.10. If h_Y is $\mathbb{A}^1$-naive, then

$$[U,Y]_N \cong [U,Y]_{\mathbb{A}^1}$$

for every affine $U \in \mathrm{Sm}_k^{\mathrm{aff}}$.

Proposition 3.15.11. [AHW18, 2.1.3] A simplicial presheaf F is $\mathbb{A}^1$-naive if and only if $L_{\mathbb{A}^1}F$ is a Nisnevich sheaf on $\mathrm{Sm}_k^{\mathrm{aff}}$.

Example 3.15.12. The following schemes are $\mathbb{A}^1$-naive over k:

1. The smooth quadric [AHW18, 4.2.1]

$$Q_{2n-1} = \left\{\sum_{i=1}^n x_i y_i = 1\right\} \subseteq \mathbb{A}^{2n}_k.$$

2. The projective line $\mathbb{P}^1$. [Bar+23, 127].
3. Any $\mathbb{A}^1$-rigid scheme.
4. $\mathbb{A}^n \setminus 0$ [AHW18, 4.2.6]
5. Any isotropic reductive group scheme, assuming k is infinite [AHW18, 4.3.1]

3.15.1 Cazanave's theorem

In general naive and genuine $\mathbb{A}^1$-homotopy classes of maps are quite different, but making statements relating the two has been a topic of interest throughout the history of motivic homotopy theory. As a particular example, we have seen that $[\mathbb{P}^1, \mathbb{P}^1]_*$ is an abelian group.

Theorem 3.15.13. (Cazanave) The set $[\mathbb{P}^1, \mathbb{P}^1]_N$ of pointed naive homotopy classes of maps admits a monoid structure, and the natural map

$$[\mathbb{P}^1, \mathbb{P}^1]_N \to [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1}$$

is a group completion.

Cazanave's monoidal structure is defined explicitly in terms of rational maps.

3.16 Unstable connectivity

We'll take the following theorem without proof.

Theorem 3.16.1. [MV99, §.3.22 (p.94)] (c.f. [AD09, 2.7]) For any $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$, the map to its singularization $F \to \mathrm{Sing}(F)$ induces a surjection on π_0.

This has an immediate corollary, which is that connectivity is preserved in passing from the sheaf topos to motivic spaces.

Corollary 3.16.2. (c.f. [Bac24, 1.2]) If $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ is connected, then so is $L_{\mathrm{mot}}F$.

Proof. Homotopy sheaves commute with filtered colimits, hence connected objects are closed under filtered colimits. By cofinality (Remark 3.1.24) it suffices to argue that $\pi_0 L_{\mathrm{Nis}} L_{\mathbb{A}^1} F$ is trivial for every connected $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$. Since $\pi_0(F) \to \pi_0(L_{\mathrm{Nis}} L_{\mathbb{A}^1} F)$ is surjective by Theorem 3.16.1, the result follows. □

This is the $n = 0$ of the *unstable connectivity theorem.*

Theorem 3.16.3. (Unstable connectivity theorem, Morel) If F is a simplicial sheaf which is n-connected, then $L_{\mathrm{mot}}F$ is n-connected.

The goal of this section is to prove this.

Lemma 3.16.4. [Bac24, 1.9] Let $F \to E \to B$ be a fiber sequence in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$, and let B be connected and $\mathbb{A}^1$-local. Then F is $\mathbb{A}^1$-local if and only if E is $\mathbb{A}^1$-local. In particular if these conditions are satisfied, it is also a fiber sequence in $\mathrm{Spc}(k)$.

Proposition 3.16.5. [Bac24, 1.8(2)] If $X \in \mathrm{Spc}(k)_*$ is a connected pointed motivic space, then for each $n \geq 1$, the truncation $\tau_{\leq n} X \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-local, and hence a motivic space.

Proof. We have fiber sequences

$$K(\pi_{n+1}X, n+1) \to X_{\leq n+1} \to X_{\leq n}.$$

Since X is a motivic space, its homotopy sheaves are strongly invariant, hence $K(\pi_{n+1}X)$ is $\mathbb{A}^1$-local. So $X_{\leq n}$ being $\mathbb{A}^1$-local will imply that $X_{\leq n+1}$ is $\mathbb{A}^1$-local. This is the inductive step.
For the base case it suffices to check that $X_{\leq 0}$ is $\mathbb{A}^1$-local. However we note that $X_{\leq 0} = \pi_0 X$, which is trivial since X is connected, and hence $\mathbb{A}^1$-local. □

We can now prove the connectivity theorem, following [Bac24, 1.10].

Proof of Theorem 3.16.3. We already proved this when $n = 0$ (Corollary 3.16.2), so we may assume $n > 0$. Consider the fiber sequence in the sheaf topos

$$0 \to (L_{\mathrm{mot}}F)_{>n} \to L_{\mathrm{mot}}F \to (L_{\mathrm{mot}}F)_{\leq n} \to 0.$$

The middle space is $\mathbb{A}^1$-local, and $L_{\mathrm{mot}}F$ has $\mathbb{A}^1$-local truncation by Proposition 3.16.5. Now by Lemma 3.16.4 the fiber $(L_{\mathrm{mot}}F)_{>n}$ is $\mathbb{A}^1$-local as well. Consider the diagram

$$\begin{tikzcd} & F \arrow[dl, dashed] \arrow[d] \arrow[dr] & \\ (L_{\mathrm{mot}}F)_{>n} \arrow[r] & L_{\mathrm{mot}}F \arrow[r] & (L_{\mathrm{mot}}F)_{\leq n} \end{tikzcd}$$

By null homotopy of the composite map of the map to the bottom right we get a lift $F \to (L_{\mathrm{mot}}F)_{>n}$. Since the target is local, the lift factors through $L_{\mathrm{mot}}F$, giving a section s. Hence we have a retract:[6]

$$\begin{tikzcd} L_{\mathrm{mot}}F \arrow[r, "s"] \arrow[dr, equal] & (L_{\mathrm{mot}}F)_{>n} \arrow[d] \\ & L_{\mathrm{mot}}F. \end{tikzcd}$$

The property of being n-connected is closed under retracts,[7] so the result follows. □

Corollary 3.16.6. [Bac24, 1.11] Let $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ be a connected sheaf. Then F is a motivic space if and only if $\pi_i F$ is strongly $\mathbb{A}^1$-invariant for all $i \geq 1$.

Proof. The forward direction is clear. For the reverse, we can inductively prove that $X_{\leq n}$ is local. Since $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ is Postnikov complete, we have that

$$X = \lim_n X_{\leq n},$$

and the result follows from observing limits of $\mathbb{A}^1$-invariant sheaves are still $\mathbb{A}^1$-invariant. □

Proposition 3.16.7. Suppose that $X \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ has $\pi_i X = 0$ for $i < n$. Then the map

$$\pi_n X \to \pi_n L_{\mathrm{mot}} X$$

is the initial map to a strongly invariant sheaf of groups (if $n = 1$) and of abelian groups (if $n \geq 2$).

Proof. If $n = 1$, let M be any strongly invariant sheaf of groups, and for $n \geq 2$, suppose it is a sheaf of abelian groups. Then take any map $\pi_n X \to M$. This is equivalent to the data of a map

$$K(\pi_n X, n) \to K(M, n).$$

[6] Letting r denote the map $(L_{\mathrm{mot}}F)_{>n} \to L_{\mathrm{mot}}F$, it is perhaps not obviously immediate why $rs = \mathrm{id}$. It is true after precomposing with $F \to L_{\mathrm{mot}}F$, then applying L_{mot} we have that $L_{\mathrm{mot}}(rs) = \mathrm{id}$. Since r and s are already maps between motivic spaces, the result follows.

[7] Apply π_i to the diagram and see that the identity on $\pi_i L_{\mathrm{mot}}F$ factors through zero.

By our assumptions on X, we have that $\tau_{\leq n}X = K(\pi_n X, n)$, and by adjunction, we get equivalences

$$[\pi_n(X), M] \cong [\tau_{\leq n}X, K(M,n)] = [X, K(M,n)] \, .$$

By our hypotheses on M, the EM space $K(M,n)$ is $\mathbb{A}^1$-local, hence any map $X \to K(M,n)$ factors through $L_{\mathrm{mot}}X$. Applying $\pi_n(-)$ to the factorization gives the universal property. □

Remark 3.16.8. This result can be improved, via induction on the Postnikov tower, to assume that $\pi_0 X = 0$ and that $\pi_i X$ is strongly invariant for $0 < i < n$, see [Mor12, 6.60].

3.17 Simplicial Freudenthal

Recall: A map $f\colon X \to Y$ is said to be *n-connective* if its homotopy fiber is n-connective, i.e. $\pi_i(\mathrm{fib}(f)) = 0$ for $i < n$.

- ▷ *n-connective* means $\pi_i = 0$ for $i < n$
- ▷ *n-connected* means $\pi_i = 0$ for $i \leq n$.

Theorem 3.17.1. [Mor12, 6.61], [ABH24, 3.7] Let $n \geq 0$ and $X \in \mathrm{Spc}(k)_*$, and suppose that X is $\mathbb{A}^1$-n-connected. Then for any $i \geq 1$ the map

$$X \to \Omega^{i,0}\Sigma^{i,0}X$$

has $2n$-connected fibers.[8]

Proof. Let F be the fiber, computed in the sheaf topos

$$F \to X \to \Omega\Sigma X.$$

Since L_{mot} preserves this fiber sequence, and the middle and rightmost spaces were already local, then F is local as well. Since F is already $2n$-connected in the sheaf topos by classical connectivity results, we have by unstable connectivity (Theorem 3.16.3) that $L_{\mathrm{mot}}F$ is as well. □

Corollary 3.17.2. If X is $\mathbb{A}^1$-n-connected, then the suspension morphisms

$$\pi_i^{\mathbb{A}^1}(X) \to \pi_{i+1}^{\mathbb{A}^1}(\Sigma^{1,0}X)$$

are isomorphisms for $i \leq 2n$ and an epimorphism for $i = 2n+1$.

3.18 Motivic homotopy groups of spheres

Considering $\mathbb{G}_m^{\wedge n}$ as a sheaf of pointed sets, it is discrete, hence we have that

$$\pi_2(\Sigma^2\mathbb{G}_m^{\wedge n}) \xrightarrow{\sim} \pi_3(\Sigma^3\mathbb{G}_m^{\wedge n}) \xrightarrow{\sim} \cdots$$

by Freudenthal. Note that we already sort of knew this (modulo the black boxing we've been doing), since we identified the free strongly invariant sheaf of abelian groups on a sheaf of pointed sets S with $\pi_n(\Sigma^n S)$ for any $n \geq 2$. Identifying this with Milnor–Witt K-theory allows us to conclude that

$$\pi_{n,n}\mathbb{S} \cong \mathbf{K}_n^{\mathrm{MW}}$$

for $n \leq 0$. We'll offer a slightly different perspective.

Intuition 3.18.1. In classical homotopy theory, the Freudenthal suspension theorem, together with the Hurewicz theorem, identifies the first nonvanishing homotopy group of S^n with the *free discrete abelian group* on S^0:

$$\pi_n(S^n) \cong H_n(S^n) \cong H_{n-1}(S^{n-1}) \cong \cdots \cong H_0(S^0) = \mathbb{Z}[S^0].$$

[8] Note that Morel initially assumed $n \geq 2$, but this hypothesis is removed in [ABH24].

In motivic homotopy theory, the first nonvanishing homotopy sheaf of $\Sigma^{n-1}\mathbb{G}_m^{\wedge n} = S^{2n-1,n}$ is the free strongly invariant sheaf of abelian groups on the sheaf of pointed sets $\mathbb{G}_m^{\wedge n}$ [Mor12, 1.23].

We can identify many groups now just using contraction:

Corollary 3.18.2. We have a canonical isomorphism

$$\pi_n\left(S^{n+i,i}\right) \cong \mathbf{K}_i^{\mathrm{MW}},$$

for $n \geq 2$ and $i \geq 1$ [Mor12, p. 167].

The proof of the theorem follows the classical proof — we develop a notion of $\mathbb{A}^1$-homology and a Hurewicz theorem, which we are omitting in this class. Instead we'll look at the generators in $\mathbf{K}_*^{\mathrm{MW}}$ as maps of motivic spheres and verify all the relations hold.
Recall the universal symbol map

$$\sigma_n \colon \mathbb{G}_m^{\wedge n} \to \mathbf{K}_n^{\mathrm{MW}}.$$

In degree 1 it send an element $u \in k^\times$ to $[u] \in \mathbf{K}_1^{\mathrm{MW}}$. Explicitly, by taking global sections $H^0(\mathrm{Spec}\,(k)_+, -)$ of the symbol map, we see that $[u] \in \mathbf{K}_1^{\mathrm{MW}}$ is represented by the map

$$S^0 \xrightarrow{[u]} \mathbb{G}_m,$$

sending the basepoint to the basepoint $1 \in \mathbb{G}_m$, and the other point to $u \in \mathbb{G}_m$. The element η is the *Hopf map*

$$\begin{aligned} \eta \colon \mathbb{A}^2 \setminus 0 &\to \mathbb{P}^1 \\ (x,y) &\mapsto [x:y]\,. \end{aligned}$$

This is a map $S^{3,2} \to S^{2,1}$, so it occurs in bidegree $\pi_{-1,-1}\mathbb{1}$.

Proposition 3.18.3. The following diagram commutes in Sm_k:

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \arrow[rr, "{(\mathrm{id},\mathrm{id})}"] \arrow[dr, "\mu"] \arrow[dd, "{(\mathrm{inv},\mathrm{inv})}"'] & & \mathbb{G}_m \times \mathbb{A}^1 \arrow[dr, "\mu"] \arrow[dd] & \\
& \mathbb{G}_m \arrow[rr, "\mathrm{id}"] \arrow[dd] & & \mathbb{A}^1 \arrow[dd] \\
\mathbb{G}_m \times \mathbb{A}^1 \arrow[rr, "\mathrm{inv}"] \arrow[dr] & & \mathbb{A}^2 \setminus 0 \arrow[dr, dashed, "\eta"] & \\
& \mathbb{A}^1 \arrow[rr] & & \mathbb{P}^1
\end{tikzcd}$$

Corollary 3.18.4. Rearranging the diagram, we have that the Hopf map factors in $\mathrm{Spc}(k)_*$ as

$$\mathbb{A}^2 \setminus 0 \to \Sigma\left(\mathbb{G}_m \times \mathbb{G}_m\right) \xrightarrow{\Sigma\mu} \Sigma\mathbb{G}_m. \tag{3.5}$$

We'll see a slightly different perspective on this soon.

3.18.1 The Steinberg relation

Recall the first relation **MW1** was $[u][1-u] = 0$. This is often called the *Steinberg relation*. The statement that it holds at the level of maps of motivic spheres was first claimed by Hu and Kriz, however their paper contained an error. It was proven by Druzhinin in the stable setting, and reproven in the unstable setting by Hoyois (see [Hoy18] for details).

Consider the pushout diagram

$$\begin{tikzcd} \mathbb{A}^1 - \{0,1\} \arrow[r] \arrow[d] & \mathbb{A}^1 - 1 \arrow[d] \\ \mathbb{A}^1 - 0 \arrow[r] & \mathbb{A}^1. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By taking the pullback of the bottom right cospan, we obtain a natural map of unpointed spaces, where each $\mathbb{G}_m$ here is now $\mathbb{A}^1 - 0$:

$$\begin{aligned} \mathbb{A}^1 - \{0,1\} &\to \mathbb{G}_m \times \mathbb{G}_m \\ a &\mapsto (a, 1-a). \end{aligned}$$

Given any unit $u \in k^\times$, the map $\operatorname{Spec}(k) \xrightarrow{(u,1-u)} \mathbb{G}_m \wedge \mathbb{G}_m$ factors through the unpointed map above

$$\begin{tikzcd} \operatorname{Spec}(k) \arrow[rr] \arrow[dr] & & \mathbb{G}_m \wedge \mathbb{G}_m \\ & \mathbb{A}^1 - \{0,1\} \arrow[ur] & \end{tikzcd}$$

Note that $\mathbb{G}_m \times \mathbb{G}_m$ is naturally pointed, but $\mathbb{A}^1 - \{0,1\}$ isn't, so we add a disjoint basepoint to get a diagram of pointed spaces

$$\begin{tikzcd} S^0 \arrow[rr] \arrow[dr] & & \mathbb{G}_m \wedge \mathbb{G}_m \\ & (\mathbb{A}^1 - \{0,1\})_+ \arrow[ur, swap, "\mathfrak{st}"] & \end{tikzcd}$$

Hence it suffices to show that the *pointed map* $\left(\mathbb{A}^1 - \{0,1\}\right)_+ \to \mathbb{G}_m \wedge \mathbb{G}_m$ is null-homotopic after suspension. The mistake in Hu-Kriz was not considering the image of the additional basepoint one must add.

Theorem 3.18.6. [Hoy18] The S^1-suspension of the Steinberg map

$$\mathfrak{st}\colon \left(\mathbb{A}^1 - \{0,1\}\right)_+ \to \mathbb{G}_m \wedge \mathbb{G}_m$$

is null-homotopic.

Sketch. The main idea is to blow up $\mathbb{A}^2$ at $(1,0)$ and $(0,1)$, and show that the map above factors after 1-fold suspension through an induced map from the union of three lines, one connecting $(0,1)$ and $(1,1)$, one connecting $(0,1)$ and $(1,0)$, and the exceptional divisor over $(0,1)$. $\square$

3.18.2 Digression: the Hopf construction

Let X be a pointed space (motivic or topological). If X has a comultiplication $X \to X \vee X$, then $\operatorname{Hom}(X,Y)$ has a monoid structure, induced by precomposition with the monoidal structure

$$\operatorname{Hom}(X,Y) \times \operatorname{Hom}(X,Y) \cong \operatorname{Hom}(X \vee X, Y) \to \operatorname{Hom}(X,Y).$$

We say that the comultiplication is *cocommutative* if the diagram commutes (up to homotopy)

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X \vee X \arrow[d, "\mathrm{swap}"] \\ & X \vee X. \end{tikzcd}$$

Example 3.18.7. The pinch maps

$$S^n \to S^n \vee S^n,$$

defined by collapsing a great circle containing the basepoint, define a coassociative comultiplication co-H-space structure on the n-sphere. For $n \geq 2$ this operation is cocommutative.

The comultiplication on S^n defines a group structure on $[S^n, X]_*$, and the cocommutative property is exactly the statement that $\pi_n(X)$ is abelian for $n \geq 2$.

Example 3.18.8. We have that ΣX admits a comultiplication for any $X \in \mathrm{Spc}_*$, induced by the pinch map $S^1 \to S^1 \vee S^1$.

Suppose now that $G \in \mathrm{Spc}_*$ is a group object. Then $[X, G]_*$ admits a group structure, and an equation of the form $f = f_1 + f_2$ in $[X, G]_*$ means exactly that f factors as

$$\begin{tikzcd} X \arrow[r, "{f_1, f_2}"] \arrow[dr, "f"'] & G \times G \arrow[d, "\mu"] \\ & G. \end{tikzcd}$$

If G is a group object, this does *not* imply that ΣG is a group object, and in general this won't be true. So what happens to the equation $f = f_1 + f_2$ when we suspend? In other words can we express Σf in terms of Σf_1 and Σf_2 in $[\Sigma X, \Sigma G]_*$?

3.18.3 Splitting cofiber sequences

The following we learned from [DI13, Appendix A].

Suppose we have a cofiber sequence in $\mathrm{Spc}(k)_*$

$$A \xrightarrow{j} B \xrightarrow{p} B/A,$$

and suppose that j is split after suspension – in other words there exists some map $f \colon \Sigma B \to \Sigma A$ so that the following diagram commutes up to homotopy

$$\begin{tikzcd} \Sigma A \arrow[r] \arrow[dr, "\mathrm{id}"'] & \Sigma B \arrow[d, "f"] \\ & \Sigma A. \end{tikzcd}$$

Applying $[-, X]_*$, we get a sequence of groups, which turns out to be short exact

$$1 \to [\Sigma(B/A), X]_* \to [\Sigma B, X]_* \to [\Sigma A, X]_* .$$

This is not necessarily split exact – if f were a suspension of a map, then it would be a map of H-spaces and induce a splitting, but the best we can say here is that the sequence is exact.

Case: Let's take $X = \Sigma B$. Then we get a short exact sequence

$$0 \to [\Sigma(B/A), \Sigma B] \to [\Sigma B, \Sigma B] \to [\Sigma A, \Sigma B] .$$

Note that since $\mathrm{id}_{\Sigma B} = (\Sigma j) f$, the map $\mathrm{id}_{\Sigma B} - (\Sigma j) f$ is in the kernel of the rightmost map, hence is uniquely in the image of $[\Sigma(B/A), \Sigma B]$.

Let $\chi \colon \Sigma(B/A) \to \Sigma B$ be the unique map sent to $\mathrm{id}_{\Sigma B} - (\Sigma j) f$.

Lemma 3.18.9. [DI13, A.3] We have that the diagram commutes

$$\begin{tikzcd} \Sigma B/A \arrow[r, "\chi"] \arrow[dr, equal] & \Sigma B \arrow[d, "\Sigma p"] \\ & \Sigma B/A, \end{tikzcd}$$

and the composite

$$\Sigma B/A \xrightarrow{\chi} \Sigma B \xrightarrow{f} \Sigma A$$

is null-homotopic.

Proposition 3.18.10. For any pointed spaces $X, Y \in \mathrm{Spc}(k)_*$, the cofiber sequence
$$X \vee Y \to X \times Y \to X \wedge Y$$
admits a canonical splitting after suspension:
$$\Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y) \cong \Sigma(X \times Y).$$
We claim further that this is an equivalence of co-H-spaces, hence the identity on $\Sigma(X \times Y)$ is a sum of the projection maps.

Proof. Suspension is a left adjoint and hence preserves the above coproduct. Since the suspension of anything is a co-group object (by the pinch map), we have an induced map
$$\Sigma(X \times Y) \to \Sigma(X \times Y) \vee \Sigma(X \times Y).$$
Post-composing with the suspensions of the two projections $X \times Y \to X$ and $X \times Y \to Y$ yields a map $\Sigma(X \times Y) \to \Sigma(X) \vee \Sigma(Y) = \Sigma(X \vee Y)$ which we claim is left inverse to the map in the suspended cofiber sequence. □

Corollary 3.18.11. If G is a group object, we can suspend the multiplication $G \times G \to G$, and use the splitting above to get three maps
$$\Sigma G \vee \Sigma G \vee \Sigma(G^{\wedge 2}) \to \Sigma G.$$
The first two component functions are the identity, and the latter is the *Hopf construction* $\Sigma(G \wedge G) \to \Sigma G$.

Example 3.18.12. The Hopf construction $\Sigma(S^1 \wedge S^1) \to \Sigma S^1$ can be identified with the Hopf map $\eta\colon S^3 \to S^2$ classically.

Proposition 3.18.13. The Hopf construction $\Sigma\mathbb{G}_m \wedge \mathbb{G}_m \to \Sigma\mathbb{G}_m$ from the multiplication on $\mathbb{G}_m$ can be identified with (or be defined as) the Hopf map η [Mor12, p. 71]. In particular since $\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) = \mathbb{A}^2 \smallsetminus 0$, we have that the suspension of the natural map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$ admits a splitting s, hence we obtain a composite
$$\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \xrightarrow{s} \Sigma(\mathbb{G}_m \times \mathbb{G}_m) \xrightarrow{\Sigma\mu} \Sigma\mathbb{G}_m,$$
which is precisely the factorization of the Hopf map we found in Equation 3.5.

Proposition 3.18.14. The splitting
$$\Sigma(\mathbb{G}_m \times \mathbb{G}_m) \xrightarrow{\pi_1 \vee \pi_2} \Sigma(\mathbb{G}_m \vee \mathbb{G}_m) \vee \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)$$
has the property that the two projections sum to the identity on $\Sigma(\mathbb{G}_m \times \mathbb{G}_m)$.

We can now prove **MW2**:

Proposition 3.18.15. The following relation holds in $\pi_{*,*}\mathbb{S}$: $[ab] = [a] + [b] + \eta[a][b]$.

Proof. The map
$$S^0 \xrightarrow{[ab]} \mathbb{G}_m$$
factors as
$$S^0 \xrightarrow{[a],[b]} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m.$$

Suspending, and leveraging the splitting above, we get a commutative diagram

$$\begin{tikzcd}
S^1 \arrow[r, "\Sigma({[a]},{[b]})"] \arrow[rr, bend left, "\Sigma[ab]"] \arrow[dr] & \Sigma(\mathbb{G}_m \times \mathbb{G}_m) \arrow[r, "\Sigma\mu"] \arrow[d, "\cong"] & \Sigma\mathbb{G}_m \\
& \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \vee \Sigma\mathbb{G}_m \vee \Sigma\mathbb{G}_m \arrow[ur, "\Sigma[a]\vee\Sigma[b]\vee\eta[a][b]"'] &
\end{tikzcd}$$

□

3.18.4 The remaining relations

For any $a \in k^\times$, we can consider the map

$$\mathbb{G}_m \to \mathbb{G}_m$$
$$x \mapsto ax.$$

Since $\mathbb{G}_m$ is pointed at 1, we note that this map is *not a pointed map*. Nevertheless, we can take its unreduced suspension, explicitly, coming from the diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[rr, "\mathrm{id}"] \arrow[dr, "a"] \arrow[dd, "\mathrm{inv}"'] & & \mathbb{A}^1 \arrow[dr, "a"] \arrow[dd] & \\
& \mathbb{G}_m \arrow[rr, "\mathrm{id}"] \arrow[dd] & & \mathbb{A}^1 \arrow[dd] \\
\mathbb{A}^1 \arrow[rr, "\mathrm{inv}"] \arrow[dr, "a^{-1}"'] & & \mathbb{P}^1 \arrow[dr, dashed] & \\
& \mathbb{A}^1 \arrow[rr] & & \mathbb{P}^1.
\end{tikzcd}$$

This induced map

$$\mathbb{P}^1 \to \mathbb{P}^1$$
$$[x : y] \mapsto [ax : y]$$

is pointed at ∞. We call this resulting map $\langle a \rangle$ ([Mor04a, 6.3.4], [Mor12, p. 74]).

Lemma 3.18.16. We have that $\langle a \rangle = 1 + \eta[a]$ [Mor12, 3.43(1)].

Proof. We can factor the endomorphism of $\mathbb{G}_m$ to get

$$\mathbb{G}_m \xrightarrow{\mathrm{id}\times a} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m.$$

Suspending, we get our map $[x : y] \mapsto [ax : y]$. Applying our factorization, we see that this decomposes as

$$1 + \eta[a].$$

□

Remark 3.18.17. Any map $\mathbb{G}_m \to \mathbb{G}_m$ is of the form $t \mapsto ut^n$ for $u \in k^\times$ and $n \in \mathbb{Z}$. Its S^1-suspension we can verify is given by

$$\mathbb{P}^1 \to \mathbb{P}^1$$
$$[x : y] \mapsto [ux^n : y^n].$$

Proposition 3.18.18. The map

$$\begin{aligned}\mathbb{G}_m &\to \mathbb{G}_m\\ t &\mapsto t^2\end{aligned}$$

corresponds to h.

Proof. We can conclude this by reference to Cazanave's theorem although this is super ahistorical. There's a more direct way to argue that it decomposes to $1 + \langle -1\rangle$ that we should add to the notes. □

Proposition 3.18.19. The swap map

$$S^1 \wedge S^1 \xrightarrow{\text{swap}} S^1 \wedge S^1$$

corresponds to the class -1.

Proposition 3.18.20. The swap map

$$\mathbb{G}_m \wedge \mathbb{G}_m \xrightarrow{\text{swap}} \mathbb{G}_m \wedge \mathbb{G}_m$$

corresponds stably to ϵ [Mor12, 3.43], [Mor04a, 6.1.1(2)].

Proof. After suspending, we recall that $\Sigma(\mathbb{G}_m \wedge \mathbb{G}_m) \simeq \mathbb{A}^2 \smallsetminus 0$, and the map is of the form $(x, y) \mapsto (y, x)$. This is induced by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Up to SL_2-equivalence, this agrees with $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so we see that this is the S^1-suspension of

$$\begin{aligned}\mathbb{P}^1 &\to \mathbb{P}^1\\ [x : y] &\mapsto [-x : y].\end{aligned}$$

This is $\langle -1\rangle$ by Lemma 3.18.16, and the swap map on the S^1-suspension picks up a negative sign. □

Proposition 3.18.21. Property **MW4** that $\epsilon\eta = \eta$ holds.

Proof. Consider multiplication on the first two factors of $\mathbb{G}_m^{\wedge 3}$, followed by the swap map

$$\begin{aligned}\mathbb{G}_m^{\wedge 3} \xrightarrow{\eta\wedge \mathrm{id}} \mathbb{G}_m^{\wedge 2} &\xrightarrow{\epsilon} \mathbb{G}_m^{\wedge 2}\\ x \wedge y \wedge z \mapsto xy \wedge z &\mapsto z \wedge xy.\end{aligned}$$

This models $\epsilon\eta$. We could alternatively cycle the third factor to the first slot (corresponding to $\epsilon^2 = 1$, and then multiply:

$$\begin{aligned}\mathbb{G}_m^{\wedge 3} \xrightarrow{\epsilon^2} \mathbb{G}_m^{\wedge 3} &\xrightarrow{\mathrm{id}\wedge\eta} \mathbb{G}_m^{\wedge 2}\\ x \wedge y \wedge z \mapsto z \wedge x \wedge y &\mapsto z \wedge xy.\end{aligned}$$

Hence $\epsilon\eta = \epsilon^2\eta = \eta$. □

Proposition 3.18.22. Property **MW3** that $\eta[a] = [a]\eta$ holds.

Proof. We can write $a[\eta]$ as the smash product

$$\mathbb{G}_m^{\wedge 3} \xrightarrow{[a]\cdot\eta} \mathbb{G}_m^{\wedge 2}.$$

This is the same as smashing them in the other order and then swapping

$$\mathbb{G}_m^{\wedge 3} \xrightarrow{\eta[a]} \mathbb{G}_m^{\wedge 2} \xrightarrow{\epsilon} \mathbb{G}_m^{\wedge 2}.$$

This is $\epsilon\eta[a]$, which by **MW4** is equal to $\eta[a]$. □

We should technically be a lot more careful about composition versus smash product, multiplication versus η, etc. For very concrete detail of the verification of the axioms, see [Dru21, §3].

3.18.5 On nilpotence

Notation 3.18.23. We denote by

$$\pi_{a,b}\mathbb{1} := \mathrm{colim}_{k\to\infty}\pi_{a+2k,b+k}(S^{2k,k}).$$

We've now argued that there is an isomorphism of graded algebras

$$\oplus_{n\in\mathbb{Z}}\pi_{n,n}\mathbb{1} = \oplus_{n\in\mathbb{Z}}K_n^{\mathrm{MW}}(k).$$

Recall the Nishida nilpotence theorem in classical homotopy theory.

Theorem 3.18.24. (Nilpotence) Any element in $\pi_k\mathbb{S}$ for $k > 0$ is nilpotent.

Note that $\eta \in \pi_{1,1}\mathbb{1}$ in motivic homotopy. This is **not** nilpotent! Multiplication by η induces an isomorphism $\mathbf{K}_{-n}^{\mathrm{MW}} \to \mathbf{K}_{-n-1}^{\mathrm{MW}}$ for any $n > 0$. Moreover, η is not the only non-nilpotent element – see [GI15] for more details.

3.19 Stable motivic homotopy theory

(This section was very directly inspired by Tom Bachmann and Viktor Burghardt's lectures at IWOAT 2024).

Recall we've seen that $\mathrm{Spc}(k)_*$ is a symmetric monoidal presentable ∞-category. Thinking about this as analogous to pointed spaces, we want to develop a notion of *spectra*, where $\mathbb{A}^1$-invariant cohomology theories of schemes will be representable.

3.19.1 Motivation: from spaces to spectra

Suppose we are starting with the ∞-category of spaces, and we're wanting to build spectra, which will be some home for things like stable homotopy groups, $\mathbb{Z}$-graded cohomology theories, etc. How do we do it?

The naive way was to consider what are called *sequential spectra*, being sequences of spaces $\{X_n\}_{n\geq 0}$ and bonding maps $\Sigma X_n \to X_{n+1}$. We might want a few things to be preserved under this construction:

1. $\mathcal{S}$ was *presentable*, and we would like spectra to have the same property
2. $\mathcal{S}$ has a symmetric monoidal structure, and we'd like spectra to have one as well.

Historically, this latter point was the hardest to achieve. Endowing spectra with a model structure dates back to work of Bousfield and Friedlander, however inducing a well-behaved smash product is difficult. This is the main reason for the development of models of spectra like symmetric and orthogonal spectra. While these model structures are important to know for explicit computations, we will be content here with endowing spectra with a presentably symmetric monoidal structure.

Recall Pr^L is the category of presentable ∞-categories and left adjoints between them. It admits a symmetric monoidal structure, due to Lurie, so that $\mathrm{CAlg}(\mathrm{Pr}^L)$ consists of *presentably symmetric monoidal* ∞-categories. Informally these are categories which are both presentable and symmetric monoidal, and these structures behave well with each other. Explicitly, it means the tensor product preserves colimits in each variable.

We might define spectra naively to be the colimit

$$\mathrm{Sp}^{\mathbb{N}}(\mathscr{C}, f) := \operatorname{colim}\left(\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \cdots\right).$$

We could try to compute this colimit in Pr^L, however we note that $\mathrm{Pr}^L \subseteq \mathrm{Cat}_\infty$ isn't necessarily closed under colimits. There is an infamous trick to get around this.

Proposition 3.19.1. Suppose $\mathscr{C} \in \mathrm{Pr}^L$ and $f\colon \mathscr{C} \to \mathscr{C}$ is an endomorphism of $\mathscr{C}$ in Pr^L. Then the colimit

$$\operatorname{colim}\left(\mathscr{C} \xrightarrow{f} \mathscr{C} \xrightarrow{f} \cdots\right)$$

is canonically equivalent to the limit

$$\lim\left(\cdots \xrightarrow{g} \mathscr{C} \xrightarrow{g} \mathscr{C}\right)$$

where $f \dashv g$.

Sketch. The equivalence between Pr^L and Pr^R is in [Lur09, 5.5.3.4], and we use the crucial fact that $\mathrm{Pr}^R \subseteq \mathrm{Cat}_\infty$ preserves limits [Lur09, 5.5.3.18]. □

Definition 3.19.2. If $\mathscr{C}$ is presentable, then the ∞-category of *spectrum objects* is the limit

$$\mathrm{Sp}(\mathscr{C}) = \lim\left(\cdots \mathscr{C} \xrightarrow{\Omega} \mathscr{C} \xrightarrow{\Omega} \mathscr{C}\right).$$

A special case comes when Σ is given by smashing with a particular object. We've seen this statement earlier, when we cooked up a symmetric monoidal structure on homotopy modules.

Theorem 3.19.3. (Robalo) For $\mathscr{C}$ a presentably symmetric monoidal category, and $X \in \mathscr{C}$, there is a functor

$$\mathrm{Sp}^{\mathbb{N}}(\mathscr{C}, X \otimes -) \to \mathscr{C}[X^{-1}]$$

which is an equivalence if X is symmetric.

Example 3.19.4. The 1-sphere $S^1 \in \mathcal{S}$ is symmetric, since the cyclic permutation

$$(1\ 2\ 3)\colon S^3 \to S^3$$

is homotopic to the identity.

Corollary 3.19.5. We have that the category of spectra is a presentably symmetric monoidal category, equivalently presented in any of the three ways:

$$\begin{aligned}\mathrm{Sp} &= \mathcal{S}_*[(S^1)^{-1}] \\ &= \operatorname{colim}_{\mathrm{Pr}^L}\left(\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \cdots\right) \\ &= \lim_{\mathrm{Cat}_\infty}\left(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*\right).\end{aligned}$$

3.19.2 Spectrum objects

Let $\mathscr{C}$ be a pointed ∞-category with finite limits (in particular it has pullbacks and a terminal object). Then the ∞-category of *spectrum objects* in $\mathscr{C}$, denoted $\mathrm{Sp}(\mathscr{C})$, is the limit

$$\mathrm{Sp}(\mathscr{C}) = \lim\left(\cdots \mathscr{C} \xrightarrow{\Omega} \mathscr{C} \xrightarrow{\Omega} \mathscr{C}\right).$$

Objects in $\mathrm{Sp}(\mathscr{C})$ are given by sequences $(X_n)_{n\geq 0}$ where we have structure maps which are equivalences

$$X_n \xrightarrow{\sim} \Omega X_{n+1}.$$

There is an obvious functor $\Omega^\infty\colon \mathrm{Sp}(\mathscr{C}) \to \mathscr{C}$ given by sending $(X_n)_{n\geq 0}$ to X_0. If $\mathscr{C}$ is presentable, then $\mathrm{Stab}(\mathscr{C})$ will be as well by construction. Hence since Ω^∞ is limit-preserving, it admits a left adjoint

$$\Sigma^\infty\colon \mathscr{C} \rightleftarrows \mathrm{Sp}(\mathscr{C}) :\Omega^\infty.$$

The mapping spaces $\mathrm{Map}_{\mathrm{Stab}(\mathscr{C})}(E_\bullet, F_\bullet)$ are by definition the limit

$$\mathrm{Map}_{\mathrm{Stab}(\mathscr{C})}(E_\bullet, F_\bullet) = \lim \mathrm{Hom}_{\mathscr{C}}(E_n, F_n),$$

where the maps in the limit are given by looping and using the structure maps

$$\mathrm{Map}_{\mathscr{C}}(E_{n+1}, F_{n+1}) \xrightarrow{\Omega} \mathrm{Map}_{\mathscr{C}}(\Omega E_{n+1}, \Omega F_{n+1}) \cong \mathrm{Map}_{\mathscr{C}}(E_n, F_n).$$

Remark 3.19.6. This category $\mathrm{Sp}(\mathscr{C})$ has a universal property, in that it is the *stabilization* of $\mathscr{C}$. In particular a category $\mathscr{C}$ is stable if and only if $\Omega^\infty\colon \mathrm{Sp}(\mathscr{C}) \to \mathscr{C}$ is an equivalence of categories [Lur17, 1.4.2.21]. See [Lur17, §1.4.2] for more details on this construction.

Example 3.19.7. The classical category of spectra Sp is the category of spectrum objects in $\mathcal{S}$.

Given our category of presheaves $\mathrm{PSh}(\mathrm{Sm}_k)$, we can stabilize it and we obtain $\mathrm{Fun}(\mathrm{Sm}_k^{\mathrm{op}}, \mathrm{Sp})$, i.e. presheaves of spectra. It turns out the descent properties descent through stabilization, in other words the stabilization of $\mathrm{PSh}(\mathrm{Sm}_k)$ is exactly Nisnevich sheaves of spectra. We denote this by

$$\mathrm{Sp}(k) := \mathrm{Sp}\left(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)\right) = \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k; \mathrm{Sp}).$$

3.19.3 Sheaves of spectra

A sequence of objects in the sheaf category give rise to a spectrum if they deloop to one another.

Example 3.19.8. For any Nisnevich sheaf of abelian groups A, we have an object $HA \in \mathrm{Sp}(k)$ given by the Eilenberg–MacLane spaces $\{K(A,n)\}_{n\geq 0}$.

Proposition 3.19.9. Cohomology is representable in the sense that for any $X \in \mathrm{Sm}_k$ and any sheaf of abelian groups A we have that

$$H^n_{\mathrm{Nis}}(X, A) = [\Sigma^\infty \Sigma^n X_+, \mathrm{HA}]_{\mathrm{Sp}(k)}.$$

We can define π_n of a spectrum E to be the sheafification of the presheaf of abelian groups

$$U \mapsto [\Sigma^\infty \Sigma^n U_+, E]_{\mathrm{Sp}(k)}.$$

Example 3.19.10. We have that $\pi_n \mathrm{HA}$ is zero except in $n = 0$ when it is A.

Remark 3.19.11. This category admits a t-structure, whose heart is precisely equivalent to $\mathrm{Ab}(\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)_{\leq 0})$, the sheaves of abelian groups.

3.19.4 $\mathbb{A}^1$-invariant sheaves of spectra

Definition 3.19.12. We say that $E \in \mathrm{Sp}(k)$ is $\mathbb{A}^1$-*invariant* if for every $X \in \mathrm{Sm}_k$ the projection $X \times \mathbb{A}^1 \to X$ induces an equivalence

$$E(X \times \mathbb{A}^1) \xrightarrow{\sim} E(X).$$

Proposition 3.19.13. We denote by $\mathrm{SH}_{S^1}(k) \subseteq \mathrm{Sp}(k)$ the category of $\mathbb{A}^1$-invariant sheaves of spectra. This is an accessible subcategory, hence the inclusion admits a left adjoint.

Remark 3.19.14. This is poor notation — technically $\mathrm{SH}_{S^1}(k)$ should denote the *homotopy category*, rather than the ∞-category itself. This is a pervasive notational inconsistency in the literature. This is denoted by $\mathrm{SH}^{S^1}(k)$ in [Mor04b, 3.2.1].

The category $\mathrm{SH}^{S^1}(k)$ admits a *homotopy t-structure* with heart given by strictly invariant sheaves of abelian groups.

3.19.5 Motivic spectra

The category of motivic spectra is defined to be

$$\mathrm{SH}(k) := \mathrm{Spc}(k)_* \left[(\mathbb{P}^1)^{-1}\right].$$

By the result of Robalo, it suffices to argue that $\mathbb{P}^1$ is symmetric in order to deduce formal properties about this category.

Proposition 3.19.15. We have that $\mathbb{P}^1$ is symmetric, therefore $\mathrm{SH}(k)$ is presentably symmetric monoidal.

Proof. By identifying $\mathbb{P}^1 = \mathbb{A}^1/(\mathbb{A}^1 - 0)$, we can identify

$$(\mathbb{P}^1)^{\otimes 3} \simeq \mathbb{A}^3/(\mathbb{A}^3 - 0),$$

and the cyclic permutation $(1\ 2\ 3)$ corresponds to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This is the product of elementary matrices, each of which is $\mathbb{A}^1$-homotopic to the identity. $\square$

Say A is a sheaf of abelian groups. How would we build HA as a spectrum here? We begin with $K(A, 0) = A$. To be the zeroth space of a spectrum, we need some $K(A', 1)$ so that $K(A, 0) \to \Omega^{2,1}K(A', 1)$ is an equivalence. The S^1-loop space decreases the topological index, while the $\mathbb{G}_m$-loop space contracts the sheaf, so we get that $(A')_{-1} = A$. We see that A will yield an Eilenberg–MacLane spectrum if:

1. A admits infinitely many deloopings
2. all the deloopings are strongly invariant.

In other words we need A to be a *homotopy module.* We collect this discussion into a proposition for convenience.

Proposition 3.19.16. Any *homotopy module* A_* gives rise to an Eilenberg–MacLane spectrum $HA_* \in \mathrm{SH}(k)$.

This correspondence identifies the heart of the homotopy t-structure on motivic spectra with homotopy modules

$$\mathrm{SH}(k)^{\heartsuit} \simeq \mathrm{HM}(k).$$

3.19.6 Cohomology and stable homotopy groups

By a completely analogous argument to Proposition 3.19.9, we have for any $X \in \mathrm{Sm}_k$ and $A \in \mathrm{HM}(k)$:

$$H^n_{\mathrm{Nis}}(X, A_{-k}) = \left[\Sigma^\infty \Sigma^{n+k,k} X_+, \mathrm{HA}\right]_{\mathrm{SH}(k)}.$$

Similarly we can define stable homotopy groups which are bigraded via the following definition.

Definition 3.19.17. For $a, b \in \mathbb{Z}$ and $E \in \mathrm{SH}(k)$, we define $\pi_{a,b}(E)$ to be the sheaf attached to the presheaf

$$U \mapsto \left[\Sigma^\infty \Sigma^{a,b} U_+, E\right]_{\mathrm{SH}(k)}.$$

3.20 Representability of algebraic K-theory

Goal 3.20.1. We want to show that algebraic K-theory is represented by a motivic spectrum KGL.

3.20.1 What is group completion?

Recall if $(M, \amalg)$ is a commutative monoid, its *group completion* is the universal group receiving a monoid map from M. This is often denoted M^{gp} or classically $K(M)$.

Definition 3.20.2. If $\mathscr{C}$ is any ∞-category with finite limits, then a *commutative monoid* is precisely a product preserving functor

$$\mathrm{CMon}(\mathscr{C}) = \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathscr{C}).$$

We recover the monoid structure by considering certain spans. Let $\underline{n}$ denote a finite set with n elements, then the span $\underline{2} = \underline{2} \to \underline{1}$ induces $M \times M \to M$, while $\varnothing \leftarrow \varnothing \to \underline{1}$ yields the unit $* \to M$. Larger spans encode various combinations of multiplications and identities, and the data of an ∞-functor $\mathrm{Span}(\mathrm{Fin}) \to \mathscr{C}$ imposes all the associativity and unitality conditions.

Consider a distinguished span $\underline{2} \leftarrow \underline{3} \to \underline{2}$ of the form

$$\begin{tikzcd} & \{x,y,z\} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \{m_1, m_2\} & & \{a,b\}\}, \end{tikzcd}$$

where $f(x) = m_1$ and $f(y) = f(z) = m_2$, and $g(x) = g(y) = a$ and $g(z) = b$. As a monoid map, this is of the shape

$$\begin{aligned} M \times M &\to M \times M \\ (m_1, m_2) &\mapsto (m_1 + m_2, m_2)\,. \end{aligned}$$

The preimage of $(0, m)$ will be $(-m, m)$, so we see that the above is an abelian group if and only if this map is an equivalence

Definition 3.20.3. We define the subcategory of *abelian group objects*

$$\mathrm{Ab}(\mathscr{C}) \subseteq \mathrm{CMon}(\mathscr{C})$$

to be the full subcategory on which the distinguished span $\underline{2} \leftarrow \underline{3} \to \underline{2}$ is an equivalence.

Group completion is about exhibiting an inverse to this.

Proposition 3.20.4. Suppose $\mathscr{C}$ is presentable.

1. Both $\mathrm{CMon}(\mathscr{C})$ and $\mathrm{Ab}(\mathscr{C})$ are presentable
2. The inclusion $\mathrm{Ab}(\mathscr{C}) \subseteq \mathrm{CMon}(\mathscr{C})$ preserves all limits and filtered colimits, and hence admits a left adjoint.

We denote by

$$(-)^{\mathrm{gp}} \colon \mathrm{CMon}(\mathscr{C}) \to \mathrm{Ab}(\mathscr{C})$$

the group completion which is left adjoint to the inclusion.

Proposition 3.20.5. If $F \colon \mathscr{C} \to \mathscr{D}$ preserves finite products, then it preserves commutative monoids and abelian group objects, inducing a commutative diagram

$$\begin{tikzcd} \mathrm{CMon}(\mathscr{C}) \arrow[r] \arrow[d, "(-)^{\mathrm{gp}}"'] & \mathrm{CMon}(\mathscr{D}) \arrow[d, "(-)^{\mathrm{gp}}"] \\ \mathrm{Ab}(\mathscr{C}) \arrow[r] & \mathrm{Ab}(\mathscr{D}). \end{tikzcd}$$

Example 3.20.6. If $\mathscr{C} = \mathrm{Set}$, then this is classical group completion.

A primary case of interest is when $\mathscr{C} = \mathcal{S}$ is the category of spaces. In this case if M is a commutative monoid, it will be a commutative group if and only if it is a loop space. There is a natural candidate for the delooping of M, namely BM. There is a natural map

$$M \to \Omega BM,$$

which can be studied at the level of homology [MS76a; Nik17].

Definition 3.20.7. Take a collection of generators for $\pi_0 M$, and denote by

$$M_\infty$$

the colimit of multiplying by finite subsets of the generators infinitely many times. In nice settings, this is precisely $M\left[\pi_0(M)^{-1}\right]$.

Since every element in $\pi_0(M)$ is invertible in $B\Omega M$, the universal map factors as

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & \Omega BM \\ & M_\infty \arrow[ur] & \end{tikzcd}$$

Theorem 3.20.8. The map

$$M_\infty \to \Omega BM = M^{\mathrm{gp}}$$

is a plus construction (see [Nik17, Theorem 9], also Randal-Williams).

There is a natural map

$$B\Sigma_n \to (M^{\times n})_{h\Sigma_n} \to M,$$

inducing a homomorphisms $\Sigma_n \to \pi_1(M)$.

Theorem 3.20.9. [Nik17, Prop. 6] The following are equivalent:

1. The natural map $M_\infty \to M^{\mathrm{gp}}$ is an equivalence (i.e. M_∞ *is* the group completion of M)
2. The cyclic permutation $(1\ 2\ 3)$ lies in the kernel

$$\Sigma_3 \to \pi_1(M) \to \pi_1(M_\infty).$$

3.20.2 Algebraic K-theory

The nerve provides an inclusion

$$\mathrm{Grpd} \hookrightarrow \mathcal{S},$$

which preserves finite products, and hence preserves commutative monoid objects.

Definition 3.20.10. For a ring R, we define $K(R) \in \mathrm{Ab}(\mathcal{S})$ to be the group completion of the monoid of finitely generated projective R-modules $\mathrm{Vect}(R)$ under direct sum.

Note that $\mathrm{Vect}(-)$ satisfies Nisnevich descent, hence it gives a sheaf of groupoids (a stack):

$$\mathrm{Vect}(-)\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Grpd} \subseteq \mathcal{S}.$$

Recall that finitely generated projective R-modules of rank n are classified by homotopy classes of maps into BGL_n in the sheaf topos, hence there is an equivalence of stacks

$$\amalg_{n=0}^{\infty} \mathrm{BGL}_n \to \mathrm{Vect}.$$

Since each of BGL_n is connected, we have that $\pi_0\left(\amalg_{n\geq 0}\mathrm{BGL}_n\right) = \mathbb{N}$. So to create Vect_∞, we have to take the colimit of "multiplication by 1." This is exactly given by the shift map

$$\amalg_{n\geq 0}\mathrm{BGL}_n \xrightarrow{+1} \amalg_{n\geq 1}\mathrm{BGL}_n \subseteq \amalg_{n\geq 0}\mathrm{BGL}_n,$$

induced by adding a free vector bundle of rank one: $\mathrm{BGL}_n \to \mathrm{BGL}_{n+1}$.

Proposition 3.20.11. We have that
$$\mathrm{Vect}_\infty = \mathbb{Z} \times \mathrm{BGL}.$$

Proof. We can pull the disjoint union out of the colimit, and we get
$$\begin{aligned} &\operatorname{colim}\left(\amalg_{n\geq 0}\mathrm{BGL}_n \xrightarrow{+1} \amalg_{n\geq 0}\mathrm{BGL}_n \xrightarrow{+1} \cdots\right) \\ &= \amalg_{n\in\mathbb{Z}}\operatorname{colim}\left(\mathrm{BGL}_n \to \mathrm{BGL}_{n+1} \to \cdots\right) \\ &= \mathbb{Z} \times \mathrm{BGL}. \end{aligned}$$
□

Since $K = \mathrm{Vect}^{\mathrm{gp}}$, the factorization $\mathrm{Vect} \to \mathrm{Vect}_\infty \to \mathrm{Vect}^{\mathrm{gp}}$ is of the form
$$\begin{tikzcd} \mathrm{Vect} \arrow[rr] \arrow[dr] & & K \\ & \mathbb{Z} \times \mathrm{BGL} \arrow[ur] & \end{tikzcd}$$

Theorem 3.20.12. We have that $\mathbb{Z} \times \mathrm{BGL} \to K$ is a motivic equivalence.

Proof. Since L_{mot} preserves finite products, it clearly preserves commutative monoids, and will preserve group objects since it preserves equivalences. It also preserves colimits, so we can commute L_{mot} with both $(-)_\infty$ and $(-)^{\mathrm{gp}}$. We are trying to show $\mathrm{Vect}_\infty \to \mathrm{Vect}^{\mathrm{gp}}$ is a motivic equivalence, and commuting past L_{mot}, we see this is equivalent to showing that $L_{\mathrm{mot}}\mathrm{Vect}$ satisfies the properties of the theorem above.

The cyclic condition reduces to checking that $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ acts trivially on $\mathcal{O}^{\oplus 3}$ after motivic localization, which is immediate. □

We have seen above that $\mathrm{Vect}^{\mathrm{gp}}$ is represented by a motivic space $\mathbb{Z} \times \mathrm{BGL}$. This corresponds to what we were calling K-theory, but should really be called *connective* K-theory.

Notation 3.20.13. We denote by $K^{\mathrm{cn}} = \tau_{\geq 0}K$ the connective cover of K-theory.

We'll show algebraic K-theory is a Nisnevich sheaf, but first we provide a more general description of algebraic K-theory of a scheme X.

3.20.3 Interlude: perfect complexes

3.20.4 Algebraic K-theory as a Nisnevich sheaf

We'd like to argue that (non-connective) algebraic K-theory is a Nisnevich sheaf of spectra. For this, a more general perspective is helpful.

Theorem 3.20.14. (Thomason–Trobaugh) Algebraic K-theory is a Nisnevich sheaf of spectra.

Given a distinguished Nisnevich square of the form
$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d] \\ U \arrow[r] & X, \end{tikzcd}$$

we want to argue that K maps it to a homotopy pullback square of spectra.

Given a scheme X, we can form its ∞-category of perfect complexes of $\mathcal{O}_X$-modules $\mathrm{Perf}(X)$. This is a stable ∞-category, giving a functor

$$\mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st}}.$$

This is an fppf sheaf (reference needed). In particular we get a pullback square

$$\begin{tikzcd} \mathrm{Perf}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Perf}(U) \arrow[d] \\ \mathrm{Perf}(V) \arrow[r] & \mathrm{Perf}(W). \end{tikzcd}$$

We want to say that $K(-)$ preserves this pullback square. To make this more clear, we look at the cofibers of the induced maps.

Notation 3.20.15. If $U \hookrightarrow X$ is an open immersion with closed complement Z, we denote by $\mathrm{Perf}(Z \text{ on } X)$ the cofiber

$$\mathrm{Perf}(Z \text{ on } X) \to \mathrm{Perf}(X) \to \mathrm{Perf}(U),$$

which is precisely the perfect complexes on X which are supported on Z.

Example 3.20.16. If $U = D(f) \subseteq X = \mathrm{Spec}(R)$, then there is an exact sequence

$$\mathrm{Perf}(\mathrm{Spec}(R) \text{ on } V(f)) \to \mathrm{Perf}(R) \to \mathrm{Perf}(R_f),$$

where the former is f-torsion modules on R.

So altogether we get a diagram

$$\begin{tikzcd} \mathrm{Perf}(X \text{ on } Z) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Perf}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Perf}(U) \arrow[d] \\ \mathrm{Perf}(W \text{ on Z}) \arrow[r] & \mathrm{Perf}(V) \arrow[r] & \mathrm{Perf}(W). \end{tikzcd}$$

To show K preserves the pullback square, it suffices to show that it preserves cofiber sequences (modulo some subtlety about what precisely that is). This follows from algebraic K-theory, viewed as a functor

$$K\colon \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}$$

being *localizing*.

Corollary 3.20.17. Any localizing invariant is a Nisnevich sheaf.

3.20.5 Algebraic K-theory as $\mathbb{A}^1$-invariant

Theorem 3.20.18. [Qui73, p. 38] (Quillen) If R is a regular Noetherian ring, the map $R \to R[t]$ induces an equivalence

$$K(R) \xrightarrow{\sim} K(R[t]).$$

Theorem 3.20.19. [TT90, 6.8] If X is a regular Noetherian scheme, then the projection map $X \times \mathbb{A}^1 \to X$ induces an equivalence

$$K(X) \xrightarrow{\sim} K(X \times \mathbb{A}^1).$$

Since every smooth finite scheme over a regular Noetherian scheme is itself regular and Noetherian, we get that K is $\mathbb{A}^1$-invariant over Sm_S for any S regular and Noetherian.

3.20.6 Algebraic K-theory spectrum

We have that $K \in \mathrm{Sp}_{\mathbb{A}^1}(k)$ is $\mathbb{A}^1$-invariant by the fundamental theorem of algebraic K-theory. Thus to produce a spectrum, we just have to give bonding maps for the projective line. This is an immediate consequence of the projective bundle formula. From this we obtain $\mathrm{KGL} \in \mathrm{SH}(k)$ given by $\{\mathrm{BGL} \times \mathbb{Z}, \mathrm{BGL} \times \mathbb{Z}, \dots\}$.

Corollary 3.20.20. [AE17, 6.15] We have that

$$\pi_i \mathrm{BGL}_n = K_i$$

for $2 \le i \le n-1$.

3.21 Obstruction theory

We will carry out some basic obstruction theory computations with an eye towards classifying GL_n-torsors.

3.21.1 Postnikov towers, basic theory

Let $X \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$ be a space in the sheaf topos. Then we recall its *Postnikov tower* is of the form

$$\tau_{\le n+1} X \to \tau_{\le n} X \to \dots \to \tau_{\le 0} X,$$

and as we are working over a field, the sheaf topos is hypercomplete, meaning

$$X = \tau_{\le \infty} X = \lim_n \tau_{\le n}.$$

Proposition 3.21.1. If X is a pointed connected sheaf, we have fiber sequences of the form

$$K(\pi_{n+1} X, n+1) \to \tau_{\le n+1} X \xrightarrow{f_n} \tau_{\le n} X.$$

Proof. By Equation 2.47, we can apply the truncation fiber sequence to $\tau_{\le n+1} X$, and we get

$$\tau_{\ge n+1} \tau_{\le n+1} X \to \tau_{\le n+1} X \to \tau_{\le n} \tau_{\le n+1} X,$$

which is of the form above. □

We can ask whether f_i is a *principal* fibration, in that it is pulled back by a map into a classifying space. This turns out to almost be true, it is what is called a *twisted principal fibration.*

Definition 3.21.2. [Mor12, B.2] If G is a sheaf of groups and M is a sheaf of G-modules, we define the *twisted Eilenberg–MacLane space* $K^G(M, n)$ to be the balanced product

$$EG \times_G K(M, n).$$

Proposition 3.21.3. Twisted Eilenberg MacLane spaces have the following properties

1. If M and G are strongly $\mathbb{A}^1$-invariant and $n \ge 2$, we have a fiber sequence [ABH24, 4.17]
$$K(M, n) \to K^G(M, n) \to BG.$$
2. Twisted Eilenberg–MacLane spaces represent cohomology with twisted coefficients
$$\left[X, K^G(M, n)\right] = H^n(X, M(G)).$$

Proposition 3.21.4. (see [AF14a, 6.1]) If X is a pointed connected motivic space, then $f_n \colon \tau_{\leq n+1}X \to \tau_{\leq n}X$ is a twisted $\mathbb{A}^1$-principal fibration, classified by a map

$$\begin{tikzcd} \tau_{\leq n+1}X \arrow[r] \arrow[d, "f_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & B\pi_1(X) \arrow[d] \\ \tau_{\leq n}X \arrow[r, "k_n"'] & K^{\pi_1(X)}(\pi_{n+1}X, n+2). \end{tikzcd}$$

This latter map is called a *k-invariant.*

Example 3.21.5. Suppose we're trying to cook up a map $X \to Y$. Then we can inductively define maps $X \to \tau_{\geq n}Y$, and show they lift along the f_n's. In particular, we get a unique lift

$$\begin{tikzcd} & \tau_{\leq n+1}Y \arrow[d, "f_n"] \\ X \arrow[r] \arrow[ur, dashed] & \tau_{\leq n}Y \end{tikzcd}$$

if and only if the composite

$$X \to \tau_{\leq n}Y \xrightarrow{k_n} K^{\pi_1(Y)}\left(\pi_{n+1}Y, n+2\right).$$

vanishes. In other words, we get *obstructions to lifting* lying in

$$H^{n+2}(X, \pi_{n+1}(Y)\left(\pi_1(Y)\right)).$$

Theorem 3.21.6. If X has Krull dimension d, then any map $X \to Y$ is uniquely determined by $X \to \tau_{\leq d-1}Y$.

Proof. The Nisnevich cohomological dimension is bounded above by the Krull dimension, so $H^k(X, A) = 0$ for any $k > d$ and any sheaf of abelian groups A. In particular, k_{d-1} and higher are trivial, so any map $X \to \tau_{\leq d-1}Y$ lifts to $\tau_{\leq d}Y$, and then all the way to Y. $\square$

Remark 3.21.7. Constructing a map $X \to Y$ by lifting up the Postnikov tower of Y can be thought of as a *relative* lift, by asking to lift along the map

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & *. \end{tikzcd}$$

More generally, given a map $Y \to Z$, we can break the lifting problem

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

into a sequence of analogous stages. This relative situation is often called the *Postnikov–Moore tower* attached to the map $Y \xrightarrow{f} Z$. Its theory is outlined in [Mor12, §B].

The key point is that if $f \colon Y \to Z$ is a fibration with fiber F, then the k-invariants for lifting $X \to Z$ along f are valued in Eilenberg–MacLane spaces for the homotopy sheaves of F.

A key application of this is the classification of torsors.

3.21.2 Grassmannians

Recall for any rank r and number n, we have a scheme $\mathrm{Gr}(r,n) \in \mathrm{Sm}_{\mathbb{Z}}$. As a functor $\mathrm{Sm}_{\mathbb{Z}}^{\mathrm{aff,op}} \to \mathrm{Set}$, it sends R to the collection of epimorphisms of the form
$$R^{\oplus n} \twoheadrightarrow P,$$
where P is a projective module of rank r over R. In other words, a map
$$\mathrm{Spec}\,(R) \to \mathrm{Gr}_{r,n}$$
classifies a finitely generated projective R-module P of rank r, with n generators.

Theorem 3.21.8. (Serre) If R is a Noetherian ring, there is a bijection between algebraic vector bundles on $\mathrm{Spec}\,(R)$ and finitely generated projective R-modules.

From this perspective, we might ask for some algebraic version of a Pontryagin–Steenrod theorem, i.e. we might want some infinite Grassmannian for which homotopy classes of maps into it will classify algebraic vector bundles. The following result is a step in this direction.

Definition 3.21.9. Let $\mathrm{Gr}(r,\infty)$ denote the colimit $\mathrm{colim}_n \mathrm{Gr}(r,n)$, considered as a presheaf.

Proposition 3.21.10. (Morel–Voevodsky) There is an equivalence of motivic spaces $\mathrm{Gr}(r,\infty) \simeq \mathrm{BGL}_r$.

The following result is the strongest current version of an algebraic Pontryagin–Steenrod theorem.

Theorem 3.21.11. (Affine representability) If $X \in \mathrm{Sm}_k^{\mathrm{aff}}$ is any affine, then there is a bijection
$$[X, \mathrm{BGL}_r]_{\mathbb{A}^1} \cong \mathrm{Vect}_r^{\mathrm{alg}}(X)^{\simeq}$$
between $\mathbb{A}^1$-homotopy classes of maps $X \to \mathrm{BGL}_r$ and rank r algebraic vector bundles over X up to isomorphism.

So if R is a smooth affine k-algebra, *every* finitely generated projective module over R is classified by a map $\mathrm{Spec}\,(R) \to \mathrm{BGL}_r$. Moreover we have access to obstruction theory in order to study them. Implicit in this result (and indeed used in this result) is $\mathbb{A}^1$-invariance, which says that every finitely generated $R[t]$-module is extended from R. In the context of smooth affine algebras over a field, this result is originally due to Lindel.

3.21.3 Rank two oriented bundles

The short exact sequence of groups
$$\mathrm{SL}_n \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m$$
gives rise to a fiber sequence
$$\mathrm{BSL}_n \to \mathrm{BGL}_n \to B\mathbb{G}_m.$$
In particular, if X is affine, the composite
$$X \to \mathrm{BGL}_n \xrightarrow{\det} B\mathbb{G}_m = \mathbb{P}^\infty$$
is exactly the first Chern class $c_1 \in \mathrm{Pic}(X)$. Hence SL_n-torsors are algebraic vector bundles with $c_1 = 0$.

Theorem 3.21.12. If $X = \mathrm{Spec}\,(R)$ is a smooth affine k-scheme of dimension 2, then a rank two finitely generated projective module over X is completely determined by its c_2.

Proof. We can analyze the Postnikov tower for BSL_2. We have that $\pi_1(\mathrm{BSL}_2) = *$, so the first nontrivial stage occurs at degree 2, which is $\tau_{\leq 2}\mathrm{BSL}_2 = K(\mathbf{K}_2^{\mathrm{MW}}, 2)$. Since X has Krull dimension two, we have that

$$[X, \mathrm{BSL}_2] \cong [X, \tau_{\leq 2}\mathrm{BSL}_2] = \left[X, K(\mathbf{K}_2^{\mathrm{MW}}, 2)\right] = \widetilde{\mathrm{CH}}^2(X).$$

We can now ask to what extent this is determined by c_2. There is a universal c_2 map

$$\mathrm{BSL}_2 \to K(\mathbf{K}_2^{\mathrm{M}}, 2),$$

which comes from the natural map $\mathbf{K}_2^{\mathrm{MW}} \to \mathbf{K}_2^{\mathrm{M}}$ after 2-truncation. From the fiber sequence

$$0 \to \mathbf{I}^3 \to \mathbf{K}_2^{\mathrm{MW}} \to \mathbf{K}_2^{\mathrm{M}} \to 0,$$

we get part of a long exact sequence

$$\cdots \to H^2(X, \mathbf{I}^3) \to H^2(X, \mathbf{K}_2^{\mathrm{MW}}) \to \mathrm{CH}^2(X) \to 0.$$

So the number of algebraic vector bundles over X with fixed c_2 is acted upon transitively by $H^2(X, \mathbf{I}^3)$. If $k = \bar{k}$, this set vanishes [AF14a, 5.1]. If k is not algebraically closed, $H^2(X, \mathbf{I}^3)$ need not vanish, and it may in fact contain more data governing the structure of algebraic vector bundles. □

Example 3.21.13. (Fasel) Over the algebraic 2-sphere, there are infinitely many non-isomorphic rank two algebraic vector bundles with vanishing c_1 and c_2.

3.21.4 The Swan–Forster theorem

This section is from soon-to-appear work of Asok, Opie, Shin and Syed.

Suppose we're given a vector bundle of rank r over a smooth affine d-fold X. This is classified by a map

$$X \to \mathrm{BGL}_r = \mathrm{Gr}(r, \infty).$$

We can ask – *how many generators do we need to describe each module of rank r over X*? In other words, where do the obstructions live to lifting

$$\begin{tikzcd} & \mathrm{Gr}(r,N) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \mathrm{Gr}(r,\infty). \end{tikzcd}$$

Our intuition might be that once N is large enough, we can obtain all rank r finitely generated projective modules over X via maps $X \to \mathrm{Gr}(r, N)$. We can make this precise with algebraic representability and motivic obstruction theory.

Proposition 3.21.14. The stabilization map

$$\mathrm{Gr}(r, N) \to \mathrm{Gr}(r, N + 1)$$

is $N - r$-connective. That is, its fiber has homotopy sheaves vanishing in degrees below $N - r$.

Theorem 3.21.15. If A is a smooth k-algebra of dimension d, then every finitely generated projective A-module of rank r needs at most $r + d$ generators.

Proof. We want to study the lifting problem

$$\begin{tikzcd} & \mathrm{Gr}(r,N) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \mathrm{Gr}(r,\infty). \end{tikzcd}$$

Let F denote the fiber

$$F \to \mathrm{Gr}(r, N) \to \mathrm{Gr}(r, \infty).$$

Then the k-invariants are valued in

$$H^{t+2}(X, \pi_{t+1}F).$$

Hence k_t vanishes for $t+2 \geq d+1$, i.e. $t \geq d-1$. Alternatively, we note that $\pi_t F = 0$ for $t \leq N-r-1$. So we obtain vanishing in the range

$$d-1 \leq t \leq N-r-1.$$

If $N \geq r+d$, then all the k-invariants vanish, and there are no obstructions to lifting! □

Remark 3.21.16. The more general result can be found in [Swa67].

3.21.5 Fiber sequences involving BGL_r

Proposition 3.21.17. There is an $\mathbb{A}^1$-fiber sequence

$$\mathrm{GL}_{r-1} \to \mathrm{GL}_r \to \mathbb{A}^r \smallsetminus 0,$$

where the rightmost map sends an $r \times r$ matrix to its last column.

Corollary 3.21.18. We obtain an $\mathbb{A}^1$-fiber sequence by delooping

$$\mathbb{A}^r \smallsetminus 0 \to \mathrm{BGL}_{r-1} \to \mathrm{BGL}_r.$$

Now suppose we have some $X \in \mathrm{Sm}_k^{\mathrm{aff}}$, and we have a rank r vector bundle over it. When does it split off a trivial summand? In other words when does it admit a lift of the form:

$$\begin{tikzcd} & \mathrm{BGL}_{r-1} \arrow[d] \\ X \arrow[ur, dashed] \arrow[r] & \mathrm{BGL}_r. \end{tikzcd}$$

Theorem 3.21.19. (Murthy, 1994) If X is a smooth affine r-fold over an algebraically closed field $k = \bar{k}$, then an algebraic vector bundle $E \to X$ of rank r splits off a trivial rank one summand if and only if the top Chern class $c_r(E) \in \mathrm{CH}^r(X)$ is zero.

Proof. The k-invariants are valued in

$$k_t \in H^{t+2}(X, \pi_{t+1}(\mathbb{A}^r \smallsetminus 0)).$$

This vanishes for $t \geq r-1$, and $\pi_{t+1}(\mathbb{A}^r \smallsetminus 0)$ is zero for $t \leq r-3$, since the first nonvanishing homotopy sheaf of $\mathbb{A}^r \smallsetminus 0$ is $\pi_{r-1}(\mathbb{A}^2 \smallsetminus 0)$. Hence there is only one nonvanishing k-invariant, which is

$$k_{r-2} \in H^r(X, \pi_{r-1}(\mathbb{A}^r \smallsetminus 0)) = H^r(X, \mathbf{K}_r^{\mathrm{MW}}) = \widetilde{\mathrm{CH}}^r(X).$$

Again we have an exact sequence

$$\cdots \to H^r(X, \mathbf{I}^{r+1}) \to \widetilde{\mathrm{CH}}^r(X) \to \mathrm{CH}^r(X) \to 0,$$

and $H^r(X, \mathbf{I}^{r+1}) = 0$ over an algebraically closed field. □

Over algebraically closed field, the difficulty of these sorts of splitting problems is governed by the *corank* of the associated bundle.

Definition 3.21.20. If $E \to X$ is an algebraic vector bundle of rank r over a d-dimensional variety, we say its *corank* is the quantity $d - r$.

A famous result called *Serre splitting* says that splitting problems are trivial in negative corank.

Theorem 3.21.21. *(Serre splitting)* Let R be a Noetherian ring of dimension d, and M a projective R-module of rank $r > d$. Then M decomposes as $M \cong M' \oplus R$ for some projective module M' [Ser58, Theorem 1].

Murthy's theorem can be rephrased as "over an algebraically closed field, corank zero bundles split if and only if their top Chern class vanishes."

Remark 3.21.22. (Historical background) For $r = d = 2$, this was proven by Murthy and Swan [MS76b]. Murthy and Mohan Kumar showed this for $r = d = 3$ in [KM82], and finally Murthy proved the general result [Mur94].

The complexity of studying these sorts of splitting problems is governed by two factors:

1. the 2-cohomological dimension of the base field
2. the *corank* of the bundle.

The next interesting case to study is corank one. Murthy conjectured essentially that an analogous result holds.

Conjecture 3.21.23. (Murthy, 1999) Over an algebraically closed field, if X is a smooth affine variety of dimension d,[9] then a corank one bundle $E \to X$ splits if and only if its top Chern class $c_{d-1}(E)$ is zero.

A quarter of a century later, this is now a theorem in characteristic zero.

Theorem 3.21.24. [ABH23, Theorem 3] (Asok–Bachmann–Hopkins) Murthy's splitting conjecture holds in characteristic zero.

The source of the assumption on the characteristic is essentially the same as in the work on the motivic Wilson space hypothesis we've been discussing in the Thursday seminar — the existence of workable models for motivic Eilenberg–MacLane spaces due to Voevodsky.

Our goal here is to give a rough explanation of how the characteristic zero Murthy's conjecture is resolved in this paper, and how it follows from the primary result of that paper, which is a $\mathbb{P}^1$-Freudenthal suspension theorem.

3.21.6 Murthy's splitting conjecture: the k-invariants

Recall by Corollary 3.21.18 we have a fiber sequence

$$S^{2d-1,d} \to \mathrm{BGL}_{d-1} \to \mathrm{BGL}_d.$$

If X has dimension $d+1$, then this is the lifting problem we're interested in studying:

$$\begin{tikzcd} & \mathrm{BGL}_{d-1} \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "E"'] & \mathrm{BGL}_d. \end{tikzcd}$$

The k-invariants as we have seen live in the (twisted) homology of the fiber, where the twist is by $\pi_1(\mathrm{BGL}_d)$, giving $\det E$. That is, our k-invariants are

$$k_t \in H^{t+2}(X, \pi_{t+1}(S^{2d-1,d})(\det E)).$$

So we have that

[9] Note that this conjecture only makes sense when $d \geq 2$. If $d = 1$, then a corank one bundle is rank zero and cannot split any further.

▷ k_d and the higher k-invariants are all zero since $\dim(X) = d+1$
▷ k_{d-3} and the lower k-invariants are zero because of the connectivity of $S^{2d-1,d}$ (todo: check the numbers here)

So we have two interesting obstruction classes:

1. the *primary obstruction* k_{d-2}, which can be identified with the *Euler class* of the bundle
$$k_{d-2}(E) = e(E) \in H^d(X, \pi_{d-1}(S^{2d-1,d})(\det E))$$
2. the *secondary obstruction* k_{d-1}, which is denoted by $o_2(E)$ in [ABH23]:
$$k_{d-1}(E) = o_2(E) \in H^{d+1}(X, \pi_d(S^{2d-1,d})(\det E)).$$

Recall by Proposition 3.3.11 we have that $S^{2d-1,d} \simeq \mathbb{A}^d \setminus \{0\}$, and by Example 3.4.11, we have that $S^{2d-1,d}$ is $\mathbb{A}^1$-$(d-2)$-connected. In particular as we have seen (c.f. Corollary 3.18.2), we get that
$$\pi_n S^{2d-1,d} = \begin{cases} 0 & n \leq d-2 \\ \mathbf{K}_d^{\mathrm{MW}} & n = d-1 \\ \text{mostly unknown} & n \geq d-2. \end{cases}$$
We now see that the primary obstruction is a bit easier to manage.

Proposition 3.21.25. [AF15, 6.3.1] For a corank one bundle over an algebraically closed field, its Euler class is zero if and only if its top Chern class is zero.

Proof. We can leverage the (twisted) short exact sequence
$$0 \to \mathbf{I}^{d+1}(\det E) \to \mathbf{K}_d^{\mathrm{MW}}(\det E) \to \mathbf{K}_d^{\mathrm{M}} \to 0,$$
and we get a long exact sequence, sending $e(E) \mapsto c_{d-1}(E)$:
$$\cdots \to H^d(X, \mathbf{I}^{d+1}(\det E)) \to H^d(X, \mathbf{K}_d^{\mathrm{MW}}(\det E)) \to H^d(X, \mathbf{K}_d^{\mathrm{M}}) \to H^{d+1}(X, \mathbf{I}^{d+1}(\det E)) \to \cdots$$
The content of this proposition is then the vanishing of these $\mathbf{I}^j$ cohomology groups in the long exact sequence. This vanishing is proven in [AF14a, 5.2], where the algebraic closure of the field is used. Note that this vanishing only holds for $\dim(X) \geq 2$, but this is the scope of Murthy's conjecture. □

Hence proving Murthy's conjecture reduces to showing the secondary obstruction vanishes. The hard part about this obstruction is that it is valued in $\pi_{d,0}(S^{2d-1,d})$. We understand a bit about the first stable homotopy groups of motivic spheres by [RSO19]. Unfortunately, however, our sheaf of study lies outside the stable range.

The issue then comes from understanding the stabilization map, and we note this is a $\mathbb{P}^1$-*stabilization*!

3.21.7 $\mathbb{P}^1$-Freudenthal suspension

The Freudenthal suspension theorem we have seen deals with suspending by the simplicial sphere S^1. It is really barely motivic in flavor, it's just the classical (sheaf topos) Freudenthal suspension theorem, combined with the unstable connectivity theorem which tells us how connectivity statements behave under motivic localization. As the projective line has motivic weight, people have long been interested in proving a $\mathbb{P}^1$ (or equivalently a $\mathbb{G}_m$) Freudenthal suspension theorem.

The first thing is to come up with a more refined notion of connectivity.

Definition 3.21.26. We say X is $S^{p,q}$-*null* if $\pi_n(X)_{-q} = 0$ for all $n \geq p-q$.

Equivalently, there is a notion of unstable localization at spaces, due to Dror, and X is $S^{p,q}$-null if $X \to L_{p,q}X$ is an equivalence.

Definition 3.21.27. We denote by $O(A)$ the collection of *weakly A-cellular* spaces, meaning those spaces X for which $L_A X \simeq *$.

There is a universal process called *cellularization*, denoted $\tau_{\geq(p,q)}(-)$. This has the universal property that $\mathrm{Map}(X,Y) \cong \mathrm{Map}(X, \tau_{\geq(p,q)}Y)$ if $X \in O(S^{p,q})$.

Example 3.21.28. If $A = S^{n,0}$, then X is $S^{n,0}$-null if and only if it is $(n-1)$-truncated. $X \in O(S^{n,0})$ if and only if X is $(n-1)$-connected.

Theorem 3.21.29. (Motivic Freudenthal) If $X \in O(S^{2n,n})$ for $n \geq 2$ and k of characteristic zero, then the fiber of

$$X \to \Omega^{2,1}\Sigma^{2,1}X$$

is in $O(S^{4n-1,2n})$.

As an application, we can consider $X = S^{2d-1,d}$, and apply π_d to get

$$\pi_d(S^{2d-1,d}) \to \pi_d(\Omega^{2,1}S^{2d+1,d+1}) = \pi_{d+1}(S^{2d+1,d+1})_{-1},$$

where the latter sheaf is in the stable range, and hence understandable by [RSO19].

Theorem 3.21.30. [ABH23, 7.2.1] For $d \geq 4$ there is an exact sequence of sheaves

$$0 \to \mathbf{K}^{\mathrm{M}}_{d+2}/24 \to \pi_d(S^{2d-1,d}) \to \pi_d(\tau_{\geq(2d-1,d)}\Omega^{2d,d}O) \to 0.$$

Proof. Uses the main theorem of [RSO19] together with the motivic Freudenthal theorem to say something about stabilization. $\square$

Hence we get a sequence

$$\cdots \to H^{d+1}(X, \mathbf{K}^{\mathrm{M}}_{d+2}/24) \to H^{d+1}(X, \pi_d(S^{2d-1,d})) \to H^{d+1}(X, \pi_d(\tau_{\geq(2d-1,d)}\Omega^{2d,d}O)) \to \cdots$$

If k is algebraically closed, and X is a $(d+1)$-fold as assumed, we have that $\mathbf{K}^{\mathrm{M}}_{d+2}/24 = 0$. We have that the map

$$\pi_d(\tau_{\geq(2d-1,d)}\Omega^{2d,d}O) \to \pi_d(\Omega^{2d,d}O)$$

is an isomorphism after $d-3$ contractions, hence their $d+1$st cohomology groups agree. It suffices to argue then that $H^{d+1}(X, \pi_d(\Omega^{2d,d}O)) = 0$. By analysis of the Gersten resolution, we see this group is a quotient of $\mathrm{CH}^{d+1}(X)/2$:

$$\mathrm{CH}^{d+1}(X)/2 \twoheadrightarrow H^{d+1}(X, \pi_d(\Omega^{2d,d}O)).$$

Note that $\mathrm{CH}^{d+1}(X)$ will be divisible coprime to the characteristic, hence $\mathrm{CH}^{d+1}(X)/2$ is zero.

Chapter 4

Appendices

4.1 Jonathan Buchanan: Matsumoto's Theorem

The goal of this appendix is to prove the following:

Theorem 4.1.1 (Matsumoto's Theorem)**.** If F is a field, there is an isomorphism $K_2^M(F) \to K_2(F)$ from Milnor K-theory to Quillen K-theory.

Note that this is the $n = 2$ case of Proposition 3.6.12.

4.1.1 An Algebraic Description of K_2

A reference for the material in this section is [Wei13], especially III.5 and IV.1. From now on, we will use K_i and K_i^M to denote $K_i(F)$ and $K_i^M(F)$, respectively. Also, SL_n and GL_n will denote $\mathrm{SL}_n(F)$ and $\mathrm{GL}_n(F)$. We will also write all groups, including abelian groups, multiplicatively, unless otherwise stated.

Definition 4.1.2. A group G is **perfect** if G is equal to its commutator subgroup $[G, G]$.

Example 4.1.3. The commutator subgroup of GL_n (assume $n \geq 3$) is $\mathrm{SL}_n \subset \mathrm{GL}_n$, and SL_n is perfect.

Recall that we defined

$$K_n = \pi_n(\mathrm{Vect}^{\mathrm{gp}}).$$

To understand these groups for low n, let $M = \mathrm{Vect}$ considered as a commutative monoid in spaces. Then there is a map $M_\infty \to M^{\mathrm{gp}}$ that is a plus construction. It is then easy to understand K_n for $n = 0, 1$ using this fact.

Proposition 4.1.4. We have

$$K_0 \cong \mathbb{Z},$$
$$K_1 \cong F^\times.$$

Proof. Since $\mathrm{Vect}^{\mathrm{gp}}$ and Vect_∞ have the same H_0,

$$K_0 = \pi_0(\mathrm{Vect}^{\mathrm{gp}}) \cong \pi_0(\mathrm{Vect}_\infty) \cong \mathbb{Z}.$$

Because $\mathrm{Vect}^{\mathrm{gp}}$ has abelian fundamental group, and $\mathrm{Vect}_\infty \to \mathrm{Vect}^{\mathrm{gp}}$ is a plus construction

$$K_1 = \pi_1(\mathrm{Vect}^{\mathrm{gp}}) \cong H_1(\mathrm{Vect}^{\mathrm{gp}}; \mathbb{Z}) \cong H_1(\mathrm{Vect}_\infty; \mathbb{Z}) \cong \pi_1(\mathrm{Vect}_\infty)^{\mathrm{ab}} \cong GL^{\mathrm{ab}}.$$

And $GL^{\mathrm{ab}} \cong \mathrm{GL}/\mathrm{SL} \cong F^\times$. $\square$

It is harder to compute the higher groups because the plus construction alters the homotopy groups in a more complicated way for $n \geq 2$, but there is still a convenient algebraic description when $n = 2$. If we form the homotopy fiber X of $\mathrm{Vect}_\infty \to \mathrm{Vect}^{\mathrm{gp}}$, then we get a long exact sequence of homotopy groups

$$\pi_2(\mathrm{Vect}_\infty) \longrightarrow K_2 \longrightarrow \pi_1(X) \longrightarrow \pi_1(\mathrm{Vect}_\infty) \longrightarrow K_1 \longrightarrow \pi_0(X)$$

But $\pi_2(\mathrm{Vect}_\infty)$ is zero because $\pi_2(\mathrm{Vect}_\infty) \cong \pi_2(\mathrm{BGL}) \cong 0$. Also, the map $\pi_1(\mathrm{Vect}_\infty) \to \pi_1(\mathrm{Vect}^{\mathrm{gp}}) = K_1$ is abelianization, and therefore surjective. Hence we have an exact sequence

$$1 \longrightarrow K_2 \longrightarrow \pi_1(X) \longrightarrow \mathrm{GL} \longrightarrow F^\times \longrightarrow 1$$

It is a general fact that for the long exact sequence of a fibration like this, the image of the map $K_2 \to \pi_1(X)$ is in the center. So we get a central extension

$$1 \longrightarrow K_2 \longrightarrow \pi_1(X) \longrightarrow \mathrm{SL} \longrightarrow 1$$

The fiber X has the homology of a point because $\mathrm{Vect}_\infty \to \mathrm{Vect}^{\mathrm{gp}}$ is a plus construction. It follows that $G = \pi_1(X)$ is perfect because its abelianization vanishes. Take the map $X \to BG$ and let Y be the homotopy fiber. Then Y is the universal cover of X, and we get a Serre spectral sequence $E^2_{p,q} \cong H_p(BG; H_q(Y;\mathbb{Z})) \cong H_p(G; H_q(Y;\mathbb{Z}))$ converging to $H_{p+q}(X;\mathbb{Z})$. But for $H_2(X;\mathbb{Z})$ to vanish, every term of total degree two in the spectral sequence eventually needs to be killed. But $E^2_{2,0} \cong H_2(G;\mathbb{Z})$ must be killed by the differential d_2 since all other differentials vanish by degree reasons, and this differential lands in $E^2_{0,1} \cong 0$ because Y is simply connected. So $H_2(G;\mathbb{Z}) \cong 0$. So, we have the following:

Proposition 4.1.5. There is a central extension

$$1 \longrightarrow K_2 \longrightarrow H \longrightarrow \mathrm{SL} \longrightarrow 1$$

such that H is perfect and $H_2(F;\mathbb{Z}) \cong 0$.

It turns out that this is enough to completely characterize $H \to \mathrm{SL}$ (and therefore K_2).

Definition 4.1.6. A **universal central extension** of a group G is an initial object in the category of central extensions of G, i.e. a central extension $\phi : X \to G$ such that for any other central extension $\psi : Y \to G$, there is a unique map $f : X \to Y$ such that $\phi = \psi f$.

Here is the main result characterizing universal central extensions:

Theorem 4.1.7. The following are equivalent:
- ▹ $X \to G$ is a universal central extension.
- ▹ $H_1(X;\mathbb{Z}) \cong H_2(X;\mathbb{Z}) \cong 0$.
- ▹ X is a perfect group and every central extension of X splits.

From this, we easily obtain the following algebraic characterization of K_2:

Corollary 4.1.8. The perfect group H is the universal central extension of SL and K_2 is the kernel.

4.1.2 The Steinberg Group and K_2

The material in the following two sections comes from Chapters 5, 8, 9, 10, and 11 of [Mil69].

To get closer to our goal, we should try to write down a universal central extension of SL.

Definition 4.1.9. For $n \geq 3$, the **Steinberg group** is the group St_n generated by the symbols x^a_{ij}, where $a \in F$ and $1 \leq i, j \leq n$ are distinct, subject to the relations $x^a_{ij} x^b_{ij} = x^{a+b}_{ij}$, $[x^a_{ij}, x^b_{jk}] = x^{ab}_{ik}$ if $i \neq k$, and $[x^a_{ij}, x^b_{k\ell}] = 1$ if $j \neq k$ and $i \neq \ell$.

There is a unique map $\mathrm{St}_n \to \mathrm{St}_{n+k}$ with $x^a_{ij} \mapsto x^a_{ij}$, so we can let $\mathrm{St} := \mathrm{colim}(\mathrm{St}_3 \to \mathrm{St}_4 \to \mathrm{St}_5 \to \ldots)$. It is easy to see that St is the group generated by the symbols x^a_{ij}, where $a \in F$ and i and j are distinct positive integers, subject to relations of the form above. The Steinberg group models some of the algebra of elementary matrices:

Lemma 4.1.10. There are maps from St_n to SL_n defined by sending x^a_{ij} to the elementary matrix that has 1s on the diagonal and a single nonzero off-diagonal entry at (i, j) equal to a.

There are surjective maps from St_n to SL_n defined by sending x^a_{ij} to the elementary matrix in SL_n that has 1s on the diagonal, a in the ijth place, and zeroes everywhere else, since the elementary matrices satisfy the relations above. It is easy to see that these maps are compatible as we increase the dimension, so that we get a map $\mathrm{St} \to \mathrm{SL}$ between the colimits.

Theorem 4.1.11. The map $\mathrm{St} \to \mathrm{SL}$ is a universal central extension for SL, and the kernel is exactly the center of St.

Proof. First, we need to show that $\mathrm{St} \to \mathrm{SL}$ is a central extension. If x is in the kernel, we need to show it commutes with every x_{ij}^a. Let n be sufficiently large so that $x \in \mathrm{St}_{n-1}$. Let P be the subgroup of St_n generated by x_{in}^a for $i < n$ and $a \in F$. By the first and third class of relations of the Steinberg group, we see that P is commutative and every element can be written uniquely as $x_{1n}^{a_1} \dots x_{(n-1)n}^{a_{n-1}}$. So P maps injectively into SL_n. Also, $x_{ij}^a P x_{ij}^{-1} \subseteq P$ if $i, j < n$, so $xPx^{-1} \subseteq P$. But then x commutes with every element of P, since P is mapped injectively to SL_n. So x commutes with every x_{in}^a where $i < n$. Using an automorphism of St, we see that x also commutes with every x_{nj}^a for $j < n$. Hence x commutes with $[x_{in}^a, x_{nj}^1] = x_{ij}^a$ if $i, j < n$. But n was just some sufficiently large number, so x commutes with every generator of St.

All of the center is in the kernel because $\mathrm{St} \to \mathrm{SL}$ is surjective and SL has trivial center, because if a matrix in SL is in the center, then it will be a multiple of the identity, and the only such matrix in SL is the identity.

Then, we need to show that St is perfect. This is easy, because $[\mathrm{St}, \mathrm{St}]$ is a subgroup of St and each generator x_{ij}^a can be written as the commutator $[x_{ik}^a, x_{kj}^1] = x_{ij}^a$, where k is some index distinct from i and j.

Finally, we must show that every central extension of St splits. Suppose we have a central extension

$$1 \longrightarrow K \longrightarrow G \xrightarrow{f} \mathrm{St} \longrightarrow 1$$

To split the map f, we need to find elements $s_{ij}^a \in G$ satisfying the relations of the Steinberg group such that $f(s_{ij}^a) = x_{ij}^a$. Given x_{ij}^a, let $y_{ij}^a \in G$ map to x_{ij}^a. Then let $s_{ij}^a = [y_{ik}^1, y_{kj}^a]$ where k is distinct from i and j. Clearly this does not depend on the choice of y_{ik}^1 and y_{kj}^a, since any other choices would differ by an element of K, and this element would not change the commutator because K is in the center of G.

If $j \neq k$ and $i \neq \ell$, let h be an index distinct from i, j, k, and ℓ. By the Steinberg relations, $[y_{ih}^1, y_{k\ell}^b], [y_{hj}^a, y_{k\ell}^b] \in K$ are in the center, so $[y_{ih}^1, y_{hj}^a]$ commutes with $y_{k\ell}^b$, since moving $y_{k\ell}^b$ past each factor in $[y_{ih}^1, y_{jh}^a]$ introduces $[y_{ih}^1, y_{k\ell}^b]$ or $[y_{hj}^a, y_{k\ell}^b]$ or their inverses exactly once, and since these are central, they all cancel. Hence $[y_{ij}^a, y_{k\ell}^b] = 1$, since y_{ij}^a differs from $[y_{ih}^1, y_{hj}^a]$ by an element of K. So we have the third Steinberg relation:

$$\begin{aligned} [s_{ij}^a, s_{k\ell}^b] &= [[y_{ih}^1, y_{hj}^a], [y_{kg}^1, y_{g\ell}^b]] \\ &= [y_{ij}^a, y_{k\ell}^b] \\ &= 1 \end{aligned}$$

when $i \neq \ell$ and $j \neq k$.

Now we recall some general group theory facts. If we let $X'' = [[X, X], [X, X]]$, where X is some group and $x, y, z \in X$, then there is a Jacobi identity $[x, [y, z]][y, [z, x]][z, [x, y]] \in X''$. Also $[x, y][x, z] = [x, yz][y, [z, x]]$.

If i, j, k, and ℓ are all distinct, let X be the subgroup of G generated by $y_{\ell i}^1$, y_{ij}^a, and y_{jk}^b. Then the commutator subgroup $[X, X]$ of X is generated by elements mapping to $x_{\ell j}^a$, x_{ik}^{ab}, and $x_{\ell k}^{ab}$, and so X'' is the trivial group. Since $y_{\ell i}^1$ and y_{jk}^b commute, the Jacobi identity implies

$$[[y_{\ell i}^1, y_{ij}^a], y_{jk}^b] = [y_{\ell i}^1, [y_{ij}^a, y_{jk}^b]].$$

Using the Steinberg relations, this is the same as

$$[y_{\ell j}^a, y_{jk}^b] = [y_{\ell i}^1, y_{ik}^{ab}].$$

If we let $a = 1$, we see that
$$s^b_{\ell k} = [y^1_{\ell i}, y^b_{ik}].$$
So the second Steinberg relation is true because
$$\begin{aligned}[s^a_{ij}, s^b_{jk}] &= [[y^1_{ik}, y^a_{kj}], [y^1_{j\ell}, y^b_{\ell k}]] \\ &= [y^a_{ij}, y^b_{jk}] \\ &= s^{ab}_{ik}\end{aligned}$$
if $i \neq \ell$. And finally, the first Steinberg relation follows because
$$\begin{aligned}s^a_{ij} s^b_{ij} &= [y^1_{ik}, y^a_{kj}][y^1_{ik}, y^b_{kj}] \\ &= [y^1_{ik}, y^a_{kj} y^b_{kj}][y^a_{kj}, [y^b_{kj}, y^1_{ik}]] \\ &= [y^1_{ik}, y^{a+b}_{kj}][y^a_{kj}, [y^b_{kj}, y^1_{ik}]] \\ &= s^{a+b}_{ij}[y^a_{kj}, [y^b_{kj}, y^1_{ik}]] \\ &= s^{a+b}_{ij}.\end{aligned}$$
□

A slight modification of this proof shows that $\mathrm{St}_n \to \mathrm{SL}_n$ is a universal central extension whenever $n \geq 5$.

Corollary 4.1.12. The kernel of $\mathrm{St} \to \mathrm{SL}$ is isomorphic to K_2.

4.1.3 Matsumoto's Theorem

Proving Matsumoto's theorem now just amounts to describing the kernel of the map $\mathrm{St} \to \mathrm{SL}$.

Definition 4.1.13. If A is an abelian group (with composition written as multiplication), a **Steinberg symbol** valued in A is a map $c : F^\times \times F^\times \to A$ that preserves multiplication in each variable separately and satisfies the identity $c(x, 1-x) = 1$.

Said differently, a Steinberg symbol is a group homomorphism $F^\times \otimes F^\times \to A$ satisfying $x \otimes (1-x) \mapsto 1$ for all $x \in F^\times$ not equal to 1. Here are some of the algebraic properties of Steinberg symbols. For any $a, b \in F^\times$:
- ▷ $c(a, 1) = c(1, a) = 1$.
- ▷ $c(a, b) = c(b, a)^{-1}$.
- ▷ $c(a, -a) = 1$.

The following Steinberg symbol will be central to the proof of Matsumoto's theorem:

Example 4.1.14. There is a Steinberg symbol $\{-, -\}$ with values in K_2. If $a, b \in F^\times$, consider the matrices
$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^{-1} \end{bmatrix}.$$
These lie in SL_3, so can be lifted to elements $\tilde{A}, \tilde{B} \in \mathrm{St}_3$. We then let $\{a, b\}$ be the commutator of $\tilde{A}$ and $\tilde{B}$, so that $\{a, b\} = \tilde{A}\tilde{B}\tilde{A}^{-1}\tilde{B}^{-1}$. We have $\{a, b\} \in K_2$ because A and B commute. The choice of representatives $\tilde{A}$ and $\tilde{B}$ do not matter because any other choices would differ by elements of K_2, which is the center, so the commutator is not affected.

This Steinberg symbol has some particularly nice properties:

Lemma 4.1.15. The subgroup $K_2 \subseteq \mathrm{St}$ is generated by $\{a, b\}$ as a and b range over $F^\times$.

Lemma 4.1.16. The Steinberg symbols $\{a, b\}$ are all trivial when F is a finite field.

Corollary 4.1.17. When F is a finite field, $K_2 = 1$.

See Corollary 9.9 and Corollary 9.13 in [Mil69].

Matsumoto's theorem hinges on the following result:

Proposition 4.1.18. If c is a Steinberg symbol with values in A, there are central extensions $G \to \mathrm{SL}_n$ with kernel A, for $n \geq 3$, such that if $\tilde{C}, \tilde{D} \in G$ map to diagonal matrices with entries $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$, respectively, then the commutator of $\tilde{C}$ and $\tilde{D}$ lies in A and is $c(c_1, d_1) \ldots c(c_n, d_n)$. As n varies, these extensions are compatible with the maps $\mathrm{SL}_n \to \mathrm{SL}_{n+1}$.

The proof of this is quite technical and involved, so we will only sketch the main ideas. The extension is built in stages over subgroups of SL_n. First, we get a central extension

$$1 \longrightarrow A \longrightarrow H \xrightarrow{\phi} D_n \longrightarrow 1$$

where $D_n \subseteq \mathrm{SL}_n$ is the subgroup of diagonal matrices, H is defined as the set $D \times A$ with the group operation $(A, a)(B, b) = \Big(AB, ab \prod_{i \geq j} c(A_{ii}, B_{jj})\Big)$, and ϕ is the projection onto D_n. The proposition will be true if this extension is $G \times_{\mathrm{SL}_n} D_n$ (and the extensions G are compatible as n varies), since commutators in H have the desired form.

The next stage of the extension is built over the subgroup of monomial matrices in SL_n. Recall that a **monomial matrix** is a matrix with one nonzero entry in every column and row. Let $M_0 \subseteq \mathrm{SL}_n$ be the subgroup consisting of monomial matrices where all entries are 0 or ± 1. If $c(-1, -1) = 1 \in A$, let $W_0 = M_0$. Note that this must be the case if F has positive characteristic, by Lemma 4.1.16. If $c(-1, -1) = -1$, then F has characteristic zero, and we let W_0 be the subgroup of St_n that is the preimage of $M_0 \subseteq \mathrm{SL}_n$. In either case, we get a map $\phi_0 : W_0 \to M_0$ (the identity or the restriction of $\mathrm{St}_n \to \mathrm{SL}_n$). The reason we do this is so that we can identify a certain subgroup of W_0 with a subgroup of H and this will be necessary for the relations to work out.

Now we set up this identification. If $i \neq j$, and $a \in F^\times$, let d_{ij}^a be the diagonal matrix with a at the ith diagonal entry and a^{-1} at the jth and all other diagonal entries equal to 1. If $i < j$, let $h_{ij}^a = (d_{ij}^a, 1)$, and if $i > j$, let $h_{ij}^a = (d_{ij}^a, c(a, a))$. These elements satisfy the identities $h_{ji}^a = (h_{ij}^a)^{-1} = (h_{ik}^a)^{-1}(h_{kj}^a)^{-1}$ and $h_{ij}^a h_{ij}^b = c(a, b) h_{ij}^{ab}$. If we let $H_0 \subseteq H$ be the subgroup generated by the elements h_{ij}^{-1}, it is isomorphic to either the subgroup of M_0 generated by d_{ij}^{-1} in the case $W_0 = M_0$, or the subgroup of $W_0 \subseteq \mathrm{St}_n$ generated by $(x_{ij}^{-1} x_{ji}^1 x_{ij}^{-1})^2$.

Every monomial matrix can be written uniquely as a product PD where P is a permutation matrix and D is a diagonal matrix.

Lemma 4.1.19. For any monomial matrix PD, where P corresponds to $\sigma \in S_n$ and the diagonal entries of D are $d_1, \ldots, d_n$, there is a unique automorphism of H that acts trivially on $A \subseteq H$ and satisfies $h_{ij}^a \mapsto c(d_i d_j^{-1}, a) h_{\sigma(i)\sigma(j)}^a$. If $P = 1$, this automorphism is the inner automorphism $x \mapsto yxy^{-1}$ whenever $\phi(y) = D$. The map assigning this automorphism to each monomial matrix is a homomorphism from the group of monomial matrices to the automorphism group of H.

Then W is defined to be the quotient of $H \times W_0$ by the equivalence relation identifying (xy, z) and (x, yz) for $y \in H_0$. The product of W is defined to be $[x, y][z, w] = [x(yzy^{-1}), yw]$, where yzy^{-1} denotes the action of the automorphism induced by $y \in W_0$ of Lemma 4.1.19 on $x \in H$. This is indeed a group law, and we get a central extension of the desired form

$$1 \longrightarrow A \longrightarrow W \xrightarrow{\phi} M_n \longrightarrow 1$$

compatible with the previous stage by mapping $[x, y]$ to $\phi(x)\phi_0(y)$.

Finally, the extension is constructed over SL_n as follows. Let m_i^a be the monomial matrix with a as the $(i, i+1)$th entry, $-a^{-1}$ as the $(i+1, i)$th entry, and 1 on each diagonal entry that is not the ith or $(i+1)$th entry. The following properties of SL_n will be important:

Lemma 4.1.20. Any matrix $x \in \mathrm{SL}_n$ can be written as $y\rho(x)z$ where y and z are upper triangular matrices and $\rho(x)$ is a monomial matrix, with $\rho(x)$ being the unique monomial matrix having this property. The map $\rho : \mathrm{SL}_n \to M_n$ satisfies the properties:

- $\rho(dx) = d\rho(x)$ and $\rho(xd) = \rho(x)d$ whenever d is diagonal.
- $\rho(m_i(1)x)$ is either $m_i(1)\rho(x)$ or $d_{i,i+1}(a)^{-1}\rho(x)$ for some unique $a \in F^\times$.
- $\rho(xm_i(-1))$ is either $\rho(x)m_i(-1)$ or $\rho(x)d_{i,i+1}(a)$ for a unique $a \in F^\times$.

Let X be the pullback (of sets)

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d, "\phi"] \\ \mathrm{SL}_n \arrow[r, "\rho"] & M_n \end{tikzcd}$$

and let G be the subgroup of permutations of X generated by the permutations $\lambda(h)(x, w) = (\phi(h)x, hw)$ for $h \in H$, $\mu(t)(x, w) = (tx, w)$ for t an upper triangular matrix, and η_i, where $\eta_i(x, w)$ is $(m_i(1)x, w_{i,i+1}(1)w)$ if $\rho(m_i(1)x) = m_i(1)\rho(x)$ and $(m_i(1)x, (h^a_{i,i+1})^{-1}w$ if $\rho(m_i(1)x) = d_{i,i+1}(a)^{-1}\rho(s)$. Here $w^a_{ij} = x^a_{ij}x^{-a^{-1}}_{ji}x^a_{ij} \in \mathrm{St}_n$. Then these generators endow G with the following property that allows us to prove it is the desired extension:

Lemma 4.1.21. The action of G on X is simply transitive.

From this we get a homomorphism $\phi : G \to \mathrm{SL}_n$ defined by sending $\sigma \in G$ to the unique $\phi(\sigma) \in \mathrm{SL}_n$ such that σ acts on the first factor of pairs in X by left multiplication by $\phi(\sigma)$. This exists because such an element of SL_n exists for all the generators of G. This is a surjective homomorphism since the action is transitive, and the kernel is A, because if σ is in the kernel, then $\sigma(x, w) = (x, w_0)$ for every $(x, w) \in X$. But $\rho(w) = \rho(w_0)$, so $w_0 = aw$ for some $a \in A$, and therefore $\sigma = \lambda(a)$ because the action of G on X is simply transitive. Hence we have the desired central extension

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\phi} \mathrm{SL}_n \longrightarrow 1$$

Proof of Theorem 4.1.1. There is a "universal Steinberg symbol" $c : F^\times \times F^\times \to A$, where A is the quotient of the group $F^\times \otimes F^\times$ by the subgroup generated by all elements of the form $x \otimes (1-x)$. It is easy to see that the target of the universal Steinberg symbol is K_2^M. The Steinberg symbol $\{-,-\}$ is classified by a map $\phi : A \to K_2$ taking $c(a, b)$ to $\{a, b\}$. Form the central extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow \mathrm{SL} \longrightarrow 1$$

of Proposition 4.1.18. Then by the universal property of St, there is a unique map $\psi : \mathrm{St} \to G$ making the diagram

$$\begin{tikzcd} \mathrm{St} \arrow[r] \arrow[d, "\psi"] & \mathrm{SL} \arrow[d, "\mathrm{id}"] \\ G \arrow[r] & \mathrm{SL} \end{tikzcd}$$

commute. Also K_2 gets mapped to A, since K_2 is the kernel of $\mathrm{St} \to \mathrm{SL}$. For $a, b \in F^\times$, if we consider the elements $\tilde{A}, \tilde{B} \in \mathrm{St}$ as in Example 4.1.14, we see that by Proposition 4.1.18 that ψ must map their commutator $\{a, b\} \in K_2$ to the commutator of $\psi(\tilde{A})$ and $\psi(\tilde{B})$ in G. Upon taking the commutator, we see that $\psi(\{a, b\}) = c(a, b)c(a^{-1}, 1)c(1, b^{-1}) = c(a, b)$. But K_2 is generated by elements of the form $\{a, b\}$, so ϕ and ψ are inverses, and therefore we get an isomorphism $K_2 \cong A \cong K_2^M$. $\square$

The statement of Proposition 3.6.12 was a little more precise than what we have proved. There, we stated that there is a *symbol map* $K_n^M \to K_n$ defined for all $n \geq 0$ and it is an isomorphism in degrees $n \in \{0, 1, 2\}$. This is defined using the ring structure on $K_\bullet$, which we have not defined, but since $K_\bullet^M$ is a graded-commutative ring generated in degree one, such a map arises from a map $K_1^M \to K_1$ and taking products. And the product $K_1 \otimes K_1 \to K_2$ corresponds to the Steinberg symbol $\{-,-\}$ we introduced above, so the map $K_2^M \to K_2$ in our proof of Matsumoto's theorem is indeed the symbol map.

Recall that *Gersten's conjecture* (Corollary 3.12.7) states

$$H^n(X, \mathbf{K}_n^M) \cong H^n(X, \mathbf{K}_n) \cong \mathrm{CH}^n(X)$$

for smooth k-schemes X. This followed from inspecting the Rost-Schmidt complexes calculating these cohomology groups and the isomorphism $K_n^M(F) \cong K_n(F)$ for $n = 0, 1$. Using Matsumoto's theorem and the Rost-Schmidt complex at

$$\ldots \longrightarrow \bigoplus_{x \in X^{(n-2)}} K_2^M(\kappa(x)) \longrightarrow \bigoplus_{x \in X^{(n-1)}} K_1^M(\kappa(x)) \longrightarrow \bigoplus_{x \in X^{(n)}} K_0^M(\kappa(x)) \longrightarrow 0$$

it follows that $H^{n-1}(X, \mathbf{K}_n^M) \cong H^{n-1}(X, \mathbf{K}_n)$ as well.

With the exception of the sketch of the proof of Matsumoto's theorem, most constructions involving K_2 and Steinberg groups that we outlined here work over general rings. Details of this can be found in [Wei13] and [Mil69]. There is also a proof of the theorem relying heavily on group homology given in [Hut90]

4.2 Hamilton Wan: Unimodular Rows and Motivic Homotopy Theory

4.2.1 Motivation and Background

Why Unimodular Rows?

To motivate the study of unimodular rows, we first review a foundational question in algebraic K-theory that we encountered at the start of this course.

Question 4.2.1 (Serre's Problem)**.** Let k be a field. Is every finitely generated projective $k[t_1,\dots,t_n]$-module free? In other words, is the base change functor

$$\mathrm{Mod}_k^{\text{f.g., proj}} \to \mathrm{Mod}_{k[t_1,\dots,t_n]}^{\text{f.g., proj}}, \quad M \mapsto M \otimes_k k[t_1,\dots,t_n]$$

essentially surjective?

Note that this problem can be reinterpreted geometrically as asking whether every algebraic vector bundle on $\mathbb{A}^n_k$ is trivial. On the other hand, from a motivic perspective, Serre's problem asks about $\mathbb{A}^1$-invariance of the sheaf $H^1_{\mathrm{Nis}}(-,\mathrm{GL}_*)$ taking an affine scheme $\mathrm{Spec}(R)$ to the graded abelian group of its algebraic vector bundles. When $n=1$, this problem is straightforward, as $k[t_1]$ is a PID. For arbitrary n, our intuition from topology tells us that vector bundles on $\mathbb{A}^n_k$ should all be trivial because $\mathbb{A}^n_k$ is somehow a "contractible" space. Of course, this intuition is far from a rigorous proof, and the first solutions to Serre's problem required deeper mathematics. In 1976, Quillen [Qui76] and Suslin [Sus76] produced independent proofs of the following theorem, answering Serre's question in the positive.

Theorem 4.2.2 (Quillen–Suslin)**.** Every finitely generated projective $k[t_1,\dots,t_n]$-module is free.

Quillen's proof of this theorem was quite sophisticated, introducing an ingenious technique called Quillen patching. Following Suslin and Quillen's original proofs in 1976, Suslin and Vaserstein independently produced new, elementary proofs of the Quillen–Suslin theorem. Roughly speaking, Suslin (in a letter to Bass dated May 2, 1976, apparently[1]) provided a "linear-algebraic" proof showing that the ring $k[t_1,\dots,t_n]$ is a *Hermite ring*, which means that unimodular rows valued in the ring can be completed to invertible matrices; it is known that finitely generated projective modules over a Hermite ring are free. Soon after, Vaserstein[2] provided another elementary proof (colloquially known as "Vaserstein's 8-line proof") of the Quillen–Suslin theorem using the theory of unimodular rows. The details of Suslin's and Vaserstein's proofs are beyond the scope of this appendix (in particular, not germane to motivic homotopy theory), and we refer the reader to [Lam06, Chapter III] for an excellent exposition of these proofs. However, we use their proofs as motivation for studying unimodular rows. At a very high level, unimodular rows in a k-algebra A control the surjective ring homomorphisms $A^{\oplus n} \to A$ and are thus ubiquitous in algebraic K-theory. We will discuss toward the end of this note some applications to the study of stably free modules over a smooth algebra, for instance.

Background

Let A be a commutative algebra over a field k, and choose a positive integer n.

[1] at least, this is how the proof was cited in [Lam06, Chapter III]

[2] in unpublished notes cited by [Lam06, Chapter III]

Definition 4.2.3. A *unimodular row* of length n valued in A is a sequence $(a_1, \ldots, a_n)$ of elements of A that collectively generate the unit ideal in A.

Observe that unimodular rows classify surjective A-module homomorphisms $A^n \to A$: the unimodular row $(a_1, \ldots, a_n)$ corresponds to the A-module morphism sending the ith standard basis vector to a_i. From a geometric perspective, we can think of unimodular rows as corresponding to the morphisms $\mathrm{Spec}(A) \to \mathbb{A}^n_k$ that factor (uniquely) through the Zariski open subset $\mathbb{A}^n_k \setminus \{0\}$. Indeed, if we identify $\mathbb{A}^n_k$ with the variety of $1 \times n$ row matrices, then $\mathbb{A}^n_k \setminus \{0\}$ consists precisely of the full rank matrices, i.e., the ones parameterizing surjective homomorphisms $A^n \to A$. Thus, the following proposition is manifest.

Proposition 4.2.4. The scheme $\mathbb{A}^n_k \setminus \{0\}$ represents the functor $\mathrm{AffSm}_k^{\mathrm{op}} \to \mathsf{Set}$ given by $\mathrm{Spec}\, A \mapsto \mathrm{Um}_n(A)$.

In fact, even more is true. Observe that the set of unimodular rows, which we denote by $\mathrm{Um}_n(A)$, carries a natural action of $\mathrm{GL}_n(A)$ on the right. We consider the subgroup $E_n(A) \subset \mathrm{GL}_n(A)$ generated by the *elementary matrices*, that is, those matrices implementing elementary row operations. The main object of interest in this appendix is the set of orbits $\mathrm{Um}_n(A)/E_n(A)$. Since unimodular rows belonging to the same orbit give rise to the "same" surjective homomorphism $A^n \to A$, we may instead be interested in understanding the functor $\mathrm{Spec}\, A \mapsto \mathrm{Um}_n(A)/E_n(A)$.

Question 4.2.5. Is there an equivalence relation $\sim$ on $\mathrm{Hom}(\mathrm{Spec}\, A, \mathbb{A}^n \setminus \{0\})$ so that $\mathrm{Hom}(\mathrm{Spec}\, A, \mathbb{A}^n \setminus \{0\})/\sim$ is naturally in bijection with $\mathrm{Um}_n(A)/E_n(A)$?

It turns out that this question can be answered in the language of motivic homotopy theory. Moreover, if n is sufficiently large, $\mathrm{Um}_n(A)/E_n(A)$ can be endowed with the structure of a group using elementary but slightly convoluted means. Motivic homotopy theory sheds light on this group structure, giving it a concrete geometric interpretation. In this appendix, we explore these motivic perspectives on unimodular rows.

Some Topological Motivation

To motivate the results we discuss in this appendix, we make a quick topological digression. For a finite CW complex X, let $C(X)$ denote the ring of continuous functions $X \to \mathbb{R}$. Under the assumption $\dim(X) \le 2n - 4$, van der Kallen shows that the set $\mathrm{Um}_n(C(X))/E_n(C(X))$ carries the natural structure of an (abelian) group. On the other hand, recall that the topological m-sphere S^m carries the natural structure of a co-H-space thanks to the natural fold map $S^m \vee S^m \to S^m$. Under suitable conditions on the space X, the structure above endows the set of homotopy classes of pointed maps $X \to S^m$, denoted $\pi^m(X)$, with the structure of a group. For instance, a classical theorem of Borsuk [Bor62] shows that $\dim X \le 2n - 4$ is sufficient for $\pi^n(X)$ to carry the aforementioned group structure. In this topological setting, van der Kallen [Kal89, Theorem 7.7] proves

Theorem 4.2.6 (van der Kallen, 1989)**.** Let X be a d-dimensional CW complex. Then, there exists a natural bijection

$$\mathrm{Um}_n(C(X))/E_n(C(X)) \cong \pi^{n-1}(X),$$

which can be upgraded to a group isomorphism whenever $d \le 2n - 4$.

Van der Kallen asked whether his theorem above can be extended to the algebraic setting. At the time, there was no suitable scheme-theoretic notion of cohomotopy groups, or even spheres. However, once the tools of $\mathbb{A}^1$-invariant motivic homotopy theory were developed, van der Kallen's question found a resolution in motivic homotopy theory, owing to Fasel [Fas10] and Lerbet [Ler24]. Let's set up the algebraic analogue of van der Kallen's result. Clearly, we want to replace X with the spectrum of a commutative k-algebra A. On the other hand, the stable dimension of A ought to play the role of the dimension of the CW complex X. The most subtle step is determining the appropriate replacement for the topological n-sphere S^n. It turns out that the correct choice is the motivic sphere $S^{2n-1,n} = \mathbb{A}^n \setminus \{0\}$. The most basic result that we study in this note is the proof of the following theorem of Fasel, which in some sense, provides an aswer to a motivic version of Question 4.2.5.

Theorem 4.2.7 (Fasel, 2010)**.** There exists a natural bijection

$$\mathrm{Um}_n(A)/E_n(A) \xrightarrow{\sim} [\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_{\mathbb{A}^1}, \tag{4.8}$$

where the right-hand side denotes the set of $\mathbb{A}^1$-homotopy classes of maps $\operatorname{Spec} A \to \mathbb{A}^n_k \setminus \{0\}$.

In this sense, we can say that $\mathbb{A}^n_k \setminus \{0\}$ represents the functor $k-\mathsf{Alg} \to \mathsf{Set}$ given by $A \mapsto \mathrm{Um}_n(A)/E_n(A)$ in the category of motivic spaces. Under some dimensionality assumptions on the ring A, van der Kallen endows the left-hand side of (4.8) with the structure of an abelian group. With the same assumptions, the right-hand side can also be equipped with the structure of an abelian group coming from the co-H-space structure on $\mathbb{A}^n_k \setminus \{0\}$. Completing the analogy with van der Kallen's topological result, Lerbet [Ler24] shows that Fasel's bijection is in fact a group isomorphism whenever these group structures can be defined. That is, our goal is to build up to the following theorem.

Theorem 4.2.9 (Lerbet, 2024)**.** Suppose A has Krull dimension at most $2n-4$. Then, the bijection (4.8) is a group isomorphism.

Since the group structure on the right-hand side of (4.8) can be interpreted geometrically, we can think of Lerbet's theorem as a motivic re-interpretation of van der Kallen's group law.
A natural question is to ask for an explicit computation of the group $\mathrm{Um}_n(A)/E_n(A)$. In the special case $n = \dim(A) + 1$, Fasel [Fas10] produces a cohomological description of this group.

Theorem 4.2.10 (Fasel, 2010)**.** Let k be a perfect field of characteristic not equal to two, and suppose A is a smooth k-algebra of Krull dimension $d = n - 1 \geq 2$. There exists a natural isomorphism

$$\mathrm{Um}_n(A)/E_n(A) \xrightarrow{\sim} H^{n-1}(\operatorname{Spec} A, \mathbf{K}^{\mathrm{MW}}_n),$$

where $\mathbf{K}^{\mathrm{MW}}_*$ is the Milnor–Witt sheaf.

Using this theorem, Fasel explicitly computes the group $\mathrm{Um}_n(A)/E_n(A)$ in some exceptional cases. Fasel's work in this direction, building on previous work of Morel, also has some applications to the study of stably free modules, which we briefly mention at the end of this note.

Outline

Before we proceed, we provide a brief overview of the structure of this note. In Subsection 4.2.2, we will first prove Fasel's bijection (4.8). In Subsection 4.2.3, we discuss the group structure on $\mathrm{Um}_n(A)/E_n(A)$ discovered by van der Kallen. In Subsection 4.2.4, we provide a brief exposition of Lerbet's proof that Fasel's bijection is actually a group isomorphism whenever the object involved carry a group structure. Finally, in Subsection 4.2.5, we discuss Fasel's cohomological interpretation of unimodular rows, some explicit calculations of the group structure, and applications to stably free modules.

4.2.2 Unimodular Rows and $\mathbb{A}^1$-Homotopy Classes

In this relatively brief section, we provide an exposition of Fasel's proof of Theorem 4.2.7, which establishes the following bijection for any commutative k-algebra A:

$$\mathrm{Um}_n(A)/E_n(A) \xrightarrow{\sim} [\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_{\mathbb{A}^1},$$

Note that Theorem 4.2.7 holds without any restrictions on the stable dimension of A.

Strictly speaking, Fasel establishes a more concrete version of the bijection above, in terms of naive homotopy classes of maps $\operatorname{Spec} A \to \mathbb{A}^n_k \setminus \{0\}$. As we will soon see, his result is readily reframed in terms of honest $\mathbb{A}^1$-homotopy classes, and we choose to reframe his result in these terms since (1) it keeps in line with the content of this course and (2) more importantly, it sets up the framework for Lerbet's group structure theorem. Recall that a naive $\mathbb{A}^1$-homotopy between maps $f, g\colon X \to Y$ of k-schemes is a morphism $H\colon X \times_k \mathbb{A}^1_k \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \arrow[dr, hook] \arrow[drr, "f\times \mathrm{id}_{\{0\}}"] & & \\
& X \times \mathbb{A}^1_k \arrow[r, "H"] & Y \\
X \times \{1\} \arrow[ur, hook] \arrow[urr, "g\times \mathrm{id}_{\{1\}}"'] & &
\end{tikzcd}$$

In ring-theoretic terms, if $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, a naive $\mathbb{A}^1$-homotopy between the corresponding ring maps $f^\sharp, g^\sharp\colon B \to A$ amounts to a map $H^\sharp\colon B \to A[t]$ such that $H^\sharp(0) = f^\sharp$ and $H^\sharp(1) = g^\sharp$. Here (and henceforth), for any $\alpha \in A$, we use the notation $H^\sharp(\alpha)\colon B \to A$ to denote the composition of $H^\sharp$ with the quotient map $A[t] \to A[t]/(t-\alpha) \cong A$.

Proposition 4.2.11. [Fas10, Theorem 2.1] There exists natural bijection

$$\mathrm{Um}_n(A)/E_n(A) \xrightarrow{\sim} [\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_N,$$

where the right-hand side denotes the set of *naive* $\mathbb{A}^1$-homotopy classes of maps $\operatorname{Spec} A \to \mathbb{A}^n_k \setminus \{0\}$.

Remark 4.2.12. In fact, Fasel proves this proposition in greater generality. Let $\mathrm{Um}_{m,n}(A)$ denote the set of surjective A-module morphisms $A^n \to A^m$ and let $D(m,n) = \mathbb{A}^{m\times n} \setminus V(m,n)$, where $V(m,n)$ is the vanishing locus of all $m \times m$ minors in $\mathbb{A}^{m\times n} = \mathrm{Mat}_{m\times n}$. In [Fas10, Theorem 2.1], he establishes a natural bijection

$$\mathrm{Um}_{m,n}(A)/E_n(A) \xrightarrow{\sim} [\operatorname{Spec} A, D(m,n)]_N.$$

Note that the case $m = 1$ is our Proposition 4.2.11.

Proof of Proposition 4.2.11. We follow the proof given by Fasel in [Fas10, Theorem 2.1], adding more detail where appropriate. Thanks to Proposition 4.2.4, it suffices to prove that (1) unimodular rows belonging to the same orbit define naively homotopic morphisms and (2) if two unimodular rows define naively homotopic morphisms, then they belong to the same orbit. Given a unimodular row $u \in \mathrm{Um}_n(A)$, let $\psi_u\colon \operatorname{Spec} A \to \mathbb{A}^n \setminus \{0\}$ denote the induced morphism.

Claim (1) is relatively straightforward. Letting E_n denote the group scheme of elementary matrices, we can understand elements of $E_n(A)$ as morphisms $M\colon \operatorname{Spec} A \to E_n$. Given a unimodular row $u \in \mathrm{Um}_n(A)$, the morphism $\psi_{uM}\colon \operatorname{Spec} A \to \mathbb{A}^n \setminus \{0\}$ is the composition

$$\operatorname{Spec} A \xrightarrow{\Delta} \operatorname{Spec} A \times \operatorname{Spec} A \xrightarrow{\psi_u \times M} \mathbb{A}^n \setminus \{0\} \times E_n \to \mathbb{A}^n \setminus \{0\},$$

where Δ is the diagonal morphism and the last morphism the natural right action of E_n on $\mathbb{A}^n \setminus \{0\}$. An explicit construction shows that any elementary is naively homotopic to the identity. Thus, the composition above is naively homotopic to ψ_u, as desired.

Claim (2) is a bit trickier. Observe that a naive homotopy between unimodular rows u_0 and u_1 amounts to the data of a unimodular row $U(t) \in A[t]$ such that $U(0) = u_0$ and $U(1) = u_1$. Consider the corresponding (split) short exact sequence of $A[t]$-module maps

$$0 \to B \to A[t]^n \xrightarrow{U(t)} A[t] \to 0,$$

where B is the kernel of $U(t)$. Since the short exact sequence above splits, we see that the submodule $B \subset A[t]$ is a finitely-generated projective $A[t]$-module. A result of Lindel [Lin82, Lemma 3] shows that B is extended from A in the sense that $B \cong B(0)[t]$, where $B(0)$ denotes the quotient B/tB. Thus, we have a split short exact sequence

$$0 \to B(0) \to A^n \xrightarrow{U(0)=u_0} A \to 0.$$

Tensor this sequence up by $A[t]$. Observe that the *same* copy of $B(0)[t] = B \subset A$ is the kernel of both $U(t)\colon A[t]^n \to A[t]$ and the constant map $u_0\colon A[t]^n \to A[t]$. Since both maps induce a splitting $A[t]^n \cong A[t] \oplus B$, there must exist an automorphism $\alpha\colon A[t]^n \to A[t]^n$ fixing B and satisfying $u_0 \circ \alpha = U(t)$. Moreover, note that ψ_0 is the identity since $U(0) = u_0$. A result of Vorst [Vor81] shows that α can be identified with an elementary matrix $E(t) \in E_n(A[t])$. It follows that $u_0 \circ E(1) = U(1) = u_1$, as desired. □

To re-express Fasel's result about naive homotopy classes as a result about $\mathbb{A}^1$-homotopy classes, we will use the fact that $\mathbb{A}^n \setminus \{0\}$ is $\mathbb{A}^1$-naive. Recall by Definition 3.15.9 that a simplicial presheaf $F \in \mathrm{PSh}(\mathrm{Sm}_k)$ is called $\mathbb{A}^1$-*naive* if for every smooth affine k-scheme U, we have an equivalence

$$L_{\mathbb{A}^1}(F)(U) \xrightarrow{\sim} L_{\mathrm{mot}}(F)(U).$$

For our purposes, the most important consequence is that naive homotopy classes of maps to $\mathbb{A}^1$-naive motivic spaces are in bijection with $\mathbb{A}^1$-homotopy classes of maps: that is, there exists a natural bijection

$$[\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_N \xrightarrow{\sim} [\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_{\mathbb{A}^1}.$$

Lemma 4.2.13. Let k be a field. The scheme $\mathbb{A}^n_k \setminus \{0\}$ is $\mathbb{A}^1$-naive.

Proof. This is a direct application of [AHW18, Corollary 4.2.6]. □

Combining the lemma above with Proposition 4.2.11, we have proven

Corollary 4.2.14 (Theorem 4.2.7)**.** There exists a natural bijection

$$\mathrm{Um}_n(A)/E_n(A) \xrightarrow{\sim} [\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_{\mathbb{A}^1}.$$

4.2.3 Van Der Kallen's Group Structure

In this section, we discuss the natural group structure on $\mathrm{Um}_n(A)/E_n(A)$ discovered by van der Kallen [Kal89]. As we mentioned in the introduction, van der Kallen discovered this group structure while studying the orbits of unimodular rows for the ring of continuous functions on a finite CW complex. The exposition in this section closely follows [Ler24, Section 3]. Van der Kallen's group structure was used by Lerbet to fully generalize Theorem 4.2.6 to the algebraic setting. The key idea here is to construct a natural correspondence between $\mathrm{Um}_n(A)/E_n(A)$ and a certain universal group of maps from $\mathrm{Um}_n(A)$. Although the structure of this universal group seems quite daunting *a priori*, we will introduce a convenient trick (the Mennicke–Newman lemma) for computing products of orbits of unimodular rows at the end of this section.

Weak Mennicke Symbols

Definition 4.2.15. Let G be a group. A *weak Mennicke symbol* on $\mathrm{Um}_n(A)$ is a map of sets $\mu\colon \mathrm{Um}_n(A) \to G$ satisfying the following relations:

(i) The map μ is invariant under the right-action of $E_n(A)$. That is, for any elementary matrix E and for any $u \in \mathrm{Um}_n(A)$, we have $\mu(uE) = \mu(u)$. In other words, the map μ factors through the quotient $\mathrm{Um}_n(A) \to \mathrm{Um}_n(A)/E_n(A)$.

(ii) For any pair of unimodular rows $u, u' \in \mathrm{Um}_n(A)$ of the form $u = (a, u_2, \ldots, u_n)$ and $u' = (1 + a, u_2, \ldots, u_n)$ and for any $r \in A$ such that $r(1 + a) = a$ modulo the ideal $\langle u_2, \ldots, u_n\rangle$, we have $\phi(u) = \phi(u'')\phi(u')$, where $u'' = (r, u_2, \ldots, u_n)$ (it can be shown that this sequence is also a unimodular row).

Remark 4.2.16. Van der Kallen uses the terminology "weak" Mennicke symbol because the axioms above do indeed give a weaker definition of the Mennicke symbols introduced by Suslin [Sus06] for the algebraic K-theory of fields.

A basic, yet fundamental, insight is the existence of a universal group $\mathrm{WMS}_n(A)$ of weak Mennicke symbols.

Definition 4.2.17. For any positive integer n and any k-algebra A, define the group $\mathrm{WMS}_n(A)$ as the quotient of the free group on $\mathrm{Um}_n(A)$ by the relations generated by Conditions (i) and (ii) in Definition 4.2.15. Denote the class of the element corresponding to $u \in \mathrm{Um}_n(A)$ by $[u]$.

There exists a universal weak Mennicke symbol $\hat{\mu}\colon \mathrm{Um}_n(A) \to \mathrm{WMS}_n(A)$ given by $u \mapsto [u]$. This symbol is universal in the sense that for any weak Mennicke symbol $\mu\colon \mathrm{Um}_n(A) \to G$, there exists a unique group homomorphism $\mu'\colon \mathrm{WMS}_n(A) \to G$ such that $\mu' \circ \hat{\mu} = \mu$. The key insight of van der Kallen is that the group structure on $\mathrm{WMS}_n(A)$ can be transferred to the set $\mathrm{Um}_n(A)/E_n(A)$ under some assumptions on A. To properly state these assumptions, we need to define the *stable dimension* of A.

Definition 4.2.18. Let A be a commutative k-algebra. The stable rank of A, denoted $\mathrm{srank}(A)$, is defined as the smallest integer r such that for any unimodular row $(a_1, \ldots, a_{r+1})$, there exist $b_1, \ldots, b_r$ such that $(a_1 + b_1 a_{r+1}, a_1 + b_2 a_{r+1}, \ldots, a_r + b_{r+1} a_{r+1})$ is also unimodular. If no such integer r exists, then we declare $\mathrm{srank}(A) = \infty$. The stable dimension of A, denoted $\mathrm{sdim}(A)$, is defined as $\mathrm{srank}(A) - 1$.

Remark 4.2.19. For our purposes, we simply need an upper bound on the stable dimension of A. Bass [Bas75, Theorem 1] shows that the stable dimension of a Noetherian ring is bounded above by its Krull dimension, and thus, in applications, we can (and will) use the Krull dimension of A instead. We record the definition of the stable dimension for completeness.

When the stable dimension of A is bounded, van der Kallen [Kal89, Theorems 3.6 and 4.1] equips $\mathrm{Um}_n(A)/E_n(A)$ with the structure of an abelian group. His results are summarized in the following theorem. The proof, while completely "elementary," is quite long and computational, so we omit it from our exposition and refer the interested reader to [Kal89, Sections 3 and 4].

Theorem 4.2.20 (Van der Kallen, 1989)**.** Suppose $\mathrm{sdim}(A) \leq 2n - 4$. Then, the group $\mathrm{WMS}_n(A)$ is abelian, and the universal map $\mathrm{Um}_n(A)/E_n(A) \to \mathrm{WMS}_n(A)$ is a bijection. In particular, $\mathrm{Um}_n(A)/E_n(A)$ inherits the structure of an abelian group.

The Mennicke–Newmann Lemma: Expliciting the Group Law

Despite the abstract definition of $\mathrm{WMS}_n(A)$, it turns out that the group law on $\mathrm{Um}_n(A)/E_n(A)$ can be computed rather explicitly using two facts.

First, [Kal89, Lemma 3.5(v)] and [Kal02, Lemma 3.1] imply that the following relation holds in $\mathrm{WMS}_n(A)$:

For any pair of unimodular rows of the form $u = (a, u_2, \ldots, u_n)$ and $u' = (1-a, u_2, \ldots, u_n)$, we have (4.21)

$$[u][u'] = [(a(1-a), u_2, \ldots, u_n)]. \tag{4.22}$$

Second, we have the so-called Mennicke–Newman lemma. The proof of this lemma will give us a good handle on elementary calculations involving unimodular rows, and thus, we provide it for pedagogical reasons.

Lemma 4.2.23 (Mennicke–Newman)**.** Assume that A has stable dimension $d \leq 2n-3$, and let $U, U' \in \mathrm{Um}_n(A)$ be unimodular rows. Then, there exist elementary matrices $E, E' \in E_n(A)$ such that $UE = (a, r_2, \ldots, r_n)$ and $U'E' = (1-a, r_2, \ldots, r_n)$ for some $a, r_2, \ldots, r_n \in A$.

This version of the Mennicke–Newman lemma and its proof appears as Proposition 3.6 in [Ler24].

Proof. As mentioned above, we follow the proof of [Ler24, Proposition 3.6]. The proof is accomplished in three steps. Fix unimodular rows $U = (u_1, \ldots, u_n)$ and $U' = (v_1, \ldots, v_n)$.

Step 1. We claim that it suffices to assume that $(u_2, \ldots, u_n, v_2, \ldots, v_n)$ is a unimodular row of length $2n-2$. Observe first that $(u_2, \ldots, u_n, v_2, \ldots, v_n, u_1v_1)$ is certainly a unimodular row – indeed, if $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ are such that $\sum_i b_iu_i = 1$ and $\sum_i c_iv_i = 1$, then

$$\sum_{j=2}^{n}\sum_{i=2}^{n}(b_iu_ic_j)v_j + b_1c_1(u_1v_1) = \sum_{j=1}^{n} c_j\left(\sum_{i=1}^{n} b_iu_i\right)v_j = 1.$$

Since A has stable dimension at most $2n-3$, it has stable rank at most $2n-2$. In particular, by definition, there exist $\alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_n \in A$ such that

$$(u_2 + \alpha_2u_1v_1, \ldots, u_n + \alpha_nu_1v_1, v_2 + \beta_2u_1v_1, \ldots, v_n + \beta_nu_1v_1)$$

is a unimodular row. Note that $(u_1, u_2 + \alpha_2u_1v_1, \ldots, u_n + \alpha_nu_1v_1)$ belongs to the $E_n(A)$-orbit of $(u_1, \ldots, u_n)$, so we may as well replace $(u_1, \ldots, u_n)$ by $(u_1, u_2 + \alpha_2u_1v_1, \ldots, u_n + \alpha_nu_1v_1)$. Similarly, we may replace $(v_1, \ldots, v_n)$ by $(v_1, v_2 + \beta_2u_1v_1, \ldots, v_n + \beta_nu_1v_1)$. Ultimately, we are only concerned with the $E_n(A)$-orbits of U and U', so these substitutions complete the first step.

Step 2. Assuming Step 1, we claim that it suffices to assume $u_1 + v_1 = 1$. The unimodularity of the row $(u_2, \ldots, u_n, v_2, \ldots, v_n)$ implies that there exist $\gamma_i, \delta_i \in A$ such that

$$\sum_{i=2}^{n}(\gamma_iu_i + \delta_iv_i) = u_1 + v_1 - 1.$$

Observe that $(u_1 - \sum_{i=1}^n \gamma_iu_i, u_2, \ldots, u_n)$ and $(v_1 - \sum_{i=1}^n \beta_iv_i, v_2, \ldots, v_n)$ belong to the $E_n(A)$-orbits of $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, respectively, and that they satisfy the conditions of Steps 1 and 2. Thus, we may replace $(u_1, \ldots, u_n)$ by $(u_1 - \sum_{i=1}^n \gamma_iu_i, u_2, \ldots, u_n)$ and replace $(v_1, \ldots, v_n)$ by $(v_1 - \sum_{i=1}^n \beta_iv_i, v_2, \ldots, v_n)$ to complete the second step.

Step 3. Finally, we establish the claim of the Mennicke–Newman Lemma under the simplifying assumptions of Steps 1 and 2. Set $a := u_1$. Since $u_1 + v_1 = 1$, we have

$$u_i + v_i - (u_i + v_i)(u_1 + v_1) = 0$$

for each $i = 2, \ldots, n$. In particular, we can define $r_i \in A$ to be the element

$$u_i - u_1(u_i - v_i) = v_i + v_1(u_i - v_i).$$

Then,

$$(a, r_2, \dots, r_n) = (u_1, u_2 - u_1(u_2 - v_2), \dots, u_n - u_1(u_n - v_n))$$

is obtained from $(u_1, \dots, u_n)$ by the action of the elementary lower triangular matrix with ones on the diagonal, $v_i - u_i$ in the ith entry of the first column for $i \geq 2$, and zeros elsewhere. Similarly,

$$(1 - a, r_2, \dots, r_n) = (v_1, v_2 + v_1(u_i - v_i), \dots, v_n + v_1(u_i - v_i)$$

belongs to the $E_n(A)$-orbit of $(v_1, \dots, v_n)$. This completes the proof. $\square$

Remark 4.2.24. Observe that Step 1 was the only step that used the fact that $\operatorname{sdim}(A) \leq 2n - 3$. In particular, if $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ are unimodular rows such that $(u_2, \dots, u_n, v_2, \dots, v_n)$ has length $2n - 2$, then the conclusion of the Mennicke–Newman lemma holds for these rows as well. This fact will be particularly useful in the proof of Lemma 4.2.30.

Let's explain how the preceding two facts allow us to compute van der Kallen's group law on $\mathrm{Um}_n(A)/E_n(A)$. First, the Mennicke–Newman lemma shows that *any* pair of orbits in $\mathrm{Um}_n(A)/E_n(A)$ can be represented by unimodular rows of the form $(a, u_2, \dots, u_n)$ and $(1 - a, u_2, \dots, u_n)$. Thus, by relation (4.21)-(4.22), the sum of the corresponding orbits is simply the orbit of the unimodular row $(a(1 - a), u_2, \dots, u_n)$.

4.2.4 The Group Isomorphism

In this section, we prove [Ler24, Theorem 5.1], showing that Fasel's natural bijection

$$\mathrm{Um}_n(A)/E_n(A) \xrightarrow{\sim} [\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_{\mathbb{A}^1}$$

has the structure of a group homomorphism whenever both sides of the bijection above carry a group structure. The exposition in this section closely follows that of [Ler24, Section 5]. In Subsection 4.2.3, we equipped $\mathrm{Um}_n(A)/E_n(A)$ with the structure of an abelian group whenever the stable dimension of A is at most $2n - 4$. In particular, this structure also exists when the *Krull dimension* of A is at most $2n - 4$.

Motivic Cohomotopy Groups

Let's first demonstrate how to equip $[\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_{\mathbb{A}^1}$ with the structure of an abelian group under the same assumptions. Intuitively, since we should think of $\mathbb{A}^n_k \setminus \{0\}$ as a motivic analog of the sphere S^{n-1}, this construction is a motivic analog of Borsuk's cohomotopy group structure on $\pi^{n-1}(X)$. This group law is thus aptly called the *motivic Borsuk's group law.* We follow the exposition in [Ler24, Section 4].

Let X and Y be pointed motivic spaces over k, with morphisms $f, g : X \to Y$. These morphisms induce a product map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y,$$

where $\Delta : X \to X \times X$ is the diagonal. Our goal is to define a "sum" of the (classes of the) maps f and g in $[X, Y]_{\mathbb{A}^1}$. In other words, we want an (associative) map

$$[X, Y]_{\mathbb{A}^1} \times [X, Y]_{\mathbb{A}^1} \to [X, Y]_{\mathbb{A}^1}.$$

We have a natural fold map $Y \vee Y \to Y$, and on the other hand, we have a natural embedding $Y \vee Y \to Y \times Y$. Thus, if the map $f \times g$ were to factor through $Y \vee Y \to Y \times Y$, we could try to define $f + g$ as the following composition

$$X \xrightarrow{\Delta} X \times X \dashrightarrow Y \vee Y \to Y,$$

where the dashed right arrow is the map $X \times X \to Y \vee Y$ induced by $f \times g : X \times X \to Y \times Y$. If this scenario were to hold, a straightforward exercise, using the properties of the fold map $Y \vee Y \to Y$, verifies that the assignment above does indeed give $[X, Y]_{\mathbb{A}^1}$ the structure of an abelian group, whose identity element is given by the constant map $X \to Y$ (recall that all motivic spaces are pointed in our discussion).

The following result of Asok and Fasel [AF22, Proposition 1.2.3] gives us a condition for which our hopes may hold. Recall that the $\mathbb{A}^1$-cohomological dimension of a smooth k-scheme X is the largest positive integer d such that $H^d_{\mathrm{Nis}}(X, \mathcal{F}) = 0$ for any Nisnevich sheaf $\mathcal{F}$ of abelian groups on X.

Theorem 4.2.25 (Asok–Fasel, 2022)**.** *Let Y be a pointed motivic space over k that is also defined over some perfect subfield $k' \subset k$. That is, Y is the base change of some k'-motivic space Y'. Moreover, suppose $n \geq 2$ is a positive integer. If Y is $(n-1)$-$\mathbb{A}^1$-connected, then the morphism $Y \vee Y \to Y \times Y$ induces a bijection*

$$[X, Y \vee Y]_{\mathbb{A}^1} \xrightarrow{\sim} [X, Y \times Y]_{\mathbb{A}^1}$$

for any smooth k-scheme X with cohomological dimension at most $2n-2$.

Proof. For the sake of brevity, we omit the proof, referring the reader to [Ler24, Proposition 4.1], for instance, for a proof. The point is to show that the map $f : Y \vee Y \to Y \times Y$ is $\mathbb{A}^1$-$(2n-2)$-connected, at which point [Ler24, Lemma 2.22] would give the desired result. The connectedness of f is computed through an application of the Blakers–Massey theorem. □

In fact, the results of Asok and Fasel [AF22, Proposition 1.2.5] also show that the group structure on $[X, Y]_{\mathbb{A}^1}$ defined in Theorem 4.2.25 is abelian. While we will not prove this fact here, we record the following corollary for convenience.

Corollary 4.2.26. *Let $\mathrm{Sm}_k^{\leq d}$ denote the full subcategory of smooth k-schemes of cohomological dimension $\leq d$. The assignment $Y \mapsto [-, Y]_{\mathbb{A}^1}$ defines a functor from the full subcategory of pointed motivic spaces satisfying the assumptions of Theorem 4.2.25 to the category of presheaves on $\mathrm{Sm}_k^{\leq 2n-2}$ valued in abelian groups.*

Remark 4.2.27. If Y is a pointed motivic space satisfying the assumptions of Theorem 4.2.25, the functor $[-, Y]_{\mathbb{A}^1} : \mathrm{Sm}_k^{\leq d} \to \mathsf{Ab}$ is called the *motivic cohomotopy theory defined by Y*.

The upshot of this discussion is that the space $\mathbb{A}^n_k \setminus \{0\}$, with base point $(1, 0, \ldots, 0)$, satisfies all assumptions of Theorem 4.2.25 (with some modifications to indices). Indeed, $\mathbb{A}^n_k \setminus \{0\}$ is certainly defined over any subfield of k, and we showed earlier this semester that it is $(2n-2)$-$\mathbb{A}^1$-connected. Thus, Theorem 4.2.25 equips $[X, \mathbb{A}^n_k \setminus \{0\}]$ with the structure of an abelian group for any smooth k-scheme X of cohomological dimension at most $2n-4$. Since the cohomological dimension of a ring A is bounded above by its Krull dimension, this group structure also exists when X has *Krull dimension* at most $2n-4$.

Lerbet's Theorem – Preliminaries

With all these pieces in place, let's discuss the proof of the following theorem of Lerbet.

Theorem 4.2.28 (Lerbet, 2024)**.** *Let A be a smooth k-algebra with Krull dimension $\leq 2n-4$. The natural bijection*

$$\mathrm{Um}_n(A)/E_n(A) \xrightarrow{\sim} [\operatorname{Spec} A, \mathbb{A}^n_k \setminus \{0\}]_{\mathbb{A}^1}$$

is a group isomorphism, where the left-hand side is equipped with van der Kallen's group structure and the right-hand side is equipped with the cohomotopical group structure.

In particular, Lerbet's theorem provides what he terms as a cohomotopical re-interpretation of van der Kallen's group law.

We break the proof of this theorem into several more digestible chunks. For concision, we write $Y_n := \mathbb{A}^n \setminus \{0\}$. We give Y_n the structure of a pointed motivic space by the basepoint $(0, \ldots, 1)$. Similarly, equip $Y_n \times Y_n$ with the basepoint $(0, \ldots, 1, 0, \ldots, 1)$, so that the induced map $Y_n \vee Y_n \to Y_n \times Y_n$ is pointed.

Lerbet's Theorem – Unimodular Description of Fold Map

The main ingredient in the proof of Theorem 4.2.28 is an explicit description of the fold map $Y_n \vee Y_n \to Y_n$ that uses the language of unimodular rows. Let's adopt the notation $U_n := \mathbb{A}^1 \times Y_{n-1} \subset Y_n$ and consider the subscheme

$$Z_n := (U_n \times Y_n) \cup (Y_n \times U_n) \subset Y_n \times Y_n.$$

Here is an elementary but crucial fact about Z_n: the map $\iota : Y_n \vee Y_n \to Y_n \times Y_n$ factors through $Z_n \to Y_n \times Y_n$. Indeed, in the first factor, the map ι is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 1) \in Y_n \times U_n \subset Z_n$, and a similar argument can be made for the second factor.

Our goal now is to construct a map $\pi_n : Z_n \to Y_n$ in the category of motivic spaces such that the fold map $Y_n \vee Y_n \to Y_n$ factors as a composition of the inclusion $Y_n \vee Y_n \to Z_n$ and π_n. Here is the motivation for this goal. Fix $f, g \in [\operatorname{Spec} A, Y_n]_{\mathbb{A}^1}$ in the category of motivic spaces. As we will soon establish, the product $f \times g : \operatorname{Spec} A \to Y_n \times Y_n$ factors through $Y_n \vee Y_n \to Y_n \times Y_n$, and the cohomotopical group law defines $f + g$ as the composition of $f \times g$ with the fold map $Y_n \vee Y_n \to Y_n$. If we were to construct the map π_n described above, then observe that the sum $f + g$ can equivalently be described as the composition $\pi_n \circ (f \times g)$. In particular, if we can obtain a concrete description of the map π_n in terms of unimodular rows, then we may hope to relate the sum $f + g$ with a sum in van der Kallen's group law.

To construct π_n, we construct a Jouanolou device for Z_n. First, we produce a Jouanolou device for Y_{2n-2}. For any $n \geq 0$, define the smooth integral quadric hypersurfaces

$$Q_{2n+1} := \operatorname{Spec} k[x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]/\langle x_1 y_1 + \cdots + x_{n+1} y_{n+1} - 1 \rangle \subset \mathbb{A}^{2n+2},$$

$$Q_{2n} := \operatorname{Spec} k[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/\langle x_1 y_1 + \cdots + x_n y_n + z(1 - z) \rangle \subset \mathbb{A}^{2n+1}.$$

Proposition 4.2.29. [Ler24, Lemma 2.8] The map $Q_{2n+1} \to Y_{n+1}$ given by projection to the first $n + 1$ coordinates is a Jouanolou device.

Proof. Let $\mathcal{O}$ denote the structure sheaf of Y_{n+1}. For fixed $(x_1, \ldots, x_{n+1}) \in Y_{n+1}$, consider the epimorphism

$$\mathcal{O}^{\oplus n+1} \twoheadrightarrow \mathcal{O}, \quad (a_1, \ldots, a_{n+1}) \mapsto a_1 x_1 + \cdots + a_{n+1} x_{n+1}.$$

The kernel of this morphism is a locally free $\mathcal{O}$-module and thus defines an algebraic vector bundle over Y_{n+1}. This vector bundle is precisely $Q_{2n+1} \to Y_{n+1}$. $\square$

We use $Q_{2n+1} \to Y_{n+1}$ to construct a Jouanolou device for Z_n. Consider the affine scheme Z'_n defined by the following Cartesian square

$$\begin{tikzcd} Z'_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q_{2n-1} \times Q_{2n-1} \arrow[d] \\ Q_{4n-5} \arrow[r] & \mathbb{A}^{2n-2} \end{tikzcd}$$

where the bottom horizontal map is the composition $Q_{4n-5} \to Y_{2n-2} \to \mathbb{A}^{2n-2}$ and the right vertical map is the composition $Q_{2n-1} \times Q_{2n-1} \to Y_n \times Y_n \to \mathbb{A}^{2n-2}$. Note that Z'_n is affine because it is the fibered product of affine schemes over an affine base. On the other hand, we have a Cartesian square

$$\begin{tikzcd} Z_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A}^n \times \mathbb{A}^n \arrow[d] \\ Y_{2n-2} \arrow[r] & \mathbb{A}^{2n-2} \end{tikzcd}$$

where the right-vertical map $\mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^{2n-2} = \mathbb{A}^{n-1} \times \mathbb{A}^{n-1}$ given by projection to the first $n-1$ coordinates in each factor. Observe that the map $Q_{4n-5} \to \mathbb{A}^{2n-2}$ factors through Y_{2n-2} by definition, so one can show that the composition $Z'_n \to Q_{2n-1} \times Q_{2n-1} \to \mathbb{A}^{2n-2}$ must factor through Z_n (the square above shows that Z_n is the scheme-theoretic preimage of Y_{2n-2} under $\mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^{2n-2}$). From this discussion, we see that there is a Cartesian square

$$\begin{tikzcd} Z'_n \arrow[r] \arrow[d] & Z_n \times_{Y_n \times Y_n} (Q_{2n-1} \times Q_{2n-1}) \arrow[d] \\ Z_n \times_{Y_{2n-2}} Q_{4n-5} \arrow[r] & Z_n \end{tikzcd}$$

The bottom and right maps in the square above are vector bundles because they are base changes of the Jouanolou devices $Q_{4n-5} \to Y_{2n-2}$ and $Q_{2n-1} \times Q_{2n-1} \to Y_n \times Y_n$, respectively. Hence, the top and left maps are also vector bundles. We deduce that either composition $Z'_n \to Z_n$ in the diagram above is a Jouanolou device for Z_n. Let's return to the construction of the map π_n. Since $Z'_n \to Z_n$ induces a motivic equivalence, it suffices to study maps $Z'_n \to Y_n$.

The key reason that we want to consider the scheme Z'_n is due to its explicit connection to unimodular rows. Lerbet [Ler24, pg. 27] describes the coordinate ring of Z'_n as the quotient

$$k[Z'_n] = k[x_1, \dots, x_n y_1, \dots, y_n, u_1, \dots, u_n, v_1, \dots, v_n, r_2, \dots, r_n, s_2, \dots, s_n]/I$$

where I is the ideal generated by relations

$$\begin{aligned} x_1y_1 + x_2y_2 + \cdots + x_ny_n &= 1 \\ u_1v_1 + u_2v_2 + \cdots + u_nv_n &= 1 \\ x_2r_2 + \cdots + x_nr_n + u_2s_2 + \cdots + u_ns_n &= 1. \end{aligned}$$

As Lerbet remarks, we should think of $(x_1, \dots, x_n, u_1, \dots, u_n)$ as the universal vector such that $(x_1, \dots, x_n)$, $(u_1, \dots, u_n)$, and $(x_2, \dots, x_n, u_2, \dots, u_n)$ are unimodular. In other words, for any k-algebra R and any morphism $R^{2n+2} \to R$ given by a vector $(x'_1, \dots, x'_n, u'_1, \dots, u'_n)$ satisfying the conditions above, there exists a unique map $k[Z'_n] \to R^{2n+2}$ given by $x_i \mapsto x'_i$ and $u_i \mapsto u'_i$. This observation will prove crucial when we proceed to the proof of Theorem 4.2.28.

Lemma 4.2.30. There exist $\gamma, a_2, \dots, a_n \in k[Z'_n]$ and elementary matrices $E_x, E_u \in E_n(k[Z'_n])$ such that $(x_1, \dots, x_n)E_x = (\gamma, a_2, \dots, a_n)$ and $(u_1, \dots, u_n)E_u = (1-\gamma, a_2, \dots, a_n)$.

Proof. We should think of this lemma as a "universal" analogue of the Mennicke–Newman Lemma 4.2.23, as $(x_1, \dots, x_n)$ and $(u_1, \dots, u_n)$ are unimodular rows. More precisely, we are almost in the situation of the Mennicke–Newman lemma, except we do not have the needed dimensionality hypotheses on $k[Z'_n]$. We circumvent this issue by Remark 4.2.24. That is, observe that the dimensionality hypothesis in the proof of the Mennicke–Newman lemma is used only when proving that $(x_2, \dots, x_n, u_2, \dots, u_n)$ is also unimodular. However, thanks to the defining relations of $k[Z'_n]$, we automatically have that $(x_2, \dots, x_n, u_2, \dots, u_n)$ is unimodular. Hence, the rest of the proof of the Mennicke–Newman lemma applies, giving us the result. $\square$

In particular, thanks to the relation (4.21)-(4.22) for unimodular rows, we see that $(\gamma(1-\gamma), a_2, \dots, a_n)$ is also a unimodular row for $k[Z'_n]$, and hence, we obtain a morphism $\pi'_n : Z'_n \to Y_n$

of schemes. Since $Z'_n \to Z_n$ is a motivic equivalence, the map π'_n uniquely determines a morphism $Z_n \to Y_n$ in the category of motivic spaces. It remains to describe the fold map $Y_n \vee Y_n \to Y_n$ in terms of π_n.

Proposition 4.2.31. [Ler24, Lemma 5.8] There is a commutative diagram in the category of pointed motivic spaces

$$\begin{tikzcd} Y_n \vee Y_n \arrow[r] \arrow[dr, "\nabla"] & Z_n \arrow[d, "\pi_n"] \\ & Y_n \end{tikzcd}$$

where the diagonal map is the fold map $\nabla : Y_n \vee Y_n \to Y_n$ and the horizontal map is the inclusion $Y_n \vee Y_n \to Z_n$.

Proof. Refer to the proof of [Ler24, Lemma 5.8], relying entirely on elementary unimodular row operations. □

Lerbet's Theorem – Motivic Ingredients

The upshot of our work in the preceding subsubsection is quite substantial. To rigorously formulate the consequences, we record some additional motivic facts.

Lemma 4.2.32. Let A be a smooth commutative k-algebra with Krull dimension at most $2n-4$. The inclusion $Y_n \vee Y_n \to Y_n \times Y_n$ induces a bijection $[\operatorname{Spec} A, Y_n \vee Y_n]_{\mathbb{A}^1} \to [\operatorname{Spec} A, Y_n \times Y_n]_{\mathbb{A}^1}$.

Proof. An immediate consequence of Theorem 4.2.25, since Y_n is $(n-1)$-$\mathbb{A}^1$-connected and the cohomological dimension of A is bounded by its Krull dimension. □

We would like to prove a similar result for the inclusion $Z_n \to Y_n \times Y_n$. We need a sequence of lemmas to achieve this result.

Lemma 4.2.33. [Ler24, Lemma 4.5] If $n \geq 2$, then Z_n is $\mathbb{A}^1$-connected.

Proof. We provide a sketch of the proof, referring the reader to [Ler24, Lemma 5.5] for details. First, one shows that $Y_n = \mathbb{A}^n \setminus \{0\}$ is $\mathbb{A}^1$-chain connected for $n \geq 2$. That is, for any finite separable extension L/k and any points $x, x' \in Y_n(L)$, there exists a morphism $g : \mathbb{A}^1_L \to Y_n(L)$ such that $g(0) = x$ and $g(1) = x'$. This step is elementary and explicit. Next, the product of $\mathbb{A}^1$-chain connected schemes is also $\mathbb{A}^1$-chain connected, so $U_n \times Y_n$ and $Y_n \times U_n$ are both $\mathbb{A}^1$-chain connected. Finally, the union $Z_n = U_n \times Y_n \cup Y_n \times U_n$ is $\mathbb{A}^1$-chain connected because the schemes $U_n \times Y_n$ and $Y_n \times U_n$ share a base point. Since $\mathbb{A}^1$-chain connectedness implies $\mathbb{A}^1$-connectedness, the result follows. □

Lemma 4.2.34. [Ler24, Lemma 5.6] Suppose $n \geq 3$. The inclusion $Z_n \to Y_n \times Y_n$ is $\mathbb{A}^1$-$(2n-4)$-connected.

Proof. The first step of the proof is to show that Z_n is $\mathbb{A}^1$-simply connected. We already kow that Z_n is $\mathbb{A}^1$-connected. On the other hand, we have

$$\pi_1^{\mathbb{A}^1}(U_n \times Y_n) = \pi_1^{\mathbb{A}^1}(Y_n) \times \pi_1^{\mathbb{A}^1}(Y_{n-1}) \times \pi_1^{\mathbb{A}^1}(\mathbb{A}^1) = \pi_1^{\mathbb{A}^1}(Y_n) \times \pi_1^{\mathbb{A}^1}(Y_{n-1}) = 1$$

because $Y_m = \mathbb{A}^m \setminus \{0\}$ is $\mathbb{A}^1$-simply connected for $m \geq 2$. Similalry, $\pi_1^{\mathbb{A}^1}(Y_n \times U_n)$ is trivial. On the other hand, the intersection $(Y_n \times U_n) \cap (U_n \times Y_n) = U_n \times U_n$ is also $\mathbb{A}^1$-simply connected. We conclude with an application of the motivic van Kampen theorem cited by Lerbet in [Ler24, Lemma 5.6].

Next, Lerbet explains that we have a cofiber sequence

$$Z_n \to Y_n \times Y_n \to (\mathbb{G}_m \times \mathbb{G}_m)_+ \wedge (\mathbb{P}^1)^{\wedge 2n-2},$$

so the homotopy cofiber of the map $Z_n \to Y_n \times Y_n$ is $\mathbb{A}^1$-$(2n-3)$-connected. From here, the result follows from an application of the Blakers–Massey theorem also cited by Lerbet. We omit the details for the sake of brevity and refer the reader to [Ler24, Lemma 5.6]. □

A standard motivic obstruction theory result [Ler24, Lemma 2.22] allows us to conclude the desired

Corollary 4.2.35. The map $Z_n \to Y_n \times Y_n$ induces a bijection $[\operatorname{Spec} A, Z_n]_{\mathbb{A}^1} \to [\operatorname{Spec} A, Y_n \times Y_n]_{\mathbb{A}^1}$.

Let's discuss the consequences of this subsubsection and subsubsection 4.2.4. Fix morphisms $f, g : \operatorname{Spec} A \to Y_n$ and define $[f,g] : \operatorname{Spec} A \to Y_n \times Y_n$ as the composition $(f \times g) \circ \Delta$, where $\Delta : \operatorname{Spec} A \to \operatorname{Spec} A \times \operatorname{Spec} A$ is the diagonal embedding. Recall that the morphism $Y_n \vee Y_n \to Y_n \times Y_n$ factors through the inclusion $Z_n \to Y_n \times Y_n$. By Lemma 4.2.32 and Corollary 4.2.35, there exists unique lifts $(f,g) : \operatorname{Spec} A \to Y_n \vee Y_n$ and $\langle f, g\rangle : \operatorname{Spec} A \to Z_n$ so that the following diagram commutes

$$\begin{tikzcd}
 & Z_n \arrow[dr] & \\
 & Y_n \vee Y_n \arrow[u] \arrow[dr] & \\
\operatorname{Spec} A \arrow[uur, "\langle f,g\rangle"] \arrow[ur, "(f,g)"'] \arrow[rr, "{[f,g]}"] & & Y_n \times Y_n
\end{tikzcd}$$

The sum $f + g \in [\operatorname{Spec} A, Y_n]_{\mathbb{A}^1}$ in the cohomotopical group law is defined as the composition $f + g := \nabla \circ (f,g)$, where $(f,g) : \operatorname{Spec} A \to Y_n \vee Y_n$ is the unique lift of the composition $(f \times g) \circ \Delta : \operatorname{Spec} A \to Y_n \times Y_n$. By Proposition 4.2.31 and the uniqueness of the lift $\langle f, g\rangle$, the following diagram commutes:

$$\begin{tikzcd}
 & Z_n \arrow[dr, "\pi_n"] & \\
\operatorname{Spec} A \arrow[ur, "\langle f,g\rangle"] \arrow[r, "(f,g)"] & Y_n \vee Y_n \arrow[u] \arrow[r, "\nabla"] & Y_n
\end{tikzcd}$$

Thus, we can also compute $f + g$ as the composition

$$f + g = \pi_n \circ \langle f, g\rangle.$$

Since π_n has an explicit description in terms of a Mennicke–Newman-like relation for unimodular rows, we can now expect some relationship between the cohomotopical and the van der Kallen group laws on $[\operatorname{Spec} A, X]_{\mathbb{A}^1}$.

Lerbet's Theorem – Proof

It remains to prove the following assertion.

Proposition 4.2.36. Suppose A is a smooth k-algebra of Krull dimension at most $2n-4$. The natural map $\varphi : \mathrm{Um}_n(A)/E_n(A) \to [\operatorname{Spec} A, Y_n]_{\mathbb{A}^1}$ is a group homomorphism.

Proof. Take $[u], [u'] \in \mathrm{Um}_n(A)/E_n(A)$. Thanks to the Mennicke–Newman lemma, we may assume that there representatives have the form $u = (x, a_2, \ldots, a_n)$ and $u' = ((1-x), a_2, \ldots, a_n)$ for some $x, a_2, \ldots, a_n \in A$. Van der Kallen's group law then gives us

$$[u] + [u'] = [(x(1-x), a_2, \ldots, a_n)].$$

For any unimodular row $(v_1, \dots, v_n)$, the morphism $\varphi(v) : \operatorname{Spec} A \to Y_n$ is the A-point
$$(v_1, \dots, v_n) \in Y_n(A) = \mathbb{A}^n(A) \setminus \{0\}.$$
Thus, the unimodular rows u and u' jointly define the morphism
$$\langle \varphi(u), \varphi(u') \rangle : \operatorname{Spec} A \to Z_n$$
given by the A-point
$$(x, a_2, \dots, a_n, 1 - x, a_2, \dots, a_n) \in Z_n \subset Y_n \times Y_n.$$
The map π_n was defined so that
$$\pi_n(x, a_2, \dots, a_n, 1 - x, a_2, \dots, a_n) = (x(1 - x), a_2, \dots, a_n) = \varphi([u] + [u']).$$
The left-hand side is the composition $\pi_n \circ \langle \varphi(u), \varphi(u') \rangle$. At the end of subsubsection 4.2.4, we noted that this composition is precisely the sum $\varphi([u]) + \varphi([u'])$ in the cohomotopical group law, so we are done. □

Without further ado, we can wrap up the proof of Theorem 4.2.28.

Proof of Theorem 4.2.28. A direct consequence of Theorem 4.2.7 and Proposition 4.2.36. □

4.2.5 Cohomological Interpretation of Unimodular Rows

We conclude this appendix with a discussion of some other results on unimodular rows explored from a motivic perspective. First, we discuss Fasel's cohomological interpretation of the group structure on $\mathrm{Um}_n(A)/E_n(A)$ (i.e., Theorem 4.2.10). Following Fasel, we will give an explicit computation of this group for some examples of $\mathbb{R}$-algebras. We concludediscuss some related applications of this computation to the study of stably free modules, per Morel and Fasel.

Cohomology and Unimodular Rows

In contrast with Lerbet's geometric interpretation van der Kallen's group structure, Fasel offers a cohomological perspective on the group $\mathrm{Um}_n(A)/E_n(A)$, though only for $n = \dim(A) + 1$. In this subsubsection, we draw our material from [Fas10, Sections 3, 4]. His results relate this group to the cohomology of the Milnor–Witt sheaf on $\operatorname{Spec} A$.

Theorem 4.2.37. [Fas10, Theorem 4.9] Let k be a perfect field of characteristic not equal to two, and suppose A is a smooth k-algebra of Krull dimension $d = n - 1 \geq 2$. There exists a natural isomorphism
$$\mathrm{Um}_n(A)/E_n(A) \xrightarrow{\sim} H^{n-1}(\operatorname{Spec} A, K_n^{MW}),$$
where K_*^{MW} is the Milnor–Witt sheaf.

Remark 4.2.38. Strictly speaking, Fasel proves the result above for a sheaf G^n of abelian groups instead of K_*^{MW}. However, as he remarks in the introduction to his paper, we have isomorphisms of cohomology groups $H^{n-1}(\operatorname{Spec} A, K_n^{MW}) \cong H^{n-1}(\operatorname{Spec} A, G^n)$ (using the notation above) whenever $\mathrm{char}(k) \neq 2$.

Instead of providing the full proof of the theorem above, which requires a long technical detour, we simply direct the reader to [Fas10, Section 4] for details. Instead, let's give a brief description of how the homomorphism in the theorem is constructed. Given a smooth k-algebra A, Fasel produces a map
$$\varphi : \mathrm{Um}_n(A)/E_n(A) \to H^{n-1}(\operatorname{Spec} A, K_n^{MW})$$

as follows. We have a natural map

$$\mathrm{Hom}_{k-\mathrm{Sch}}(\mathrm{Spec}\, A, \mathbb{A}^{m+1} \setminus \{0\}) \to H^m(A, K_{m+1}^{MW})$$

given by $f \mapsto f^*(\xi)$, where $\xi \in H^m(\mathbb{A}^m \setminus \{0\}, K_{m+1}^{MW})$ is some distinguished class defined in [Fas10, Section 3]. One can show that this map is $\mathbb{A}^1$-naive homotopy invariant, so that it factors through $\mathrm{Um}_n(A)/E_n(A)$ by Proposition 4.2.11. The fact that this map is a homomorphism is the content of the proof of [Fas10, Theorem 4.1]. There, Fasel notes that it suffices to verify the relation (4.21)-(4.22) in $H^m(A, K_{m+1}^{MW})$ and reduces the verification of this relation to a simple computation using the defining relations of the Milnor–Witt group $K_1^{MW}(k(t))$. The remainder of [Fas10, Section 4] is dedicated to proving that this homomorphism is an isomorphism when $n-1 = \dim(A) \geq 2$ (in fact, the cases $\dim(A) = 2, 3$ must be handled separately), and we refer the reader to that section for details of this proof, which are quite technical and perhaps beyond the scope of this appendix. As a consequence of this theorem, however, Fasel is able to explicitly compute the group $\mathrm{Um}_n(A)/E_n(A)$ in some special cases. He works over the field $\mathbb{R}$ in particular. Recall that a smooth $\mathbb{R}$-variety X is called rational if the base change to $\mathbb{C}$ is birational to $\mathbb{P}^d_{\mathbb{C}}$ for some d. As a result of some rather technical computations (we refer the reader to [Fas10, Section 5] for details), Fasel proves the following result. In particular, under some restrictive assumptions, he shows that the group $\mathrm{Um}_n(A)/E_n(A)$ is free abelian with an explicitly determined indexing set.

Theorem 4.2.39. [Fas10, Theorem 5.7, Remark 5.8] Suppose A is a smooth $\mathbb{R}$-algebra of Krull dimension $d = n-1 \geq 2$. Moreover, assume that $X = \mathrm{Spec}(A)$ is rational and has trivial canonical bundle. Then, we have an isomorphism

$$\mathrm{Um}_n(A)/E_n(A) \cong H^{n-1}(X, K_n^{MW}) \cong \mathbb{Z}^{\oplus \pi_0^c(X(\mathbb{R}))}$$

where $\pi_0^c(X(\mathbb{R}))$ denotes the set of compact connected components of the Euclidean space $X(\mathbb{R})$. In particular, if $d \geq 3$, we have an isomorphism

$$\mathrm{Um}_n(A)/E_n(A) \cong \pi^d(X(\mathbb{R})),$$

where $\pi^d(X(\mathbb{R}))$ is the dth cohomotopy group.

Applications to Stably Free Modules

In this section, we discuss some applications of the motivic perspective on unimodular rows to the study of stably free modules, following the work of Fasel [Fas10, Section 5.2]. Recall that an A-module M is called *stably free* if there exists some $r \geq 0$ such that $M \oplus A^{\oplus r}$ is a finite rank free A-module. Observe that a stably free module is necessarily projective. The study of stably free modules arises as a special case of the following question in commutative algebra.

Question 4.2.40. Let P and Q be projective modules such that $P \oplus A^{\oplus n} \cong Q \oplus A^{\oplus n}$ for some $n \geq 0$. Are P and Q isomorphic?

This question finds a partial resolution through a result of Bass and Schanuel [BS62, Theorem 2]:

Theorem 4.2.41. (Bass–Schanuel, 1962) If P is a finitely generated projective A-module of f-rank at least the Krull dimension of A, and Q is any finitely generated projective A-module satisfying $P \oplus A^{\oplus r} \cong Q \oplus A^{\oplus r}$ for some $r \geq 0$, then $P \cong Q$.

Observe that the Bass–Schanuel theorem reduces the study of stably free modules to the special case of finitely generated projective modules P such that $P \oplus A \cong A^{\oplus n}$, where $n-1$ is the Krull dimension of A. It turns out that the study of these modules can be handled with the theory of unimodular rows, and this source of motivation actually was the impetus for Fasel's study of unimodular rows in [Fas10].

Indeed, we claim that a finitely generated projective A-module P satisfying $P \oplus A \cong A^{\oplus n}$ for some $n \geq 0$ is the same data as a unimodular row $u \in \mathrm{Um}_n(A)$. Given a unimodular row $(u_1, \dots, u_n)$, we take $P \subset A^{\oplus n}$ as the kernel of the corresponding surjection $A^{\oplus n} \to A$. Note that the resulting short exact sequence

$$0 \to P \to A^{\oplus n} \to A \to 0$$

splits, as desired. Conversely, a split short exact sequence as above contains a surjective homomorphism $A^{\oplus n} \to A$, which gives us a unimodular row of length n. Of course, note that isomorphisms of the modules P correspond to equivalences of unimodular rows under the natural action of $\mathrm{GL}_n(A)$. Thus, we are actually interested in the orbit set

$$\mathrm{Um}_n(A)/\mathrm{GL}_n(A) \simeq \mathrm{Um}_n(A)/\mathrm{SL}_n(A),$$

a quotient of $\mathrm{Um}_n(A)/E_n(A)$. Given the results of Fasel that we presented the previous subsubsection, it is perhaps unsurprising that Morel[3] found a cohomological interpretation of the orbit set above. Strictly speaking, Morel's result only applies to smooth algebras A of Krull dimension at least three. Fasel [Fas10, Theorem 4.11] extends Morel's result to the dimension two case, resulting in the following theorem.

Theorem 4.2.42 (Fasel–Morel)**.** Let A be a smooth k-algebra of Krull dimension $d = n - 1 \geq 2$ over a perfect field k with characteristic not equal to two. There is a natural bijection

$$\mathrm{Um}_n(A)/\mathrm{SL}_n(A) \cong H^{n-1}(\operatorname{Spec} A, K_n^{MW})/\mathrm{SL}_n(A).$$

The proof of this theorem relies on the study of some short exact sequences related to the Chow–Witt group. In order to avoid a long technical discussion, we omit the proof and refer the reader to [Fas10, Section 4]. Similar to the case of elementary orbits of unimodular rows, Fasel uses the theorem above to explicitly compute the set $\mathrm{Um}_n(A)/\mathrm{SL}_n(A)$ in the case of some exceptional real algebras A.

Theorem 4.2.43. [Fas10, Theorem 5.9] Let A be a smooth $\mathbb{R}$-algebra of even Krull dimension d such that $\operatorname{Spec} A$ has trivial canonical bundle and is rational. The set of isomorphism classes of stably free A-modules is isomorphic to $\mathbb{Z}^{\oplus \pi_0^c(X(\mathbb{R}))}$, where $\pi_0^c(X(\mathbb{R}))$ is the set of compact connected components of $X(\mathbb{R})$.

The proof of this theorem crucially uses Theorem 4.2.39 and thus the cohomological/motivic machinery employed by Fasel. As a consequence of Theorem 4.2.43, Fasel gives a criterion for some stably free modules to be free.

Theorem 4.2.44. [Fas10, Theorem 5.10] Suppose A is an $\mathbb{R}$-algebra satisfying the assumptions of Theorem 4.2.43. A stably free A-module of rank $\dim(A)$ is free if and only if its Euler class vanishes.

Fasel's classification result above finds the following curious application.

Corollary 4.2.45. [Fas10, Corollary 5.12] The set of isomorphism classes of stably free rank $2d$ modules on the sphere S^{2d} (i.e., stably free vector bundles on S^{2d}) is isomorphic to $\mathbb{Z}$ and is generated by the tangent bundle.

[3] I am unable to find the reference cited by Fasel's paper

4.3 Howard Beck: Real Betti realization

In this section, we cover an interesting connection between motivic homotopy theory and C_2-equivariant homotopy theory, following the works of [Bac18] and [BS20]. We will assume comfort with the language and notation of equivariant (stable) homotopy theory – for a somewhat old-school but still sufficient (and way too extensive for our purposes here) reference, see [GM95].

4.3.1 Introduction

We start with the Betti realization functor, which associates to a smooth scheme over $\mathbb{C}$ a topological space given by its $\mathbb{C}$-points, endowed with the complex analytic topology:

$$\mathrm{Be} : \mathrm{Sm}_{\mathbb{C}} \to \mathrm{Top}$$
$$Z \mapsto Z(\mathbb{C})$$

We may similar form a C_2-Betti realization, that takes a smooth scheme over $\mathbb{R}$ and sends it also to its $\mathbb{C}$-points. However, here we get a C_2-action via complex conjugation, so we actually get a map into C_2-equivariant topological spaces:

$$\mathrm{Be}^{C_2} : \mathrm{Sm}_{\mathbb{R}} \to \mathrm{Top}^{C_2}$$
$$Z \mapsto \{C_2 \curvearrowright Z(\mathbb{C})\}$$

We will soon pass to the stable setting, but we should first fix notation. We formed the stable motivic homotopy category $\mathrm{SH}(\mathbb{R})$ by inverting the loop functor $\Omega^{2,1}$ corresponding to $\mathbb{P}_1$ which has the effect of inverting all loop functors $\Omega^{i,j}$ with respect to all motivic spheres $S^{i,j}$. Equivariantly, we form genuine C_2-spectra by inverting loop functors for all representation spheres S^ρ, where ρ is a finite-dimensional, continuous, real, orthogonal representation of C_2. We may similarly only invert the loop functor $\Omega^{\mathrm{triv}+\sigma}$, where triv is the one-dimensional trivial representation and σ is the sign representation. Doing so will also invert all loop functors, which are given by $\Omega^{(\mathrm{triv})i+(\sigma)j}$. The upshot is that we may view motivic spectra in $\mathrm{SH}(\mathbb{R})$ or equivariant spectra in Sp^{C_2} as being either mono- or bi-graded.

Our convention will be to take the bi-graded approach, and then have to make a choice about indexing. It is not difficult to see that we may take C_2-Betti realization on the space level of our spectra, and we get a functor:

$$\mathrm{Be}^{C_2} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}^{C_2}$$

Using our convention for bigrading of motivic spheres, C_2-Betti realization will send the motivic sphere spectra $\mathbb{S}^{i,j}$ to the genuine C_2-equivariant sphere spectra $\mathbb{S}_{C_2}^{(i-j)\mathrm{triv}+j\sigma}$. Therefore, we will choose to the $\mathrm{RO}(C_2)$ bigrading where (i,j) corresponds to $(\mathrm{triv})i + (\sigma - \mathrm{triv})j$, since we allow virtual representations such as $\sigma - 1$.

We have an inclusion map $S^{0,0} = \pm 1 \hookrightarrow \mathbb{G}_m = S^{1,1}$. By stabilizing and desuspending in both dimensions, we get a map $\rho : \mathbb{S}^{-1,-1} \to \mathbb{S}^{0,0}$. Work by [Bac18] showed that C_2-Betti realization acts like ρ-localization:

Theorem 4.3.1 ([Bac18], Proposition 31 – as interpreted by [BS20], Theorem 1.5)**.** Be^{C_2} induces an equivalence of ∞-categories $\mathrm{SH}(\mathbb{R})[\rho^{-1}] \simeq \mathrm{Sp}$:

$$\begin{tikzcd} \mathrm{SH}(\mathbb{R}) \arrow[r, "\mathrm{Be}^{C_2}"] \arrow[d, "\mathrm{loc}"'] & \mathrm{Sp} \\ \mathrm{SH}(\mathbb{R})[\rho^{-1}] \arrow[ur, "\simeq"] & \end{tikzcd}$$

Our goal is to find a similar statement that localizes $\mathrm{SH}(\mathbb{R})$ into Sp^{C_2}. This will be slightly too much to ask for, but we do get such a phenomenon when we p-complete everything and restrict ourselves to cellular motivic spectra. By X cellular – or $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})$ – we mean that we may construct X using motivic sphere spectra via cofiber sequences and filtered homotopy colimits. We may then take p-completed Betti realization:

$$\begin{tikzcd} \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^{\wedge}_p \arrow[r, "\mathrm{Be}^{C_2}"] \arrow[rr, bend right, "\widehat{\mathrm{Be}}_p^{C_2}"'] & \mathrm{Sp}^{C_2} \arrow[r, squiggly, "p\text{-completion}"] & (\mathrm{Sp}^{C_2})^{\wedge}_p \end{tikzcd}$$

The main result of [BS20] is the following:

Theorem 4.3.2 ([BS20], Theorem 1.12)**.** *p-completed cellular C_2-Betti realization – $\widehat{\mathrm{Be}}_p^{C_2}$ – is a localization functor.*

4.3.2 Isotropy Separation

In order to prove the theorem, we will consider different "parts" of X via the isotropy separation square. For a genuine G-spectrum, we have the following definitions:

Definition 4.3.3. X^h is the *homotopy completion* of X, given by:

$$X^h = \mathrm{Map}_*^G((EG)_+, X)$$

We similarly define the *geometric localization* X^Φ as:

$$X^\Phi = X \wedge \widetilde{EG}$$

where $\widetilde{EG}$ is the cofiber of the isotropy separation sequence:

$$EG_+ \to S^0 \to \widetilde{EG}$$

and the *equivariant Tate spectrum* X^t as:

$$X^t = (Y^h)^\Phi$$

Note that after taking G-fixed points, we recover the homotopy fixed points, geometric fixed points, and Tate fixed points.

These constructions fit into the *isotropy separation square*, a homotopy pullback diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^\Phi \arrow[d] \\ X^h \arrow[r] & X^t \end{tikzcd}$$

Bachmann's work has already done a lot for us, as it describes geometric localization of $\mathrm{Be}^{C_2}(X)$ as ρ-localization. In doing this, we are identifying Sp with geometrically local C_2-spectra $\mathrm{Sp}^{\Phi C_2}$, and this identification is an isomorphism given by taking C_2-geometric fixed points. If we can also describe homotopy completion of $\mathrm{Be}^{C_2}(X)$ as a localization, then we may apply some higher algebra nonsense and learn that $\mathrm{Be}^{C_2}(X)$ itself is a localization. We will not be able to achieve this, but we do get it after p-completion at any prime for cellular motivic spectra.

The correct localizing behavior for homotopy completion is given as follows. We know by [TODO] that:

$$\pi^{\mathbb{C}}_{*,*}(H\mathbb{F}_p)_{\mathbb{C}} \simeq \mathbb{F}_p[\tau]$$

where $(H\mathbb{F}_p)_K$ is the mod p motivic Eilenberg-MacLane spectrum. We may view τ as a homotopy class of maps from $\mathbb{S}^{0,-1} \to (H\mathbb{F}_p)_K \simeq (\mathbb{S}^{0,0})^\wedge_p$. In this case, we have the following localization:

Theorem 4.3.4 ([BS20], Theorem 1.11)**.** p-complete C_2-Betti realization provides a localization of p-complete cellular real motivic spectra into homotopy complete genuine C_2-spectra, by inverting τ after ρ-completion:

$$\begin{tikzcd}
\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p \arrow[r, "\widehat{\mathrm{Be}}_p^{C_2}"] \arrow[d, "\rho\text{-completion, }\tau\text{-localization}"'] & \mathrm{Sp}^{hC_2} \\
\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_{p,\rho}[\tau^{-1}] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

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